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A Simple Model for the In-Plane Rotational Response of a Disk Resonator in Liquid: Resonant Frequency, Quality Factor, and Optimal Geometry

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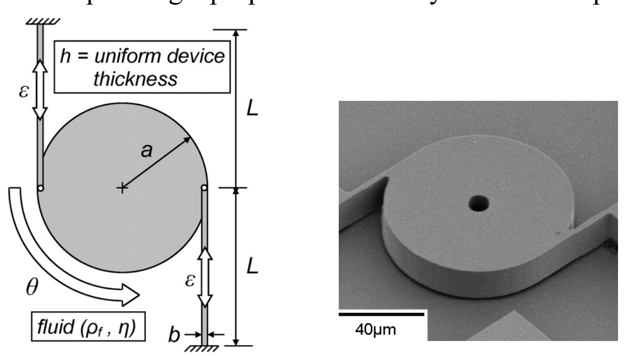
Introduction and Motivation: Micromechanical resonators have a wide variety of applications, including biological and chemical sensing, which often require the device to operate in a viscous liquid environment in which energy dissipation and fluid inertia play a key role in the device's performance. Therefore, many investigations of MEMS resonators in liquids have focused on how to improve the quality factor (Q) by reducing energy losses to the surroundings through the use of new device geometries or unconventional modes. In the present study we explore potential advantages of an "all-shear interaction device" (ASID), consisting of a microdisk supported by two tangentially oriented microcantilevers, or "legs" (Fig. 1a). When the device is excited by imparting a harmonic axial strain ε to the legs (e.g., via electrothermal actuation), an in-plane rotational oscillation of the disk will ensue. This mode of vibration will engage the fluid only through shear-type interaction, as the motions of all surfaces of the ASID will have no normal component – hence, our use of the term "all-shear interaction device." This type of interaction is expected to minimize fluid motion and, thus, viscous dissipation and fluid inertial forces, thereby yielding higher values of resonant frequency and Q in liquids. Another advantage of the ASID in sensors applications is the potentially large functionalized surface area afforded by the disk. To investigate the potential advantages of such a device and the effects of system parameters on the resonant behavior, a simple theoretical model is developed herein to quantify the damped "eigenproperties" of the system. More specifically, the primary objective is to derive simple analytical expressions relating the (in-fluid) natural frequency and corresponding quality factor to the geometric and material parameters of the device/fluid system for the case of a free vibration in the mode indicated in Fig. 1a. Such information is also extremely relevant when the device is driven as a resonator, as the forced-vibration resonant response will reflect the underlying eigenproperties of the system.



(a) (b)

Fig. 1. (a) Schematic of ASID Concept; (b) Silicon disk resonator with tangential support beams [3] (© 2010 IEEE).

The primary motivation for the present study is that previous theoretical and experimental studies of in-plane flexural vibrations of microcantilevers in water [1,2] showed that Q values approaching 100 are possible for such devices, but further improvements are limited by the fluid resistance on the microcantilever's smaller faces (those moving in their normal directions). The proposed design of Fig. 1a virtually eliminates this problem. A second motivation is of the *ex post facto* variety: the ASID modeling described herein was undertaken without knowledge of two very relevant recent papers [3,4] in which designs similar to that of Fig. 1a, shown in Fig. 1b, were fabricated and tested, and proof-of-concept was demonstrated through Q measurements in excess of 300 in heptane. Because those studies were experimental and included limited modeling (in-vacuum modal analysis using COMSOL was performed in [4]), the present study may provide some theoretical basis for the encouraging experimental results and shed some light on how one might achieve optimal designs of ASID-type resonators.

Equation of Motion, Natural Frequency, *Q* Factor: The major assumptions employed in the present model are (a) the disk is rigid and the legs are elastic; (b) the legs are much smaller than the disk so that their mass and fluid resistance are negligible relative to their counterparts for the disk; (c) the legs provide only axial force to the disk; (d) the local fluid resistance on all surfaces of the disk is given by the classical solution of Stokes's second problem for a rigid plane oscillating harmonically in a viscous fluid; and (e) the vibration amplitude is small (nonlinear effects are negligible). These assumptions enable one to derive the equation of motion for the free vibration of the idealized model of Fig. 1a in terms of the disk rotation $\theta = \theta(\bar{t})$, $\bar{t} \equiv \omega_0 t$ being a dimensionless time:

$$\left(1+\frac{2\zeta}{\sqrt{\pi}}\frac{1}{\sqrt{\bar{\omega}}}\right)\ddot{\theta}+\frac{2\zeta}{\sqrt{\pi}}\sqrt{\bar{\omega}}\dot{\theta}+\frac{16}{\pi^3\alpha}\theta=0 \ , \qquad (1)$$

in which

$$\bar{\omega}\equiv\omega/\omega_0 \ , \quad \zeta\equiv\left(1+\frac{2h}{a}\right)\left(\frac{L^2}{h^4}\frac{\rho_f{}^2\eta^2}{E\rho^3}\right)^{1/4} , \quad \alpha\equiv\frac{a^2}{bL} \ . \qquad (2a\text{-}c)$$

