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The (1,2)-Step Competition Graph of a Tournament

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The $(1, 2)$-step competition graph of a tournament

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Abstract

The competition graph of a digraph, introduced by Cohen in 1968, has been extensively studied. More recently, in 2000, Cho, Kim, and Nam defined the m-step competition graph. In this paper, we offer another generalization of the competition graph. We define the $(1, 2)$-step competition graph of a digraph D, denoted $C_{1,2}(D)$, as the graph on $V(D)$ where $\{x, y\} \in E(C_{1,2}(D))$ if and only if there exists a vertex $z \neq x, y$, such that either $d_{D-y}(x, z) = 1$ and $d_{D-x}(y, z) \leq 2$ or $d_{D-x}(y, z) = 1$ and $d_{D-y}(x, z) \leq 2$. In this paper, we characterize the $(1, 2)$-step competition graphs of tournaments and extend our results to the (i, k)-step competition graph of a tournament.

Key words: competition graph, tournament, digraph, m-step graph

1. Introduction

Competition graphs, created in connection to a biological model, have a forty year history of study. For a comprehensive introduction to competition graphs, see Brigham and Dutton [3] or Lundgren [10]. Recent generalizations of competition graphs include Kim and Roberts [8] and Helleloid [7]. Closely related to the $(1, 2)$-step competition graph of this paper is the m-step competition graph introduced by Cho, Kim, and Nam [2]. The *m-step competition graph* of a digraph D is created on the vertex set of D with an edge $\{x, y\}$ if there is a vertex z in D such that both an (x, z)-path and a (y, z)-path of length m exists.

For notation and terms not defined here, see Bang-Jensen and Gutin [1]. A *tournament* is an oriented complete graph. An *n-tournament* is a tournament on n vertices. The vertex and edge sets of graph G are denoted by $V(G)$ and $E(G)$ respectively. The vertex and arc sets of digraph D are denoted by $V(D)$ and $A(D)$ respectively. We say x and y are *adjacent* in a digraph if $(x, y) \in A(D)$ or $(y, x) \in A(D)$. If $x \in V(D)$, then the *outset* of x is $N^+(x) = \{y : (x, y) \in A(D)\}$. The *out-degree* of x, $|N^+(x)|$, is denoted by $d^+(x)$.

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An (x, y)-walk is defined as a sequence of arcs and vertices

$$x, (x, v_1), v_1, (v_1, v_2), v_2, \ldots, v_{k-1}(v_{k-1}, v_k), v_k, (v_k, y), y.$$

The *distance* from x to y, denoted $dist(x, y)$, is defined as the minimum number of arcs in an (x, y)-walk. The distance from x to y in digraph D is denoted by $d_D(x, y)$. The digraph $D - x$ is the digraph obtained from D by removing vertex x and all arcs incident with x.

Recall that the competition graph of a digraph D is obtained by using vertex set $V(D)$ and adding edge $\{x, y\}$ whenever $N^+(x) \cap N^+(y) \neq \emptyset$. The $(1,2)$-*step competition graph* of a digraph D, denoted $C_{1,2}(D)$, is a graph on $V(D)$ where $\{x, y\} \in E(C_{1,2}(D))$ if and only if there exists a vertex $z \neq x, y$, such that either $d_{D-y}(x, z) \leq 1$ and $d_{D-x}(y, z) \leq 2$ or $d_{D-x}(y, z) \leq 1$ and $d_{D-y}(x, z) \leq 2$. For example, all 4-tournaments and their $(1,2)$-step competition graphs are shown in Figure 1.

It should be noted that in 1991, Hefner (Factor) et al. [6] defined the (i, j) competition graph. In that paper, i was the maximum indegree and j was the maximum outdegree of vertices in the digraph. In 2008, Hedetniemi et al. [5] introduced $(1,2)$-domination. This was followed by Factor and Langley's introduction of the $(1,2)$-domination graph [4]. Because of the similarities between our construction and those of [4] and [2], we refer to the $(1,2)$-step competition graph of a digraph.

