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Gamma-Set Domination Graphs. I: Complete Biorientations of q-Extended Stars and Wounded Spider Graphs

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Gamma-set Domination Graphs I: Complete biorientations of q-extended stars and wounded spider graphs

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Abstract

The domination number of a graph G, $\gamma(G)$, and the domination graph of a digraph D, $dom(D)$ are integrated in this paper. The γ-set domination graph of the complete biorientation of a graph G, $dom_\gamma(\overleftrightarrow{G})$ is created. All γ-sets of specific trees T are found, and $dom_\gamma(\overleftrightarrow{T})$ is characterized for those classes.

Keywords: *γ-set domination graphs, domination, unipathic digraphs, wounded spider graphs, q-extended stars, complete biorientations, trees.*

1 Introduction

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. If $(x, y) \in A(D)$, then x *dominates* y. A vertex is also considered to dominate itself. The *domination graph of* D, $dom(D)$, is the graph where $V(dom(D)) = V(D)$ and $\{x, y\} \in E(dom(D))$ whenever x and y dominate all other vertices in D.

Fisher, et al. ([8],[9],[10],[11],[12]) first introduced dominated graphs in terms of diagraphs that are tournaments. Further tournament related research includes papers from Cho, et al. ([1], [2]) along with Lundgren and Jimenez [16]. Recently, Factor and Factor [6], Factor [7], and Cocking and Factor [4] have extended these concepts to tournaments that may include ties, and the biorientation of graphs.

It is the nature of domination graphs to represent pairs of vertices that dominate all others in a digraph. This is done without observance of the minimum number of vertices required to dominate in the digraph. For anything other than digraphs where exactly two vertices are needed to dominate, we have either an over-representation of domination or a null domination graph that gives no insight as to the true nature of domination in the digraph it represents.

An easy example of the over-representation of domination is that of the orientation of $K_{1,n-1}$ where the center is oriented toward the other vertices. Clearly, the center vertex is able to dominate alone. However, the domination graph of this digraph is $K_{1,n-1}$ since the center forms a dominant pair with each of the other vertices. The information regarding how many vertices are needed to dominate is not available in the traditional domination graph where only dominating pairs are considered. In order to represent this relationship accurately, a new concept must be developed. Here, the minimum-set domination graph is introduced.

For a diagraph D, let M be a subset of the vertices in $V(D)$ where $\forall v \in V(D)$, $v \in M$ or $(u,v) \in A(D)$ for $u \in M$, and M has the minimum cardinality of all such subsets. The set M is referred to as a *minimum dominating set* D. The *minimum-set domination graph of* D is created using the vertices of D, with a copy of $K_{|M|}$ formed by the vertices of each minimum dominating set.

In the case where D is the complete biorientation of a graph $G, D = \overleftrightarrow{G}$, the minimum-set domination graph depends upon the domination number of G, $\gamma(G)$. The *complete biorientation* of a graph G is created by replacing every edge $\{u, v\}$ in G with arcs (u, v) and (v, u). In graphs, $\gamma(G)$ represents the minimum number of vertices necessary to dominate all vertices in the graph.

A wide range of results have been obtained regarding the domination number of a graph. Haynes, Hedetniemi and Slater [14] have brought together many of the basic concepts and results of domination in graphs and in [15] examine advanced results. $\gamma(G)$ translates into the minimum number of vertices needed to dominate all other vertices in $\overleftrightarrow{G}$. Thus, the minimum-set domination of D in general can be referred to as the γ-set domination of $\overleftrightarrow{G}$ for biorientations of graphs. The resulting γ-set domination graph is denoted $dom_\gamma(\overleftrightarrow{G})$ where each γ-set in G forms a copy of K_γ in $dom_\gamma(\overleftrightarrow{G})$. Although other digraphs may have the characteristic that the cardinality of their minimum-domination set is $\gamma(G)$ for the underlying graph G, it is not generally the case. Therefore, $dom_\gamma(\overleftrightarrow{G})$ will be used only when an entire class has that characteristic.

The problem of finding the domination number of a graph is generally NP-complete, as first shown by Johnson [17]. Since that number is used to determine the size of dominating sets for $dom_\gamma(\overleftrightarrow{G})$, it becomes a problem of selecting classes of graphs where $\gamma(G)$ can be determined. Fortunately, there are a variety of linear algorithms available that will find $\gamma(T)$ for a tree T. These include a linear-time algorithm by Mitchell, Cockayne, and Hedetniemi [18]. This makes the class of trees a highly desirable place to begin to explore γ-set domination graphs.

This paper characterizes γ-set domination graphs specifically for the classes of wounded spider graphs and q-extended stars. Each class is formed by special subdivisions of the branches of a star and are defined in Sections 3 and 4 respectively. In conclusion, the biorientations of trees for the special cases of $\gamma = 1, 2$ are examined.

2 Results governing the general structure of γ-set domination graphs

First, we explore general concepts that will be used in characterizing further results for γ-set domination graphs. Any tree is isomorphic to a rooted tree of minimum height. Here, the notation T_R will represent a rooted tree of minimum height that is isomorphic to a tree T. For example, the star is isomorphic to a rooted tree of height 1.

In a tree, pendant vertices can be referred to as *leaves*. A vertex that is adjacent to another vertex, but is one level closer to the root will be called the *parent*. The leaves and parents of leaves play a major role in constructing γ-sets for a tree. Only one vertex other than the leaf itself dominates the leaf, and that is its parent.

Proposition 2.1 *For any vertex $p \in V(T)$ where $deg(p) = 2$ and p is the parent of exactly one leaf, then p or its adjacent leaf must be in any dominating set of T.*

Proof: At least one vertex in the dominating set must dominate the leaf $\Rightarrow p$ is in the dominating set or the leaf is in the dominating set. □

Corollary 2.2 *For any vertex $p \in V(T)$ where $\deg(p) = 2$ and p is the parent of exactly one leaf, either p or its adjacent leaf (but not both) must be in any minimal dominating set.*

Let $W \subset V$ be any subset of vertices in a graph $G = (V, E)$, and let $v \in W$. The vertex $u \in V - (W - \{v\})$ is a *private neighbor* of v if any vertex other than v in W does not dominate u. Note that u can be equal to v. The set of private neighbors of a vertex u with respect to a set S of vertices is denoted $pn[u, S]$. A set W is considered *irredundant* if every vertex in W has a private neighbor. Cockayne, Hedetniemi and Miller used these concepts to characterize a minimum dominating set for a graph G.

Proposition 2.3 [3] *A dominating set S is a minimal dominating set if and only if it is dominating and irredundant.*

The restriction that p have degree 2 and be the parent of exactly one leaf placed upon p in Corollary 2.2 is necessary in part because the existence of more than one leaf will not allow for any of the leaves to be placed in a minimal dominating set.

Proposition 2.4 *If $p \in V(T)$ where p is the parent of at least two leaves, then p must be in any minimal dominating set and none of the leaves will be in the set.*

Proof: Suppose that p is not in the dominating set S. Each leaf adjacent with p must then be in S. But each leaf dominates only $p \Rightarrow$ at least two elements in S do not have a private neighbor, and S is not irredundant. Thus, by Proposition 2.3, S is not a minimal dominating set. So p must be in S. For similar reasons, if one of the leaves is in S also, S is not a minimal dominating set. $\square$

Further observations regarding the general structure of any γ-set domination graph of the biorientation of a tree are given in the next two results. The first shows that any leaf vertex and its parent vertex will not form an edge in $dom_\gamma(\overleftrightarrow{T})$. Second is a theorem regarding the absence of any copy of a K_{k+1} subgraph in $dom_\gamma(\overleftrightarrow{T})$ where $\gamma(T) = k$, regardless of how many copies of K_k are in the γ-set domination graph.

Proposition 2.5 *For any vertex $p \in V(T)$ where p is adjacent to a vertex v of degree 1, the edge $\{p, v\} \notin E(dom_\gamma(\overleftrightarrow{T}))$.*

Proof: Corollary 2.2 and Proposition 2.4 insure that either a parent of a leaf or in some cases the leaf itself, but not both, will be in any γ-set of a graph. Since they never appear together in a γ-set, they will never be part of a copy of K_γ, which is the only construct in $dom_\gamma(\overleftrightarrow{T})$. $\square$

Remark 2.6 *K_1 and $K_{1,n-1}$ are the only trees with $\gamma(T) = 1$.*

Proof: For $\gamma(T) = 1$, there must be one vertex with degree of $n-1$. In a tree, this can only occur if T is K_1 or T is a star. $\square$

Theorem 2.7 *If T is a tree on n vertices where $\gamma(T) = k \geq 1$, then K_{k+1} is not a subgraph of $dom_\gamma(\overleftrightarrow{T})$.*

Proof: (PMI) 1) Let $n = 1$. Then by Remark 2.6, $T = K_1$ or $T = K_{1,n-1} \Rightarrow dom_\gamma(\overleftrightarrow{T})$ is the null graph (by definition of $dom_\gamma(\overleftrightarrow{T})$), and K_2 is not a subgraph thereof.

