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# Lower Semimodular Inverse Semigroups, II

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# Lower semimodular inverse semigroups, II

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## Abstract

The description by the authors of the inverse semigroups  $S$  for which the lattice  $\mathcal{LF}(S)$  of full inverse subsemigroups is lower semimodular is used to describe those for which (a) the lattice  $\mathcal{L}(S)$  of all inverse subsemigroups or (b) the lattice  $\mathcal{Co}(S)$  of convex inverse subsemigroups has that property. In each case, we show that this occurs if and only if the entire lattice is a subdirect product of  $\mathcal{LF}(S)$  with  $\mathcal{L}(E_S)$ , or  $\mathcal{Co}(E_S)$ , respectively, where  $E_S$  is the semilattice of idempotents of  $S$ ; a simple necessary and sufficient condition is found for each decomposition. For a semilattice  $E$ ,  $\mathcal{L}(E)$  is in fact always lower semimodular, and  $\mathcal{Co}(E)$  is lower semimodular if and only if  $E$  is a tree. The conjunction of these results leads to quite a divergence between the ultimate descriptions in the two cases,  $\mathcal{L}(S)$  and  $\mathcal{Co}(S)$ , with the latter being substantially richer.

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It has long been known (see [17] for a survey) that only in very restrictive cases does the lattice  $\mathcal{L}(S)$  of all inverse subsemigroups of an inverse semigroup  $S$  satisfy any lattice-theoretic property of general interest. This is also true of the lattice  $\mathcal{Co}(S)$  comprised of the inverse subsemigroups that are *convex* with respect to the natural partial order on  $S$ , the study of which was initiated by the authors in [2, 3]. For that reason, most research in this area has focused on their common sublattice  $\mathcal{LF}(S)$ , of full inverse subsemigroups, where a rich theory has been developed. However we show that lower semimodularity determines interesting classes of inverse semigroups in the general context. This is especially true in the convex case.

To a large extent, the paucity of interesting results obtained heretofore is a result of the restrictions imposed on a semilattice  $E$  by lattice-theoretic properties of  $\mathcal{L}(E)$  and  $\mathcal{Co}(E)$  (see §2.1). The prospect that more interesting results might be obtained in the case of lower semimodularity arises from the long-known fact that the lattice  $\mathcal{L}(E)$  always has that property, and the fact [1] that  $\mathcal{Co}(E)$  has that property if (and only if)  $E$  is a tree, that is, a semilattice in which every principal ideal is a chain.

Our study of lower semimodularity of  $\mathcal{L}(S)$  and  $\mathcal{Co}(S)$  proceeds in parallel. Apart from the differences between  $\mathcal{L}(S)$  and  $\mathcal{Co}(S)$  stated above in the case of semilattices, the final characterizations, Theorems 4.9 and 4.5, respectively, turn out to have a somewhat different character due to the contrast in the case of simple semigroups, the case to which the study of

the lattice  $\mathcal{LF}(S)$  essentially reduces. It turns out that in the simple case, if  $\mathcal{LF}(S)$  is lower semimodular and  $E_S$  is a tree, then  $\mathcal{Co}(S)$  is also lower semimodular. However, in the case of  $\mathcal{L}(S)$ , lower semimodularity forces  $S$  to be a group in this situation.

The essence of this paper is carried in the fact (Proposition 4.1) that lower semimodularity of either  $\mathcal{L}(S)$  or  $\mathcal{Co}(S)$  implies that the semilattice  $E_S$  of idempotents of  $S$  is a neutral element in the lattice, and hence the lattice is a subdirect product of the ideal  $\mathcal{L}(E_S)$  or  $\mathcal{Co}(E_S)$ , respectively, with their common filter  $[E_S, S] = \mathcal{LF}(S)$ . In §3, we conduct an in-depth analysis of the neutrality of  $E_S$ , with a view to future application in other contexts. The simple “archimedean” properties (2) and (2C), respectively, of Proposition 3.2 are the precise conditions required for neutrality to be valid in the respective lattices.

Thus describing those  $S$  for which  $\mathcal{L}(S)$  or  $\mathcal{Co}(S)$  is lower semimodular reduces to the corresponding question for semilattices — answered above — and for the lattice  $\mathcal{LF}(S)$ , answered in [4]: see §2.2. Combining that information with that associated with the decomposition leads to the final descriptions in Theorems 4.9 and 4.5.

In a sequel [5], the authors will develop further the methods of this paper, with application to join semidistributivity of  $\mathcal{L}(S)$  and  $\mathcal{Co}(S)$ .

## 1 Preliminaries.

### 1.1 Background on lattices and posets.

We use [8] as a general reference. In any poset  $P$ , if  $a \leq b$  then  $[a, b]$  denotes the interval  $\{c \in P : a \leq c \leq b\}$ ; open and half-open intervals then have their usual meaning. The notation  $b \succ a$  means that  $b$  covers  $a$ , that is,  $b > a$  and  $[a, b] = \{a, b\}$ ;  $a \parallel b$  means that  $a$  and  $b$  are incomparable in the natural order. For  $X \subseteq P$ ,  $X \downarrow = \{a \in P : a \leq x \text{ for some } x \in X\}$  and  $X \uparrow$  is its order dual; if  $X = \{x\}$ , we may instead write  $x \downarrow$  and  $x \uparrow$ . A subset  $X$  of  $P$  is an (*order*) *ideal* if  $X \downarrow \subseteq X$ , and an (*order*) *filter* if the dual relation holds.

A lattice  $L$  is *lower semimodular* if whenever  $a \vee b \succ a$  in  $L$  then  $b \succ a \wedge b$ . This property is preserved by interval sublattices and subdirect products [18, Theorem 1.7.6]. For further information on semimodularity and its many variations, see the monograph by Stern [18].

A lattice is *modular* if the modularity relation  $M$  is the universal relation. Here  $a M b$  if  $(a \vee x) \wedge b = (a \wedge b) \vee x$  for all  $x \leq b$ ; equivalently,  $(a \vee x) \wedge b = x$  for all  $x \in [a \wedge b, b]$ . Its order dual is denoted  $M^*$ .

A variation of lower semimodularity is  $M^*$ -symmetry: the property that the relation  $M^*$  is symmetric. Similarly to lower semimodularity, it is preserved by interval sublattices and subdirect products. Every  $M^*$ -symmetric lattice is lower semimodular; for lattices of finite length, the two properties are equivalent. Every modular lattice is  $M^*$ -symmetric and thus lower semimodular.

The following terms are useful in the analysis of lattice decompositions (see [8]). An element  $a$  of a lattice  $L$  is *distributive* in  $L$  if  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$  for all  $b, c \in L$ . If  $L$  is a complete lattice then  $a$  is *completely distributive* if the binary meets may be replaced by arbitrary ones. Define *dual distributivity* and *complete dual distributivity* in the obvious way. The element  $a$

separates  $L$  if  $a \wedge b = a \wedge c$  and  $a \vee b = a \vee c$  together imply  $b = c$ , for all  $b, c \in L$ .

An element is *neutral* in the lattice if it is distributive, dually distributive and separating. Clearly  $a$  is neutral if and only if the map  $x \rightarrow (a \wedge x, a \vee x)$  embeds  $L$  in the (subdirect) product of the principal ideal  $a \downarrow$  and the principal filter  $a \uparrow$ . In a complete lattice, it may be possible to describe the image more precisely, as follows.

**LEMMA 1.1** *Let  $L$  be a complete lattice  $L$ , with least and greatest elements  $0$  and  $1$ , respectively. If  $a$  is a neutral element of  $L$  that, in addition, is completely dually distributive, then the image of  $L$  in the direct product  $[0, a] \times [a, 1]$  is  $\{(f, u) : u \leq a \vee f^{\max}\}$ , where for  $f \leq a$ ,  $f^{\max}$  is the greatest element of  $L$  having  $f$  as its meet with  $a$ .*

**Proof.** Complete dual distributivity of  $a$  in  $L$  guarantees the existence of  $f^{\max}$ . Let  $M$  denote the specified subset of  $[0, a] \times [a, 1]$ . For any  $x \in L$ ,  $x \leq (a \wedge x)^{\max}$ , by definition. Hence  $a \vee x \leq a \vee (a \wedge x)^{\max}$  and so  $(a \wedge x, a \vee x) \in M$ . Conversely, if  $(f, u) \in M$ , let  $x = u \wedge f^{\max}$ . Then it is straightforward to check that  $a \wedge x = f$  and  $a \vee x = u$ .  $\square$

The following elementary result will also be useful.

