1-1-1983

Topological Extensions and Subspaces of $\eta\alpha$-sets

Paul Bankston
Marquette University, paul.bankston@marquette.edu

Published version. *Fundamenta Mathematicae*, Vol. 118, No. 3 (1983): 191-199. Publisher Link.

FUNDAMENTA
MATHEMATICAE
CXVIII (1983)

Topological extensions and subspaces of η_α-sets

by

Paul Bankston (Milwaukee, Wi.)

Abstract. The η_α-sets of Hausdorff have large compactifications (of cardinality $\geqslant \exp(\alpha)$; and of cardinality $\geqslant \exp(\exp(2^{<\alpha}))$ in the Stone-Čech case). If Q_α denotes the unique (when it exists) η_α-set of cardinality α, then Q_α can be decomposed (= partitioned) into homeomorphs of any prescribed nonempty subspace; moreover the subspaces of Q_α can be characterized as those which are regular T_1, of cardinality and weight $\leqslant \alpha$, whose topologies are closed under $< \alpha$ intersections.

Let $\langle A, <\rangle$ be a linearly ordered set. If B and C are subsets of A, we use the notation $B < C$ to mean that $b < c$ for all $b \in B$, $c \in C$. If α is an infinite cardinal number, we say that $\langle A, <\rangle$ is an *η_α-set* if whenever $B, C \subseteq A$ have cardinality $< \alpha$ and $B < C$ then there is an element $a \in A$ with $B < \{a\} < C$. Such ordered sets, invented by Hausdorff [8] (see also [5, 6, 7]), are the forerunners and prototypical examples of saturated relational structures in model theory (see [5, 6]). Our interest in the present note centers on topological issues related to η_α-sets, considered as linearly ordered topological spaces (LOTS's) with the open interval topology.

Roughly stated, our results are these: (i) certain Hausdorff extensions of η_α-sets must have cardinality $\geqslant 2^\alpha$, and some (the compact C^*-extensions) must have cardinality $\geqslant \exp(\exp(2^{<\alpha}))$; (ii) the (unique when it exists; i.e., when $\alpha = \alpha^{<\alpha}$) η_α-set Q_α of cardinality α can be decomposed (= partitioned) into homeomorphs of any prescribed nonempty subspace; and (iii) the subspaces of Q_α are precisely the regular T_1 spaces, of cardinality and weight $\leqslant \alpha$, whose topologies are closed under $< \alpha$ intersections.

1. Preliminaries. We follow the convention that ordinal numbers are the sets of their predecessors and that cardinals are initial ordinals. If α is an infinite cardinal, α^+ denotes the cardinal successor of α ($\omega = \{0, 1, 2, \ldots\}$, $\omega_1 = \omega^+$, etc.) If A is a set, $|A|$ denotes the cardinality of A. If B is another set then ${}^B A$ is the set of all functions $f\colon B \to A$. For cardinals α, β, we let $\alpha^\beta = |{}^\beta\alpha|$ and $\alpha^{<\beta} = \operatorname{Sup}\{\alpha^\gamma\colon \gamma < \beta\}$. $\exp(\alpha)$ sometimes denotes 2^α, especially in interactions: $\exp^2(\alpha) = \exp(\exp(\alpha))$, etc. A useful application of Kőnig's Lemma is the following.

1.1. LEMMA (see [6]). *Let α be an infinite cardinal. Then $\alpha = \alpha^{<\alpha}$ iff α is regular and $\alpha = 2^{<\alpha}$.*

The basic properties of η_α-sets can be summarized as follows.

1.2 THEOREM. *Let α be an infinite cardinal.*

(i) *If $\langle A, <\rangle$ is an η_α-set then the ordering is dense without endpoints. Moreover, no co-initial or cofinal sequence can have cardinality $< \alpha$.*

(ii) (Hausdorff) *There exists an η_{α^+}-set of cardinality 2^α.*

(iii) (Hausdorff) *Any two η_α-sets of cardinality α are order isomorphic (call this set Q_α, when it exists).*

(iv) (Gillman) *If $\alpha^+ < 2^\alpha$ then there are at least two nonisomorphic η_{α^+}-sets of cardinality 2^α.*

(v) (Gillman, B. Jónsson) *Q_α exists iff $\alpha = \alpha^{<\alpha}$.*

1.3. Remark. Q_ω is, of course, the rational line Q. From (v) above plus Lemma 1.1 we can also infer that Q_{α^+} exists iff $\alpha^+ = 2^\alpha$, Q_α does not exist for α a singular cardinal, and Q_α always exists for α strongly inaccessible. Thus if the Generalized Continuum Hypothesis holds then Q_α exists iff α is a regular cardinal.

