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Varieties of Restriction Semigroups and Varieties of Categories

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Abstract

The variety of restriction semigroups may be most simply described as that generated from inverse semigroups $(S, \cdot, ^{-1})$ by forgetting the inverse operation and retaining the two operations $x^+ = xx^{-1}$ and $x^* = x^{-1}x$. The subvariety $\mathbf{B}$ of *strict* restriction semigroups is that generated by the Brandt semigroups. At the top of its lattice of subvarieties are the two intervals $[\mathbf{B_2}, \mathbf{B_2M} = \mathbf{B}]$ and $[\mathbf{B_0}, \mathbf{B_0M}]$. Here $\mathbf{B_2}$ and $\mathbf{B_0}$ are respectively generated by the five-element Brandt semigroup and that obtained by removing one of its nonidempotents. The other two varieties are their joins with the variety of all monoids. It is shown here that the interval $[\mathbf{B_2}, \mathbf{B}]$ is isomorphic to the lattice of varieties of categories, as introduced by Tilson in a seminal paper on this topic. Important concepts, such as the local and global varieties associated with monoids, are readily identified under this isomorphism. Two of Tilson's major theorems have natural interpretations and application to the interval $[\mathbf{B_2}, \mathbf{B}]$ and, with modification, to the interval $[\mathbf{B_0}, \mathbf{B_0M}]$ that lies below it. Further exploration may lead to applications in the reverse direction.

Keywords: restriction semigroup; strict restriction semigroup; variety of categories

Mathematics Subject Classification: 20M07

1 Introduction

In one sense this paper is a continuation of the study of the lattice of varieties of restriction semigroups, initiated by the author in [7] and narrowed in [8], where attention was focused on characterizations of the subvariety $\mathbf{B}$ of strict restriction semigroups and the subvarieties $\mathbf{B_2}$ and $\mathbf{B_0}$ generated respectively by B_2 and B_0. Here B_2 is the five-element Brandt semigroup and B_0 is the restriction subsemigroup obtained by deleting one of its nonidempotents. In this paper the focus is on the lattice $\mathcal{L}(\mathbf{B})$ of subvarieties itself. Figure 1 summarizes the results of this aspect of our study.

However, it turns out that there is a rich interplay between the subvarieties of $\mathbf{B}$ and varieties of categories as introduced by Tilson [17] in order to study varieties and pseudovarieties of monoids and of semigroups. In particular, we show that the interval sublattice $[\mathbf{B_2}, \mathbf{B}]$ is isomorphic to the lattice of varieties of categories (and similarly for the respective lattices of pseudovarieties). Concepts introduced by Tilson, such as the global and local varieties of categories corresponding to a variety $\mathbf{N}$ of monoids, denoted here by $g\mathbf{N}$ and $\ell\mathbf{N}$, respectively, have natural intepretations under this isomorphism. Tilson's

cited paper was followed by one with Steinberg [15] (see Section 10, in particular, devoted to monoids and categories), much of which found its way into the monograph of Rhodes and the latter author [13].

In a sequel, we will extend the correspondence to relate the structure of free restriction semigroups in the various varieties to that of free categories in associated varieties and, likewise, to relate defining identities for the semigroup varieties to those of the category varieties.

Restriction semigroups have been introduced in various fashions, largely through the efforts of the so-called 'York' school, and have a long history, evidenced in the term 'weakly E-ample' by which they were until fairly recently known (see the surveys [4, 5]). From our universal-algebraic perspective, we may view them succinctly as follows. They are biunary semigroups in the signature $(\,\cdot\,,^+,^*\,)$. Any inverse semigroup $(S, \cdot, ^{-1})$ may be regarded as a restriction semigroup under the induced operations $x \mapsto x^+ = xx^{-1}$ and $x \mapsto x^* = x^{-1}x$, forgetting the inverse operation entirely. The restriction semigroups are the members of the variety generated by the semigroups induced from inverse semigroups in this way. A key point to note is that the role that groups play in the theory of inverse semigroups is now played by monoids. In this work, the term *monoid* will always refer either to restriction semigroups containing just one projection, or to categories with one object. In either case, for our purposes they are, in essence, 'plain' monoids.

From that perspective, the variety $\mathbf{B}$ is that generated by the Brandt semigroups. Elaborating on results from [7], the lattice $\mathcal{L}(\mathbf{B})$ of subvarieties decomposes as a union of 'slices' corresponding to fixed intersections with the variety $\mathbf{M}$ of monoids. Within the interval $[\mathbf{B}_2, \mathbf{B}]$, the slice corresponding to the monoid variety $\mathbf{N}$ is the interval $[\mathbf{B}_2\mathbf{N}, \mathbf{B} \cap \mathbf{mon}(\mathbf{N})]$. Here, and similarly below, $\mathbf{B}_2\mathbf{N}$ is shorthand for $\mathbf{B}_2 \vee \mathbf{N}$ and $\mathbf{B} \cap \mathbf{mon}(\mathbf{N})$ consists of the strict restriction semigroups all of whose submonoids belong to $\mathbf{N}$. Under the isomorphism with the lattice of varieties of categories, this interval corresponds to $[g\mathbf{N}, \ell\mathbf{N}]$.

The theorem of Tilson that the variety $\ell\mathbf{T}$ of locally trivial categories is the unique atom in the lattice of varieties translates into the statement that $\mathbf{B} \cap \mathbf{mon}(\mathbf{T})$ covers $\mathbf{B}_2$, where $\mathbf{T}$ is the variety of trivial monoids.

It was shown in [8] that the lattice $\mathcal{L}(\mathbf{B}_2)$ consists of the sequence of coverings $\mathbf{B}_2 \succ \mathbf{B}_0 \succ \mathbf{S} \succ \mathbf{T}$, where $\mathbf{S}$ denotes the variety of semilattices (considered as restriction semigroups). This provides a second decomposition of $\mathcal{L}(\mathbf{B})$ into four 'layers', the top one being $[\mathbf{B}_2, \mathbf{B}]$, the next $[\mathbf{B}_0, \mathbf{B}_0\mathbf{M}]$, then $[\mathbf{S}, \mathbf{S}\mathbf{M}]$ and $[\mathbf{T}, \mathbf{M}]$, the last two forming the decomposition of the lattice of varieties of semilattices of monoids determined in [7].

The techniques that we develop to study the interval $[\mathbf{B}_2, \mathbf{B}]$ also apply to the interval $[\mathbf{B}_0, \mathbf{B}_0\mathbf{M}]$, since the restriction semigroups belonging to $\mathbf{B}_0\mathbf{M}$ were characterized in [8] in a manner that translates easily into category-theoretic terms. The proof of the theorem of Tilson cited above enables the analogous covering $\mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{T}) \succ \mathbf{B}_0$ to be proved. More profoundly, the Bonded Component Theorem shows that, except in the case just demonstrated, the intersection of this layer with any slice, determined by a monoid variety $\mathbf{N}$ is trivial, consisting merely of $\mathbf{B}_0\mathbf{N}$. Thus the interval $[\mathbf{B}_0, \mathbf{B}_0\mathbf{M}]$ is isomorphic to the lattice of varieties of monoids, with a zero adjoined.

Our results are summarized in Figure 1. In Section 6, by verifying that finiteness may be preserved in all relevant steps, it is shown that the above results carry over to pseudovarieties of strict restriction semigroups and of categories, with Figure 1 again representing a summary.

The tool behind the varietal connection is the identification of categories with 'primitive' restriction semigroups. While this identification has long been known (a very special case of general work of Lawson [9]), what to us is most important is identification of the connected non-monoid categories with the 'completely 0-r-simple' restriction semigroups introduced by the author in [8]. Based on the concept of 'r-ideal', it was shown there that the strict restriction semigroups are precisely the subdirect products of such semigroups (and monoids).

In the final section, we determine the varieties of strict restriction semigroups that are closed under

proper covers (in the sense used in the semigroup-theoretic literature). Cornock [3, Theorem 10.1.6] showed that a variety has this property if and only if its free members are proper. Since the latter property heightens the prospects of a structural description of the free members, this has been considered of interest by researchers in the area. While everything in this section could have been stated and proved entirely in semigroup-theoretic terms, it is instructive and illuminating to interpret it in category-theoretic terms. The key concept in the latter terms is that of a 'star-division' of categories: a division $\phi : C \longrightarrow D$ that is injective on the set of all out-arrows and on the set of all in-arrows for each vertex of C.