**LEMMA 1.2** *If  $g$  is a dually distributive element of a lattice  $L$ , then whenever  $a \succ b$  in  $L$ ,  $g \wedge a \succeq g \wedge b$ .*

**Proof.** If  $g \wedge a > g \wedge b$ , choose  $x$  such that  $g \wedge b < x \leq g \wedge a$ . Then  $b < b \vee x \leq a$  and so, since  $a \succ b$ ,  $b \vee x = a$ . By dual distributivity,  $g \wedge a = (g \wedge b) \vee (g \wedge x) = x$ . Hence  $g \wedge a \succ g \wedge b$ .  $\square$

## 1.2 The lattice $\mathcal{C}o(S)$ .

The *natural partial order* on an inverse semigroup  $S$  is given by  $a \leq b$  if  $a = aa^{-1}b$ , with many equivalent conditions to be found in [9, Proposition 5.2.1]. In particular, if  $E_S$  denotes the semilattice of idempotents of  $S$ , then  $a \downarrow = E_S a = a E_S$  for any  $a \in S$ . That the natural partial order is compatible with the product and with inversion is easily seen, as is the fact that it is respected by homomorphic images.

An inverse subsemigroup of  $S$  is *convex* if whenever it contains  $a$  and  $b$ , with  $a \leq b$ , then it contains  $[a, b]$ . Since convexity is preserved by arbitrary intersections, the convex inverse subsemigroups of  $S$  form a complete lattice,  $\mathcal{C}o(S)$ , with the empty subsemigroup as its least element. (Note that in [1, 2, 3], the notation  $\mathcal{LCV}(S)$  was used instead of  $\mathcal{C}o(S)$ .) The lattice of all inverse subsemigroups is denoted  $\mathcal{L}(S)$ . In general,  $\mathcal{C}o(S)$  is not a sublattice of  $\mathcal{L}(S)$ : in fact [2, p. 53] this holds if and only if the length of  $E_S$  is at most 2, in which case the two lattices coincide. If  $X \subseteq S$ , we denote the inverse subsemigroup that it generates by  $\langle X \rangle$  (but be warned that in papers that are strictly on the lattice of *full* inverse subsemigroups, such as [4], this notation is used for the *full* inverse subsemigroup generated by  $S$ ). The convex inverse subsemigroup that it generates is denoted by  $\langle\langle X \rangle\rangle$ . If  $X = \{x_1, x_2, \dots, x_n\}$  we may instead write  $\langle x_1, x_2, \dots, x_n \rangle$  and  $\langle\langle x_1, x_2, \dots, x_n \rangle\rangle$ , respectively. If  $U, V \in \mathcal{C}o(S)$ , we denote their join

in  $\mathcal{L}(S)$  by  $U \vee V$  and their join in  $\mathcal{C}o(S)$  by  $U \diamond V$ . Clearly  $U \diamond V = \langle\langle U \vee V \rangle\rangle$ . The following result will find frequent application.

**RESULT 1.3** [2, Proposition 2.2] *For any inverse subsemigroup  $U$  of an inverse semigroup,  $\langle\langle U \rangle\rangle$  is the union of the intervals  $[a, b]$ ,  $a, b \in U$ ,  $a \leq b$ . Therefore  $E_{\langle\langle U \rangle\rangle} = \langle\langle E_U \rangle\rangle$ .*

The semilattice of idempotents of any inverse semigroup  $S$  is a convex inverse subsemigroup of  $S$ . Hence the lattice  $\mathcal{C}o(E_S) = [\emptyset, E_S]$  is an ideal in the lattice  $\mathcal{C}o(S)$ . In a complementary fashion, every *full* inverse subsemigroup — one that contains  $E_S$  — is convex, and the full inverse subsemigroups form the filter  $[E_S, S]$  in the lattice  $\mathcal{C}o(S)$ .

Note that since any group is unordered under the natural partial order, its convex inverse subsemigroups comprise its subgroups together with its empty subsemigroup, which acts as an adjoined zero. For properties of subgroup lattices, see [19, 16]. In particular, we need the fact that a group has distributive lattice of subgroups if and only if it is locally cyclic (whence abelian). The finite groups with lower semimodular subgroup lattice are completely determined; however, little is known in the infinite situation. The situation is similar for modularity.

### 1.3 Further semigroup-theoretic background.

An inverse semigroup is *combinatorial* (also termed *aperiodic*) if Green's relation  $\mathcal{H}$  is the identity relation, equivalently, each of its subgroups is trivial. We call a subgroup *isolated* if it comprises an entire  $\mathcal{D}$ -class, and thus an entire  $\mathcal{J}$ -class. The notation  $G_S$  refers to the union of the subgroups of  $S$ . An inverse semigroup  $S$  is  *$E$ -unitary* if  $E_S \uparrow = E_S$ . Denote by  $\sigma$  the least group congruence on  $S$ .

With each  $\mathcal{J}$ -class  $J$  of an inverse semigroup  $S$  is associated its *principal factor*  $PF(J)$ , which is either a 0-simple semigroup or, in case  $J$  is the minimum ideal (the *kernel* of  $S$ , a simple semigroup. (See [6]). The definition used in earlier work by Jones varied slightly from this standard one, in that in the case of a minimum ideal, a zero element was adjoined.)

A 0-simple semigroup is completely 0-simple if every nonzero idempotent is minimal among such idempotents. The completely 0-simple inverse semigroups are the *Brandt* semigroups. Denote by  $B_n$  the combinatorial Brandt semigroup with  $n$  nonzero idempotents.

Any 0-simple inverse semigroup that is not completely 0-simple contains (a copy of) the *bicyclic* semigroup (again, see [6]): the inverse monoid presented by  $B = \langle a \mid aa^{-1} > a^{-1}a \rangle$ . Its identity element is  $e = aa^{-1}$  and  $E_B = \{e > a^{-1}a > \dots > a^{-n}a^n > \dots\}$ , isomorphic to the chain  $C_\omega$  of nonnegative integers under the reverse of the usual order. It is well known (and easily verified) that  $B$  is  *$E$ -unitary* and combinatorial. In any inverse semigroup  $S$ ,  $E_S$  is said to be *archimedean in  $S$*  if for any element  $a$  of  $S$  such that  $aa^{-1} > a^{-1}a$ , and for any idempotent  $f$  of  $S$ ,  $a^{-n}a^n \leq f$  for some positive integer  $n$ .

The bicyclic semigroup is isomorphic to the *Munn semigroup*  $T_{C_\omega}$ . In general, given a semilattice  $Y$ ,  $T_Y$  consists of the isomorphisms between principal ideals of  $Y$ , under composition of partial mappings. Its idempotents are the identity automorphisms  $1_{e_l}, e \in Y$ ; thus its semilattice of idempotents is isomorphic to  $Y$  and, in general, we shall identify  $1_{e_l}$  with  $e$  itself.

Such a semigroup is bisimple if and only if  $Y$  is *uniform*: all its principal ideals are isomorphic. See [9, Chapter 5] for these and further properties.

Finally, we will need some technical details regarding monogenic inverse semigroups. According to [15, Theorem IX.3.11], each such semigroup is defined by exactly one of the following relations, where  $k, \ell$  are positive integers: (i)  $a^k = a^{-1}a^{k+1}$ ; (ii)  $a^k a^{-1} = a^{-1}a^k$ ; (iii)  $a^k = a^{k+\ell}$ ; (iv)  $a = a$ . Those in (iv) are freely generated. Those in (i) — (iii) possess a kernel that is bicyclic, infinite cyclic or finite cyclic, respectively. If  $k = 1$  then that kernel is the entire semigroup; if  $k \geq 2$ , then the semigroup is an extension of its kernel by the quotient of the free monogenic inverse semigroup modulo the ideal generated by  $a^k$  (the quotient being a semigroup of type (iii), with trivial kernel).

The elements of any monogenic inverse semigroup are expressible in the form  $(a^{-m}a^m)a^i(a^n a^{-n})$ , where  $m, n \geq 0$  and  $m + i + n \geq 1$  (with  $a^0$  representing an adjoined identity element here and elsewhere in this paper), the expression being unique in the free case. Note that every nonidempotent of such a semigroup is below some nonzero power of  $a$ , in the natural partial order.

**LEMMA 1.4** *In any monogenic inverse semigroup  $\langle a \rangle$ , the generator  $a$  is a maximal element in the partial order.*

**Proof.** If  $a$  is idempotent, this is clear. Otherwise, suppose  $a \leq b \in \langle a \rangle$ . Then, in view of the description of the elements of  $\langle a \rangle$  above,  $b \leq a^i$  for some nonzero integer  $i$ . If  $i < 0$ , then since  $a^{-1} \leq b^{-1} \leq a^{-i}$ ,  $a = aa^{-1}a \leq a^{2-i}$ , so we may assume  $i > 0$ . But then  $a = (aa^{-1})a^i = a^i$  and so  $a = b$ .  $\square$

In the sequel, the semilattice of idempotents will frequently be a tree.

