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# Prospective K-8 Teachers' Knowledge of Relational Thinking

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# Prospective K-8 Teachers' Knowledge of Relational Thinking

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**Abstract:** The goal of this study was to examine two issues: First, pre-service teachers' ability and inclination to think relationally prior to instruction about the role relational thinking plays in the K-8 mathematics curriculum. Second, to examine task specific variables possibly associated with pre-service teachers' inclination to engage in relational thinking. The results revealed that preservice teachers engage in relational thinking about equality, however, their inclination to do so is rather limited. Furthermore, they tend to engage in relational thinking more frequently in the context of arithmetic than algebra-related tasks. Pre-service teachers' inclination to engage in relational thinking appeared to also relate to the overall task complexity and the use of

variables. Implications of these findings for pre-service teacher education are provided.

**Key words:** Relational thinking, early algebra Instruction, pre-service teacher preparation

## Introduction

The results of international assessments (i.e., the Trends in International Mathematics and Science Study, TIMSS) consistently document U.S. students' low performance in mathematics. Although the results of the 2007 TIMSS assessment revealed improvement in U.S. students' relative mathematics performance overall, interpreting U.S. students' algebra skills over several years shows that they have an insufficient understanding of the knowledge and skills of algebra, characterized by the ability to apply basic mathematical knowledge only in straightforward situations (Gonzales, Williams, Jocelyn, Roey, Kastberg, Brenwald, 2008). Given that algebra knowledge and skills are considered essential for educational and employment opportunities, students' low algebra performance has been a long and growing concern of mathematics educators and policymakers. For that reason an emphasis of recent reform efforts in mathematics education has been placed not only on algebra curricula and algebra instruction but also on the preparation of mathematics teachers (e.g., National Council of Teachers of Mathematics, 1997, 2000; National Research Council, 1998, RAND Mathematics Study Panel, 2003). In particular, at the K-8 level, inclusion of algebra-based concepts into the mathematics curriculum necessitates drastic changes in how mathematics is being taught in the elementary and middle grades. The instructional changes draw attention to the adequate preparation of K-8 mathematics teachers' to effectively implement early algebra instruction.

Teachers' knowledge has been recognized as an increasingly complex phenomenon that extends well beyond knowing mathematical content well. Teachers' own mathematical competency as well as their abilities to identify mathematical ideas in the context of different solution approaches, to extend and generalize different mathematical concepts within the mathematics curriculum, to select mathematically

rich task, and to analyze students' mathematical thinking are but a few examples of aspects of teachers' broad knowledge identified as essential for effective mathematics teaching (Ball, Lubienski & Mewborn, 2001; Hill & Ball, 2004; Ball, Hill & Bass, 2005; Hill 2010; Usiskin, 2001). This paper addresses one aspect of teachers' broad knowledge, namely pre-service teachers' knowledge of relational thinking.

### **Transition from Arithmetic to Algebra: Relational Thinking about Equality**

Mathematics education researchers interpret students' transition from arithmetic to algebra as a continuum along which students progress from considering numerical relationships for a problem or mathematical situation to generalizing and representing these relationships with the symbols of algebra. Warren (2003) states that while making this transition, students map the abstract processes of operating on or with unknowns (algebra) onto their preexisting models of arithmetic. The essential shift from arithmetic to algebraic thinking is marked by students' ability to investigate and analyze relationships. The notion of relational thinking about equality is at the heart of this process. Carpenter, Levi, Franke, and Zeringue (2005) described relational thinking about equality as examining relations among quantities using the fundamental properties of equality, numbers, and operations, rather than, examining quantities as a sequence of steps or procedures. The understanding of equality, on which relational thinking hinges, and in particular the understanding of the equal sign, is viewed as fundamental for the learning of algebra (Alibali, 1999; Kieran, 1981; McNeil & Alibali, 2005a, b). For example, Knuth, Stephens, McNeil, & Alibali, (2006) found a strong positive relationship between students' understanding of equality and their performance on solving equations. Research documents that students who consider equality in a relational way are flexible in connecting their numerical thinking (i.e. thinking centered on analyzing numbers and operations to produce a single number answer) to algebraic thinking (i.e. thinking centered on analyzing patterns and relations). Relational thinking about equality serves as a bridge between arithmetic and algebraic structures. Relational understanding of equality, including interpreting the equal sign as a symbol of equivalence, in contrast to thinking

about the equal sign as a signal to write down an answer (an operator symbol), supports students in making a successful transition from the study of arithmetic to the study of algebra (Kieran, 1981; Knuth, Alibali, Weinberg, McNeil, & Stephens, 2005; Knuth et al, 2006; Hunter, 2007).

Traditionally introduced in the early elementary grades, the concept of the equal sign is given little explicit attention in the subsequent grades and many elementary students demonstrate a limited understanding of the concept of equality (Knuth et al. 2006). Research shows that many elementary students interpret the equal symbol as an operator symbol, i.e., an invitation to perform an operation (Falkner, Levi & Carpenter, 1999; Molina & Ambrose, 2008; Knuth et al., 2005, 2006; Barody & Ginsburg, 1983; Carpenter, Franke & Levi, 2003). Although students' understanding of the equal sign is less known beyond the elementary grades, documentation exists, that some middle and high school students also display a tendency to think about the equal sign as an operator (Behr, Erlwanger, & Nichols, 1980; McNeil & Alibali, 2005a, b; McNeil, Grandau, Knuth, Alibali, Stephens, Hattikudur, Krill, 2006; Alibali, Knuth, Hattikudur, McNeil & Stephens, 2007).

Despite that relational thinking about equality is identified as essential to students' successful transition from arithmetic to algebra, very little attention has been paid to teachers' knowledge of relational thinking about equality. However, research done by Stephens (2006) and Asquith, Stephens, Knuth and Alibali (2007) strongly support the need to focus on, and strengthen, teachers' knowledge of relational thinking. In particular, both groups of authors uncovered that practicing and pre-service teachers alike demonstrate a limited awareness of students' understanding of the concept of equality, and of the implications that an insufficient understanding of equality has on students' learning of mathematics. The research described in this paper sought to provide further understanding of teachers' knowledge in the domain of relational thinking. This research is part of a larger study which examined three dimensions of pre-service K-8 teachers' knowledge of relational thinking, broadly defined: (1) their own relational thinking ability, (2) their ability to identify and analyze students' thinking about equality, and (3) their ability to analyze a

task's potential to engage students in relational thinking about equality. The discussion of pre-service teachers' relational thinking in this paper is limited to the first of the three dimensions, for which we investigated pre-service K-8 teachers' *ability* and *inclination* to engage in relational thinking about equality. Reaching beyond merely identifying pre-service teachers' ability to think relationally about equality, our goal was to identify: (1) pre-service teachers' inclination to engage in relational thinking in the context of arithmetic- and algebra-based tasks; and, (2) task specific variable(s) that appear to be associated with pre-service teachers' selection of relational thinking as a viable strategy for solving a task.

