9-1-2011

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# Robust and Resilient State Dependent Control of Discrete-Time Nonlinear Systems with General Performance Criteria

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**Abstract:** A novel state dependent control approach for discrete-time nonlinear systems with general performance criteria is presented. This controller is robust for unstructured model uncertainties, resilient against bounded feedback control gain perturbations in achieving optimality for general performance criteria to secure quadratic optimality with inherent asymptotic stability property together with quadratic dissipative type of disturbance reduction. For the system model, unstructured uncertainty description is assumed, which incorporates commonly used types of uncertainties, such as norm-bounded and positive real uncertainties as special cases. By solving a state dependent linear matrix inequality at each time step, sufficient condition for the control solution can be found which satisfies the general performance criteria. The results of this paper unify existing results on nonlinear quadratic regulator, $H_\infty$ and positive real control to provide a novel robust control design. The effectiveness of the proposed technique is demonstrated by simulation of the control of inverted pendulum.

## 1. INTRODUCTION

Optimal control of nonlinear systems is traditionally characterized in terms of Hamilton Jacobi Equations (HJEs). The solution of the HJEs provides the necessary and sufficient optimal control condition for nonlinear systems. Furthermore, when the controlled system is linear time invariant and the performance index is Linear Quadratic Regulator (LQR), the HJEs reduced to Algebraic Riccati Equations (AREs). As for $H_\infty$ nonlinear control problem, the optimal control solution is equivalent to solving the corresponding Hamilton Jacobi Inequalities (HJIs). However, HJEs and HJIs, which are first order partial differential equations and inequalities, cannot be solved for more than a few state variables. In the past few years, it has been shown that the problems of quadratic regulation and $H_\infty$ nonlinear control can be effectively solved by state dependent Riccati equation (SDRE) and nonlinear matrix inequality (NLMI) techniques (Huang and Lu 1996). The state dependent LMI control of nonlinear systems, as pointed out in (Wang and Yaz 2009, Wang and Yaz 2010), synthesizes a controller to achieve mixed nonlinear quadratic regulator (NLQR) and $H_\infty$ control.

Dissipative control for linear systems has also received considerable attention over the past two decades. The concept of dissipative system was first introduced in by Willems (1972a, b), and further generalized by Hill and Moylan (1976, 1980), playing an important role in systems, circuits and controls. The theory of dissipative systems generalizes the basic tools including the passivity theorem, bounded real lemma, Kalman-Yakubovich lemma and circle criterion. Dissipativity performance includes $H_\infty$ performance, passivity, positive realness, sector bounded constraint as special case. Research addressing the problems of $H_\infty$ and positive real control systems can be found in (Safonov *et al.* 1987, Doyle *et al.* 1989, Haddad and Bernstein 1991, Sun *et al.* 1994). Control of uncertain linear systems with $L_2$-bounded structured uncertainty satisfying $H_\infty$ and passivity criteria have been tackled in (Peterson 1987, Khargonekar *et al.* 1990). More recent development involving the quadratic dissipative control for linear systems problem has been tackled in (Xie *et al.* 1998, Tan *et al.* 2000).

In this paper, we further consider the problem of optimal, robust and resilient linear matrix inequality control of discrete-time nonlinear systems with general performance criteria. The controller is robust for model uncertainties and resilient for gain perturbations. As for the uncertain nonlinear systems, we consider a general form of $l_2$-bounded uncertainty description, without any standard structure, incorporating commonly used types of uncertainty, such as norm-bounded and positive real uncertainties as special cases. The purpose behind this novel approach is to convert a nonlinear system control problem into a convex optimization problem which is solved by state dependent LMI. The recent development in convex optimization provides very efficient algorithms for solving LMIs. If a solution can be expressed in a LMI form, then there exist optimization algorithms providing efficient global numerical solutions (Boyd 1994). Therefore if the LMI is feasible, then LMI control technique provides asymptotically stable solutions satisfying various general performance criteria. We further propose to employ general performance criteria to design the controller guaranteeing the quadratic sub-optimality with inherent stability property in combination with dissipativity type of disturbance attenuation. The general performance criteria is a generalization of the nonlinear quadratic regulator, $H_\infty$, positive realness and sector bounded constraint. The results of the paper unify existing results on nonlinear quadratic regulator, $H_\infty$ and positive real control and provide a novel robust control design. The paper is organized as follows: in section 2, we will present the general performance criteria including the performance of nonlinear quadratic regulator, $H_\infty$, positive realness and sector bounded constraint. Section 3 presents state dependent LMI based control for nonlinear systems achieving general performance criteria. Finally, inverted pendulum on a cart system is used for to examining the effectiveness and robustness of the new approach in section 4.

