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Various Characterizations of Modified Weibull and Log-Modified Weibull Distributions

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Abstract: Various characterizations of the well-known modified Weibull and log-modified Weibull distributions are presented. These characterizations are based on a simple relationship between two truncated moments; on the hazard function and on functions of the order statistics.

Zusammenfassung: Diverse Charakteristika der gut bekannten modifizierten Weibull und log-modifizierten Weibull Verteilungen werden angeführt. Diese beruhen auf der einfachen Beziehung zwischen zwei abgeschnittenen Momenten; auf der Hazardfunktion und auf Funktionen der Ordnungsstatistiken.

Keywords: Hazard Function, Order Statistics, Truncated Moments.

1 Introduction

As we have mentioned in our previous characterization works, the problem of characterizing a distribution is an important problem which has recently attracted the attention of many researchers. Thus, various characterizations have been established in many different directions.

The modified Weibull (MW) distribution is one of the most important distributions in lifetime modeling, and some well-known distributions are special cases of it. This distribution was introduced by Lai, Xie, and Murthy (2003) to which we refer the reader for a detailed discussion as well as applications of the MW distribution (in particular, the use of a real data set representing failure times to illustrate the modeling and estimation procedure). The MW distribution depends on scale, shape and accelerating parameters. Recently, Carrasco, Ortega, and Cordeiro (2008), Ortega, Cordeiro, and Carrasco (2011) extended the MW distribution by adding another shape parameter and introducing a four-parameter generalized MW (GMW) and log-GMW (LGMW). In Section 8 of Carrasco et al. (2008) two applications of GMW in *serum-reversal data* and *radiotherapy data* are presented to which we refer the interested reader for details. In Section 6 of Ortega et al. (2011), an application of LGMW in *survival times for the golden shiner data* is presented. Although in many applications an increase in the number of parameters provides a more suitable model, in the characterization problem a lower number of parameters (without affecting the suitability of the model) is mathematically more appealing (see Glänzel and Hamedani, 2001), especially in the MW case which already has a shape parameter. So, we restrict our attention to the MW and log-MW (LMW) distributions. In the applications where the underlying distribution is assumed to be MW (or LMW), the investigator needs to verify that the underlying distribution is in fact the MW (or LMW). To this end the investigator has to rely on the characterizations of these distributions and determine if the corresponding conditions are satisfied. Thus, the problem of characterizing the MW (or

LMW) become essential. Our objective here is to present characterizations of MW as well as LMW distributions. We shall do this in three different directions as discussed in Section 2 below.

The probability density function (pdf) and cumulative distribution function (cdf) of the MW distribution are given, respectively, by

$$f(x) = f(x;\mu,\delta,\gamma) = \mu(\delta+\gamma x)x^{\delta-1}\exp\{\gamma x - \mu x^{\delta}e^{\gamma x}\}\,, \qquad x > 0 \tag{1}$$

and

$$F(x) = 1 - \exp\{-\mu x^{\delta}e^{\gamma x}\}\,, \qquad x \geq 0\,, \tag{2}$$

where $\mu > 0$, $\delta \geq 0$ and $\gamma > 0$ are parameters. The parameters μ, δ and γ are scale, shape and accelerating, respectively.

Let X have pdf (1) and define $Y = \log X$, then Y is said to have a LMW distribution with parameters $\mu > 0$, $\delta \geq 0$ and $\gamma > 0$. The pdf and cdf of Y are given, respectively, by

$$f(y) = \mu(\delta+\gamma e^{y})\exp\{\delta y + \gamma e^{y} - \mu e^{\delta y+\gamma e^{y}}\}\,, \qquad y \in \mathbb{R} \tag{3}$$

and

$$F(y) = 1 - \exp\{-\mu e^{\delta y+\gamma e^{y}}\}\,, \qquad y \in \mathbb{R}\,. \tag{4}$$

2 Characterization Results

As we mentioned in the previous section, the MW and LMW distributions have applications in many fields of study, in particular in life time modeling. So, an investigator will be vitally interested to know if their model fits the requirements of MW or LMW distributions. To this end, the investigator relies on characterizations of these distributions, which provide conditions under which the underlying distribution is indeed a MW or a LMW distribution. In this section we will present various characterizations of these distributions.

Throughout this section we assume that the distribution function F is twice differentiable on its support.

2.1 Characterization Based on two Truncated Moments

In this subsection we present characterizations of the MW and LMW distributions in terms of truncated moments. We like to mention here the works of Galambos and Kotz (1978), Kotz and Shanbhag (1980), Glänzel (1988, 1987, 1990, 1994); Glänzel, Telcs, and Schubert (1984); Glänzel and Hamedani (2001) and Hamedani (1993, 2002, 2006) in this direction. Our characterization results presented here will employ an interesting result due to Glänzel (1987) (Theorem G below).