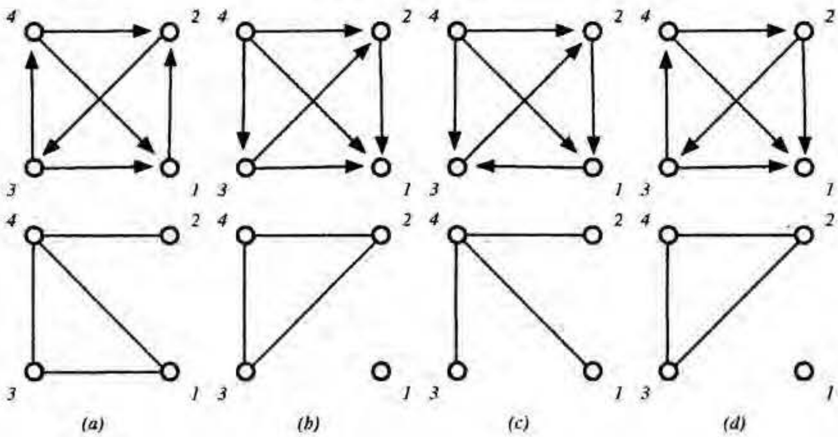

Figure 1: All 4-tournamnents and their $(1,2)$-step competition graphs.

We say that x and y $(1,2)$-*compete* provided there exists $z \neq x, y$ such that either $d_{D-y}(x, z) = 1$ and $d_{D-x}(y, z) = 2$ or $d_{D-x}(y, z) = 1$ and $d_{D-y}(x, z) = 2$. We say that x and y *compete* provided there exists $z \in N^+(x) \cap N^+(y)$. Thus, $\{x, y\} \in E(C_{1,2}(D))$ provided x and y compete or $(1,2)$-compete. For example, in Figure 1(a), vertices 4 and 2 $(1,2)$-compete, but do not compete.

In 1998, Merz et al. [12] determined the competition graphs of tournaments. A significant result from that paper is that the minimum number of edges in the competition graph of a tournament is $\binom{n}{2} - n$ edges. Observe that the competition graph of a digraph D is a subgraph of the $(1,2)$-step competition graph of D. It is easier for two vertices to be adjacent in the $(1,2)$-step competition graph as compared to the competition graph. Thus it makes sense to ask: what is the minimum number of edges in the $(1,2)$-step competition graph of a tournament?

Recall that vertex x in a tournament is a king provided for all y, $dist(x, y) \leq 2$. Additionally, it is left to the reader to show the following result about kings.

Remark 1. If T is an n-tournament, $n > 3$, and x and y are kings with $d^+(x)$, $d^+(y) > 1$, then $\{x, y\}$ is an edge in $C_{1,2}(T)$.

Moon [13] generally stated and Maurer [11] specifically proved that in almost all tournaments, every vertex is a king. Since, in an n-tournament with $n > 3$, there is at most one king x with $d^+(x) = 1$, we conclude that the $(1,2)$-step competition graphs of most tournaments are complete. Thus we ask: under what circumstances is an edge missing in the $(1,2)$-step competition graph of a tournament?

Digraph D is called *strongly connected* or *strong* provided there is an (x, y)-walk for each pair of vertices x and y. In Section 1, we consider the $(1,2)$-step competition graphs of strong tournaments. In Section 2, we extend these results to all tournaments. In Section 3, we consider the (i,k)-step competition graph, where $i > 1$ and $k > 2$.

2. Strongly connected tournaments

We begin with a lemma. Observe in Figure 1(a), $N^+(1) = \{2\}$ and $\{1, 2\}$ is missing from $C_{1,2}(T)$. In a strong tournament, this is the only way an edge can be missing in $C_{1,2}(T)$.

Lemma 1. *Let T be a strong tournament. Then $\{x, y\} \notin E(C_{1,2}(T))$ if and only if $N^+(x) = \{y\}$ or $N^+(y) = \{x\}$.*

Proof. $(\Leftarrow)$ Assume $N^+(x) = \{y\}$. Suppose $\{x, y\} \in E(C_{1,2}(T))$. Since there is no $\{z\} \in N^+(x) \cap N^+(y)$, x and y must $(1,2)$-compete. This is a contradiction, since $N^+(x) = \{y\}$ means that $d_{D-y}(x, z) \neq 1, 2$ for all $z \in V(D - y)$.

$(\Rightarrow)$ Conversely, assume that $\{x, y\} \notin E(C_{1,2}(T))$. Since T is a tournament, x and y are adjacent. Without loss of generality, say $y \in N^+(x)$. We claim that $N^+(x) = \{y\}$. Suppose not. Let z be another vertex in $N^+(x)$. Since x and y do not compete for z, $(z, y) \in A(T)$. T is strongly connected, so let w denote a vertex in $N^+(y)$. If $(w, z) \in A(T)$ then $(x, z), (y, w)$, and $(w, z) \in A(T)$ implies that $\{x, y\} \in E(C_{1,2}(T))$, a contradiction. Thus, $(z, w) \in A(T)$. But then $(y, w), (x, z)$, and $(z, w) \in A(T)$ implies that $\{x, y\} \in E(C_{1,2}(T))$, a contradiction. Thus, $N^+(x) = \{y\}$. ■

Figure 1(b) illustrates that Lemma 1 is not the case for every tournament (consider $\{1, 4\}$). From the previous proof, we can see that in any digraph D, $N^+(x) = \{y\}$ implies that $\{x, y\} \notin E(C_{1,2}(D))$.