Research-based information related to pre-service K-8 teachers' relational thinking is essential for the design of teacher-education programs that focus on effective ways to prepare K-8 teachers for the challenges of early algebra instruction. The accounts of pre-service teachers' relational thinking described in this report characterize pre-service teachers' ability and inclination for relational thinking examined prior to instruction in a teacher preparation program that emphasizes the role relational thinking plays in the learning of K-8 mathematics.

## **Conceptual Framework**

Our conception of pre-service teachers' knowledge of relational thinking about equality draws on the existing research on teachers' knowledge and on early algebra instruction. Given that relational thinking is fundamental for making meaningful connections between the concepts of arithmetic and the concepts of algebra, pre-service teachers (1) must not only be able to think relationally about equality, but also (2) should spontaneously consider relational thinking about equality as a viable strategy for solving a task. For the purpose of this study, therefore, we conceptualized pre-service teachers' knowledge of relational thinking in terms of their *ability* and *inclination* to engage in relational thinking about equality in the context of arithmetic and algebra-related tasks. Our perspective is that pre-service teachers' preparedness to engage students in relational thinking might depend not only on pre-service teachers' ability to think relationally but also on their inclination to do so. Research shows that most pre-service teachers demonstrate rote and procedural understanding of school

mathematics (Ball, 1990; Van Dooren, Verschaffel, & Onghena, 2002). If pre-service teachers ought to foster relational thinking in their future students, they have to themselves engage in relational thinking spontaneously, and move beyond their inclinations for procedural (computational) thinking about equality.

To operationalize pre-service teachers' relational thinking about equality we examined the existing mathematical education literature (Carpenter, Levi, Franke, & Zerinque, 2005; Stephens, A., 2006; Stephens, M. 2006) for descriptive accounts of relational thinking about equality. Our operational definition is summarized in Table 1.

*Table 1*

**Framework for Analyzing Pre-service Teachers' Relational Thinking**

Pre-service Teachers' Relational Thinking	Operational Description
Ability to engage in relational thinking	(a) Demonstrates relational understanding of the equal sign in the context of solving arithmetic and algebra-based problems, and (b) Analyzes equations and expressions as a whole rather than a set of procedures to be followed while considering <ul style="list-style-type: none"><li>-properties of equality</li><li>-properties of numbers</li><li>-properties of operations</li></ul>

## Research Questions

The following two questions guided our investigation:

- (1) To what extent do pre-service teachers spontaneously engage in relational thinking in the context of arithmetic and algebra-related tasks prior to the instruction on relational thinking?
  - (a) How do they explain the meaning of the equal sign when directly asked?
  - (b) What is their ability and inclination to engage in relational thinking?
- (2) What task variables might affect pre-service teachers' inclination to engage in relational thinking?

## Methodology

### Participants

Featured in this report are 32 undergraduate students (31 females and 1 male) enrolled in a K-8 teacher preparation program in a large private university in the Midwest. Ten were enrolled in a Number Systems and Operations course and 22 were enrolled in an Algebra and Geometry course. In both mathematics courses the pre-service teachers examined relevant mathematics content and engaged in activities concerned with analyzing students' mathematical thinking. In the context of the mathematics problems, they also discussed ways of mathematical thinking that a given problem might evoke. Our decision to select both courses for the study of pre-service teachers' knowledge of relational thinking was motivated by the fact that both courses were integrated with an education field-experience course. The education field-experience course coupled with the Number Systems and Operations mathematics course focused on elementary students' development of arithmetic ideas. During the fieldwork related to this course the pre-service teachers interacted with a small group (3-4) of 3rd and 4th grade students in the University After-School Learning Lab studying elementary students' conceptions of equality. The focus of the education field-experience course in the Algebra and Geometry mathematics content course was on middle school students' difficulties in making the transition from arithmetic to algebra. The pre-service teachers conducted weekly class observations of middle school mathematics instruction and one-on-one tutoring sessions with a selected middle school student.

### Data Sources

The data reported in this paper came from a written test administered at the beginning of the semester in the two content courses. The test was administered prior to any class discussion about relational thinking about equality and its role in the K-8 mathematics curriculum. From the existing literature on relational thinking about equality, we purposefully selected ten tasks to (1) probe pre-service teachers' interpretations of the meaning of the equal sign, and (2) probe their ability and inclination to engage in relational thinking.



While one of the ten tasks explicitly asked for an explanation of the meaning of the equal sign, the remaining nine tasks were selected to identify (a) strategies that pre-service teachers implement and (b) whether or not pre-service teachers engage in relational thinking spontaneously. Each of the nine tasks prompted for providing at least two different solution strategies. We assumed that pre-service teachers who spontaneously engage in relational thinking respond to these tasks using relational thinking strategy as their first. (None of the participants provided more than two strategies for any individual task).

## **Data Analysis**

In this section we provide details about the coding of pre-service teachers' responses to each of the ten tasks. We use examples of pre-service teachers' responses to selected tasks to illustrate how we applied the coding schema.

Our coding schema drew on the operational definition of relational thinking about equality (Table 1). We analyzed and coded responses our pre-service teachers in order to identify: (a) how pre-service teachers' interpret the equal sign when explicitly asked, (b) whether pre-service teachers think about a given task relationally or not (c) whether or not pre-service teachers engaged in relational thinking about equality spontaneously or not, and (d) which specific relational thinking strategy was used. To do so, we rated pre-service teachers' responses on a 3-point scale.

### **Coding responses to Task 1: The meaning of the equal sign.**

Task 1 was adapted from Knuth et al., (2005) to explicitly prompt pre-service teachers' explanations about the meaning of the equal sign: "*The arrow points to a symbol. What is the name of this symbol, what does this symbol mean?*  $5+3 = 8$ ," (Task 1).

A response was coded (3) *relational* if the pre-service teacher explained the meaning of the equal sign referring to "sameness" in general terms, without explicitly focusing on computing and comparing values (quantities) on both sides of the equation. The explanation of the meaning of the equal sign was coded (2) *computational* if the pre-

service teacher focused on "sameness" of computational results, that is, emphasized the meaning of the equal sign in terms of producing the "same" results by operating on both sides of the equation. Finally, a response was coded (1) *operational*, when the included explanation carried a notion of an operational view of the equal sign. That is, an indication that the equal sign served as a prompt to perform an operation. Included in Figure 1 are selected responses of three pre-service teachers (#31, #8, and #6) to illustrate the coding of interpretations of equal sign responses.

<p>"[the equals symbol] means that the amounts are relationally the same, no matter what combination of symbols, or numbers you put on either side," (PST #31).</p> <p>"The symbol means that each side of the equal sign holds the same value. The sides are usually in different form e.g., <math>5 + 3 = 4 + 4</math> but an answer to each side is the same" (PST #8).</p> <p>"The symbol means what the sum of the numbers is," (PST # 6).</p>
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*Figure 1.* Pre-service teacher #31, #8, and #6's explanations of the meaning of the equal sign.