Let X be a topological space. Then:

(i) X is *α-compact* if every open cover has a subcover of cardinality $< \alpha$ (compact $= \omega$-compact, Lindelöf $= \omega_1$-compact);

(ii) X is *α-additive* [10] ($=$ a *P_α-space* [6]) if intersections of $< \alpha$ open sets are open (P-space $= P_{\omega_1}$-space); and

(iii) X is *α-Baire* if intersections of $< \alpha$ dense open sets are dense (Baire $= \omega_1$-Baire).

2. Extensions. A topological space X is an *η_α-LOTS* if there is a linear ordering on the underlying set of X which makes that set an η_α-set and whose open intervals basically generate the topology of X. The following is an easy application of the definition of η_α-set.

2.1 LEMMA. *Let X be an η_α-LOTS, and let $\mathscr{B}$ be the open interval basis arising from a suitable η_α-order on X.*

(i) (*α-intersection condition*) *If $\mathscr{U} \subseteq \mathscr{B}$ has cardinality $< \alpha$ and $\bigcap \mathscr{U} = \emptyset$ then $\bigcap \mathscr{U}_0 = \emptyset$ for some finite $\mathscr{U}_0 \subseteq \mathscr{U}$.*

(ii) (*α-cover condition*) *If $\mathscr{U} \subseteq \mathscr{B}$ has cardinality $< \alpha$ and $\bigcup \mathscr{U} = X$ then $\bigcup \mathscr{U}_0 = X$ for some finite $\mathscr{U}_0 \subseteq \mathscr{U}$.*

(iii) (*α-additivity condition*) *If $\mathscr{U} \subseteq \mathscr{B}$ has cardinality $< \alpha$ then $\bigcap \mathscr{U}$ is an open set.*

2.2 THEOREM. *Let X be an η_α-LOTS.*

(i) *X is α-additive, hence strongly zero-dimensional when $\alpha > \omega$.*

(ii) *X is α-Baire.*

Proof. (i) That X is α-additive follows immediately from Lemma 2.1 (iii). If $\alpha > \omega$, X is a regular T_1 P-space; and such spaces are well known to be strongly zero-dimensional.

(ii) This is essentially the proof of Theorem 2.2 in [3]: use Lemma 2.1 (i, iii). ∎

2.3. THEOREM. *Let X be an η_α-LOTS.*

(i) *There is a family of $2^{<\alpha}$ pairwise disjoint open subsets of X.*

(ii) *X has a closed discrete subset of cardinality $2^{<\alpha}$.*

(iii) *X is not $2^{<\alpha}$-compact.*

Proof. (i) Pick an appropriate η_α-ordering for X and let $\langle x_\xi\colon \xi < \lambda\rangle$ be an increasing well ordered cofinal sequence in X. Then $\lambda \geqslant \alpha$, so let $I_\xi = \{x \in X\colon x_\xi < x < x_{\xi+1}\}$ for $\xi < \alpha$. For each $\xi < \alpha$ use the properties of the open interval basis given in Lemma 2.1 to construct in I_ξ a binary tree, of height α, consisting of $2^{|\gamma|}$ pairwise disjoint open sub-intervals at each level $\gamma < \alpha$ (ordering is reverse inclusion). This gives $\operatorname{Sup}\{2^{|\xi|}\colon \xi < \alpha\} = 2^{<\alpha}$ pairwise disjoint open subsets of X.

(ii) If $\alpha = \omega$, use the increasing sequence from (i) above and stop at ω; i.e., use the closed discrete set $S = \langle x_\xi\colon \xi < \omega\rangle$, of cardinality $\omega = 2^{<\omega}$. If $\alpha > \omega$ use zero-dimensionality: by (i) there is a set $\mathscr{U}$ of $2^{<\alpha}$ pairwise disjoint clopen subsets of X. Let S consist of one point from each member of $\mathscr{U}$.

