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Common Hypercyclic Vectors for the Conjugate Class of a Hypercyclic Operator

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Abstract

Given a separable, infinite dimensional Hilbert space, it was recently shown by the authors that there is a path of chaotic operators, which is dense in the operator algebra with the strong operator topology, and along which every operator has the exact same dense G_δ set of hypercyclic vectors. In the present work, we show that the conjugate set of any hypercyclic operator on a separable, infinite dimensional Banach space always contains a path of operators which is dense with the strong operator topology, and yet the set of common hypercyclic vectors for the entire path is a dense G_δ set. As a corollary, the hypercyclic operators on such a Banach space form a connected subset of the operator algebra with the strong operator topology.

1. Introduction

Throughout the present paper, let X be a separable, infinite dimensional Banach space over the scalar field $\mathbb{C}$ or $\mathbb{R}$, and let $B(X)$ denote the algebra of bounded linear operators $T: X \to X$. An operator T in $B(X)$ is *hypercyclic* if there is a vector x in X for which its orbit, $Orb(T, x) = \{Tnx: n \geqslant 0\}$, is dense in X. Such a vector x is called a *hypercyclic vector* for T. An operator T in $B(X)$ is hypercyclic if and only if the set of hypercyclic vectors for T, denoted by $\mathcal{HC}(T)$,is a dense G_δ set; see Kitai [23]. For a countable family $\mathcal{F}$ of hypercyclic operators, a direct application of the Baire Category Theorem implies that the set $\bigcap_{T\in\mathcal{F}}\mathcal{HC}(T)$ of common hypercyclic vectors is also a dense G_δ set. However, for the situation when $\mathcal{F}$ is an uncountable family of hypercyclic operators, we cannot apply this Baire Category Theorem argument to show the set $\bigcap_{T\in\mathcal{F}}\mathcal{HC}(T)$ of common hypercyclic vectors is a dense G_δ set, or is even nonempty. This observation has prompted research on the existence of common hypercyclic vectors for uncountable families of hypercyclic operators. Bayart and Matheron [5], Chan and Sanders [13], and Costakis and Sambarino [19] have separately developed different sufficient conditions for an uncountable family of operators to have a dense G_δ set of common hypercyclic vectors. Other results on common hypercyclic vectors include the work of Abakumov and Gordon [1], Aron, Bès, León, and Peris [3], Bayart [4], Bayart and Grivaux [7], Conejero, Müller, and Peris [18], and León and Müller [24].

In much of the above work on common hypercyclic vectors, the uncountable family of operators maintains some sort of continuity within the family. This brings us to the definition of a path of operators. A family of operators $\{F_t \in B(X): t \in I\}$, where I is an interval of real numbers, is a *path of operators* if the map $F: I \to (B(X), \|\cdot\|)$, defined by $F(t) = F_t$,is continuous with respect to the usual topology on the interval I and the operator norm topology on $B(X)$. If the interval $I = [a, b]$, then the path $\{F_t \in B(X): t \in I\}$ is a *path of operators between* F_a *and* F_b. For any path, a vector x in X is called a *common hypercyclic vector* for the path if $x \in \bigcap_{T\in I} \mathcal{HC}(F_t)$.

In the present paper, we examine common hypercyclic vectors for a family of operators which consists of the conjugates of a single hypercyclic operator. For notation, let $S(T) = \{L^{-1}TL: L \text{ invertible}\}$ be the conjugate set of the operator T. The conjugate set $S(T)$ is also often referred to as the similarity orbit of T. A standard similarity argument shows that an operator T in $B(X)$ is hypercyclic if and only if each operator in the conjugate set $S(T)$ is hypercyclic. From this observation, one can ask whether the set $\bigcap_{A\in S(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the entire conjugate set $S(T)$ of a hypercyclic operator T is a dense G_δ set. In Proposition 2.1 below, we show this set of common hypercyclic vectors has only two possibilities. If every nonzero vector in X is a hypercyclic vector for T, then the set $\bigcap_{A\in S(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the conjugate set S(T) contains every nonzero vector also. Otherwise, the set $\bigcap_{A\in S(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the conjugate set $S(T)$ is empty.

Not only does the conjugate set $S(T)$ of a hypercyclic operator T consist entirely of hypercyclic operators, those hyper-cyclic operators are dense in $B(X)$ with respect to the strong operator topology, or SOT for short. This result was proved by Bès and Chan [9] by applying a fundamental property of the strong operator topology established by Hadwin, Nordgren, Radjavi, and Rosenthal [22]. As we have mentioned above, if $\mathcal{HC}(T) \neq X \setminus \{0\}$, then the set $\bigcap_{A\in S(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the conjugate set must be empty. Regardless of this, we show the conjugate set $S(T)$ must contain a path $\{F_t \in B(X): t \in [1, \infty)\}$ of operators which is SOT-dense in $B(X)$, and yet the set $\bigcap_{t\in[1,\infty)} \mathcal{HC}(F_t)$ of common hypercyclic vectors for the whole path is a dense G_δ set; see Theorem 2.4 below. As a corollary, we show the hypercyclic operators in $B(X)$ form an SOT-connected subset of $B(X)$; see Corollary 3.3 below. Also using Theorem 2.4, we show that for any nonzero vector g in X, the set $\{T \in B(X): g \in \mathcal{HC}(T)\}$ is SOT-dense, as well as SOT-connected in $B(X)$; see Corollary 3.2 below.

A hypercyclic operator clearly has orbits which exhibit wild behavior. It may also possess orbits with simple behavior. A vector x in X is a *periodic point* of the operator T if $T^n x = x$ for some positive integer n. An operator T in $B(X)$ is called *chaotic* if it is hypercyclic and the set of periodic points for T is dense in X. Recently, Chan and Sanders [15] showed that every separable, infinite dimensional Hilbert space H over the scalar field $\mathbb{C}$ admits a path of chaotic operators which is SOT-dense in $B(H)$, and yet each operator along the path shares the exact same set of hypercyclic vectors. However, Bonet, Martínez-Giménez, and Peris [11] provided examples of separable, infinite dimensional Banach spaces which fail to support even a single chaotic operator. Hence, the techniques in [15] do not work for an arbitrary separable, infinite dimensional Banach space. For this general setting, even though we are not able to show that there is an SOT-dense path of hypercyclic operators, each of which has the exact same set of hypercyclic vectors, Theorem 2.4 below exhibits such a path with a dense G_δ set of common hypercyclic vectors. In the case where the Banach space does support a chaotic operator, using Theorem 2.4, we show there does exist a path of chaotic operators which is SOT-dense in $B(X)$, and for which the set of common hypercyclic vectors for the whole path is a dense G_δ set. Furthermore, the chaotic operators in $B(X)$ form a connected subset of $B(X)$; see Corollary 3.4 below.

