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Robust and Resilient Finite-Time Control of a Class of Discrete-Time Nonlinear Systems

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Abstract: In this paper, we address the finite-time state-feedback stabilization of a class of discrete-time nonlinear systems with conic type nonlinearities, bounded feedback control gain perturbations, and additive disturbances. Sufficient conditions for the existence of a *robust* and *resilient* linear state-feedback controller for this class of systems are derived. Then, using linear matrix inequality techniques, a solution for the controller gain is obtained. The developed controller is robust for all unknown nonlinearities lying in a hyper-sphere and all admissible disturbances. Moreover, it is resilient against any bounded perturbations that may alter the controller's gain by at most a prescribed amount. We conclude the paper with a numerical example showcasing the applicability of the main result.

Keywords: Robustness, Resilience, linear state-feedback controller, nonlinear systems, finite-time stability

1. INTRODUCTION

Finite-time stabilization via state feedback of discrete-time nonlinear systems with conic type nonlinearities and additive disturbances is presented. Generally, when addressing a stability problem, the main concern is usually the Lyapunov asymptotic stability of the system over an infinite-time interval. However, several applications necessitate that the transient states of a system remain within a bounded region over a finite-time interval. Therefore, the concept of Finite (or Short)-Time Stability, FTS, was introduced (Dorato, 1961; Weiss and Infante, 1967). A system is said to be FTS if, for an initial state within a given bound, the state of the system does not exceed a prescribed threshold over a finite-time interval. Various developments and extensions in the field of FTS have been implemented and most of which have been applied to linear systems. For instance, Dorato (1997) presents the design of a robust finite-time controller of continuous linear systems with polytopic uncertainties. Furthermore, in quite a number of his works, Amato, et al. (2001, 2005, 2006, 2010a) address the problem of FTS and finite-time control of linear systems with several variations.

However, to the best of our knowledge, the study of Finite-Time Stabilization, FTS, of nonlinear systems is rarely addressed in the literature. Yang, et al. (2009) consider nonlinear systems that are hybrid and stochastic. Other works have studied the FTS of nonlinear quadratic systems (Amato, et al., 2010b). Zhuang and Liu (2010) present the stabilization of a class of uncertain nonlinear systems with time-delay. In this work, we introduce the robust and resilient FTS, or more precisely, the finite-time state-feedback stabilization of discrete-time nonlinear systems with conic type nonlinearities, feedback gain perturbations, and additive disturbances. The significance of the controller design developed is that it requires the knowledge of a linear dynamical bound on the system's nonlinearity rather than its exact dynamics. Thus, the controller design developed is applicable to all nonlinear systems which are locally Lipschitz (Khalil, 2002).

Table 1. Notation

Notation	Definition
$x \in R^n$	An n-dimensional real vector
$\lVert x \rVert = \left(x^T x\right)^{1/2}$	Euclidean norm
$(.)^T$	Matrix transpose
$A \in R^{m\times n}$	An $m\times n$ real matrix
A^{-1}	Inverse of matrix A
$A > 0 (A < 0)$	A is a positive (negative) definite matrix
I	Identity matrix of appropriate dimensions
$\lambda_{\min}(A)(\lambda_{\max}(A))$	Minimum (Maximum) eigenvalue of the symmetric matrix A
$\mathbb{N}_0$	Set of nonnegative integers

Recall that a controller design is said to be robust if a variation in the original design parameters and uncertainties does not affect the performance intended for the closed-loop system. Hence, in this work, the controller is robust for all nonlinearities lying within the conic bound and all admissible disturbances. A linear state-feedback controller is considered and the controller gain is solved for via Linear Matrix Inequality, LMI, techniques.

Since Keel and Battacharyya's (1997) study of the non-fragility or resilience of some common controllers, several authors have developed controller designs that are first and foremost resilient (Dorato, 1998; Takabashi, et al., 2000). A controller design is said to be resilient if its performance remains unaltered despite a slight variation in the controller's structure. Therefore, conditions for the resilience of the controller developed against any perturbations which may alter the controller's gain and, consequently, destabilize the closed-loop system are derived and a bound on the controller gain perturbations is solved for.