2) Assume that for $n = k$, $k \geq 1$ that K_{k+1} is not a subgraph of $dom_\gamma(\overleftrightarrow{T})$ when $\gamma(T) = k$.

3) Consider $n = k+1$. Suppose that K_{k+2} is a subgraph of $dom_\gamma(\overleftrightarrow{T})$ for some tree T where $\gamma(T) = k+1$. Let T_R be the rooted tree of minimum height h that is isomorphic to T. There exists a vertex u such that u

is a leaf on level h of T_R. Either u or its parent w is in every γ-set of T (Proposition 2.2). Remove w from T_R with all adjacent leaves and all edges incident with w. Let the new tree be known as T_R^*. $\gamma(T_R^*) = k$, as only one parent vertex and adjacent pendant vertices was removed, giving a net total of one less cover vertex needed for the vertices of T_R^*. By the induction hypothesis, K_{k+1} is not a subgraph of $dom_\gamma(\overleftrightarrow{T}_R^*) \Rightarrow K_k$ is the largest complete subgraph of $dom_\gamma(\overleftrightarrow{T}_R^*)$. Adding in u, its adjacent pendant vertices, and incident edges to T_R^* when we recreate T_R, does not create new relationships between vertices in $T_R^* \Rightarrow$ no new edges are created in the domination graph between those vertices in $T_R^* \Rightarrow K_{k+2}$ must contain two of the vertices v_1, v_2 in $V(T_R) \setminus V(T_R^*) \Rightarrow v_1$ and v_2 are both leaves, or $v_1 = u$ and v_2 is a leaf. The former possibility contradicts Proposition 2.4, and the latter contradicts Proposition 2.5. Thus, K_{k+2} cannot be a subgraph of $dom_\gamma(\overleftrightarrow{T})$ when $\gamma(T) = k+1$. □

3 Wounded Spider Graphs

Now that the basic set of rules have been formed that govern the construction of γ-set domination graphs of the complete biorientations of trees, we can begin to examine the class of graph known as the *wounded spider graph*.

A *subdivision* of an edge $\{u, v\}$ is achieved by removing the edge and replacing it with a new vertex w and the edges $\{u, w\}$ and $\{w, v\}$. Domke, Dunbar and Markus [5] create a wounded spider graph by subdividing 0 to $t-1$ edges of the star $K_{1,t}$. Various wounded spider graphs are shown in Figure 1.

(a) (b) (c) (d)

Figure 1. Examples of wounded spider graphs

A wounded spider graph with $t-1$ subdivisions can be achieved with the corona graph $K_{1,t} \circ K_1$. The *corona graph* is defined by Frucht and Harary

[13] to be the graph $G = G_1 \circ G_2$, which is formed by taking one copy of $G_1, |V(G_1)|$ copies of G_2, and adding edges from each vertex $v_1 \in V(G_1)$ and every vertex in the i^{th} copy of G_2. Figure 1(d) represents the corona graph $K_{1,3} \circ K_1$.

Wounded spider graphs, W have a structure that makes it possible to characterize the γ-set domination graph of $\overleftrightarrow{W}$. By design, there is always at least one leaf adjacent to the root. The remaining leaves each have a different parent, which is adjacent to the root. The domination number, $\gamma(T)$ is known for a wounded spider graph and is related to the maximum vertex degree, $\Delta(T)$, by the following theorem.

Theorem 3.1 [5] *For any tree T, $\gamma(T) = n - \Delta(T)$ if and only if T is a wounded spider.*

The statement of Theorem 3.1 is written in terms of n and $\Delta(T)$. Since the objective here is to count the γ-sets in addition to determining $\gamma(T)$, it is necessary to know how many leaves are not adjacent to the root. Corollary 2.2 states that either a leaf or its parent must be in every dominating set, so the number of these leaves will be used in the calculation of the number of γ-sets. The following proposition makes use of the information in Theorem 3.1.

Proposition 3.2 *Let T be a wounded spider graph. T has $m = n - \Delta(T) - 1$ leaves that are not adjacent to the center vertex.*

Proof: Let T be a wounded spider graph. $\gamma(T) = n - \Delta(T)$ (Theorem 3.1). By construction, there exists at least one leaf adjacent to the center vertex $r \Rightarrow r$ is in at least one γ-set. For that γ-set, there are $n - \Delta(T) - 1$ vertices that are not r and not adjacent to r, so are not covered by $r \Rightarrow$ there are $n - \Delta(T) - 1$ leaves not adjacent to r. □

Corollary 3.3 *Let T be a wounded spider graph and $m = n - \Delta(T) - 1$. $\gamma(T) = m + 1$.*

Now we can take the information provided by Corollary 3.3 and characterize the γ-set domination graphs of the complete biorientations of wounded spider graphs. For ease in delineating the three possibilities in the following theorem, the cases can be seen in Figure 1. Figures 1(a) and 1(b) are examples of the first case, Figure 1(d) shows the corona graph in case 2, and the third case where there is more than one vertex of degree 1 on level 1 of the graph T_R is shown in 1(c).

Theorem 3.4 *Let W be a wounded spider graph on n vertices with $m = n - \Delta(T) - 1$ leaves not adjacent to the center r.*

1. *If $m = 0$, then $dom_\gamma(\overleftrightarrow{W})$ is the null graph.*

2. *If $W = K_{1,t} \circ K_1$ so that $m = t$, then $dom_\gamma(\overleftrightarrow{W})$ is the connected graph on $2(m+1)$ vertices consisting of 2^{m+1} copies of K_{m+1}, 2^m of which are incident with the root r and 2^m of which are incident with the one leaf adjacent to r.*

3. *If there are at least two vertices of degree 1 adjacent to the root r in T_R, then $dom_\gamma(\overleftrightarrow{W})$ is the graph of $(n-2m)$ components on n vertices consisting of 2^m copies of K_{m+1}, of which all are incident with r, and $(\Delta(T) - m)$ isolated vertices: namely, all leaves adjacent to r.*

Further, K_{m+2} is not a subgraph of $dom_\gamma(\overleftrightarrow{W})$, and for all edges $\{u,v\} \in E(W)$ such that v is a leaf vertex and u is its parent vertex, $\{u,v\} \notin E(dom_\gamma(\overleftrightarrow{W}))$.

Proof: $\gamma(W) = m+1$ (Corollary 3.3) 1) If $m = 0$, then $W = K_1$ or $W = K_{1,n-1}$. By definition, $dom_\gamma(\overleftrightarrow{W})$ has only copies of K_1. Thus, it is the null graph.

2) If $W = K_{1,t} \circ K_1$, then $m = t$. Let T_R be the rooted tree where the root r is a vertex with maximum degree. By construction, there is exactly one leaf v adjacent to r. Corollary 2.2 dictates that either vertex r or vertex v will be in every γ-set, but not both. In either case, r and v are dominated by the selection. The remaining m vertices of degree 1 are on level 2 of T_R. Again, either the leaf or its parent must be in every γ-set. Thus, a γ-set will contain anywhere from 0 to m of these m leaves. There are $\sum_{i=0}^{m} \binom{m}{i} = 2^m$ ways to select the m leaves for γ-sets. With the choice of either r or v for each γ-set, we have 2^{m+1} γ-sets in W. By definition, each forms a copy of K_{m+1} in $dom_\gamma(\overleftrightarrow{W})$. Vertex r is in 2^m copies and vertex v is in the other 2^mcopies by virtue of each of the vertices being in 2^m γ-sets. All vertices are in γ-sets with both r and v, so the 2^{m+1} copies of K_{m+1} form one component.