The relative lengths of the histories of categories and of restriction semigroups lends itself to application of the former to the latter; however it is to be hoped that the different perspective in our work will ultimately lead to new insights into the former as well. It is natural to ask whether varieties of semigroupoids ('categories without identities') might be similarly linked with varieties of other, appropriately modified semigroups.
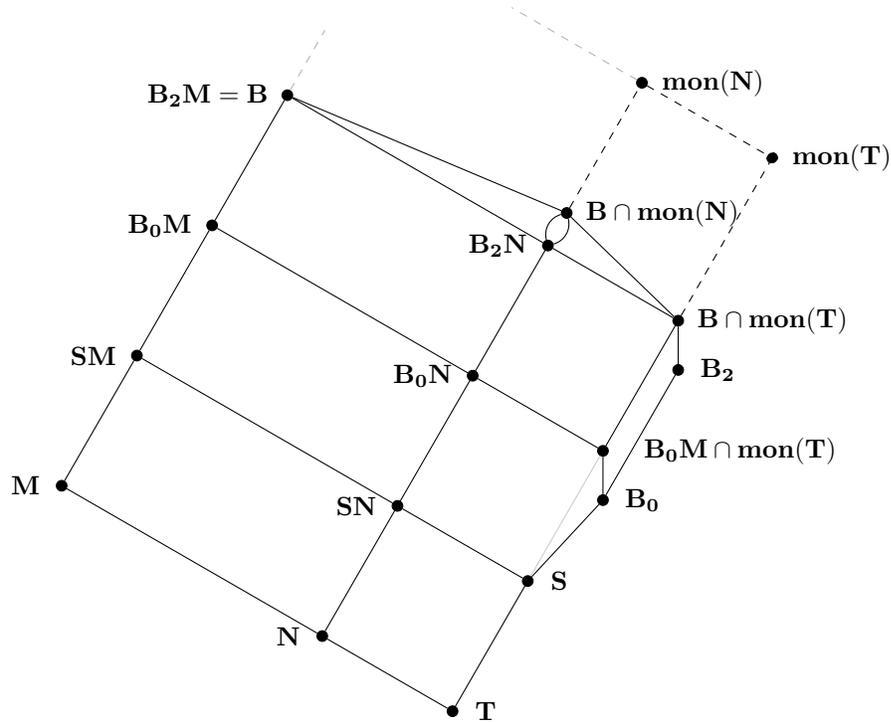

Figure 1: The lattice $\mathcal{L}(\mathbf{B})$.

2 Background on restriction semigroups

We first introduce restriction semigroups more formally, along with their basic properties and related definitions. In addition to referring the reader to the general citations [4, 5], especially for historical background, we do likewise to the 'prequels' [7, 8] for more concise general background and for more information about strict restriction semigroups and their varieties. Some of the material below is repeated,

for convenience, from the latter.

For the purposes of this work, it is appropriate to define these semigroups by means of their identities. A *restriction* semigroup is a biunary semigroup $(S, \cdot, ^+, ^*)$ that satisfies the 'left restriction' identities

$$x^+ x = x; \qquad (x^+ y)^+ = x^+ y^+; \qquad x^+ y^+ = y^+ x^+; \qquad xy^+ = (xy)^+ x,$$

the 'dual' identities, obtained by replacing $^+$ by * and reversing the order of each expression,

$$xx^* = x; \qquad (yx^*)^* = y^* x^*; \qquad x^* y^* = y^* x^*; \qquad y^* x = x(yx)^*,$$

along with $(x^+)^* = x^+$ and $(x^*)^+ = x^*$.

From the identities it follows that for all $x \in S$, x^+ is idempotent and $(x^+)^+ = x^+$. We term these idempotents the *projections* of S. Denote the set of projections by P_S and the set of all idempotents by E_S. Although, by the third identity, P_S is a semilattice, this need by no means be true of E_S. In the usual way, E_S is partially ordered by $e \leq f$ if $e = ef = fe$.

We refer the reader to standard texts such as [6] for general semigroup theory and [11] for background on inverse semigroups and their varieties.

If S is a restriction semigroup with zero, put $S^* = S \backslash \{0\}$ and $P^* = P_S \backslash \{0\}$.

In general, the terms 'homomorphism', 'congruence' and 'divides' will be used appropriate to context; that is, they should respect both unary operations for restriction semigroups (and the inverse operation in an inverse semigroup). Thus, for instance, a restriction semigroup divides a restriction semigroup T if it is a (biunary) homomorphic image of a (biunary) subsemigroup of T.

If S and T are restriction semigroups, a *relational morphism* (often termed a subhomomorphism in the literature, e.g. [3]) is a fully-defined relation ϕ from S to T such that $(a\phi)(b\phi) \subseteq (ab)\phi$, $(a\phi)^+ \subseteq a^+ \phi$ and $(a\phi)^* \subseteq a^* \phi$; it is a *division* if whenever $a\phi \cap b\phi \neq \emptyset$, then $a = b$. Such a division exists if and only if S divides T, written $S \prec T$, as in the previous paragraph.

We will use the term *monoid* (sometimes 'restriction monoid', for the sake of clarity) for a restriction semigroup having precisely one projection, since any semigroup with identity element 1 may be regarded as a restriction semigroup by setting $a^+ = a^* = 1$ for all a, and conversely. In this paper, S^1 will always denote the restriction monoid obtained by adjoining an identity element (its only projection, now) if it is not a monoid to begin with. A *submonoid* of a restriction semigroup S is a restriction subsemigroup that is a monoid, that is, contains a unique projection of S. We should note that in other work, the term 'restriction monoid' simply means a restriction semigroup that is a monoid. Since that context never arises in this paper, there should be no ambiguity.

Any inverse semigroup $(S, \cdot, ^{-1})$ may be regarded as a restriction semigroup by setting $x^+ = xx^{-1}$ and $x^* = x^{-1}x$ and 'forgetting' the inverse operation. In that case, $P_S = E_S$. This source of examples may be expanded upon by noting that any subsemigroup that is full (that is, contains all its idempotents) again induces such a restriction subsemigroup. The inverse semigroups of particular relevance to this paper are those that are completely 0-simple: the *Brandt semigroups*. Following [11, II.3], Brandt semigroups are semigroups representable in the form $B(G, I)$, where G is a group and I a nonempty set: $B(G, I) = (I \times G \times I) \cup \{0\}$, where $(i, g, j)(j, h, \ell) = (i, gh, \ell)$ and all other products are zero. In this terminology, the semigroup with parameters $|G| = 1$ and $|I| = n$ is usually denoted B_n. More generally, the *strict* inverse semigroups are those isomorphic to subdirect products of Brandt semigroups and groups, equivalently those that satisfy $\mathcal{D}$-majorization [11, II.4]. More generally still, the *completely semisimple* inverse semigroups are those whose principal factors are Brandt semigroups or groups. These terms will be extended to restriction semigroups in the next subsection.

As mentioned in the introductions, B_2 denotes the five-element, combinatorial Brandt semigroup $\{a, b, ab, ba, 0\}$ and B_0 its full subsemigroup $\{a, ab, ba, 0\}$. The semigroup B_0 is in a very natural way a

restriction semigroup that is already known to play an important role in the study of varieties of such semigroups [7, 8].

2.1 The generalized Green relations and strict restriction semigroups

In any restriction semigroup, put $\widetilde{\mathcal{R}} = \{(a,b) : a^+ = b^+\}$, $\widetilde{\mathcal{L}} = \{(a,b) : a^* = b^*\}$, $\widetilde{\mathcal{H}} = \widetilde{\mathcal{L}} \cap \widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{D}} = \widetilde{\mathcal{L}} \vee \widetilde{\mathcal{R}}$. Each contains the corresponding usual Green relation, $\widetilde{\mathcal{R}}$ is a left congruence and $\widetilde{\mathcal{L}}$ is a right congruence. In the standard literature, these relations would have been denoted $\widetilde{\mathcal{R}}_{P_S}$, $\widetilde{\mathcal{L}}_{P_S}$ and $\widetilde{\mathcal{H}}_{P_S}$, respectively, the notation above referring to a rather special case. Due to the potential for conflict, the author has used $\mathbb{R}, \mathbb{L}$ and $\mathbb{H}$ instead in the recent cited work [7, 8] but hopes the notation used here will prove standard in future. By analogy with the term 'combinatorial' for inverse semigroups in which $\mathcal{H}$ is the identical relation ι, a restriction semigroup will be called $\widetilde{\mathcal{H}}$-*combinatorial* if $\widetilde{\mathcal{H}} = \iota$. It is well known that the $\widetilde{\mathcal{H}}$-classes $\widetilde{\mathcal{H}}_e$, where $e \in P_S$, are precisely the maximal submonoids.