**LEMMA 1.5** *The semilattice of idempotents of a monogenic inverse semigroup  $A = \langle a \rangle$  is a tree if and only if it is of type (i), (ii) or (iii) above, with  $k \leq 2$ .*

**Proof.** If  $k = 1$  then  $A$  is either a (cyclic) group or bicyclic (so that  $E_A$  is a chain in the latter case). If  $k = 2$  then  $A$  is an ideal extension of a kernel  $K$ , which is a group or bicyclic semigroup, by  $\langle a \mid a^2 = 0 \rangle \cong B_2$ . Thus  $E_A$  is obtained by adjoining two maximal idempotents to  $E_K$  and is therefore a tree.

To prove the converse, observe from the description of the monogenic inverse semigroups provided above that in case (iv), and hence also when  $k \geq 3$  in cases (i) — (iii), the idempotent  $aa^{-1}$  is strictly above the distinct idempotents  $(a^{-1}a)(aa^{-1})$  and  $a^2a^{-1}$ , so  $E_A$  is not a tree.  $\square$

## 2 Subsemilattices and full inverse subsemigroups.

### 2.1 Subsemilattices.

The unproven statements in the following are either well known or to be found in [1].

**RESULT 2.1** *Let  $E$  be a semilattice.*

(i) If  $A, B \in \mathcal{L}(E)$ , then  $AM^*B$  in  $\mathcal{L}(E)$  if and only if  $A \vee B = A \cup B$ ;  $A \succ B$  in  $\mathcal{L}(E)$  if and only if  $B \subset A$  and  $|A - B| = 1$ ;

(ii) hence  $\mathcal{L}(E)$  is always  $M^*$ -symmetric, whence lower semimodular;

(iii) lower semimodularity and  $M^*$ -symmetry of  $\mathcal{C}o(E)$  are both equivalent to  $E$  being a tree, and to the property that if  $A, B \in \mathcal{C}o(E)$ , then  $A \succ B$  in  $\mathcal{C}o(E)$  if and only if  $B \subset A$  and  $|A - B| = 1$ .

**Proof.** Only (i) needs proof. Suppose  $AM^*B$  in  $\mathcal{L}(E)$  and  $x \in A \vee B$ . Now  $B \vee \{x\} \in [B, A \vee B]$  so by  $M^*$ -symmetry,  $(A \cap (B \vee \{x\})) \vee B = B \vee \{x\}$ . If  $x \notin B$ , it follows that  $x = ab$ , for some  $a \in A \cap (B \vee \{x\})$  and  $b \in B^1$ . Now  $a \notin B$ , so  $a \leq x$  and thus  $x = a \in A$ .  $\square$

**RESULT 2.2** *Let  $E$  be a semilattice. Modularity and distributivity of  $\mathcal{L}(E)$  are both equivalent to  $E$  being a chain. Modularity and distributivity of  $\mathcal{C}o(E)$  are both equivalent to  $|E| \leq 2$ .*

Ershova [7] showed that any inverse semigroup  $S$  for which  $\mathcal{L}(S)$  is modular must be a Clifford semigroup with trivial structure mappings. In the case of  $\mathcal{C}o(S)$ , if  $|E_S| \leq 2$ , then  $\mathcal{C}o(S) = \mathcal{L}(S)$ , as remarked earlier, and so the same conclusion holds.

## 2.2 The lattice of full inverse subsemigroups.

The next result is the main general tool in the study of the lattice of full inverse subsemigroups.

**RESULT 2.3** [11] *Let  $S$  be an inverse semigroup. Then  $\mathcal{LF}(S)$  is isomorphic to a subdirect product of the lattices of full inverse subsemigroups of its principal factors, each of which is isomorphic to an interval sublattice of  $\mathcal{LF}(S)$ .*

The focus may therefore be on the simple and 0-simple cases. We begin with the completely 0-simple case. All these results may be found in [4], although the specializations to modularity and distributivity were found much earlier.

**RESULT 2.4** *Let  $S$  be a completely 0-simple inverse semigroup that is not just a group with adjoined zero. Then lower semimodularity,  $M^*$ -symmetry and modularity of  $\mathcal{LF}(S)$  are all equivalent to  $S$  being isomorphic to either  $B_2$  or  $B_3$ . In the case of distributivity, only  $B_2$  is allowed.*

**RESULT 2.5** *If  $S$  is a 0-simple inverse semigroup that is not completely 0-simple, and  $\mathcal{LF}(S)$  is lower semimodular, then  $S$  has no zero divisors and  $\mathcal{LF}(S) \cong \mathcal{LF}(S - 0)$ , where  $S - 0$  is simple.*

Recall the definitions of isolated subgroups and of the archimedean property from §1.3.

**RESULT 2.6** *If  $S$  is a simple inverse semigroup that is not a group, then  $\mathcal{LF}(S)$  is lower semimodular if and only if*

- (a)  $\mathcal{L}(H)$  is lower semimodular for every isolated subgroup  $H$  of  $S$ ;
- (b) every nonisolated subgroup of  $S$  is trivial;
- (c)  $E_S$  is archimedean in  $S$  and any  $\mathcal{D}$ -class  $D$  of  $S$  contains at most two mutually incomparable idempotents, each of which is maximal in the poset  $E_D$ .

In that event, the maximum group quotient  $G = S/\sigma$  is abelian, in fact isomorphic to a subgroup of the reals under addition.

The lattice  $\mathcal{LF}(S)$  is  $M^*$ -symmetric [resp. modular] if and only if, in addition to the above,  $\mathcal{L}(H)$  is  $M^*$ -symmetric [resp. modular] for each isolated subgroup  $H$ , and  $G$  is locally cyclic. In that event,  $G$  is isomorphic to a subgroup of the rationals.

The lattice  $\mathcal{LF}(S)$  is distributive if and only if, in addition to all of the above, each isolated subgroup is locally cyclic and  $S$  is  $E$ -unitary (equivalently, the poset  $E_D$  is a chain for any  $\mathcal{D}$ -class  $D$  of  $S$ ).

The bicyclic semigroup  $B$  is an example of a bisimple semigroup for which  $\mathcal{LF}(B)$  is distributive.

### 3 Decompositions based on $E_S$ .

In the next section we shall show that if either of the lattices  $\mathcal{L}(S)$  or  $\mathcal{Co}(S)$  is lower semimodular, then  $E_S$  is neutral in that lattice. While there are many direct correspondences between the two lattices, there are also some distinctions with significant consequences.

Throughout the sequel,  $S$  will be an inverse semigroup.

**LEMMA 3.1** *If  $x = e_1 a_1 \cdots e_n a_n$  for some  $e_1, \dots, e_n \in E_S, a_1, \dots, a_n \in S$ , then  $x \leq a_1 \cdots a_n$ . Hence  $E_S \vee A = E_S \cup A \downarrow$  for any  $A \in \mathcal{L}(S)$ . Thus  $E_S \vee A$  is an order ideal of  $S$ , whence convex, and if  $A \in \mathcal{Co}(S)$ , then  $E_S \diamond A = E_S \vee A$ . The subsemigroup  $E_S$  separates  $\mathcal{L}(S)$  and therefore also separates  $\mathcal{Co}(S)$ .*

**Proof.** The first statement, which is well known, follows from iteration of the equation  $ae = (aea^{-1})a$ , for any  $e \in E_S, a \in S$ . The second statement is an immediate consequence. Since  $E_S$  is itself an order ideal, so is  $E_S \cup A \downarrow$ .

Observe that for any  $x \in S$ , if  $x \leq a \in A$  and  $xx^{-1} \in E_A$ , then  $x = (xx^{-1})a \in A$ . Thus  $A = A \downarrow \cap E_A \mathcal{R} = (E_S \cup A \downarrow) \cap E_A \mathcal{R} = (E_S \vee A) \cap E_A \mathcal{R}$  (where  $\mathcal{R}$  denotes Green's relation). That  $E_S$  separates  $\mathcal{L}(S)$ , and so also  $\mathcal{Co}(S)$ , is now clear.  $\square$

**PROPOSITION 3.2** *The following are equivalent:*

- (1)  $E_S$  is distributive in  $\mathcal{L}(S)$ , that is,  $E_S \vee (A \cap B) = (E_S \vee A) \cap (E_S \vee B)$  for all  $A, B \in \mathcal{L}(S)$ ;
- (2) for all  $a \in S$ ,  $a \downarrow \subseteq E_S \cup \langle a \rangle$ ;
- (3) for every  $A \in \mathcal{L}(S)$ ,  $E_S \vee A = E_S \cup A$ ;



(4)  $E_S$  is completely distributive in  $\mathcal{L}(S)$ .

