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Some Remarks on Recent Characterizations of Continuous Distributions

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SOME REMARKS ON RECENT CHARACTERIZATIONS OF CONTINUOUS DISTRIBUTIONS

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ABSTRACT

We would closely look at two recent published papers dealing with characterizations of certain univariate continuous distributions. We shall explain that the main results reported lack important assumptions. These results nevertheless are based on the conditional expectations of monotone functions of the generalized order statistics. We will also mention that similar results have recently been reported without the assumption of monotonicity of the functions of the generalized order statistics.

REMARKS

Haque and Faizan (2010), stated the following lemma, which we call it "Lemma HF".

Lemma HF:

Let $F(x)$ be a *df* (distribution function) such that $F(0)=0$ and has a continuous second order derivative on $(0,\infty)$ with $F(x)>0$ for all $x>0$ (so that $F(x)<1$ for all x, in particular). If it satisfies the differential equation

$$\frac{\bar{F}''(x)}{\bar{F}(x)}+(\gamma_{r+1}-1)\left[\frac{\bar{F}'(x)}{\bar{F}(x)}\right]^2-\frac{(p-1)}{x}\left[\frac{\bar{F}'(x)}{\bar{F}(x)}\right]-\gamma_{r+1}\Theta^2 p^2 x^{2(p-1)}=0, \tag{1}$$

where $\bar{F}(x)=1-F(x)$. Then $\bar{F}(x)=e^{-\Theta x^p}$ for all $x>0$ where Θ, p, γ_{r+1} are all positive constants.

Remark 1:

It is easy to see that if $F(x)$ is *df* of a Weibull distribution with parameters Θ, p, then $\bar{F}(x)=e^{-\Theta x^p}$ satisfies equation (1). The hazard function, $\lambda_F(x)$ of the Weibull distribution is

$$\lambda_F(x)=\Theta p x^{p-1}, x>0 \text{ with } \lambda_F(1)=\Theta p \tag{2}$$

which satisfies the following differential equation

$$\lambda'_F(x)-(p-1)x^{-1}\lambda_F(x)=0\,. \tag{3}$$

For a continuous *df* $F(x)$ and $\lambda_F(x) = -\frac{\bar{F}'(x)}{\bar{F}(x)}$, we obtain from (1)

$$\lambda_F'(x) - (p-1)x^{-1}\lambda_F(x) - \gamma_{r+1}\lambda_F^2(x) + \gamma_{r+1}\Theta^2 p^2 x^{2(p-1)} = 0 \quad (4)$$

In view of (2) and (3), equation (4) is in fact the same as equation (3), the last two terms cancel out each other for the above mentioned Weibull *df*.

Haque and Faizan (2010), attempted to show that the unique *df* solution of (1) is $\bar{F}(x) = e^{-\Theta x^p}$ by looking at a change of variables

$$t = p\gamma_{r+1}x^{p-1}\left[\frac{\bar{F}'(x)}{\bar{F}(x)}\right]^{-1} < 0, \text{ for } x > 0. \quad (5)$$

In view of (5), they obtain from (1)

$$\frac{\gamma_{r+1} + \Theta t}{\gamma_{r+1} - \Theta t} = Ae^{2\gamma_{r+1}\Theta x^p}, \quad (6)$$

where A is the constant of integration. By introducing a new variable $u = e^{2\gamma_{r+1}\Theta x^p}$, they arrive at

$$\bar{F}(x) = B\left[Ae^{\gamma_{r+1}\Theta x^p} - e^{-\gamma_{r+1}\Theta x^p}\right]^{\frac{1}{\gamma_{r+1}}}, \quad (7)$$

where B is also a constant to be determined. They conclude that since $F(x)$ is bounded, then $\bar{F}(x) = e^{-\Theta x^p}$, in view of the initial conditions on x.

Although (7) is a solution of (1), we cannot obtain $\bar{F}(x) = e^{-\Theta x^p}$ from (7) without proper assumption. One can see that if $\bar{F}(x) = e^{-\Theta x^p}$ is inserted on the left hand side of (7), one obtains

$$\frac{1 + B^{-\gamma_{r+1}}}{AB^{-\gamma_{r+1}}} = e^{2\gamma_{r+1}\Theta x^p},$$

which is simply impossible for $x > 0$.