The paper is divided into five sections. Next, we introduce the model and control problem. In section 3, we recall the basic definition of Finite-Time Boundedness, FTB. In section 4, we present the main results on finite-time control and derive the sufficient LMI conditions. In section 5, a simulation study to illustrate the use of these results is presented.

Table 1 shows the notation used in this work.

2. SYSTEM MODEL AND CONTROL

Consider the following discrete-time nonlinear system:

$$x_{k+1} = Ax_k + Bu_k + Fw_k + \Im_k \tag{1a}$$

$$w_{k+1} = \Phi w_k \tag{1b}$$

where $x_k \in W_n \subset R^n$ is the system state vector, $u_k \in W_m \subset R^m$ is the input vector, $w_k \in W_r \subset R^r$ is the disturbance state, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $F \in R^{n\times r}$, and $\Phi \in R^{r\times r}$. W_n, W_r, and W_m are open and connected sets. Note that the disturbance is one of known waveform but it does not have to be of finite-energy type. $\Im_k$ is an unknown nonlinearity whose dynamics have the following conic sector description:

$$\left\|\Im_k\right\| \le \left\|C_f x_k + D_f u_k + F_f w_k\right\| \tag{2}$$

for all time $k \in \mathbb{N}_0$, $x_k \in W_n$, $u_k \in W_m$, and $w_k \in W_r$.

Even though the added nonlinearity $\Im_k$ is assumed to be unknown, the matrices A, B, F, C_f, D_f, and F_f are assumed to be known for the system in consideration. The inequality shown in (2) implies that the unknown nonlinearity lies in an n-dimensional hypersphere whose center is the linear system $Ax_k + Bu_k + Fw_k$ and whose radius is bounded by the right hand side term of (2).

Moreover, given system (1), a linear state-feedback controller

$$u_k = Kx_k \tag{3}$$

is considered where $K \in R^{m\times n}$ is the controller gain. We first derive the conditions that guarantee the finite-time boundedness, FTB, which is an extension of the definition of FTS to systems with additive disturbances, of the resulting closed-loop system. Then, the controller gain is perturbed and the conditions are extended to obtain a resilient controller maintaining the boundedness property of the closed-loop system. But before we delve into the theory of the work presented in this paper, we recall the basic definition of FTB in the following section.

3. DEFINITIONS

Generally, a system is said to be Finite-Time Bounded, FTB, if, given a bound on the initial state of the system and the disturbance input, the state of the system does not exceed a given bound over a fixed time interval and for all admissible additive disturbances. In this work, the definitions stated in the work of Amato, et al., (2005) are adopted here and are generalized to include nonlinear systems.

Definition: (Finite-Time Boundedness)

Consider a system that is described by the following dynamics:

$$x_{k+1} = f\left(x_k, u_k, w_k\right) \tag{4}$$

where f is the vector function which is in general nonlinear.

System (4) *is said to be FTB with respect to* $\left(\alpha_x, \alpha_w, \beta, R, N\right)$ *where* $R > 0$, $\alpha_w \ge 0$, $0 \le \alpha_x \le \beta$, *and* $N \in \mathbb{N}_0$ *if*

$$\begin{cases} x_0^T R x_0 \le \alpha_x^2 \\ w_0^T w_0 \le \alpha_w^2 \end{cases} \Rightarrow x_k^T R x_k \le \beta^2 \quad \forall k = 1,\ldots,N$$

Now, we proceed to present the main results of this paper.

4. MAIN RESULTS

The problem to be solved is to find a robust and resilient state feedback controller that will render the closed-loop system (5) FTB as long as the nonlinearity is within the hypersphere defined by (2). This section will be divided into two subsections. First, we present the sufficient conditions for the existence of the robust finite-time controller. Then, we extend the obtained conditions to derive the sufficient conditions of the robust and resilient finite-time controller.