Denote by (1C) to (4C) the analogous statements with respect to  $\mathcal{C}o(S)$ . Then they are also equivalent.

**Proof.** (1)  $\Rightarrow$  (2). Suppose  $a > b$ . Put  $A = \langle a \rangle$  and  $B = \langle b \rangle$ . Since  $b = (bb^{-1})a$ ,  $b \in (E_S \vee A) \cap (E_S \vee B) = E_S \vee (A \cap B)$ . If  $b \notin E_S$ , then by Lemma 3.1,  $b \in (A \cap B)\downarrow$ . But, according to Lemma 1.4,  $b$  is also maximal in  $B$ , so  $b \in A$ .

The implication (2)  $\Rightarrow$  (3) is clear from Lemma 3.1. The remaining implications are obvious.

For  $\mathcal{C}o(S)$ , recall first that  $E_S \diamond A = E_S \vee A$  for all  $A \in \mathcal{C}o(S)$ . Again, the only implication that is not clear is (1C)  $\Rightarrow$  (2C). Putting  $A = \langle\langle a \rangle\rangle$  and  $B = \langle\langle b \rangle\rangle$  again yields  $b \in (A \cap B)\downarrow$ . But  $b$  is maximal in  $\langle\langle b \rangle\rangle$ , by Result 1.3, so again  $b \in A$ .  $\square$

In any inverse semigroup  $S$ , the implication (2)  $\Rightarrow$  (2C) holds, since for any  $a \in S$ ,  $\langle a \rangle \subseteq \langle\langle a \rangle\rangle$ . Since  $\mathcal{C}o(S) = \mathcal{L}(S)$  whenever  $E_S$  has height at most 2, the following example therefore exhibits a minimal inverse semigroup in which (2) and (2C) are inequivalent.

**EXAMPLE 3.3** Let  $Y$  be the semilattice obtained from two three-element chains by amalgamating their zeroes. Then the Munn semigroup  $T_Y$  satisfies (2C) but not (2). In the sequel it will be shown that  $\mathcal{C}o(T_Y)$  is lower semimodular but  $\mathcal{L}(T_Y)$  is not.

**Proof.** Let  $Y = \{e_0 > e_1 > 0\} \cup \{f_0 > f_1 > 0\}$ . As mentioned in the introduction, we identify each idempotent  $1_{e\downarrow}$  with  $e$  itself. Let  $\alpha$  be the isomorphism of  $e_0\downarrow$  upon  $f_0\downarrow$ . Since  $\alpha^2 = 0$ ,  $\langle \alpha \rangle = \{e_0, f_0, 0, \alpha, \alpha^{-1}\}$ . Then  $T_Y = Y \cup \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$ , where  $\beta$  is the restriction of  $\alpha$  to  $e_1\downarrow$ . Since  $\alpha > \beta > 0$ ,  $\beta \in \langle \alpha \rangle$ . Together with  $\alpha^{-1} > \beta^{-1} > 0$ , which is handled similarly, these are only nontrivial orderings needing verification in order for (2C) to hold. On the other hand, since  $\beta \notin \langle \alpha \rangle$ ,  $T_Y$  does not satisfy (2).  $\square$

There will be further discussion of the consequences of these properties, and further such examples, in the discussion of dual distributivity, to which we now turn. Note that it follows from the finitariness of the operations on an inverse semigroup that dual distributivity implies complete dual distributivity.

**PROPOSITION 3.4** *The following are equivalent:*

(1')  $E_S$  is dually distributive in  $\mathcal{L}(S)$ , that is,  $E_{A \vee B} = E_A \vee E_B$  for all  $A, B \in \mathcal{L}(S)$ ;

(2') for all  $a \in S$ ,  $a\downarrow \subseteq G_S \cup \langle a \rangle$ ;

(3') for all  $A \in \mathcal{L}(S)$ ,  $E_S \vee A \subseteq G_S \cup A$ .

(4') if  $A \succ B$  in  $\mathcal{L}(S)$  then  $E_A \succeq E_B$  in  $\mathcal{L}(E_S)$ .

Denote by (1C') to (4C') the analogous statements with respect to  $\mathcal{C}o(S)$ . If  $E_S$  is a tree, then they are also equivalent. In fact the implications (2C')  $\Leftrightarrow$  (3C')  $\Rightarrow$  (1C')  $\Rightarrow$  (4C') hold in any inverse semigroup.

**Proof.** We first treat  $\mathcal{L}(S)$ . (1')  $\Rightarrow$  (4') is an instance of Lemma 1.2.

(4')  $\Rightarrow$  (2'). Let  $a \in S$  and suppose  $b \in a\downarrow$  with  $b \notin \langle a \rangle$ . Put  $T = \langle a, b \rangle$ . By Zorn's lemma, there exists  $U \in \mathcal{L}(T)$ , maximal with respect to containing  $a$  but not  $b$ . Hence  $T \succ U$ . By (4'),  $E_T \succeq E_U$ . Now  $bb^{-1} \notin U$  for, otherwise,  $b = bb^{-1}a \in U$ ; similarly  $b^{-1}b \notin U$ . By Result 2.1,  $|E_T - E_U| = 1$  and thus  $bb^{-1} = b^{-1}b$ .

(2')  $\Leftrightarrow$  (3'). This is clear from Lemma 3.1.

(3')  $\Rightarrow$  (1'). Let  $A, B \in \mathcal{L}(S)$ . Any idempotent in  $A \vee B$  can be expressed in the form  $x_1 \cdots x_n = (x_n^{-1} \cdots x_1^{-1})(x_1 \cdots x_n)$ , where each  $x_j$  belongs to  $A \cup B$ . We prove by induction that for  $1 \leq j \leq n$  the idempotent  $e_j = (x_j^{-1} \cdots x_1^{-1})(x_1 \cdots x_j)$  belongs to  $E_A \vee E_B$ . Since  $e_1 = x_1 x_1^{-1}$ , the basis step is clear. For  $j > 1$ ,  $e_j = x_j^{-1} e_{j-1} x_j$ . Now by (3') either  $e_{j-1} x_j \in G_S$  or  $e_{j-1} x_j \in A \cup B$ . Since  $e_j = (e_{j-1} x_j)^{-1} (e_{j-1} x_j)$ , then in the former case  $e_j = e_{j-1} (x_j x_j^{-1})$  and the induction hypothesis applies; and in the latter case  $e_j \in A \cup B$ . This yields one of the required inclusions; the other one is obvious.

Next we treat  $\mathcal{C}o(S)$ . Once more, (1C')  $\Rightarrow$  (4C') is an instance of Lemma 1.2 and (2C')  $\Leftrightarrow$  (3C') is immediate from Lemma 3.1.

The argument for (4C')  $\Rightarrow$  (2C') also proceeds analogously, except that in the case of  $\mathcal{C}o(E_S)$ , the implication  $E_T \succ E_U \Rightarrow |E_T - E_U| = 1$  may require that  $E_S$  be a tree, by virtue of Result 2.1.

Finally, assume (3C') holds. By Lemma 3.1, if  $A \in \mathcal{C}o(S)$  then  $E_S \diamond A = E_S \vee A$ . So (3') holds when restricted to  $\mathcal{C}o(S)$ , and thus (1') holds likewise. It follows that if  $A, B \in \mathcal{C}o(S)$ , then  $E_{A \vee B} = E_A \vee E_B$ . Then Result 1.3 gives  $E_{A \diamond B} = E_A \diamond E_B$ .  $\square$

Similarly to the remark following Proposition 3.2, the implication (2')  $\Rightarrow$  (2C') holds in any inverse semigroup. Thus, combining the first and last statements of the last proposition, each of the statements (1'),  $\dots$ , (4') implies the analogous statement (1C'),  $\dots$ , (4C').

For the purposes of this paper, the hypothesis that  $E_S$  be a tree in the second part of the proposition is not restrictive. It will be shown in Example 3.10 that, without that hypothesis, the implications in the final statement cannot be replaced by equivalences.