The response of pre-service teacher #31 expresses a general idea of "sameness" on both sides of the equation without a specific focus on computing answers. Thus we coded this response as (3) proficient. In contrast, pre-service teacher #8 explained the meaning of the equal sign with a strong focus on computational "sameness" emphasizing that the "...answer on each side is the same." Given the notion of computational sameness, we assessed this explanation as (2) emerging. Finally the notion of the meaning of the equal sign given in the explanation of preservice teacher #6 was that of an operator symbol, ". . . what the sum of the numbers is." Thus we scored this response as (1) operational.

**Coding responses to Tasks 2-10: Arithmetic and algebra-related tasks.** For arithmetic- and algebra-related (equation solving) tasks, we coded the strategies that led to a task solution. A response was coded (3) *spontaneous* to denote that a pre-service teacher demonstrated the ability to think relationally about equality and did so spontaneously by selecting a relational strategy as his or her first solution strategy. A response was coded (2) *prompted* to denote that, in the context of a task, the pre-service teacher engaged in relational thinking about equality, however did not use relational thinking

spontaneously (i.e., the second strategy used was relational, but not the first). Finally, a response was coded (1) *not evident* to denote that a pre-service teacher did not engage in relational thinking in the context of a given task (neither the first nor second strategy was relational). In addition to the 3-point coding schema we used the existing literature concerned with relational thinking strategies to define a set of *a priori codes* and used these definitions to code specific relational thinking strategies recognized in the pre-service teachers' responses. The examples of pre-service teacher #13's and #18's responses to Task 6 (Figures 2 and 3) serve to illustrate how we applied our coding system to rate responses to the remaining nine tasks (Task 2 -10).

6. Find at least two different ways to solve this problem:  $99 + 87 = 98 + 86 + c$   
Clearly explain your reasoning for each solution approach.

①  $99$  is 1 more than  $98$ .  $87$  is 1 more than  $86$ .  
so  $1+1=2$   $c=2$

② solve problem:  
$$\begin{array}{r} 99+87 = 98+86 + c \\ \hline 186 = 184 + c \\ -184 \quad -184 \\ \hline 2 = c \end{array}$$
  
using standard algebra

Figure 2. PST #13 responses to Task 6

Pre-service teacher #13's written solution reveals that her first strategy was based on the examination of the relationships among the quantities on both sides of the equation. The answer she provides for this problem results from examining the relationship between the magnitudes of the numbers on both sides of the equation, rather than from computing and comparing the quantities on both sides, as in the case of her second solution. Given that this pre-service teacher demonstrated her ability to think relationally and did so spontaneously (first solution relational), we rated her relational thinking ability and inclination for this task as (3) spontaneous. We also coded the pre-service teachers' specific relational strategy as thinking about the differences in the magnitudes of the numbers (DM) that transpired from her relational thinking response. Similarly, the relational thinking

of a pre-service teacher who would have used a relational strategy as his or her second strategy would have been rated as (2) prompted, and followed up with a specific code to characterize the specific relational thinking strategy used.

The response of pre-service teacher #18 (Figure 3) contrasts that of pre-service teacher #13 (Figure 2). Both strategies generated by pre-service teacher #18 indicate that this participant engaged in thinking about a standard set of procedures to find the missing value  $c$ , rather than thinking about the relationship between the quantities on both sides of the equation. According to our scoring rubric we rated this response as (1) not evident (of relational thinking).

6. Find at least two different ways to solve this problem:  $99 + 87 = 98 + 86 + c$   
Clearly explain your reasoning for each solution approach.

$99 + 87 = 98 + 86 + c$ $186 = 184 + c$ $-184$ $2 = c$ Add $99 + 87 = 186$ Add $98 + 86 = 184$ subtract 184 to isolate $c$ $2 = c$	$99 + 87 = 98 + 86 + c$ $\frac{99}{+87} = \frac{98}{+86}$ $\frac{186}{186} = \frac{184 + c}{184 + c}$ $186 = 184 + c$ $\frac{184 + c = 186}{184 + 0 = 184}$ $184 + 1 = 185$ $184 + 2 = 186$ using a list I found that $184 + 2 = 186$ so $c = 2$
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Figure 3. PST #18 Responses to Written Task 6

To assess the reliability of the coding schema, each of the three authors coded approximately 20% of the data. The results and the interpretation of the coding schema were discussed until 100% agreement was reached. Following this process, the first author coded and recoded the remaining parts of the data.

## Results

### Research Question 1

For our first research question we investigated the extent to which pre-service teachers engage in relational thinking while responding to arithmetic- and algebra-related tasks. We also analyzed how pre-service teachers explain the meaning of the equal sign when directly asked. We begin our presentation of results by describing the pre-service teachers' interpretations of the equal sign. Subsequently, we present the results concerning the pre-service teachers' relational thinking ability and inclination.

**Equal sign.** Figure 4 gives a summary of the pre-service teachers' interpretations of the equal sign. When directly asked, 56% of the pre-service teachers explained the meaning of the equal sign in terms of computational sameness, emphasizing the need to compute and compare answers on both sides of the equation. We coded these responses "Computational."

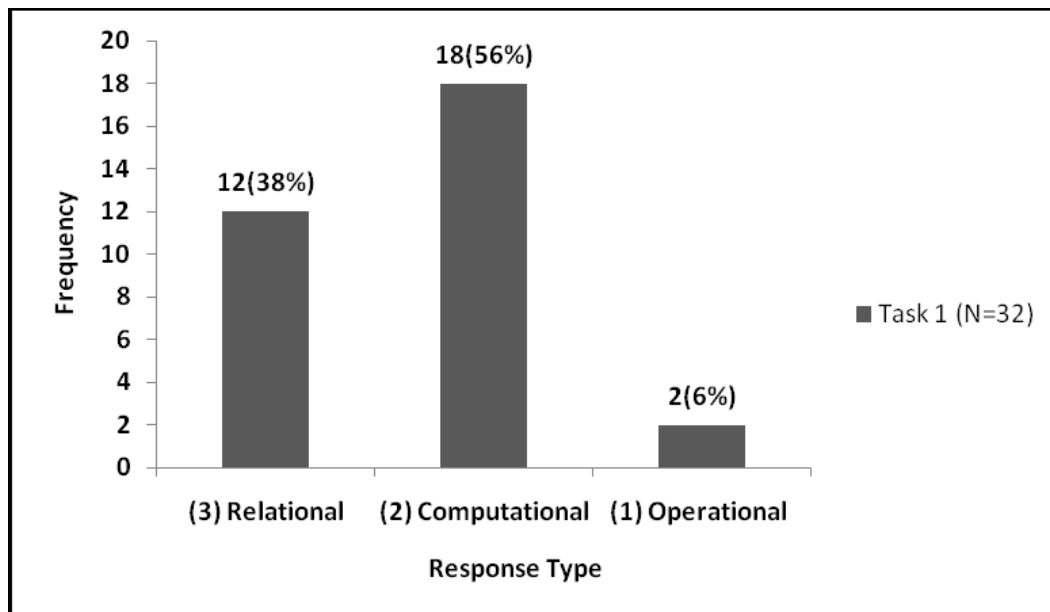


Figure 4. Distribution of equal sign interpretations.