(iii) This is immediate from (ii). ∎

The main result in this section can now be stated.

2.4. THEOREM. *Let X be an η_α-LOTS and let Y be an α-compact, regular T_1 topological extension of X.*

(i) $|Y| \geqslant 2^\alpha$.

(ii) *If X is dense in Y then Y is α^+-Baire.*

(iii) *If Y is compact Hausdorff and X is C^*-embedded in Y then $|Y| \geqslant \exp^2(2^{<\alpha})$.*

Proof. (i) Since α-compactness is a closed-hereditary property, we can assume X is dense in Y. For each $S \subseteq Y$ let S^-, S^0 denote respectively the closure and the interior of S in Y. Using Lemma 2.1, let $\mathscr{B}$ be an open basis for X with the α-intersection condition. We show first that for $U \subseteq Y$ open, $U \cap X \subseteq (U \cap X)^{-0}$. To see this, let $x \in U^-$. Since X is dense and every open V containing x intersects U, we have $x \in (U \cap X)^-$. Thus $(U \cap X)^- = U^-$ and we get $U \cap X \subseteq U \subseteq U^{-0} = (U \cap X)^{-0}$.

We build an α-level tree T in Y by induction satisfying: (a) T is a binary tree of sets, ordered by reverse inclusion; (b) each member of T is of the form B^- where $\emptyset \neq B \in \mathscr{B}$; (c) the members of each level of T are pairwise disjoint; and (d) whenever $B_1^- \subseteq B_2^-$ in T, it is also true that $B_1 \subseteq B_2$.

For each ordinal $\xi < \alpha$ define the ξth level T_ξ inductively: $T_0 = \{B_0^-\}$ where $\emptyset \neq B_0 \in \mathscr{B}$. Assuming $T \restriction (\xi+1) = \bigcup\{T_\gamma\colon \gamma \leqslant \xi\}$ has been defined, define $T_{\xi+1}$ as follows: Let $B^- \in T_\xi$. Then there is an open $U \subseteq Y$ with $B = U \cap X \subseteq (U \cap X)^{-0} = B^{-0}$. Use regularity to find open sets $U_1, U_2 \neq \emptyset$ (all nonempty open sets are self-dense) with $U_1^- \cup U_2^- \subseteq B^{-0}$ and $U_1^- \cap U_2^- = \emptyset$. Since B is dense in B^-, there are $B_1, B_2 \in \mathscr{B}$, nonempty, such that

$B_i \subseteq U_i \cap B$, $i = 1, 2$. So define $T_{\xi+1} = \bigcup\{\{B_1^-, B_2^-\}: B^- \in T_\xi\}$. In the case where ξ is a limit ordinal and $T \restriction \xi$ is already constructed, let $\mathscr{B}_\xi = \langle B_\gamma^- : \gamma < \xi\rangle$ be a branch in $T \restriction \xi$. By the inductive hypothesis, $\bigcap_{\gamma<\xi} B_\gamma \neq \emptyset$ and contains a nonempty $B_\xi \in \mathscr{B}$. Let $T_\xi = \{B_\xi^- : \mathscr{B}_\xi$ is a branch of $T \restriction \xi\}$, and let $T = \bigcup_{\xi<\alpha} T_\xi$. By α-compactness, each branch of T has nonempty intersection. Since T has 2^α branches, we conclude that $|Y| \geqslant 2^\alpha$.

(ii) Let X, Y, $\mathscr{B}$ be as above. Let $\langle U_\xi : \xi < \alpha\rangle$ be a family of α dense open subsets of Y, with $S = \bigcap_{\xi<\alpha} U_\xi$. We show S is dense in Y. To this end let $V \subseteq Y$ be nonempty open. To show $V \cap S \neq \emptyset$, use induction on α. We construct a decreasing chain $\langle B_\xi^- : \xi < \alpha\rangle$ where $B_\gamma \supseteq B_\xi$ for $\gamma < \xi < \alpha$, $\emptyset \neq B_\xi \in \mathscr{B}$ for $\xi < \alpha$, and $B_\xi^- \subseteq V \cap (\bigcap_{\gamma<\xi} U_\gamma)$. This is possible since X is dense in Y, Y is regular T_1, and $\mathscr{B}$ satisfies the α-intersection condition. Using α-compactness we get $\emptyset \neq \bigcap_{\xi<\alpha} B_\xi^- \subseteq V \cap S$.