**THEOREM 3.5** *Neutrality of  $E_S$  in  $\mathcal{L}(S)$  is equivalent to (2): for all  $a \in S$ ,  $a\downarrow \subseteq E_S \cup \langle a \rangle$ . In that case,  $\mathcal{L}(S)$  is isomorphic to the sublattice  $\{(F, U) : U \subseteq E_S \cup F^{\max}\}$  of  $\mathcal{L}(E_S) \times \mathcal{L}\mathcal{F}(S)$ , where  $F^{\max}$  is the greatest inverse subsemigroup of  $S$  whose semilattice of idempotents is  $F$ .*

*The entirely analogous statement holds with respect to  $\mathcal{C}o(S)$ , with (2C) in place of (2),  $\langle\langle a \rangle\rangle$  in place of  $\langle a \rangle$ , and  $\mathcal{C}o(E_S)$  in place of  $\mathcal{L}(E_S)$ .*

**Proof.** It is clear that conditions (2) and (2C) in Proposition 3.2 imply the respective conditions (2') and (2C') in Proposition 3.4. In view of the validity of the implication (2C')  $\Rightarrow$  (1C') in any inverse semigroup, distributivity of  $E_S$  in either  $\mathcal{L}(S)$  or  $\mathcal{C}o(S)$  always implies dual distributivity, and thus by Lemma 3.1 is equivalent to neutrality, in the respective lattices.

In addition, noting the comment preceding Proposition 3.4,  $E_S$  is actually completely dually distributive in each case. The description of the images in the respective direct products is then an immediate consequence of Lemma 1.1.  $\square$

An alternative description of the images in the respective direct products may also be found by applying the order dual of Lemma 1.1.

As noted above, in inverse semigroups for which  $E_S$  has height 2, the lattices  $\mathcal{L}(S)$  and  $\text{Co}(S)$  coincide. Moreover in that case if  $a > b$ , then  $b$  necessarily belongs to the group kernel, so (2') is automatically satisfied. Any such semigroup is an ideal extension of its group kernel by a primitive inverse semigroup. (A primitive inverse semigroup is a union of completely 0-simple inverse semigroups, identifying the common zero elements.)

In the case of  $\mathcal{L}(S)$ , (2') imposes significant restrictions on  $S$ .

**PROPOSITION 3.6** *For a monogenic inverse semigroup  $A = \langle c \rangle$ , (2') and (2) are equivalent, and equivalent to the property that  $c^3$  belongs to a subgroup of  $A$  (that is, in the terminology of §1.3,  $A$  is of type (ii) or (iii) with  $k \leq 3$ ).*

*Hence a 0-simple inverse semigroup satisfies (2') if and only if it is completely 0-simple; and a simple inverse semigroup satisfies (2') if and only if it is a group.*

**Proof.** Put  $b = (c^{-1}c)c^2 \leq c^2$ . If  $c \in G_A$  or  $c^2 \in G_A$ , then  $c^3 \in G_A$ , so assume otherwise. Supposing  $A$  satisfies (2'), then  $b \in G_A \cup \langle c^2 \rangle$ . On the one hand, if  $b \in G_A$  then  $c^{-3}c^3 = b^{-1}b = bb^{-1} = (c^{-1}c)(c^2c^{-2})$ . Conjugating by  $c^3$  yields  $c^3c^{-3} = c^5c^{-5}$ ; conjugating by  $c^{-2}$  yields  $c^{-5}c^5 = c^{-3}c^3$ . In combination,  $c^3 \mathcal{H} c^5$  and so  $c^3 \in G_A$ . On the other hand, if  $b \in \langle c^2 \rangle$ , then  $c^{-3}c^3 = b^{-1}b \in \langle c^2 \rangle$ . Now in this case since  $c^2 \notin G_A$  we have  $c^{-4}c^4 \prec c^{-2}c^2$  in  $E_{\langle c^2 \rangle}$ . Since  $c^{-3}c^3 = c^{-2}c^2$  or  $c^{-3}c^3 = c^{-4}c^4$  and since  $\mathcal{L}$  is a right congruence, in either case  $c^3 \mathcal{L} c^4$ . Dualizing this entire argument, either  $c^3 \in G_A$  or  $c^3 \mathcal{R} c^4$ . But if  $c^3 \mathcal{H} c^4$ , then again  $c^3 \in G_A$ .

Conversely, suppose  $c^3 \in G_A$  and denote by  $e$  the identity of the subgroup  $H_{c^3}$ . We prove (2), which then implies (2'). If  $A$  is a group, this is obvious. Otherwise, either  $J_c > J_{c^2} = H_e$  or  $J_c > J_{c^2} > J_{c^3} = H_e$ . Suppose  $a > b$  in  $A$ , with  $b \notin E_A$ . In either case, if  $a \in J_c$ , then  $a = c$  or  $a = c^{-1}$ , whence  $b \in \langle a \rangle$ . This covers the former case. In the latter, there remains only the situation that  $a \in J_{c^2}$  and  $b \in H_e$ . But then  $a^2 \in H_e$ , so  $e \in \langle a \rangle$  and thus the same is true of  $b = ea$ .

Turning to the final statements, it is well known that any 0-simple semigroup in which some power of every element belongs to a subgroup is completely 0-simple (because otherwise it contains a bicyclic subsemigroup). That every completely 0-simple semigroup satisfies (2') was shown prior to this proposition.  $\square$

Comparing the next two results with the previous one demonstrates how much stronger is (2') than (2C').

**PROPOSITION 3.7** *Every monogenic inverse semigroup  $A = \langle c \rangle$  satisfies (2C), and therefore (2C'). Thus  $E_A$  is neutral in  $\text{Co}(A)$ .*

**Proof.** Suppose  $a > b \in A$ , with  $b \notin E_A$ . Thus  $a \notin E_A$  and so  $a = ec^n$  for some  $e \in E_A$ ,  $n \neq 0$ . Further, since  $a^{-1} > b^{-1}$  and (2C) is (left-right) self-dual, we may assume that  $n > 0$ . We show that  $E_A \subseteq E_{\langle a \rangle} \uparrow$ . Since  $bb^{-1} \leq aa^{-1}$ , this shows that  $bb^{-1} \in \langle\langle a \rangle\rangle$ , whence the same is true of  $b = (bb^{-1})a$ . Consider first an idempotent of the form  $c^m c^{-m}$ , where  $m > 0$ .

Now from  $a \leq c^n$  it follows that  $a^m \leq c^{mn}$  and so  $a^m a^{-m} = f c^{mn} c^{-mn}$  for some  $f \in E_A$ . Then  $a^m a^{-m} \leq c^m c^{-m}$ . Similarly  $a^{-k} a^k \leq c^{-k} c^k$  for every  $k > 0$ . But every idempotent of  $\langle c \rangle$  is either of one of these forms or a product of one of each, and so above an idempotent of  $\langle\langle a \rangle\rangle$ .  $\square$

The properties (2C) and (2C') belong to a class of ‘‘archimedean’’ properties that have arisen in a number of situations associated with lattices of inverse subsemigroups. In the case of particular relevance in the sequel, where  $E_S$  is a tree and  $S$  is combinatorial, the two of particular significance are the following, which were introduced in [13] (also see [3]):  $S$  is *pseudoarchimedean* if no idempotent is strictly below every idempotent of a free monogenic or bicyclic inverse subsemigroup of  $S$ ;  $S$  is *faintly archimedean* if whenever an idempotent  $e$  is strictly below every idempotent of a bicyclic or free monogenic inverse subsemigroup  $\langle a \rangle$  of  $S$ , then  $e < a$ . Note that if  $E_S$  is a tree,  $S$  cannot contain a free monogenic inverse subsemigroup (see Lemma 1.5). Clearly if  $S$  is pseudoarchimedean then it is faintly archimedean.

**PROPOSITION 3.8** *If  $E_S$  is a tree, then  $E_S$  is archimedean in  $S$  if and only if  $S$  is pseudoarchimedean. If, in addition,  $S$  is combinatorial, then  $S$  satisfies (2C) if and only if it is faintly archimedean and, therefore, whenever  $E_S$  is archimedean in  $S$ .*

*Hence, under the additional assumption that its semilattice of idempotents is a tree, any simple inverse semigroup  $S$  for which  $\mathcal{LF}(S)$  is lower semimodular satisfies (2C).*

**Proof.** Throughout, we assume that  $E_S$  is a tree. Let  $a \in S$ ,  $a \notin E_S$ . As noted above,  $\langle a \rangle$  cannot be free. Let  $e \in E_S$ . Then since  $e(aa^{-1}) \leq aa^{-1}$  and  $a^n a^{-n} \leq aa^{-1}$  for all  $n > 0$ , either  $e \in E_{\langle a \rangle} \uparrow$  or  $e(aa^{-1}) < a^n a^{-n}$  for all such  $n$ . In combination with the dual argument, it follows that either  $e \in E_{\langle a \rangle} \uparrow$  or  $e(aa^{-1})(a^{-1}a)$  is strictly below every idempotent of  $\langle a \rangle$ . The first statement is now immediate from the respective definitions.