Only 12 of the 32 participants (38%) interpreted the equal sign as an indicator of general “sameness” without emphasizing computation and comparison. We coded their responses “Relational.” Two of the 32 pre-service teachers (6%) interpreted the equal sign as an operator symbol. We coded their responses “Operational.” There was no significant difference between the frequencies of the relational and computational responses. As expected, however, there were significantly fewer operational responses than relational ( $z=2.72$ ,  $p<0.01$ ) or computational responses ( $z=4.05$ ,  $p<0.01$ ).

**Relational thinking ability and inclination.** Figure 5 presents the collection of tasks that we used to study the pre-service teachers’ ability and inclination to engage in relational thinking about equality. We analyzed 283 responses from the 32 pre-service teachers on the nine tasks (omissions were excluded from the analysis).

<b>Task 2:</b> Find at least two different ways to solve this problem: $16 + 15 = 31$ is true. Is $16 + 15 - 9 = 31 - 9$ true or false? Clearly explain your reasoning for each solution approach.
<b>Task 3:</b> Find at least two different ways to solve this problem: $36 + 53 = a + 55$ . Clearly explain your reasoning for each solution approach.
<b>Task 4:</b> Find at least two different ways to solve this problem: $44 + 29 = 23 + 45 + a$ . Clearly explain your reasoning for each solution approach.
<b>Task 5:</b> Find at least two different ways to solve this problem: $65 + 38 = 62 + 39 + b$ . Clearly explain your reasoning for each solution approach.
<b>Task 6:</b> Find at least two different ways to solve this problem: $99 + 87 = 98 + 86 + c$ . Clearly explain your reasoning for each solution approach.
<b>Task 7:</b> Find at least two different ways to solve this problem: The solution to the equation $2k + 17 = 35$ is $k = 9$ . What is the solution to the equation $2k + 17 - 8 = 35 - 8$ ? Clearly explain your reasoning for each solution approach.
<b>Task 8:</b> Find at least two different ways, other than by using the standard algorithms, to solve this problem: $178 + 99$ . Clearly explain your reasoning for each solution approach.
<b>Task 9:</b> Find at least two different ways, other than by using the standard algorithms, to solve this problem: $500 - 199$ . Clearly explain your reasoning for each solution approach.
<b>Task 10:</b> Find at least two different ways, other than by using the standard algorithms, to solve this problem: $153 - 70$ . Clearly explain your reasoning for each solution approach.

Figure 5. Written test tasks.

Figure 6 shows the distribution of the 283 analyzed responses. Relational thinking about equality (Spontaneous or Prompted) was evident in 67.5% of the pre-service teachers’ solutions. However, only

38.5% of the responses were spontaneously relational. In eighty-two of the 283 responses (29%), the prospective teachers responded relationally only in order to generate a second solution. Finally, 92 of the 283 responses (32.5%) did not provide any evidence that the pre-service teachers engaged in relational thinking. These results suggest that although the preservice teachers are able to think relationally about equality they might not always consider relational thinking about equality as a viable strategy for solving a task.

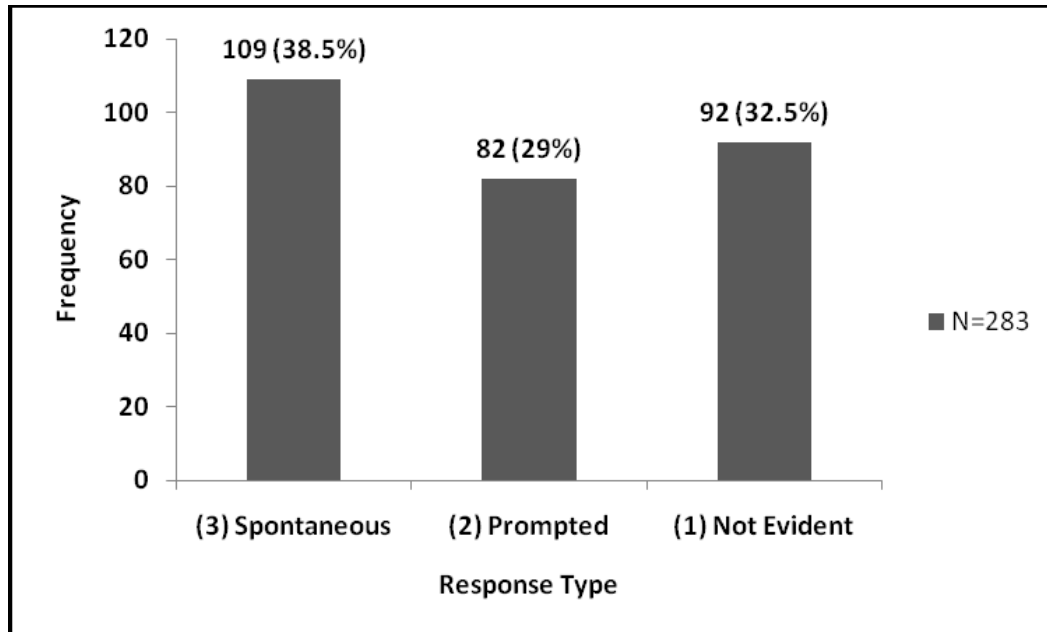


Figure 6. Distribution of pre-service teachers' responses across the collection of tasks.

When comparing instances of spontaneous and prompted relational thinking, a promising result was that overall, across the collection of tasks, prospective teachers engaged in relational thinking about equality significantly more often on their own (spontaneously) than when prompted ( $z=2.31, p<0.01$ ). They also demonstrated relational thinking (score 3 or 2) significantly more often than not (score 1) ( $z=8.2, p<0.01$ ). The overall mean relational thinking (RT) score, computed as an average of ratings across tasks and all participants, was  $M=2.02, SD=0.40$ .



## Research Question 2

For our second research question we tried to determine the task variables that influenced the pre-service teachers' inclination to think relationally. In particular, we investigated whether pre-service teachers were more inclined to engage in relational thinking about equality when they solved algebra-related tasks than when they solved arithmetic-related tasks. We also investigated how pre-service teachers' use of relational thinking differed depending on whether they solved tasks for which relational thinking about equality necessitated the use of properties of equality, properties of operations, or the analysis of the differences in the magnitude of the numbers.

Included in Table 2 is a task-specific summary of the pre-service teachers' responses to each of the nine tasks. In the discussion that follows, we analyze similarities and differences in the problems that may account for the similarities and differences in the distribution of the preservice teachers' use of relational thinking to solve the tasks.