In contrast with the usual Green relations, $\widetilde{\mathcal{D}} \neq \widetilde{\mathcal{L}} \circ \widetilde{\mathcal{R}}$: it is the transitive closure of $\widetilde{\mathcal{L}} \cup \widetilde{\mathcal{R}}$. The next material in this section is extracted from [8, Sections 5 and 6].

To define $\widetilde{\mathcal{J}}$, we need the notion [8, Section 5] of an *r-ideal* (for 'restriction ideal') in S: an ideal I of S that is also a restriction subsemigroup. It is easily verified that the Rees factor semigroup S/I is again a restriction semigroup. (As usual, for technical reasons it is convenient to allow the empty set to be an r-ideal and, in that case, to put $S/I = S$.) A restriction semigroup S without zero is *r-simple* if S is the only r-ideal; a restriction semigroup with zero is *0-r-simple* if $\{0\}$ and S are its only r-ideals.

If $A \subseteq S$, denote by $rI(A)$ the r-ideal it generates. Since for any $a \in S$, $a = a^+ a = aa^*$, $rI(a) = rI(a^+) = rI(a^*)$. So if either $a \,\widetilde{\mathcal{L}}\, b$ or $a \,\widetilde{\mathcal{R}}\, b$, then $rI(a) = rI(b)$, and thus the same holds if $a \,\widetilde{\mathcal{D}}\, b$.

Define $\widetilde{\mathcal{J}}$ on S by $a \,\widetilde{\mathcal{J}}\, b$ if $rI(a) = rI(b)$. The set $rQ(a) = rI(a) \backslash \widetilde{\mathcal{J}}_a$ is an r-ideal of $rI(a)$ and the Rees factor semigroup $rI(a)/rQ(a)$ is the r-*principal factor* associated with a. It is 0-r-simple, or r-simple in case $rQ(a)$ is empty. As in the semigroup case, the r-principal factor is essentially the $\widetilde{\mathcal{J}}$-class itself, with products that do not lie within the class (if any) sent to zero. A restriction semigroup S without zero is r-simple if and only if it has only one $\widetilde{\mathcal{J}}$-class; a restriction semigroup with zero is 0-r-simple if and only if $\{0\}$ and $S \backslash \{0\}$ are its only $\widetilde{\mathcal{J}}$-classes.

By analogy with inverse semigroups, a restriction semigroup is *completely r-semisimple* [8] if the distinct projections within any $\widetilde{\mathcal{D}}$-class are incomparable. In such a semigroup, $\widetilde{\mathcal{J}} = \widetilde{\mathcal{D}}$.

Following [8], a restriction semigroup without zero is *primitive* if each projection is minimal. These are simply the monoids (regarded as restriction semigroups). A restriction semigroup S with zero is *primitive* if each of its nonzero projections is minimal; it is *completely 0-r-simple* if it is primitive and 0-r-simple. Since any such semigroup is completely r-semisimple, $\widetilde{\mathcal{J}} = \widetilde{\mathcal{D}}$.

Thus a restriction semigroup is completely r-semisimple if and only if each r-principal factor is completely 0-r-simple or a monoid. In that case, if it has a zero it is primitive if and only if it is the 0-direct sum of its r-principal factors. The following property of the multiplication will frequently be used.

LEMMA 2.1. *In any completely 0-r-simple semigroup, if $a, b \neq 0$, then $ab \neq 0$ if and only if $a^* = b^+$. Thus in any completely r-semisimple semigroup, if $a \,\widetilde{\mathcal{D}}\, b$, then $ab \,\widetilde{\mathcal{D}}\, a$ if and only if $a^* = b^+$, in which case $a \,\widetilde{\mathcal{R}}\, ab \,\widetilde{\mathcal{L}}\, b$.*

Similarly to the case of Brandt semigroups [11, Section II.3], homomorphisms on completely 0-r-simple restriction semigroups are constrained.

LEMMA 2.2. *Let S be a completely 0-r-simple semigroup and $\phi : S \longrightarrow T$ a homomorphism. Either ϕ is projection-separating or $a\phi = 0\phi$ for all $a \in S$.*

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Proof. Clearly $\{a \in S : a\phi = 0\phi\}$ is an r-ideal of S and so is either $\{0\}$ or all of S, by 0-r-simplicity. If $e\phi = f\phi$ for distinct projections e, f, then $e\phi = (ef)\phi = 0\phi$, by primitivity, and the result follows. $\square$

Again by analogy with inverse semigroups [11], a restriction semigroup is *strict* if it is a subdirect product of completely 0-r-simple semigroups and monoids. Likewise, a restriction semigroup satisfies $\widetilde{\mathcal{D}}$-*majorization* if whenever f, g, h are projections such that $f > g$, $f > h$ and $g \; \widetilde{\mathcal{D}} \; h$, then $g = h$. This property was investigated in [8, Section 6]. It was shown that any such semigroup is completely r-semisimple.

Further, it was shown in the proof of [8, Proposition 6.3] that for each $\widetilde{\mathcal{J}}$-class $J = \widetilde{\mathcal{J}}_a$, say, there is a homomorphism π_J of S upon the associated r-principal factor: if $s \in S$ and $rI(a) \nsubseteq rI(s)$, then $s\pi_J = 0$; otherwise, $s\pi_J = es$, where e is the unique projection in J that is below s^+. These homomorphisms separate S and therefore it is a subdirect product of its r-principal factors; further, $\widetilde{\mathcal{D}}$-majorization implies that the latter are primitive. Hence S is strict.

The main structural result in [8] was the following, contained in Theorem 8.1 (see the subsection on varieties for further equivalences).

RESULT 2.3. *A restriction semigroup is strict if and only if it satisfies $\widetilde{\mathcal{D}}$-majorization.*

PROPOSITION 2.4. *On any strict restriction semigroup S, $\widetilde{\mathcal{H}}$ is a projection-separating congruence and $S/\widetilde{\mathcal{H}}$ is locally finite. Thus if S is finitely generated, then P_S is finite.*

Proof. We use the results of [8, Section 9] (also see Result 2.12). It was shown there that $\widetilde{\mathcal{H}}$ is a congruence on any strict restriction semigroup S and that $S/\widetilde{\mathcal{H}}$ belongs to the variety of restriction semigroups generated by the finite semigroup B_2, which is a locally finite variety by standard universal algebraic arguments [2]. So if S is finitely generated, $S/\widetilde{\mathcal{H}}$ is finite. Now it is clear from its definition that $\widetilde{\mathcal{H}}$ separates projections and thus $P_S \cong P_{S/\widetilde{\mathcal{H}}}$. $\square$

The following technical lemma will find key uses in the sequel.

LEMMA 2.5. *Let S be a finitely generated strict restriction semigroup, T a completely 0-r-simple semigroup and $\theta : S \longrightarrow T$ a surjective homomorphism. Then there is a $\widetilde{\mathcal{J}}$-class J of S and a projection-separating, surjective homomorphism ϕ from the r-principal factor associated with J upon T such that $\theta = \pi_J \phi$.*

Proof. By the proposition above, P_S is finite. For each nonzero projection $e \in T$, let $\overline{e}$ denote the least projection among its preimages in the semilattice P_S.

Let t be an arbitrary nonzero element of T and let s be any preimage in S. Put $x = \overline{t^+} s \overline{t^*}$. Then $x\phi = t^+ t t^* = t$ and so $x^+\phi = t^+$. Therefore $x^+ \geq \overline{t^+}$. In fact equality holds, since $x^+ = \overline{t^+}(s\overline{t^*})^+ \leq \overline{t^+}$, using the defining axioms. Similarly, $x^* = \overline{t^*}$. As a result, if e and f are nonzero projections of T such that $e \; \widetilde{\mathcal{R}} \; t \; \widetilde{\mathcal{L}} \; f$, then $\overline{e} \; \widetilde{\mathcal{R}} \; x \; \widetilde{\mathcal{L}} \; \overline{f}$. But any two nonzero projections of T are connected by a sequence of relations of this sort (a zigzag, in the language of [8]), so their preimages are likewise connected. In other words, all the projections $\overline{e}$ lie in a single $\widetilde{\mathcal{D}} = \widetilde{\mathcal{J}}$-class J, say, of S.

Let P be the r-principal factor associated with J and define $\phi : P \longrightarrow T$ by putting $a\phi = a\theta$ for $a \in J$ and $0\phi = 0$. The map is well defined, since for any $a \in J$, a is $\widetilde{\mathcal{D}}$-related to some preimage of a nonzero element of T, and so its image is also nonzero. If $a, b \in J$ and $ab = 0$ in P, then $a^* \neq b^+$ and so $(a\theta)^* \neq (b\theta)^+$, whence $(a\theta)(b\theta) = 0$ in T. Thus ϕ is a homomorphism, which is projection-separating and surjective, by the preceding paragraph.

