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CHARACTERIZATIONS OF CERTAIN CONTINUOUS UNIVARIATE DISTRIBUTIONS BASED ON THE CONDITIONAL DISTRIBUTION OF GENERALIZED ORDER STATISTICS

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ABSTRACT

The problem of characterizing probability distributions is an interesting problem which has recently attracted the attention of many researchers. Various characterization results have been established in different directions as reported in the literature. We present here, various characterizations of certain univariate continuous distributions based on the conditional distribution of generalized order statistics.

1. INTRODUCTION

The concept of generalized order statistics (gos) from a univariate continuous distribution was introduced by Kamps $[9]$ in terms of their joint pdf (probability density function). The order statistics, record values, k-record values, Pfeifer's records and progressive type II order statistics are special cases of the gos. The $rv's$ (random variables) $X(1,n,m,k)$, $X(2,n,m,k)$,..., $X(n,n,m,k)$, $k>0$, $m\in \mathrm{R}$, are n gos from an absolutely continuous cdf (cumulative distribution function) F with corresponding pdf f if their joint pdf, $f_{1,2,\dots,n}(x_1,x_2,\dots,x_n)$, can be written as

$$f_{1,2,\dots,n}(x_1,x_2,\dots,x_n)=k\left(\Pi_{j=1}^{n-1}\gamma_j\right)\left[\Pi_{j=1}^{n-1}\left(1-F\left(x_j\right)\right)^m f\left(x_j\right)\right]$$
$$\times\left(1-F\left(x_n\right)\right)^{k-1} f\left(x_n\right),\ \ F^{-1}(0+)<x_1<x_2<\dots<x_n<F^{-1}(1-), \qquad (1.1)$$

where $\gamma_j=k+(n-j)(m+1)$ for all j, $1\le j\le n$, k is a positive integer and $m\ge -1$.

If $k=1$ and $m=0$, then $X(r,n,m,k)$ reduces to the ordinary rth order statistic and (1.1) will be the joint pdf of order statistics $X_{1,n}\le X_{2,n}\le\dots\le X_{n,n}$ from F. If $k=1$ and $m=-1$, then (1.1) will be the joint pdf of the first n upper record values of the $i.i.d.$ (independent and identically distributed) $rv's$ with cdf F and pdf f.

Integrating out $x_1, x_2, ..., x_{r-1}, x_{r+1}, ..., x_n$ from (1.1) we obtain the *pdf* $f_{r,n,m,k}$ of $X(r,n,m,k)$

$$f_{r,n,m,k}(x) = \frac{c_r}{\Gamma(r)} \left(1-F(x)\right)^{\gamma_r - 1} f(x)\, g_m^{r-1}\left(F(x)\right), \tag{1.2}$$

where $c_r = \Pi_{j=1}^{r} \gamma_j$ and

$$g_m(x) = \frac{1}{m+1}\left[1-(1-x)^{m+1}\right], \quad m \neq -1$$
$$= -\ln(1-x), \quad m = -1, \quad x \in (0,1).$$

Since $\lim_{m \to -1} \frac{1}{m+1}\left[1-(1-x)^{m+1}\right] = -\ln(1-x)$, we will write

$$g_m(x) = \frac{1}{m+1}\left[1-(1-x)^{m+1}\right], \ \textit{for all } x \in (0,1) \textit{ and all } m \textit{ with}$$
$$g_{-1}(x) = \lim_{m \to -1} g_m(x).$$

The joint *pdf* of $X(r,n,m,k)$ and $X(r+1,n,m,k)$, $1 \le r < n$, is given by (see Kamps [9], p.68)

$$f_{r,r+1,n,m,k}(x,y)$$
$$= \frac{c_{r+1}}{\Gamma(r)}\left(1-F(x)\right)^m f(x)\, g_m^{r-1}\left(F(x)\right)\left(1-F(y)\right)^{\gamma_{r+1}-1} f(y), \quad x < y$$

and consequently the conditional *pdf* of $X(r+1,n,m,k)$ given $X(r,n,m,k) = x$, for $m \ge -1$, is

$$f_{r+1|r,n,m,k}(y \mid x) = \gamma_{r+1}\left(\frac{1-F((y))}{1-F(x)}\right)^{\gamma_{r+1}-1} \cdot \frac{f(y)}{\left(1-F(x)\right)}, \quad y > x, \tag{1.3}$$

where $\gamma_{r+1} = \gamma_r - 1 - m$ (see [2], p. 383).

2. CHARACTERIZATION RESULTS

In this section we present characterizations of certain univariate continuous distributions which include, as special cases, exponential, Pareto, power function, Rayleigh and Weibull distributions. We refer the interested readers to the references [3]−[8],[10] and [11] for further characterizations of distributions via linearity of regression or conditional distributions of order statistics, record values and *gos*.