2. Conjugate Class of a Hypercyclic Operator

As stated in the introduction, an operator T in $B(X)$ is hypercyclic if and only if every operator in the conjugate set $S(T) = \{L^{-1}TL: L \text{ invertible}\}$ is hypercyclic. In fact, one can easily verify that

$$x \in \mathcal{HC}(L^{-1}TL) \quad \text{if and only if} \quad Lx \in \mathcal{HC}(T). \tag{2.1}$$

Using this observation, we show that the set of common hypercyclic vectors for the conjugate set $S(T)$ of an operator T has only two possibilities, either the set of all nonzero vectors or the empty set.

Proposition 2.1. *Let T be an operator in $B(X)$.*

(i) *If $\mathcal{HC}(T) = X \setminus \{0\}$, then the set $\cap_{A \in S(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the conjugate set $S(T)$ is also $X \setminus \{0\}$.*

(ii) *If $\mathcal{HC}(T) \neq X \setminus \{0\}$, then the set $\cap_{A \in S(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the conjugate set $S(T)$ is empty.*

Proof. Part (i) follows directly from the statement given in (2.1). For part (ii), let y be any nonzero vector in X which fails to be a hypercyclic vector for the operator T. For any nonzero vector x in X, there exists an invertible operator L such that $Lx = y$. For instance, if x and y are linearly independent, we may take $Lx = y$ and $Ly = x$ and $L = I$ on a closed subspace complementary to the finite dimensional subspace spanned by x and y. If $y = \alpha x$ for some nonzero scalar α, then let $L = \alpha I$ on X. Since $Lx = y \notin \mathcal{HC}(T)$, by (2.1), we have $x \notin \mathcal{HC}(L^{-1}TL)$. Therefore, $\bigcap_{A\in S(T)} \mathcal{HC}(A) = \emptyset$.

Read [26] provided an example of an operator T on ℓ^1 for which every nonzero vector is a hypercyclic vector. Thus, it is possible for the set of common hypercyclic vectors for a conjugate set to be nonempty. On the other hand, every separable, infinite dimensional Banach space X admits a hypercyclic operator T for which $\mathcal{HC}(T) = X \setminus \{0\}$; see the hypercyclic operator constructed by Ansari in [2] or by Bernal in [8]. Since the conjugate set $S(T)$ of this particular hypercyclic operator fails to have a single hypercyclic vector in common, it follows trivially that the set of all hypercyclic operators in $B(X)$ fails to have a single hypercyclic vector in common.

The conjugate set $S(T)$ of a hypercyclic operator T is SOT-dense in the operator algebra $B(X)$. However, in many cases, this SOT-dense set fails to have a single common hypercyclic vector. On the positive side, it does contain a path of operators which is SOT-dense in $B(X)$, and for which the set of common hypercyclic vectors for the whole path is a dense G_δ set. For this, we need two technical results.

Lemma 2.2. *Let $x_1, x_2, \ldots, x_k$ be k linearly independent vectors in X, and let*

$$d = \min_{1\leqslant j\leqslant k} \operatorname{dist}\big(x_j, \operatorname{span}\{x_i\colon\ i \neq j\}\big).$$

There exists a $\delta > 0$ such that whenever $y_1, y_2, \ldots, y_k$ are k vectors in X satisfying $\| x_j - y_j \| < \delta$ for each integer j with 1 $\leqslant$ k, we have

$$\min_{1\leqslant j\leqslant k} \operatorname{dist}\big(y_j, \operatorname{span}\{y_i\colon\ i \neq j\}\big) \geqslant \frac{d}{2}.$$

Proof. Since all norms are equivalent on the finite dimensional space span $\{x_1, x_2, \ldots, x_k\}$, there is a constant $C > 0$ such that

$$\sum_{i=1}^{k} |\alpha_i| \leqslant C \left\| \sum_{i=1}^{k} \alpha_i x_i \right\| \tag{2.2}$$

for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$. Choose a $\delta > 0$ such that

$$(1 - C\delta) \geqslant \frac{1}{2}. \tag{2.2}$$

Let $y_1, y_2, \ldots, y_k$ be any k vectors in X satisfying $\|x_j - y_j\| < \delta$ for 1 $\leqslant$ k. For any integer j with 1 $\leqslant$ k and for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_k$, we have

$$\begin{aligned}
\left\| y_j - \sum_{i \neq j} \alpha_i y_i \right\| &= \left\| \left(x_j - \sum_{i \neq j} \alpha_i x_i \right) + (y_j - x_j) + \sum_{i \neq j} \alpha_i (x_i - y_i) \right\| \\
&\geqslant \left\| x_j - \sum_{i \neq j} \alpha_i x_i \right\| - \|x_j - y_j\| - \sum_{i \neq j} |\alpha_i| \|x_i - y_i\| \\
&> \left\| x_j - \sum_{i \neq j} \alpha_i x_i \right\| - \delta \left(1 + \sum_{i \neq j} |\alpha_i| \right) \\
&\geqslant \left\| x_j - \sum_{i \neq j} \alpha_i x_i \right\| - \delta C \left\| x_j - \sum_{i \neq j} \alpha_i x_i \right\|, \quad \text{by (2.2)} \\
&\geqslant (1 - C\delta) d \\
&\geqslant \frac{d}{2}, \quad \text{by (2.3)}.
\end{aligned}$$

Thus, our result follows. □

The second result involves the union of a finite linearly independent set with the tail end of an orbit generated by a hypercyclic vector.