Theorem G. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $a < b$ ($a = -\infty$, $b = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with distribution function F and let g and h be two real functions defined on H such that

$$\mathrm{E}[g(X)|X \geq x] = \mathrm{E}[h(X)|X \geq x]\eta(x)\,, \qquad x \in H\,,$$

is defined with some real function η. Assume that $g,\, h \in C^1(H)$, $\eta \in C^2(H)$ and F is a twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and η, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)h(u) - g(u)} \right| \exp(-s(u))du\,,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' h}{\eta h - g}$ and C is a constant, chosen to ensure $\int_H dF = 1$.

Remark 2.1.1. (a) In Theorem G, the interval H need not be closed. (b) The goal is to have the function η as simple as possible. For a more detailed discussion on the choice of η we refer the reader to Glänzel and Hamedani (2001) and Hamedani (1993, 2002, 2006).

Proposition 2.1.2. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) = 1$ and $g(x) = x - \frac{1}{\mu}(\delta + \gamma x)^{-1} x^{1-\delta} e^{-\gamma x}$ for $x \in (0, \infty)$. The pdf of X is (1) if and only if the function η defined in Theorem G has the form $\eta(x) = x$, $x > 0$.
Proof. Let X have pdf (1), then

$$(1 - F(x))\mathrm{E}[h(X)|X \geq x] = \exp\{-\mu x^\delta e^{\gamma x}\}\,, \qquad x > 0\,,$$

and

$$\begin{aligned}
&(1 - F(x))\mathrm{E}[g(X)|X \geq x] \\
&= \int_x^\infty \left(u - \frac{1}{\mu}(\delta + \gamma u)^{-1} u^{1-\delta} e^{-\gamma u}\right) \left(\mu(\delta + \gamma u)u^{\delta - 1}\right) \exp\{\gamma u - \mu u^\delta e^{\gamma u}\}du \\
&= \int_x^\infty (\mu(\delta + \gamma u)u^\delta e^{\gamma u} - 1) \exp\{-\mu u^\delta e^{\gamma u}\}du \\
&= x \exp\{-\mu x^\delta e^{\gamma x}\}\,, \qquad x > 0\,,
\end{aligned}$$

and finally

$$\eta(x)h(x) - g(x) = \frac{1}{\mu}(\delta + \gamma x)^{-1} x^{1-\delta} e^{-\gamma x} > 0\,, \qquad \text{for } x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)h(x)}{\eta(x)h(x) - g(x)} = \mu(\delta + \gamma x)x^{\delta - 1} e^{\gamma x}\,, \qquad x > 0\,,$$

and hence

$$s(x) = \mu x^\delta e^{\gamma x}\,, \qquad x > 0.$$

Now, in view of Theorem G, X has cdf (2) and pdf (1).

Corollary 2.1.3. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) = 1$ for $x \in (0, \infty)$. The pdf of X is (1) if and only if there exist functions g and η defined in Theorem G satisfying the differential equation

$$\frac{\eta'(x)}{\eta(x) - g(x)} = \mu(\delta + \gamma x)x^{\delta - 1} e^{\gamma x}\,, \qquad x > 0\,.$$

Remark 2.1.4. The general solution of the differential equation given in Corollary 2.1.3 is

$$\eta(x) = \exp\{\mu x^{\delta} e^{\gamma x}\} \left[-\int g(x)\mu(\delta+\gamma x)x^{\delta-1}e^{\gamma x} \exp\{-\mu x^{\delta} e^{\gamma x}\} dx + D \right],$$

for $x > 0$, where D is a constant. One set of appropriate functions is given in Proposition 2.1.2 with $D = 0$.

Proposition 2.1.5. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $h(x) = 1$ and $g(x) = x - \frac{1}{\mu}(\delta + \gamma e^{x})^{-1} \exp(-\delta x - \gamma e^{x})$ for $x \in \mathbb{R}$. The pdf of X is (3) if and only if the function η defined in Theorem G has the form $\eta(x) = x$, $x \in \mathbb{R}$.
The Proof is similar to that of Proposition 2.1.2.

Corollary 2.1.6. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $h(x) = 1$ for $x \in \mathbb{R}$. The pdf of X is (3) if and only if there exist functions g and η defined in Theorem G satisfying the differential equation

$$\frac{\eta'(x)}{\eta(x) - g(x)} = \mu(\delta + \gamma e^{x}) \exp(\delta x + \gamma e^{x}), \qquad x \in \mathbb{R}.$$

Remark 2.1.7. The general solution of the differential equation given in Corollary 2.1.6 is

$$\eta(x) = \exp\{\mu e^{\delta x} + \gamma e^{x}\} \left[-\int g(x)\mu(\delta+\gamma e^{x}) \exp\{\delta x + \gamma e^{x} - \mu e^{\delta x + \gamma e^{x}}\} dx + D \right],$$

for $x \in \mathbb{R}$, where D is a constant. One set of appropriate functions is given in Proposition 2.1.5 with $D = 0$.