3) This case is quite similar to that in part (2), except that root r must be in every γ-set (Proposition 2.4). Thus, there are half as many copies of K_{m+1} in the $dom_\gamma(\overleftrightarrow{W})$. This gives 2^m copies that are all adjacent to r since r is in every γ-set, forming one component in $dom_\gamma(\overleftrightarrow{W})$. No leaf adjacent to r is in any γ-set (Proposition 2.4), so each of these is an isolated vertex in $dom_\gamma(\overleftrightarrow{W})$. There are m leaves on level 2 of T_R with one parent each, so there are $n - 2m - 1 = \Delta(T) - m$ leaves adjacent to r. None of these leaves will be in a γ-set, so are isolated vertices in the $dom_\gamma(\overleftrightarrow{W})$.

The further restrictions follow directly from Theorem 2.7 and Proposition 2.5. □

4 q-extended stars

If each pendant edge of a star is subdivided one time, the resulting graph is referred to as a *1-extended star* and is isomorphic to a rooted tree of height 2. If the pendant edges of a 1-extended star are each subdivided one time, the resulting graph will be referred to as a *2-extended star* and is isomorphic to a rooted tree of height 3. In general, if the subdivisions occur q times on each edge, we have a *q-extended star* isomorphic to a rooted tree of height $q+1$. Examples of 1-extended and 2-extended stars represented by rooted trees are shown in Figure 2.

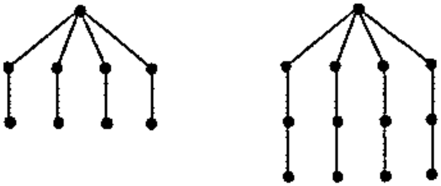


Figure 2. 1-extended and 2-extended stars with 4 branches each

The unique structure of a path lends itself well to the task of counting the number of γ-sets in a graph. Thus, the structure of a q-extended star can be examined as a collection of paths of length $q+2$ joined by a single vertex. This vertex can be referred to as the center of the star or, as will be the case more often in this paper while using rooted trees, as the *root* of the star.

We will see that results for all q-extended stars can be generated from the basis graphs of 1-, 2-, and 3-extended stars. Each subclass has a different γ-set structure that affect the characterization of their associated γ-set domination graphs. To separate into these three subclasses, it is advantageous to represent q in the following manner: $q = q_0 + 3m$ for $q_0 = 1, 2, 3$ and $m \geq 0$ a nonnegative integer. For all but the case when $q_0 = 3$, the approach is to find the domination number for a basis graph, count the number of γ-sets in the graph, then create the subclass of extended stars generated using each basis graph by extending each branch of the star with three vertices and three edges. The general results for a subclass will then be used to classify the associated γ-set domination graphs of the complete biorientations of the graphs.

For ease in understanding the development of this section, the following notation and concepts will be adopted. The *basis graphs* for the q-extended star are the 1-extended, 2-extended and 3-extended stars that will generate all possible q-extended stars. An *extension block* of a q-extended star for a given q_0 denotes a block of three additional vertices and edges being adjoined to every branch of an existing q-extended star. The first extension block occurs when $m = 1$, and in general the k^{th} extension block occurs when $m = k$. Each star will be said to have b branches as the number of rays of the star, where b_i refers to the i^{th} branch. A branch has $q + 2$ vertices, which includes the root.

In a q-extended star, all branches are of equal length, making the root the exact middle of the union of two branches. As a result, there are an odd number of vertices in the union of any pair of branches in the graph. The *eccentricity* of a vertex u is the distance of the vertex farthest from u. In this graph, the eccentricity of each end vertex is the length of the longest path in the graph: $e = 2(q + 1)$. Thus, e is an even length and the longest path in the graph, P_{2k+1}, is obtained by the union of any two branches. This is important when determining the γ-sets, as the number and selection of vertices needed to cover the longest path in our rooted tree, will bound general results.

4.1 $q_0 = 3$

To begin the generation of the subclasses, let us examine the case where $q = 3 + 3m = 3(1 + m)$, $m \geq 0$. This case begins the investigation because it possesses only one γ-set. The example given in Figure 3 shows a 3-extended star where $q_0 = 3$, $b = 4$, $m = 0$, $\gamma = 5$, and the longest path is $P_{2k+1} = P_9$. The vertices forming the unique γ-set are circled. Following are results pertaining to this subclass.

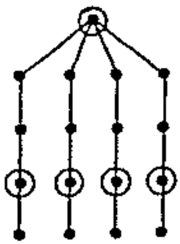


Figure 3. 3-extended star with 4 branches and a unique γ-set

Proposition 4.1.1 *Let T be a q-extended star where $q = q_0 + 3m$ for $q_0 = 3$, $m \geq 0$. If P_{2k+1} is the longest path in T, then $2k + 1 = 6m + 9$.*

Proof: The length of the longest path in T is the eccentricity of an end vertex, so has an odd number of vertices, $2k + 1$, as discussed earlier. There are $q + 2$ vertices in each branch of T, each including the root, giving $2q + 3$ vertices in the longest path. Since $q = q_0 + 3m$ and $q_0 = 3$, $m \geq 0 \Rightarrow q =$

$3 + 3m$ and $2q + 3 = 6m + 9$. Thus, $2k + 1 = 6m + 9$. □

To begin establishing the uniqueness of the γ-set for the $3(m+1)$-extended star, the following proposition begins by examining the unique γ-set of P_{3k}.

Proposition 4.1.2 *A path on $3k$ vertices where $k \geq 1$ has a unique γ-set, and $\gamma = k$.*

Proof: In a path, a vertex can dominate at most 3 vertices: itself and its two neighbors $\Rightarrow$ there are at least k vertices in any γ-set of P_{3k}. Label the vertices of the path: $1, 2, \ldots, 3k$. To form a γ-set, select vertices $\{3i-1\}_{i=1}^{k}$. Each vertex covers exactly 3 vertices, thus maximizing the coverage. There are k vertices in the set, so $\gamma = k$. Suppose that there is another γ-set. Every vertex in this set must dominate 3 vertices - all 3 must be its private neighbors - in order for the $3k$ vertices to be covered in k. Both sets must have vertex 2 to cover vertex 1, because 1 would not cover 3 vertices and is not an option for any γ-set. Remove vertices 1, 2, and 3 since they are already represented in the γ-set. Consider the remaining path. Vertex 5 must be chosen for the same reasons as the previous choice of vertex 2. Continuing in this fashion leads to the only possible γ-set being comprised of vertices $\{3i-1\}_{i=1}^{k}$. □

The previous two results lead to the following corollary that stands as a foundation for this subclass of q-extended stars.

Corollary 4.1.3 *Any $3(m+1)$-extended star where $m \geq 0$ with $b = 2$ branches has a unique γ-set, and $\gamma = 2m + 3$.*

From the uniqueness of the γ-set and the structure of the rooted tree associated with the $3(m+1)$-extended star, an important characteristic surfaces regarding the root of the star. This result is one that separates it from the other subclasses of q-extended stars.

Lemma 4.1.4 *If T is a $3(m+1)$-extended star where $b = 2$ and $m \geq 0$, then the root of T is in the unique γ-set.*

Proof: By labeling the vertices and selecting the unique γ-set constructed in the proof of Proposition 4.1.2, consider the labeling of the root vertex. The root vertex has an odd label, so is $\lceil \frac{6m+9}{2} \rceil$, or $3m + 5$. This can be written as $3(m+2) - 1$, which is included in $\{3i-1\}_{i=1}^{k}$. Thus, the root is in the γ-set. □

The previous result is important in establishing the general result for the $3(m+1)$-extended star for the general value of b.

Theorem 4.1.5 *If T is a $3(m+1)$-extended star where $b \geq 2$ is the number of branches of T and $m \geq 0$, then the root of T is in the unique γ-set. Further, $\gamma(T) = b(m+1)+1$.*

Proof: Let T be a tree as described above. By the principle of mathematical induction, for the case when $b = 2$, Lemma 4.1.4 insures that the root is in the γ-set. Also, Corollary 4.1.3 says that $\gamma = 2m + 3$, giving $2m + 3 = 2(m+1) + 1 = b(m+1) + 1$. If we assume for $b = k$ that T has a unique γ-set consisting of $k(m+1)+1$ vertices of which the root is a member, then consider the case when $b = k+1$. Examine any k branches of T. They have a unique γ-set consisting of $k(m+1)+1$ vertices, including the root. The $(k+1)^{st}$ branch has $q = 3(m+1)$ vertices that cannot be covered by any vertices in the previous set, and which cannot cover any vertices outside of those on the branch. It will take a minimum of $(m+1)$ vertices to dominate those on this branch. The branch with $3(m+1)$ vertices has a unique γ-set (Proposition 4.1.2). Therefore, the union of these creates a unique γ-set. Further, it takes $k(m+1)+1+(m+1)$ vertices to dominate this graph, giving $(k+1)(m+1)+1$ vertices needed in the γ-set. □

All of the preceding results give support to the ultimate goal of describing the γ-domination graph associated with the biorientations of these $3(m+1)$-extended stars. Together with the outcome obtained in the next two subclasses of graphs, the characterization for all γ-domination graphs associated with q-extended stars is made.