## 2. SYSTEM MODEL AND GENERAL PERFORMANCE CRITERIA

The following notation is used in this work: $\Re_+$ stands for the set of non-negative real numbers, $\Re^n$ stands for the n-dimensional Euclidean space. $x_k \in \Re^n$ denotes n-dimensional real vector with norm $\|x_k\| = (x_k^T x_k)^{1/2}$ where $(\cdot)^T$ indicates transpose. $\Re^{n\times m}$ is the set of $n\times m$ real matrices. $I_n$ is the $n\times n$ identity matrix. $A \ge 0$ for a symmetric matrix denotes a positive semi-definite matrix. $l_2$ is the space of finite dimensional vectors with finite energy: $\sum_{k=0}^{\infty}\|x_k\|^2 < \infty$. The inner product on $\Re^n$ is defined by $\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i$.

Consider the nonlinear dynamical system and performance output equation as following:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, w_k) \\ &= \left(A(x_k)+\Delta_A(x_k)\right)x_k + \left(B(x_k)+\Delta_B(x_k)\right)\cdot u_k \\ &\quad + \left(E(x_k)+\Delta_E(x_k)\right)w_k \\ &= \left(A_k+\Delta_A\right)x_k + \left(B_k+\Delta_B\right)u_k + \left(E_k+\Delta_E\right)w_k \end{aligned} \tag{1}$$

$$z_k = g(x_k, u_k) = C_k\cdot x_k + D_k\cdot w_k \tag{2}$$

where

| | |
|---|---|
| $x_k \in \Re^n$ | state of the dynamical system |
| $u_k \in \Re^m$ | applied input |
| $w_k \in \Re^p$ | $l_2$ type of disturbance |
| $z_k \in \Re^r$ | performance output |
| $f, g$ | nonlinear vector functions |

$A_k \in \Re^{n\times n}$, $B_k \in \Re^{n\times m}$, $E_k \in \Re^{n\times p}$, $C_k \in \Re^{r\times n}$, $D_k \in \Re^{r\times p}$
state dependent coefficient matrices

$\Delta_A \in \Re^{n\times n}$, $\Delta_B \in \Re^{n\times m}$, $\Delta_E \in \Re^{n\times p}$
state dependent uncertainty matrices

It is assumed that the full state is available for feedback and the state feedback control input is given by

$$u_k = \left(K(x_k)+\Delta_K(x_k)\right)x_k = \left(K_k+\Delta_K\right)x_k \tag{3}$$

where there is additive (possibly state dependent) perturbation on the feedback gain. Introducing the quadratic energy supply function $E$ associated with the system equations, defined by (Hill and Moylan 1976, 1980) as:

$$E(z_k, w_k) = \langle z_k, Qz_k\rangle + 2\langle z_k, Sw_k\rangle + \langle w_k, Rw_k\rangle \tag{4}$$

where $Q \in \Re^{r\times r}$, $S \in \Re^{r\times p}$, $R \in \Re^{p\times p}$ are the chosen weighing matrices. Next, from the definition of dissipativity, we have:

**Definition 1:** Given matrices $Q \in \Re^{r\times r}$, $S \in \Re^{r\times p}$, $R \in \Re^{p\times p}$ with $Q, R$ symmetric, the system (1), (2) with energy function (4) is said to be ($Q, S, R$)-dissipative if for some real function $\beta(\cdot)$ with $\beta(0) = 0$,

$$E(z_k, w_k) + \beta(x_0) \ge 0, \forall w \in l_2, \forall k \ge 0 \tag{5}$$

Furthermore, if for some scalar $\alpha > 0$,

$$E(z_k, w_k) + \beta(x_0) \ge \alpha\langle w_k, w_k\rangle, \forall w \in l_2, \forall k \ge 0 \tag{6}$$

The system (1) (2) is said to be strictly ($Q, S, R$)-dissipative.

