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CHARACTERIZATIONS OF DISTRIBUTION OF RATIO OF RAYLEIGH RANDOM VARIABLES

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ABSTRACT

Various characterizations of the distribution of the ratio of two independent Rayleigh random variables are presented. These characterizations are based, on a truncated moment; on hazard function; and on certain functions of order statistics.

1. INTRODUCTION

The problem of characterizing a distribution is an important problem which has recently attracted the attention of many researchers. Thus, various characterizations have been established in many different directions. As pointed out by Shakil and Ahsanullah (2011), the distribution of the ratio of independent random variables arises in many fields of studies such as biology, economics, engineering, genetics and order statistics, to name a few. For the detailed explanation of the importance of the distribution of the ratio of independent random variables, we refer the interested reader to Shakil and Ahsanullah (2011) where they consider the distributional properties of record values of the ratio of independent Rayleigh random variables. In Section 2, we present characterizations of the distribution of the ratio of two independent Rayleigh random variables in three different directions. Our results in subsections 2.1-2.3, will be based: on a truncated moment; on hazard function; and on truncated moments of certain functions of order statistics, respectively.

The *pdf* (probability density function) f, *cdf* (cumulative distribution function) F and hazard function λ_F of the distribution of the ratio of two independent Rayleigh random variables are given, respectively, by

$$f(x) = f(x;\alpha,\beta) = 2\alpha\beta x\left(\beta x^2+\alpha\right)^{-2}, \qquad x>0 \tag{1.1}$$

and

$$F(x) = 1-\alpha\left(\beta x^2+\alpha\right)^{-1}, \quad x \geq 0, \tag{1.2}$$

$$\lambda_F(x) = 2\beta x\left(\beta x^2+\alpha\right)^{-1}, \quad x>0 \tag{1.3}$$

where $\alpha>0$ and $\beta>0$ are parameters. We like to mention that *pdf* (1.1) is also the *pdf* of the square root of the Pareto random variable with the shape parameter 1.

Throughout this work we denote the ratio of two independent Rayleigh random variables with RR.

2. CHARACTERIZATION RESULTS

The distribution of the ratio of independent Rayleigh random variables has applications in many fields of study, in particular in life modeling. So, an investigator will be vitally interested to know if their model fits the requirements of RR distribution. To this end, the investigator relies on characterizations of RR distribution, which provide conditions under which the underlying distribution is indeed that of RR. In this section we will present several characterizations of this distribution.

Throughout this section we assume, where necessary, that the distribution function F is twice differentiable on its support.

2.1 Characterization Based on a Truncated Moment

In this subsection we present characterizations of RR distribution in terms of a truncated moment. We like to mention here the works of Galambos and Kotz (1978), Kotz and Shanbhag (1980), Glänzel (1987, 1988, 1990), Glänzel et al. (1984, 1994), Glänzel and Hamedani (2001) and Hamedani (1993, 2002, 2006, 2011) in this direction. Our characterization results presented here will employ a special version of an interesting result due to Glänzel (1987) (Theorem 2.1.1 below).

Theorem 2.1.1

Let (Ω, F, P) be a given probability space and let $H=[a,b]$ be an interval for some $a<b$ $(a=-\infty, b=\infty$ might as well be allowed). Let $X:\Omega \to H$ be a continuous random variable with the distribution function F and let g be a real function defined on H such that

$$E\left[g(X) \middle| X \geq x\right] = \eta(x),\ x \in H,$$

is defined with some real function η. Assume that $g \in C^1(H), \eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g and η, particularly

$$F(x) = \int_0^x C\left|\frac{\eta'(u)}{\eta(u)-g(u)}\right| \exp(-s(u))\,du\ ,$$

where the function s is a solution of the differential equation $s' = \dfrac{\eta'}{\eta - g}$ and C is a constant, chosen to make $\int_H dF = 1$.

Remarks 2.1.2

(a) In Theorem 2.1.1, the interval H need not be closed since the restriction is on the interior of H. (b) The goal is to have the function η as simple as possible. For a more detailed discussion on the choice of η, we refer the reader to Glänzel and Hamedani (2001) and Hamedani (1993, 2002, 2006).