Proposition 2.3. *Let $T \in B(X)$ be a hypercyclic operator. If $\mathrm{g} \in \mathcal{HC}(T)$ and $x_1, x_2, \ldots, x_k$ are k linearly independent vectors in X, then there is an integer $N \geqslant 0$ such that the set $\{x_1, x_2, \ldots, x_k\} \cup \{T^n \mathrm{g} : n \geqslant N\}$ is linearly independent.*

Proof. By way of contradiction, we suppose that no such integer N exists; that is, the set $\{x_1, x_2, \ldots, x_k\} \cup \{T^n \mathrm{g} : n \geqslant N\}$ is linearly dependent for each integer $N \geqslant 0$. If we take $N = 1$, then by the linear independence of the vectors $x_1, x_2, \ldots, x_k$ and the linear independence of the orbit of a hypercyclic vector (see, for example Bourdon [12]), we obtain a nonzero polynomial p_1 for which $p_1(T)g \in \operatorname{span}\{x_1, x_2, \ldots, x_k\}$. For similar reasons, by taking $N = 1 + \deg p_1$, we obtain a nonzero polynomial p_2 with $deg\, p_2 > deg\, p_1$ and $p_2(T)\mathrm{g} \in \operatorname{span}\{x_1, x_2, \ldots, x_k\}$. After $k+1$ steps, we obtain nonzero polynomial p_{k+1} with $\deg p_{k+1} > \deg p_k$ and $p_{k+1}(T)\mathrm{g} \in \operatorname{span}\{x_1, x_2, \ldots, x_k\}$. Since the $k+1$ vectors $p_1(T)\mathrm{g}, p_2(T)\mathrm{g}, \ldots, p_{k+1}(T)\mathrm{g}$ lie in the subspace $\operatorname{span}\{x_1, x_2, \ldots, x_k\}$, which is k-dimensional, it follows that they must be linearly dependent. However, this contradicts the fact that g is a hypercyclic vector. □

We are now ready to prove that every conjugate set of a hypercyclic operator contains a path of operators which is SOT-dense in $B(X)$.

Theorem 2.4. *Let T be a hypercyclic operator in $B(X)$. The conjugate set $S(T) = \{L^{-1}TL\colon L \text{ invertible}\}$ contains a path $\{F_t \in B(X)\colon t \in [1,\infty)\}$ of operators which is SOT-dense in $B(X)$, and for which the set $\bigcap_{t\in[1,\infty)}\mathcal{HC}(F_t)$ of common hypercyclic vectors for the whole path is a dense G_δ set.*

Proof. We begin with an outline of the construction of the desired path of hypercyclic operators. The path must contain a hypercyclic operator in every nonempty SOT-basic open set $\mathcal{O}$ in $B(X)$, which is of the form

$$\mathcal{O} = \left\{A \in B(X)\colon \left\|A_{x_l} - B_{x_l}\right\| < \epsilon \text{ for } 1 \leqslant l \leqslant k\right\},$$

where $B \in B(X)$, $\epsilon > 0$, and $x_l \in X$. The vectors x_l and Bx_l provide a starting point of our construction of an invertible operator L so that $L^{-1}TL$ is in $\mathcal{O}$ and it can be joined to the given hypercyclic operator T with a path having a dense G_δ set of common hypercyclic vectors. For that, we may assume that the vectors x_l are linearly independent and use Proposition 2.3 to choose appropriate powers of T on a hypercyclic vector $g \in \mathcal{HC}(T)$ that can approximate x_l and Bx_l .Then we use Lemma 2.2 to control the norms of L and L^{-1} so that the terms $L^{-1}TL(x_l) - Bx_l$ qualify $L^{-1}TL$ to be in the set $\mathcal{O}$. Furthermore, to create the desired path we first note that we can trivially write T as $I^{-1}TI$, where I is the identity, and so we have to join I with L with an appropriate path of invertible operators. The operator L takes the form of the sum of the identity and a finite rank operator K whose range is the linear span of carefully chosen powers $T^m g$. The path will then be in the form of $I + tK$, where t in $[0,1]$ is the parameter for the path. However, in order to carefully select vectors $T^m g$ to make our argument work, we need to have good estimations on their distances from each other and separate them in terms of linear functionals.

To this end, let $g \in \mathcal{HC}(T)$. Let $\mathcal{E}$ be the collection of all sets E of the form

$$E = \{T^{m_1}g, T^{m_2}g, \dots, T^{m_{2k}}g, T^N g, T^{N+1}g, \dots T^{N+2k-1}g\} \tag{2.4}$$

where N, k are integers with $N \geq 0$ and $k \geq 1$ and $m_1, m_2, \dots, m_{2k}$ are distinct integers with each $m_j \geq N + 2k$. Note that the collection $\mathcal{E}$ is countable. For the set E given in (2.4) and for each integer j with $1 \leqslant j \leqslant 2k$, define

$$d_{j,E} = \operatorname{dist}(T^{m_j}\mathrm{g}, \operatorname{span}(E \setminus \{T^{m_j}\mathrm{g}\})) \quad \text{and} \quad D_{j,E} = \operatorname{dist}\left(T^{N+j-1}\mathrm{g}, \operatorname{span}(E \setminus \{T^{m_j}\mathrm{g}\})\right).$$

Then define Δ_E by

$$\Delta_E = \min\{d_{1,E}, d_{2,E}, \dots, d_{2k,E}, D_{2,E}, \dots, D_{2k,E}\}.$$

Since the orbit of the hypercyclic vector g must be linearly independent, each set $E \in \mathcal{E}$ is linearly independent, and so $\Delta_E > 0$.