It also follows that if  $a > b$ , then either  $b \in \langle\langle a \rangle\rangle$ , or  $bb^{-1} < a^n a^{-n}$  and  $b^{-1}b < a^{-n} a^n$  for all  $n > 0$  (these options being mutually exclusive). Now according to [13, Proposition 3.3], if  $S$  is combinatorial, then it is faintly archimedean if and only if whenever  $bb^{-1} < a^n a^{-n}$  and  $b^{-1}b < a^{-n} a^n$  for all  $n > 0$ , then  $b \in E_S$ . Hence this property is equivalent to (2C).

To deduce the final statement, observe that such a semigroup is combinatorial and  $E_S$  is archimedean in  $S$ , by Result 2.6.  $\square$

**EXAMPLE 3.9** *Property (2C) is not equivalent to the property that  $E_S$  be archimedean in  $S$ , even in combinatorial, bisimple inverse semigroups whose semilattices of idempotents are totally ordered.*

**Proof.** Let  $S$  be the Munn semigroup  $T_E$ , where  $E = C_\omega \times C_\omega$ , with  $(i, j) \geq (k, l)$  if and only if either  $i = k$  and  $j \leq l$ , or  $i < k$ . (That is,  $E$  is the ordinal product of two copies of  $C_\omega$ .) Note that  $E$  is a chain satisfying the ACC, so by [14, Theorem 1.4],  $T_E$  is combinatorial. For each  $i \in C_\omega$ , put  $E_i = \{(i, j) : j \in C_\omega\}$ .

Let  $\alpha \in S$ , with domain  $(i, j) \downarrow$  and range  $(k, l) \downarrow$ . Since  $E$  is a chain and  $S$  is combinatorial, if  $\alpha$  is nonidempotent then either  $\alpha\alpha^{-1} > \alpha^{-1}\alpha$  or the opposite inequality holds. Without generality we may assume the former.

If  $k = i$  then  $j < l$ . There is a unique member of the bicyclic subsemigroup  $T_{E_i}$  that maps  $(i, j)$  to  $(i, l)$ ; it extends to an isomorphism of  $(i, j) \downarrow$  upon  $(i, l) \downarrow$ , by fixing every  $E_j, j > i$ . By the combinatorial property, this extension must be  $\alpha$  itself. Hence for every idempotent  $g = 1_{(m,n)}$  of  $S$ , either  $m \leq i$  and  $g \geq \alpha^{-r} \alpha^r$  for some positive integer  $r$  (using the properties of the bicyclic semigroup  $T_{E_i}$ ), so that  $g\alpha \in \langle\langle \alpha \rangle\rangle$ , or  $m > i$  and  $g\alpha = g$ .

Hence (2C') (equivalently (2C)) holds in this case. But at the same time, this calculation proves that  $E_S$  is not archimedean in  $S$  since, for  $m > i$ ,  $g < \alpha^{-r} \alpha^r$  for every positive integer  $r$ .

Next, consider  $\alpha$  for which  $k > i$ . Then for  $r > 0$ , the range of  $\alpha^r$  is  $(i + r(k - i), \ell') \downarrow$  for some  $\ell'$ . Hence for any idempotent  $g$  as above,  $g > \alpha^{-r} \alpha^r$  for some  $r$ . So property (2C') also holds in this case.

Clearly, every principal ideal of  $E$  is isomorphic to  $E$  itself, so  $E$  is uniform and  $T_E$  is bisimple.  $\square$

The following example was cited after Proposition 3.4.

**EXAMPLE 3.10** Let  $C = \langle c \mid c^3 = c^2 \rangle$ , let  $Y$  be the two-element semilattice  $1 > 0$ , and put  $S = C \times Y$ . The implication  $(1C') \Rightarrow (3C')$  fails to hold in  $S$ .

**Proof.** In the terminology of §1.3,  $C \cong B_2$ ;  $C$  consists of the nonidempotents  $c$  and  $c^{-1}$  and the idempotents  $cc^{-1}$ ,  $c^{-1}c$  and  $c^2 = (cc^{-1})(c^{-1}c)$ . To simplify notation, put  $A = C \times \{1\}$  and  $a = (c, 1)$ ; and put  $B = C \times \{0\}$  and  $b = (c, 0)$ . Note that  $a > b$ , but  $b \notin G_S \cup \langle\langle a \rangle\rangle$ , so (2C') (and thus (3C')) fails in  $S$ . Now let  $U, V \in \mathcal{C}o(S)$  and assume that  $U$  and  $V$  are incomparable under inclusion. If  $a \in U$ , then  $A \subseteq U$  and so some idempotent  $e$  of  $B$  belongs to  $V$ . Therefore  $b^2 = ea^2 \in U \vee V$  and  $b \in [b^2, a] \in U \diamond V$ , that is,  $U \diamond V = S$ ; a slight modification of this argument yields  $E_U \diamond E_V = E_S = E_{U \diamond V}$ . If  $a \in V$ , the same conclusion holds. In the case that  $a \notin U \cup V$ , then in fact  $U, V \subseteq B \cup E_S$ . But  $B \cup E_S$  satisfies (2C'), its only nonidempotents being the generators  $b$  and  $b^{-1}$  of  $B$ , so by the last sentence of Proposition 3.4,  $E_{U \diamond V} = E_U \diamond E_V$  in this case as well.  $\square$

Finally, it is worth remarking that (2') and (2C') imply that  $S$  is cryptic, that is, Green's relation  $\mathcal{H}$  is a congruence. It follows that  $G_S$ , the union of the subgroups of  $S$ , is an inverse subsemigroup and order ideal. In a continuation [5], the authors will pursue a similar investigation of the role of  $G_S$  within  $\mathcal{L}(S)$  and  $\mathcal{C}o(S)$ , with applications to join semidistributivity. It will also be shown there that if either of these lattices is lower semimodular, then an alternative decomposition of the lattice exists, in terms of Clifford semigroups and combinatorial inverse semigroups.

## 4 Semimodularity of $\mathcal{C}o(S)$ or $\mathcal{L}(S)$ .

**PROPOSITION 4.1** Let  $S$  be any inverse semigroup. If  $\mathcal{L}(S)$  is lower semimodular, then  $E_S$  is distributive, whence neutral, in  $\mathcal{L}(S)$ . If  $\mathcal{C}o(S)$  is lower semimodular, then the same conclusion holds in  $\mathcal{C}o(S)$ .

**Proof.** We first consider  $\mathcal{L}(S)$ . It suffices to prove that (2) holds in Proposition 3.2. Observe first that  $S$  satisfies (4') of Proposition 3.4, for if  $A \succ B$  and  $E_A \neq E_B$ , then  $A = B \vee E_A \succ B$  and so  $E_A \succ E_A \cap B = E_B$ . By that proposition, if  $a \in S$  and  $a > b$ , then  $b \in H_e$  for some  $e \in E_S$ . If  $b \notin \langle a \rangle$ , let  $T = \langle a, b \rangle$  so that, as in the proof of (4')  $\Rightarrow$  (2') in Proposition 3.4, there exists  $U \in \mathcal{L}(S)$  such that  $T \succ U$ ,  $b \notin U$ , and by lower semimodularity,  $\langle b \rangle \succ U \cap \langle b \rangle$ . But since  $\langle b \rangle$  is a subgroup and  $e \notin U$  (otherwise  $b = ea \in U$ ), so that  $U \cap \langle b \rangle = \emptyset$ , this is possible only if  $\langle b \rangle = \{e\}$ , that is,  $b \in E_S$ .

The argument in the case of  $\mathcal{C}o(S)$  proceeds for the most part analogously, noting first that by Result 2.1,  $E_S$  is a tree and so all the properties in Proposition 3.4 are equivalent. In the notation of the previous paragraph,  $b \in H_e$  and so  $\langle\langle b \rangle\rangle = \langle b \rangle$  and  $\mathcal{C}o(H_e) = \mathcal{L}(H_e)$ . Hence when we arrive at the analogous covering  $\langle\langle b \rangle\rangle \succ U \cap \langle\langle b \rangle\rangle$ , the rest of the proof follows verbatim.  $\square$

The following theorem combines the previous proposition with Theorem 3.5 and Result 2.1.