**Theorem 1:** Consider the quadratic function $V_k = x_k^T P_k x_k > 0$, matrices $Q \in \Re^{r\times r}$, $S \in \Re^{r\times p}$, $R \in \Re^{p\times p}$ with $Q, R$ symmetric, $M \in \Re^{n\times n}, M > 0$, $N \in \Re^{m\times m}, N > 0$ with $M, N$ symmetric, the system (1) (2) control will achieve mixed NLQR and dissipative performance if the following condition holds:

$$V_{k+1} - V_k + x_k^T M x_k + u_k^T N u_k - \left(z_k^T Q z_k + 2 z_k^T S w_k + w_k^T R w_k\right) < 0, \\ \forall k \ge 0 \tag{7}$$

**Proof:**
Note that upon summation over $k$, we have

$$\begin{aligned} &\sum_{i=0}^{N-1}\left[z_k^T Q z_k + 2z_k^T S w_k + w_k^T R w_k\right] > \\ &\sum_{i=0}^{N-1}\left[x_k^T M x_k + u_k^T N u_k\right] + V_N - V_0 \end{aligned} \tag{8}$$

Let $\beta(x_0) = V_0$, $V_k(x) = x_k^T P_k x_k$, $V_N \ge 0$, (8) implies

$$\sum_{i=0}^{N-1}\left(z_k^T Q z_k + 2z_k^T S w_k + w_k^T R w_k\right) + \beta(x_0) > 0 \tag{9}$$

which is the condition for ($Q, S, R$)-dissipativity.

**Remark 1:** By adding the terms $x_k^T M x_k + u_k^T N u_k$, we include the nonlinear quadratic regulator control performance into the original ($Q, S, R$)-dissipative criteria.

**Remark 2:** Notice that both $H_\infty$ and passivity are special cases of ($Q, S, R$)-dissipativity.

The special cases are summarized as follows:

Case 1: $Q = -I, S = 0, R = \gamma^2 I$, the strict ($Q, S, R$)-dissipativity reduces $H_\infty$ Design (Doyle *et al.* 1989). The overall control design satisfies mixed NLQR-$H_\infty$ performance.

Case 2: $Q = 0, S = I, R = 0$, the strict ($Q, S, R$)-dissipativity reduces to strict positive realness (Sun *et al.* 1994). The overall control design satisfies mixed NLQR-strict positive realness performance.

Case 3: $Q=-\theta I, S=(1-\theta)I, R=\theta\gamma^2 I$ , the strict ($Q,S,R$)-dissipativity reduces to mixed $H_\infty$ and positive real performance design, when $\theta\in(0,1)$. The overall control design satisfies mixed NLQR-$H_\infty$-positive real performance.

Case 4: $Q=-I, S=\frac{1}{2}(K_1+K_2)^T, R=-\frac{1}{2}\left(K_1^TK_2+K_2^TK_1\right)^T$, where $K_1$ and $K_2$ are constant matrices of appropriate dimensions, the strict ($Q,S,R$)-dissipativity reduces to a sector-bounded constraint (Gupta and Joshi 1994). The overall control design satisfies mixed NLQR-sector bounded constraint performance.

Before introducing the main result of the paper, the following model of uncertainties is introduced.

**Assumption 1:** The following general form of $l_2$-bounded unstructured uncertainties is considered:

$$\begin{cases}\Delta_A\Delta_A^T\le\gamma_A I\\ \Delta_B\Delta_B^T\le\gamma_B I\\ \Delta_E\Delta_E^T\le\gamma_E I\\ \Delta_K\Delta_K^T\le\gamma_K I\end{cases}\tag{10}$$

for $\forall x_k\in\Re^n$ and $k\ge0$.

## 3. STATE DEPENDENT LINEAR MATRIX INEQUALITY CONTROL

**Lemma 1:** $AB^T+BA^T\le\alpha AA^T+\alpha^{-1}BB^T$ (11)

This can be proven easily by considering

$$\left(\alpha^{1/2}A-\alpha^{-1/2}B\right)\left(\alpha^{1/2}A-\alpha^{-1/2}B\right)^T\ge0\tag{12}$$

Also, by choosing *A, B* matrices as $A=\begin{bmatrix}a^T\\0\end{bmatrix}, B=\begin{bmatrix}0\\b^T\end{bmatrix}$, we have

$$\begin{bmatrix}0 & a^Tb\\ b^Ta & 0\end{bmatrix}\le\begin{bmatrix}\zeta a^Ta & 0\\ 0 & \zeta^{-1}b^Tb\end{bmatrix}\tag{13}$$

The following theorem summarizes the main results of the paper:

**Theorem 2:** Given the system equation (1), performance output (2) and control input (3), if there exist matrices $X_k=P_k^{-1}>0$ and $Y_k$ for all $k>0$, such that the following state dependent linear matrix inequality holds:

If $Q<0$,

$$\begin{bmatrix}X_k & \Upsilon_{12} & \Upsilon_{13} & Y_k^T & \Upsilon_{15} & X_k\\ * & \Upsilon_{22} & E^T & 0 & 0 & 0\\ * & * & \Upsilon_{33} & 0 & 0 & 0\\ * & * & * & \Upsilon_{44} & 0 & 0\\ * & * & * & * & \Upsilon_{55} & 0\\ * & * & * & * & * & \Upsilon_{66}\end{bmatrix}>0\tag{14}$$

If $Q=0$,

$$\begin{bmatrix}X_k & \Upsilon_{12} & \Upsilon_{13} & Y_k^T & X_k\\ * & \Upsilon_{22} & E^T & 0 & 0\\ * & * & \Upsilon_{33} & 0 & 0\\ * & * & * & \Upsilon_{44} & 0\\ * & * & * & * & \Upsilon_{66}\end{bmatrix}>0\tag{15}$$

where

$$\begin{aligned}\Upsilon_{12}&=X_kC_k^TQD_k+X_kC_k^TS\\ \Upsilon_{13}&=X_kA_k^T+Y_k^TB_k^T\\ \Upsilon_{15}&=X_kC_k^T\\ \Upsilon_{22}&=D_k^TS+S^TD_k+D_k^TQD_k+R+I\\ \Upsilon_{33}&=X_k+(2\gamma_B+\gamma_E+1)I+B_kB_k^T\\ \Upsilon_{44}&=N^{-1}\\ \Upsilon_{55}&=-Q^{-1}\\ \Upsilon_{66}&=M^{-1}-(\gamma_A+2\gamma_K)^{-1}I\end{aligned}\tag{16}$$

Then the performance index inequality (7) is satisfied. The nonlinear feedback control gain is given by

$$K_k=Y_k\cdot P_k\tag{17}$$

**Proof:**

In the proof below, the time and state argument will be dropped for notational simplicity. By applying system and performance output equations (1), (2), and state feedback input equation (3), the performance index can be formed as follows:

$$\begin{aligned}&\left\{x_k^T\left[A_k+\Delta_A+(B_k+\Delta_B)(K_k+\Delta_K)\right]^T+w_k^T\left[E_k+\Delta_E\right]^T\right\}\cdot P_{k+1}\cdot\\ &\left\{\left[A_k+\Delta_A+(B_k+\Delta_B)(K_k+\Delta_K)\right]x_k+\left[E_k+\Delta_E\right]w_k\right\}+\\ &-x_k^TP_kx_k+x_k^TMx_k+x_k^T\left[K_k+\Delta_K\right]^TN\left[K_k+\Delta_K\right]x_k\\ &-\left[C_kx_k+Dw_k\right]^TQ\left[C_kx_k+Dw_k\right]\\ &-2\left[C_kx_k+D_kw_k\right]^TSw_k-w_k^TRw_k<0\end{aligned}\tag{18}$$

By grouping the terms, we have

$$\begin{bmatrix}x_k^T & w_k^T\end{bmatrix}\Psi\begin{bmatrix}x_k & w_k\end{bmatrix}^T=\begin{bmatrix}x_k^T & w_k^T\end{bmatrix}\begin{bmatrix}\Psi_{11} & \Psi_{12}\\ * & \Psi_{22}\end{bmatrix}\begin{bmatrix}x_k\\ w_k\end{bmatrix}<0\tag{19}$$

where

$$\begin{aligned}\Psi_{11}&=\left\{(A_k+\Delta_A)+(B_k+\Delta_B)(K_k+\Delta_K)\right\}^T\cdot P_{k+1}\cdot\\ &\left\{(A_k+\Delta_A)+(B_k+\Delta_B)(K_k+\Delta_K)\right\}+M-P_k+\\ &\left[K_k+\Delta_K\right]^TN\left[K_k+\Delta_K\right]-C_k^TQC_k\\ \Psi_{12}&=\left\{(A_k+\Delta_A)+(B_k+\Delta_B)(K_k+\Delta_K)\right\}^TP_{k+1}\left[E_k+\Delta_E\right]\\ &-C_k^TQD_k-C_k^TS\\ \Psi_{22}&=\left[E_k+\Delta_E\right]^TP_{k+1}\left[E_k+\Delta_E\right]-D_k^TQD_k\\ &-\left(D_k^TS+S^TD_k\right)-R\end{aligned}\tag{20}$$