Claim 1. *For the set* $E \in \varepsilon$ *given in* (2.4)*, there are* $2k$ *linear functionals* $\lambda_{1,E}, \lambda_{2,E}, \dots, \lambda_{2k,E}$ *in the dual space* X^* *such that for any integers* i,j *with* $1 \leqslant i,j \leqslant 2k$*, we have* $\|\lambda_{j,E}\|$ *and*

$$\lambda_{j,E}(T^{m_i}\mathrm{g}) = \lambda_{j,E}(T^{N+i-1}\mathrm{g}) = \begin{cases} 1, & if\ i = j, \\ 0, & if\ i \neq j. \end{cases}$$

Proof of Claim 1. As a corollary of the Hahn–Banach Theorem, there exist linear functionals $\varphi_{1,E}, \dots, \varphi_{2k,E}$ and $\psi_{1,E}, \dots, \psi_{2k,E}$ in the dual space X^{**} such that for integers i,j with $1 \leqslant i,j \leqslant 2k$, we have $\varphi_{j,E}(T^{m_j}\mathrm{g}) = 1$ and $\varphi_{j,E}(x) = 0$ for all $x \in span(E \setminus \{T^{m_j}\mathrm{g}\})$, and $\psi_{j,E}(T^{N+j-1}\mathrm{g})$ and $\psi_{j,E}(x) = 0$ for all $x \in span(E \setminus \{T^{m_j}\mathrm{g}\})$. Furthermore, $\|\varphi_{j,E}\| = \frac{1}{d_{j,E}} \leqslant \frac{1}{\Delta_E}$ and $\|\psi_{j,E}\| = \frac{1}{D_{j,E}} \leqslant \frac{1}{\Delta_E}$. Letting $\lambda_{j,E} = \varphi_{j,E} + \psi_{j,E}$ for each integer j with $1 \leqslant j \leqslant 2k$ completes the proof of Claim 1. □

We now use Claim 1 to form a countable collection of invertible operators in $B(X)$.For theset $E \in \mathcal{E}$ given in (2.4), define the operator $L_E : X \to X$ by

$$L_E(x) = x + \sum_{j=1}^{2k} \lambda_{j,E}(x)\left(T^{N+j-1}\mathrm{g} - T^{m_j}g\right). \tag{2.5}$$

To see that the operator L_E is invertible, define the operator $A_E : X \to X$ by

$$A_E(x) = x + \sum_{i=1}^{2k} \lambda_{i,E}(x)\left(T^{m_i}\mathrm{g} - T^{N+i-1}\mathrm{g}\right).$$

For any $x \in X$, observe that

$$L_E A_E(x) = L_E(x) + \sum_{i=1}^{2k} \lambda_{i,E}(x) L_E\left(T^{m_i}\mathrm{g} - T^{N+i-1}\mathrm{g}\right).$$

By Claim 1, for any integers i, j with $1 \leqslant i, j \leqslant 2k$, we have $\lambda_{j,E}\left(T^{m_i}\mathrm{g} - T^{N+i-1}\mathrm{g}\right) = 0$, and so by (2.5),

$$L_E\left(T^{m_i}\mathrm{g} - T^{N+i-1}\mathrm{g}\right) = T^{m_i}\mathrm{g} - \mathrm{T}^{N+i-1}\mathrm{g}.$$

Thus,

$$\begin{aligned} L_E A_E(x) &= L_E(x) + \sum_{i=1}^{2k} \lambda_{i,E}(x) L_E\left(T^{m_i}\mathrm{g} - T^{N+i-1}\mathrm{g}\right) \\ &= x + \sum_{j=1}^{2k} \lambda_{i,E}(x)\left(T^{N+j-1}\mathrm{g} - T^{m_j}\mathrm{g}\right) + \sum_{j=1}^{2k} \lambda_{i,E}(x)\left(T^{m_i}\mathrm{g} - T^{N+j-1}\mathrm{g}\right) \\ &= x. \end{aligned}$$

Likewise, $L_E A_E(x) = x$ for any $x \in X$. Therefore, the operator L_E is invertible and $L_E^{-1} = A_E$. Moreover, by definitions of L_E, L_E^{-1} and by Claim 1, both operators L_E, L_E^{-1} satisfy the inequality

$$\left\|L_E^{-1}\right\|, \left\|L_E\right\| \leqslant 1 + \sum_{j=1}^{2k} \left\|\lambda_{j,E}\right\| \left\|T^{N+j-1}\mathrm{g} - T^{m_j}\mathrm{g}\right\| \leqslant 1 + \frac{2}{\Delta_E} \sum_{j=1}^{2k} \left\|T^{N+j-1}\mathrm{g} - T^{m_j}\mathrm{g}\right\|. \tag{2.6}$$

Using the countable collection $\{L_E : E \in \mathcal{E}\}$ of invertible operators, we generate a countable SOT-dense subset of $S(T)$.

Claim 2. *The countable collection* $\{L_E^{-1} T L_E : E \in \mathcal{E}\}$ *is SOT-dense in* $B(X)$.

Proof of Claim 2. Let $\mathcal{U}$ be a nonempty SOT-open set in $B(X)$. Then there exists an operator $B \in B(X)$, an $\epsilon > 0$, and nonzero vectors $x_1, x_2, \ldots, x_k$ in X such that

$$\left\{A \in B(X) : \left\|A_{x_l} - B_{x_l}\right\| < \epsilon \text{ for } 1 \leqslant l \leqslant k\right\} \subseteq \mathcal{U}.$$

Without loss of generality, we may assume the set $\{x_1, x_2, \ldots, x_k\}$ is linearly independent. By Proposition 2.3, there is an integer $N \geqslant 0$ such that the set $\{x_1, x_2, \ldots, x_k\} \cup \{T^n\mathrm{g} : n \geqslant N\}$ is linearly independent. Since $\mathrm{g} \in \mathcal{HC}(T)$, we can choose k distinct integers $m_2, m_4, \ldots, m_{2k}$ satisfying

$$m_{2l} \geqslant N + 2k \quad \text{and} \quad \left\|T^{m_{2l}}\mathrm{g} - B_{x_l}\right\| < \frac{\epsilon}{2} \text{ for } 1 \leqslant l \leqslant k. \tag{2.7}$$

Consider the linearly independent set

$$\tilde{E} = \{x_1, T^{m_2}g, x_2, T^{m_4}g, \dots, x_k, T^{m_{2k}}g, T^N g, T^{N+1}g, \dots, T^{N+2k-1}g\},$$

and define

$$\tilde{d}_{2l-1} = \operatorname{dist}\left(x_l, \operatorname{span}\left(\tilde{E} \setminus \{x_l\}\right)\right) \text{ for } 1 \leqslant l \leqslant k,$$

$$\tilde{d}_{2l} = \operatorname{dist}\left(T^{m_{2l}}g, \operatorname{span}\left(\tilde{E} \setminus \{T^{m_{2l}}g\}\right)\right) \text{ for } 1 \leqslant l \leqslant k,$$

$$\widetilde{D}_j = \operatorname{dist}\left(T^{N+j-1}g, \operatorname{span}\left(\tilde{E} \setminus \{T^{N+j-1}g\}\right)\right) \text{ for } 1 \leqslant j \leqslant 2k,$$

$$\tilde{\Delta} = \min\{\widetilde{d_1}, \widetilde{d_2}, \dots, \widetilde{d_{2k}}, \widetilde{D_1}, \widetilde{D_2}, \dots, \widetilde{D_{2k}}\}.$$