4.1 Sufficient Conditions for Robust Finite-Time Controller

Consider the closed-loop system resulting from applying controller (3) to system (1):

$$x_{k+1} = \left(A + BK\right)x_k + Fw_k + \Im_k \tag{5a}$$

$$w_{k+1} = \Phi w_k \tag{5b}$$

Lemma 1: System (5) is FTB with respect to $\left(\alpha_x, \alpha_w, \beta, R, N\right)$ if there exist positive-definite matrices $Q_1 \in R^{n\times n}$ and $Q_2 \in R^{r\times r}$, a matrix $Y \in R^{m\times n}$, and positive scalars $\gamma \ge 1$, b_1, and δ such that

$$\begin{bmatrix} \gamma Q_1 & 0 & Q_1 A^T + Y^T B^T & Q_1 C_f^T + Y^T D_f^T & 0 \\ * & \gamma Q_2 & Q_2 F^T & Q_2 F_f^T & Q_2 \Phi^T \\ * & * & Q_1 - b_1 I & 0 & 0 \\ * & * & * & b_1 I & 0 \\ * & * & * & * & Q_2 \end{bmatrix} > 0 \tag{6}$$

$$\begin{bmatrix} Q_1 - \delta R^{-1} & 0 \\ 0 & Q_2 - \delta I \end{bmatrix} > 0 \tag{7}$$

$$\delta R^{-1} \frac{\beta^2 \gamma^{-N}}{\alpha_x^2 + \alpha_w^2} - Q_1 > 0 \tag{8}$$

where * denotes the elements of the matrix that need to be added to make the matrix symmetric. The controller gain is given by $K = YQ_1^{-1}$.

Proof of Lemma 1:

Assume that $x_0^T R x_0 \le \alpha_x^2$, $w_0^T w_0 \le \alpha_w^2$, and that $x_k^T R x_k \le \beta^2$ $\forall k = 1,...,N$. Consider the energy function,

$$V_k = x_k^T P_1 x_k + w_k^T P_2 w_k \tag{9}$$

such that

$$V_{k+1} < \gamma V_k \tag{10}$$

where $P_1 > 0$, $P_2 > 0$ and $\gamma \ge 1$

Moreover, consider the inequality shown in (2) which can be rewritten as follows:

$$\Im_k^T \Im_k \le \left(A_f x_k + F_f w_k\right)^T \left(A_f x_k + F_f w_k\right) \tag{11}$$

where $A_f = C_f + D_f K$.

Substituting (9) into (10), then replacing x_{k+1} and w_{k+1} with the equations of system (5), and applying Schur's complement (Boyd, et al., 1994), the following matrix inequality is obtained.

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{12}^T & h_{22} \end{bmatrix} > \begin{bmatrix} 0 & -\Im_k^T P_1 \\ -P_1 \Im_k & 0 \end{bmatrix} \tag{12}$$

where

$h_{11} = \gamma\left(x_k^T P_1 x_k + w_k^T P_2 w_k\right) - w_k^T \Phi^T P_2 \Phi w_k$, $h_{22} = P_1$,

and $h_{12} = \left(A_c x_k + F w_k\right)^T P_1$ and $A_c = A + BK$

For any $b_1 > 0$, it is true that

$$\begin{bmatrix} b_1^{-1/2} \Im_k^T \\ b_1^{1/2} P_1 \end{bmatrix} \begin{bmatrix} b_1^{-1/2} \Im_k & b_1^{1/2} P_1 \end{bmatrix} \ge 0 \tag{13}$$

which can be rewritten as follows:

$$\begin{bmatrix} b_1^{-1} \Im_k^T \Im_k & 0 \\ 0 & b_1 P_1^2 \end{bmatrix} \ge \begin{bmatrix} 0 & -\Im_k^T P_1 \\ -P_1 \Im_k & 0 \end{bmatrix} \tag{14}$$