Denote the following terms:

$$\mathrm{A}=\left(A_k+\Delta_A\right)+\left(B_k+\Delta_B\right)\left(K_k+\Delta_K\right)$$
$$\mathrm{K}=K_k+\Delta_K \tag{21}$$
$$\mathrm{E}=E_k+\Delta_E$$

Then (19) is equivalent to

$$\begin{bmatrix} \mathrm{A}^T P_{k+1}\mathrm{A}-P_k & \mathrm{A}^T P_{k+1}\mathrm{E} \\ * & \mathrm{E}^T P_{k+1}\mathrm{E} \end{bmatrix}+$$
$$\begin{bmatrix} M+\mathrm{K}^T N\mathrm{K}-C_k^T Q C_k & -C_k^T Q D_k - C_k^T S \\ * & -D_k^T S - S^T D_k - D_k^T Q D_k - R \end{bmatrix}<0 \tag{22}$$

By adding and subtracting $P_k$ term, we have

$$\begin{bmatrix} \mathrm{A}^T \\ \mathrm{E}^T \end{bmatrix}\left(P_{k+1}-P_k+P_k\right)\left[\mathrm{A} \quad \mathrm{E}\right]-\begin{bmatrix} I \\ 0 \end{bmatrix}P_k\left[I \quad 0\right]$$
$$+\begin{bmatrix} M+\mathrm{K}^T N\mathrm{K}-C_k^T Q C_k & -C_k^T Q D_k - C_k^T S \\ * & -D_k^T S - S^T D_k - D_k^T Q D_k - R \end{bmatrix}<0 \tag{23}$$

Imposing the property $P_{k+1}\le P_k$, the sufficient condition for (23) is given as follows:

$$\begin{bmatrix} \mathrm{A}^T \\ \mathrm{E}^T \end{bmatrix}P_k\left[\mathrm{A} \quad \mathrm{E}\right]-\begin{bmatrix} I \\ 0 \end{bmatrix}P_k\left[I \quad 0\right]$$
$$+\begin{bmatrix} M+\mathrm{K}^T N\mathrm{K}-C_k^T Q C_k & -C_k^T Q D_k - C_k^T S \\ * & -D_k^T S - S^T D_k - D_k^T Q D_k - R \end{bmatrix}<0 \tag{24}$$

Equivalently, we obtain

$$\begin{bmatrix} P_k - M-\mathrm{K}^T N\mathrm{K}+C_k^T Q C_k & C_k^T Q D_k + C_k^T S \\ * & D_k^T S + S^T D_k + D_k^T Q D_k + R \end{bmatrix}$$
$$-\begin{bmatrix} \mathrm{A}^T \\ \mathrm{E}^T \end{bmatrix}P_k\left[\mathrm{A} \quad \mathrm{E}\right]>0 \tag{25}$$

Applying the Schur complement (Boyd, et al, 1994), we have

$$\begin{bmatrix} \begin{pmatrix} P_k - M-\mathrm{K}^T N\mathrm{K} \\ +C_k^T Q C_k \end{pmatrix} & C_k^T Q D_k + C_k^T S & \mathrm{A}^T \\ * & \begin{pmatrix} D_k^T S + S^T D_k \\ +D_k^T Q D_k + R \end{pmatrix} & \mathrm{E}^T \\ * & * & P_k^{-1} \end{bmatrix}>0 \tag{26}$$

Taking $Q<0$ (the case where $Q=0$ will be considered later), we apply Schur complement twice to (26), then

$$\begin{bmatrix} P_k - M & C_k^T Q D_k + C_k^T S & \mathrm{A}^T & \mathrm{K}^T & C_k^T \\ * & D_k^T S + S^T D_k + D_k^T Q D_k + R & \mathrm{E}^T & 0 & 0 \\ * & * & P_k^{-1} & 0 & 0 \\ * & * & * & N^{-1} & 0 \\ * & * & * & * & -Q^{-1} \end{bmatrix}>0 \tag{27}$$