By the definition of π_J, if $s \in S$ and $rJ \subseteq rI(s)$, then $s\pi_J = es \in J$, where e is the unique projection of J such that $e \leq s^+$. So $(s\pi_J)\phi = (es)\theta = s\theta$ (since $e\theta \leq s^+\theta$ and both are nonzero). Otherwise, for any nonzero element t of T, the element $x = \overline{t^+} s \overline{t^*}$ has $x^+ \leq s^+$ and so $x \notin J$. Therefore $s\theta \neq t$ and so $s\theta = 0 = s\pi_J\phi$. $\square$

2.2 Varieties of restriction semigroups

Denote by $\mathbf{R}$ the variety of restriction semigroups, considered as biunary semigroups. If $\mathbf{V}$ is any subvariety then $\mathcal{L}(\mathbf{V})$ denotes its lattice of subvarieties. The varieties of restriction semigroups consisting of trivial semigroups, monoids, and semilattices, respectively, will be denoted $\mathbf{T}$, $\mathbf{M}$ and $\mathbf{S}$. (The last of these was denoted $\mathbf{SL}$ in [7, 8].) As a subvariety of $\mathbf{R}$, $\mathbf{M}$ may be defined by the identity $x^+ = y^+$. Note that subvarieties of $\mathbf{M}$ are essentially varieties of monoids and we shall treat them as such. The variety $\mathbf{S}$ may be defined by the identity $x = x^+$.

The first precursor [7] to this paper provided an overview of the lattice $\mathcal{L}(\mathbf{R})$. While containing several general results, it focused on the 'bottom' of the lattice of varieties of restriction semigroups (and the analogues for left restriction semigroups). We summarize the relevant theorems. In the first result, μ denotes the greatest projection-separating congruence.

RESULT 2.6. *[7, Theorems 3.1, 3.3] If $\mathbf{V} \in \mathcal{L}(\mathbf{R})$, then $\mathbf{V} \vee \mathbf{M} = \{S \in \mathbf{R} : S/\mu \in \mathbf{V}\}$. Thus the map $\mathbf{V} \mapsto \mathbf{V} \vee \mathbf{M}$ is a complete lattice homomorphism.*

We shall in general abbreviate $\mathbf{V} \vee \mathbf{M}$ to $\mathbf{VM}$. In fact, it is convenient to denote the join $\mathbf{V} \vee \mathbf{N}$ by $\mathbf{VN}$ for *any* variety $\mathbf{N}$ of monoids.

We turn next to background from [8], the second precursor, which is essentially devoted to characterizations of the variety $\mathbf{B}$ of restriction semigroups that is generated by the Brandt semigroups (equivalently, by the strict inverse semigroups), together with some remarkable properties of certain of its subvarieties.

The proof of the equivalence of strictness with $\widetilde{\mathcal{D}}$-majorization noted in Result 2.3, one direction of which was demonstrated there, was completed by means of the discovery of a (necessarily infinite) basis of identities for $\mathbf{B}$.

RESULT 2.7. *Further to Result 2.3, the following are equivalent for a restriction semigroup S:*

(1) $S \in \mathbf{B}$;

(2) S satisfies any one of the infinite sequences of identities listed in [8, Proposition 7.2];

(3) S is strict.

PROPOSITION 2.8. *On $\mathcal{L}(\mathbf{B})$, the map $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{M}$ is a complete lattice homomorphism.*

Proof. It only need be shown that the map respects arbitrary joins. Suppose $\{\mathbf{V}_i : i \in I\}$ is a family of strict restriction varieties and that the monoid M belongs to their join. We may assume M to be finitely generated. Then M is a homomorphic image of a finitely generated restriction subsemigroup S of a product $\prod_{i \in I} S_i$, where each $S_i \in \mathbf{V}_i$. Apply Lemma 2.5 to the completely 0-r-simple semigroup T obtained by adjoining a zero to M. There is $\widetilde{\mathcal{J}}$-class D of S and a projection-separating homomorphism of its r-principal factor upon T. From the projection-separating property it follows that the inverse image of M is a submonoid N of D and therefore of $\prod_{i \in I} S_i$. It is easily seen, then, that N embeds in a product of submonoids of the respective semigroups S_i. As a result, $M \in \bigvee_{i \in I}(\mathbf{V}_i \cap \mathbf{M})$. $\square$

The 'base' of $\mathcal{L}(\mathbf{B})$ consists of two layers, the intervals $[\mathbf{T}, \mathbf{M}]$ and $[\mathbf{S}, \mathbf{SM}]$, as follows.

RESULT 2.9. *[7, Theorems 3.6, 3.8] (1) The following are equivalent for a restriction semigroup S: (a) $S \in \mathbf{SM}$; (b) S satisfies $x^+ = x^*$; (c) S is a [strong] semilattice of monoids.*

(2) The lattice $\mathcal{L}(\mathbf{SM})$ is isomorphic to $\mathcal{L}(\mathbf{S}) \times \mathcal{L}(\mathbf{M})$ under the map $\mathbf{V} \mapsto (\mathbf{V} \cap \mathbf{S}) \vee (\mathbf{V} \cap \mathbf{M})$. If $\mathbf{V}$ is not simply a variety of monoids, then $\mathbf{V} = \mathbf{SN}$, where $\mathbf{N} = \mathbf{V} \cap \mathbf{M}$.

For any variety $\mathbf{V}$ of restriction semigroups it is easily verified that the class $\mathbf{loc}(\mathbf{V}) = \{S \in \mathbf{R} : eSe \in \mathbf{V} \ \forall e \in P_S\}$ is again a variety. We say that its members are 'locally' in $\mathbf{V}$. In contrast with the situation for strict inverse semigroups [11], $\mathbf{B}$ is strictly contained in $\mathbf{loc}(\mathbf{SM})$.

Given a variety $\mathbf{N}$ of monoids, let $\mathbf{mon}(\mathbf{N})$ consist of the restriction semigroups S all of whose (maximal) submonoids $\widetilde{\mathcal{H}}_e$, $e \in P_S$, belong to $\mathbf{N}$. In general, $\mathbf{mon}(\mathbf{N})$ is not a variety. However, it was shown in [8, Proposition 8.3] that $\mathbf{B} \cap \mathbf{mon}(\mathbf{N}) = \mathbf{B} \cap \mathbf{loc}(\mathbf{SN})$ and is therefore a subvariety of $\mathbf{B}$. It is then clearly the maximum such variety whose intersection with $\mathbf{M}$ is $\mathbf{N}$. In particular, the class of strict restriction semigroups, all of whose submonoids are trivial, is the subvariety $\mathbf{B} \cap \mathbf{loc}(\mathbf{S})$, defined within $\mathbf{B}$ by the additional identity $exe = (exe)^2$.

COROLLARY 2.10. *The lattice $\mathcal{L}(\mathbf{B})$ is the disjoint union of the intervals $[\mathbf{N}, \mathbf{B} \cap \mathbf{mon}(\mathbf{N})]$, $\mathbf{N} \in \mathcal{L}(\mathbf{M})$.*

We next consider the join with $\mathbf{M}$. Recall the notation $\mathbf{VM}$ for $\mathbf{V} \vee \mathbf{M}$. Also recall from Proposition 2.4 that $\widetilde{\mathcal{H}}$ is a projection-separating congruence on any strict restriction semigroup S (and $S/\widetilde{\mathcal{H}}$ is $\widetilde{\mathcal{H}}$-combinatorial). Therefore $\mu = \widetilde{\mathcal{H}}$ and so Result 2.6 yields the following.

COROLLARY 2.11. *For any variety $\mathbf{V}$ of strict restriction semigroups, $\mathbf{VM} = \{S \in \mathbf{B} : S/\widetilde{\mathcal{H}} \in \mathbf{V}\}$.*

Let $\mathbf{B}_2$ denote the variety generated by B_2 (equivalently, by the combinatorial strict inverse semigroups) and $\mathbf{B}_0$ that generated by B_0. Rather than restate the stand-alone theorems from [8], we characterize $\mathbf{B}_2$ within the context of strict restriction semigroups and then $\mathbf{B}_0$ within the larger variety $\mathbf{B}_2$.

RESULT 2.12. *[8, Theorem 9.3, Corollaries 9.4, 9.5, Theorem 10.3, Corollary 10.2] (1) The following are equivalent for a strict restriction semigroup S: $S \in \mathbf{B}_2$; S satisfies the identity $x(x^+y)^* = y(y^+x)^*$; S is $\widetilde{\mathcal{H}}$-combinatorial. Thus $\mathbf{B} = \mathbf{B}_2\mathbf{M}$.*

(2) The following are equivalent for a member S of $\mathbf{B}_2$: $S \in \mathbf{B}_0$; S satisfies the identity $xyx = x^2y^2$; S contains no regular elements other than projections.