Set

$$M = k + \sum_{l=1}^{k} \left\|T^{N+2l-1}g - T^{m_{2l}}g\right\| + \sum_{l=1}^{k} \left\|T^{N+2l-2}g - x_l\right\|. \tag{2.8}$$

Since the set $\tilde{E}$ is linearly independent, by Lemma 2.2, there is a $\delta > 0$ such that whenever $m_1, m_3, \dots, m_{2k-1}$ are k distinct integers with

$$E = \{T^{m_1}g, T^{m_2}g, \dots, T^{m_{2k-1}}g, T^{m_{2k}}g, T^N g, \dots, T^{N+2k-1}g\} \in \mathcal{E}$$

and

$$\|T^{m_{2l-1}}g - x_l\| < \delta \text{ for } 1 \leqslant l \leqslant k, \tag{2.9}$$

we get

$$\Delta_E \geqslant \frac{\tilde{\Delta}}{2}. \tag{2.10}$$

We may further assume that δ satisfies

$$\delta < \min\left\{1, \frac{\epsilon}{2\|T\|\left(1 + \frac{4}{\tilde{\Delta}}M\right)^2}\right\}. \tag{2.11}$$

Note that there exists such an $E \in \mathcal{E}$ because g is a hypercyclic vector for T. Moreover, for any such $E \in \mathcal{E}$ and for any integer l with $1 \leqslant l \leqslant k$, we have

$$\|L_E^{-1}TL_E(x_l) - Bx_l\| \leqslant \|L_E^{-1}TL_E(x_l) - L_E^{-1}TL_E(T^{m_{2l-1}}g)\| + \|L_E^{-1}TL_E(T^{m_{2l-1}}g) - Bx_l\|. \tag{2.12}$$

To estimate the first summand on the right-hand side of (2.12), note that

$$\|L_E^{-1}TL_E(x_l) - L_E^{-1}TL_E(T^{m_{2l-1}}g)\| \leqslant \|L_E^{-1}\|\|T\|\|L_E\|\|x_l - T^{m_{2l-1}}g\| < \|L_E^{-1}\|\|T\|\|L_E\|\delta, \text{ by (2.9)}$$

$$\leqslant \left(1+\frac{2}{\Delta_E}\sum_{j=1}^{2k}\left\|T^{N+j-1}g-T^{m_j}g\right\|\right)^2 \|T\|\delta, \qquad \text{by (2.6)}$$

$$\leqslant \left(1+\frac{4}{\tilde{\Delta}}\sum_{j=1}^{2k}\left\|T^{N+j-1}g-T^{m_j}g\right\|\right)^2 \|T\|\delta, \qquad \text{by (2.10)}. \tag{2.13}$$

We now estimate the above summation

$$\sum_{j=1}^{2k}\left\|T^{N+j-1}g-T^{m_j}g\right\| = \sum_{l=1}^{k}\|T^{N+2l-1}g-T^{m_{2l}}g\| + \sum_{l=1}^{k}\|T^{N+2l-2}g-T^{m_{2l-1}}g\|$$

$$\leqslant \sum_{l=1}^{k}\|T^{N+2l-1}g-T^{m_{2l}}g\| + \sum_{l=1}^{k}\|T^{N+2l-1}g-x_l\| + \sum_{l=1}^{k}\|x_l-T^{m_{2l-1}}g\|$$

$$< k + \sum_{l=1}^{k}\|T^{N+2l-1}g-T^{m_{2l}}g\| + \sum_{l=1}^{k}\|T^{N+2l-2}g-x_l\|, \qquad \text{by (2.9), (2.11)}$$

$$= M, \qquad \text{by (2.8)}.$$

Combining inequality (2.13) with the above inequality gives us

$$\left\|L_E^{-1}TL_E(x_l)-L_E^{-1}TL_E(T^{m_{2l-1}}g)\right\| < \left(1+\frac{4}{\tilde{\Delta}}M\right)^2 \|T\|\delta < \frac{\epsilon}{2}, \qquad \text{by (2.11)}. \tag{2.14}$$

To estimate the second summand on the right-hand side of (2.12), observe that for each integer i with $1 \leqslant i \leqslant 2k$, we have

$$L_E(T^{m_i}g) = T^{m_i}g + \sum_{j=1}^{2k}\lambda_{j,E}(T^{m_i}g)\left(T^{N+j-1}g-T^{m_i}g\right), \qquad \text{by (2.5)}$$

$$= T^{m_i}g + \left(T^{N+i-1}g-T^{m_i}g\right), \qquad \text{by Claim 1}$$

$$= T^{N+i-1}g. \tag{2.15}$$

Thus,

$$\left\|L_E^{-1}TL_E(T^{m_{2l-1}}g)-Bx_l\right\| = \left\|L_E^{-1}TT^{N+2l-2}g-Bx_l\right\|, \qquad \text{by (2.15)}$$

$$= \left\|L_E^{-1}T^{N+2l-1}g-Bx_l\right\|$$

$$= \|T^{m_{2l}}g-Bx_l\|, \qquad \text{by (2.15)}$$

$$< \frac{\epsilon}{2}, \qquad \text{by (2.7)}. \tag{2.16}$$

Combining inequality (2.12) with inequalities (2.14) and (2.16) yields $\left\|L_E^{-1}TL_E(x_l)-Bx_l\right\| < \epsilon$, and so $L_E^{-1}TL_E \in \mathcal{U}$ which completes the proof of Claim 2. □

We now construct a path of operators between T and $L_E^{-1}TL_E$ which lies entirely within the conjugate set $\mathcal{S}(T)$ of the operator T.