Let $X_k = P_k^{-1}$, by pre- and post-multiplying the above matrix inequality by $diag\left\{X_k \quad I \quad I \quad I \quad I\right\}$, we have

$$\begin{bmatrix} X_k - X_k M X_k & X_k C^T Q D_k + X_k C^T S & X_k \mathrm{A}^T & X_k \mathrm{K}^T & X_k C_k^T \\ * & \begin{pmatrix} D_k^T S + S^T D_k + \\ D_k^T Q D_k + R \end{pmatrix} & \mathrm{E}^T & 0 & 0 \\ * & * & X_k & 0 & 0 \\ * & * & * & N^{-1} & 0 \\ * & * & * & * & -Q^{-1} \end{bmatrix}>0 \tag{28}$$

By applying the Schur complement again, we have

$$\begin{bmatrix} X_k & X_k C_k^T Q D_k + X_k C_k^T S & X_k \mathrm{A}^T & X_k \mathrm{K}^T & X_k C^T & X_k \\ * & \begin{pmatrix} D_k^T S + S^T D_k + \\ D_k^T Q D_k + R \end{pmatrix} & \mathrm{E}^T & 0 & 0 & 0 \\ * & * & X_k & 0 & 0 & 0 \\ * & * & * & N^{-1} & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 \\ * & * & * & * & * & M^{-1} \end{bmatrix}>0 \tag{29}$$

Denote $Y_k = K_k X_k$ (30)

By replacing the variables with (21) and applying Lemma 1 and Assumption 1, the sufficient condition for inequality (29) is given below

$$\begin{bmatrix} X_k & \begin{pmatrix} X_k C_k^T Q D_k + \\ X_k C_k^T S \end{pmatrix} & \begin{pmatrix} X_k A_k^T + \\ Y_k^T B_k^T \end{pmatrix} & X_k K_k^T & X_k C_k^T & X_k \\ * & \begin{pmatrix} D_k^T S + S^T D_k + \\ D_k^T Q D_k + R \end{pmatrix} & E_k^T & 0 & 0 & 0 \\ * & * & X_k & 0 & 0 & 0 \\ * & * & * & N^{-1} & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 \\ * & * & * & * & * & M^{-1} \end{bmatrix}+$$
$$\begin{bmatrix} \Omega_{11} & 0 & 0 & 0 & 0 & 0 \\ * & \Omega_{22} & 0 & 0 & 0 & 0 \\ * & * & \Omega_{33} & 0 & 0 & 0 \\ * & * & * & \alpha_1^{-1} I & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}>0 \tag{31}$$

where

$$\Omega_{11}=\left(\alpha_1\gamma_K+\alpha_1\gamma_A+\alpha_4\gamma_K\right)X_k X_k+\alpha_2 Y_k^T Y_k$$
$$\Omega_{22}=\alpha_2 I$$
$$\Omega_{33}=\alpha_1^{-1}\gamma_B I+\alpha_2^{-1}\left(\gamma_B+\gamma_E\right)I+\alpha_3^{-1} I+\alpha_4^{-1} B_k B_k^T \tag{32}$$

Finally, by applying Schur complement twice, we have

$$\begin{bmatrix} X_k & \Upsilon_{12} & \Upsilon_{13} & Y_k^T & \Upsilon_{15} & X_k \\ * & \Upsilon_{22} & E^T & 0 & 0 & 0 \\ * & * & \Upsilon_{33} & 0 & 0 & 0 \\ * & * & * & \Upsilon_{44} & 0 & 0 \\ * & * & * & * & \Upsilon_{55} & 0 \\ * & * & * & * & * & \Upsilon_{66} \end{bmatrix}>0 \tag{33}$$

where

$$
\begin{aligned}
\Upsilon_{12} &= X_k C_k^T Q D_k + X_k C_k^T S \\
\Upsilon_{13} &= X_k A_k^T + Y_k^T B_k^T \\
\Upsilon_{15} &= X_k C_k^T \\
\Upsilon_{22} &= D_k^T S + S^T D_k + D_k^T Q D_k + R + \alpha_2 I \\
\Upsilon_{33} &= X_k + \alpha_1^{-1}\gamma_B I + \alpha_2^{-1}(\gamma_B + \gamma_E) I + \alpha_3^{-1} I + \alpha_4^{-1} B_k B_k^T \\
\Upsilon_{44} &= N^{-1} + (\alpha_1^{-1} - \alpha_2^{-1}) I \\
\Upsilon_{55} &= -Q^{-1} \\
\Upsilon_{66} &= M^{-1} - (\alpha_1\gamma_A + \alpha_2\gamma_K + \alpha_4\gamma_K)^{-1} I
\end{aligned}
\tag{34}
$$