Proposition 2.1:

Let $X : \Omega \to (0,\infty)$ be a continuous random variable with *cdf* F and g be a continuous function on $(0,\infty)$ with $\lim_{x\to 0} g(x) = 1$ and $\lim_{x\to\infty} g(x) = 0$. If

$$P\left(X(r+1,n,m,k) \geq x \mid X(r,n,m,k) = t\right) = \left(g(x-t)\right)^{\gamma_{r+1}}, \quad x \geq t, \tag{2.1}$$

then $F(x) = 1 - e^{-\lambda x}$, $x > 0$ for some $\lambda > 0$.

Proof:

In view of (1.3), from (2.1) we have

$$\left(\frac{1-F(x)}{1-F(t)}\right)^{\gamma_{r+1}} = \left(g(x-t)\right)^{\gamma_{r+1}}, \quad x \geq t,$$

or

$$\frac{1-F(x)}{1-F(t)} = g(x-t), \quad x \geq t. \tag{2.2}$$

Now, letting $t \to 0$, we arrive at $1 - F(x) = g(x)$ for all $x > 0$ and again in view of (2.2) we will have

$$1 - F(x) = \left(1 - F(x-t)\right)\left(1 - F(t)\right), \quad x \geq t. \tag{2.3}$$

Equation (2.3) is the well-known Cauchy functional equation. The non-zero solution of this equation is (see, $[1]$, Theorem 1 page 38) $1 - F(x) = e^{c\,x}$, where c is a constant. Using the boundary conditions $\lim_{x\to 0} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$ we obtain $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$ for some $\lambda > 0$.

Proposition 2.2:

Let $X : \Omega \to [1,\infty)$ be a continuous random variable with *cdf* F, $F(1) \neq 1$ and g be a continuous function on $[1,\infty)$ with $g(1) = 1$ and $\lim_{x\to\infty} g(x) = 0$. If

$$P\left(X(r+1,n,m,k) \geq x \mid X(r,n,m,k) = t\right) = \left(g\left(\tfrac{x}{t}\right)\right)^{\gamma_{r+1}}, \quad x \geq t, \tag{2.4}$$

then $F(x) = 1 - x^{\lambda}$, $x \geq 1$ for some $\lambda < 0$.

Proof:

As in the proof of Proposition 2.1, from (1.3) and (2.4) we have

$$\left(\frac{1-F(x)}{1-F(t)}\right)^{\gamma_{r+1}} = \left(g\left(\frac{x}{t}\right)\right)^{\gamma_{r+1}}, \quad x \geq t,$$

or

$$\frac{1-F(x)}{1-F(t)}=g\left(\frac{x}{t}\right)\ ,\ \ x\geq t\ . \tag{2.5}$$

Now, letting $t=1$, we arrive at $1-F(x)=g(x)$ for all $x\geq 1$ and hence

$$1-F(x)=\left(1-F\left(\frac{x}{t}\right)\right)\left(1-F(t)\right)\ ,\ \ x\geq t\ . \tag{2.6}$$

Equation (2.6) is a Cauchy functional equation. The non-zero solution of this equation is (see, [1], page 39) $1-F(x)=cx^{\lambda}$, where c is a constant. Using the boundary conditions $\lim_{x\to 1}F(x)=0$ and $\lim_{x\to\infty}F(x)=1$ we obtain $F(x)=1-x^{\lambda}$, $x\geq 1$ for some $\lambda<0$.

Proposition 2.3:

Let $X:\Omega\to[0,1]$ be a continuous random variable with *cdf* F and g be a continuous function on $[0,1]$ with $g(0)=1$ and $g(1)=0$. If

$$P\left(X(r+1,n,m,k)\geq x\ |\ X(r,n,m,k)=t\right)=\left(g\left(\tfrac{1-x}{1-t}\right)\right)^{\gamma_{r+1}}\ ,\ \ x\geq t\ , \tag{2.7}$$

then $F(x)=1-(1-x)^{\lambda}$, $x\geq 0$ for some $\lambda>0$.