Claim 3. *For each $E \in E$, there is a path of operators between T and $L_E^{-1}TL_E$ contained in the conjugate set $\mathcal{S}(T)$ for which the set of common hypercyclic vectors for the whole path is a dense G_δ set.*

Proof of Claim 3. Let E be the set in $\mathcal{E}$ given in (2.4). For each $t \in [0,1]$, define an operator $L_{t,E} : X \to X$ by

$$L_{t,E}(x) = x + \sum_{j=1}^{2k} t\lambda_{j,E}(x)\big(T^{N+j-1}\mathrm{g} - T^{m_j}\mathrm{g}\big).$$

Using computations similar to those before Claim 1, each operator $L_{t,E}$ is invertible, and its inverse $L_{t,E}^{-1} : X \to X$ is given by

$$L_{t,E}^{-1}(x) = x + \sum_{i=1}^{2k} t\lambda_{i,E}(x)\big(T^{m_i}\mathrm{g} - T^{N+i-1}\mathrm{g}\big).$$

Consider the path of operators $\{G_t \in B(X) : t \in [0,1]\}$, where $G_t = L_{t,E}^{-1}TL_{t,E}$. Clearly this path of operators is between T and $L_E^{-1}TL_E$ and lies entirely in the conjugate set $S(T)$. To show $\bigcap_{t\in[0,1]}\mathcal{HC}(G_t)$ is a dense G_δ set, first note that by Corollary 2.3 in [13], the set $\bigcap_{t\in[0,1]}\mathcal{HC}(G_t)$ is a G_δ set, and so it suffices to show this set is also dense. We do this by proving $\{\mathrm{spanOrb}(T,\mathrm{g})\} \setminus \{0\}$ is contained inside the set $\bigcap_{t\in[0,1]}\mathcal{HC}(G_t)$. To begin, note that given any nonzero polynomial p and any $t \in [0,1]$, we have $L_{t,E}p(T)\mathrm{g} \neq 0$ because $p(T)\mathrm{g} \neq 0$ by the linear independence of the orbit of g, and because the operator $L_{t,E}$ is invertible. Furthermore,

$$L_{t,E}p(T)\mathrm{g} = p(T)\mathrm{g} + \sum_{j=1}^{2k} t\lambda_{j,E}(p(T)\mathrm{g})\big(T^{N+j-1}\mathrm{g} - T^{m_j}\mathrm{g}\big) \in \{\mathrm{spanOrb}(T,\mathrm{g})\} \setminus \{0\} \subseteq \mathcal{HC}(T),$$

because every nonzero vector from the linear span of a dense orbit is a hypercyclic vector; see Bourdon [12] and Bès [10]. Therefore, by statement (2.1), we get $p(T)\mathrm{g} \in \mathcal{HC}\big(L_{t,E}^{-1}TL_{t,E}\big) = \mathcal{HC}(G_t)$. Hence, $\{\mathrm{spanOrb}(T,\mathrm{g})\} \setminus \{0\} \subseteq \bigcap_{t\in[0,1]}\mathcal{HC}(G_t)$, and this concludes the proof of Claim 3.
□

We are now ready to construct the desired SOT-dense path of operators in the conjugate set $\mathcal{S}(T)$. Let $\{E_n: n \geqslant 1\}$ be an enumeration of the countable set $\mathcal{E}$. By Claim 3, if for each integer $n \geqslant 1$, let $G_{t,n} = L_{2t,E_n}^{-1} T L_{2t,E_n}$ for $t \in [0, 1/2]$ and $G_{t,n} = L_{2-2t,E_n}^{-1} T L_{2-2t,E_n}$ for each $t \in [1/2, 1]$, then $\{G_{t,n} \in B(X): t \in [0, 1]\}$ is a path of operators in the conjugate set $\mathcal{S}(T)$ such that $G_{0,n} = G_{1,n} = T$ and $L_{E_n}^{-1} T L_{E_n} = G_{1/2,n} \in \{G_{t,n} \in B(X): t \in [0, 1]\}$, and in addition, the set $\bigcap_{t\in[0,1]} \mathcal{HC}(G_{t,n})$ is a dense G_δ set. For each $t \in [n, n+1]$, let $F_t = G_{t-n,n}$. Then $\{F_t \in B(X): t \in [1, \infty)\}$ is a path of operators in the conjugate set $\mathcal{S}(T)$ which is SOT-dense by Claim 2, and for which the set $\bigcap_{t\in[1,\infty)} \mathcal{HC}(F_t) = \bigcap_{n=1}^{\infty} \bigcap_{t\in[0,1]} \mathcal{HC}(G_{t,n})$ is a dense G_δ set. □

The SOT-dense path $\{F_t \in B(X): t \in [1, \infty)\}$ in the previous theorem consists of operators of the form $L^{-1}TL$, which share many properties that each other has; in fact, any properties preserved by similarity. For instance, if one of them is chaotic, then every operator in the whole path is chaotic. If one of them has a nontrivial kernel, then every one in the whole path has one. If one of them is surjective, then every one is. If one of them has a nontrivial invariant subspace, then every one has one. If every nonzero vector is a hypercyclic vector for one single operator in the path, then the same holds true for every operator in the path. If one of them has a hypercyclic subspace, which is an infinite dimensional closed subspace consisting, except the zero vector, of hypercyclic vector of the operator, then every one in the path has such a subspace. In the next section, we study more common properties that a path of operators may share.

3. Corollaries of the Main Result

Theorem 2.4 has several interesting corollaries. First, let us examine the linear structure within the dense G_δ set of common hypercyclic vectors for the path of operators given within the proof of Theorem 2.4. For each set $E \in \mathcal{E}$, consider the path of operators $\{G_t \in B(X): t \in [0,1]\}$ between the operators T and $L_E^{-1} T L_E$ given in the proof of Claim 3. To show the set $\bigcap_{t\in[0,1]} \mathcal{HC}(G_t)$ of common hypercyclic is dense, we prove that if g is a hypercyclic vector for T, then $\{\mathrm{spanOrb}(T, \mathrm{g})\} \setminus \{0\}$ is contained inside the set $\bigcap_{t\in[0,1]} \mathcal{HC}(G_t)$. Furthermore, these paths of operators are the building blocks for the desired SOT-dense path of operators. Thus, the set of common hypercyclic vectors for the path of operators constructed within the proof of Theorem 2.4 contains some natural linear structure.

