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On Semigroups with Lower Semimodular Lattice of Subsemigroups

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Abstract

The question of which semigroups have lower semimodular lattice of subsemigroups has been open since the early 1960's, when the corresponding question was answered for modularity and for upper semimodularity. We provide a characterization of such semigroups in the language of principal factors. Since it is easily seen (and has long been known) that semigroups for which Green's relation $\mathcal{J}$ is trivial have this property, a description in such terms is natural. In the case of periodic semigroups — a case that turns out to include all eventually regular semigroups — the characterization becomes quite explicit and yields interesting consequences. In the general case, it remains an open question whether there exists a simple, but not completely simple, semigroup with this property. Any such semigroup must at least be idempotent-free and $\mathcal{D}$-trivial.

1 Introduction.

The lattice $\mathcal{L}(S)$ of subsemigroups of a semigroup S has been a topic of intense study since the 1960's [11]. Those semigroups for which this lattice satisfies common lattice-theoretic properties such as distributivity, modularity and upper semimodularity were determined in the early years of that decade. As noted in [11, §5.14], little is known — or at least little is published — about lower semimodularity in this context, other than that an apparently diverse array of semigroups do have subsemigroup lattices with this property. To this author's mind, the fact that these include the free semigroups, free commutative semigroups, nilpotent semigroups, etc, is strong motivation for a general study.

Such a study is the purpose of this paper. In Theorem 1.1 below, we characterize the semigroups whose lattice of subsemigroups is lower semimodular in terms of their principal factors and certain relations between them. That Green's relations, in particular the relation $\mathcal{J}$, should play a central role to this study is to us self-evident, for the feature common to the known examples, other than groups, is that $\mathcal{J}$ is trivial. It is almost a triviality (see Lemma 1.3) that any such semigroup has the property that we study.

In the case of periodic semigroups, the theorem simplifies considerably (Theorem 5.3) to provide a description that can readily be used to test a given semigroup for the property under consideration. This restriction is not as narrow as might appear, since any regular semigroup with lower semimodular subsemigroup lattice is necessarily periodic, for instance. Within the class of periodic semigroups, we identify some further special cases of importance. These allow us, for instance, to determine all the semigroup varieties in which every member has lower semimodular subsemigroup lattice. In a related topic (also motivated by the examples cited above), we determine the varieties for which all the relatively free semigroups have this same property.

In the general situation, the key difficulty lies in the case of [0-] simple semigroups. In that case, the main theorem reduces to a simply stated criterion for lower semimodularity of the subsemigroup lattice. We show that such semigroups must be $\mathcal{D}$-trivial, idempotent-free and non-cancellative, and conjecture that in fact there are no such semigroups at all. However this remains an open question.

Naturally, this study must consider subsemigroup lattices of groups. It is easily seen that if G is a group and $\mathcal{L}(G)$ is lower semimodular, then G is periodic and so $\mathcal{L}(G)$ is simply the subgroup lattice of G (with the empty subsemigroup adjoined). Although the finite groups with this property were determined by Ito in 1953 (see [10, Theorem 5.3.11]), a complete description is not known even in the periodic case. However, if $\mathcal{L}(S)$ is lower semimodular for a semigroup S, there is essentially no interaction between the nontrivial subgroups of S and the rest of the semigroup (see Corollary 1.2(ii)).

One might wonder why so little progress has previously been made in the study of lower semimodularity, while so much is known about semigroups whose subsemigroup lattice is upper semimodular, modular, distributive, etc. (See [11, Chapter 2].) In §5.1, we address this issue by showing that upper semimodularity (and thus modularity) imposes a succession of conditions on the underlying semigroup, very few of which are satisfied by semigroups with lower semimodular subsemigroup lattice, as we demonstrate by a number of examples. Not least of these is periodicity; another critical distinction relates to the five-element Brandt semigroup B_2: $\mathcal{L}(B_2)$ is lower semimodular but not upper semimodular; yet another is that the semilattices with upper semimodular subsemigroup lattice must be chains. There are several further distinctions of this type.

Sections 3 and 4 are devoted to a proof of the main general theorem and its elaboration. In Section 5 we specialize to periodic semigroups; prove that within that context the property that the lattice of subsemigroups is lower semimodular is preserved under quotients; determine the varieties of semigroups, all of whose members have that property; and the varieties all of whose relatively free semigroups have that property; and consider upper semimodularity and modularity, as described above. In Section 6 we specialize to simple and 0-simple semigroups. The final section of the paper contains a series of examples demonstrating the independence and non-vacuousness of the hypotheses in this theorem and its specializations.

If X is a subset of a semigroup, then $\langle X \rangle$ denotes the subsemigroup that it generates.

THEOREM 1.1 *A semigroup S has lower semimodular lattice $\mathcal{L}(S)$ of subsemigroups if and only if*

(I) each non-null principal factor of S is either:

(a) a group with lower semimodular subgroup lattice or a singular band; such a semigroup with zero adjoined; or, up to isomorphism, the five-element combinatorial Brandt semigroup B_2; or

(b) *a $\mathcal{D}$-trivial, idempotent-free [0-] simple semigroup;*

(II) *for any nontrivial subgroup of S, with identity e, say, if $ea \in H_e$ for some $a \in S$, then either $e \in \langle a \rangle$ or $e = ea$;*

(III) *for each element x of S that does not belong to a nontrivial subgroup of S,*

(a) *if $x = xab$ for some $a, b \in S$, then either $x \in \langle a, b \rangle$ or $x = xa$; and dually,*

(b) *if the associated principal factor is null and $x = bxa$ for some $a, b \in S$, then either $x \in \langle a, b \rangle$ or $x = xa$,*

(c) *if the associated principal factor is of type I(b) and $x = a_0xa_1x \cdots xa_n$ for some $a_0, \ldots, a_n \in S^1$ and $n \geq 1$, then either $x \in \langle a_0, a_1, \ldots, a_n \rangle$ or $n = 1$ and $x = xa_1$.*

The role of subgroups in such semigroups is elaborated in the next result.

COROLLARY 1.2 *Let S be a semigroup such that $\mathcal{L}(S)$ is lower semimodular.*

(i) *If S contains a nontrivial $\mathcal{H}$-class, then it is an (isolated) subgroup;*

(ii) *if e is the identity element of a nontrivial subgroup, then for all $a \in S$ such that $e \notin \langle a \rangle$, and for all $x \in H_e$, either $xa = ea$ and $ax = ae$, or $xa = ax = x$. In particular, if e, f are the identity elements of nontrivial subgroups and $f > e$, then $H_f e = eH_f = \{e\}$.*

We conclude this section with some background material. Further background, on simple and 0-simple semigroups, follows.

A lattice L is *lower semimodular* if for all $a, b \in L$, the covering relation $a \vee b \succ b$ implies $a \succ a \wedge b$. It is easily seen that an equivalent property is that the relation $a \succ b$ implies $a \wedge c \succeq b \wedge c$ for all $c \in L$. Upper semimodularity is defined dually. Each of these properties is inherited by interval sublattices and direct products. A lattice L is *modular* if for all $a, b \in L$ and $x \leq b$, $(a \vee x) \wedge b = (a \wedge b) \vee x$. A finite lattice is modular if and only if it is both lower and upper semimodular, but this is not true for lattices in general. Little other lattice theory is required in the sequel, but the reader may refer to [13] for a comprehensive study of semimodularity and related topics.

For background on subsemigroup lattices in general, see the monograph [11]. Denote by $\mathcal{L}(S)$ the lattice of subsemigroups of a semigroup S. Its least element is always the empty subsemigroup.

For general semigroup theory, and especially ideals and Green's relations, see the monographs of Clifford and Preston [1, 2]. Denote by E_S the set of idempotents of a semigroup S. A semigroup without zero is *idempotent-free* if $E_S = \emptyset$. A semigroup with zero is called idempotent-free if $E_S = \{0\}$. A semigroup with zero is *nil* if for each $a \in S$, $a^n = 0$ for some $n \geq 1$; and *nilpotent* if $S^n = \{0\}$ for some $n \geq 1$. In the case $n = 2$, S is a *null* (or zero) semigroup. A *singular band* is a semigroup that is either a left zero or a right zero semigroup. We call a subgroup that comprises an entire $\mathcal{J}$-class of a semigroup *isolated*.

As remarked above, it was noted in [11, §5.14] that a description of the semigroups with lower semimodular subsemigroup lattice is unknown, and a diverse collection of semigroups that have this property was cited. Their common feature is that they are *$\mathcal{J}$-trivial*, that is, Green's relation $\mathcal{J}$ is trivial. That any $\mathcal{J}$-trivial semigroup has this lattice-theoretic property is almost obvious, as shown in the next lemma. More generally, we shall see that Green's relations play a fundamental role in the sequel. Along with the term $\mathcal{J}$-trivial, we shall encounter the $\mathcal{D}$-trivial semigroups and the *combinatorial* semigroups: those that are $\mathcal{H}$-trivial (sometimes also called *aperiodic*).

LEMMA 1.3 *Let S be any semigroup and $U, V \in \mathcal{L}(S)$.*

1. *Suppose $U \succ V$ in $\mathcal{L}(S)$. Then $U - V$ is contained in a single $\mathcal{J}$-class of S.*
2. *If $V \subset U$ and $|U - V| = 1$, then $U \cap W \succeq V \cap W$ for all $W \in \mathcal{L}(S)$.*
3. *Any $\mathcal{J}$-trivial semigroup has lower semimodular lattice of subsemigroups.*
4. *All nilsemigroups, semilattices, free semigroups and free commutative semigroups are $\mathcal{J}$-trivial.*
5. *[11, §3.1] For any cyclic semigroup $S = \langle a \rangle$ that is not a group, $\mathcal{L}(S)$ satisfies the single covering property, that is, there exists a unique maximal subsemigroup, namely $S - \{a\}$.*

Proof. To prove 1, suppose that $x, y \in U - V$. Since $U \succ V$, $U = V \vee \langle y \rangle$. Thus any expression for x as a product of elements of V with instances of y involves at least one y, and so $J_x \leq J_y$. Similarly, $J_y \leq J_x$. The proof of 2 is obvious, and 3 follows immediately from 1 and 2. The statements in 4 are well known and easily proved. □

With every covering $U \succ V$ there is therefore associated a unique $\mathcal{J}$-class. We shall show that whenever $\mathcal{L}(S)$ is lower semimodular, if $U \succ V$ then $|U - V| = 1$ unless the associated $\mathcal{J}$-class is an (isolated) group, in which case a covering is induced in its subgroup lattice.

While the following result will not be used directly in the sequel, it demonstrates that a wide range of semigroups with lower semimodular subsemigroup lattices may be constructed from the special cases exhibited above and later in this paper. Recall from [11] that a partition of a semigroup S into subsemigroups $\{S_\alpha\}_{\alpha \in Y}$ is a *U-partition* if $ab \in \langle a \rangle \cup \langle b \rangle$ whenever a and b belong to distinct components. If the partition induces a band congruence on S, then it is a *U-band* of the subsemigroups. In particular this holds when S a U-chain, or *ordinal sum*, of the subsemigroups: the indexing set Y is a chain and $ab = ba = b$ whenever $a \in S_\alpha, b \in S_\beta$ and $\alpha > \beta$.

RESULT 1.4 *[11, Theorem 3.6] Every direct decomposition of $\mathcal{L}(S)$ corresponds to a U-partition of S into subsemigroups $\{S_\alpha\}_{\alpha \in Y}$, in which case $\mathcal{L}(S) \cong \Pi_{\alpha \in Y} \mathcal{L}(S_\alpha)$. In that event, if each $\mathcal{L}(S_\alpha)$ is lower semimodular, then so is $\mathcal{L}(S)$.*

Finally, the following elaboration of a well known property of periodic semigroups will be used on occasion. We include a proof for completeness.

LEMMA 1.5 *Let S be a periodic semigroup. Suppose $x \neq y \in S$ and $x \mathcal{R} y$, $y = xa, x = yb$, say. Then there exist mutually inverse elements $u, v \in \langle a, b \rangle$ such that $y = xu, x = yv$.*

Proof. Let $g = (ab)^n$ be the idempotent power of ab. Then $u = ga$ and $v = b(ab)^{n-1}g$ satisfy the stipulations. $\square$

2 Background on principal factors and [0-] simple semigroups.

In view of Lemma 1.3, it is clear that the nature of the $\mathcal{J}$-classes will play a major role in the study of lower semimodularity. This is best viewed through the associated principal factors, the definition and basic properties of which we now summarize. We refer the reader to [1, Chapter 2] for further general information on principal factors and on simple and 0-simple semigroups.