(3) The interval $[\mathbf{T}, \mathbf{B}_2]$ comprises the chain of coverings $\mathbf{T} \prec \mathbf{S} \prec \mathbf{B}_0 \prec \mathbf{B}_2$.

This yields the first stage of the decomposition of $\mathcal{L}(\mathbf{B})$ shown in Figure 1.

THEOREM 2.13. *The lattice $\mathcal{L}(\mathbf{B})$ is the disjoint union of the four intervals $[\mathbf{T}, \mathbf{M}]$, $[\mathbf{S}, \mathbf{SM}]$, $[\mathbf{B}_0, \mathbf{B}_0\mathbf{M}]$ and $[\mathbf{B}_2, \mathbf{B}_2\mathbf{M} = \mathbf{B}]$.*

Proof. Let $\mathbf{V} \in \mathcal{L}(\mathbf{B})$. According to Result 2.12(3), $\mathbf{V} \cap \mathbf{B}_2$ is either $\mathbf{T}$, $\mathbf{S}$, $\mathbf{B}_0$ or $\mathbf{B}_2$; and in each case the intersection consists of the $\widetilde{\mathcal{H}}$-combinatorial members of $\mathbf{V}$. According to Corollary 2.11, $\mathbf{V} \subseteq (\mathbf{V} \cap \mathbf{B}_2)\mathbf{M}$. $\qquad\square$

The sublattice $\mathcal{L}(\mathbf{SM})$ was determined in Result 2.9. Next let us specialize Corollary 2.10 to the interval $[\mathbf{B}_2, \mathbf{B}]$. Since $B_2 \in \mathbf{mon}(\mathbf{T})$, the upper ends of each subinterval belong to $[\mathbf{B}_2, \mathbf{B}]$.

COROLLARY 2.14. *The interval $[\mathbf{B}_2, \mathbf{B}]$ is the disjoint union of the intervals $[\mathbf{B}_2\mathbf{N}, \mathbf{B} \cap \mathbf{mon}(\mathbf{N})]$, $\mathbf{N} \in \mathcal{L}(\mathbf{M})$.*

Finally, we consider $\mathbf{B}_0\mathbf{M}$. Again, the result is framed relative to $\mathbf{B}$.

RESULT 2.15. *[8, Theorem 10.6, Corollary 10.8] (1) The following are equivalent for a strict restriction semigroup S: $S \in \mathbf{B}_0\mathbf{M}$; $S/\widetilde{\mathcal{H}} \in \mathbf{B}_0$; S satisfies the identities $(xyx)^+ = (x^2y^2)^+$ and $(xyx)^* = (x^2y^2)^*$; if $e, f \in P_S$ then $\widetilde{\mathcal{R}}_e \cap \widetilde{\mathcal{L}}_f$ and $\widetilde{\mathcal{L}}_e \cap \widetilde{\mathcal{R}}_f$ cannot both be nonempty.*

(2) The interval $[\mathbf{M}, \mathbf{B}]$ comprises the chain of coverings $\mathbf{M} \prec \mathbf{SM} \prec \mathbf{B}_0\mathbf{M} \prec \mathbf{B}_2\mathbf{M} = \mathbf{B}$.

The last condition in (1) asserts that no $\widetilde{\mathcal{D}}$-class of S contains a 'square' in its eggbox picture (as used in [8]), so we may term such semigroups *square-free*.

Similarly to Corollary 2.14, we specialize Corollary 2.10 to the interval $[\mathbf{B}_2, \mathbf{B}]$.

COROLLARY 2.16. *The interval* $[\mathbf{B}_0, \mathbf{B}_0\mathbf{M}]$ *is the disjoint union of the intervals* $[\mathbf{B}_0\mathbf{N}, \mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{N})]$, $\mathbf{N} \in \mathcal{L}(\mathbf{M})$.

3 Categories

The material in this section is taken in large part from [17], to which the reader is referred for other standard graph-theoretic and category-theoretic terms, and for more depth. The work in [17] was pursued further in [15] and in the monograph [13]. The connection with restriction semigroups is new (albeit perhaps folklore in its superficial aspects).

In this context, a (directed) *graph* C comprises a set Obj C of objects and a set Arr C of arrows, where for each $e, f \in$ Obj C, $C(e, f)$ — or Arr (e, f) if the context is clear — denotes the homset consisting of the arrows from e to f, written $e :\to f$. A *category* is a graph C on which the product of consecutive arrows is defined, associative in the natural way, along with (partial) identity arrows 1_e, $e \in$ Obj C. From this point of view, a monoid may also be regarded as a one-object category. The sets $C(e) = C(e, e)$ are called the *local monoids* of C. Sometimes, the set of objects of a category is suppressed and identified with its set of identity arrows. We shall do so when convenient, in order to avoid awkward technicalities (e.g. in Proposition 4.1).

Given any nonempty graph X, the *free category* X^* on X has as its objects those of X and as its arrows the paths on X, the identity arrow at a vertex being the empty path there. The graph *generates* a category C if there is a quotient morphism $X^* \longrightarrow C$; C is finitely generated if it is generated by a finite graph.

A category is *connected* if its underlying *undirected* graph is connected, and *strongly connected* (*bonded* in [17]) if its underlying *directed* graph is connected. In each context, any category is the disjoint union (or *coproduct* in [17]) of the respective components. A category is *trivial* if each homset has at most one member. (Usually, a category is called trivial if it has only one object and one arrow, but the reason for its use in [17] will become clearer below.)

Three small categories will play important roles in the sequel. The first two are trivial categories, in the sense just used.

- Let C_0 be the category with two vertices e and f, an arrow $a : e \to f$ and the two identity arrows;
- Let C_2 be the category obtained from C_0 by adding an arrow $b : f \to e$ such that $ab = 1_e$ and $ba = 1_f$;
- Let A_2 (following [17]) be the category obtained from C_0 by instead adding another arrow $b : e \to f$.

A *relational morphism* $\phi : C \to D$ of *graphs* consists of a function $\phi : $ Obj $C \to$ Obj D and a family of fully-defined relations $\phi_{e,f} : C(e, f) \to D(e\phi, f\phi)$, $e, f \in$ Obj C. The homset relations are generally simply denoted ϕ, as well. If C and D are categories, ϕ is a *relational morphism of categories* if, further, it satisfies $1_{e\phi} \in 1_e\phi$ for each $e \in$ Obj C, and $(x\phi)(y\phi) \subseteq (xy)\phi$ for all pairs of consecutive arrows x, y.

A relational morphism ϕ is a *division* if each homset relation is injective, that is, if x and y are coterminal arrows and $x\phi \cap y\phi \neq \emptyset$, then $x = y$. We write $C \prec D$, 'C divides D', when there is a division $\phi : C \to D$.

We term a division $\phi : C \longrightarrow D$ of categories a *star division* if whenever arrows a and b have either the same initial vertex or the same terminal vertex, and $a\phi \cap b\phi \neq \emptyset$, then $a = b$. In that case we say that C *star-divides* D.

It is convenient to isolate a convenient technical tool introduced by Tilson.

RESULT 3.1. *[17, Lemma 2.6, Corollary 2.7] If C and D are categories and $\phi : C \longrightarrow D$ is a graph morphism that satisfies all the requirements to be a category division except, possibly, that $1_{c\phi} \in 1_c\phi$ for each $c \in \mathrm{Obj}\ C$, then it may be extended to a division $\phi^+ : C \longrightarrow D$ by adding $1_{c\phi}$ to the set $1_c\phi$ for each c.*

Category morphisms in the usual sense (functors, in the abstract setting) are special types of relational morphisms but do not play a significant role in this work except in the following special cases. A morphism is *quotient* if the object mapping is a bijection and each homset mapping is a surjection. These induce *congruences*, and vice versa. One rather obvious congruence, denoted τ, collapses each homset, so that the quotient category is trivial, in the sense used above. A morphism is *faithful* if each homset mapping is injective.

RESULT 3.2. *[17, Section 2] Every faithful morphism and the inverse of every quotient morphism is a division. Conversely, every division factors as the inverse of a quotient morphism, followed by a faithful morphism.*

As a consequence, when restricted to monoids, division reduces to the usual notion: $M \prec N$ if M is a quotient of a submonoid of N.