Corollary 3.1. *Let T be a hypercyclic operator in $B(X)$,and let $\mathrm{g} \in \mathcal{HC}(T)$. There exists a path*

$\{F_t \in B(X): t \in [1, \infty)\}$ *of operators, contained entirely in the conjugate set* $\mathcal{S}(T)$*, which is SOT-dense in* $B(X)$*, and for which* $\{spanOrb(T, \mathrm{g})\} \setminus \{0\}$ *is contained within the dense* G_δ *set* $\bigcap_{t \in [1,\infty)} \mathcal{HC}(F_t)$ *of common hypercyclic vectors.*

Since the orbit of a hypercyclic vector is linearly independent, the set of common hypercyclic vectors for the path of operators given in Corollary 3.1 contains an infinite dimensional linear manifold for which every nonzero vector is a common hypercyclic vector. However, the linear manifold given in Corollary 3.1 is not closed. Corollary 3.5 of Sanders [27] provides a natural sufficient condition for the set of common hypercyclic vectors for a path of operators to contain a closed, infinite dimensional subspace of which every nonzero vector is a common hypercyclic vector.

The existence of a path of hypercyclic operators that is SOT-dense in $B(X)$ gives us information about the connectedness of the hypercyclic operators in $B(X)$. Recall that if Y and Z are subsets of a topological space X satisfying $Y \subseteq Z \subseteq \bar{Y}$, and if Y is connected, then Z is also connected; see Munkres [25]. A path of operators in $B(X)$ is SOT-connected, and so any set of operators in $B(X)$, which contains an SOT-dense path of operators, is also SOT-connected. From this topological argument and Corollary 3.1, we get the next result.

Corollary 3.2. *Let* g *be any nonzero vector in a separable, infinite dimensional Banach space* X*. Then the set* $\mathcal{A} = \{T \in B(X): \mathrm{g} \in \mathcal{HC}(T)\}$ *is SOT-dense and SOT-connected in* $B(X)$*. Furthermore, its set of common hypercyclic vectors is* $\bigcap_{t \in \mathcal{A}} \mathcal{HC}(T) = (\mathrm{span}\{\mathrm{g}\}) \setminus \{0\}$.

Proof. For the first part of the proof, it suffices to show there is an operator T in $B(X)$ with $\mathrm{g} \in \mathcal{HC}(T)$. By Corollary 3.1, it follows that the set $\{T \in B(X): \mathrm{g} \in \mathcal{HC}(T)\}$ contains a path of operators which is SOT-dense in $B(X)$, and consequently SOT-connected by the topological argument above. To this end, let T_0 be a hypercyclic operator with $\mathcal{HC}(T_0) \neq X \setminus \{0\}$; see Ansari in [2] or by Bernal in [8]. Let $\mathrm{g}_0 \in \mathcal{HC}(T_0)$. Choose an invertible map $L: X \to X$ such that $L\mathrm{g} = \mathrm{g}_0$, and set $T = L^{-1}T_0L$. Since $L\mathrm{g} = \mathrm{g}_0 \in \mathcal{HC}(T_0)$, by statement (2.1) we get $\mathrm{g} \in \mathcal{HC}(L^{-1}T_0L) = \mathcal{HC}(T)$.

For the second part, observe that $(\mathrm{span}\{\mathrm{g}\}) \setminus \{0\} \subseteq \bigcap_{T \in \mathcal{A}} \mathcal{HC}(T)$ because $g \in \bigcap_{T \in \mathcal{A}} \mathcal{HC}(T)$. To establish the reverse set inequality, let $h_0 \notin \mathcal{HC}(T_0)$ with $h_0 \neq 0$. For any $h \notin \mathrm{span}\{\mathrm{g}\}$, the sets $\{\mathrm{g}, h\}$ and $\{\mathrm{g}_0, h_0\}$ are each linearly independent, and so there is an invertible map $L_0: X \to X$ with $L_0\mathrm{g} = \mathrm{g}_0$ and $L_0h = h_0$. Again by statement (2.1), this implies

$g \in \mathcal{HC}(L_0^{-1}T_0L_0)$ and $h \notin \mathcal{HC}(L_0^{-1}T_0L_0)$.Thus, $h \notin \bigcap_{T\in\mathcal{A}} \mathcal{HC}(T)$. □

In many of the known cases, the set of common hypercyclic vectors for an uncountable family of hypercyclic operators is either empty or a dense G_δ set. Corollary 3.2 provides an example of a set of common hypercyclic vectors which is a G_δ set that fails to be dense.

When H is a separable, infinite dimensional Hilbert space over the scalar field $\mathbb{C}$, the invertible operators are path connected; see Douglas [21]. Thus, the conjugate set of a hypercyclic operator is both SOT-dense and SOT-connected in $B(H)$. By the topological argument given before Corollary 3.2, the hypercyclic operators in $B(H)$ then form an SOT-connected subset of $B(H)$; see [15]. For the Banach space version of the result, we can combine Theorem 2.4 and the topological argument given before Corollary 3.2.

Corollary 3.3. *Let X be a separable, infinite dimensional Banach space. The set of all hypercyclic operators is SOT-connected in $B(X)$.*

The argument used in Corollary 3.2 can be used to show certain well-known classes of hypercyclic operators are SOT-connected in $B(X)$. For example, from the definition of a chaotic operator, one can easily see that an operator is chaotic if and only if each operator in its conjugate set is chaotic. Using the same argument as with Corollary 3.3, we get the following result.

Corollary 3.4. *Let X be a separable, infinite dimensional Banach space which admits a chaotic operator. The set of all chaotic operators is SOT-connected in $B(X)$.*

For another example, an operator T in $B(X)$ satisfies the Hypercyclicity Criterion if and only if each operator in its conjugate set satisfies the criterion. Moreover, every separable, infinite dimensional Banach space admits an operator which satisfies the Hypercyclicity Criterion; see the hypercyclic operator constructed by Ansari in [2] or by Bernal in [8]. Thus, the collection of all hypercyclic operators in $B(X)$ which satisfies the Hypercyclicity Criterion is SOT-connected in $B(X)$. De la Rosa and Read [20] provided an example of a Banach space which admits a hypercyclic operator that fails to satisfy the Hypercyclicity Criterion. Using techniques inspired by De la Rosa and Read, Bayart and Matheron [6] showed some common Banach spaces, including the sequence Hilbert space ℓ^2, also admit such hypercyclic operators. Since a hypercyclic operator fails to satisfy the Hypercyclicity Criterion if and only if each operator in its conjugate set fails to satisfy the criterion, we get that whenever a Banach space X admits a hypercyclic operator that fails to satisfy the Hypercyclicity Criterion, then the collection of all such operators is SOT-connected in $B(X)$. Again, by a similar argument, if the Banach space X admits an operator with no nontrivial, closed, invariant subset, then the collection of all such operators is SOT-connected in $B(X)$. Recently, Chan and Seceleanu [16,17] provided classes of operators for each of which having one orbit with a nonzero limit point imply the operator be hypercyclic. An operator has this property if and only if each operator in the conjugate set also has this property. By our topological argument, the collection of all operators having this property is an SOT-connected subset of $B(X)$.