Theorem 4.1.6 *Let T be a $3(m+1)$-extended star where $b \geq 2$ is the number of branches of T and $m \geq 0$. The $dom_\gamma(\overleftrightarrow{T})$ consists of one copy of $K_{b(m+1)+1}$, which includes the root, and the rest isolated vertices.*

Proof: From Theorem 4.1.5 it is known that there is a unique γ-set of size $b(m+1)+1$, and that the root is in the set. By definition, these vertices form a copy of $K_{b(m+1)+1}$ in $dom_\gamma(\overleftrightarrow{T})$. There will be no other edges in $dom_\gamma(\overleftrightarrow{T})$. Thus, all vertices not included in $K_{b(m+1)+1}$ are isolated. □

Remark 4.1.7 *If at least one parent in a 1-extended star is adjacent to more than one leaf, the resulting graph is referred to as a* 1-extended star with multiple leaves. *Any parent of a leaf in a $3(m+1)$-extended star may be adjacent to more than one leaf without changing the γ value, as the parent vertices are all included in the γ-set and will dominate all leaves. The only addition to the γ-set domination graph is the addition of more isolated vertices representing those additional pendant vertices.*

4.2 $q_0 = 1$

The $3(m+1)$-extended stars have a unique γ-set. This makes the results of the general case easier to obtain without first determining all results for $m = 0$, and then extending them to the case for general m. However, in both the cases for $q_0 = 1$ and $q_0 = 2$ there are many γ-sets to count in the basis block prior to examining the extension block representatives in each γ-set. Fortunately, the extension blocks are less varied in the vertex selection, so work well as an expansion of the γ-sets in the 1- and 2-extended stars. Thus, the method for both the $q = 1 + 3m$ and $q = 2 + 3m$ cases is to count all possible γ-sets for the basis blocks, then generalize the results to any $m \geq 0$.

In this section, we will consider the case where $q_0 = 1$. An example of a 1-extended star with its biorientation and associated γ-set domination graph is shown in Figure 4. Here, $\gamma(T) = 3$ and $b = 3$. The γ-sets for T are given as follows: $\{1,2,3\}, \{1,2,6\}, \{1,5,3\}, \{4,2,3\}, \{1,5,6\}, \{4,2,6\}, \{4,5,3\}$.

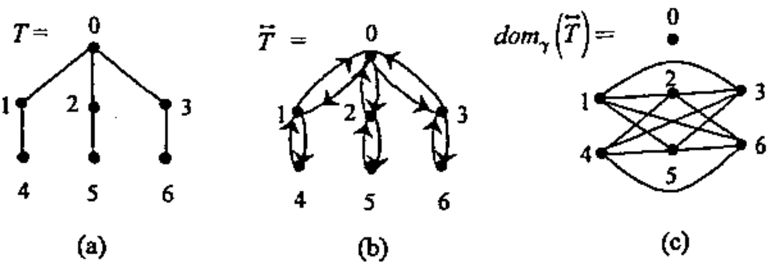


Figure 4. 1-extended star, its complete biorientation and the associated γ-set domination graph

The domination number of a 1-extended star is easy to observe and given in the following proposition.

Proposition 4.2.1 *If T is a 1-extended star with $b \geq 2$, with or without multiple leaves, where r is the center of the star, then $\gamma(T) = b$.*

Proof: The collection of b vertices adjacent to r cover all vertices of T, so $\gamma(T) \leq b$. Corollaries 2.2 and 2.4 require at least one parent of a leaf or its adjacent leaf be in every γ-set of T, so $\gamma(T) \geq b$. Thus, $\gamma(T) = b$. $\square$

As opposed to the subclass of $3(m+1)$-extended stars, when T is a 1-extended star the root is not an option in any γ-set. Its inclusion forces the set containing it to have a cardinality of $\gamma + 1$. The following proposition formalizes the exclusion of the root in any γ-set of a 1-extended star with or without multiple leaves.

Proposition 4.2.2 *If T is a 1-extended star with $b \geq 2$, with or without multiple leaves, then the center of the star is not in any γ-set of T.*

Proof: Let T be a 1-extended star with or without multiple leaves where $b \geq 2$. Label the center of the star r. Then $\deg(r) = b$. Proposition 4.2.1 gives $\gamma(T) = b$. If r is chosen for a domination set, then there are at least b branches with leaves not covered $\Rightarrow$ the cardinality of that minimal domination set is $b+1$ and it does not form a γ-set. Thus, r is not in any γ-set of T. □

With the absence of the root in a γ-set, one parent of a leaf must always be included so that the root is covered.

Proposition 4.2.3 *If T is a 1-extended star with $b \geq 2$, with or without multiple leaves, then every γ-set of T must contain at least one vertex that is the parent of a leaf.*

Proof: The center of T, vertex r, is not in any γ-set of T. Therefore, at least one vertex adjacent to r must be in every γ-set in order to cover r. These vertices are all parents of leaves. □

With the structure of our γ-sets now defined, we can commence to count how many there are.

Lemma 4.2.4 *Let T be a 1-extended star with $b \geq 2$ branches where $\gamma(T) = b$. Then T has $2^b - 1$ γ-sets.*

Proof: Here, the number of vertices is $2b+1$ and $\gamma(T) = b$. Corollary 2.2 guarantees that each leaf or the parent of the leaf is a member of every γ-set. The center of the star, vertex r, is not in any γ-set (Proposition 4.2.2), and one parent of a leaf must be in every γ-set. Thus, we can choose 1 up to b of the parents of leaves for each γ-set. This equals $\sum_{i=1}^{b} \binom{b}{i} =$ $\left(\sum_{i=0}^{b} \binom{b}{i}\right) - 1 = 2^b - 1$ unique γ-sets. □

Remark 4.2.5 *If T is a 1-extended star with $b \geq 2$ branches where $\gamma(T) = b$ and B parents of leaves have multiple leaves, then the number of γ-sets for T is reduced to $2^{b-B} - 1$. This is because only the parent vertex for those B parents can be selected for any γ-set.*

The characterization of γ-set domination graphs for 1-extended stars is included in the general result at the end of this subsection.

If T is a 1-extended star with b branches, Proposition 4.2.1 tells us that $\gamma(T) = b$. What happens when an extension block is added to the basis block? Each branch has 3 new vertices. At best, a vertex in any γ-set chosen in the basis block can cover only one of these new vertices. Any leaf or parent of a leaf chosen in the extension will not dominate any of the vertices in the basis block. Therefore, each branch in the extension must have at least one vertex in the γ-set. By choosing the parent of a leaf in the extension, all three vertices can be dominated by one vertex independent of what is happening in the other blocks. When m extensions are added, each branch will require one vertex per each of the m extensions to dominate all vertices not covered by a vertex in the basis block. Figure 5 shows a 4-extended star consisting of one basis block (a 1-extended star) and one extension block. The circled vertices represent one of the γ-sets.

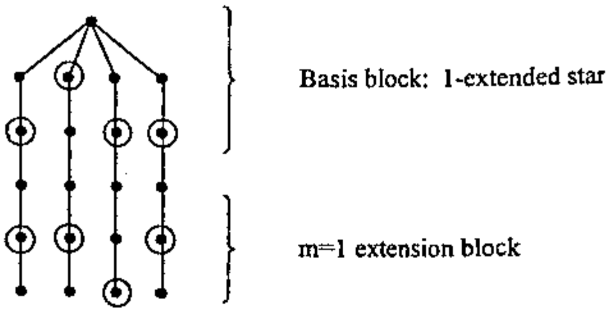


Figure 5. A 4-extended star and one γ-set

Lemma 4.2.6 *If T is a $(1+3m)$-extended star with $b \geq 2$ branches and $m \geq 0$, then $\gamma(T) = b(m+1)$.*

Proof: Using induction on m, let $m = 0$. Proposition 4.2.1 states that $\gamma(T) = b(0+1)$. For $m = k$, assume that $\gamma(T) = b(k+1)$. Now consider the case where $m = k+1$. The $[1+3(k+1)]$-extended star is obtained by taking a $(1+3k)$-extended star and adding an extension block of three vertices. $\gamma(T) = b(k+1)$ for the $(1+3k)$-extended star. Each new extension branch requires at least one vertex to dominate. By choosing the parent vertex of each leaf, this can be done in exactly one vertex per branch. Each selected vertex covers two vertices that cannot be covered by any vertex in the γ-set of the $(1+3k)$-extended star, and cannot cover any of the vertices in that star $\Rightarrow b$ additional vertices are necessary to cover the $(k+1)^{st}$ extension $\Rightarrow \gamma = b(k+1) + b = b[(k+1)+1]$. □

Now that $\gamma(T)$ has been identified for the $(1+3m)$-extended star, we must count the number of γ-sets in this subclass. In order to do this successfully, it is important to understand the nature of vertex selection in the basis block and all extension blocks. In each branch of a block, the selection of a leaf L or the parent of a leaf P determines possible selections in the subsequent level. To illustrate the process, a decision tree consisting of outcomes L and P can be used. An example of this representation is shown in Figure 6 where a decision tree is constructed for one branch of a 10-extended star when a leaf is chosen in the basis block.