**THEOREM 4.2** *Let  $S$  be an inverse semigroup. Then  $\mathcal{L}(S)$  is lower semimodular if and only if  $\mathcal{LF}(S)$  is lower semimodular and  $S$  satisfies (2): for all  $a \in S$ ,  $a \downarrow \subseteq E_S \cup \langle a \rangle$ . In that case,  $\mathcal{L}(S)$  is isomorphic to the sublattice  $\{(F, U) : U \subseteq E_S \cup F^{\max}\}$  of  $\mathcal{L}(E_S) \times \mathcal{LF}(S)$ .*

*The lattice  $\mathcal{C}o(S)$  is lower semimodular if and only if  $E_S$  is a tree,  $\mathcal{LF}(S)$  is lower semimodular and  $S$  satisfies (2C): for all  $a \in S$ ,  $a \downarrow \subseteq E_S \cup \langle\langle a \rangle\rangle$ . The analogous decomposition of  $\mathcal{C}o(S)$  holds if  $\mathcal{L}(E_S)$  is replaced by  $\mathcal{C}o(E_S)$ .*

Independence of the respective conditions stated in this theorem is readily verified by considering Clifford semigroups. For such a semigroup  $S$ , applying Result 2.3,  $\mathcal{LF}(S)$  is lower semimodular if and only if the same is true for each subgroup. The properties (2) and (2C) are each equivalent to the property that every structure homomorphism be trivial (that is, maps to the identity element). Example 3.9 demonstrates that, even in the bisimple case, (2C) and the property that  $E_S$  be a tree (even a chain) do not together imply lower semimodularity of  $\mathcal{LF}(S)$ . Since (2) implies (2C), we immediately obtain the following.

**COROLLARY 4.3** *If  $\mathcal{L}(S)$  is lower semimodular and  $E_S$  is a tree, then  $\mathcal{C}o(S)$  is lower semimodular.*

#### 4.1 Lower semimodularity of $\mathcal{C}o(S)$ .

It remains to combine the above decomposition and the additional restrictions on the structure of  $S$  imposed by the interaction of (2) and (2C) with the principal factors of  $S$ .

In the case of  $\mathcal{C}o(S)$ , observe that the monogenic inverse subsemigroups are severely restricted, by virtue of Lemma 1.5. We separate the technical details in a lemma, since it will find application elsewhere. Recall that the set of  $\mathcal{J}$ -classes of any semigroup is partially ordered under the relation  $J_1 \leq J_2$  if  $J_1 \subseteq S^1 J_2 S^1$ .

**LEMMA 4.4** *Let  $S$  be an inverse semigroup for which  $E_S$  is a tree and  $\mathcal{LF}(S)$  is lower semimodular. Then  $S$  satisfies (2C) [resp. (2C')] if and only if whenever  $e \in E_S$ ,  $J_e \in \mathcal{L}(S)$ ,  $f < e$  and  $J_f < J_e$ , then  $af = fa = f$  [resp.  $fa \in H_f$ ] for all  $a \in J_e$ .*

**Proof.** To prove necessity, suppose  $e, a, f$  are as given. Now  $e(aa^{-1}) \leq e$ , so since  $E_S$  is a tree,  $e(aa^{-1}) > f$ , whence  $fa \mathcal{R} f$ . Since  $J_e \leq S$ ,  $\langle\langle a \rangle\rangle \subseteq J_e$ . Thus if  $S$  satisfies (2C') then  $fa \in G_S$  and so  $fa \in H_f$ . If  $S$  satisfies (2C), then  $fa \in E_S$  and so  $fa = f$ .

Conversely, suppose  $a > b$  in  $S$ . If  $J_a = J_b$ , then the principal factor is not completely 0-simple so  $J_a \in \mathcal{L}(S)$  by Result 2.5 and, according to the last part of Proposition 3.8,  $J_a$  satisfies (2C), that is,  $b \in E_S \cup \langle\langle a \rangle\rangle$ . Otherwise,  $J_a > J_b$ . If  $J_a < S$ , then the stated hypotheses, with  $e = aa^{-1}$  and  $f = bb^{-1}$ , imply that since  $b = (bb^{-1})a$ , either  $b = bb^{-1} \in E_S$  or  $b \in H_{bb^{-1}}$ , respectively. The remaining case is where  $J_a \notin \mathcal{L}(S)$ . Then  $aa^{-1} \parallel a^{-1}a$  and, according to Lemma 1.5, the kernel  $K$  of  $\langle a \rangle$  is either a group or a bicyclic semigroup, which is easily seen to be generated in either case by  $a^2a^{-1}$  or by  $a^{-1}a^2$ . In either case, by Results 2.4 and 2.5, the associated  $\mathcal{J}$ -class is a subsemigroup.

Assume  $K = \langle a^2a^{-1} \rangle$ , the other case being dual. Now  $aa^{-1} > bb^{-1}$  and  $aa^{-1} > a^2a^{-2}$ , so  $bb^{-1}$  and  $a^2a^{-2}$  are comparable, since  $E_S$  is a tree. If  $bb^{-1} \geq a^2a^{-2}$ , then  $b \in \langle\langle a \rangle\rangle$ . Otherwise  $b \leq (a^2a^{-2})a = a^2a^{-1}$  and so  $b \in E_S$  or  $b \in G_S$ , as shown in the previous paragraph.  $\square$

This immediately yields the main result of this paper for  $Co(S)$ .

**THEOREM 4.5** *The lattice  $Co(S)$  is lower semimodular if and only if:  $E_S$  is a tree;  $\mathcal{LF}(S)$  is lower semimodular, so that for each  $\mathcal{J}$ -class  $J$  of  $S$ , either its principal factor  $PF(J)$  is isomorphic to  $B_2$  or  $B_3$ , or  $J$  itself is a simple inverse subsemigroup that is either a group or is as described in Result 2.6 (with its semilattice a tree); whenever  $e \in E_S$ ,  $J_e \in \mathcal{L}(S)$ ,  $f < e$  and  $J_f < J_e$ , then  $af = fa = f$  for all  $a \in J_e$ .*

Theorem 4.5 gives an explicit description in many cases of interest. For instance, if  $S$  is finite then  $\mathcal{J}$ -classes that are subsemigroups must be groups. More generally, this is the case for *completely semisimple* inverse semigroups: those that contain no bicyclic subsemigroups. If, further,  $S$  is combinatorial, then the last condition in the theorem is automatically satisfied, yielding the following simple characterization.

**COROLLARY 4.6** *If  $S$  is both combinatorial and completely semisimple, then  $Co(S)$  is lower semimodular if and only if  $E_S$  is a tree and every  $\mathcal{J}$ -class of  $S$  has at most three idempotents.*

Theorem 4.5 shows that for the semigroups under consideration, the  $\mathcal{J}$ -classes that are subsemigroups act trivially on the order ideals that they generate. Further properties of the products associated with such  $\mathcal{J}$ -classes are provided by the following.

**COROLLARY 4.7** *Suppose  $Co(S)$  is lower semimodular and  $a, b \in S$ . If  $J_a, J_b \in \mathcal{L}(S)$  and  $J_a \parallel J_b$  in the poset of  $\mathcal{J}$ -classes, then  $ab \in E_S$ .*

**Proof.** Write  $ab = \bar{a}\bar{b}$ , where  $\bar{a} = a(bb^{-1}) \leq a$  and  $\bar{b} = (a^{-1}a)b \leq b$ . By Theorem 4.5 (or direct from (2C)),  $\bar{a}, \bar{b} \in E_S$ , so  $ab \in E_S$ .  $\square$

The following example shows that when the  $\mathcal{J}$ -classes are not inverse subsemigroups, the products are less constrained.

**EXAMPLE 4.8** Let  $Z$  be the tree in Figure 1 and let  $U$  be the inverse subsemigroup of  $T_Z$  generated by  $\alpha$  and  $\gamma$ , the isomorphisms of  $e_0\downarrow$  upon  $f_0\downarrow$ , and  $g_0\downarrow$  upon  $h_0\downarrow$ , respectively. Then  $\mathcal{C}o(U)$  is lower semimodular,  $J_\alpha \parallel J_\gamma$ , and  $\alpha\gamma \notin E_S \cup \langle\langle\alpha\rangle\rangle \cup \langle\langle\gamma\rangle\rangle$ . The lattice  $\mathcal{C}o(T_Z)$  is not lower semimodular.

**Proof.** Clearly  $\langle\langle\alpha\rangle\rangle \cong \langle\langle\gamma\rangle\rangle \cong T_Y$ , where  $Y$  is the semilattice in Example 3.3. A straightforward calculation, invoking the details from that example, reveals that  $U = \langle\langle\alpha\rangle\rangle \cup \langle\langle\gamma\rangle\rangle \cup \{\alpha\gamma, (\alpha\gamma)^{-1}\}$ , where  $\alpha\gamma$  is the isomorphism of  $e_1\downarrow$  upon  $h_1\downarrow$ . The only intersection of the three components of this union consists of the idempotents 0 and  $f_1 = g_1$ .

Then  $PF(J_\alpha) \cong PF(J_\gamma) \cong B_2$  and  $PF(J_{\alpha\gamma}) \cong B_3$ . According to Corollary 4.6,  $\mathcal{C}o(U)$  is therefore lower semimodular.