Using (14), the following is a sufficient condition for (12):

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{12}^T & h_{22} \end{bmatrix} > \begin{bmatrix} b_1^{-1} \Im_k^T \Im_k & 0 \\ 0 & b_1 P_1^2 \end{bmatrix} \tag{15}$$

Moreover, based on (11), (15) will still be satisfied if the following inequality holds.

$$\begin{bmatrix} h_{11} - b_1^{-1}\left(A_f x_k + F_f w_k\right)^T \left(A_f x_k + F_f w_k\right) & h_{12} \\ h_{12}^T & h_{22} - b_1 P_1^2 \end{bmatrix} > 0 \tag{16}$$

Now, apply Schur's complement to (16) to obtain

$$\begin{aligned} h_{11} - b_1^{-1}\left(A_f x_k + F_f w_k\right)^T \left(A_f x_k + F_f w_k\right) \\ - h_{12}\left(h_{22} - b_1 P_1^2\right)^{-1} h_{12}^T > 0 \end{aligned} \tag{17}$$

Substitute the expressions of h_{11}, h_{12}, and h_{22} in (17) and then rearrange the obtained expression in a quadratic format as shown in (18).

$$\begin{bmatrix} x_k^T & w_k^T \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{12}^T & d_{22} \end{bmatrix} \begin{bmatrix} x_k^T \\ w_k^T \end{bmatrix} > 0 \tag{18}$$

where $d_{11} = \gamma P_1 - b_1^{-1} A_f^T A_f - A_c^T P_1 \left(P_1 - b_1 P_1^2\right)^{-1} P_1 A_c$,

$d_{12} = -A_c^T P_1 \left(P_1 - b_1 P_1^2\right)^{-1} P_1 F - b_1^{-1} A_f^T F_f$, and

$d_{22} = \gamma P_2 - \Phi^T P_2 \Phi - b_1^{-1} F_f^T F_f - F^T P_1 \left(P_1 - b_1 P_1^2\right)^{-1} P_1 F$

Inequality (18) implies that matrix $\begin{bmatrix} d_{11} & d_{12} \\ d_{12}^T & d_{22} \end{bmatrix} > 0$, which can be rewritten as

$$\begin{aligned} \begin{bmatrix} \gamma P_1 - b_1^{-1} A_f^T A_f & -b_1^{-1} A_f^T F_f \\ -b_1^{-1} F_f^T A_f & \gamma P_2 - \Phi^T P_2 \Phi - b_1^{-1} F_f^T F_f \end{bmatrix} \\ - \begin{bmatrix} A_c^T P_1 \\ F^T P_1 \end{bmatrix} \left(P_1 - b_1 P_1^2\right)^{-1} \begin{bmatrix} P_1 A_c & P_1 F \end{bmatrix} > 0 \end{aligned} \tag{19}$$

By applying Schur's complement to (19), we obtain

$$\begin{bmatrix} \gamma P_1 - b_1^{-1} A_f^T A_f & -b_1^{-1} A_f^T F_f & A_c^T P_1 \\ -b_1^{-1} F_f^T A_f & \gamma P_2 - \Phi^T P_2 \Phi - b_1^{-1} F_f^T F_f & F^T P_1 \\ P_1 A_c & P_1 F & P_1 - b_1 P_1^2 \end{bmatrix} > 0 \tag{20}$$

Now, pre and post multiply (20) by

$$\begin{bmatrix} P_1^{-1} & 0 & 0 \\ 0 & P_2^{-1} & 0 \\ 0 & 0 & P_1^{-1} \end{bmatrix} \tag{21}$$

and, again, apply Schur's complement to the resulting matrix after rewriting it in an appropriate form. We, then, obtain the following inequality:

$$\begin{bmatrix} \gamma P_1^{-1} & 0 & P_1^{-1} A_c^T & P_1^{-1} A_f^T \\ 0 & \gamma P_2^{-1} - P_2^{-1} \Phi^T P_2 \Phi P_2^{-1} & P_2^{-1} F^T & P_2^{-1} F_f^T \\ A_c P_1^{-1} & F P_2^{-1} & P_1^{-1} - b_1 I & 0 \\ A_f P_1^{-1} & F_f P_2^{-1} & 0 & b_1 I \end{bmatrix} > 0 \tag{22}$$