(iii) Assume Y is compact Hausdorff and X is C^*-embedded in Y. Using Theorem 2.3 (ii), let S be any closed discrete subset of X of cardinality $2^{<\alpha}$. Since X is normal, S is C^*-embedded in X, hence in Y. Therefore S^- is homeomorphic to the Stone–Čech compactification of S, so $|Y| \geqslant |S^-| = \exp^2(2^{<\alpha})$. ∎

2.5. Remark. Both estimates in Theorem 2.4 (i, iii) can be realized as follows: (i) the order compactification (= Dedekind completion-plus-endpoints) of $\boldsymbol{Q}_\alpha$ has cardinality 2^α; (ii) the Stone–Čech compactification of $\boldsymbol{Q}_\alpha$ has cardinality $\exp^2(\alpha) = \exp^2(2^{<\alpha})$.

3. Subspaces. In this section we will focus on topological subspaces of the spaces $\boldsymbol{Q}_\alpha$.

A space X *partitions* a space Y (see [4]) if there is a family of embeddings of X into Y whose images form a cover of Y by pairwise disjoint sets. Our first aim is to show that any nonempty subspace of $\boldsymbol{Q}_\alpha$ partitions $\boldsymbol{Q}_\alpha$ (a property shared by the space of irrational numbers and the Cantor discontinuum, but not the real line [4]). The proof for $\alpha = \omega$ is quite easy and rests on the following well known result [9].

3.1. LEMMA (Sierpiński). *Let X be countable, first countable, regular T_1, and self-dense. Then X is homeomorphic to $\boldsymbol{Q}$ ($X \simeq \boldsymbol{Q}$).*

3.2. THEOREM. *Let X be a nonempty subspace of $\boldsymbol{Q}_\omega = \boldsymbol{Q}$. Then X partitions $\boldsymbol{Q}_\omega$.*

Proof. Simply note that by Lemma 3.1, $X \times \boldsymbol{Q} \simeq \boldsymbol{Q}$. ∎

To prove an analogue to Theorem 3.2 for $\alpha > \omega$, we will need some machinery a bit more involved, namely the ultraproduct construction [1, 3, 5] of which we give only a sketch here.

Let $\langle A_i : i \in I\rangle$ be an indexed family of sets, with D an ultrafilter of subsets of I. $\Pi_D A_i$ (respectively $A^{(D)}$, when $A_i = A$ for all $i \in I$) denotes the *D-ultraproduct* (respectively *D-ultrapower*), namely the set of equivalence classes $[\bar{a}]_D$; where $\bar{a} \in \prod_{i\in I} A_i$, and $\bar{a} \underset{D}{\sim} \bar{b}$ iff $\{i: \bar{a}(i) = \bar{b}(i)\} \in D$. If $R_i \subseteq A_i^n$ is an n-ary relation on A_i, $0 < n < \omega$, then $\Pi_D R_i = \{\langle [\bar{a}_1]_D, \ldots, [\bar{a}_n]_D \rangle \in (\Pi_D R_i)^n$: $\{i: \langle \bar{a}_1(i), \ldots, \bar{a}_n(i) \rangle \in R_i\} \in D\}$. In the ultrapower case, there is a natural *D-diagonal* map, denoted by $\Delta_D: A \to A^{(D)}$, which takes $a \in A$ to $[\text{const } a]_D$. When A carries additional finitary relations (e.g., order structure, algebraic structure) this mapping is an "elementary embedding", in the parlance of model theory.

Since some of the following arguments use techniques from model theory, in particular the theory of ultraproducts and saturated models, we refer the reader to [5] for the basic theory and terminology. Regrettably we cannot make the paper self-contained for topologists who do not have some grounding in model theory.