4. Final Remarks

In the final section of this paper, we discuss some natural questions which arise from the results in the previous sections. To begin, Proposition 2.1 states that the set of common hypercyclic vectors for the entire conjugate set is either all nonzero vectors or the empty set. In the Hilbert space setting, the unitary orbit, $\mathcal{U}(T) = \{U^{-1}TU: U \text{ unitary}\}$, of an operator T is a well-studied subset of the conjugate set $\mathcal{S}(T)$. Obviously, the unitary orbit $\mathcal{U}(T)$ is strictly smaller than the conjugate set $\mathcal{S}(T)$. Hence, in view of Proposition 2.1, one may ask whether the set $\bigcap_{\mathcal{A}\in\mathcal{U}(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the unitary orbit $\mathcal{U}(T)$ is always a dense G_δ set if T is hypercyclic.

As it turns out, the answer is still negative, unless we have the trivial case that every nonzero vector is a hypercyclic vector for T and hence the set of common hypercyclic vectors is $H \setminus \{0\}$. Otherwise, we have a unit vector y that is not a hypercyclic vector for T. Extend the singleton set $\{y\}$ to an orthonormal basis of H. For any unit vector x, extend the singleton set $\{x\}$ to an orthonormal basis of H. Let $V: H \to H$ be a unitary operator taking the second orthonormal basis one-to-one and onto the first orthonormal basis with $Vx = y$. Since $y \notin \mathcal{HC}(T)$, by statement (2.1), we get $x \notin \mathcal{HC}(V^{-1}TV)$. From this, we can conclude $\bigcap_{A\in\mathcal{U}(T)} \mathcal{HC}(T) = \emptyset$.

Of course, the unitary orbit $\mathcal{U}(T)$ cannot be SOT-dense in $B(H)$ because every operator in the unitary orbit $\mathcal{U}(T)$ has the same norm as the operator T. Along that line, a question one may ask is whether we can have a path of operators in the unitary orbit $U(T)$ of a hypercyclic operator T that is SOT-dense in $\|T\| \cdot \mathrm{Sph}(H)$, where $\mathrm{Sph}(H)$ denotes the unit sphere of H. This may appear to have a positive answer, given the results in the previous sections. However, the answer is negative because the unitary orbit $\mathrm{U}(T)$ is not necessarily

SOT-dense in $\|T\| \cdot \mathrm{Sph}(H)$. One can easily construct the following counterexample in the sequence space $\ell^2(\mathbb{Z}) = \{\sum_{-\infty}^{\infty} a_n e_n : \sum |a_n|^2 < \infty\}$, where $\{e_n : n \in \mathbb{Z}\}$ is the canonical orthonormal basis of $\ell^2(\mathbb{Z})$. Let $T: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the bilateral weighted backward shift on the sequence space defined by

$$T\left(\sum_{n=-\infty}^{\infty} a_n e_n\right) = \sum_{n=-\infty}^{-1} \frac{1}{2} a_n e_{n-1} + \sum_{n=0}^{\infty} 2 a_n e_{n-1}.$$

The above formula defines a hypercyclic shift T due to a result of Salas [28, Theorem 2.1]. Let $A: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be defined by

$$A\left(\sum_{n=-\infty}^{\infty} a_n e_n\right) = 2 a_0 e_0.$$

Clearly $\|T\| = \|A\| = 2$. If we let

$$\mathcal{O} = \left\{S \in B\big(\ell^2(\mathbb{Z})\big) : \|Se_1 - Ae_1\| < \frac{1}{4}\right\},$$

be an SOT-open set containing A, then one can easily show that no operator in the unitary orbit $\mathcal{U}(T)$ is in $\mathcal{O}$. In fact, $\|Tf\| \geqslant \|f\|/2$ for every vector f in $\ell^2(\mathbb{Z})$, and so $\|U^{-1}TUf\| \geqslant \|f\|/2$ for any unitary operator U. Hence we have

$$\|U^{-1}TUe_1 - Ae_1\| \geqslant \frac{1}{2},$$

and so $U^{-1}TU \notin \mathcal{O}$.

We now switch our focus to weak hypercyclicity in the setting of a separable, infinite dimensional Banach space X. An operator T in $B(X)$ is *weakly hypercyclic* if there is a vector x in X for which its orbit $\mathrm{Orb}(T, x)$ is dense in X with respect to the weak topology. Any such vector x is called a *weakly hypercyclic vector* for T, and we use the notation $WHC(T)$ to denote the set of all weakly hypercyclic vectors for the operator T. By a similarity argument, an operator is weakly hypercyclic but not hypercyclic if and only if the same is true for each operator in the conjugate set; see Chan and Sanders [14] or Shkarin [29] for the existence of such operators. Bès and Chan [9] showed that the conjugate set of a weakly hypercyclic operator is SOT-dense in the operator algebra $B(X)$. Combining above discussion with Theorem 2.4 naturally leads to the question:

Question 4.1. If T is a weakly hypercyclic operator in $B(X)$ which fails to be hypercyclic, does the conjugate set $S(T)$ contain a path $\{F_t \in B(X): t \in [1,\infty)\}$ of operators which is SOT-dense in $B(X)$, and for which the set $\bigcap_{t\in[1,\infty)} WHC(F_t)$ of common weakly hypercyclic vectors is dense?

The techniques in the proof of Theorem 2.4 cannot be applied to the above problem because the existence of a norm dense orbit is vital to the construction of the desired SOT-dense path.

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