2.2 Characterization Based on the Hazard Function

For the sake of completeness, we state the following definition.
Definition 2.2.1. Let F be an absolutely continuous distribution with the corresponding pdf f. The hazard function corresponding to F is denoted by λ_F and is defined by

$$\lambda_F(x) = \frac{f(x)}{1 - F(x)}, \qquad x \in \text{support}(F). \tag{5}$$

It is obvious that the hazard function of a twice differentiable distribution function satisfies the first order differential equation

$$\frac{\lambda_F'(x)}{\lambda_F(x)} - \lambda_F(x) = k(x), \tag{6}$$

where $k(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f'(x)}{f(x)} = \frac{\lambda_F'(x)}{\lambda_F(x)} - \lambda_F(x),$$

for many univariate continuous distributions (6) seems to be the only differential equation in terms of the hazard function. The goal here is to establish a differential equation which has as simple form as possible and is not of the trivial form (6). For some general families of distributions this may not be possible. Here are our characterization results for MW and LMW distributions.

Proposition 2.2.2. Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is (1), if and only if its hazard function λ_F satisfies the differential equation

$$\lambda_F'(x) - \gamma \lambda_F(x) = \mu\delta((\delta-1)+\gamma x)x^{\delta-2}e^{\gamma x}\,, \qquad x > 0\,. \tag{7}$$

Proof. If X has pdf (1), then obviously (7) holds. If λ_F satisfies (7) , then

$$\frac{d}{dx}(e^{-\gamma x}\lambda_F(x)) = \mu\frac{d}{dx}((\delta+\gamma x)x^{\delta-1})\,,$$

or

$$\lambda_F(x) = \frac{f(x)}{1-F(x)} = \mu((\delta+\gamma x)x^{\delta-1}e^{\gamma x})\,.$$

Integrating both sides of the above equation with respect to x from 0 to x and after some computations, we arrive at (2).

Proposition 2.2.3. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The pdf of X is (3) if and only if its hazard function λ_F satisfies the differential equation

$$\lambda_F'(x) - \gamma e^x \lambda_F(x) = \mu(\delta^2+\gamma(1+\delta)e^x)e^{\delta x+\gamma e^x}\,, \qquad x \in \mathbb{R}\,. \tag{8}$$

The proof is similar to that of Proposition 2.2.2.

Remark 2.2.4. For characterizations of other well-known continuous distributions based on the hazard function, we refer the reader to Hamedani (2004) and Hamedani and Ahsanullah (2005).

2.3 Characterization Based on Truncated Moments of Certain Functions of Order Statistics

Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be n order statistics from a continuous cdf F. We present here characterization results based on some functions of these order statistics. We refer the reader to Ahsanullah and Hamedani (2007), and Hamedani, Ahsanullah, and Sheng (2008); Hamedani (2010), among others, for characterizations of other well-known continuous distributions in this direction.

Proposition 2.3.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable with cdf F such that $\lim_{x\to\infty} x^\delta e^{\gamma x}(1-F(x))^n = 0$, for some $\delta \ge 0$, $\gamma > 0$. Then

$$\mathrm{E}[X_{1:n}^\delta \exp(\gamma X_{1:n}) | X_{1:n} > t] = t^\delta e^{\gamma t} + \frac{1}{n\mu}\,, \qquad t > 0\,, \tag{9}$$

if and only if X has cdf (2).

Proof. If X has cdf (2), then clearly (9) is satisfied. Now, if (9) holds, then using integration by parts on the left hand side of (9), in view of the assumption $\lim_{x\to\infty} x^{\delta}e^{\gamma x}(1-F(x))^n = 0$, we have

$$\int_t^{\infty} (\delta+\gamma x)x^{\delta-1}e^{\gamma x}(1-F(x))^n dx = \left(t^{\delta}e^{\gamma t} + \frac{1}{n\mu}\right)(1-F(t))^n\,, \qquad t>0\,. \quad (10)$$

Differentiating both sides of (10) with respect to t, we arrive at

$$\frac{f(t)}{1-F(t)} = \mu(\delta+\gamma t)t^{\delta-1}e^{\gamma t}\,, \qquad t>0\,. \quad (11)$$