When X is a topological space ("X" also stands for the underlying point set) and $\mathscr{B}$ is a basis for the open sets of X, we use $\langle X; \mathscr{B} \rangle$ to denote the relational structure whose universe is $X \cup \mathscr{B}$, and whose distinguished relations are X (unary for points), $\mathscr{B}$ (unary for basic open sets), and $\in$ (binary for membership between members of X and members of $\mathscr{B}$). Thus $\langle X; \mathscr{B} \rangle = \langle X \cup \mathscr{B}, X, \mathscr{B}, \in \rangle$. If D is an ultrafilter then $\langle X^{(D)}; \mathscr{B}^{(D)} \rangle = \langle (X \cup \mathscr{B})^{(D)}, X^{(D)}, \mathscr{B}^{(D)}, \in^{(D)} \rangle$. Note that $(X \cup \mathscr{B})^{(D)}$ and $X^{(D)} \cup \mathscr{B}^{(D)}$ can be put in a natural one-to-one correspondence, and that $\mathscr{B}^{(D)}$ is a topological basis for $X^{(D)}$ (see also [1, 3] for a more complete treatment of topological ultraproducts).

A very simple but important result from [1] is that if $<$ is a linear order on X, $\mathscr{B}$ is a topological basis for the order topology, and D is any ultrafilter then $\mathscr{B}^{(D)}$ is a topological basis for the order topology on $X^{(D)}$ arising from $<^{(D)}$.

By way of a brief digression into general model theory, suppose $A = \langle A, \ldots, R, \ldots \rangle$ is a relational structure (over a countable language). If D is an ultrafilter then Δ_D is an elementary embedding. In particular, if $\langle X; \mathscr{B} \rangle$ is a topological basis structure then for each $B \in \mathscr{B}$, $\Delta_D[B] = \Delta_D(B) \cap \Delta_D[X]$. Thus $\Delta_D \restriction X$ is a topological embedding, provided it is continuous. We will come back to this later.

We assume the reader to be familiar with what it means for a relational structure A to be α-saturated, for α a cardinal number. In particular, the η_α-sets are precisely the α-saturated dense linearly ordered sets without endpoints.

Of major importance to us are the following well known results.

3.3. LEMMA. *Let A be a relational structure and let D be a β^+-good countably incomplete ultrafilter on a set of cardinality β. Then $A^{(D)}$ is β^+-saturated, and of cardinality $|A|^\beta$.*

3.4. LEMMA. *Any two α-saturated elementarily equivalent relational structures of cardinality α are isomorphic.*

Fix $\alpha = \alpha^{<\alpha}$. If α is a successor we fix $\alpha = \beta^+ = 2^\beta$ and let D be a β^+-good countably incomplete ultrafilter on β. If α is a limit cardinal, we let $\langle \beta_\xi : \xi < \alpha \rangle$ be a fixed increasing sequence of cardinals which is cofinal in α (note: α is regular; and for $\xi < \alpha$, $\beta_\xi^+ < \alpha$, and $2^{\beta_\xi} \leqslant \alpha$.); and for each $\xi < \alpha$ we let D_ξ be a β_ξ^+-good countably incomplete ultrafilter on β_ξ. If A is a relational structure of cardinality $\leqslant \alpha$ we form an elementary extension $A^{(\alpha)}$ of A, which is α-saturated and of cardinality α, as follows: If $\alpha = \beta^+$, set $A^{(\alpha)} = A^{(D)}$. If $\alpha = \operatorname{Sup}\{\beta_\xi : \xi < \alpha\}$ let $A^{(\alpha)}$ be the union of the elementary chain $\{A^{(\xi)} : \xi < \alpha\}$ where $A^{(0)} = A^{(D_0)}$, $A^{(\xi+1)} = A^{(\xi)(D_{\xi+1})}$, and $A^{(\gamma)} = \bigcup_{\xi<\gamma} A^{(\xi)}$ where γ is a limit ordinal.

3.5. THEOREM. *Let $\alpha = \alpha^{<\alpha}$. Then $\boldsymbol{Q}_\alpha$ is an "η_α-field" (i.e., a field which is ordered by an η_α-set). Hence $\boldsymbol{Q}_\alpha$ is a homogeneous LOTS.*

Proof. Letting A be the ordered field of rational numbers, we obtain $A^{(\alpha)}$ via the machinery outlined above. Then the order structure on $A^{(\alpha)}$ is an η_α-set of cardinality α, hence $\boldsymbol{Q}_\alpha$. To get (point) homogeneity, we use the additive abelian group structure on $\boldsymbol{Q}_\alpha$ to translate points. ∎