Table 2  
*Task specific summary of pre-service teachers' responses*

Task	N*(100%)	(3) Spontaneous	(2) Prompted	(1) Not evident
2	31	14 (45.2%)	9 (29%)	8 (25.5%)
3	32	5 (15.6%)	18 (56.3 %)	9 (28.1%)
4	32	0 (0%)	16 (50%)	16 (50/0%)
5	32	5 (15.6%)	13 (40.6%)	14 (43.8%)
6	32	10 (31.3%)	11 (34.4%)	11 (34.4%)
7	32	7 (21.9%)	6 (18.8%)	19 (59.4%)
8	31	23 (74.2%)	4 (12.9%)	4 (12.9%)
9	31	23 (74.2%)	2 (6.5%)	6 (19.4%)
10	30	22 (73.3%)	3 (10%)	5 (16.7%)
Total	283	109 (38.5%)	82 (29%)	92 (32.5%)

\*Number of prospective teachers responding to each task

**Tasks 2 and 7: Recognizing the additive property of equality.** Thinking relationally about Tasks 2 and 7 (Figure 5) requires one to recognize the additive property of equality. For Task 2, one needs to realize that the true statement remains true after adding the same quantity (-9) to both sides of the equation. A similar way of reasoning suffices for Task 7: the solution to the equation in Task 7 remains the same ( $k=9$ ) after adding -8 to both sides of the equation. Fifty-seven percent of the aggregated responses to these two tasks included the evidence that the preservice teachers' used the additive



property of equality to support relational thinking (Figure 7). However, only 33% of responses were Spontaneous (score 3). There was no significant difference between the proportions of Spontaneous and Prompted responses. However, there were significantly fewer Prompted responses than Not Evident responses ( $z=2.08, p<0.01$ ).

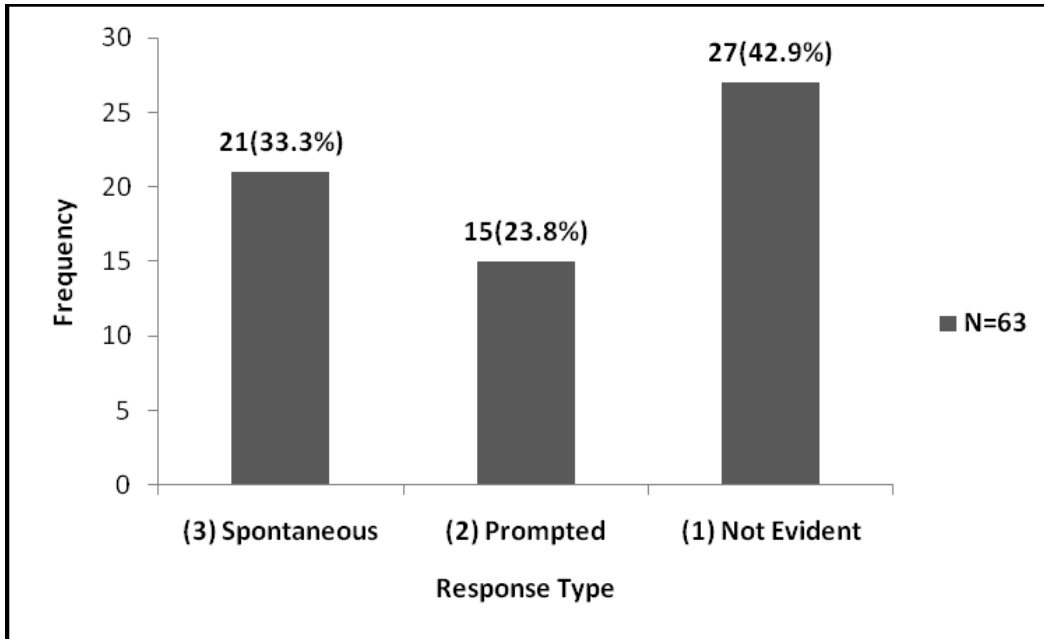


Figure 7. Distribution of response types for Tasks 2 and 7, combined.

Despite that both tasks fostered use of the same property of equality, pre-service teachers engaged in relational thinking for Task 2 far more often than they did for Task 7. As illustrated in Figure 8, 23 (74.2%) responses to Task 2 used relational thinking (Spontaneous or Prompted) in contrast to only 13 (40.1%) responses to Task 7. The difference in the proportions of relational thinking responses to these two tasks was significant ( $z=2.44, p<0.01$ ). The same was true of the Spontaneous responses to these two tasks: a greater proportion of the responses to Task 2 were Spontaneous than were the responses to Task 7 ( $z=1.69, p <0.05$ ).

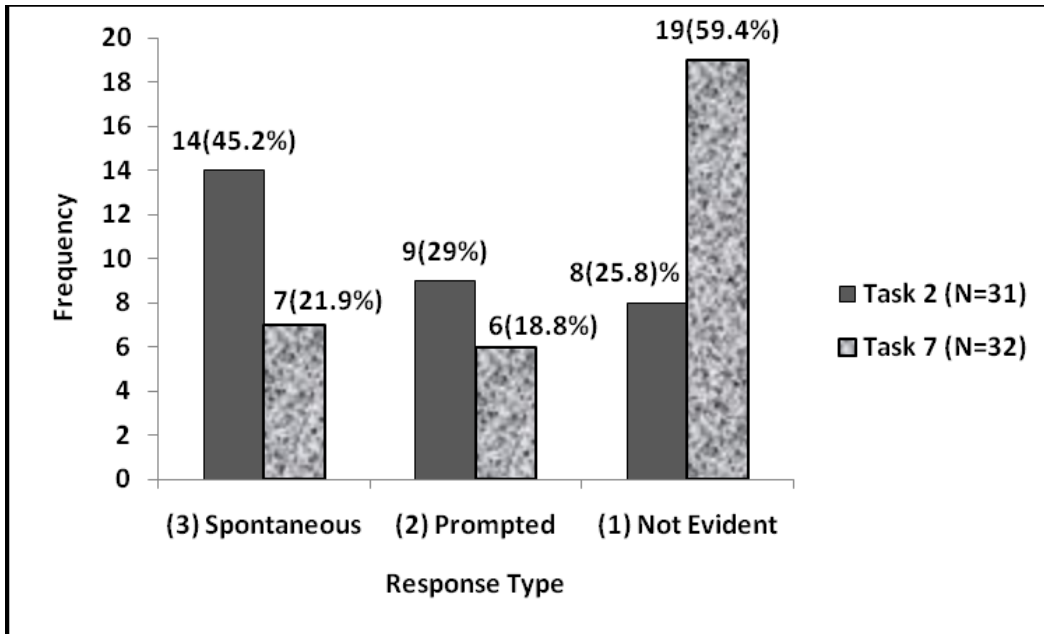


Figure 8. Comparison of pre-service teachers' responses to Tasks 2 and 7.

Given that Task 2 appears to be more arithmetic-related while Task 7 appears more algebra-related, these results might indicate that pre-service teachers are less likely to consider relational thinking when solving a task that includes a variable. Relational thinking was not evident in 59% of the responses. Figure 9 shows a typical Not Evident response, since neither of pre-service teacher #8's two strategies uses relational thinking about equality. While both solutions provide evidence of an awareness of the additive property of equality and additive inverse, they both give a rather strong sense that pre-service teacher #8 interprets equality as computational sameness. For Task 2, similar reasoning (based on computation) was found in only 25.5% of responses.