3.1   Varieties of categories

Again following [17], a *variety* of categories is a family $\mathbf{W}$ of (small) categories that is closed under products and under category division. In view of Result 3.2, this is the case if and only if (i) $\mathbf{W}$ is closed under products and quotient morphisms, and (ii) whenever ϕ is a faithful morphism from a category C to a member of $\mathbf{W}$, then C again belongs to $\mathbf{W}$. Tilson showed that varieties of categories are precisely the classes that are definable by laws on graphs. In this paper, we do not need that perspective on the topic.

Denote by $\mathbf{Cat}$ the variety of all categories. If $\mathbf{W}$ is a variety of categories, then its subvarieties once again form a lattice $\mathcal{L}(\mathbf{W})$. With each variety $\mathbf{N}$ of monoids, Tilson associated two varieties of categories:

- $\ell\mathbf{N}$ consists of the categories for which each local monoid belongs to $\mathbf{N}$ (and are said to be 'locally' in $\mathbf{N}$).
- $g\mathbf{N}$ consists of the categories that divide a member of $\mathbf{N}$ (and are said to be 'globally' in $\mathbf{N}$). This was denoted $\mathbf{N}_C$ in [17].

It is clear that $g\mathbf{M} = \ell\mathbf{M} = \mathbf{Cat}$. Moreover, it is easily seen that $g\mathbf{N} \subseteq \ell\mathbf{N}$ in general. Varieties of monoids for which equality holds are said to be *local* and (especially in the context of pseudovarieties) play an important role in the study of finite semigroups.

The variety $\mathbf{I} = g\mathbf{T}$ consists precisely of the trivial categories [17, Lemma 3.1] and, since it is generated by the one-element monoid, is the least element in the lattice $\mathcal{L}(\mathbf{Cat})$. Note that any other variety of categories contains the locally trivial category A_2 introduced above and so $g\mathbf{T}$ is properly contained in $\ell\mathbf{T}$, the variety of locally trivial categories. Tilson proved the following nontrivial result.

RESULT 3.3. *[17, Theorems 7.1 and 8.1] Every locally trivial category divides a power of the category A_2 (and divides a finite power if it is itself finite). Therefore the variety $\ell\mathbf{T}$ is generated by A_2 and is contained in every nontrivial variety of categories. Thus $\ell\mathbf{T} \succ \mathbf{I}$.*

According to [17, (3.9)], C divides the product of its connected components. As a result, any variety of categories is generated by its connected members. Decidedly nontrivial is the "Bonded Component Theorem":

RESULT 3.4. *[17, Theorem 11.3] Let C be a category that is not locally trivial. Then C divides a product of its strongly connected components (and divides a finite product if it is itself finite). Thus any such variety is generated by its strongly connected members.*

We shall call a category C *anticyclic* if for any two distinct vertices e and f of C, either $C(e, f)$ or $C(f, e)$ is empty. In terms of the underlying graph, this means that there is no closed path other than a loop at a vertex. Equivalently, its strongly connected components are just its local monoids. (There appears to be no common name for this class of graphs in the literature. This term was used in [10].)

Note (cf [17, Section 7]) that C is anticyclic if and only if Obj C can be partially ordered by setting $e \geq f$ if there is a path (and so an edge) from e to f in the underlying graph. Thus the trivial, anticyclic categories are precisely the posets, when regarded in this fashion. (Since the class of anticyclic categories contains the trivial monoid, it is therefore not closed under division.) It follows that a category C is anticyclic if and only if the quotient category C/τ is a poset.

The locally trivial, anticyclic graphs are called *loop-free* in [17]. (The definition there is different, but [17, Lemma 7.3] demonstrates the equivalence.)

Although Result 3.4 has implications for varieties of strict restriction semigroups in general, we shall only apply it to anticyclic categories where, as noted above, the strongly connected components are simply the local monoids.

COROLLARY 3.5. *Let C be anticyclic category that is not locally trivial. Then C divides a product of its local monoids (and divides a finite product if it is itself finite).*

4   Primitive restriction semigroups and categories

In this section we formalize and go far beyond the folklore knowledge that primitive restriction semigroups and (small) categories are in essence the same thing.

If S is a primitive restriction semigroup *without* zero, then it is a monoid. Define $C(S) = S$, now treating the monoid S as a category. If S is a primitive restriction semigroup *with* zero, the category $C(S)$ is defined as follows:

- Obj $C(S) = P^*$;
- Arr $C(S) = S^*$, where Arr (e, f) is the $\widetilde{\mathcal{H}}$-class $\widetilde{\mathcal{R}}_e \cap \widetilde{\mathcal{L}}_f$, if nonempty, and otherwise empty, so that $a : a^+ \to a^*$, for any $a \in S^*$;
- if $a \in$ Arr (e, f) and $b \in$ Arr (f, g), then the product ab is that in S^*.

As noted in Lemma 2.1, in this context, the product of two nonzero elements a and b in S is again nonzero if and only if $a^* = b^+$. Thus the product in $C(S)$ is well defined. (Note that our definition is just the *trace* of S [9], modified by omitting the zero element entirely.) The local monoids of $C(S)$ are the maximal nonzero submonoids of S.

Note that if $e, f \in$ Obj $C(S)$, then Arr $(e.f) \neq \emptyset$ if and only if $e \; \widetilde{\mathcal{R}} \; a \; \widetilde{\mathcal{L}} \; f$ in S for some $a \in S^*$. It follows that the connected components of $C(S)$ correspond to the nonzero $\widetilde{\mathcal{D}}$-classes of S itself. In fact, the connected components are precisely the categories $C(F)$ associated with the nonzero principal factors F of S. In particular, $C(S)$ is connected if and only if S is either completely 0-r-simple or a monoid.

Clearly S is $\widetilde{\mathcal{H}}$-*combinatorial* if and only if $C(S)$ is trivial; and S is square-free if and only if $C(S)$ is anticyclic.

Given a category C, $R(C)$ is its *consolidation*, as termed by Tilson [17] (but there denoted C_{cd} and expressed in somewhat different terms):

- if C is simply a monoid, $\mathrm{R}(C) = C$, regarded as a restriction semigroup;
- otherwise, $\mathrm{R}(C) = \mathrm{Arr}\, C \cup \{0\}$, using the previously defined products and setting all previously undefined products to zero.

It is easily verified that $\mathrm{R}(C)$ is a restriction semigroup, where for $a \in C(e, f)$, then $a^+ = 1_e$ and $a^* = 1_f$ (and $0^+ = 0^* = 0$). We shall generally follow the practice, noted in Section 3, of identifying objects e with their identity arrows 1_e. Thus $P_{\mathrm{R}(C)}$ is the semilattice $\mathrm{Obj}\, C \cup \{0\}$, considered as an antichain with zero adjoined.

Consider the three examples of categories itemized in the preceding section:

- $\mathrm{R}(C_0) = B_0$;
- $\mathrm{R}(C_2) = B_2$;
- $\mathrm{R}(A_2)$ is the primitive restriction semigroup TR_2 introduced in [8, Proposition 3.1], which is locally a semilattice but has nontrivial $\widetilde{\mathcal{H}}$-class $\widetilde{\mathcal{R}}_e \cap \widetilde{\mathcal{L}}_f = \{a, b\}$.

It is immediate from the definitions that $\mathrm{R}(\mathrm{C}(S)) = S$ for all primitive restrictions semigroups S, and $\mathrm{C}(\mathrm{R}(C)) = C$ for all categories C. In the following summary, those connections that have not already been stated are easily verified.

PROPOSITION 4.1. *The mappings* $\mathrm{R} : C \mapsto \mathrm{R}(C)$ *and* $\mathrm{C} : S \mapsto \mathrm{C}(S)$ *are mutually inverse bijections between the classes of all (small) categories and of all primitive restriction semigroups. Under this correspondence:*

(1) monoids correspond to monoids;

(2) connected categories that are not monoids correspond to completely 0-r-simple restriction semigroups;

(3) coproducts correspond to 0-direct unions;

(4) finitely generated categories correspond to finitely generated restriction semigroups;

(5) trivial categories correspond to $\widetilde{\mathcal{H}}$-combinatorial primitive restriction semigroups, that is, to primitive members of $\mathbf{B}_2$;

(6) anticyclic [loop-free] categories correspond to square-free [square-free and locally trivial] primitive restriction semigroup; posets correspond to primitive members of $\mathbf{B}_0$.

Extending the proposition to include the appropriate homomorphisms could indeed be turned into a categorical equivalence, but the simplest connection – and the one we shall use in the sequel – is in the 'connected' case. In the case of monoids, there is nothing to do. Turning to the case of completely 0-r-simple restriction semigroups, recall from Lemma 2.2 that a homomorphism on such a semigroup is either projection-separating or induces the universal congruence.