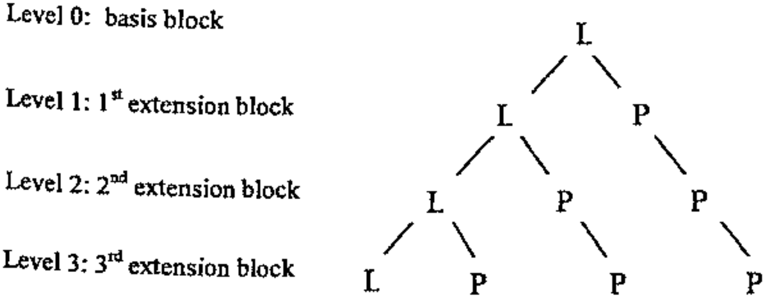


Figure 6. Decision tree representing the choices of vertices for a γ-set in one branch of a 10-extended star where a leaf is chosen in the basis block

Both the previous two figures give a clear idea of what is happening in our vertex selection. In the first, Figure 5, one branch has a parent P chosen in the basis block. There is no other way to cover the vertices in the first extension of that branch other than to choose a parent vertex in the extension as well. This is borne out in Figure 6 where once a parent is chosen, the path never branches again.

Remark 4.2.7 *Once a parent of a leaf, P is chosen in a branch, all future selections in that path must also be P. This is true because once P is chosen in extension m, no vertices in extension $m+1$ are dominated, thus leaving 3 vertices to be covered, and only the middle vertex P can cover all three.*

The selection of a leaf increases the number of choices at each level of the decision tree by 1. A leaf generates the possible selection of an L or P. Only the L branches out at each level. It increases the choices by 1 each time as it replicates itself once and generates a new P. The relationship between the decision tree and selection of vertices for a γ-set is given in the following remark. Future collections of vertices for γ-sets use this relationship to count the sets.

Remark 4.2.8 *Let T be a decision tree representing the choices of L or P that follows the restrictions of vertex selection for a γ-set. The collections of vertices generated by each path from root to leaf of T represents a selection for the γ-sets of a $(1+3m)$-extended star.*

Proposition 4.2.9 *Let T be a $(1+3m)$-extended star with $b \geq 2$ branches and $m \geq 0$. There are $m+1$ collections of vertices generated in a branch where a leaf in the basis block has been chosen for a γ-set.*

Proof: There are two possibilities, L and P, generated in the first extension where $m = 1$, so $2 = m + 1$. One more is generated in each successive extension by each L branch $\Rightarrow$ there are $m + 1$ branches generated by the leaf in the basis block when $q = 1 + 3m$. □

At this time, we can pull together the results garnered for the 1-extended star and those relationships determined for the extensions to generalize the domination number of the $(1+3m)$-extended star. The γ-sets will be chosen for the 1-extended star, then decision trees constructed for each branch that has a leaf in the basic block. Since choosing a leaf is the only option that gives a branch, and thus more choices for vertex selection, the combinatorial argument centers around the selection of leaves. Proposition 4.2.3 insists that one parent from the basis block be chosen also.

Theorem 4.2.10 *If T is a $(1 + 3m)$-extended star with $b \geq 2$ branches and $m \geq 0$, then there are $(m+2)^b - (m+1)^b$ γ-sets of T.*

Proof: Let T be a $(1+3m)$-extended star with $b \geq 2$ branches and $m \geq 0$. For a leaf chosen in branch b_i in the basis block, there are $m+1$ selections of vertices that can be used in a γ-set for T (Proposition 4.2.9). There are $\binom{b}{i}$ ways to choose i leaves in the basis block. There must be at least one parent chosen in the γ-set for the basis block, so the number of leaves chosen can be at most $b-1$. Thus, $0 \leq i \leq b-1$. The number of γ-sets for a $(1+3m)$-extended star is $\sum_{i=0}^{b-1}(m+1)^i \binom{b}{i} = \left[\sum_{i=0}^{b}(m+1)^i \binom{b}{i}\right] - (m+1)^b = (m+2)^b - (m+1)^b$. □

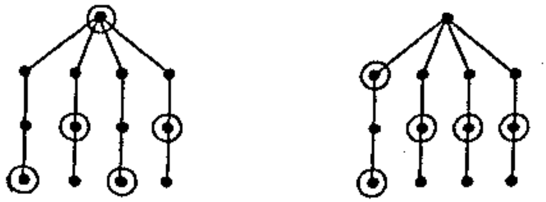
Corollary 4.2.11 *Let T be a $(1+3m)$-extended star with $b \geq 2$ branches and $m \geq 0$. The $dom_\gamma(\overleftrightarrow{T})$ consists of $(m+2)^b - (m+1)^b$ copies of $K_{b(m+1)}$ and $mb+1$ isolated vertices. Further, Theorem 2.7 guarantees no copy of a larger complete graph.*

Note that the first vertex in each extension of a branch will never be chosen in a γ-set, as it cannot dominate all three of the vertices in one extension. There are m extension blocks, b branches and one root, which gives the number of isolated vertices listed in the corollary.

4.3 $q_0 = 2$

To complete the discussion of the class of q-extended stars, we now turn to the final subclass: $q_0 = 2$. This has been saved as the final example because γ-sets for this subclass may include the root or may not. This makes for an interesting counting experience using a variety of techniques.

As in the case where $q_0 = 1$, we will begin by finding the number of γ-sets in the 2-extended star, which is the basis block. Then we will look at the general $(2+3m)$-extended star. Figure 7 illustrates two γ-sets associated with the 2-extended star possessing 4 branches. 7(a) shows a collection of vertices that includes the root, while the root is not a member of the collection in 7(b).



(a) (b)

Figure 7. Two γ-sets of a 2-extended star with 4 branches

To begin, the following proposition gives the domination number of a 2-extended star.

Proposition 4.3.1 *If T is a 2-extended star with $b \geq 2$ branches, then $\gamma(T) = b+1$.*

Proof: There are $3b+1$ vertices in T. If the root r is in a γ-set, then there are b leaves not covered by r. Therefore, at least b more vertices are needed in the dominating set. If r is not in the γ-set, the maximum number of vertices that any other vertex can dominate is 3. Thus, it will take at least

$\lceil \frac{3b+1}{3} \rceil = b+1$ vertices to dominate in $T \Rightarrow \gamma(T) \geq b+1$. Select r and all leaf vertices for a dominating set $\Rightarrow \gamma(T) \leq b+1$. Therefore, $\gamma(T) = b+1$. □

Now to the central question: How many γ-sets are in a 2-extended star? This can be split into the problem of counting the sets containing the root r and those not containing r. In the case where r is not in a γ-set, the following lemma describes the other vertices that must be contained therein.

Lemma 4.3.2 *Let T be a 2-extended star with $b \geq 2$ branches and root r, and let γ^* be a γ-set of T where $r \notin \gamma^* \Leftrightarrow \gamma^*$ consists of 1) the parents of leaves for at least $b-1$ branches of T, and 2) exactly one vertex u adjacent to r.*

Proof: Let T be a 2-extended star with $b \leq 2$ branches and root r, γ^* is a γ-set of T,

($\Rightarrow$) $r \notin \gamma^* \Rightarrow$ there is a vertex $u \in \gamma^*$ in branch b_i such that $\{u, r\} \in E(T)$. The vertex u does not dominate any leaf of $T \Rightarrow$ a parent or leaf vertex from each of the b branches must be an element of γ^* (Corollary 2.2) $\Rightarrow u$ and one other vertex from b_i must be in $\gamma^* \Rightarrow$ only $(b-1)$ other vertices can be used to dominate the remaining vertices in the $b-1$ branches not incident with u. These vertices must cover both the leaves of the branches and the vertices of the branches that are adjacent to $r \Rightarrow$ the parent of the leaves of each of the $b-1$ branches must be in γ^*.