Proposition 2.1.3

Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $g(x) = 2\left(\beta x^2 + \alpha\right)^{-1}$ for $x \in (0, \infty)$. The *pdf* of X is (1.1) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \left(\beta x^2 + \alpha\right)^{-1}, \qquad x > 0 .$$

Proof:

Let X have *pdf* (1.1), then

$$E\left[g(X) \middle| X \geq x\right] = \left(\beta x^2 + \alpha\right)^{-1}, \qquad x > 0,$$

and

$$\eta(x) - g(x) = -\left(\beta x^2 + \alpha\right)^{-1} < 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)}{\eta(x) - g(x)} = 2\beta x\left(\beta x^2 + \alpha\right)^{-1}, \qquad x > 0,$$

and hence

$$s(x) = \ln\left(\beta x^2 + \alpha\right).$$

Now, in view of Theorem 2.1.1 (with $C = \alpha$), X has *cdf* (1.2) and *pdf* (1.1) .

Corollary 2.1.4

Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The *pdf* of X is (1.1) if and only if there exist functions g and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)}{\eta(x) - g(x)} = 2\beta x\left(\beta x^2 + \alpha\right)^{-1}, \qquad x > 0.$$

Remarks 2.1.5

(a) The general solution of the differential equation given in Corollary 2.1.4 is

$$\eta(x)=\left(\beta x^2+\alpha\right)\left[-\int g(x)2\beta x\left(\beta x^2+\alpha\right)^{-2}dx+D\right],$$

for $x>0$, where D is a constant. One set of appropriate functions is given in Proposition 2.1.3 with $D=0$.

(b) Clearly there are other pairs of functions (g,η) satisfying conditions of Theorem 2.1.1. We presented one such pair of functions in Proposition 2.1.3.

2.2 Characterization Based on Hazard Function

For the sake of completeness, we state the following simple fact.

Let F be an absolutely continuous distribution with the corresponding *pdf* f. The hazard function corresponding to F is

$$\lambda F(x)=\frac{f(x)}{1-F(x)}, \qquad x\in Supp\ F, \tag{2.2.1}$$

where *Supp* F is the support of F.

It is obvious that the hazard function of a twice differentiable distribution function satisfies the first order differential equation

$$\frac{\lambda_F'(x)}{\lambda_F(x)}-\lambda_F(x)=k(x), \tag{2.2.2}$$

where $k(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f'(x)}{f(x)}=\frac{\lambda_F'(x)}{\lambda_F(x)}-\lambda_F(x),$$

for many univariate continuous distributions $(2.2.2)$ seems to be the only differential equation in terms of the hazard function. The goal here is to establish a differential equation which has as simple form as possible and is not of the trivial form $(2.2.2)$. For some general families of distributions this may not be possible. Here is our characterization result for RR distribution.

Proposition 2.2.1

Let $X:\Omega\to(0,\infty)$ be a continuous random variable. The *pdf* of X is (1.1) if and only if its hazard function λ_F, with boundary condition $\lambda_F(0)=0$, satisfies the differential equation

$$\lambda'_F(x) - x^{-1}\lambda_F(x) + \lambda_F^2(x) = 0, \qquad x > 0. \tag{2.2.3}$$

Proof:

If X has *pdf* (1.1), then obviously (2.2.3) holds. If λ_F satisfies (2.2.3), then

$$\frac{d}{dx}\left(x\lambda_F^{-1}(x)\right) = x,$$

or

$$\lambda_F(x) = \frac{f(x)}{1-F(x)} = \frac{2x}{x^2+c},$$

where c is a constant.

Integrating both sides of the above equation with respect to x from 0 to x and after some computations, we arrive at (1.2) with $c = \frac{\alpha}{\beta}$.

Remark 2.2.2

For characterizations of other well-known continuous distributions based on the hazard function, we refer the reader to Hamedani (2004) and Hamedani and Ahsanullah (2005).

2.3 Characterization based on Truncated Moment of Certain Functions of Order Statistics

Let $X_{1:n} \le X_{2:n} \le .... \le X_{n:n}$ be n order statistics from a continuous *cdf* F. We present here characterization results base on some functions of these order statistics. We refer the reader to Ahsanullah and Hamedani (2007), Hamedani et al. (2008) and Hamedani (2010), among others, for characterizations of other well-known continuous distributions in this direction.

Proposition 2.3.1.