PROPOSITION 4.2. *Let S and T be completely 0-r-simple restriction semigroups and $\phi : S \longrightarrow T$ a surjective, projection-separating homomorphism. Then ϕ induces a quotient morphism $\mathrm{C}(\phi) : \mathrm{C}(S) \longrightarrow \mathrm{C}(T)$.*

Let C and D be connected categories that are not monoids, and $\theta : C \longrightarrow D$ be a quotient morphism. Then θ induces a surjective, projection-separating homomorphism $\mathrm{R}(\theta) : \mathrm{R}(C) \longrightarrow \mathrm{R}(D)$.

The mappings $\phi \mapsto \mathrm{C}(\phi)$ and $\theta \mapsto \mathrm{R}(\theta)$ are mutually inverse.

Proof. In the first case, the object map is the restriction of ϕ to P_S^* and the arrow map is the restriction to P^*. In the second case, $R(\theta)$ is defined on S^* by the arrow maps, and 0 is mapped to 0. The details are routine to check. □

The connection between relational morphisms could also be stated generally, but the only case that we need is the following.

PROPOSITION 4.3. *Let M be a monoid. Any relational morphism from a completely 0-r-simple restriction semigroup S to M induces, by restriction and the correspondence of Proposition 4.1, a relational morphism $C(S) \longrightarrow M$. Conversely, any relational morphism from a category C to M induces a relational morphism $R(C) \longrightarrow M$.*

Proof. In the first case, suppose ϕ is such a relational morphism. It is straightforward to check that the restriction of ϕ to S^* (and to P^* for the object map) yields a relational morphism of categories.

In the second case, suppose θ is such a relational morphism. Put $S = R(C)$. The homset relations together induce a relation $\bar{\theta}$ on S^*, which we extend to S by putting $0\,\bar{\theta} = M$. If $a, b \in S^*$ and $ab \neq 0$, then a and b are consecutive arrows in C and so $(a\bar{\theta})(b\bar{\theta}) \subseteq (ab)\bar{\theta}$. If $ab = 0$, the same inclusion holds automatically. Again, the remaining requirements are straightforwardly checked. □

We collect further properties of these operators that will be needed in the sequel. The references to star-division (defined in the previous section) are needed only in Section 7 and we prove here slightly more than is even needed, due to its prominent role there. Recall that if S is a restriction semigroup, then S^1 is the restriction monoid obtained from S (that is, with 1 its only projection).

LEMMA 4.4. *Every category C star-divides the monoid $R(C)^1$.*

Proof. This is essentially (3.11) of [17]: map all the objects of C to 1 and extend the identity map $\kappa : \text{Arr } C \longrightarrow R(C)$ to the division κ^+ specified in Result 3.1. □

In light of Proposition 4.1, our most important tool is the following modification of a result of Tilson [17, Proposition 3.3, cf Corollary 3.4].

LEMMA 4.5. *Let C and D be categories. If $D \prec C$, then $R(D) \prec R(C) \times R(D/\tau)$, where τ is the congruence on D that identifies coterminal arrows.*
Since D/τ is a trivial category, $R(D/\tau) \in \mathbf{B}_2$. If, further, D is anticyclic, then $R(D/\tau) \in \mathbf{B}_0$.

Proof. Let $\phi : D \to C$ be the category division. Define a relational morphism $\theta : R(D) \to R(C) \times R(D/\tau)$ as follows. If $a \in C(e, f)$, put $a\theta = a\phi \times a\tau$. Put $0\theta = R(C) \times \{0\}$. If a and b are consecutive arrows of D, then $(a\phi)(b\phi) \subseteq (ab)\phi$ and $(ab)\tau = (a\tau)(b\tau)$, where the latter arrow corresponds to a nonzero element of $R(D/\tau)$. Thus θ is a relational morphism of semigroups.

We next show that θ respects the unary operation $^+$, the other operation begin similar. Let $a \in R(D)^*$, so $a \in D(e, f)$, say, and $a^+ = 1_e$. Suppose $b \in (a\theta)^+$, that is, $b = (x, a\tau)$ for some $x \in a\phi$. Now $x \in C(e\phi, f\phi)$, so $x^+ = 1_{e\phi} \in (1_e)\phi$. Therefore $b^+ = (x^+, (a\tau)^+) = (1_{e\phi}, 1_{e\tau}) \in 1_e\phi \times 1_e\tau = a^+\theta$, as required.

Finally, suppose $a, b \in R(D)$ and $a\theta \cap b\theta \neq \emptyset$. Then either $a = b = 0$ or $a, b \in R(D)^*$, in which case $a\tau = b\tau$, so that, considered as arrows in D, a and b are coterminal, whence equal since ϕ is a division on D. Thus θ is a division of restriction semigroups. □

Recall that for any nonempty set X, B_X denotes the combinatorial Brandt semigroup with $|X|$ nonzero idempotents.

COROLLARY 4.6. *[17, Corollary 3.4] Let D be a category and M a monoid. Put $X = $ Obj D. If $D \prec M$, then $\mathrm{R}(D)^1 \prec M \times B_X^1$. Hence D star-divides $M \times B_X^1$.*

Proof. Tilson provides an explicit division. Alternatively, by the previous result, $\mathrm{R}(D) \prec M \times \mathrm{R}(D/\tau)$, as restriction semigroups. Either directly, or by noting that $\mathrm{R}(D/\tau)$ is $\widetilde{\mathcal{H}}$-combinatorial and applying [8, Proposition 5.8], this semigroup embeds in a combinatorial Brandt semigroup. Turn the latter into a monoid and employ Result 3.1 to obtain a division. The last statement follows from Lemma 4.4. $\square$

The last statement of the corollary could, of course, have been shown directly. We shall also use the following related result that is probably already known, though we know of no reference.

PROPOSITION 4.7. *Let D be a category and M a monoid. Put $X = $ Obj D. If $D \prec M$ then D star-divides $M \times RB_X^1$, where RB_X^1 is the monoid obtained by adjoining an identity to the $|X| \times |X|$ rectangular band.*

Proof. Let $\theta : D \longrightarrow M$ be the division. First define $\theta' : D \longrightarrow M \times RB_X^1$ as follows: for $d \in $ Obj D, put $d\theta' = \{(1,1)\}$; for $a \in D(e,f)$, put $a\theta' = a\theta \times \{(e,f)\}$. If, also, $b \in D(f,g)$, then $(a\theta')(b\theta') = (a\theta)(b\theta) \times \{(e,f)(f,g)\} = (a\theta)(b\theta) \times \{(e,g)\} \subseteq (ab)\theta'$. If $a\theta' \cap b\theta' \neq \emptyset$, then a and b are coterminal and thus equal, since θ is a division. Finally, again extend θ' to a star division $(\theta')^+$ of categories, using Result 3.1. $\square$

Finally, we prove two general statements regarding Proposition 4.1.

LEMMA 4.8. *Let $\{C_i\}_{i \in I}$ be a family of categories. Then $\mathrm{R}(\prod_{i \in I} C_i) \prec \prod_{i \in I} \mathrm{R}(C_i)$.*

Proof. Define a map $\prod_{i \in I} \mathrm{R}(C_i) \longrightarrow \mathrm{R}(\prod_{i \in I} C_i)$ as follows. Let $f \in \prod_{i \in I} \mathrm{R}(C_i)$. If $f(i) \neq 0$ for all $i \in I$, map f to itself, regarded as an element of $\prod_{i \in I} C_i$ (and therefore as a nonzero element of $\mathrm{R}(\prod_{i \in I} C_i)$). Otherwise, map f to the zero element of that semigroup. This map is clearly a surjective semigroup morphism. $\square$

LEMMA 4.9. *Let $\{C_i\}_{i \in I}$ and C be connected categories, with C finitely generated. If $\mathrm{R}(C) \prec \prod_{i \in I} \mathrm{R}(C_i)$, then there is a subset J of I such that $C \prec \prod_{j \in J} C_j$.*

Proof. The semigroup $\mathrm{R}(C)$ is a homomorphic image of a finitely generated restriction subsemigroup S of $P = \prod_{i \in I} \mathrm{R}(C_i)$. According to Lemma 2.5, $\mathrm{R}(C)$ is a projection-separating homomorphic image of the r-principal factor U, say, associated with some $\widetilde{\mathcal{J}}$-class D of S. By Proposition 4.2, the latter homomorphism induces a quotient category morphism $\mathrm{C}(U) \longrightarrow \mathrm{C}(\mathrm{R}(C)) = C$.