If J is a $\mathcal{J}$-class of a semigroup S, then it generates the principal ideal S^1JS^1, which in turn contains the (possibly empty) ideal $I(J) = S^1JS^1 - J$. The associated principal factor $PF(J)$ is the Rees quotient $S^1JS^1/I(J)$ (with the understanding that if $I(J)$ is empty, then the quotient is S^1JS^1 itself, in which case J is the *kernel* of S, that is, its minimum ideal). The principal factors of any semigroup either are null or are [0-] simple: either 0-simple or (in the case where $I(J)$ is empty) simple.

We review the definition and properties of completely 0-simple semigroups, following [1, Section 2.7]. Deleting reference to the zero yields the corresponding definition and properties of completely simple semigroups that we shall need. Although we will not make use of the Rees representation of such semigroups, all of these properties may easily be seen in terms of that representation, for readers so inclined. A 0-simple semigroup S is *completely* 0-simple if it contains an idempotent that is 0-minimal in the natural partial order on E_S; equivalently, S contains a 0-minimal left ideal and a 0-minimal right ideal. In that case, S is the union of its 0-minimal left ideals, and dually; and S is 0-bisimple and regular. In fact, for any nonzero element a of S, $S^1a = L_a \cup \{0\}$ and $aS^1 = R_a \cup \{0\}$.

From the description of the nonzero principal left and right ideals above, it follows that for any nonzero elements a, b of S, if $ab \neq 0$, then $ab \in R_a \cap L_b$. Thus $\langle a, b \rangle$ is contained within the union of the four $\mathcal{H}$-classes $H_a, H_b, R_a \cap L_b, R_b \cap L_a$ and $\{0\}$. According to [1, Theorem 2.17], ab is nonzero if and only if the $\mathcal{H}$-class $R_b \cap L_a$ contains an idempotent. Thus if $a^2 \neq 0$, H_a contains an idempotent; in that event, H_a is necessarily a subgroup [1, Theorem 2.52].

If each $\mathcal{R}$-class and each $\mathcal{L}$-class of S contains a unique idempotent, then S is an inverse semigroup, called a *Brandt* semigroup. See [1, Section 3.4]. If a Brandt semigroup is combinatorial, it is uniquely determined by its cardinality. Denote by B_n the combinatorial Brandt semigroup with n nonzero idempotents. A presentation for B_2 is $\langle a, b \mid a = aba, b = bab, a^2 = b^2 = 0 \rangle$.

Finally, we observe that a completely simple semigroup is the union of its maximal subgroups. Hence any combinatorial, completely simple semigroup is a rectangular band.

We now turn to the more complex situation of [0-] simple semigroups that are not completely [0-] simple. If such a semigroup contains a nonzero idempotent, then it contains a pair of comparable such idempotents. This leads to an important result of O. Anderson, regarding the key role played by the bicyclic semigroup. Before stating this result, we review the definition and properties of this semigroup and of four semigroups studied by the author in [5]. Three of these play a similar role in [0-] simple, idempotent-free semigroups to that played by the bicyclic semigroup in [0-] simple semigroups with [nonzero] idempotents.

First, for specificity, we let G denote the infinite cyclic group, which may be presented as a monoid by $\langle a, b \mid ab = ba = 1\rangle$. We may write a^{-1} for b, as usual.

The bicyclic semigroup B may be defined by the monoid presentation $\langle a, b \mid ab = 1\rangle$. Each nonidentity element of B is uniquely expressible as a nonempty product of the form $b^n a^m$, where m, n are nonnegative integers.

The semigroup A is defined by the semigroup presentation $\langle a, b \mid a(ab) = a\rangle$. According to [5, Theorem 2.7], each element of A is uniquely expressible as a nonempty product of the form

$$vs^l a^m, \quad v \in Y^*, l \geq 0, m \geq 0,$$

where $s = ab$ and $Y = \{b, sb, s^2b, \ldots\}$. Here Y^* denotes the free monoid on Y (and Y^+ will denote the free semigroup on Y). Although A is not itself simple, the ideal generated by a is simple, right cancellative and $\mathcal{L}$-trivial but not $\mathcal{R}$-trivial (since it is clear that a and a^2 are distinct $\mathcal{R}$-related elements in that ideal).

The dual of A will be denoted A^d and presented as $\langle a, b \mid (ab)b = b\rangle$.

The semigroup C, introduced by L. Rédei[8], is defined by the semigroup presentation $\langle a, b \mid a(ab) = a, (ab)b = b\rangle$. According to [5, Result 2.1], each element of C is uniquely expressible as a nonempty product of the form

$$b^n s^l a^m, \quad l, m, n \geq 0,$$

where $s = ab$. This time C itself is a simple semigroup without idempotents, within which $a \mathrel{\mathcal{R}} a^2$ and $b \mathrel{\mathcal{L}} b^2$.

The semigroup D, defined by the semigroup presentation $\langle a, b \mid a(ab)^n b = ab, \forall n \geq 1\rangle$, was introduced by the author in [5, Example 6.6]. According to [5, Proposition 6.7], each element of A is uniquely expressible as a nonempty product of the form

$$vs^l u, \quad v \in Y^*, l \geq 0, u \in X^*,$$

where $s = ab$, $Y = \{b, sb, s^2b, \ldots\}$ and $X = \{a, as, as^2, \ldots\}$. The ideal generated by ab is a $\mathcal{D}$-trivial simple, idempotent-free semigroup, the kernel of D. In every proper idempotent-free quotient of D, the image of this kernel is $\mathcal{D}$-nontrivial.

From the relations defining each of the semigroups above, it is clear that there are canonical homomorphisms

$$\begin{tikzcd} & A \arrow[dr] & & & \\ D \arrow[ur] \arrow[dr] & & C \arrow[r] & B \arrow[r] & G. \\ & A^d \arrow[ur] & & & \end{tikzcd}$$

In terms of the canonical form for elements of D, under the sequence passing through A,

$$vs^l u \to vs^l a^{\ell(u)} \to b^{\ell(v)} s^l a^{\ell(u)} \to b^{\ell(v)} a^{\ell(u)} \to a^{\ell(u)-\ell(v)},$$

where $\ell(w)$ denotes the length of a word w. The expressions in A, C, B and G then also represent elements of the respective semigroups canonically. (It follows easily that D is the pullback of the appropriate section of this diagram, although this information is not used in the sequel.)

We shall show in the next section that none of the semigroups G, A, A^d, B or C has lower semimodular subsemigroup lattice. Thus the next result plays a critical role in the sequel. (The semigroup D does not play such an important role. It will be shown in the final section of the paper that $\mathcal{L}(D)$ is not lower semimodular.) As usual, when we write "contains X", where X denotes a specific semigroup, we mean "contains a subsemigroup isomorphic to X".

RESULT 2.1 *1. [1, Theorem 2.54] Any [0-] simple semigroup that contains a nonzero idempotent, but is not completely [0-] simple, contains B;*

2. [5, Theorem 4.2] Any idempotent-free [0-] simple semigroup that contains distinct $\mathcal{R}$-related elements contains either C or A; any idempotent-free [0-] simple semigroup that contains distinct $\mathcal{L}$-related elements contains either C or A^d.

The following result and its dual play a subtle but important role in the sequel.

RESULT 2.2 *[5, Proposition 4.1] A [0-] simple, idempotent-free semigroup is $\mathcal{R}$-trivial if and only if the equation $x = xy$ has no nonzero solution x.*

3 Necessity.

Throughout this section, S will be a semigroup with lower semimodular subsemigroup lattice. The numerals I, II and III will refer to the hypotheses of Theorem 1.1. Our first, elementary, result points to the need to describe the [0-] simple semigroups with this property.

LEMMA 3.1 *For every principal factor of S, $\mathcal{L}(P)$ is lower semimodular.*

Proof. If J is any $\mathcal{J}$-class of S, then it is easily observed that $\mathcal{L}(PF(J))$ is isomorphic with the interval sublattice $[I(J), S^1JS^1]$ of $\mathcal{L}(S)$.

Refer to the previous section for the definitions of, and canonical forms in, A, A^d, B, C and G.

PROPOSITION 3.2 *The subsemigroup lattice of the infinite cyclic group G is not lower semimodular. Hence a group has lower semimodular subsemigroup lattice if and only if it is periodic and has lower semimodular subgroup lattice.*

Proof. Let $M = G - \langle a \rangle = \langle b \rangle \cup \{1\}$. Then since for any $i > 1$, $a^i b^{i-1} = a$, $G \succ M$. By lower semimodularity, $\langle a \rangle \succ \langle a \rangle \cap M = \emptyset$, which is clearly impossible. $\square$

PROPOSITION 3.3 *The subsemigroup lattice of the bicyclic semigroup B is not lower semimodular.*

Proof. Let $x, y \in B$, $x = b^k a^l$, $y = b^{k'} a^{l'}$ and suppose $xy \in \langle a \rangle$. Now if $l \geq k'$, then $xy = b^k a^{l-k'+l'}$; and if $l \leq k'$, then $xy = b^{k+k'-l} a^{l'}$. Hence k must equal zero, that is, $x \in \langle a \rangle$. Hence $M = B - \langle a \rangle < B$.

Again, $a^i b^{i-1} = a$ for any $i > 1$, so $M \prec B$ and lower semimodularity leads once more to $\langle a \rangle \succ \langle a \rangle \cap M = \emptyset$, which is once more clearly impossible. $\square$

PROPOSITION 3.4 *The subsemigroup lattices of A, A^d and C are not lower semimodular.*

Proof. Clearly we only need to consider A and C. The argument follows that for B: we will show that $A - \langle a \rangle$ is a maximal subsemigroup of A, from which lower semimodularity implies the contradiction $\langle a \rangle \succ \emptyset$; then we prove the corresponding statement for C.

First let $x = vs^l a^m$, $y = v' s^{l'} a^{m'}$ be elements of A, as described in §2. Suppose $xy \in \langle a \rangle$. Then by mapping onto B, we obtain $(b^{\ell(v)} a^m)(b^{\ell(v')} a^{m'}) \in \langle a \rangle$ in B. As noted in the proof of Proposition 3.3, this requires that $\ell(v) = 0$, that is, v is empty. We again must consider the various cases for the product xy. Observe that since $as^n = a$, $as^n b = s$ for all $n \geq 1$. Hence if $v' \in Y^+$, $a^{\ell(v')} v' = s$ so that, for any v', $a^{\ell(v')+1}(v' s^{l'} a^{m'}) = a^{m'+1}$.

(i) If $m > \ell(v')$ then $xy = s^l a^{m-\ell(v')+m'}$.
(ii) If $m = \ell(v') \neq 0$, then $xy = s^{l+l'+1} a^{m'}$.
(iii) If $m = \ell(v') = 0$, then $xy = s^{l+l'} a^{m'}$.
(iv) If $m < \ell(v')$, then $xy = ws^{l'} a^{m'}$, where $w \in Y^+$ is the product of s^l with the terminal segment of v' of length $\ell(v') - m$.

In cases (ii), (iii) and (iv), $xy \notin \langle a \rangle$. (In case (iii), $l > 0$, since both v and v' are empty.) In case (i) it is clear that in order for $xy \in \langle a \rangle$, l must be zero, so that $x \in \langle a \rangle$.

Thus $A - \langle a \rangle < A$. Now for any $m > 1$, $a^m b^{m-1} = a(a^{m-1} b^{m-1}) = as = a$, and so $A - \langle a \rangle \prec A$.

Next suppose that $x = b^k s^l a^m, y = b^{k'} s^{l'} a^{m'} \in C$, in canonical form, and that $xy \in \langle a \rangle$. Under the canonical homomorphism $A \to C$, the preimage in A of the subsemigroup $\langle a \rangle$ of C is again $\langle a \rangle$. Hence by interpreting x and y as elements of A, we obtain that $C - \langle a \rangle < C$, from the corresponding result for A. That $C - \langle a \rangle \prec C$ also follows from the same calculation as for A. $\square$

Applying Result 2.1, these two propositions immediately yield the following.

COROLLARY 3.5 *Suppose S is [0-] simple. If S contains a [nonzero] idempotent, then S is completely [0-] simple. If S is idempotent-free, then S is $\mathcal{D}$-trivial.*

PROPOSITION 3.6 *If S is completely [0-] simple, then either S is combinatorial or it is a (periodic) group [with adjoined zero]. Hence any nontrivial subgroup of S is isolated. If S is completely simple and combinatorial, then it is a singular band.*

Proof. First consider the completely 0-simple case and suppose S is not a group with adjoined zero. Again we refer the reader to Section 2. Let $e \in E_S$. Then there exists $b \notin H_e^0$. Without loss of generality we may assume $b \in R_e$. Now $H_eH_b = H_b$; if H_b is a subgroup then $H_bH_b = H_b$ and $H_bH_e = H_e$; and otherwise $H_bH_b = H_bH_e = \{0\}$. Hence the subset $T = H_e \cup H_b \cup \{0\}$ is a subsemigroup of S. From $H_eH_b = H_b$ it is immediate that $T \succ H_e \cup \{0\}$. Hence from lower semimodularity, $H_b^0 \succ H_b^0 \cap H_e^0 = \{0\}$. But H_b^0 contains a two-element subsemigroup — either $\{0, b\}$ if H_b is not a subgroup, or $\{0, f\}$ if H_b is a subgroup with identity f. Hence $|H_e| = |H_b| = 1$, as required.