Notice that (33) is derived under the condition that $Q < 0$. However, when strict positive realness criteria are chosen for control design, then condition $Q = 0$ must be satisfied. In this case, matrix inequality (33) should be replaced by

$$
\begin{bmatrix}
X_k & \Upsilon_{12} & \Upsilon_{13} & Y_k^T & X_k \\
* & \Upsilon_{22} & E^T & 0 & 0 \\
* & * & \Upsilon_{33} & 0 & 0 \\
* & * & * & \Upsilon_{44} & 0 \\
* & * & * & * & \Upsilon_{66}
\end{bmatrix} > 0 \tag{35}
$$

Since positive constants $\alpha_1,\ldots,\alpha_5$ are arbitrary, choosing all of them as 1, we obtain (14) and (15). Therefore, if LMI (14) or (15) holds under different conditions on $Q$, the inequality (7) is satisfied. This concludes the proof.

**Remark 3:** At this point, it is to be noted that other choices of constants $\alpha_1,\ldots,\alpha_4$ are possible and can be tried if the value 1 for all these constants does not work.

## 4. INVERTED PENDULUM CONTROL WITH GENERAL PERFORMANCE CRITERIA

The inverted pendulum on a cart problem is a classical control problem used widely as a benchmark for testing control algorithms. The control objective is to find the state dependent LMI control to set cart position $x$, velocity of the cart $\dot{x}$, angle of the beam $\theta$ and angular velocity $\dot{\theta}$ all to zero while satisfying some chosen optimality criteria. A model of the inverted pendulum problem dynamical equation is given as follows:

$$
\begin{cases}
(M+m)\ddot{x} + b\dot{x} + mL\ddot{\theta}\cos(\theta) - mL\dot{\theta}^2\sin(\theta) = F \\
(I + mL^2)\ddot{\theta} + mgL\sin(\theta) + mL\ddot{x}\cos(\theta) = 0
\end{cases}
\tag{36}
$$

The following system parameters are assumed

$$
M = 0.5kg, m = 0.5kg, b = 0.1N \cdot \frac{\sec}{m}, L = 0.3m, I = 0.06kg \cdot m^2
$$

Sampling Time: $T = 0.01\sec$

The control performance results are shown in the Fig.1-5, in comparison with the traditional Linear Quadratic Regulator (LQR) technique based on linearization. From these figures, we find that the novel state dependent LMI control has better performance compared with the traditional LQR technique based on linearization. Especially, Fig.1 and Fig.2 show that the traditional LQR technique loses control of the position and velocity of the cart, respectively. It should also be noted that predominant NLQR and predominant $H_\infty$ control techniques lead to faster response times than the NLQR-passivity technique.

## 5. CONCLUSION

This paper has addressed nonlinear control system design with general nonlinear quadratic regulator and quadratic dissipative criteria to achieve asymptotic stability, quadratic optimality and strict quadratic dissipativeness. For systems with unstructured but bounded uncertainty, the linear matrix inequality based sufficient conditions are derived for the control solution. These results unify the existing results on State Dependent Riccati Equation control, robust $H_\infty$, positive real control. The relative weighting matrices of these criteria can be achieved by choosing different coefficient matrices. The optimal control can be obtained by solving LMI at each time step. The inverted pendulum on a cart is used as an example to demonstrate the effectiveness and robustness of the proposed method. The simulation studies show that the proposed method provides a satisfactory alternative to the existing nonlinear control approaches.

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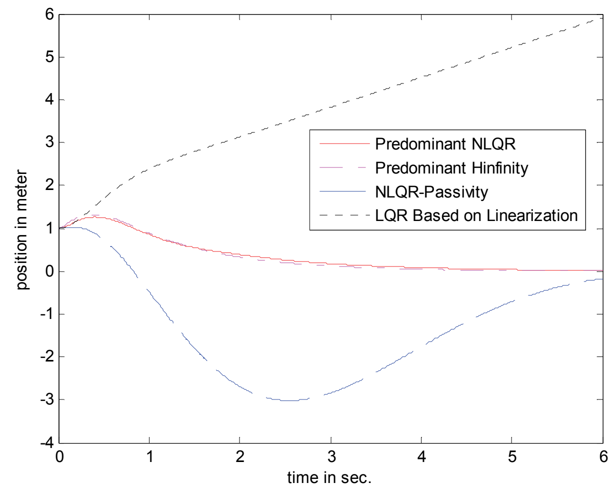