Proof:

As before, from (1.3) and (2.7) we have

$$\frac{1-F(x)}{1-F(t)}=g\left(\frac{1-x}{1-t}\right)\ ,\ \ x\geq t\ . \tag{2.8}$$

Now, letting $t=0$, we arrive at $1-F(x)=g(1-x)$ for all $x\geq 0$ and hence

$$1-F(x)=\left(1-F\left(1-\frac{1-x}{1-t}\right)\right)\left(1-F(t)\right)\ ,\ \ x\geq t\ ,$$

or

$$1-F(x)=\left(1-F\left(\frac{x-t}{1-t}\right)\right)\left(1-F(t)\right)\ ,\ \ x\geq t\ . \tag{2.9}$$

With $g(1-y)=1-F(y)$, equation (2.9) will be equivalent to

$$g(1-y)=g\left(\frac{1-y}{1-t}\right)g(1-t).$$

Letting $z = 1 - y$, we have $g(z) = g\left(\frac{z}{1-t}\right) g(1-t)$ which a is Cauchy functional equation. The non-zero solution of this equation is (see, [1], page 39) $g(x) = cx^{\lambda}$, where c is a constant, and hence $F(x) = 1 - c(1-x)^{\lambda}$. Using the boundary conditions $\lim_{x \to 0} F(x) = 0$ and $\lim_{x \to 1} F(x) = 1$, we obtain $F(x) = 1 - (1-x)^{\lambda}$, $0 \le x \le 1$, for some $\lambda > 0$.

Proposition 2.4:

Let $X : \Omega \to (0, \infty)$ be a continuous random variable with *cdf* F and g be a continuous function on $(0, \infty)$ with $\lim_{x \to 0} g(x) = 1$ and $\lim_{x \to \infty} g(x) = 0$. If

$$P\left(X(r+1, n, m, k) \ge x \mid X(r, n, m, k) = t\right) = \left(g\left(x^{\beta} - t^{\beta}\right)\right)^{\gamma_{r+1}}, \quad x \ge t, \qquad (2.10)$$

where $\beta > 0$, then $F(x) = 1 - e^{-\lambda x^{\beta}}$, $x > 0$ for some $\lambda > 0$.

Proof:

From (1.3) and (2.10) we have

$$\frac{1 - F(x)}{1 - F(t)} = g\left(x^{\beta} - t^{\beta}\right), \quad x \ge t. \qquad (2.11)$$

Now, letting $t \to 0$, we arrive at $1 - F(x) = g\left(x^{\beta}\right)$ for all $x > 0$ and in view of (2.11) we will have

$$\frac{1 - F(x)}{1 - F(t)} = 1 - F\left(\left(x^{\beta} - t^{\beta}\right)^{1/\beta}\right), \quad x \ge t. \qquad (2.12)$$

Now, let $\varphi(x) = \ln\left(1 - F(x)\right)$, then from (2.12), we have

$$\exp\left\{\varphi(x) - \varphi(t)\right\} = \exp\left\{\varphi\left(\left(x^{\beta} - t^{\beta}\right)^{1/\beta}\right)\right\}, \quad x \ge t. \qquad (2.13)$$

Differentiating both sides of (2.13) with respect to x, we obtain

$$\varphi'(x) \exp\left\{\varphi(x) - \varphi(t)\right\} = \varphi'\left(\left(x^{\beta} - t^{\beta}\right)^{1/\beta}\right) \frac{x^{\beta-1}}{\left(x^{\beta} - t^{\beta}\right)^{1-1/\beta}} \exp\left\{\varphi\left(\left(x^{\beta} - t^{\beta}\right)^{1/\beta}\right)\right\},$$

and in view of (2.13)

$$\frac{\varphi'(x)}{\varphi'\left(\left(x^{\beta} - t^{\beta}\right)^{1/\beta}\right)} = \frac{x^{\beta-1}}{\left(x^{\beta} - t^{\beta}\right)^{1-1/\beta}}, \quad x > t. \qquad (2.14)$$

From (2.14), we have

$$\frac{\varphi'(x)}{x^{\beta-1}} = \frac{\varphi'\left(\left(x^{\beta}-t^{\beta}\right)^{1/\beta}\right)}{\left(x^{\beta}-t^{\beta}\right)^{1-1/\beta}} .$$

Letting $h(x) = \frac{\varphi'(x)}{x^{\beta-1}}$, then in view of the last equation above, $h(x) = h\left(\left(x^{\beta}-t^{\beta}\right)^{1/\beta}\right)$ for all $x > t \geq 0$ and all $\beta > 0$. Setting $y = x^{\beta} - t^{\beta}$, we have $h(\left(y+t^{\beta}\right)^{1/\beta}) = h\left(y^{1/\beta}\right)$ for all $y \geq 0$, $t \geq 0$ and all $\beta > 0$. Thus $h(y) = c$, where c is a constant and hence $\varphi'(x) = cx^{\beta-1}$. In view of the boundary conditions on F, we obtain $F(x) = 1 - e^{-\lambda\, x^{\beta}}$, $x > 0$ for some $\lambda > 0$.

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