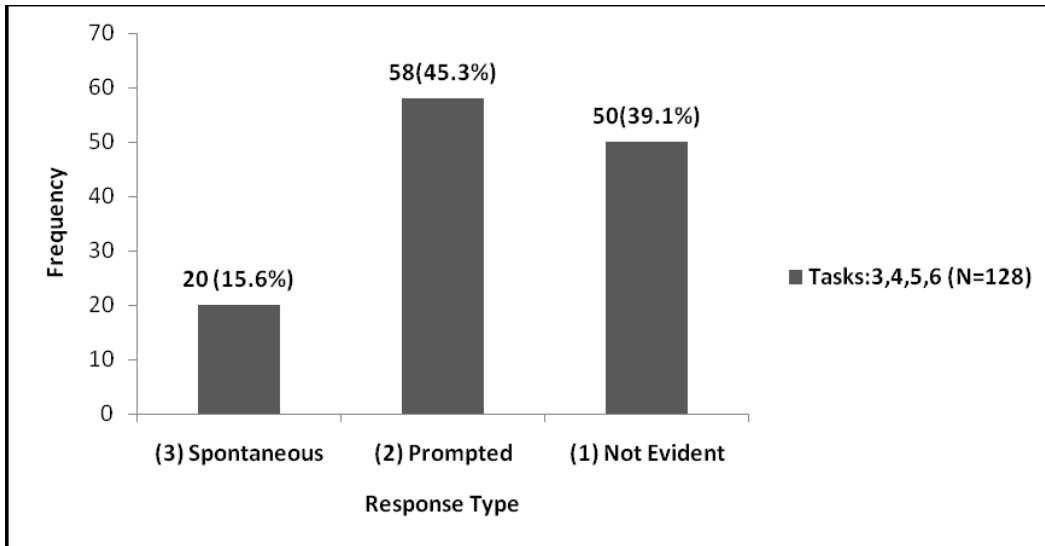


Figure 10. Overall distribution of responses across tasks promoting thinking about the differences in the magnitude of the numbers

Included in Figure 11 is a distribution of responses for each of these four tasks.

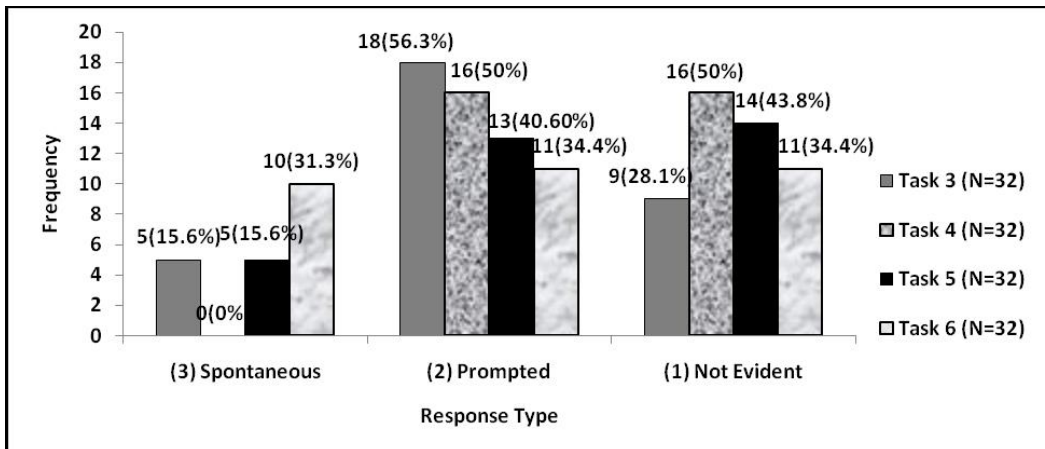


Figure 11. Comparison of responses across the four tasks

Pre-service teachers' engaged in relational thinking spontaneously more frequently for Task 3 than they did for Task 4 ( $z=1.86, p<0.01$ ). Similarly, they spontaneously considered relational thinking about equality more frequently for Task 5 than they did for Task 4 ( $z=1.86, p<0.05$ ) and for Task 6 than they did for Task 4 ( $z=3.10, p <0.01$ ). Interestingly, within this subgroup of tasks only

Task 4 necessitated considering the commutative property of addition to analyze the differences in the magnitude of the numbers. Thus, the pre-service teachers' greater inclination to think relationally about Tasks 3, 5, and 6 could be attributed to the reduced complexity of the tasks. From the opposite point of view, the reason that there were no significant differences in the pre-service teachers' inclination to use relational thinking to solve Tasks 3, 5, and 6 may have been that none of these tasks required a consideration of both commutativity and differences in the magnitude of numbers. A possible association between the structure (and a complexity) of a task and one's inclination to engage in relational thinking about equality in the context of that task deserves further study. In particular, a follow up interview conducted with pre-service teachers could uncover some mechanisms that possibly explain preservice teachers' strategy selection (relational or not) within this group of tasks.

**Tasks 8, 9, and 10: Relational thinking as a tool for computation.** Although not explicitly associated with the equal sign, Tasks 8, 9, and 10 were selected to foster relational thinking about equality in the context of mental computation. For example, to find the sum of  $178 + 99$  (Task 8) one could reason that the sum of 178 and 99 will remain the same after decreasing 178 by 1 and increasing 99 by 1:  $178 + 99 = (178-1) + (99+1) = 177 + 100$ . One could use a similar, but arguably more difficult, line of reasoning to find  $500 - 199$  (Task 9). In this case, the difference remains the same after increasing 500 by 1 and also increasing 199 by 1:  $500 - 199 = (500 + 1) - (199 + 1) = 501 - 200$ . For these tasks pre-service teachers were restricted from the use of the standard addition or subtraction algorithms.

For this group of tasks as a whole, 83.7% of the aggregated responses indicated that the pre-service teachers used relational thinking about equality (Spontaneous or Prompted). Relational thinking was Not Evident in only 15 of the 92 aggregated responses (16.3%) (Figure 12).