ω and $\bar{\omega}$ are the in-fluid natural frequency and its normalized counterpart, the latter rendered dimensionless by $\omega_0\equiv(\pi/2L)\sqrt{E/\rho}$ (i.e., by the in-vacuum fundamental axial frequency of a single leg of density ρ); a and ρ are the radius and density of the disk; L, b, and E are the length, width, and Young's modulus of the legs; h is the device thickness; and ρ_f and η are the density and viscosity of the fluid (Fig. 1a). The system is described completely by two parameters: ζ (fluid resistance) and α (normalized disk area). Assuming that $\bar{\omega}$ is insensitive to the damping term in (1), the stiffness and ($\bar{\omega}$-dependent) mass in (1) dictate that $\bar{\omega}$ must satisfy

$$\left(\sqrt{\bar{\omega}}\right)^4+\frac{2\zeta}{\sqrt{\pi}}\left(\sqrt{\bar{\omega}}\right)^3-\frac{16}{\pi^3\alpha}=0 \ , \qquad (3)$$

which may easily be solved numerically for general values of α and ζ. In practice, however, ζ is often so small that the truncated Taylor's series solution to (3) may be used:

$$\bar{\omega}\approx\frac{4}{\sqrt{\pi^3\alpha}}\left[1-\frac{(\pi\alpha)^{1/4}\zeta}{2}\right] \ , \quad \zeta\ll 1 \ . \qquad (4)$$

Placing (4) into (1) permits one to calculate the damping ratio, ξ, of the system, and thus the quality factor, Q:

$$Q\equiv\frac{1}{2\xi}\approx\frac{1}{(\pi\alpha)^{1/4}\zeta}=\frac{1}{\pi^{1/4}}\left(\frac{h}{L_0}\right)^{1/2}\left(\frac{b}{L}\right)^{1/4}\frac{\sqrt{a/h}}{2+a/h} \ , \qquad (5)$$

where $L_0\equiv(\eta^2\rho_f{}^2/E\rho^3)^{1/2}$ is a characteristic length that incorporates all of the model's material properties. From (4) and (5) the relative drop in ω (with respect to vacuum) is

$$\left|\frac{\Delta\omega}{\omega}\right|\equiv\left|\frac{\omega-\omega_{vac.}}{\omega_{vac.}}\right|=\frac{1}{2Q} \ . \qquad (6)$$

Equations (4)-(6) are explicit analytical results for the in-fluid values of natural frequency, Q, and the relative decrease in frequency of the ASID in the free-vibration mode considered. Thus, they also furnish corresponding estimates for the *forced*-vibration case when the ASID is resonating in the same mode. (Simply interpret "natural frequency" as "resonant frequency," i.e., the exciting frequency causing peak response.)

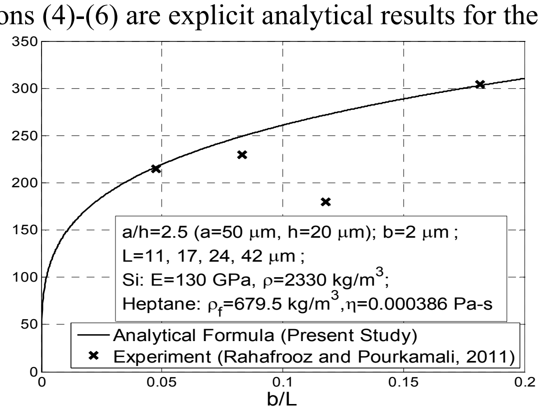


Fig. 2. Comparison between Q predictions using Eq. (5) of current model and experimental data [4].

Discussion: Equations (5) and (6) indicate that higher Q and smaller $|\Delta\omega/\omega|$ may be achieved by (a) decreasing L_0, i.e., decreasing ρ_f or η or increasing the modulus or density of the ASID material, or (b) increasing the leg stiffness by increasing b or h or decreasing L. Of particular note are (a) the small influence of the liquid on the natural frequency as given by (4) or (6), and (b) the fact that (5) implies the existence of a relative maximum for Q with respect to disk size (and (6) a relative minimum for $|\Delta\omega/\omega|$) at the theoretically optimal value of a/h=2:

$$Q_{\max}=Q\big|_{a/h=2}=\frac{\sqrt{2}}{4\pi^{1/4}}\left(\frac{h}{L_0}\right)^{1/2}\left(\frac{b}{L}\right)^{1/4} , \quad \left|\frac{\Delta\omega}{\omega}\right|_{\min}=\frac{1}{2Q_{\max}} \ . \qquad (7a,b)$$

The optimality of a/h=2 to yield maximum Q is somewhat supported by experimental results [4] showing that, of 23 specimens tested of similar design as in Fig. 1b and spanning the range a/h=[2.5, 20], the highest Q corresponded to a/h=2.5. (Smaller a/h values were not considered.) Moreover, when the results of formula (5) are compared with limited Q data for a/h=2.5 [4], Fig. 2 shows that (5) not only captures the qualitative trend with respect to b/L but also yields excellent quantitative estimates for Q, provided that the third point is deemed an "outlier." Thus, the simple model proposed here may be useful in helping one to understand the behavior of ASID resonators of the type considered and in achieving optimal designs for such devices. This may be especially valuable given that these devices have recently been shown to yield Q values that are unprecedented in liquids [4].

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