3.6. THEOREM. *Let $\alpha = \alpha^{<\alpha}$ and let X be a regular T_1 space which is self-dense, and of cardinality and weight $\leqslant \alpha$. Then $X^{(\alpha)} \simeq \boldsymbol{Q}_\alpha$.*

Proof. Choose a basis $\mathscr{B}$ for X which has cardinality $\leqslant \alpha$, and let $\langle X_0; \mathscr{B}_0 \rangle$ be a countable elementary substructure of $\langle X; \mathscr{B} \rangle$. Then $\langle X_0; \mathscr{B}_0 \rangle$ (more precisely $\langle X_0; \{B \cap X_0 : B \in \mathscr{B}_0\} \rangle$) generates a regular T_1 space which is self-dense, and of countable cardinality and weight. By Lemma 3.1 there is a basis $\mathscr{C}$ for the open sets of $\boldsymbol{Q}$ such that $\langle X_0; \mathscr{B}_0 \rangle \cong \langle \boldsymbol{Q}; \mathscr{C} \rangle$. Therefore $\langle X; \mathscr{B} \rangle$ and $\langle \boldsymbol{Q}; \mathscr{C} \rangle$ are elementarily equivalent. So we use Lemma 3.4, plus the machine for constructing the $A^{(\alpha)}$'s, and conclude that $\langle X; \mathscr{B} \rangle^{(\alpha)} \cong \langle \boldsymbol{Q}; C \rangle^{(\alpha)}$. Thus $X^{(\alpha)}$ is homeomorphic with $\boldsymbol{Q}_\alpha$. ∎

3.7. COROLLARY. $\boldsymbol{Q}_\alpha \simeq \boldsymbol{Q}_\alpha^2$ *(with the usual product topology).*

Proof. Simply use Theorem 3.6 to conclude $\boldsymbol{Q}_\alpha \simeq \boldsymbol{Q}^{2(\alpha)}$. It is then easy to verify (since ultraproducts commute with finite cartesian products) that $\boldsymbol{Q}^{2(\alpha)} \simeq \boldsymbol{Q}^{(\alpha)2} \simeq \boldsymbol{Q}_\alpha^2$. ∎

We can now prove our analogue to Theorem 3.2 for uncountable α.

3.8. THEOREM. *Let $X \subseteq \boldsymbol{Q}_\alpha$ be nonempty. Then X partitions $\boldsymbol{Q}_\alpha$.*

Proof. We actually prove that X partitions $\boldsymbol{Q}_\alpha^3$ and then invoke Corollary 3.7. We use the technique of Theorem 2.5 in [3], in analogy with the question of subsets of the real line partitioning Euclidean 3-space.

By Theorem 3.5 we can use the η_α-field structure of $\boldsymbol{Q}_\alpha$ to treat $\boldsymbol{Q}_\alpha^3$ as affine 3-space. Thus we can talk of affine lines and planes in $\boldsymbol{Q}_\alpha^3$ as if we were in Euclidean space. In particular, lines are (affine) homeomorphs of $\boldsymbol{Q}_\alpha$, a line L not contained in a plane P must intersect P in at most one point, each point $p \in P$ is contained in α distinct lines in P, and each point $p \in \mathbf{Q}_\alpha^3$ is contained in α distinct planes.

Let $\emptyset \neq X \subseteq \mathbf{Q}_\alpha$, and let $\langle p_\xi: \xi < \alpha \rangle$ be a well ordering of the points of $\mathbf{Q}_\alpha^3$. Inducting on α, we assume that p_ξ is the first point not covered by a copy of X, that $p_\gamma \in X_\gamma \simeq X$ for $\gamma < \xi$, and that distinct X_γ's are disjoint and embedded in affine lines $L_\gamma \subseteq \mathbf{Q}_\alpha^3$. Since $|\xi| < \alpha$ there is a plane P_ξ containing p_ξ but failing to contain any L_γ for $\gamma < \xi$.