In the completely simple case, it is clear that $\mathcal{L}(S^0)$ is the direct product of $\mathcal{L}(S)$ with a two-element lattice and is therefore once more lower semimodular, so the conclusion in the first statement of the proposition follows from the previous paragraph. If S is combinatorial, then it is a rectangular band. If it is not a singular band, then there exist x, y such that $\{x, y, xy, yx\}$ forms a subband U, say, the union of the two right zero subsemigroups $N = \{x, xy\}$ and $P = \{y, yx\}$. Now $U \succ N$ (since $yx = y \cdot x$ and $y = yx \cdot xy$). Lower semimodularity then would imply that $P \succ P \cap N = \emptyset$, contradicting the inclusions $\emptyset \subset \{y\} \subset P$. □

In view of Lemma 3.1, Corollary 3.5, Proposition 3.6 and the facts stated in Section 1, only an analysis of the combinatorial, completely 0-simple case remains in order to complete the proof of necessity of I. Conducting that analysis requires that we first prove III(a). First, however, we prove necessity of II.

LEMMA 3.7 *Suppose $e \in E_S$, $a \in S$ and $ea \,\mathcal{H}\, e$. Then either $ea = e$ or $e \in \langle a \rangle$.*

Proof. By the preceding corollary and proposition, H_e is trivial unless it constitutes an isolated subgroup, so we shall assume the latter. Put $y = ea$ and suppose $y \neq e$. By Proposition 3.2, H_e is periodic and so $e \in \langle y \rangle$. Since H_e is isolated, $H_ea^n, a^nH_e \subseteq H_e$ for all $n > 0$. Now $ya^n = yea^n = y(ea)^n = y^{n+1}$ and dually, so $\langle a, y \rangle = \langle a \rangle \cup \langle y \rangle$. Further, for any $n > 0$, $e \in \langle y^n \rangle$ and so $y \in \langle a, y^n \rangle$, that is, $\langle a, y \rangle \succ \langle a \rangle$. By lower semimodularity, $\langle y \rangle \succ \langle a \rangle \cap \langle y \rangle$. Since $y \neq e$, $\langle e \rangle \cap \langle y \rangle \neq \emptyset$, and so $e \in \langle a \rangle$, completing the proof. □

The key to the proof of III is knowledge of the maximal subsemigroups of two-generated semigroups with lower semimodular subsemigroup lattice.

LEMMA 3.8 *Suppose $S = \langle x, y \rangle$, $x \neq y$. If neither x nor y belongs to a nontrivial subgroup, then either $S - \{x\} < S$ or $S - \{y\} < S$. Hence any maximal subsemigroup of S contains either x or y.*

Proof. By Zorn's Lemma, there exists a subsemigroup M of S, maximal such that $x \in M, y \notin M$. Clearly $S \succ M$. If $|S - M| = 1$, then $M = S - \{y\}$ and so $S - \{y\} < S$. Otherwise, choose $z \in S - M, z \neq y$. By Lemma 1.3, $z \mathcal{J} y$. Note that $z \notin \langle y \rangle$ for, in the event that y is nonidempotent, then by lower semimodularity, $\langle y \rangle \succ \langle y \rangle \cap M$ and by the single covering property of $\mathcal{L}(\langle y \rangle)$, $y^i \in M$ for all $i \geq 2$. Since $S = \langle x, y \rangle$, $J_z \leq J_x$. Hence $J_y \leq J_x$. Similarly, either $S - \{x\} < S$ or $J_x \leq J_y$.

Suppose, then, that neither $S - \{x\} < S$ nor $S - \{y\} < S$. Then $J_x = J_y$ and this $\mathcal{J}$-class is the maximum $\mathcal{J}$-class of S. In fact, since all relevant products lie in this $\mathcal{J}$-class, we may pass to the associated principal factor and, without loss of generality, assume that S is [0]-simple (clearly being non-null). By Corollary 3.5, S is therefore either a completely [0-] simple semigroup or a $\mathcal{D}$-trivial, idempotent-free [0-] simple semigroup. But the latter case is ruled out by [5, Corollary 5.2], according to which no finitely generated, idempotent-free [0-] simple semigroup is $\mathcal{D}$-trivial.

If S is completely simple, then from the last statement of Proposition 3.6, S is the singular band $\{x, y\}$, with the obvious contradiction $S - \{x\} = \{y\} < S$.

In the final, combinatorial, completely [0-] simple case, we refer the reader once more to §2. Now x, y cannot be $\mathcal{R}$- or $\mathcal{L}$-related, for S will not then be regular, so S comprises the union of the four singleton $\mathcal{H}$-classes H_x, H_y, $R_x \cap L_y$ and $R_y \cap L_x$ with $\{0\}$. Since $R_x \cap L_y \subset xS^1y$, the product xy is nonzero and therefore is the unique element in that $\mathcal{H}$-class; similarly, $R_y \cap L_x = \{yx\}$. Thus $S = \{x, y, xy, yx, 0\}$. Moreover from $xy, yx \neq 0$ it follows that yx, xy, respectively, are idempotents. Now from $S - \{y\} \not< S$, y can be expressed as a product of elements from $\{x, xy, yx\}$, a product that must start with yx and end with xy. But then $x^2 \neq 0$, so that H_x is a subgroup and x is idempotent. Similarly, y is idempotent, yielding that S is a rectangular band, again a contradiction. This completes the proof in this case. □

Now we can we prove necessity of III. Suppose x does not belong to any nontrivial subgroup of S. To prove III(a), suppose $x = xab$ but $xa \neq x$. Put $T = \langle a, b, x \rangle$. Then by Zorn's lemma, there exists a subsemigroup M, maximal such that $a, b \in M, x \notin M$. Clearly $T \succ M$. Note that since $x = (xa)b$, $xa \notin M$. Thus, similarly to the argument used in the last lemma, $xa \notin \langle x \rangle$. Also, xa again belongs to no nontrivial subgroup of S, since $J_x = J_{xa}$ in S and so this $\mathcal{J}$-class cannot consist of an isolated group. Thus, $x \notin \langle xa \rangle$, as well. Put $U = \langle x, xa \rangle$. By lower semimodularity, $U \succ U \cap M$, contradicting Lemma 3.8.

To prove III(b), suppose $x = bxa$ and that $x \neq xa$. The proof is almost identical to that of the previous case. (Note that the conclusion holds in the case of completely [0-] simple principal factors as well, and that this also covers the exceptional case in III(c), where $n = 1$).

To prove III(c) for $n > 1$, suppose $x = a_0xa_1 \cdots xa_n$, with $a_i \in S^1$, $0 \leq i \leq n$, but $x \notin \langle a_0, a_1, \ldots, a_n \rangle$. The proof is similar to the two previous ones. Put $T = \langle a_0, a_1, \ldots, a_n, x \rangle$. Then there exists a subsemigroup M, maximal such that $a_0, a_1, \ldots, a_n \in M, x \notin M$. Clearly $T \succ M$ and hence $\langle x \rangle \succ \langle x \rangle \cap M$. By Lemma 1.3(5) and $\mathcal{D}$-triviality, $x^i \in M$ for all $i > 1$. Let $y = xa_1 \cdots xa_n$. Clearly $y \mathcal{J} x$ in S. Moreover, by hypothesis, $n > 1$ and so $a_1 \cdots xa_n \mathcal{J} x$. By Result 2.2, no equation of the form $x = xz$ can hold in S, whence $y \neq x$. Since $x = a_0y$, $y \notin M$, and so $y \notin \langle x \rangle$. Similarly, $x \notin \langle y \rangle$. Now put $U = \langle x, y \rangle$. By lower semimodularity, $U \succ U \cap M$, again contradicting Lemma 3.8. □

Finally, we complete the proof of necessity of I(a) and thereby of necessity in Theorem 1.1. The given formulation allows some simplification in verifying that specific examples satisfy (I).

PROPOSITION 3.9 *If T is combinatorial, completely 0-simple, and satisfies III(a), then either T is a singular band with zero adjoined or $T \cong B_2$.*

Proof. Suppose that T contains distinct $\mathcal{R}$-related idempotents e, f. In this situation, $\langle e, f\rangle = \{e, f\}$. Now if $T = R_f \cup \{0\}$, then R_f is a right zero subsemigroup. Otherwise, there exists an element x, say, in L_f, distinct from f. Now $x = xf = (xe)f$, where $xe \in R_x \cap L_e$, so that $xe \neq x$. But $x \notin \langle e, f\rangle$, contradicting III(a).

In conjunction with the dual of the previous paragraph, it follows that if T is not a singular band with zero adjoined, then T is inverse. Suppose T contains three distinct nonzero idempotents, e, f, g. Again, refer to the discussion of products within completely 0-simple semigroups in Section 2. Let $x \in R_e \cap L_g, a \in R_g \cap L_f, b \in R_f \cap L_g$. Then since $g \in L_x \cap R_a, xa \in R_x \cap L_f$, whence $xa \neq x$. Similarly, since $f \in L_{xa} \cap R_b, (xa)b \in R_{xa} \cap L_b = H_x$. Thus $x = xab$. But $\langle a, b\rangle = \{a, b, f, g, 0\}$. This also contradicts III(a). Hence $T \cong B_2$. □

Before proving Corollary 1.2, we prove some useful technical results. The first of these will permit III(a) to be replaced by a weakened version (cf the statement of Theorem 5.3(3)).

LEMMA 3.10 *In any semigroup T satisfying I, III(a) holds whenever $ab \in J_x$ (or whenever $ba \in J_x$, in the dual statement).*

Proof. Let $x \in T$, assume that x does not belong to a nontrivial subgroup of T and that $x = xab$ for some $a, b \in T$, $ab \in J_x$. (The dual result is proved similarly.) Since $x, ab \in J_x$, the associated principal factor cannot be null; by Result 2.2, nor can I(b) hold. Thus by I, either J_x is a singular band or $PF(J_x) \cong B_2$. If J_x is right zero, then $x = x(ab) = ab$. If it is left zero, then since $xa \mathcal{R} x$, $x = xa$.

So suppose $PF(J_x) \cong B_2$. Note that from $x = xab$ we obtain $x = x(aba)(bab)$, where aba, bab are mutually inverse members of J_x and $x(aba) = xa$. Without generality, then, we may assume that a, b themselves are mutually inverse members of J_x. Now if $a \neq b$, then the four elements a, b, ab, ba are distinct and thus comprise J_x. If $a = b$, then this element is idempotent and $x = xab = xa$. □

From the next lemma, we shall deduce both part (i) of Corollary 1.2 (in fact a slightly stronger statement) and, in Lemma 5.2, a simplification of Theorem 1.1 under a certain finiteness condition.

LEMMA 3.11 *Suppose that a semigroup T satisfies III(a), that $x = bxa$ for some $a, b \in T$ and that the associated principal factor is null. If $x \mathcal{R} xa$ in T, then $x = xa$.*

Proof. From $x \mathcal{R} xa$, we obtain $x = xac$ for some $c \in T$. Observe that $bx = bxac = xc$; thus $b^n x = xc^n$ for all $n \geq 1$. It follows that for any $m, n \geq 1$, $xc^m a^n = b^m xa^n$, with value x when $m = n$, xc^{m-n} when $m > n$, or xa^{n-m} when $n > m$.

Suppose that $x \neq xa$. By III(a), $x \in \langle a, c\rangle$. We shall prove that for any $w \in \langle a, c\rangle$, $b^k(xw)a^\ell = x$ for some $k, \ell \geq 0$, so that $xw \mathrel{\mathcal{J}} x$. With $w = x$, this contradicts the assumption that $PF(J_x)$ is null.

We prove the assertion by induction on the minimum number of alternations of a's and c's in any expression for w as a product in $\langle a, c\rangle$. In the basis case, where there are no alternations, either $w = c^\ell$ for some $\ell \geq 1$, in which case $(xw)a^\ell = x$, or $w = a^k$ for some $k \geq 1$, in which case $b^k(xw) = x$. Otherwise, express w as a product with the miminum number of alternations and suppose the assertion is true for products with fewer alternations. If w begins with c, then $w = c^m a^n u$, where u may be empty. As noted above, either $xw = xu$ or $xw = xc^{m-n}u$ or $xw = xa^{n-m}u$. In any event, the respective terms u, $c^{m-n}u$ and $a^{n-m}u$ each have fewer alternations and so the induction hypothesis applies. If w begins with a, then $w = a^m c^n u$, similarly, and now $b^m(xw) = xc^n u$. The induction hypothesis once more yields the desired conclusion. □

COROLLARY 3.12 *Suppose x, y belong to a $\mathcal{J}$-class of S whose associated principal factor is null. If $x \mathrel{\mathcal{R}} y$ and $L_x \leq L_y$ in S, then $x = y$. Hence any nontrivial $\mathcal{H}$-class of S is a subgroup.*

Proof. Given such x, y, suppose $x \neq y$. Then $y = xa$ and $x = by$ for some $a, b \in S$, so that the hypotheses of the lemma are satisfied and the contradiction $x = y$ is obtained. Now suppose H is a nontrivial $\mathcal{H}$-class of S. Then by the first statement of the corollary, the associated principal factor is non-null. Thus the conclusion is clear from I. □

We now prove part (ii) of Corollary 1.2.