($\Leftarrow$) Let $u \in \gamma^*$ where $\{u, r\} \in E(T)$, and let $P_1, P_2, \ldots, P_{b-1} \in \gamma^*$ where the $P_i \in \gamma^*$ are the parents of the i leaves in branches not incident with vertex u. Suppose that r is in γ^*. The vertices u, r, and $P_1, \ldots, P_{b-1}$ account for all $b+1$ vertices in γ^*. However, the leaf in the branch incident with u is not covered by any of these vertices. Thus, γ^* is not a γ-set, and r cannot be in any γ-set under such conditions. □

Through the proof of Lemma 4.3.2, the template is now set for counting the number of γ-sets that do not include the root of a 2-extended star. The following theorem uses this information in setting forth the number of γ-sets in a 2-extended star.

Theorem 4.3.3 *If T is a 2-extended star with $b \geq 2$ branches, then T has $2^b + 2b$ γ-sets.*

Proof: Let T be as described above, and let γ^* be a γ-set of T.

1. If $r \in \gamma^* \Rightarrow$ the b leaves in T are not covered by r and there must be b other vertices to cover them $\Rightarrow$ one parent or leaf from each branch

of T must be in γ^*. Since all vertices adjacent to r are dominated by r, no other vertex needs to cover them $\Rightarrow 2^b$ ways to choose a parent or a leaf from each of the b branches.

2. If $r \notin \gamma^* \Rightarrow$ exactly one vertex u adjacent to r is in γ^* and all branches not incident with u have only one choice for γ^* (Lemma 4.3.2). There are 2 ways to choose a parent or a leaf from the one branch incident with u, and b ways to choose u, giving 2b choices.

Thus, there are $2^b + 2b$ γ-sets for T. □

As in the previous subclass, allowing multiple leaves will reduce the total number γ-sets. Wherever there is a parent of multiple leaves, that parent must be in every γ-set.

Remark 4.3.4 *If B of the parents of leaves in a 2-extended star with $b \geq 2$ branches have multiple leaves, our number of γ-sets is reduced to $2^{b-B} + 2(b - B)$.*

The characterization of the $dom_\gamma(\overleftrightarrow{T})$ when T is a 2-extended star is included in the general result later in this subsection. Now the outcomes for the 2-extended star are used to develop those general results where $q = 2 + 3m$. First, the domination number of T can be obtained.

Lemma 4.3.5 *Let T be a $(2+3m)$-extended star with $b \geq 2$ branches and $m \geq 0$. $\gamma(T) = b(m+1) + 1$.*

Proof: Let T be as described above. Create a dominating set for T by 1) selecting any γ-set of $b+1$ vertices for the 2-extended star basis block, and 2) adding to that collection the parent vertex for each branch in every one of the m extension blocks. This set contains $b + 1 + mb = b(m+1) + 1$ vertices $\Rightarrow \gamma(T) \leq b(m+1) + 1$.

1. If $r \in \gamma^* \Rightarrow$ there are b leaves that need to be covered in every block, including the basis block. This requires at least an additional$(m+1)b$ vertices consisting of parents or their leaves $\Rightarrow \gamma(T) \geq b(m+1) + 1$.

2. If $r \notin \gamma^* \Rightarrow$ every vertex in γ^* can dominate at most 3 vertices. $|V(T)| = 3b(m+1) + 1 \Rightarrow$ at least $\lceil \frac{3b(m+1)+1}{3} \rceil = b(m+1) + 1$ vertices are needed in $\gamma^* \Rightarrow \gamma(T) \geq b(m+1) + 1$.

Therefore, $\gamma(T) = b(m+1) + 1$. □

Consider the 2- and 5-extended stars in Figure 8 below. The 2-extended star in 8(a) has two γ-sets represented by different circle types for each of the different sets. In 8(b) and 8(c), possible γ-sets formed using the seed sets in the 2-extended star are given. Arrows between vertices in the extension block indicate a choice of vertex for the γ-set. Note that the choice of a P vertex at any level still dictates that only P vertices can be chosen in the subsequent extension blocks.

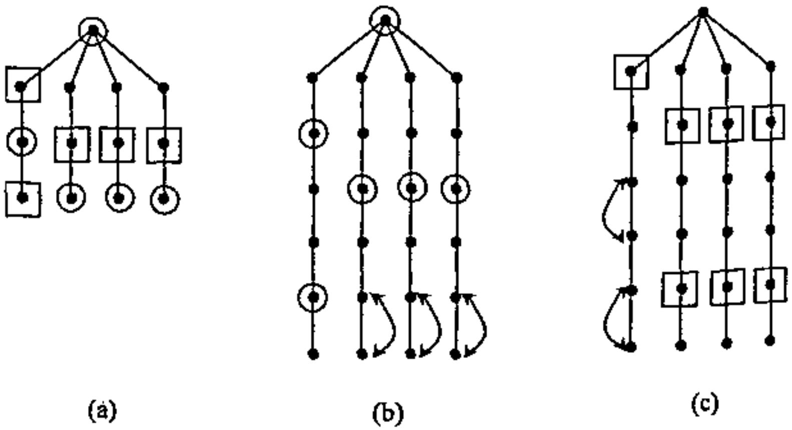


Figure 8. γ -set possibilities in selections using a 2-extended star

The interesting new choice is illustrated in part 8(c) in the first branch. When the non-root vertex u is chosen in the 2-extended star, a choice becomes available when an extension block is attached. It is clear that if u in the first branch is chosen and the leaf is chosen as well, the leaf does not have to be in the expanded tree. The first vertex in the first extension, call it V_1 could be chosen instead to cover the leaf without disrupting any of the other selections of vertices in the basis block. This third choice, which happens only when the vertex $u \neq r$ is in the basis block γ-set, adds a third branch to the decision tree method developed in the last subsection. Figure 9 shows the decision tree for one branch of an 8-extended star that is associated with choosing u in the γ-set of the 2-extended star. Notice that V_i is selected in extension block i, but a leaf, L or parent, P vertex in that same block may be selected as well. If neither is selected for a γ-set, the next V_{i+1} must be used in order to cover the leaf of extension block i. This, of course, cannot continue in the last extension block, as a P or L must be included in the set in order to cover the leaf in that branch.

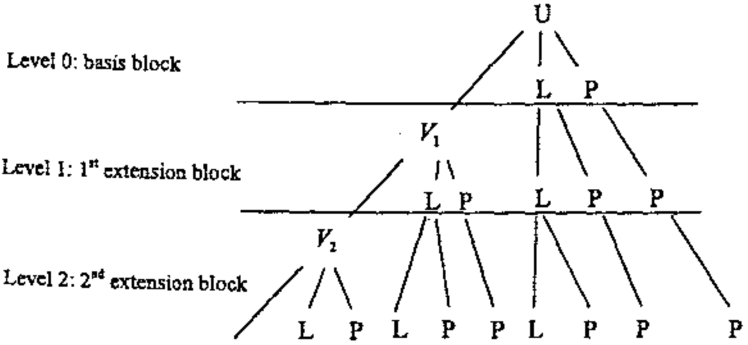


Figure 9. Decision tree representing choices for γ-set members in one branch of an 8-extended star where a vertex U adjacent to the root is selected

The results regarding the number of γ-sets generated by a $(2+3m)$-extended star can now be formulated. First, we must determine the number of γ-sets possible when a vertex u that is adjacent to the root is chosen for the set. The proof and computation rely upon a vertex V_i always producing the possibility of a leaf, L or a parent P as a choice in extension block i.

Lemma 4.3.6 *If T is a $(2+3m)$-extended star with $b \geq 2$ branches and $m \geq 0$, γ^* is a γ-set of T, and for a vertex u that is adjacent to root r, $u \in \gamma^*$, then there are $\frac{(m+1)(m+4)}{2}$ possible γ^* sets.*

Proof: Let T be as described above, and let $u \in \gamma^*$, a γ-set of T, where $u \neq r$ and $\{u, r\} \in E(T)$. Say that u is incident with branch b_i in T. All other branches have only once choice for elements in γ^* (Lemma 4.3.2), therefore we will count only the possibilities in the branch with u. If $m = 0$, there are only two choices in branch b_i: P or L. For $m \geq 1$, the choice expands to include a vertex V_1, which is the first vertex in the first extension block. For choices of P or L in the basis block, previous results find that P gives us one path in the decision tree, and L gives us $m+1$ paths.