Thus $|P_\xi \cap X_\gamma| \leqslant 1$ for $\gamma < \xi$, so $|P_\xi \cap \bigcup_{\gamma < \xi} X_\gamma| < \alpha$. Since there are α lines in P_ξ containing p_ξ, there is one, say L_ξ, which misses $\bigcup_{\gamma<\xi} X_\gamma$ altogether. Since $L_\xi \simeq \mathbf{Q}_\alpha$ is homogeneous, there is a copy X_ξ of X such that $p_\xi \in X_\xi \subseteq L_\xi$, and the induction is complete. ■

We next turn to characterizing those topological spaces X which embed as subspaces of $\mathbf{Q}_\alpha$, for $\alpha = \alpha^{<\alpha}$. Clearly if X does embed in $\mathbf{Q}_\alpha$ then (i) X is regular T_1; (ii) both the cardinality and the weight of X are $\leqslant \alpha$; and (iii) X is α-additive. We will show that these three conditions suffice for X to embed in $\mathbf{Q}_\alpha$. When $\alpha = \omega$, a simple application of Lemma 3.1 does the trick. For uncountable α, however, it seems necessary to resort again to model-theoretic methods.

3.9. THEOREM. *Assume $\alpha = \alpha^{<\alpha}$ and suppose X is a space which is regular T_1, both of whose cardinality and weight are $\leqslant \alpha$, and which is α-additive. Then X embeds in $\mathbf{Q}_\alpha$.*

Proof. First let $Y = X \times \mathbf{Q}_\alpha$. Then Y has all of the above properties and is self-dense as well. By Theorem 3.6, then, $Y^{(\alpha)} \simeq \mathbf{Q}_\alpha$; so it remains to show that Y embeds in $Y^{(\alpha)}$. This will suffice since X clearly embeds in Y.

Suppose $\alpha = \beta^+ = 2^\beta$. Then $Y^{(\alpha)} = Y^{(D)}$. To show that $\Delta_D\colon Y \to Y^{(D)}$ is a topological embedding we need only show continuity. Let $\mathscr{B}$ be a basis for the topology on X, and let $[\bar{B}]_D \in \mathscr{B}^{(D)}$. Then $\Delta_D^{-1}[[\bar{B}]_D] = \bigcup_{J \in D} \bigcap_{\xi \in J} \bar{B}(\xi)$, an open set in Y since Y is α-additive.

Suppose $\alpha = \operatorname{Sup}\{\beta_\xi: \xi < \alpha\}$, and $Y^{(\alpha)}$ is constructed as a chain union of the $Y^{(\xi)}$'s using the ultrafilters D_ξ, $\xi < \alpha$. For each $\xi < \alpha$, let $d_\xi = \Delta_{D_\xi}\colon Y^{(\xi)} \to Y^{(\xi+1)}$, and let $e_\xi\colon Y \to Y^{(\xi)}$ be the natural elementary embedding. Since α is a limit ordinal, $e_\alpha\colon Y \to Y^{(\alpha)}$ will be continuous provided the same is true for each ξ. The only difficulty in a proof by induction on α is at the successor stages, but that case has essentially been taken care of: Let $\mathscr{B}$ be a basis for the topology on $Y^{(\xi)}$, and let $[\bar{B}]_{D_\xi} \in \mathscr{B}^{(D_\xi)}$. Then

$$\begin{aligned} e_{\xi+1}^{-1}[[\bar{B}]_{D_\xi}] &= e_\xi^{-1}\left[d_\xi^{-1}[[\bar{B}]_{D_\xi}]\right] \\ &= e_\xi^{-1}\Big[\bigcup_{J \in D_\xi} \bigcap_{\gamma \in J} \bar{B}(\gamma)\Big] \\ &= \bigcup_{J \in D_\xi} \bigcap_{\gamma \in J} e_\xi^{-1}[\bar{B}(\gamma)], \end{aligned}$$

an open set in Y since e_ξ is continuous by the inductive hypothesis, $|J| \leqslant \beta_\xi < \alpha$ for all $J \in D_\xi$, $\xi < \alpha$, and Y is α-additive (note: The maps d_ξ are generally not continuous.). ∎

3.10. Remark. Although the subspaces of $\boldsymbol{Q}_\alpha$ can be characterized in a purely topological manner, it seems the same cannot be said for $\boldsymbol{Q}_\alpha$ itself: some sort of saturatedness condition must be imposed; and that involves the semantics of artificial language. The following example dashes any hope of achieving the obvious analogue to Lemma 3.1 for uncountable α.