LEMMA 3.13 *If e is the identity element of a nontrivial subgroup of S, then for all $a \in S$ such that $e \notin \langle a\rangle$, and for all $x \in H_e$, either $xa = ea$ and $ax = ae$, or $xa = ax = x$. In particular, if e, f are the identity elements of nontrivial subgroups and $f > e$, then $H_f e = eH_f = \{e\}$ (so that $xa = ax = x$ for all $x \in H_e, a \in H_f$).*

Proof. First suppose $xa \in H_e$. Since H_e is isolated, ea, ae and ax also belong to H_e. By II, $ea = e$, so $xa = x$. Dually, $ax = x$. Now the final statement of the lemma also follows immediately.

Next suppose $xa \notin H_e$. If xa belongs to a nontrivial subgroup, with identity element f, say, then since $xa = fxa$, $fx \in H_f$ (recalling that H_f is isolated). By II, $fx = f$ and therefore $xa = fxa = fa$. Now since $e \mathrel{\mathcal{L}} x, ea \mathrel{\mathcal{L}} xa$ and, since H_f is isolated, $ea \in H_f$. Hence $ea = fa$, similarly.

If xa does not belong to a nontrivial subgroup, then we may apply the dual statement in III(a) to the equation $xa = xx^{-1}(xa)$, where x^{-1} is the inverse of x in H_e. Since $\langle x, x^{-1}\rangle \subseteq H_e$, $xa = x^{-1}xa = ea$. The dual case follows similarly. □

4 Sufficiency.

The proof is divided into two parts. Throughout this section S satisfies the hypotheses of Theorem 1.1. Suppose that $U \succ V$ in $\mathcal{L}(S)$. We use Lemma 1.3(1) without comment and once more refer the reader to §2 for calculations in completely [0-] semigroups.

LEMMA 4.1 *1. If $U - V$ is not contained within a subgroup of S, then $|U - V| = 1$.*

2. If $U - V$ is contained within the subgroup H of S, then $U \cap H \succ V \cap H$ in $\mathcal{L}(H)$.

Proof. (1) By hypothesis, if $U - V$ is not contained within a subgroup of S, then the associated principal factor P is either (i) a combinatorial completely [0-] simple semigroup, (ii) null, or (iii) a $\mathcal{D}$-trivial, idempotent-free [0-] simple semigroup.

We first observe that III(b) is always satisfied in case (i). For if $x = bxa$ then [$xa \neq 0$ in P and] $R_{xa} \leq R_x$, so that $R_{xa} = R_x$; and from $x = b(xa)$ it follows that $L_x \leq L_{xa}$, so $L_{xa} = L_x$. Since P is combinatorial, $xa = x$.

Returning to the proof itself, suppose on the contrary that $U - V$ contains distinct elements x and y. Since $U \succ V$, $x \in V \vee \langle y \rangle$ and so $x = a_0ya_1 \cdots ya_n$ for some $n \geq 1$ and $a_i \in V^1$, $0 \leq i \leq n$. Similarly, $y = b_0xb_1 \cdots xb_m$ for some $m \geq 1$ and $b_j \in V^1$, $0 \leq j \leq m$. Substituting for y in the equation for x yields an equation $x = a_0b_0x \cdots xb_ma_n$.

In case (i), if $m > 1$ temporarily put $z = xb_1 \cdots b_{m-1}x = x$. Then [$z \neq 0$ in P and] $R_z \leq R_x$, $L_z \leq L_x$ so, similarly to the above, $z = x$. In other words, we may without loss of generality assume that $m = 1$ and that, similarly, $n = 1$. In case (ii), necessarily $m = n = 1$: for instance if $m > 1$, then $y = (b_0x)(b_1 \cdots xb_m)$, a [nonzero] product in P of [nonzero] elements. In case (iii), from III(c) it follows that $x \in \langle a_0, \ldots, a_n, b_0, \ldots, b_m \rangle \subseteq V$, contradicting the assumption, unless once more $m = n = 1$.

In any case we obtain $x = a_0b_0xb_1a_1$. Given that $x \notin V$, from III(b) or III(c), as appropriate, we obtain $x = a_0b_0x = xa_1b_1$. Suppose $a_1b_1 \neq 1$. Then by III(a), $x = xa_1$. If $a_0b_0 \neq 1$, then by a dual argument, $x = a_0x$: but then $y = a_0xa_1 = x$, a contradiction. But if $a_0b_0 = 1$, then $y = xa_1 = x$ also. If $a_0b_0 \neq 1$, then a similar contradiction is reached.

(2) Denote by e the identity element of H. By hypothesis, H is isolated, that is, an entire $\mathcal{D}$=$\mathcal{J}$-class of S. Let Y be a subgroup of H such that $V \cap H \subset Y \subseteq U \cap H$. We will show that $Y = (V \vee Y) \cap H$. Since $V \subset V \vee Y \subseteq U$, $V \vee Y = U$ and $Y = U \cap H$ as required.

Let $x \in (V \vee Y) \cap H$. So $x = v_0y_1v_1 \cdots y_nv_n$ for some $n \geq 1$, $y_1, \ldots, y_n \in Y$ and $v_0, \ldots, v_n \in V^1$.

If $e \in V$, then since $x = exe$ and $y_i = ey_ie$ for each i, we may replace each $v_i \in V^1$ by $ev_ie \in (V \cap H)^1$, whence $x \in (V \cap H) \vee Y = Y$.

Otherwise, consider the product y_nv_n, which lies in H since the subgroup is isolated. Suppose $v_n \in V$ (that is, $v_n \neq 1$). Then $ev_n = (y_n^{-1}y_n)v_n \in H$ and so by II, $ev_n = e$, whence $y_nv_n = y_n$. Thus if $n > 1$, $xy_n^{-1} = v_0y_1 \cdots y_{n-1}v_{n-1}$. Repeating this argument leads to $xy_n^{-1} \cdots y_1^{-1} = v_0y_1y_1^{-1} = v_0e$. Finally, II being self-dual, $v_0e = e$ and so $x = y_1 \cdots y_n \in Y$, as required. $\square$

Now let $W \in \mathcal{L}(S)$. In the first case above, it is clear that $U \cap W \succeq V \cap W$. In the second case, suppose there are distinct elements x, y of $(U \cap W) - (V \cap W)$. Then $x, y \in H$ and $U \cap H \succ V \cap H$. By lower semimodularity of $\mathcal{L}(H)$, $U \cap W \cap H \succ V \cap W \cap H$. Thus $y \in (V \cap W \cap H) \vee \langle x \rangle \subseteq (V \cap W) \vee \langle x \rangle$; similarly $x \in (V \cap W) \vee \langle y \rangle$. Hence $U \cap W \succ V \cap W$, completing the proof of sufficiency in Theorem 1.1. $\square$

5 Periodic semigroups and regular semigroups.

Theorem 1.1 simplifies considerably in the case of periodic semigroups, not least because any [0-] simple periodic semigroup is necessarily completely [0-] simple [1, Corollary 2.56], so that hypotheses I(b) and III(c) of the theorem may be omitted. Periodicity is not as restrictive an assumption as might first appear. Recall, firstly, that by Proposition 3.2, any group whose subsemigroup lattice is lower semimodular is periodic. As we now show, this property extends to all *eventually regular* semigroups: semigroups in which each element has a power that is a regular element of the semigroup. (The term π*-regular* is also commonly used.) Along with regular semigroups, this class includes all *epigroups* (also termed *group-bound* semigroups): semigroups in which each element has a power that belongs to a subgroup.

PROPOSITION 5.1 *If $\mathcal{L}(S)$ is lower semimodular then*

1. *for any regular element a of S, if a^n is again regular for some $n > 1$, then a^n belongs to a subgroup of S and a therefore has finite order;*
2. *if S is eventually regular, it is periodic;*
3. *if S is regular, it is orthodox — that is, the product of idempotents is again an idempotent — and a^2 belongs to a subgroup for every $a \in S$.*

Proof. Let $a \in S$ and suppose a is regular. From I of Theorem 1.1, the only case in which a does not already belong to a subgroup is when its principal factor is isomorphic to B_2. We may assume that $D_a = \{a, b, e, f\}$, with $ab = e \neq f = ba$. Suppose a^n is also regular, for some $n > 1$. Similarly, a^n then belongs to a subgroup unless, possibly, its principal factor is again isomorphic to B_2, so that its $\mathcal{D}$-class consists of two distinct $\mathcal{R}$-classes (and two distinct $\mathcal{L}$-classes). But $a^n b \mathrel{\mathcal{R}} a^n$ since $a^n ba = a^n$, and so either $a^n b = a^n$ or $a^n b \in E_S$. In the former case, $a^n = a^n ba = a^{n+1}$; in the latter case, $a^{2n-1} = (a^n b)(a^n b)a = (a^n b)a = a^n$. Thus in either case, a^n belongs to a subgroup. That a has finite order follows from periodicity of the subgroups of S.

Now suppose S is eventually regular and let $a \in S$. Then a^k is regular for some $k \geq 1$. If $(a^k)^2$ is not regular, then one of its powers is regular. In any event, some proper power of a^k is regular, whence a^k has finite order, by the first part of the proposition.

Finally, if S is regular, clearly a^2 belongs to a subgroup for every $a \in S$. Note that each principal factor is either a group or a singular band, each possibly with adjoined zero, or is isomorphic to B_2. Each of these is orthodox, whence so is S, using the result of Hall [4, Lemma 1] that, in any semigroup, a regular element that is a product of idempotents is a product of idempotents in D_a. $\square$

The argument cited in the last paragraph demonstrates that, under those hypotheses, if a product of idempotents is regular, then it is idempotent. The idempotent-generated, periodic $\mathcal{J}$-trivial semigroup $\langle e, f \mid e^2 = e, f^2 = f, fe = 0\rangle$ demonstrates that, in general, E_S need not be a subsemigroup when $\mathcal{L}(S)$ is lower semimodular.

Following [2, §6.6] a semigroup T is said to be *right stable* if whenever $x = bxa$, for some $a, b \in T$, then $x \mathcal{R} bx$. Observe that since $xa = b(xa)a$, it then also follows that $xa \mathcal{R} b(xa) = x$. By [2, Lemma 6.42], a semigroup is right stable if and only if the $\mathcal{R}$-classes contained in any given $\mathcal{J}$-class of S satisfy the minimum condition. In particular, any $\mathcal{J}$-trivial semigroup is right stable. Left stability is defined dually, and stability is the conjunction of the two properties. It is easily seen that any periodic semigroup is stable (cf [2, Exercise 6.2]).

LEMMA 5.2 *In any left or right stable semigroup, and thus in any periodic semigroup, (i) there are no principal factors of type I(b), and (ii) III(b) follows from I and III(a).*

Proof. (i) Suppose, without loss of generality, that T is right stable and that J is a $\mathcal{J}$-class whose associated principal factor P is [0]-simple. Then P is $\mathcal{D}$-trivial and idempotent-free. Given $x \in J$, by [0-] simplicity there exist $a, b \in J$ such that $x = bxa$. As noted above, right stability implies that $x \mathcal{R} xa$ in T, say $x = (xa)t$, $t \in T^1$. But since $at \in J$, the equation $x = x(at)$ contradicts Result 2.2.

(ii) Suppose $x = bxa$, with $PF(J_x)$ null. As noted above, right stability implies that $x \mathcal{R} xa$. Then $x = xa$ by Lemma 3.11. The dual of the lemma yields the same conclusion if T is left stable. $\square$

Example 7.6 demonstrates that, in general, III(b) is independent of the other hypotheses of Theorem 1.1.

Under stability, therefore, I(b), III(b) and III(c) may be omitted entirely from Theorem 1.1. However, under periodicity II and III may be re-expressed in ways that warrant a separate exposition and lead to structural consequences that do not hold in stable semigroups, as examples in §7 show. We note that although assuming regularity only appears to entail deletion of the adjective “non-null” in (1), there are yet more structural consequences (cf Proposition 5.1 and the comments following Corollary 5.4), which we will not pursue here.