We believe what has taken place is that the authors took equation (3) and added and subtracted $\gamma_{r+1}\lambda_F^2(x)$ in (3) to arrive at (4) which is the same as (1) for any *df* $F(x)$ and

$\lambda_F(x) = -\frac{\bar{F}'(x)}{\bar{F}(x)}$. They, then tried to use (1) via a change of variables to obtain $\bar{F}(x) = e^{-\Theta x^p}$ which we showed above that it is not possible.

Remark 2:

The main result of Haque and Faizan (Theorem 2.1, page 520, (2010)) is based on Lemma HF. The approach taken by Haque and Faizan for the proof of the sufficiency part of their Theorem 2.1 (page 521) is the natural one, but unfortunately it comes down to a differential equation whose general solution does not provide a distribution function.

Remark 3:

Equation (4) is a Riccati equation with a particular solution $\lambda_{F,1}(x) = \Theta p x^{p-1}$. The general solution of (4), then can be shown to be

$$\lambda_F(x) = \Theta p x^{p-1} \tanh\left(C - \Theta \gamma_{r+1} x^p\right) \tag{8}$$

where C is an arbitrary constant. In the limit case $C \to \infty$ we rediscover the particular solution $\lambda_{F,1}(x) = \Theta p x^{p-1}$ and hence $\bar{F}(x) = e^{-\Theta x^p}$. This means that the limiting solution of (8) is the hazard function of the Weibull distribution. We can arrive at the same conclusion by integrating both sides of (8) with respect to x from 0 to x to obtain

$$\bar{F}(x) = \left(\frac{e^{\left(C-\Theta\gamma_{r+1}x^p\right)} + e^{-\left(C-\Theta\gamma_{r+1}x^p\right)}}{e^C + e^{-C}} \right)^{\frac{1}{\gamma_{r+1}}}. \tag{9}$$

Note that (9) does not produce a *df* solution, but letting $C \to \infty$, we rediscover $\bar{F}(x) = \left(e^{-\Theta\gamma_{r+1}x^p}\right)^{\frac{1}{\gamma_{r+1}}} = e^{-\Theta x^p}$.

Khan et al. (2010), stated the following theorem (their Theorem 2.1, page 617, (2010)), which we call it “Theorem KAC”. They use the notation $X'(s,n,m,k)$ for the generalized order statistics.

Theorem KAC:

Let $\xi(x)$ be a monotonic and continuous function of x. If for $1 \le r < s-1 < n$

$$E\left[\xi\left(X'(s,n,m,k)\right) \mid X'(s,n,m,k) = x\right] = g_{s|l}(x), \qquad l = r, r+1,$$

exist and is differentiable with respect to x, then

$$F(x) = \exp\left[-\frac{1}{\gamma_{r+1}} \int_x^\beta \frac{g'_{s|l}(t)}{g_{s|r+1}(t) - g_{s|r}(t)} dt \right], \quad \alpha < x < \beta \tag{10}$$

Remark 4:

i) Clearly the assumption $\lim_{x \to \alpha} \int_x^\beta \frac{g'_{s|l}(t)}{g_{s|r+1}(t) - g_{s|r}(t)} dt = \infty$ is missing in Theorem KAC.

ii) Similar assumptions are missing in their Theorem 2.2 (page 621) and Theorem 3.1 (page 624). As for their Theorem 3.2 (page 626), the appropriate assumptions on $B_s(y)$ as well as on $D(y)$ should be added to insure that $F(y)$ is indeed a *df*.

iii) We like to mention that there are other papers in the field of characterization of distributions which lack appropriate assumptions similar to the ones we mentioned above.

Remark 5:

All the characterization papers presenting similar results as the above mentioned two papers, as far as we have gathered, deal with the functions of the generalized order statistics that are monotone. It is shown in Ahsanullah and Hamedani (2013) that, this assumption can be dropped.

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