Now we will count the paths generated in the V_1 branch of the decision tree. Within each extension block, V_i produces L or P as a possibility. Each of these produces a new L branch or P branch that continues to the m^{th} extension block. Each P branch produces only one path. Each L branch will create $k+1$ paths, where k is the distance from the block in which L was produced to the m extension block. This creates the following number of paths created in each extension block:

$$\begin{array}{ll} 1+m & \text{(paths generated by } P \text{ and } L \text{ branches created in extension block 1)} \\ 1+(m-1) & \text{(paths generated by } P \text{ and } L \text{ branches created in extension block 2)} \\ \vdots & \\ \underline{\quad 1+1 \quad} & \text{(paths generated by } P \text{ and } L \text{ branches created in extension block } m) \end{array}$$

$$\left(m+\sum_{i=1}^{m} i\right) = m + \frac{m(m+1)}{2} = \frac{m^2+3m}{2}.$$

$\Rightarrow$ the total number of paths generated in branch b_i is $1+(m+1)+\frac{m^2+3m}{2} = \frac{(m+1)(m+4)}{2}$. □

Corollary 4.3.7 *Let T be a $(2+3m)$-extended star with $b \geq 2$ branches and $m \geq 0$. There are $b\left[\frac{(m+1)(m+4)}{2}\right]$ γ-sets of T that do not include the root r.*

Proof: This follows from the previous lemma and the fact that there are b ways to choose the vertex u. □

Next, the number of γ-sets including r is calculated. Together with Corollary 4.3.7 above, it will finalize the results for this subclass, and completely characterize the number of γ-sets in q-extended stars. Thus, the γ-set domination graphs for this class will be determined.

Lemma 4.3.8 *Let T be a $(2+3m)$-extended star with $b \geq 2$ branches and $m \geq 0$. There are $(m+2)^b$ γ-sets of T that include the root r.*

Proof: Let T be as described above, and consider the γ-sets of which r is a member. r does not cover any leaves in the basis block, so a leaf or parent must be chosen in each branch. There are $\binom{b}{i}$ ways to choose i leaves in the basis block. There can be anywhere from 0 leaves selected to b leaves selected, so $0 \leq i \leq b$. For each branch where a leaf is selected in the basis block, there are $(m+1)$ possible selections of vertices produced for inclusion in the γ-set. This gives $\sum_{i=0}^{b}(m+1)^i \binom{b}{i} = (m+2)^b$ paths generated by the choice of leaves in the basis block of a $(2+3m)$-extended star. No additional choices are available from the parent branches of the basis block. Thus, there are $(m+2)^b$ γ-sets containing r. □

Theorem 4.3.9 *Let T be a $(2+3m)$-extended star with $b \geq 2$ branches and $m \geq 0$. T has $\frac{b(m+1)(m+4)}{2} + (m+2)^b$ γ-sets.*

Proof: Corollary 4.3.7 and Lemma 4.3.8 together produce this result. □

Corollary 4.3.10 *Let T be a $(2+3m)$-extended star with $b \geq 2$ branches and $m \geq 0$. $dom_\gamma(\overleftrightarrow{T})$ consists of $\frac{b(m+1)(m+4)}{2} + (m+2)^b$ copies of $K_{b(m+1)+1}$. There are no isolated vertices and no copy of $K_{b(m+1)+2}$.*

Proof: The first part of the Corollary comes directly from Lemma 4.3.5 and Theorem 4.3.9. Every vertex can be in some γ-set as the root, a parent, a leaf or a $v_i \Rightarrow$ there are no isolated vertices. Theorem 2.7 dictates that no γ-set domination graph of a tree will have a copy of a complete graph of order higher than γ. □

Notice that $\gamma(T)$ is the same for the cases when T is a $(2+3m)$-extended star or a $3(m+1)$-extended star, and only differs by 1 from the domination number of a $(1+3m)$-extended star. However, the change in just one vertex per branch of the basis blocks creates structures producing greatly different numbers of γ-sets.

The following corollary summarizes the collection of γ-set results.

Corollary 4.3.11 *Let T be a q-extended star with $b \geq 2$ branches, where $q = q_0 + 3m$ for $q_0 = 1, 2, 3$ and $m \geq 0$. T has the following number of γ-sets:*

1. *1 if $q \equiv 0 \pmod 3$,*
2. *$(m+2)^b - (m+1)^b$ if $q \equiv 1 \pmod 3$, or*
3. *$\frac{b(m+1)(m+4)}{2} + (m+2)^b$ if $q \equiv 2 \pmod 3$.*

5 Special cases of $\gamma : \gamma = 1, 2$

5.1 $\gamma = 1$

The case where $\gamma = 1$ is simple, yet brings to the forefront the only area in γ-set domination graphs where there is a choice to make as to representation. In Section 3, Theorem 3.4 described the γ-set domination graphs of wounded spider graphs where $n - \Delta(T) = 1$ as null graphs. To the domination graph traditionalist, this may seem contrary to previous dogma. It is. However, by definition, only copies of $K_\gamma = K_1$ will be in the graph. The question then becomes: How many vertices will there be in the null graph? The answer must depend upon the application and upon the individual using the graph. If n vertices are used, then there will be more copies of K_1 in $dom_\gamma(\overleftrightarrow{T})$ than the number of dominating vertices. If fewer than n vertices are used, then the vertex set will not be that of T.

What biorientations are subject to this choice? The answer is: all biorientations of graphs on n vertices that possess at least one vertex of degree $n-1$. Thus, $\overleftrightarrow{K}_1, \overleftrightarrow{K}_{1,n-1}$ and even $\overleftrightarrow{K}_n$ will have null γ-set domination graphs.

Remark 5.1.1 *If G is a graph on n vertices with vertex $u \in V(G)$ where* $\deg(u) = n-1$, *then $dom_\gamma(\overleftrightarrow{G})$ is the null graph.*

In this paper, the biorientations of trees are the only graphs being considered. As seen earlier, only K_1 and $K_{1,m-1}$ are trees with $\gamma = 1$. Of these, only $dom_\gamma\left(\overleftrightarrow{K}_{1,m-1}\right)$ will allow for a choice in representation. It can be represented either with one vertex as the one dominating vertex, or as the null graph on n vertices.

5.2 $\gamma(\mathrm{T}) = 2$

The case where $\gamma(T) = 2$ is special because it is the only time when the γ-set domination graph is the same as the tranditional domination graph for the complete biorientation of T. In this section, both $dom_\gamma(\overleftrightarrow{T})$ and $dom(\overleftrightarrow{T})$ are characterized for $\gamma(T) = 2$. One difference between $dom_\gamma(\overleftrightarrow{T})$ and $dom(\overleftrightarrow{T})$ is that $dom_\gamma(\overleftrightarrow{T})$ will only be null when $\gamma(T) = 1$, whereas $dom(\overleftrightarrow{T})$ is null whenever $\gamma(T) \geq 3$.

Remark 5.2.1 *If T is a tree and $\gamma(T) \geq 3$, then $dom(\overleftrightarrow{T})$ is the null graph.*

Proof: If $\gamma(T) \geq 3$, then no two vertices dominate, so $dom(\overleftrightarrow{T})$ has no vertices. □

It is the nature of graphs that the further in distance two vertices, the less chance they have of dominating. In the case of a tree, this is readily seen by examining the eccentricity of a leaf. Actually, by taking the longest path in a tree, we will obtain the lower bound on the number of vertices needed to cover all of the vertices in the tree. Since we are interested in this case with $\gamma = 2$, the following two propositions aid in the development of further results.

Proposition 5.2.2 *If T is a tree and $\gamma(T) = 2$, then there are exactly two parents of leaves.*

Proof: Corollary 2.2 indicates that either a leaf or its parent must be in every γ-set, so there are at most two parents of leaves when $\gamma(T) = 2$. There are at least two parents of leaves on every tree, so there can be no more than two parents of leaves. Thus, there are exactly two parents of leaves on T. $\square$

Proposition 5.2.3 *If T is a path and $\gamma(T) = 2$, then T contains at most 6 vertices.*

Proof: Consider a path on n vertices. One vertex will dominate at most three vertices on the path. An internal vertex of the path covers the maximum number. Since each leaf or its parent must be chosen in the γ-set, and only two can be chosen, choose the parents of the leaves for the γ-set. The maximum number of vertices covered is $\lceil \frac{n}{3} \rceil = 2 = \gamma$, so $n = 6$ is the maximum number of vertices in the path. $\square$

Any number of pendant vertices can be added to the parent vertex of each end vertex in P_6 to create new trees. These all have the maximum height allowed, as no eccentricity greater than 5 will be able to generate a tree with $\gamma(T) = 2$. The following corollary applies these results to the rooted tree T_R and bounds the maximum height of the rooted tree.