Apply similar manipulations as before to (22), let $Q_1 = P_1^{-1}$, $Q_2 = P_2^{-1}$, substitute the expressions of A_f and A_c, let $Y = KQ_1$, and condition (6) is obtained.

Now, we proceed to show the derivation of conditions (7) and (8). Applying (10) iteratively and knowing that $\gamma \ge 1$, we obtain the following:

$$V_k < \gamma^N V_0 \tag{23}$$

Replace V_k and V_0 with their corresponding expressions based on (9) and since $x_k^T P_1 x_k < x_k^T P_1 x_k + w_k^T P_2 w_k$, then

$$x_k^T P_1 x_k < \gamma^N \left(x_0^T P_1 x_0 + w_0^T P_2 w_0 \right) \tag{24}$$

In (24), introduce the term $R^{1/2} R^{-1/2}$ to the left and right hand side of P_1, express the right hand side of the inequality in a quadratic format, and apply Rayleigh's inequality, which states that given $\mathrm{Q} > 0$, then $\lambda_{\min}(\mathrm{Q}) x_k^T x_k < x_k^T \mathrm{Q} x_k < \lambda_{\max}(\mathrm{Q}) x_k^T x_k$ is true. Thus, inequality (25) is obtained.

$$\begin{aligned}&\lambda_{\min}\left(R^{-1/2} P_1 R^{-1/2}\right) x_k^T R x_k < \\ &\qquad \gamma^N \lambda_{\max}\left(\begin{bmatrix} R^{-1/2} P_1 R^{-1/2} & 0 \\ 0 & P_2 \end{bmatrix}\right)\left(\alpha_x^2 + \alpha_w^2\right)\end{aligned} \tag{25}$$

In order for $x_k^T R x_k < \beta^2$ to be satisfied then

$$\lambda_{\max}\left(\begin{bmatrix} R^{-1/2} P_1 R^{-1/2} & 0 \\ 0 & P_2 \end{bmatrix}\right) < \frac{\beta^2 \gamma^{-N}}{\left(\alpha_x^2 + \alpha_w^2\right)} \lambda_{\min}\left(R^{-1/2} P_1 R^{-1/2}\right) \tag{26}$$

must hold. Let $\delta^{-1} > 0$ such that

$$\lambda_{\max}\left(\begin{bmatrix} R^{-1/2} P_1 R^{-1/2} & 0 \\ 0 & P_2 \end{bmatrix}\right) < \delta^{-1} \tag{27}$$

and

$$\delta^{-1} < \frac{\beta^2 \gamma^{-N}}{\left(\alpha_x^2 + \alpha_w^2\right)} \lambda_{\min}\left(R^{-1/2} P_1 R^{-1/2}\right) \tag{28}$$

Then, conditions (7) and (8) can be obtained from (27) and (28) respectively through basic algebraic manipulations and the proof of the proposed lemma is concluded.

4.2 Sufficient Conditions for Robust and Resilient Finite-Time Controller

In this subsection, we extend the results obtained earlier to derive sufficient conditions for the existence of a finite-time controller that is not only robust but also resilient. Consider the following system:

$$x_{k+1} = \left(A + B\tilde{K}\right) x_k + F w_k + \Im_k \tag{29a}$$

$$w_{k+1} = \Phi w_k \tag{29b}$$

where $\tilde{K} = K_r + K_\Delta$, K_r is the controller gain, and K_Δ is an additive bounded gain perturbation such that