THEOREM 5.3 *For a periodic semigroup S, $\mathcal{L}(S)$ is lower semimodular if and only if*

(1) each non-null principal factor of S is either a group with lower semimodular subgroup lattice or a singular band; such a semigroup with zero adjoined; or, up to isomorphism, B_2;

(2) if $e < f \in E_S$ and both H_e and H_f are nontrivial, then $eH_f = H_fe = \{e\}$;

(3) for each element x of S that does not belong to a nontrivial subgroup of S, if $x = xab$ for some mutually inverse elements $a, b \in S$ [such that $ab \notin J_x$], then either $x \in \langle a, b\rangle$ or $x = xa$; and dually.

Proof. Necessity of (1) is a consequence of Lemma 5.2; (2) is contained in Corollary 1.2(ii); and (3) is a special case of III(a).

For sufficiency, we first deduce II. Suppose $ea \in H_e$, where H_e is nontrivial and thus isolated. By periodicity, $\langle a \rangle$ has a group kernel H_f, where $f = a^n$, say. Now since H_e is isolated, $ef \in H_e$ and thus $ef = e$; similarly, $fe = e$. So $e \leq f$. If $f = e$ then $e \in \langle a \rangle$. Otherwise, by assumption, $eH_f = \{e\}$. But $a^{n+1} \in H_f$ and so $ea = efa = ea^{n+1} = e$.

To prove III(a), suppose $x = xab$. By Lemma 1.5, without loss of generality we may assume that a and b are mutually inverse. Thus III(a) follows from (3). That the alternative reading in (3) also suffices follows from Lemma 3.10. Finally, III(b) then follows from Lemma 5.2. □

The presence, or otherwise, of the semigroup B_2 as a principal factor plays a critical role in the structure of such semigroups, as we now show.

COROLLARY 5.4 *Suppose J is a nontrivial, irregular $\mathcal{J}$-class of a periodic semigroup for which $\mathcal{L}(S)$ lower semimodular. Then $|J| = 4$, J consists of two $\mathcal{R}$-classes and two $\mathcal{L}$-classes, and in fact $J = \{x, xa, bx, bxa\}$ for some mutually inverse elements a, b such that $PF(J_a) \cong B_2$, $x \in \langle a, b \rangle$, and $x = x(ab) = (ab)x$. In general, for any such pair a, b, every irregular $\mathcal{J}$-class of $\langle a, b \rangle$ has precisely four elements.*

Proof. Let $x \in J$ and suppose $y \in R_x, y \neq x$. By Lemma 1.5, there exist mutually inverse elements a, b of S such that $y = xa, x = yb$. Since $x = xab$ and $x \neq xa$, then by (3) of Theorem 5.3, $x \in \langle a, b \rangle$. Since J_x is irregular, $x \notin J_a$ and thus J_a is not a subsemigroup of S. By (1), $PF(J_a) \cong B_2$.

Consider for the moment *any* element x that belongs to any irregular $\mathcal{J}$-class of $\langle a, b \rangle$. Write x as a word in a and b. If x begins in b, then since $b = bab$, $x = bax$, where $ax \mathrel{\mathcal{L}} x$. By replacing x by ax, if necessary, we may presume that x begins in a, so that $x = abx$ and $bx \mathrel{\mathcal{L}} x$. Now $bx \neq x$, for otherwise $x = bx = ax$ and $x \in \langle a, b \rangle$ together yield $x^2 = x$, contradicting irregularity of J. Similarly, we may presume that x ends in b, so that $x = xab$, $x \mathrel{\mathcal{R}} xa$ and $x \neq xa$. Then $bx \mathrel{\mathcal{L}} x$ implies $bxa \mathrel{\mathcal{L}} xa$ and $bxa \neq xa$. Hence $|J_x| \geq 4$.

Returning to the first paragraph, we obtain that J contains at least the four distinct elements specified in the statement of the corollary, presuming we modify the choice of x as in the previous paragraph.

It remains to prove that $R_x = \{x, y\}$, since the dual statement will follow similarly. Suppose, then, that there exists a third element z, say, in R_x. Similarly to the above, $z = xc, x = zd$ for some mutually inverse elements c, d in S. Let $g = ab, h = cd$. By periodicity, there is an idempotent power $f = (gh)^n$, and since $x = xg = xh$, $x = xf$. Note that since $f^2 = f$ and $g \mathrel{\mathcal{R}} a$, $f \mathrel{\mathcal{R}} fg \mathrel{\mathcal{R}} fa$. Also $f = fh \mathrel{\mathcal{R}} fc$. Dually we obtain $bf \mathrel{\mathcal{L}} f$ and $df \mathrel{\mathcal{L}} f$. As above, since $x = x(fa)(bf)$, $x \neq x(fa) = xa$ and $J_f \neq J$, it is necessarily the case that $PF(J_f) \cong B_2$. However, from $|\{f, fa, fc\}| \leq 2$ we obtain the contradiction $|\{x, xa, xc\}| \leq 2$.

The final statement now follows from the second paragraph of the proof. □

Example 7.2 demonstrates that the situation described in the corollary can occur. In fact, the structure of the semigroups $\langle a, b \rangle$ that arise in this corollary may be described more completely, but we will not pursue the details here.

The situation in which the semigroup has *no* principal factors isomorphic to B_2 arises sufficiently frequently that it deserves separate discussion. In that case, by Theorem 5.3, every regular $\mathcal{D}$-class is a subsemigroup. According to [12, Theorem 3.16], a periodic semigroup has this property if and only if it is decomposable as a semilattice of archimedean subsemigroups. A semigroup S is *archimedean* if for all $a, b \in S$, $a^n \in SbS$ for some $n \geq 1$, equivalently, if it is a nilextension of a completely simple semigroup (that is, an ideal extension of a completely simple semigroup — its kernel — by a nilsemigroup). Refer to [12, §1] for a broader review of such decompositions in the context of epigroups.

COROLLARY 5.5 *For a periodic semigroup S in which no principal factor is isomorphic to B_2, $\mathcal{L}(S)$ is lower semimodular if and only if*

1. *each irregular $\mathcal{J}$-class of S (equivalently, each non-null principal factor) is trivial;*
2. *each regular $\mathcal{J}$-class of S is either a group with lower semimodular subgroup lattice or a singular band;*
3. *if $e < f$ in E_S, then $ea = ae = e$ for all $a \in D_f$.*

Proof. Necessity of 1 is immediate from Corollary 5.4; 2 is immediate from Theorem 5.3.

To prove necessity of 3, consider first the case that H_e is a nontrivial (and therefore isolated) subgroup. Clearly $e \notin \langle a \rangle$. If H_f is also a nontrivial subgroup, then the conclusion is immediate from (2) of Theorem 5.3. If J_f is right zero, then $ae = a(fe) = (af)e = fe = e$ and so $(ea)^2 = ea$, whence $ea = a$ as well. If J_f is left zero, a dual argument applies.

Next suppose J_e is right zero. If J_f is either a group or right zero, then $f = fab$ for some $b \in J_f$, so $e = eab$. Again $e \notin \langle a, b \rangle$, so by (3) of the cited theorem, $e = ea$. If J_f is left zero, then $ea = efa = ef = e$ in any event. Since J_e is left zero, $ae = e$ as well. The argument when J_e is left zero is dual.

To prove the converse, we need only derive (3) of Theorem 5.3 from properties 1 - 3, since (2) is immediate from property 3. So suppose $x = xab$, where a, b are mutually inverse. Note that $x \mathrel{\mathcal{R}} xa$ so that $x = xa$ is obvious if J_x is either left zero or irregular (using property 1 in the latter case). The remaining case is where J_x is right zero. If $J_x = J_a$, then $x = xab = b$; otherwise since $(ab)x = x$, $x < ab$ and, by property 3, $x = xa$. □

Example 7.1 shows that, without the hypothesis of periodicity, the first property in Corollary 5.5 need not hold, even for stable semigroups. Note also that in the statement of this corollary, the phrase "then either $x \in \langle a, b \rangle$ or" need never be invoked. Example 7.2 demonstrates (cf Corollary 5.4) that this is not true in general for periodic semigroups. In a slightly different direction, (3) in Corollary 5.5 does not hold for periodic semigroups in general, as demonstrated by Example 7.4.

Specializations of Corollary 5.5 to various subclasses of semigroups, such as to completely regular semigroups, result in very simple characterizations of lower semimodularity. Looking ahead to the following subsection, we note two particular cases. Recall that every *band* is a semilattice of rectangular bands (its $\mathcal{J}$-classes); and that every archimedean semigroup is a nilextension of a completely simple semigroup, which is its kernel.

COROLLARY 5.6 *(1) The subsemigroup lattice of a band is lower semimodular if and only if it is a semilattice of singular bands, with the property that whenever one idempotent is below another, then it is below every idempotent in the component of the latter.*

(2) The subsemigroup lattice of an archimedean semigroup is lower semimodular if and only if its kernel is either a singular band or a periodic group whose subgroup lattice is lower semimodular.

We remark that (1) does not imply that given two components, one below the other, every idempotent in the lower is below every idempotent in the higher, as the following example demonstrates.

EXAMPLE 5.7 *Let the band B be the union of the right zero semigroup $\{e, f\}$ with the one-element semigroup $\{g\}$, satisfying $eg = fg = e$ (and $ge = e, gf = f$). Then $e < g$ but $f \not< g$. By the corollary, $\mathcal{L}(B)$ is lower semimodular.*

5.1 Modularity.

Since every modular lattice is lower semimodular, the class of semigroups whose subsemigroup lattice is modular should be identifiable in terms of Theorem 1.1. A description of such semigroups (and those whose lattice belongs to some subvariety of modular lattices, likewise) may be found in [11, §6]. Modularity imposes far stronger restrictions on the underlying semigroup than does lower semimodularity. The purpose of this section is to demonstrate that to derive modularity from lower semimodularity requires a sequence of steps little different from a proof that begins from scratch. It is upper semimodularity that is responsible for the additional restrictions.

In addition to the definition of U-band of semigroups near the end of §1 and the terminology reviewed prior to Corollary 5.5, we need the following. A semigroup S is a *U-semigroup* if $ab \in \langle a \rangle \cup \langle b \rangle$ for all $a, b \in S$. A semigroup is *unipotent* if it contains a unique idempotent. If such a semigroup is periodic, then it is a nilextension of its group kernel (and so archimedean).

RESULT 5.8 *[11, §5] For a semigroup S, $\mathcal{L}(S)$ is upper semimodular if and only if (a) S is periodic, (b) S is a U-chain of archimedean subsemigroups and (c) each of those archimedean subsemigroups has upper semimodular lattice of subsemigroups.*

An archimedean semigroup T, with completely simple kernel K and nil quotient Q, has upper semimodular lattice of subsemigroups if and only if (d) T is a singular band of unipotent semigroups, (e) the maximal subgroups of T are periodic, with upper semimodular subgroup lattices, and (f) Q is a nilpotent U-semigroup.

[11, §6] For a semigroup S, $\mathcal{L}(S)$ is modular if and only if $\mathcal{L}(S)$ is upper semimodular and each archimedean component has the same property. For an archimedean semigroup T, $\mathcal{L}(T)$ is modular if and only if if $\mathcal{L}(T)$ is upper semimodular, the maximal subgroups of T have modular subgroup lattices and, in addition, the decomposition (d) is a U-band decomposition.

It is clear that (a) and (b) do not hold in the lower semimodular case. In fact, (b) fails in more than way. As observed in the previous section, it implies the absence of principal factors isomorphic to B_2. In the absence of such principal factors, consider the case of bands. Property (b) above implies that, under upper semimodularity, (1) B is a chain of its rectangular band components (which by (d) must again be singular bands) and (2) given two components, every idempotent in the lower is below every idempotent in the higher. Example 5.7 exhibits a three-element band B for which $\mathcal{L}(B)$ is lower semimodular but for which the latter criterion is not satisfied. The former condition fails in the lower semimodular case, since every semilattice is $\mathcal{J}$-trivial.

Next consider an archimedean semigroup T, with K and Q as above. According to Corollary 5.6, $\mathcal{L}(T)$ is lower semimodular if and only if K is either a periodic group (with lower semimodular subgroup lattice) or a singular band, Q being an arbitrary nilsemigroup. However (f) imposes stringent restrictions on the quotient semigroup. The statement (d) asserts that if $e, f \in E_K$, then $K_eK_f \subseteq K_{ef}$. Consider $R_{3,1} = \langle a, e \mid e^2 = e = ae, ea^2 = a^2 \rangle$. Here $R_{3,1}$ is the extension of the three-element right zero semigroup $\{e, ea, a^2\}$ by the two-element zero semigroup $\{a, 0\}$. Thus $\mathcal{L}(R_{3,1})$ is lower semimodular. But $e \in K_e$ and $a \in K_{a^2}$, while $ea \in K_{ea}$, so $\mathcal{L}(R_{3,1})$ is not upper semimodular.

Note that according to Result 5.8, a completely simple semigroup S has upper semimodular subsemigroup lattice if and only if S is a right group or left group (with maximal subgroups having subgroup lattice with the same property).