The tournament in Figure 1(b) is called transitive. Tournament T is *transitive* provided it is acyclic. If T is transitive, we assume its vertices are labeled $v_1, v_2, \ldots, v_n$ so that $i < j$ implies that $(v_j, v_i) \in A(T)$. Tournament T is an *upset tournament* provided it is obtained from a transitive tournament by reversing the arcs on a single (v_n, v_1)-walk, W, so that the upset tournament contains arcs (v_1, v_2) and (v_{n-1}, v_n), as well as the other arcs reversed on W. For example, the tournament in Figure 1(a) is an upset tournament.

Another useful collection is the set of all regular tournaments. Tournament T, on n vertices, is *regular* provided all vertices in the tournament have the same out-degree. Thus all regular tournaments have an odd number of vertices. We say T is *near regular* provided the largest difference between the out-degrees of any two vertices is 1. All near regular tournaments have an even number of vertices.

Recall that P_i is a path on i vertices. The graph $G - E(H)$ is obtained from G by removing the edges from a subgraph of G that is isomorphic to H. For example, in Figure 1, the graph shown in (a) is $K_4 - E(P_3)$.

Theorem 2. *A graph G on $n \geq 5$ vertices is the $(1,2)$-step competition graph of some strong tournament if and only if G is K_n, $K_n - E(P_3)$, or $K_n - E(P_2)$.*

Proof. ($\Leftarrow$) So long as $n \geq 5$, if T is regular or near regular, then $C_{1,2}(T)$ will be complete. Next, we show that if T is an upset tournament, then $C_{1,2}(T) = K_n - E(P_3)$.

Let T be an upset n-tournament, $n \geq 5$, with vertices $v_1, v_2, \ldots, v_n$ labeled as given by the definition of an upset tournament. In particular, (v_1, v_2) and (v_{n-1}, v_n) are arcs on a path P from v_1 to v_n, and for every arc not on P, $j > i$ implies $(v_j, v_i) \in A(T)$. Furthermore, label the vertices of P as $v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_m}$. So $i_1 = 1$, $i_2 = 2$, $i_{m-1} = n - 1$, and $i_m = n$. Observe that $v_1 \in N^+(v_k)$ for $3 \leq k \leq n$. Thus $\{v_3, \ldots, v_n\}$ is a complete subgraph of $C_{1,2}(T)$. Since $N^+(v_1) = \{v_2\}$ and $N^+(v_2) = \{v_{i_3}\}$, by Lemma 1, $\{v_1, v_2\}$ and $\{v_2, v_{i_3}\} \notin E(C_{1,2}(T))$. We claim that $\{v_1, v_k\} \in E(C_{1,2}(T))$ for $3 \leq k \leq n$ and that $\{v_2, v_k\} \in E(C_{1,2}(T))$ for $3 \leq k \leq n$, $k \neq i_3$.

For the first case, let $3 \leq k \leq n$ and consider v_1 and v_k. If $k \neq i_3$, then v_1 and v_k compete for v_2. If $k = i_3$ then $(v_1, v_2), (v_k, v_{i_4})$, and $(v_{i_4}, v_2) \in A(T)$. So v_1 and v_k $(1,2)$-compete. Thus, $\{v_1, v_k\} \in E(C_{1,2}(T))$.

For the second case, let $2 < k \leq n$ where $k \neq i_3$. Then $(v_2, v_{i_3}), (v_{i_3}, v_1)$, and $(v_k, v_1) \in A(T)$. Thus $\{v_2, v_k\} \in E(C_{1,2}(T))$. Thus $C_{1,2}(T)$ is $K_n - E(P_3)$. In particular, the edges missing in $C_{1,2}(T)$ are $\{v_1, v_2\}$ and $\{v_2, v_{i_3}\}$.

Finally, if T is obtained from the transitive tournament by reversing arcs (v_n, v_1) and (v_n, v_2), then v_1 is the only vertex with out-degree 1 and T is strong, so $C_{1,2}(T) = K_n - E(P_2)$.