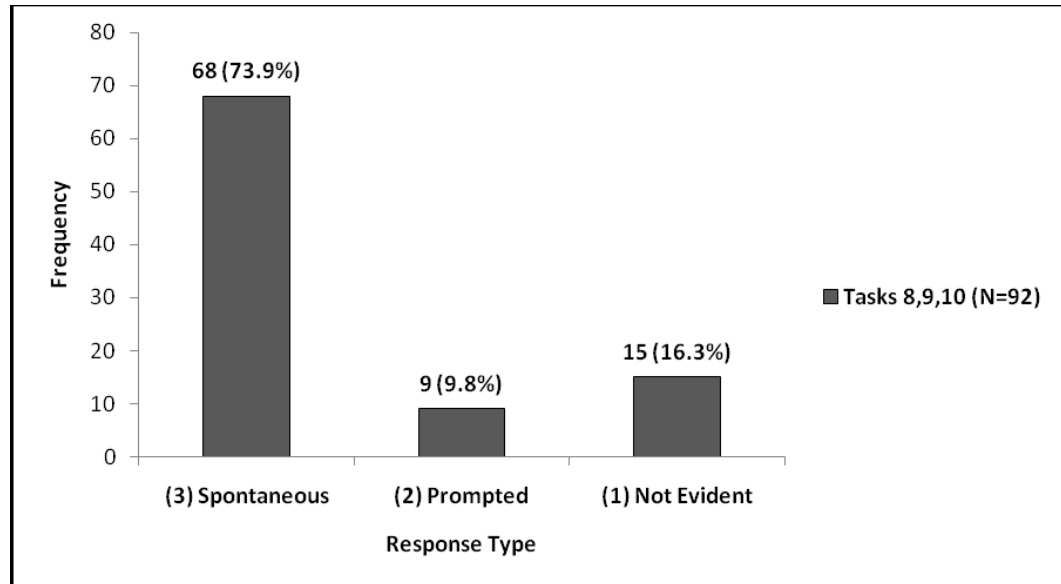


Figure 12. Distribution of responses to mental arithmetic tasks.

Overall pre-service teachers' engaged in relational thinking (score 3 or 2) in the context of these tasks significantly more often than they did not ( $z=8.99, p<0.01$ ). Moreover, for this group of tasks, pre-service teachers engaged in relational thinking spontaneously (score 3) significantly more often than they did not ( $z=6.34, p<0.01$ ). No significant differences were found within the distributions of the pre-service teachers' relational thinking responses to tasks 8, 9, or 10.

**Comparison of arithmetic- and algebra-related tasks.** We also examined possible differences in our pre-service teachers' use of relational thinking by considering whether the tasks were arithmetic- or algebra-related. We considered Tasks 2, 8, 9, and 10 as arithmetic-related (Group 1) and Tasks 3, 4, 5, 6, and 7 as algebra-related (Group 2). Figure 13 summarizes the distribution of pre-service teachers' responses to the arithmetic and algebra-related tasks as a

whole.

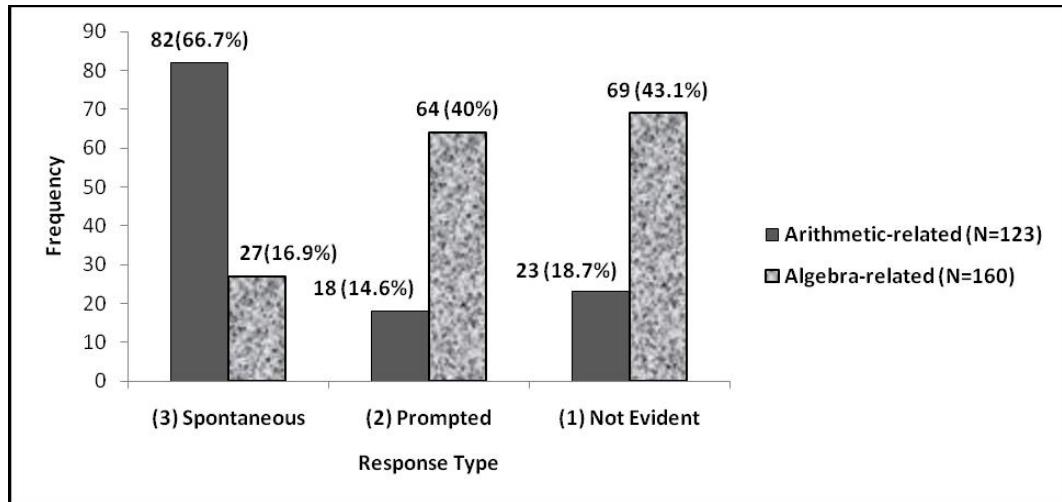


Figure 13. Distribution of responses for arithmetic- and algebra-related tasks.

Overall, the pre-service teachers' engaged in relational thinking (Spontaneous or Prompted) significantly more often when solving arithmetic-related tasks than algebra-related tasks ( $z=4.22, p<0.01$ ). They also significantly more often demonstrated spontaneous relational thinking for arithmetic- than algebra-related tasks ( $z=7.15, p<0.01$ ). Included in Table 3 is a comparison of the mean scores for arithmetic- and algebra-related tasks as a whole. The mean RT score for arithmetic-related tasks was significantly higher than the mean RT score for algebra-related tasks ( $t(31)=5.64, p<0.01$ ), however the arithmetic- and algebra-related RT scores were not correlated. This latter result suggests that pre-service teachers are more able and more inclined to engage in relational thinking when solving arithmetic-related tasks than when solving algebra-related tasks.

Table 3.

Comparison of mean RT scores for arithmetic- and algebra-related tasks

Task Nature	N	$\bar{M}$	SD
Arithmetic-related	123	2.46	0.56
Algebra-related	160	1.74	0.46

Moreover, when considering only those arithmetic and algebra-related tasks that facilitated thinking about the differences in the magnitude of the numbers (Figure 14) pre-service teachers more frequently considered thinking about the differences in the magnitude of the numbers in the context of arithmetic-related tasks (Tasks 8, 9, 10) than they did in the context of algebra-related tasks (Tasks 3, 4, 5, 6) ( $z=8.61, p<0.01$ ).

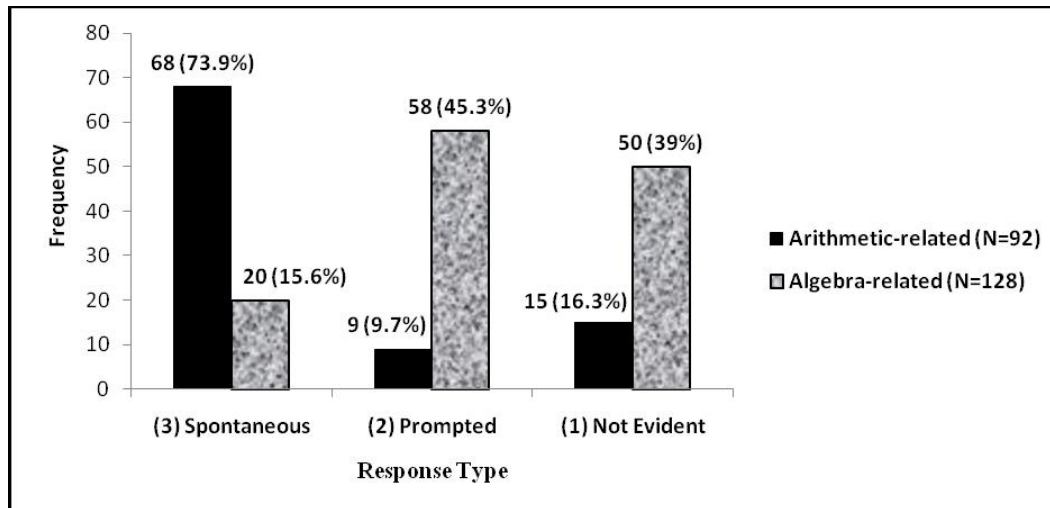


Figure 14. Distribution of responses to arithmetic- and algebra-related tasks that fostered thinking about differences in the magnitude of the numbers.