The only necessary condition imposed by modularity, above and beyond those imposed by upper semimodularity and the obvious condition on the subgroups, is that an archimedean semigroup T must be a U-band of its unipotent components, that is, if $a \in K_e, b \in K_f$, where $e \neq f \in E_K$, then $ab \in \langle a \rangle \cup \langle b \rangle$. In conjunction with upper semimodularity, this follows from lower semimodularity. For, given such a, b, then by lower semimodularity K is a singular band. If $ab \in K$, then since, by (d) $K_eK_f \subseteq K_{ef}$, $ab \in K \cap (K_e \cup K_f) = \{e, f\} \subseteq \langle a \rangle \cup \langle b \rangle$. Otherwise, $ab \neq 0$, regarded as a product in Q, and therefore belongs to $\langle a \rangle \cup \langle b \rangle$ by (f). This paragraph proves the following result.

PROPOSITION 5.9 *The lattice $\mathcal{L}(S)$ is modular if and only if it is both lower and upper semimodular and the subgroup lattice of every subgroup of S is modular.*

5.2 Homomorphic images.

Since every free semigroup has lower semimodular subsemigroup lattice, this property is not in general inherited by homomorphic images. However for periodic semigroups the situation is different.

THEOREM 5.10 *Lower semimodularity of the lattice of subsemigroups is inherited by homomorphic images within the class of periodic semigroups.*

Proof. Suppose S is periodic, with $\mathcal{L}(S)$ lower semimodular, and let $\phi : S \to T$ be a surjective homomorphism. We will show that (1), (2) and (3) of Theorem 5.3 are satisfied.

LEMMA 5.11 *Under the assumptions just stated, for each nontrivial subgroup of T, with idempotent f say, there is a unique subgroup of S that maps onto H_f, and the idempotent e of that subgroup is the minimum idempotent that maps onto f. Hence $\mathcal{L}(H_f)$ is lower semimodular and T satisfies (2).*

Proof. Since S is periodic, there is some idempotent g that maps to f; further, if $y \in H_f, y \neq f$ and $x\phi = y$, then for some $n \geq 1$, $e = (gx)^n \in E_S$ and $ex = e(gx) \in H_e$. Now $e\phi = f$ and $(ex)\phi = y$, so H_e is nontrivial. Let h be any idempotent of S such that $h\phi = f$. Then $(he)\phi = f, (hex)\phi = y$, so $he \neq h(ex)$. According to Corollary 1.2(ii), $ex = h(ex)$ and so, since H_e is isolated, $h \geq e$. Since this argument was independent of the choice of y in H_f, $H_e\phi = H_f$ and H_e is the unique subgroup with that property.

By the correspondence theorem, the subgroup lattice of H_f is isomorphic to an interval sublattice of the subgroup lattice of H_e and is therefore again lower semimodular.

Finally, suppose that $f < h$ in E_T, where H_f and H_h are nontrivial subgroups. Let e, g be the minimum idempotents that map onto f, h, as above, so that $H_e\phi = H_f, H_g\phi = H_h$. Since for some $n \geq 1$, $(ege)^n$ is an idempotent that again maps to f, $e < g$. Hence $eH_g = H_g e = e$ and so $fH_h = H_h f = f$ as required. □

To complete the proof, we make use of properties of congruences and Green's relations on eventually regular semigroups, proved by P.M. Edwards [3], that generalized earlier results of Hall on regular semigroups. For completeness, we include direct proofs for the special cases we need.

To prove (3), suppose $x, a, b \in T$, where x does not belong to a nontrivial subgroup, $x = xab$ and a, b are mutually inverse. Then there exist mutually inverse elements c, d of S such that $c\phi = a, d\phi = b$. (Suppose $u\phi = a, v\phi = b$. Let $e = (uv)^n$ be the idempotent power of uv. Put $c = eu$ and $d = v(uv)^{n-1}e$.) Put $e = cd \in E_S$ and let $y \in S$ map to x. Then $(ye)\phi = x$ and $ye = (ye)cd$. Applying (3) to S, either $ye \in \langle c, d\rangle$ or $ye = (ye)c$, yielding (3) for T.

To prove (1), let $f \in E_T$ and suppose that $J_f(= D_f)$ is neither a nontrivial subgroup nor a singular band. Then there exists $g \in E_T$, $g \notin L_f \cup R_f$. Let $a \in R_f \cap L_g$, so a has an inverse $b \in L_f \cap R_g$. As above, there exist mutually inverse elements $c, d \in S$ such that $c\phi = a, d\phi = b$. Since cd and dc are then $\mathcal{D}$-related idempotents of S that are neither $\mathcal{L}$- nor $\mathcal{R}$-related, the principal factor associated with J_c is isomorphic to B_2. As a consequence, $a, b \notin E_T$. Suppose that $y \in R_f$. Then there exist $g \in E_S$ and $u, v \in R_g$ such that $g\phi = f, u\phi = a, v\phi = y$. (Let z be the inverse of y and choose mutually inverse elements $s, t \in S$ such that $s\phi = y, t\phi = z$. With $k = cd, \ell = st \in E_S$, where $k\phi = ab = f = yz = \ell\phi$, let $g = (k\ell)^n$ be the idempotent power of $k\ell$, $u = gc$ and $v = gs$.) Again, the principal factor associated with J_g is isomorphic to B_2 and so $|R_g| = 2$. Hence $y = a$ or $y = f$ and the principal factor associated with J_f is also isomorphic with B_2. □

5.3 Varieties.

Theorem 5.10 naturally leads to consideration of varieties of semigroups all of whose members have the property that their subsemigroup lattices are lower semimodular. We call such a variety *LSM*. It was noted in [11] that free semigroups and free commutative semigroups have lower semimodular subsemigroup lattices. The connection with LSM varieties is provided by Theorem 5.12 below. First we provide some terminology and well known background that will used throughout this section. In this context, a useful reference is [9].

A variety that contains all commutative semigroups is often termed *overcommutative*. More broadly, a variety that contains all semilattices is often termed *regular* (and otherwise *irregular*). A variety is *periodic* if it satisfies some periodic identity $x^n = x^{n+k}$, where $n, k \geq 1$. We recall some elementary facts about semigroup varieties. Every variety is either overcommutative or periodic. A variety is irregular if and only if it (is periodic and) consists of nilextensions of completely simple semigroups.

THEOREM 5.12 *Let* $\mathbf{V}$ *be a variety of semigroups. Then* $\mathcal{L}(F)$ *is lower semimodular for every (relatively) free semigroup* $F \in \mathbf{V}$ *if and only if either* $\mathbf{V}$ *is overcommutative or* $\mathbf{V}$ *is a periodic LSM variety.*

Proof. It is no doubt well known that the relatively free semigroups in any overcommutative are $\mathcal{J}$-trivial, but we include a proof for completeness. Let $\mathbf{V}$ be such a variety. We shall prove that $\mathbf{V}$ is periodic, contradicting the dichotomy stated above. It suffices to prove the statement for the relatively free semigroup F over the countably infinite set X. If F is not $\mathcal{J}$-trivial, then there exist distinct $w, z \in F$ such that $w = szt, z = pwq$, where $s, t, p, q \in F^1$. Thus $w = uwv$ for some $u, v \in F^1$, not both 1. Interpreting each of w, u, v as a word over X, this equation becomes an identity satisfied in $\mathbf{V}$. Now map each variable to a new variable x, yielding an identity of the form $x^n = x^k x^n x^\ell$ satisfied in $\mathbf{V}$, with $k + n + \ell > n$.

If $\mathbf{V}$ is periodic and $\mathcal{L}(F)$ is lower semimodular for every (relatively) free semigroup $F \in \mathbf{V}$, then the same is true of every member of $\mathbf{V}$, by Theorem 5.10. $\square$

One class of obvious examples of LSM varieties comprises the *$\mathcal{J}$-trivial* ones: those that consist entirely of $\mathcal{J}$-trivial semigroups.

PROPOSITION 5.13 *The following are equivalent for a variety* $\mathbf{V}$ *of semigroups:*

1. $\mathbf{V}$ *is* $\mathcal{J}$*-trivial;*
2. $\mathbf{V}$ *consists of semilattices of nilsemigroups;*
3. $\mathbf{V}$ *satisfies the identities* $x^n = x^{n+1}, (xy)^n = (yx)^n$, *for some* $n \geq 1$*;*
4. $\mathbf{V}$ *consists of semigroups in which every regular* $\mathcal{J}$*-class is trivial.*

Proof. Suppose $\mathbf{V}$ is $\mathcal{J}$-trivial. Then it is periodic (since it is not overcommutative). Now $\mathbf{V}$ does not contain B_2, so by [9], $\mathbf{V}$ consists of semilattices of archimedean semigroups. Each component is a nilextension of a completely simple semigroup and each of the latter must be trivial.

Suppose next that $\mathbf{V}$ consists of semilattices of nilsemigroups. Then $\mathbf{V}$ is periodic and so satisfies $x^n = x^{n+k}$ for some $n, k \geq 1$; further, $\mathbf{V}$ contains no nontrivial groups and so $k = 1$. For any $x, y \in S$, $(xy)^n$ and $(yx)^n$ are idempotents that lie in the same nilsemigroup component, and are therefore equal.

Next, if $\mathbf{V}$ satisfies the given identities then for any $S \in \mathbf{V}$, $\mathcal{J}$=$\mathcal{D}$ and for any $\mathcal{D}$-related idempotents $e, f \in S$, $e = xy, f = yx$ for some $x, y \in S$, whence $e = f$. Thus every regular $\mathcal{J}$-class is a group which, further, is trivial.

Finally, a variety satisfying (iv) is clearly periodic, and (i) follows from the well known fact that a periodic semigroup is $\mathcal{J}$-trivial if its regular $\mathcal{J}$-classes are trivial. (Suppose there exist distinct $\mathcal{R}$-related elements x, y. Then by Lemma 1.5 there exist mutually inverse elements a, b of S such that $y = xa, x = yb$. Assuming J_a is trivial yields the contradiction $x = xab = xa = b$. Thus $\mathcal{R}$ is trivial and, dually, $\mathcal{L}$ is trivial.) □

LEMMA 5.14 *If a variety* $\mathbf{V}$ *is LSM then:*

(i) $\mathbf{V}$ *does not contain* B_2*;*

(ii) if $\mathbf{V}$ *contains a nontrivial semigroup from any one of the classes of semilattices, groups, left zero semigroups or right zero semigroups, then it contains nontrivial semigroups from only one of those classes;*

(iii) $\mathbf{V}$ *is periodic.*

Proof. (i) This follows from Example 7.3(d). (ii) This follows from (a)—(c) of the same example. (iii) According to the remarks above, if $\mathbf{V}$ is not periodic, then it contains all commutative semigroups. But this contradicts (ii). □

THEOREM 5.15 *Every LSM variety of semigroups is periodic. The regular LSM varieties are precisely the (regular)* $\mathcal{J}$*-trivial varieties described above. The irregular LSM varieties are (a) the variety of left zero semigroups; (b) the variety of right zero semigroups; (c) the periodic group varieties all of whose members have lower semimodular subgroup lattice; and (d) varieties comprising nilextensions of members of a variety of type (a), (b) or (c).*

Proof. Let $\mathbf{V}$ be an LSM variety. The first statement is Lemma 5.14(iii). In conjunction with (i) of that lemma, it follows that we may apply Corollary 5.5. Further, if $\mathbf{V}$ contains a nontrivial semilattice, then in conjunction with (ii) of the same lemma, it then follows that $\mathbf{V}$ is $\mathcal{J}$-trivial.

Otherwise, $\mathbf{V}$ consists of nilextensions of completely simple semigroups. The completely simple members of $\mathbf{V}$ form a subvariety of $\mathbf{V}$ and then Lemma 5.14(ii) yields the specified classification. The converse follows from Theorem 5.3. □

The author is not aware of a structural description of the LSM group varieties. It is known that the product of two finite groups with lower semimodular subgroup lattices retains that property [10, Corollary 5.3.12].

6 0-simple and simple semigroups.

As remarked in the introduction, the author does not know whether there exists a simple, or a 0-simple, semigroup S that is not completely 0-simple yet has lower semimodular lattice of subsemigroups. The criterion to be satisfied is remarkably simple: it must be $\mathcal{D}$-trivial (thus idempotent-free and satisfying no equation of the form $xy = x$ or $xy = y$) and satisfy III(c), without the necessity for the exceptional case when $n = 1$. That is, whenever an element x satisfies an equation of the form $x = a_0xa_1x \cdots xa_n$, for some $a_0, \ldots, a_n \in S^1$ (necessarily $a_0, a_n \in S$) and $n \geq 1$, then $x \in \langle a_0, a_1, \ldots, a_n \rangle$.

We should note that in the more general (and as yet hypothetical) situation of a [0-] simple principal factor within a larger semigroup, III(a) and the exceptional case in III(c) must also be verified in the situations that a, b, and a_0, a_1, respectively, lie in a higher $\mathcal{J}$-class of S.