($\Rightarrow$) To prove the converse, let G on $n \geq 5$ vertices be the $(1,2)$-step competition graph of some strong tournament T. For each $x \in V(T)$, $d^+(x) \geq 1$. If for all $x \in V(T)$, $d^+(x) > 1$, then by Lemma 1, we know that $C_{1,2}(T)$ is complete. Since $n \geq 5$, it is impossible for T to have more than two vertices with out-degree 1 and be strongly connected. Thus, $C_{1,2}(T)$ is missing at most two edges. It remains to be shown that these missing edges, if they exist, must share an endpoint.

Suppose not. Let $\{x, y\}$ and $\{u, v\}$ denote the edges missing from $C_{1,2}(T)$ where x, y, u, and v are distinct. Without loss of generality, say (x, y) and $(u, v) \in A(T)$. Then by Lemma 1, $N^+(x) = \{y\}$ and $N^+(u) = \{v\}$. This is a contradiction since x and u must be adjacent. Thus, G is either K_n, $K_n - E(P_3)$, or $K_n - E(P_2)$. ■

Thus, we know all $(1,2)$-step competition graphs of strongly connected tournaments on n vertices. The cases $n = 1, 3, 4$ are easy to check. See Figure 1 for the $(1,2)$-step competition graphs of all tournaments on 4 vertices; only (a) is strong.

3. Remaining Tournaments

If a tournament is not strong, then the vertices of T may be partitioned into $T_1, T_2, \ldots, T_k$ where each T_i is a maximally strongly connected tournament and for all i, j, if $x \in T_i$ and $y \in T_j$, then $(x, y) \in A(T)$ if and only if $i < j$. Such as partition of T is called the *strong decomposition* of T.

Lemma 3. *Let T be an n-tournament with strong decomposition $T_1, T_2, \ldots, T_k$. If $\{x, y\} \notin E(C_{1,2}(T))$, then $x, y \in V(T_k)$ or $|V(T_k)| = 1$ and $C_{1,2}(T) = K_{n-1} \cup K_1$.*

Proof. Observe that every vertex in T_i for $i < k$ has an arc to each vertex in T_k. Thus, the vertices of $T_1, T_2, \ldots, T_{k-1}$ induce a complete subgraph in $C_{1,2}(T)$. If $|V(T_k)| > 1$ then since T_k is strong, every vertex $x \in T_k$ has an arc to at least one vertex in T_k. Thus x competes with every other vertex of T_i for $i < k$. On the other hand, if $|V(T_k)| = 1$, say $x \in T_k$, then $d^+(x) = 0$, so x is isolated in $C_{1,2}(T)$. ■

Theorem 4. *G, a graph on n vertices, is the $(1,2)$-step competition graph of some tournament if and only if G is one of the following graphs:*

1. *K_n, where $n \neq 2, 3, 4$,*
2. *$K_{n-1} \cup K_1$, where $n > 1$,*
3. *$K_n - E(P_3)$ where $n > 2$,*
4. *$K_n - E(P_2)$ where $n \neq 1, 4$, or*

5. $K_n - E(K_3)$ *where* $n \geq 3$.

Proof. ($\Leftarrow$) K_1 is the $(1,2)$-step competition graph of a 1-tournament. $K_3 - E(P_3)$ is the $(1,2)$-step competition graph of the transitive 3-tournament. $K_4 - E(P_3)$ is the $(1,2)$-step competition graph of the tournament shown in Figure 1(a). $K_2 - E(P_2)$ is the $(1,2)$-step competition graph of any 2-tournament. By Theorem 2, the remaining graphs in cases (1), (3), and (4) are the $(1,2)$-step competition graphs of some tournaments. If T is transitive on 2 or more vertices, then $C_{1,2}(T) = K_{n-1} \cup K_1$. Finally, the $(1,2)$-step competition graph of a cyclic 3-tournament is $K_1 \cup K_1 \cup K_1$. So if T, an n-tournaments with $n > 3$, has strong decomposition T_1, T_2 where T_1 is any tournament and T_2 is a cyclic 3-tournament, then $C_{1,2}(T)$ is $K_n - E(K_3)$.