## Discussion and Implications

The study reported in this paper helps fill the gap in the mathematics education literature about pre-service teachers' readiness for early algebra instruction. Our primary goal was to provide an understanding of pre-service teachers' relational thinking prior to the instruction they received in a teacher education program. Our intention was to provide direction for K-8 teacher preparation programs concerned with preparing pre-service teachers for the challenges of early algebra instruction. Hill (2010) emphasized the need for fine-grained analyses of different aspects of teachers' content (and pedagogical) knowledge, making a case that this type of understanding is necessary for designing teacher preparation programs that prepare high quality mathematics teachers. In response



to the need for such fine-grained analyses, our study reached beyond identifying pre-service teachers' relational thinking ability. In our study we not only examined pre-service teachers' relational thinking ability, but also (1) identified the extent to which pre-service teachers spontaneously use relational thinking, and (2) examined possible task specific variable(s) that could relate to pre-service teachers' inclination to consider relational thinking as a viable strategy for solving tasks.

Our finding about the pre-service teachers' ability to use relational thinking was promising. The pre-service teachers in our study demonstrated a relatively high ability to think relationally. Across the 283 analyzed responses, pre-service teachers used relational thinking about equality (spontaneously or prompted) in 67.5% of their solutions.

The ability to think relationally, however, is not the same as having the inclination to use relational thinking across a wide variety of situations. In answer to research question one, we found that although our pre-service teachers demonstrated the ability to engage in relational thinking about equality on more than two-thirds of the 283 tasks. However, only 38.5% of their responses revealed that they did so spontaneously. This result suggests there is a need for teacher education programs to emphasize the value of relational thinking and the effect that relational thinking has on students' learning. Thus, it appears that an important goal of teacher education programs may well be to increase pre-service teachers' use of relational thinking strategies, as well as the benefits of using relational thinking with K-8 students.

The answer to question two of our research provides a more fine-grained understanding of our pre-service K-8 teachers' relational thinking. We investigated task specific variables that might possibly be associated with the pre-service teachers' choice of strategies (relational or not) to solve arithmetic- and algebra-related tasks. The results showed that, prior to instruction, preservice teachers' engaged in relational thinking about equality (spontaneously or not) far more often in the context of arithmetic-related tasks than algebra-related tasks. This result was consistent, whether we were comparing pre-

service teachers' strategies for solving arithmetic and algebra-related tasks that fostered thinking about differences in the magnitude of the numbers or their strategies for solving arithmetic- and algebra-related tasks that fostered thinking about properties of equality or operations.

The data in our study are not robust enough to identify all the task-specific variables that are associated with pre-service teachers' inclination to engage in relational thinking. However, the insights we gained could prove helpful in designing effective teacher preparation programs. Take, for example, our finding that task complexity (e.g. involving subtraction of negative numbers or the use of commutativity when comparing differences in the magnitude of numbers) was negatively associated with the pre-service teachers' inclination and ability to use a relational thinking strategy. This result implies that teacher education programs may want to pay closer attention to pre-service teachers' selection of strategies and emphasize relational thinking as an alternative strategy that could be considered in solving a task. Or consider our finding that the overall nature of the task (arithmetic vs. algebraic) might be associated with pre-service teachers' inclination to engage in relational thinking about equality. This result suggests that teacher educators may want to closely monitor the selection of strategies that pre-service teachers' employ to solve arithmetic and algebra-related tasks and explicitly emphasize relational thinking about equality within both domains, arithmetic and algebra. Pre-service teachers' relative lack of tendency to engage in relational thinking about equality in the context of algebra-related tasks also suggests that it might prove beneficial to explicitly engage pre-service teachers in discussions about the role relational thinking plays in students' learning of algebra.

Research shows that relational thinking about equality is essential for the successful learning of algebraic concepts (Van Ameron, 2003). However, research documents that many K-8 (and older) students have difficulty solving equations, especially equations with operations on both sides (e.g.,  $23 + 4 + 6 = 24 + a$ ). McNeil and Alibali (2005a, b) link these difficulties to students' early experiences with equality. Because these experiences usually consist of performing computations on the left side of an equation and writing the resulting answer on the right side, students often come to believe that the equal

sign is a signal to compute. Unless teachers are attuned to the possibly pernicious side effects of such seemingly benign experiences, they may unknowingly contribute to students' long-term difficulties in mathematics. Thus, in our opinion, it is not only pre-service teachers' ability to think relationally, but also their inclination to do so, that is an important predictor of pre-service teachers' success in early algebra instruction. Without the inclination to think relationally, pre-service teachers are likely be content to simply focus on the procedural aspects of arithmetic problems (Ball, 1990; Van Dooren et al. 202)), instead of challenging their students to understand the relational aspects of equality.

Stephens A. (2006) argued that not only fostering pre-service teachers' own development of relational thinking but most of all heightening their awareness of *why* a teacher might engage students in relational thinking should be emphasized in teacher education. The findings from our study support that argument. Building pre-service teachers' understanding of how fostering relational thinking in elementary and middle school students prepares these students for further study of mathematics might contribute to pre-service teachers' awareness of relational strategies they might consider modelling in their own classrooms. The pre-service teachers in our study exhibited rather strong ability to engage in relational thinking overall (spontaneously or prompted). Therefore, there is good reason to believe that pre-service teachers might benefit from attempts to help them understand the value of relational thinking for students' learning of mathematics since it might motivate them give more frequent consideration to the use of relational thinking about equality in the context of algebra and arithmetic tasks. This is necessary if the pre-service teachers' are to effectively facilitate students' relational thinking in their future work with students. Unless teacher educators address these issues, prospective teachers may not learn to use and model relational thinking about equality and as a consequence limit their students' chances for success in algebra.

Hill, Rowan, and Ball (2005) stated that students' mathematics achievement closely relates to teachers' mathematics content knowledge for teaching. Teacher education programs should then explicitly emphasize relational thinking in the context of arithmetic *and*

algebra-related tasks, to increase pre-service teachers' ability and inclination for relational thinking with respect to these two groups of tasks.

Our results indicate that pre-service teachers' inclination to engage in relational thinking about equality might relate to the overall nature of the task (arithmetic- versus algebra-related) and to other possible task-specific variables (e.g. task complexity). The data in our study do not establish a causal relationship between pre-service teachers' inclination to engage in relational thinking about equality and task specific variables that contribute to their inclination to engage in relational thinking. Further studies warrant careful examination of how the nature or complexity of a task might be associated with pre-service teachers' inclination to think relationally.

We recognize that the results of this study are limited by our selection of tasks, the small number of participants, and our methodology, which was restricted to the analysis of written accounts of pre-service teachers' task solutions. Certainly, a wider selection of tasks, a broader selection of participants, and the use of follow-up interviews could provide more detailed knowledge of pre-service teachers' ability and inclination to engage in relational thinking, as well as a more comprehensive list of task variables associated with pre-service teachers' inclination to engage in relational thinking.

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