We will focus here on the case of simple semigroups. Any idempotent-free semigroup that satisfies no equation of the form $xy = x$ or $xy = y$ can be embedded in a simple, $\mathcal{D}$-trivial semigroup [6]. Moreover, any cancellative semigroup without idempotents (which necessarily satisfies the condition on equations) can be embedded in a cancellative semigroup of that type [1]. Thus such semigroups are plentiful. Concrete examples are harder to find in the literature, however. Anderson (see [1, §2.1]) showed how to construct cancellative examples from the positive parts of ordered fields. However, none of these may serve the desired purpose, as we now show.

COROLLARY 6.1 *If a left or right cancellative, simple semigroup has lower semimodular lattice of subsemigroups, then it is a group or a singular band.*

Proof. Assume throughout that the given semigroup S is right cancellative, the alternative argument being a dual one. If S is completely simple, then it is a left group, that is, the direct product of a left zero semigroup and a group [1]. The conclusion follows from Theorem 1.1. Otherwise, S is $\mathcal{D}$-trivial and so idempotent-free. By simplicity, for any $x \in S$, $x = ax^3b$ for some $a, b \in S$. Rewriting this equation as $x = (ax)x(xb)$, then by the criterion stated above, $x \in \langle ax, xb \rangle$. Since $x \notin xS$ and $x \notin Sx$, $x = (ax)^i u (xb)^j$, where $i, j \geq 1$ and either $u = 1$ or $u \in (xb)S(ax)$, $u = xvx$, say.

From $x = (ax)x(xb)$ it follows that $x = (ax)^j x (xb)^j$ whence, by right cancellativity, $(ax)^j x = (ax)^i u$. If $u = 1$, then necessarily $j < i$, since $(ax)^i \notin (ax)^i S$. But by cancelling x we obtain $(ax)^j = (ax)^{i-1} a \in (ax)^j S$, a contradiction. If $u = xvx$, then cancelling x yields $(ax)^j = (ax)^i xv$ and, similarly, $j > i$. But then, substituting $(ax)^{j-i} x (xb)^{j-i}$ for x in $(ax)^i xv$ yields $(ax)^j \in (ax)^j S$ and, once more, a contradiction is obtained. □

The proof may be amended in the 0-simple case to cover the case that S is 0-cancellative (that is, all nonzero elements may cancelled). For semigroups in general, cancellativity does not conflict with lower semimodularity: for instance free semigroups have both properties. The author conjectures that there does exist a cancellative semigroup having a nontrivial null principal factor and lower semimodular subsemigroup lattice. None of the examples in the next section that have nontrivial null principal factors are left or right cancellative, however.

Recall from §2 that simple semigroups that are not $\mathcal{D}$-trivial were removed from consideration by identifying one of the subsemigroups A, A^d, B, C in each, and proving that no such semigroup has lower semimodular subsemigroup lattice. Unfortunately, the author knows of no analogous theorem in the $\mathcal{D}$-trivial case.

However, we may attempt to proceed as follows. In any semigroup S, each element x of a nontrivial $\mathcal{J}$-class satisfies an equation of the form $x = axb$. If S is simple, $\mathcal{D}$-trivial and idempotent-free then $x \neq ax, xb$. Under III(c), therefore, $x \in \langle a, b\rangle$. Moreover, since $x = a^n x b^n$, $T = x \in \langle a^n, b^n\rangle$, for every $n \geq 1$. Thus T satisfies a set of equations of the form $x = w_n(a^n, b^n)$, $n = 1, 2, \ldots$. (Of course, it must satisfy many further equations, either generated in a similar manner from lower semimodularity or already present in S itself.) Note that T will not itself be simple since, as observed in §2, no finitely generated simple semigroup is $\mathcal{D}$-trivial.

The following lemma provides a constraint on such a subsemigroup T, which we will then apply to eliminate a previously known example from consideration. First, recall from §2 that G denotes the infinite cyclic group, presented as a monoid by $G = \langle a, b \mid ab = ba = 1\rangle$. Alternatively, a semigroup presentation is $G = \langle a, b \mid a(ab) = (ba)a = a, (ab)b = b(ba) = b\rangle$. Thus for any semigroup $T = \langle a, b\rangle$, the congruence σ generated by the latter set of relations yields a cyclic quotient group H, in which a and b are mutually inverse.

LEMMA 6.2 *Suppose a semigroup $T = \langle a, b\rangle$ satisfies no equation of the form $xy = y$ or $xy = x$. Let $H = T/\sigma$, where σ is the cyclic group congruence defined above. Suppose that $|H| \neq 1$. Then if T satisfies an equation $x = axb$, it does not satisfy the criterion stated in III(b) (and in III(c) in the case $n = 1$).*

Proof. Suppose, to the contrary, that T satisfies the criterion stated in III(b). Since T is idempotent-free, no principal factor is regular, and so the principal factor associated with J_x is either null or [0-] simple (and so $\mathcal{D}$-trivial). Now for each $n \geq 1$, $x = a^n x b^n$ and so $x \in \langle a^n, b^n\rangle$. Hence $x\sigma \in \bigcap_{n\geq 1}\langle (a\sigma)^n, (b\sigma)^n\rangle = \{1\}$. So $x\sigma = 1$. However the element ax of T also satisfies the equation $ax = a(ax)b$, and so $(ax)\sigma = 1$, similarly, yielding $a\sigma = 1$ and thus triviality of H, contradicting the assumption. $\square$

One of the simplest candidates for such a set of equations is $x = a^n b^n$, $n = 1, 2, \ldots$, each a consequence, of course, of $ab = a(ab)b$.

PROPOSITION 6.3 *Let $T = \langle a, b \mid ab = a(ab)b\rangle$. Then $\mathcal{L}(T)$ is not lower semimodular. The principal factor associated with J_{ab} is an infinite null semigroup; T has no simple or 0-simple principal factors.*

Proof. Clearly the given relation is a consequence of the relations defining G, so $T/\sigma \cong G$. Now any words u, v in the free semigroup on $\{a, b\}$ that are equal as elements of T must begin with the same letter, end with the same letter, and contain the same number of alternations of a's and b's. So no equation $xy = x$ or $xy = y$ can hold. Hence, by the lemma, $\mathcal{L}(T)$ is not lower semimodular. The $\mathcal{J}$-class of ab consists of the elements $a^n b, ab^n, n \geq 1$ and is null, since any product contains the alternation ba. Clearly, no equation of the form $y = uy^2v$ can hold in

T. Since in any simple or 0-simple semigroup, some element is $\mathcal{J}$-related to its square, no such principal factor can occur. $\square$

In any [0-] simple semigroup, there is at least one nonzero element x such that all powers of x are nonzero, so that for each $i \geq 1$, an equation of the form $x = u_i x^i v_i$ holds. In combination with the previous discussion, setting $x = ab$, each $u_i = a$ and each $v_i = b$ yields perhaps the next simplest semigroup to consider in this context, the semigroup $D = \langle a, b \mid a(ab)^n b = ab, \forall n \geq 1\rangle$ introduced in §2.

PROPOSITION 6.4 *The lattice $\mathcal{L}(D)$ is not lower semimodular. In fact, the subsemigroup lattice of the simple, $\mathcal{D}$-trivial and noncancellative kernel $K = J_{ab}$ of D is not lower semimodular.*

Proof. Observe first that, similarly to the previous example, the map $a \to a, b \to b$ extends to a surjective morphism $D \to G$ that induces σ. When restricted to the kernel K, $ax \to a$ and $xb \to b$, where once again we put $x = ab$, so the restriction is also surjective. Thus $K/\sigma \equiv G$. The equation $x = (ax)x(xb)$ holds in K, whence the result follows from Lemma 6.2.$\square$

Lemma 6.2 also provides an alternative method of proving that the cancellative semigroups constructed by Anderson from ordered fields, cited above, do not have lower semimodular subsemigroup lattices.

7 Examples.

In this final section we present a series of examples, primarily demonstrating the independence of the hypotheses in Theorem 1.1, also demonstrating that the alternative outcomes stated in some of those hypotheses are each necessary and demonstrating that the hypotheses are satisfied nonvacuously. We shall also consider Theorem 5.3 in the same context.

Note first that all the completely simple semigroups that appear as principal factors in I(a) do have lower semimodular subsemigroup lattices. In fact, the subsemigroup lattice of a singular band is distributive. That $\mathcal{L}(B_2)$ is lower semimodular is immediate from Theorem 5.3, using the alternative reading of (3). Recall that we do not know whether there exists a semigroup in which I(b) and III(c) are satisfied nonvacuously. Subsequent statements will always be relative to that open question.

Result 1.4 allows semigroups to be constructed with all possible combinations of non-null principal factors, according to I. This remains true under the restriction of periodicity as in Theorem 5.3. Note, however, that by Corollary 5.5, if $\mathcal{L}(S)$ is lower semimodular for a periodic semigroup without principal factors isomorphic to B_2, then it has no nontrivial null principal factors. The following example demonstrates that the absence of B_2 does not imply triviality of null principal factors in general, even under the assumption of stability.

EXAMPLE 7.1 *Let $S = \langle a, b \mid b^2 = b^2ab = bab^2, b^3 = 0, a^2 = 0\rangle$. Then S has exactly one nontrivial $\mathcal{J}$-class, namely $J_{b^2} = \{b^2, b^2a, ab^2, ab^2a\}$. The associated principal factor is a null* *semigroup comprising two distinct nonzero $\mathcal{R}$-classes and two distinct nonzero $\mathcal{L}$-classes. The semigroup S is stable but not periodic, and $\mathcal{L}(S)$ is lower semimodular.*

Proof. From the relations it is clear that if a product of generators contains b^2 and is nonzero, then it reduces to one of the four forms exhibited in the description of J_{b^2}. Clearly, $b^2 \mathcal{R} b^2a \mathcal{L} ab^2a \mathcal{R} ab^2$.

Denoting by ρ the congruence on the free semigroup on $\{a, b\}$ induced by the given relations, it is straightforward to check that $\{(ba)^i b^2 (ab)^j : i, j \geq 0\} = b^2\rho$. It follows that the four words listed above represent distinct elements of S (since equality of any two of them would lead to a relation of the form $b^2 = ub^3v$); and that the associated principal factor is null.

The other nonzero elements of S are then of the form $(ab)^n$ or $(ba)^n$, for $n \geq 1$, or $b(ab)^n$ or $(ab)^n a$, for $n \geq 0$, all of which are distinct and constitute singleton $\mathcal{J}$-classes.

Clearly S satisfies I and II. Now setting $x = b^2$ first, suppose $x = xuv$ for some $u, v \in S$, with $xu \neq x$. Then, as above, $u = (ab)^i a, v = b(ab)^j$, for some $i, j \geq 0$. Now $b^2 = (ba)^j b^2 (ab)^j = b(ab)^j b(ab)^j = v^2 \in \langle u, v\rangle$. Similar arguments apply to $x = b^2a, x = ab^2$ and $x = ab^2a$. Thus S satisfies III(a). Any semigroup all of whose $\mathcal{J}$-classes are finite is stable (see §5). By Proposition 5.2, S satisfies III(b). Hence $\mathcal{L}(S)$ is lower semimodular. □

It should be noted that, with somewhat more difficulty, an example similar to the last one may be constructed in which the only nontrivial $\mathcal{J}$-class consists of one two-element $\mathcal{R}$-class (cf Corollary 5.4).

The next example demonstrates that, without the assumption on the lack of principal factors of the form B_2, periodic semigroups may indeed have nontrivial null principal factors (cf Corollary 5.4). It also demonstrates that if S is periodic but not regular then, in contrast to Proposition 5.1, the square of a regular element need not belong to a subgroup. Since the arguments are similar to, but simpler than, those of the previous example, we omit the details.

EXAMPLE 7.2 *Let $S = \langle a, b \mid aba = a, bab = b, b^3 = a^2 = 0\rangle$. Then S has two nonzero $\mathcal{J}$-classes, namely $J_a = \{a, b, ab, ba\}$ and $J_{b^2} = \{b^2, b^2a, ab^2, ab^2a\}$. The principal factor for the former is isomorphic to B_2; the principal factor for the latter is a null semigroup containing four distinct nonzero $\mathcal{H}$-classes. The lattice $\mathcal{L}(S)$ is lower semimodular.*

That Theorem 5.3(2) (and therefore also II) is independent of the other hypotheses is easily seen by consideration of the product of a two-element semilattice with a two-element group. We collect some similar results together, as follows.