$$K_\Delta^T K_\Delta \le c^2 I \tag{30}$$

Theorem 1: Given a gain perturbation described by (30), system (29) is FTB with respect to $(\alpha_x, \alpha_w, \beta, R, N)$ if there exist positive-definite matrices $Q_1 \in R^{n\times n}$ and $Q_2 \in R^{r\times r}$, a matrix $Y_r \in R^{m\times n}$, and positive scalars $\gamma \ge 1$, b_1, b_2, b_3, and δ such that

$$\begin{bmatrix} \gamma Q_1 & 0 & Q_1 A^T + Y_r^T B^T & Q_1 C_f^T + Y_r^T D_f^T & 0 & Q_1 \\ 0 & \gamma Q_2 & Q_2 F^T & Q_2 F_f^T & Q_2 \Phi^T & 0 \\ * & * & Q_1 - b_1 I - b_2 B B^T & -b_2 B D_f^T & 0 & 0 \\ * & * & * & b_1 I - b_2 D_f D_f^T & 0 & 0 \\ * & * & * & * & Q_2 & 0 \\ * & * & * & * & * & b_3 I \end{bmatrix} > 0 \tag{31}$$

and conditions (7) and (8) hold. The controller gain is given by $K_r = Y_r Q_1^{-1}$ and the controller gain perturbation bound is given by $c = \sqrt{b_2 b_3^{-1}}$.

Proof of Theorem 1:

Consider *Lemma 1* and replace Y by $\tilde{Y}$

where $\tilde{Y} = \tilde{K} Q_1 = Y_r + Y_\Delta$, $Y_r = K_r Q_1$, and $Y_\Delta = K_\Delta Q_1$. Then condition (6) can be rewritten as the equivalent condition

$$\begin{gathered}\begin{bmatrix} \gamma Q_1 & 0 & Q_1 A^T + Y_r^T B^T & Q_1 C_f^T + Y_r^T D_f^T & 0 \\ * & \gamma Q_2 & Q_2 F^T & Q_2 F_f^T & Q_2 \Phi^T \\ * & * & Q_1 - b_1 I & 0 & 0 \\ * & * & * & b_1 I & 0 \\ * & * & * & * & Q_2 \end{bmatrix} \\ > \begin{bmatrix} 0 & 0 & -Y_\Delta^T B^T & -Y_\Delta^T D_f^T & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}\end{gathered} \tag{32}$$

For an arbitrary $b_2 > 0$, it is true that

$$\begin{bmatrix} b_2^{-1/2} Y_\Delta^T \\ 0 \\ b_2^{1/2} B \\ b_2^{1/2} D_f \\ 0 \end{bmatrix} \begin{bmatrix} b_2^{-1/2} Y_\Delta & 0 & b_2^{1/2} B^T & b_2^{1/2} D_f^T & 0 \end{bmatrix} \ge 0 . \tag{33}$$

Inequality (33) can be expanded and rewritten as

$$\begin{gathered}\begin{bmatrix} b_2^{-1} Y_\Delta^T Y_\Delta & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & b_2 B B^T & b_2 B D_f^T & 0 \\ * & * & * & b_2 D_f D_f^T & 0 \\ * & * & * & * & 0 \end{bmatrix} \\ \ge \begin{bmatrix} 0 & 0 & -Y_\Delta^T B^T & -Y_\Delta^T D_f^T & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}\end{gathered} \tag{34}$$

Given condition (34), condition (32) will still hold if the following condition holds:

$$\begin{bmatrix} \gamma Q_1 & 0 & Q_1A^T + Y_r^TB^T & Q_1C_f^T + Y_r^TD_f^T & 0 \\ * & \gamma Q_2 & Q_2F^T & Q_2F_f^T & Q_2\Phi^T \\ * & * & Q_1 - b_1I & 0 & 0 \\ * & * & * & b_1I & 0 \\ * & * & * & * & Q_2 \end{bmatrix} > \begin{bmatrix} b_2^{-1}Y_\Delta^TY_\Delta & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & b_2BB^T & b_2BD_f^T & 0 \\ * & * & * & b_2D_fD_f^T & 0 \\ * & * & * & * & 0 \end{bmatrix} \quad (35)$$