Since in  $T_Z$ ,  $PF(J_\alpha) \cong B_4$ ,  $\mathcal{C}o(T_Z)$  is not lower semimodular. □

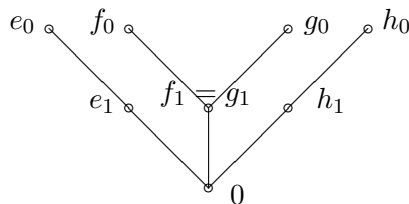


Figure 1: The tree  $Z$

In a different direction, we may consider the simple inverse semigroups that are not groups. In the case of *bisimple* inverse semigroups, it follows from Result 2.6 that when  $\mathcal{L}\mathcal{F}(S)$  is lower semimodular,  $E_S$  must always be a tree. Hence lower semimodularity of  $\mathcal{C}o(S)$  and  $\mathcal{L}\mathcal{F}(S)$  are equivalent. Such inverse semigroups  $S$  were completely determined in [4]. With precisely one exception,  $E_S$  is in fact always a chain. The lattice  $\mathcal{C}o(B)$  was explicitly constructed in [3, Example 6.9].

For *simple* inverse semigroups in general, lower semimodularity of  $\mathcal{C}o(S)$  is equivalent to the conjunction of lower semimodularity of  $\mathcal{L}\mathcal{F}(S)$  with the property that  $E_S$  be a tree. An example was given in [12] of a simple inverse semigroup  $S$  such that  $\mathcal{L}\mathcal{F}(S)$  is distributive — so that  $E_D$  is a chain for each  $\mathcal{D}$ -class — but  $E_S$  itself is not a tree. [4, Example 11.1] exhibits a simple, non-bisimple, inverse semigroup  $S$  for which  $\mathcal{L}\mathcal{F}(S)$  is modular and  $E_S$  is a tree but is not a chain. Thus  $\mathcal{C}o(S)$  is lower semimodular.

## 4.2 Lower semimodularity of $\mathcal{L}(S)$ .

We now consider  $\mathcal{L}(S)$ . In this case, both the monogenic inverse subsemigroups and the principal factors are severely restricted, by virtue of Proposition 3.6. Recall that  $B_n$  denotes the combinatorial Brandt semigroup with  $n$  nonzero idempotents. It was proved in [12] that  $\mathcal{L}\mathcal{F}(B_n) \cong \Pi_n$ , the full partition lattice on a set of  $n$  elements.

**THEOREM 4.9** *The lattice  $\mathcal{L}(S)$  is lower semimodular if and only if (i) each  $\mathcal{J}$ -class either is a group with lower semimodular subgroup lattice, or its principal factor is isomorphic to  $B_2$*



or  $B_3$ , and (ii) if  $a > b$  in  $S$ , then either  $b \in E_S$  or  $aa^{-1} \parallel a^{-1}a$ ,  $a^3 \in G_S$ , and either  $b = a^2a^{-1}$  or  $b = a^{-1}a^2$ , or  $b = a^3a^{-2}$ .

In that event,  $\mathcal{L}(S)$  is isomorphic to a subdirect product of  $\mathcal{L}(E_S)$ , the subgroup lattices of the nontrivial maximal subgroups of  $S$  (if any), and possibly copies of the full partition lattices  $\Pi_2$  and/or  $\Pi_3$ .

**Proof.** We apply the first part of Theorem 4.2. Applying Proposition 3.6 to the results of §2.2 yields the stated restriction on the principal factors. If  $a > b$  and  $aa^{-1} = a^{-1}a$ , then only  $b \in E_S$  can hold. Otherwise  $aa^{-1} \parallel a^{-1}a$ . Now by Proposition 3.6,  $a^3 \in G_S$ . Thus, considering the description of the monogenic inverse semigroups in §1.3, the only instances of (2) to be verified are those stated in the theorem.  $\square$

From this result and Result 2.2 one may easily deduce Ershova's determination of the inverse semigroups for which  $\mathcal{L}(S)$  is modular or distributive, as cited in §2.1: for in either case,  $E_S$  must be a chain and therefore, from the theorem, the principal factors are groups with zero, or groups, and  $a > b$  implies  $b \in E_S$ , equivalently, every structure homomorphism is trivial.

As was the case for Theorem 4.5, Theorem 4.9 imposes severe restrictions on the products in  $S$ .

**COROLLARY 4.10** *Suppose  $\mathcal{L}(S)$  is lower semimodular and  $a, b \in S$ . Then  $ab \in G_S \cup \langle a \rangle \cup \langle b \rangle$ , except in the case that  $a, b, ab \notin E_S$  and  $a^{-1}a = bb^{-1}$ , so that  $ab \in R_a \cap L_b$ . The exceptional case occurs if and only if  $S$  has a principal factor isomorphic to  $B_3$ .*

**Proof.** If  $a \in E_S$  or  $b \in E_S$  then  $ab \leq b$  or  $ab \leq a$ , respectively, and the desired result follows from (2). Otherwise, as in the proof of Corollary 4.7, write  $ab = \bar{a}\bar{b}$ , where  $\bar{a} = a(bb^{-1}) \leq a$ ,  $\bar{b} = (a^{-1}a)b \leq b$  and  $ab \in R_{\bar{a}} \cap L_{\bar{b}}$ . Again, if either  $\bar{a}$  or  $\bar{b}$  is idempotent, the result follows. Otherwise,  $\bar{a} \in \langle a \rangle$  and  $\bar{b} \in \langle b \rangle$ .

Suppose  $\bar{a} < a$ . If  $\bar{a} \in G_S$ , then the subgroup  $H_{\bar{a}}$  is nontrivial and so isolated, whence contains  $ab$ . If  $\bar{a} \notin G_S$  then, by the theorem, only the cases  $\bar{a} = a^2a^{-1}$  or  $\bar{a} = a^{-1}a^2$  remain. In either case, within  $\langle a \rangle$  itself,  $PF(J_{a^2}) \cong B_3$ . Applying Theorem 4.9, within  $S$  itself  $J_{\bar{a}} \subset \langle a \rangle$  and so  $ab \in \langle a \rangle$ . A similar conclusion holds if  $\bar{b} < b$ .

The only remaining case to consider is where  $\bar{a} = a$  and  $\bar{b} = b$ : but that is precisely the exceptional case in the statement. In that case, the idempotents  $aa^{-1}$ ,  $a^{-1}a = bb^{-1}$  and  $b^{-1}b$  are distinct, so  $PF(J_a) \cong B_3$ . Conversely, if  $S$  has such a principal factor, then  $a$  and  $b$  may be found with the requisite property.  $\square$

Our final example shows that a product  $ab$  may lie in  $G_S$  without lying in  $E_S \cup \langle a \rangle \cup \langle b \rangle$ . Let  $T = \langle a, b, g \mid g^7 = g, a^2a^{-1} = g^2, b^2b^{-1} = g^3, ab = ba = g \rangle$ . Then  $T$  has as its kernel the cyclic group  $K = \langle g \rangle$  of order six, and  $T/K$  is the 0-direct union of two copies of  $B_2$ . (Alternatively,  $T$  is the retract extension defined by the partial homomorphisms induced by  $a \rightarrow g^2, b \rightarrow g^3$ ).

The only nontrivial orderings in  $T$  are  $a > g^2, a^{-1} > g^4, b > g^3, b^{-1} > g^3$ . Since  $\langle a \rangle = J_a \cup \langle g^2 \rangle$  and  $\langle b \rangle = J_b \cup \langle g^3 \rangle$ , it follows that (2) is satisfied. Hence  $\mathcal{L}(T)$  is lower semimodular. Here  $ab = g^5 \notin E_T \cup \langle a \rangle \cup \langle b \rangle$ .

### 4.3 $M^*$ -symmetry.

Since  $M^*$ -symmetry implies lower semimodularity, the decompositions in Theorem 4.2 remain valid. Moreover, in the convex case,  $\mathcal{C}o(E_S)$  is  $M^*$ -symmetric if and only if  $E_S$  is a tree, once more, by Result 2.1. Therefore the description in this case is obtained from Theorem 4.5 mutatis mutandi. Note, however, that as stated in Result 2.6, the simple semigroups for which the lattice  $\mathcal{L}\mathcal{F}$  is  $M^*$ -symmetric are precisely those for which the lattice is modular. In [4], it is shown that this class is properly contained in the class of simple semigroups for which  $\mathcal{L}\mathcal{F}$  is lower semimodular.

In the case of  $\mathcal{L}(S)$ , the description is obtained from Theorem 4.9 mutatis mutandi.

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