Call a space X $\boldsymbol{Q}_\alpha$*-like* if X is regular T_1, of cardinality and weight α, which is α-additive and self-dense. Clearly, there are no $\boldsymbol{Q}_\alpha$-like spaces of singular cardinality, and $\boldsymbol{Q}$ is the only $\boldsymbol{Q}_\omega$-like space.

3.11. Example. For any regular uncountable α there exists a $\boldsymbol{Q}_\alpha$-like space which is not a Baire space.

Construction. We use a well known example due to Sikorski [10]. Let $(2^\alpha)_\alpha$ (see also [6]) denote the space formed by allowing as basis all $< \alpha$ intersections of open sets in the usual product topology on ${}^\alpha 2$; and let

$$\mathscr{D}_\alpha = \{f \in {}^\alpha 2\colon f(\xi) = 0 \text{ for all but finitely many } \xi < \alpha\} \subseteq (2^\alpha)_\alpha.$$

Then $\mathscr{D}_\alpha$ is $\boldsymbol{Q}_\alpha$-like but is the union of countably many nowhere dense subsets. ∎

3.12. Question. Are there $\boldsymbol{Q}_\alpha$-like spaces which are not homeomorphic to $\boldsymbol{Q}_\alpha$ but which have open bases with the α-intersection condition (see Lemma 2.1 (i))?

To end on a more positive note, it is easy to prove that every subspace X of $\boldsymbol{Q}$ can be embedded as a closed subspace of $\boldsymbol{Q}$: X is closed in $X \times \boldsymbol{Q} \simeq \boldsymbol{Q}$. A similar statement can be made for $\boldsymbol{Q}_\alpha$ when α is uncountable. (We are thankful to R. L. Levy for bringing this question to our attention.) We will first need a lemma, the proof of which can be easily adapted from the proof of New Theorem 7.7 in [2].

3.13. Lemma ([2]). *Let X be an α-additive regular T_1 space and let D be an ultrafilter on a set of cardinality $< \alpha$. Then Δ_D embeds X as a closed subset of $X^{(D)}$.*

3.14. Theorem. (i) *Every subspace of $\boldsymbol{Q}_\alpha$ embeds as a closed subspace of $\boldsymbol{Q}_\alpha$.*

(ii) *Let X be a nonempty subspace of $\boldsymbol{Q}_\alpha$. Then $\boldsymbol{Q}_\alpha$ can be partitioned into homeomorphs of X, each of which is closed and nowhere dense in $\boldsymbol{Q}_\alpha$.*

Proof. (i) Let $X \subseteq \boldsymbol{Q}_\alpha$ and refer to the proof of Theorem 3.9. We show that $Y = X \times \boldsymbol{Q}_\alpha$ embeds as a closed subspace of $Y^{(\alpha)}$. In the case $\alpha = \beta^+ = 2^\beta$, we apply Lemma 3.13 directly. When $\alpha = \operatorname{Sup}\{\beta_\xi\colon \xi < \alpha\}$, use induction on α: at the successor stages, the only stages where difficulties may arise, use Lemma 3.13 again.

(ii) This follows easily from (i) above plus the proof of Theorem 3.8. ∎

References

[1] P. Bankston, *Ultraproducts in topology*, Gen. Top. and Appl. 7 (1977), pp. 283–308, MR56 #16554.

[2] – *Note on 'Ultraproducts in topology'*, Gen. Top. and Appl. 10 (1979), pp. 231–232.

[3] – *Topological reduced products via good ultrafilters*, Gen. Top. and Appl. 10 (1979), pp. 121–137.

[4] – and R. J., McGovern, *Topological partitions*, Gen. Top. and Appl. 10 (1979), pp. 215–229.

[5] C. C. Chang and H. J. Keisler, *Model Theory*, North Holland, Amsterdam 1973.

[6] W. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer-Verlag, Berlin 1974.

[7] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton 1960.

[8] F. Hausdorff, *Gundzüge der Mengenlehre*, Leipzig 1914.

[9] W. Sierpiński, *Sur une propriété topologique des ensembles dénombrables dense en soi*, Fund. Math. 1 (1920), pp. 11–16.

[10] R. Sikorski, *Remarks on some topological spaces of high power*, Fund. Math. 37 (1950), pp. 125–136.

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE
MARQUETTE UNIVERSITY
Milwaukee, Wisconsin 53233

Accepté par la Rédaction le 6.4.1981