Now, using (30) and after some algebraic manipulations, it can be easily shown that the following is a sufficient condition for (35) :

$$\begin{bmatrix} \gamma Q_1 & 0 & Q_1A^T + Y_r^TB^T & Q_1C_f^T + Y_r^TD_f^T & 0 \\ * & \gamma Q_2 & Q_2F^T & Q_2F_f^T & Q_2\Phi^T \\ * & * & Q_1 - b_1I - b_2BB^T & -b_2BD_f^T & 0 \\ * & * & * & b_1I - b_2D_fD_f^T & 0 \\ * & * & * & * & Q_2 \end{bmatrix} - \begin{bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \left(b_2c^{-2}\right)^{-1} \begin{bmatrix} Q_1 & 0 & 0 & 0 & 0 \end{bmatrix} > 0 \quad (36)$$

Finally, apply Schur's complement to condition (36) and let $b_3 = b_2c^{-2}$ to obtain condition (31). The derivation of conditions (7) and (8) is the same as that shown in the previous section. This concludes the proof of the theorem.

Given $(\alpha_x, \alpha_w, \beta, R, N)$, system (1), and the coefficient matrices in (2) and for a fixed value of γ, conditions (31), (7), and (8) constitute a set of LMIs with unknown variables Q_1, Q_2, b_1, b_2, b_3, and Y_r. Thus, a controller gain and a bound on the gain perturbation for which the LMIs are feasible can be solved for. The controller gain is given by $K_r = Y_rQ_1^{-1}$ and the gain perturbation bound is given by $c = \sqrt{b_2b_3^{-1}}$. A numerical example is provided in the following section to illustrate the applicability of the developed controller design.

5. SIMULATION STUDIES

Consider the open-loop discretized state-space model corresponding to Chua's circuit (Chua, et al., 1993).

$$\begin{cases} x_{k+1}^1 = 1 - T\alpha_C(1+b)x_k^1 + T\alpha_C x_k^2 \\ \qquad\qquad + 0.5T\alpha_c(a-b)\left(\left|x_k^1+1\right| - \left|x_k^1-1\right|\right) \\ x_{k+1}^2 = Tx_k^1 + (1-T)x_k^2 + Tx_k^3 \\ x_{k+1}^3 = -T\beta_C x_k^2 + (1-T\mu)x_k^3 \end{cases} \quad (37)$$

where x_k^i is the i^{th} state variable, $\alpha_C = 9.1$, $\beta_C = 16.5811$, $\mu = 0.138083$, $a = -1.3659$, $b = -0.7408$, and $T = 0.05s$ is the sampling period.

System (37) can be rewritten in a closed-loop form with additive disturbance input, which resembles the class of nonlinear systems considered in the design criteria.

$$x_{k+1} = Ax_k + Bu_k + Fw_k + \Im_k \quad (38)$$

where

$$A = \begin{bmatrix} 1-T\alpha_C(1+b) & T\alpha_C & 0 \\ T & 1-T & T \\ 0 & -T\beta_C & 1-T\mu \end{bmatrix}, B = T\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}, x_k = \begin{bmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \end{bmatrix},$$

$$F = T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \Im_k = \begin{bmatrix} 0.5T\alpha_c(a-b)\left(\left|x_k^1+1\right| - \left|x_k^1-1\right|\right) \\ 0 \\ 0 \end{bmatrix}$$

The dynamics of the disturbance input are described by (5b) where $\Phi = 0.9$. Since $\left|\left|x_k^1+1\right| - \left|x_k^1-1\right|\right| \le \left|2x_k^1\right|$, then $\Im_k^T\Im_k \le \left(T\alpha_c(a-b)x_k^1\right)^2$ which can be rewritten in a matrix format as in (11) where

$$C_f = \begin{bmatrix} T\alpha_c(a-b) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D_f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } F_f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Given $(\alpha_x = 1.1, \alpha_w = 0.6, R = I, N = 25)$, we start with a large value of β and then we check for the feasibility of the LMIs while varying γ^{-1} over the range $(0,1]$. If there exists a value of γ for which the LMIs are feasible, the value of β is decreased until we reach infeasibility for all values of γ^{-1}. Otherwise, the value of β is increased until feasibility is attained for at least one value of γ^{-1}.