($\Rightarrow$) It is left to the reader to verify that the $(1,2)$-step competition graphs of every tournament on 4 or fewer vertices is listed. Suppose T is a tournament on $n \geq 5$ vertices. If T is strongly connected, then by Theorem 2, $C_{1,2}(T)$ is one of the graphs listed. So assume that T is not strong. Let $T_1, T_2, \ldots, T_k$ be the strong decomposition of T. By Lemma 3, any missing edges in $C_{1,2}(T)$ must occur in T_k. If $|V(T_k)| = 1$, then $C_{1,2}(T)$ is $K_{n-1} \cup K_1$. Since T_k is strong, $|V(T_k)| \neq 2$. If $|V(T_k)| = 3$, then $C_{1,2}(T_k)$ is either $K_1 \cup K_1 \cup K_1$ (so $C_{1,2}(T)$ is $K_n - E(K_3)$) or $K_3 - E(P_2)$ (in which case, $C_{1,2}(T)$ is $K_n - E(P_2)$). If $|V(T_k)| = 4$, then by Figure 1(a) and Lemma 3, $C_{1,2}(T)$ must be $K_n - E(P_2)$. Otherwise $|V(T_k)| \geq 5$. Then by Theorem 2 and Lemma 3, $C_{1,2}(T)$ must be K_n, $K_n - E(P_3)$, or $K_n - E(P_2)$. ■

Observe that for $n < 4$, the maximum number of edges missing in the $(1,2)$-step competition graph of a tournament on n vertices is n. Using Theorem 4, for $n \geq 4$, we have the following.

Corollary 5. *If T is a tournament, the maximum number of edges missing from the $(1,2)$-step competition graph of a tournament on $n \geq 4$ vertices is $n-1$.*

4. The (i,k)-step competition graph of a tournament

We can generalize the $(1,2)$-step competition graph to the (i,k)-step competition graph as follows. Let $\{x,y\}$ be an edge in the (i,k)-step competition graph, denoted $C_{i,k}(T)$, if for some $z \in V(T) - \{x,y\}$, $d_{T-y}(x,z) \leq i$ and $d_{T-x}(y,z) \leq k$ or $d_{T-x}(y,z) \leq i$ and $d_{T-y}(x,z) \leq k$.

By making the observation that for any digraph D, $i \geq 1$ and $k \geq 2$, $E(C_{1,2}(D)) \subseteq E(C_{i,k}(D))$, the proof of Lemma 1 implies the following corollary.

Corollary 6. *Let T be a strongly connected tournament with $i \geq 1$ and $k \geq 2$. Edge $\{x,y\} \notin E(C_{i,k}(T))$ if and only if $N^+(x) = \{y\}$ or $N^+(y) = \{x\}$.*

Similarly, using the proof of Lemma 3, we make the following conclusion.

Corollary 7. *Let T be an n-tournament with strong decomposition $T_1, T_2, \ldots, T_k$. If $\{x,y\} \notin E(C_{i,k}(T))$, then $x, y \in V(T_k)$ or $|V(T_k)| = 1$ and $C_{i,k}(T) = K_{n-1} \cup K_1$.*

Theorem 8. *If T is an n-tournament, $i \geq 1$ and $k \geq 2$, then $C_{i,k}(T) = C_{1,2}(T)$.*

Proof. Since $C_{1,2}(T)$ is a subgraph of $C_{i,k}(T)$, it suffices to show that $E(C_{i,k}(T)) \subseteq E(C_{1,2}(T))$. So let $\{x,y\} \in E(C_{i,k}(T))$. Suppose $\{x,y\} \notin E(C_{1,2}(T))$. If T is strongly connected, then by Lemma 1, $N^+(x) = \{y\}$ or $N^+(y) = \{x\}$. This contradicts Corollary 6. So we should assume that T is not strongly connected.

Let $T_1, T_2, \ldots, T_k$ be the strong decomposition of T. By Lemma 3, either $x, y \in V(T_k)$ or $|V(T_k)| = 1$ and $C_{1,2}(T) = K_n \cup K_1$. Suppose $x, y \in V(T_k)$. Then applying Lemma 1 to T_k, we conclude that $N^+(x) = \{y\}$ or $N^+(y) = \{x\}$. Then by Corollary 6, $\{x,y\} \notin E(C_{i,k}(T_k))$, a contradiction.

On the other hand, suppose that $|V(T_k)| = 1$ and $C_{1,2}(T) = K_{n-1} \cup K_1$. Every pair of vertices competes for the single vertex in T_k, so we know that $x \in V(T_k)$ or $y \in V(T_k)$. Without loss of generality, say $\{x\} = V(T_k)$. Then $N^+(x) = \emptyset$, so x is isolated in $C_{i,k}(T)$, a contradiction. Thus $\{x,y\} \in E(C_{1,2}(T))$. ■

Thus, even if we make it easier for vertices to compete in the tournament by increasing i and k, the (i,k)-step competition graph will never have more edges than the $(1,2)$-step competition graph.

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