Let $X : \Omega \to (0,\infty)$ be a continuous random variable with *cdf* F such that $\lim_{x\to\infty}\left(\beta x^2+\alpha\right)\left(1-F(x)\right)^n = 0, (n>1)$, for some $\alpha > 0$ and $\beta > 0$. Then

$$E\left[\left(\beta X_{1:n}^2+\alpha\right)\middle| X_{1:n} > t\right] = \frac{n}{n-1}\left(\beta t^2+\alpha\right), \qquad t > 0 \tag{2.3.1}$$

if and only if X has *cdf* (1.2).

Proof:

If X has *cdf* (1.2), then clearly (2.3.1) is satisfied. Now, if (2.3.1) holds, then using integration by parts on the left hand side of (2.3.1), in view of the assumption $\lim_{x\to\infty}\left(\beta x^2+\alpha\right)\left(1-F(x)\right)^n = 0$, we have

$$\int_t^\infty 2\beta x\left(1-F(x)\right)^n dx = \frac{1}{n-1}\left(\beta t^2+\alpha\right)\left(1-F(t)\right)^n, \quad t>0 \tag{2.3.2}$$

Differentiating both sides of (2.3.2) with respect to t, we arrive at

$$\frac{f(t)}{1-F(t)} = 2\beta t\left(\beta t^2+\alpha\right)^{-1}, \qquad t>0 \tag{2.3.3}$$

Now, integrating both sides of (2.3.3) from 0 to x, we have

$$F(x) = 1-\alpha\left(\beta x^2+\alpha\right)^{-1}, \quad x\geq 0 .$$

Proposition 2.3.2

Let $X:\Omega\to(0,\infty)$ be a continuous random variable with *cdf* F. Then

$$E=\left[X_{n:n}^{-2(n-1)}\middle| X_{n:n}<t\right]=\frac{\alpha}{\beta}t^{-2n}\left\{\left(\frac{\beta t^2+\alpha}{\alpha}\right)^n-1\right\}, \quad t>0, \tag{2.3.4}$$

for $n>1$ and some $\alpha>0,\ \beta>0$, if and only if X has *cdf* (1.2).

Proof:

Is similar to that of Proposition 2.3.1.

Let $X_j, j=1,2,...,n$ be n *i.i.d.* random variables with *cdf* F and corresponding *pdf* f and let $X_{1:n}\leq X_{2:n}\leq ...\leq X_{n:n}$ be their corresponding order statistics. Let $X_{1:n-i+1}^*$ be the 1*st* order statistic from a sample of size $n-i+1$ of random variables with *cdf* $F_t(x)=\frac{F(x)-F(t)}{1-F(t)}, x\geq t$ (t is fixed) and corresponding *pdf* $f_t(x)=\frac{f(x)}{1-F(t)}$, $x\geq t$. Then

$$\left(X_{i:n}\middle| X_{i-1:n}=t\right)\overset{d}{=}X_{1:n-i+1}^* \quad (\overset{d}{=} \text{ means equal in distribution}),$$

that is

$$f_{X_{i:n}|X_{i-1:n}}(x|t)=f_{X_{1:n-i+1}^*}(x)=(n-i+1)\left(1-F_t(x)\right)^{n-i}\frac{f(x)}{1-F(t)},\ x\geq t .$$

Now we can state the following characterization of the RR distribution in yet somewhat different direction.

Proposition 2.3.3

Let $X:\Omega\to(0,\infty)$ be a continuous random variable with *cdf* F. Then

$$E\left[\left(\beta X_{i:n}^{2}+\alpha\right)\middle| X_{i-1:n}=t\right]=\frac{n-i+1}{n-i}\left(\beta t^{2}+\alpha\right),\qquad t>0 \tag{2.3.5}$$

for $n>i$ and some $\alpha>0,\ \beta>0$ if and only if X has *cdf* (1.2).

Proof:

If X has *cdf* (1.2.), then clearly (2.3.5) is satisfied. Now, if (2.3.5) holds, then the left hand side of (2.3.5) can, in view of the previous explanation, be written as

$$\frac{1}{\left(1-F(t)\right)^{n-i+1}}\int_{t}^{\infty}\left(\beta x^{2}+\alpha\right)\left(n-i+1\right)\left(1-F(x)\right)^{n-i} f(x)\,dx\ .$$

Now, the rest of the proof is similar to that of Proposition 2.3.1.

Remark 2.3.4

For characterizations of some other continuous distributions in the direction of Proposition 2.3.3, we refer the reader to Ahsanullah (2009).

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