For the system and the set of parameters considered, a solution for the controller gain is found for $\beta = 5.5$ and $\gamma = 1.0101$ where $K = [-2.7142 \;\; -4.0836 \;\; -0.1035]$ and the gain perturbation bound is $c = 0.1449$.

The closed-loop system (38) is simulated for the controller gain solution obtained and it is compared to its open-loop counterpart. The initial values for the state and disturbance inputs are $x_0 = \begin{bmatrix} 0 & -1.09 & 0 \end{bmatrix}^T$ and $w_0 = 0.5$ respectively. Figure 1 shows the norm of the state of the system with respect to time in both the closed-loop and open-loop cases. In the closed-loop case, the controller is applied for $N = 25$ steps and then removed. The norm of the state remains within the prescribed bound $\beta = 5.5$ for every time step over the interval during which the controller is applied. Figure 2 shows the state variables of the system for the two cases.

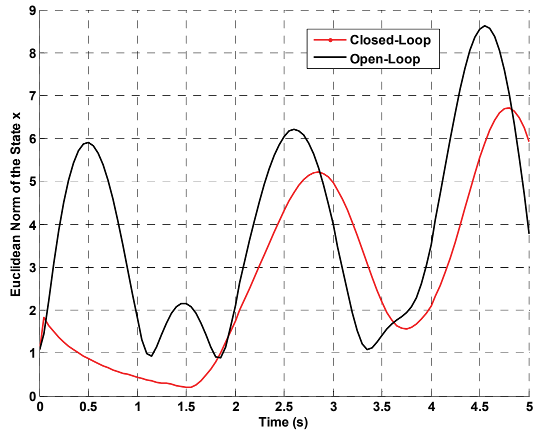

Fig. 1 Evolution of $\|x_k\|$ over time for the open-loop and closed-loop cases

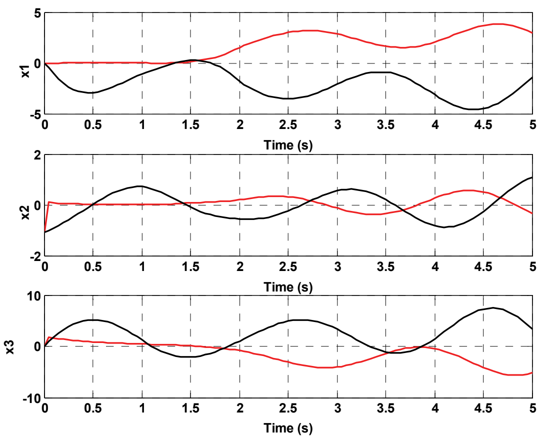

Fig. 2 System state variables for the open-loop case (black) and closed-loop case (red)

Moreover, in order to show the resilience of the controller obtained, the controller gain used in the previous simulation is perturbed with a perturbation lying within the calculated upper bound. The closed-loop system is simulated again and it is observed that the system maintains its finite-time boundedness property despite the perturbation in the controller gain.

6. CONCLUSION

In this paper, we have presented sufficient conditions for the finite-time state feedback stabilization of a class of discrete-time nonlinear systems with conic type nonlinearities, bounded feedback gain perturbations, and additive disturbances. The conditions obtained are transformed into an LMI-based feasibility problem to find a solution of the controller gain. A numerical example demonstrating a possible application of the proposed control design is presented.

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