Fig.1. Position trajectory of the inverted pendulum

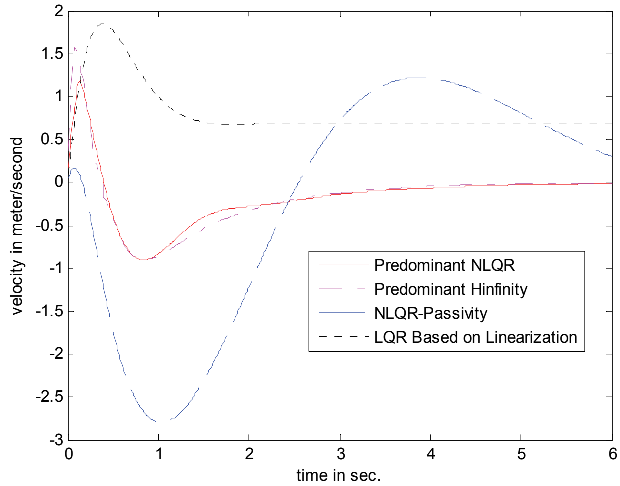


Fig.2. Velocity trajectory of the inverted pendulum

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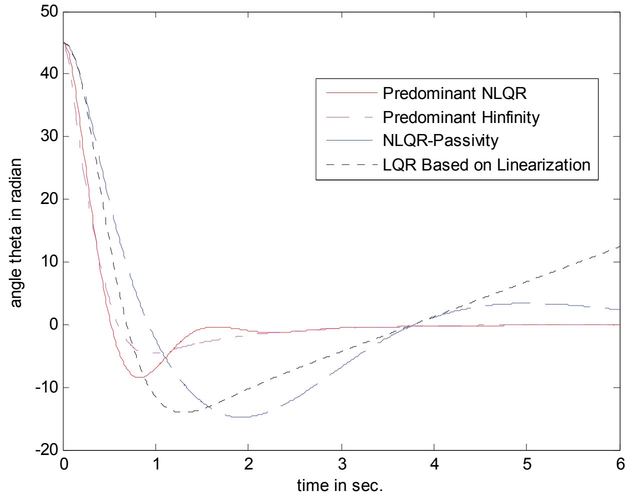


Fig.3. Angle “theta” trajectory of the inverted pendulum

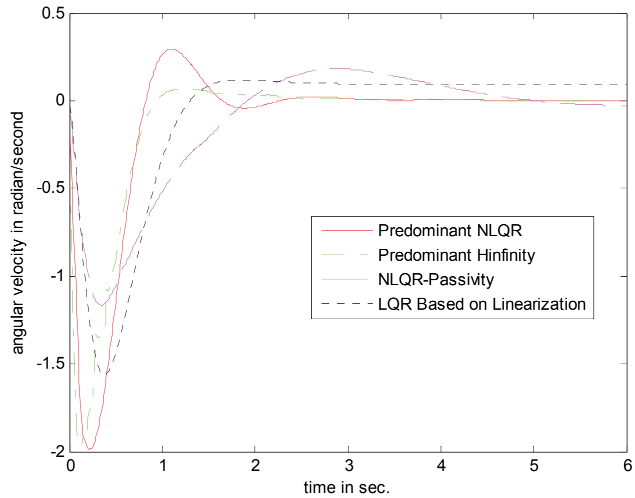


Fig.4. Angular velocity trajectory of the inverted pendulum

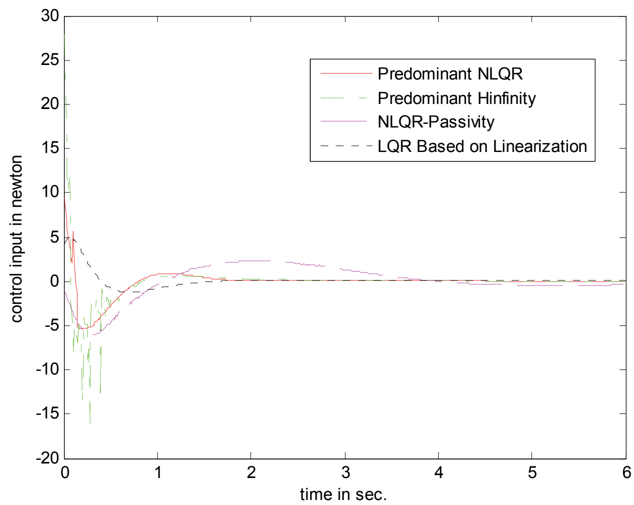


Fig.5. Control input

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