Corollary 5.2.4 *Let T_R be the rooted tree of maximum height where 1) T_R has minimal height and 2) $\gamma(T_R) = 2$. Then the height of T_R is 3.*

Proof: P_6 is the longest path with domination number of 2 (Proposition 5.2.3). If T_R is the rooted tree with one branch of length 3 and the other of length 2, then T_R is the rooted tree with minimum height that represents P_6, and the height of T_R is 3. $\square$

As for the minimum height of a rooted tree with $\gamma(T) = 2$, it is 2. Any tree with minimum height of 1 is a star and $\gamma(T) = 1$.

To summarize, we will only observe trees that are isomorphic to minimum height rooted trees of height 2 or 3, which have 2 parents of leaves. What, then of the γ-set domination graphs of the complete biorientations of these trees? To characterize what they may be, it is instructive to find the limits on the number of edges in the domination graphs themselves and the forms they take, as well as the numerical limits of these structures.

Lemma 5.2.5 *If T is a tree with $\gamma(T) = 2$, then there are at most 4 edges in $dom_\gamma(\overleftrightarrow{T})$.*

Proof: Let T be as described above, ℓ_1, ℓ_2 be leaves of different parents, and p_1, p_2 be their respective parents. Every γ-set must contain a leaf or the parent of a leaf, and $\ell_1 p_1, \ell_2 p_2$ are not dominating pairs (Corollary 2.2) $\Rightarrow$ there are at most 4 dominant pairs possible: $p_1 p_2, p_1 \ell_2, \ell_1 p_2, \ell_1 \ell_2$. Thus, there are at most 4 dominating pairs, and at most 4 edges in $dom_\gamma(\overleftrightarrow{T})$. $\square$

Corollary 5.2.6 *If T is a tree with $\gamma(T) = 2$, then there are at most 4 edges in $dom(\overleftrightarrow{T})$.*

Proposition 5.2.7 *P_4 is the only tree with $\gamma = 2$ where $p_1p_2, p_1\ell_2, \ell_1p_2$, $\ell_1\ell_2$ are dominating pairs. Further, $dom_\gamma(\overleftrightarrow{P_4}) = C_4$.*

Proof: If $\ell_1\ \ell_2$ is a dominating pair $\Rightarrow p_1$ and p_2 are the only other vertices $\Rightarrow p_1p_2, p_1\ell_2$ and ℓ_1p_2 are also dominating pairs $\Rightarrow \overleftrightarrow{P_4}$ is the only tree when $\gamma = 2$ where $p_1p_2, p_1\ell_2, \ell_1p_2, \ell_1\ell_2$ are dominating pairs. Further, these pairs form C_4 in $dom_\gamma(\overleftrightarrow{P_4})$. □

Corollary 5.2.8 *P_4 is the only tree with $\gamma = 2$ where $dom_\gamma(\overleftrightarrow{T})$ contains four edges.*

Corollary 5.2.9 *P_4 is the only tree with $\gamma = 2$ where ℓ_1, ℓ_2 is a dominating pair.*

With the limits on the number of edges in the γ-set domination graphs, it is natural to wonder if a) there is any edge that will always be in the graph, and b) if there can be more than one connected component that is not an isolated vertex. The following two results address these issues.

Remark 5.2.10 *If T is a tree where $\gamma(T) = 2$ with p_1, p_2 being the two parents of leaves in T, then $\{p_1, p_2\} \in E(dom_\gamma(\overleftrightarrow{T}))$.*

Proof: If p_1 and p_2 do not dominate, then $\gamma(T) \neq 2 \Rightarrow$ they dominate and thus, form an edge in the γ-set domination graph. □

Lemma 5.2.11 *Let T be a tree where $\gamma(T) = 2$. There exists exactly one connected component that is not an isolated vertex in $dom_\gamma(\overleftrightarrow{T})$.*

Proof: $\{p_1, p_2\} \in E(dom_\gamma(\overleftrightarrow{T}))$, so there is at least one connected component in $dom_\gamma(\overleftrightarrow{T})$ that is not an isolated vertex. Suppose there is another connected component $\Rightarrow$ there are two vertices other than p_1 and p_2 that form a dominating pair $\Rightarrow \ell_1$ and ℓ_2 must be a dominating pair since each leaf must be represented, and this is a seperate component $\Rightarrow T$ must be P_4 (Corollary 5.2.9), but $dom_\gamma(\overleftrightarrow{P_4}) = C_4$, which is one connected component. Thus, only one connected component exists that is not an isolated vertex. □

Finally, for $\gamma(T) = 2$, we can characterize both the γ-set domination graph of T and the domination graph of T.

Theorem 5.2.12 *If T is a tree where $\gamma(T) = 2$, then $dom_\gamma(\overleftrightarrow{T})$ is one of the following:*

1. *C_4 if $T = P_4$, or*
2. *P_2 with possible isolated vertices, or*
3. *P_3 with possible isolated vertices, or*
4. *P_4 with possible isolated vertices.*

Proof: Let T be as described above. There are no paths of length greater than 3 in $dom_\gamma(\overleftrightarrow{T})$ (Lemma 5.2.5 and Proposition 5.2.7). There are no copies of K_m, $m \geq 3$ in any $dom_\gamma(\overleftrightarrow{T})$ (Theorem 2.7). By definition, $dom_\gamma(\overleftrightarrow{T})$ is not a null graph, and Lemma 5.2.11 guarantees that there is only one connected component in $dom_\gamma(\overleftrightarrow{T})$ that is not an isolated vertex. Thus, only C_4, P_2, P_3, and P_4 match all of these restrictions when $\gamma(T) = 2$. Examples of these graphs are as follows: 1) $dom_\gamma(\overleftrightarrow{P_4}) = C_4$, 2) $dom_\gamma(\overleftrightarrow{P_6}) =$ P_2 with isolated vertices, 3) If T is the tree in Figure 10, then $dom_\gamma(\overleftrightarrow{T}) =$ P_3 with isolated vertices, and 4) $dom_\gamma(\overleftrightarrow{P_5}) = P_4$ with an isolated vertex. □

T =

$dom_\gamma(\tilde{T}) =$

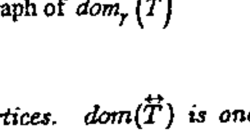


Figure 10. Tree where P_3 is a subgraph of $dom_\gamma(\tilde{T})$

Theorem 5.2.13 *Let T be a tree on n vertices. $dom(\overleftrightarrow{T})$ is one of the following:*

1) *K_n if $n \leq 3$, or*
2) *$K_{1,n-1}$ if $T = K_{1,n-1}$ for $n \geq 4$, or*
3) *C_4, or*
4) *P_2, P_3 or P_4 all with possible isolated vertices, or*
5) *the null graph on n vertices.*

Proof: Let T be a tree on n vertices.

1. If $n \leq 3$, then $T = K_1, K_2$ or $P_3 \Rightarrow dom(\overleftrightarrow{T}) = K_n$.

2. [4] If $n \geq 4$ and $T = K_{1,n-1}$, then T is a star and $dom(\overleftrightarrow{T}) = K_{1,n-1}$.

3. If $n = 4$ and $T \neq K_{1,n-1} \Rightarrow T = P_4 \Rightarrow dom(\overleftrightarrow{T}) = C_4$.

4. If $n \geq 5$ and $\gamma(T) = 2$, then $dom(\overleftrightarrow{T}) = dom_\gamma(\overleftrightarrow{T})$, which by Theorem 5.2.12 says that it will be P_2, P_3 or P_4 all with possible isolated vertices.

5. If $n \geq 5$ and $\gamma(T) > 2$, then $dom(\overleftrightarrow{T})$ is the null graph on n vertices (Remark 5.2.1). □

The algorithmic nature of some of the proofs in this paper suggests computational methods for examining other classes of trees. A general algorithm for determining all γ-sets in trees that is modeled upon these methods is anticipated in future research.

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