EXAMPLE 7.3 *In each of the following cases, the lattice of subsemigroups is not lower semimodular: (a) the product of a nontrivial semilattice and either a nontrivial group or a nontrivial singular band; (b) the product of a nontrivial group and a nontrivial singular band; (c) the product of a nontrivial left zero semigroup and a nontrivial right zero semigroup; (d) the product of two copies of B_2.*

Proof. In (a), II (or Corollary 1.2(ii)) is contradicted in the first case, and III(a) (or Corollary 5.5(3)) in the second case. In (b),(c) and (d), I(a) is contradicted. (Note that $B_2 \times B_2$

contains a copy of B_4.) □

The necessary condition Corollary 5.5(3) states that for periodic semigroups with no principal factor B_2, whenever e, f are idempotents such that $e < f$, then $ea = ae = e$ for all $a \in D_f$. The next example demonstrates that this does not extend to periodic semigroups in general (cf Theorem 5.3(2) and Corollary 1.2(ii)). It also demonstrates that the phrase "or $e \in \langle a \rangle$" cannot be removed from II.

EXAMPLE 7.4 *For any $n \geq 2$, let M_n be the inverse semigroup presented, as such, by $\langle a \mid a^2 = a^{2+n} \rangle$. (See, for instance, [7, Chapter IX].) Then the kernel K of M_n is the cyclic group H_{a^2} of order n, with identity element $e = a^n$, and $M_n/K \cong B_2$. Then M_n satisfies the hypotheses of Theorem 5.3: (1), since the subgroup lattice of a cyclic group is distributive; (2), vacuously; and (3), vacuously, using the alternative reading. Hence $\mathcal{L}(M_n)$ is lower semimodular. Note that $e < aa^{-1}$ but $ea \neq e$ (cf Corollary 5.5(3)).*

Turning now to III(a) (or Theorem 5.3(3)), Example 7.3(a) demonstrates that this is not even a consequence of the assumption that each nontrivial principal factor be a singular band. Our next example does the same under the assumption that each nontrivial principal factor be isomorphic to B_2.

EXAMPLE 7.5 *Let E be the semilattice consisting of two chains $e_1 > e_2 > 0$ and $f_1 > f_2 > 0$, amalgamated at 0; and let S be the inverse semigroup of isomorphisms between principal ideals of E (the "Munn semigroup" on E). Let $a : Ee_1 \cong Ef_1$, with inverse b, and let $g : Ee_1 \cong Ee_1$. Then the nonzero $\mathcal{J}$-classes of S are $J_a = \{a, b, ab, ba\}$ and $J_{ga} = \{ga, bg, g, bga\}$; the corresponding principal factors are each isomorphic to B_2. Note that $g = gab$. However $g \neq ga$ and since $a^2 = b^2 = 0$, $g \notin \langle a, b \rangle = \{a, b, ba, ab, 0\}$. Hence $\mathcal{L}(S)$ is not lower semimodular.*

Observe that in Corollary 5.5, the alternative conclusion "then either $x \in \langle a, b \rangle$" does not appear. Example 7.2 demonstrates that this phrase cannot be removed from Theorem 5.3(3) (or from III(a)) in general. Adjoining an identity element to that semigroup then demonstrates that the alternative "or $x = xa$" also cannot be removed.

Turning next to III(b), we recall from Lemma 5.2 that this is a consequence of the other hypotheses in the case of left or right stable semigroups. We now provide an example demonstrating that this is not true in general.

EXAMPLE 7.6 *The semigroup $S = \langle a, x, b \mid x = bxa, ab = ba = 0 \rangle$ satisfies all the hypotheses of Theorem 1.1 except III(b).*

Proof. From the given relations, it is clear that any nonzero product of generators can contain at most one instance of x; thus the only nonzero elements of S other than x itself are of the form a^n, b^n, $b^n x$ and xa^n, for $n \geq 1$. It is easily verified that these elements are distinct. The elements of $\langle a, b \rangle$ form singleton $\mathcal{J}$-classes; the remaining elements constitute J_x, the associated principal factor being null.

Then I, II, III(a) and III(c) are satisfied either trivially or vacuously. (For III(a), this is an immediate consequence of the easily verified observation that S satisfies no equation of the form $u = uv$ or $u = vu$, for nonzero u.) □

The next example demonstrates that the alternative conclusion "$x \in \langle a, b\rangle$" in III(b) cannot be removed (and thus that III(b) is not satisfied vacuously). By adjoining an identity element to this example, we see that the alternative "or $x = xa$" also cannot be removed in general.

EXAMPLE 7.7 *Let F denote the free semigroup on $\{a, b\}$. Let*

$$W = \{b^{k+t}a^{(i+1)k}b^k a^{k+t-i} : t \geq 0, k \geq 1, 0 \leq i < k\}$$

and let T be the semigroup generated by $\{a, b\}$, subject to all the relations $w_1 = w_2$, $w_1, w_2 \in W$, together with the relations $aw = wb = 0, w \in W$. Let x be the image in T of some (any) element of W. Then

(i) T contains exactly one nontrivial $\mathcal{J}$-class, namely J_x, which consists of the distinct elements $x, b^n x, xa^n, n \geq 1$.

(ii) The principal factor associated with J_x is null.

(iii) If $y \neq 0$, then the equation $y = dyc$ is satisfied only if $y \in J_x$ and $d = b^n, c = a^n$, for some $n \geq 0$.

(iv) The equation $yc = y$, $c \neq 1$, is satisfied in T only if $y = 0$.

(v) For every $y \in J_x$, $y \in \langle a^n, b^n\rangle$ for every $n \geq 1$. In conjunction with (iii), T therefore satisfies III(b).

(vi) Hence $\mathcal{L}(T)$ is lower semimodular.

Proof. By introducing an auxiliary variable x, the relations may more conveniently be expressed as the union of the sets of relations

$$x = b^k a^k b^k a^k = b^k a^{2k} b^k a^{k-1} = \cdots = b^k a^{(k-1)k} b^k a,$$

for positive integers k, together with the relations $x = bxa$ and $ax = xb = 0$.

Throughout, a and b will denote both the generators of F and their images in T. Then the use of x as a variable, above, is consistent with its use in the statement of the Example, as an element of T. Denote by ρ the congruence on F induced by the given relations; and denote by Z the set of words corresponding to 0. Let $w \in W$. Observe first that $b^n w a^n \rho w$ for all $n \geq 0$. In terms of T, $b^n x a^n = x$, and so $b^n x, xa^n \in J_x$ for all $n \geq 0$.

Secondly, if $vwu \notin Z$ for some $u, v \in F^1$, then since $aw, wb \in Z$, $v = b^m$ and $u = a^n$, for some $m, n \geq 0$. In addition, if $m = n$ then $vwu\, \rho\, w$; if $m > n$, then $vwu\, \rho\, b^{m-n}w$; and if $m < n$, then $vwu\, \rho\, wa^{n-m}$. In terms of T, therefore, $J_x = \{x, b^m x, xa^n : m, n \geq 1\}$; and $(\langle b\rangle^1 W \langle a\rangle^1)\rho = J_x$.

Since $ax = xb = 0$, (ii) holds.

Now since $b^k a^k b^k a^k \in W$, $x \in \langle a^k, b^k\rangle$, for all $k \geq 1$. Similarly, since $b^k a^{(i+1)k} b^k a^{k-i} \in W$, $xa^i \in \langle a^k, b^k\rangle$, for all $k \geq 1$ and for all i, $0 \leq i < k$. For $i \geq k$, by writing $i = qk + r$ we obtain that $xa^i \in \langle a^k, b^k\rangle$, for all $k \geq 1$ and for all $i \geq 0$. For $0 \leq i < k$ we may write $b^i x = b^k x a^{k-i}$, and so $b^i x \in \langle a^k, b^k \rangle$ for all $k \geq 1$. Similarly, this in fact holds for all $i \geq 0$. Thus the first statement of (v) holds.

Clearly $J_0 = \{0\}$. We next show that if $s \in F$, $s \notin Z$ and $s \notin \langle b \rangle^1 W \langle a \rangle^1$, then $s\rho = \{s\}$. For if $s\rho t$ for some $t \in F, t \neq s$, then t results from a sequence of elementary transitions of the form $vw_1u \to vw_2u$, where $v, u \in F^1, w_1, w_2 \in W$. In particular, $s = vw_1u$ and, since $s \notin Z$, this contradicts the second assumption on s, according to the first statement in the third paragraph of the proof.

Hence if $y \in T$, $y \neq 0$ and $y \notin J_x$, the equation $y = dyc$ in T cannot hold for any $d, c \in T^1$, not both 1. For if $y = s\rho$ then $s\,\rho\,vsu$ for some $v, u \in F^1$, not both empty, a contradiction. It immediately follows that J_x is the only nontrivial $\mathcal{J}$-class of T.

To prove (iii) and (iv), suppose $y \in T$ and $y = dyc$, where $d \in T^1$ and $c \in T$. As noted in the previous paragraph, $y \in J_x$. Thus $d = b^m$ and $c = a^n$ for some $m \geq 0, n \geq 1$. Now $y = b^i x a^j$, for some $i, j \geq 0$, and $x = b^j y a^i = b^m x a^n$. In (iv), we may take $m = 0$; in (iii), $m \geq 1$ and if $m \neq n$, we obtain $x = xa^{|n-m|}$.

Thus the proof of (iii) and (iv) will be completed once we have shown that the equation $x = xa^n$ cannot hold for $n \geq 1$. In fact, by arguments similar to those in the previous paragraph, this will also demonstrate that the elements of J_x listed in (i) are distinct. Now from the form of the words in W it may be verified that if $b^m a^n b^p a^q \in W$, then $m - q = n/p - 1$. It follows that $baba^{n+1} \notin W$. Note that $(baba^{n+1})\rho = xa^n$. That $x \neq xa^n$ will then follow from the fact that W comprises an entire ρ-class of F, as we now prove.

It suffices to show that for any $w \in W$, any elementary transition $w = vw_1u \to vw_2u$, where $v, u \in F^1, w_1, w_2 \in W$, results in another element of W. As above, necessarily $v = b^m, u = a^n$ for some $m, n \geq 0$. We will show that $m = n$, whence $vw_2u = b^n w_2 a^n \in W$, as shown in the first paragraph of the proof. Now since $w, w_1 \in W$, there exist $t \geq 0, k \geq 1, 0 \leq i < k$ such that $w = b^{k+t} a^{(i+1)k} b^k a^{k+t-i}$, and $s \geq 0, \ell \geq 1, 0 \leq j < \ell$ such that $w_1 = b^{\ell+s} a^{(j+1)\ell} b^\ell a^{\ell+s-j}$. By matching exponents of the second occurrence of b be obtain $k = \ell$; similarly, from the first occurrence of a we therefore also obtain $i = j$; from the first occurrence of b we obtain $k+t = \ell+s+m$ and so $t = s+m$; and from the second occurrence of a that $k+t-i = \ell+s-j+n$, so that $t = s + n$. Hence $m = n$, as required.

That $\mathcal{L}(T)$ is lower semimodular now follows from Theorem 1.1. $\square$

A final observation on the complexity of this example is in order. In the same notation, the simplest instance of III(b) that must be satisfied there is $x = bxa$; and the simplest solution exhibited in the example is $x = baba$. It can be shown that no example of this type can be constructed in which such a solution takes the form $x = b^m a^n$ or $x = a^m b^n$.

We conclude with a result that enables the construction of further, complex examples.

PROPOSITION 7.8 *Let T be an ideal extension of a periodic semigroup S by a periodic, $\mathcal{J}$-trivial semigroup Q (so that T is again periodic). If $\mathcal{L}(S)$ is lower semimodular, so is $\mathcal{L}(T)$.*

Proof. We apply Theorem 5.3. Since (2) and (3) involve only multiplication by regular elements, it suffices to show that if distinct elements x and y of S are $\mathcal{J}$-related in the extension T then they are $\mathcal{J}$-related in S, for then the principal factors will retain the requisite properties.

By periodicity and duality, it suffices to show this is true when $x \mathcal{R} y$ in T. But by Lemma 1.5 there exist inverse elements s, t of T such that $y = xs, x = yt$. If $s, t \notin S$, then $t = s = s^2$, yielding $x = y$. Thus $s, t \in S$ and $x \mathcal{R} y$ in S, as required. $\square$

With rather more difficulty, it may be shown that the conclusion remains true without the restriction of periodicity on S. Without periodicity of Q, however, the conclusion is false.

EXAMPLE 7.9 *Let $S = \{r, s\} \cup \{0\}$ be a null semigroup and let $F = \langle a \rangle$ be infinite cyclic. Define a product on $T = S \cup F$ by putting $ar = as = 0$ and $ra = s, sa = r$; extending these actions to F in the obvious way; and retaining the products in S and F. Then T is an ideal extension of S by $Q = F^0$, with the properties that S is periodic, $\mathcal{L}(S)$ is lower semimodular (in fact, distributive) and Q is $\mathcal{J}$-trivial. But $\mathcal{L}(T)$ is not lower semimodular since $r = ra^2$, $r \notin \langle a \rangle$, but $r \neq ra$, contradicting III(a).* $\square$

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