Now, integrating both sides of (11) from 0 to x, we have

$$F(x) = 1-\exp\{-\mu x^{\delta}e^{\gamma x}\}\,, \qquad x\geq 0\,.$$

Proposition 2.3.2. Let $X:\Omega\to(0,\infty)$ be a continuous random variable with cdf F. Then

$$\mathrm{E}[\exp(-\mu X_{n:n}^{\delta}e^{\gamma X_{n:n}})|X_{n:n}<t] = \frac{1}{n+1}(1+n\exp(-\mu x^{\delta}e^{\gamma t})\,, \qquad t>0 \quad (12)$$

for some $\mu>0$, $\delta\geq 0$ and $\gamma>0$, if and only if X has cdf (2).
The proof is similar to that of Proposition 2.3.1.

Let X_j, $j=1,\dots,n$ be n iid random variables with cdf F and corresponding pdf f and let $X_{1:n}\leq X_{2:n}\leq\cdots\leq X_{n:n}$ be their corresponding order statistics. Let $X^*_{1:n-i+1}$ be the first order statistic from a sample of size $n-i+1$ of random variables with cdf $F_t(x)=\frac{F(x)-F(t)}{1-F(t)}$, $x\geq t$ (t fixed) and corresponding pdf $f_t(x)=\frac{f(x)}{1-F(t)}$, $x\geq t$. Then

$$(X_{i:n}|X_{i-1:n}=t) \overset{d}{=} X^*_{1:n-i+1}\,,$$

where "$\overset{d}{=}$" means equality in distribution, that is

$$f_{X_{i:n}|X_{i-1:n}}(x|t) = f_{X^*_{1:n-i+1}}(x) = (n-i+1)(1-F_t(x))^{n-i}\frac{f(x)}{1-F(t)}\,, \qquad x\geq t\,.$$

Now we can state the following characterization of the MW distribution in yet somewhat different direction.

Proposition 2.3.3. Let $X:\Omega\to(0,\infty)$ be a continuous random variable with cdf F. Then

$$\mathrm{E}[X_{i:n}^{\delta}\exp(\gamma X_{i:n})|X_{i-1:n}=t] = t^{\delta}e^{\gamma t} + \frac{1}{(n-i+1)\mu}\,, \qquad t>0\,, \quad (13)$$

for some $\mu>0$, $\delta\geq 0$ and $\gamma>0$ if and only if X has cdf (2).
Proof. If X has cdf (2), then clearly (13) is satisfied. Now, if (13) holds, then in view of the above explanation the left hand side of (13) can be written as

$$\frac{1}{(1-F(t))^{n-i+1}}\int_t^{\infty} x^{\delta}e^{\gamma x}(n-i+1)(1-F(x))^{n-i}f(x)dx\,.$$

Now, the rest of the proof is similar to that of Proposition 2.3.1.

The following three propositions are stated for LMW distribution and their proofs are similar to those of the previous three propositions and hence will be omitted.

Proposition 2.3.4. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable with cdf F such that $\lim_{x\to\infty} \exp(\delta x + \gamma e^{x})(1 - F(x))^n = 0$ for some $\delta \geq 0, \gamma > 0$.

$$\mathrm{E}[\exp(\delta X_{1:n} e^{\gamma X_{1:n}})|X_{1:n} > t] = \exp(\delta t + \gamma e^{t}) + \frac{1}{n\mu}, \qquad t \in \mathbb{R},$$

if and only if X has cdf (4).

Proposition 2.3.5. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable with cdf F. Then

$$\mathrm{E}[\exp(-\mu \exp(\delta X_{n:n} + \gamma e^{X_{n:n}}))|X_{n:n} < t] = \frac{1}{n+1}(1 + n \exp(-\mu \exp(\delta t + \gamma e^{t})),$$

for $t \in \mathbb{R}$ and some $\mu > 0$, $\delta \geq 0$ and $\gamma > 0$, if and only if X has cdf (4).

Proposition 2.3.6. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable with cdf F. Then

$$\mathrm{E}[\exp(\exp \delta X_{i:n} + \gamma e^{X_{i:n}})|X_{i-1:n} = t] = \exp(\delta t + \gamma e^{t}) + \frac{1}{(n-i+1)\mu}, \qquad t \in \mathbb{R}$$

for some $\mu > 0$, $\delta \geq 0$ and $\gamma > 0$ if and only if X has cdf (4).

Remark 2.3.7. (i) For characterizations of some other continuous distributions in the direction of Proposition 2.3.3 we refer the reader to Ahsanullah (2009). (ii) Propositions 2.3.1 to 2.3.6 can be extended in a straightforward manner to include possibly other distributions by replacing the given functions of the order statistics with more general functions and of course under appropriate conditions.

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