Let $a, b \in D$. Regarded as members of P, write $a = (a_i)_{i \in I}$, $b = (b_i)_{i \in I}$. Now $a \, \widetilde{\mathcal{D}} \, b$ in S, so $a \, \widetilde{\mathcal{D}} \, b$ in the product P and $a_i \, \widetilde{\mathcal{D}} \, b_i$ in $\mathrm{R}(C_i)$, that is, $a_i = 0$ if and only if $b_i = 0$. Thus there is a nonempty subset J of I such that for each $j \in J$, the jth components of all members of D correspond to arrows of C_j; and for any $j \notin I$, the jth components of all members of D are 0.

In other words, $\mathrm{C}(U)$ embeds in $\prod_{j \in J} \mathrm{R}(C_j)$. $\square$

5   Varieties of strict restriction semigroups and varieties of categories

We define mappings between the lattice of varieties of categories and the interval $[\mathbf{B}_2, \mathbf{B}]$ in the lattice of varieties of strict restriction semigroups, consisting of those varieties that contain B_2.

For any variety $\mathbf{W}$ of categories, let $\mathcal{R}(\mathbf{W})$ be the variety of strict restriction semigroups generated by $\{\mathrm{R}(C) : C \in \mathbf{W}\}$. In view of the previous section, $\mathcal{R}(\mathbf{Cat}) = \mathbf{B}$. Further, $\mathcal{R}(\mathbf{I}) = \mathbf{B}_2$, since the latter variety consists precisely of the $\widetilde{\mathcal{H}}$-combinatorial strict restriction semigroups, according to [8, Theorem 9.3], and is generated by B_2 itself.

Conversely, for any variety $\mathbf{V}$ of strict restriction semigroups that contains $\mathbf{B}_2$, let $\mathcal{C}(\mathbf{V}) = \{\mathrm{C}(S) : S$ is a primitive member of $\mathbf{V}\}$. Using Proposition 4.1, $\mathcal{C}(\mathbf{V}) = \{C : \mathrm{R}(C) \in \mathbf{V}\}$.

PROPOSITION 5.1. *For* $\mathbf{V} \in [\mathbf{B}_2, \mathbf{B}]$, $\mathcal{C}(\mathbf{V})$ *is a variety of categories.*

Proof. It needs to be shown that $\mathcal{C}(\mathbf{V})$ is closed under division and products. Let $C \in \mathcal{C}(\mathbf{V})$ and $D \prec C$. By Lemma 4.5, $\mathrm{R}(D) \prec \mathrm{R}(C) \times \mathrm{R}(D/\tau)$, where $\mathrm{R}(C) \in \mathbf{V}$ and $\mathrm{R}(D/\tau) \in \mathbf{B}_2$, so that $\mathrm{R}(D) \in \mathbf{V}$, that is, $D \in \mathcal{C}(\mathbf{V})$. Next, if $\{C_i\}_{i \in I} \subseteq \mathcal{C}(\mathbf{V})$, then $\mathrm{R}(C_i) \in \mathbf{V}$ for each i. Applying Lemma 4.8, $\mathrm{R}(\prod_{i \in I} C_i) \in \mathbf{V}$, that is, $\prod_{i \in I} C_i \in \mathcal{C}(\mathbf{V})$. $\square$

THEOREM 5.2. *The mappings* $\mathcal{R}$ *and* $\mathcal{C}$ *are mutually inverse isomorphisms between the lattice* $\mathcal{L}(\mathbf{Cat})$ *and the interval* $[\mathbf{B}_2, \mathbf{B}]$.

Proof. It is clear that each mapping is order-preserving. Since every variety of strict restriction semigroups is generated by its primitive members, the correspondence evinced in Proposition 4.1 yields that $\mathcal{R}(\mathcal{C}(\mathbf{V})) = \mathbf{V}$ for every $\mathbf{V} \in [\mathbf{B}_2, \mathbf{B}]$.

Now let $\mathbf{W}$ be a variety of categories. If $C \in \mathbf{W}$, then $\mathrm{R}(C)$ is a primitive member of $\mathcal{R}(\mathbf{W})$ and $C = \mathrm{C}(\mathrm{R}(C))$, so $C \in \mathcal{C}(\mathcal{R}(\mathbf{W}))$.

Next let $C \in \mathcal{C}(\mathcal{R}(\mathbf{W}))$, so that $\mathrm{R}(C) \in \mathcal{R}(\mathbf{W})$. Since C belongs to the variety generated by its connected components, it suffices to assume it is itself connected. By standard arguments (or by the results in [17, Section 10]), C may also be assumed to be finitely generated.

There exist connected categories $C_i \in \mathbf{W}$, $i \in I$, such that $\mathrm{R}(C) \prec \prod_{i \in I} \mathrm{R}(C_i)$. By Lemma 4.9, there is a subset J of I such that $C \prec \prod_{j \in J} C_j$. Therefore $C \in \mathbf{W}$. $\square$

We turn now to the interval $[\mathbf{B}_0, \mathbf{B}_0 \mathbf{M}]$. As observed in Result 2.15, the members of $\mathbf{B}_0 \mathbf{M}$ are characterized within $\mathbf{B}$ by being 'square-free': for distinct projections e, f, the $\widetilde{\mathcal{H}}$-classes $\widetilde{\mathcal{R}}_e \cap \widetilde{\mathcal{L}}_f$ and $\widetilde{\mathcal{R}}_f \cap \widetilde{\mathcal{L}}_e$ cannot both be nonempty. Clearly this property holds in a strict restriction semigroup if and only if it holds in its principal factors. According to Proposition 4.1, such completely 0-r-simple restriction semigroups correspond to anticyclic categories.

Given a variety $\mathbf{W}$ of categories, the anticyclic members do not themselves form a variety (since they are not closed under division). It is possible to formulate a direct analogue to Theorem 5.2 by restricting the operators $\mathcal{C}$ and $\mathcal{R}$; the varieties in the interval $[\mathbf{B}_0, \mathbf{B}_0 \mathbf{M}]$ then correspond to classes of anticyclic categories that are closed under the operations of product and division. While it is routine to formalize this connection, it does not assist us directly.

We now make specific the correspondences between the respective lattices of varieties and show that Figure 1 faithfully represents the lattice $\mathcal{L}(\mathbf{B})$.

Starting with the top 'layer' $[\mathbf{B}_2, \mathbf{B}]$, recall from Corollary 2.14 its decomposition as the disjoint union of the intervals $[\mathbf{B}_2 \mathbf{N}, \mathbf{B} \cap \mathbf{mon}(\mathbf{N})]$, $\mathbf{N} \in \mathcal{L}(\mathbf{M})$, where $\mathbf{B}_2 \mathbf{N}$ is shorthand for $\mathbf{B}_2 \vee \mathbf{N}$.

THEOREM 5.3. *For each* $\mathbf{N} \in \mathcal{L}(\mathbf{M})$, $\mathbf{B}_2 \mathbf{N} = \mathcal{R}(g\mathbf{N})$ *and* $\mathbf{B} \cap \mathbf{mon}(\mathbf{N}) = \mathcal{R}(\ell\mathbf{N})$. *Equality holds if and only if* $\mathbf{N}$ *is a local variety of monoids.*

Proof. If a category D belongs to $g\mathbf{N}$, then $D \prec N$ for some monoid $N \in \mathbf{N}$. By Lemma 4.5, $\mathrm{R}(D) \prec N \times \mathrm{R}(D/\tau) \in \mathbf{B}_2 \mathbf{N}$. Therefore $\mathbf{B}_2 \mathbf{N} \supseteq \mathcal{R}(g\mathbf{N})$. Since $\mathcal{R}(g\mathbf{N})$ contains $\mathbf{N}$, the opposite inclusion is obvious. For any category D, $D \in \ell\mathbf{N}$ if and only if $\mathrm{R}(D) \in \mathbf{mon}(\mathbf{N})$ and so the second equality

follows from the definition of $\mathcal{R}$. □

There is a considerable literature on locality of monoid varieties. We mention here only that any nontrivial variety of monoids consisting of groups is local [17, Proposition 11.6]; the variety of semilattice monoids is local [14]; but the variety of commutative monoids is not local [16].

Next we apply Tilson's Result 3.3. Recall that $\mathbf{B} \cap \mathbf{mon}(\mathbf{T}) = \mathbf{B} \cap \mathbf{loc}(\mathbf{S})$ comprises the strict restriction semigroups with trivial submonoids.

THEOREM 5.4. *The variety* $\mathbf{B} \cap \mathbf{mon}(\mathbf{T})$ *covers* $\mathbf{B}_2$ *and is, therefore, contained in every member of the interval* $(\mathbf{B}_2, \mathbf{B}]$.

Proof. This is a direct translation of the second statement of Result 3.3, using Theorem 5.2. See the remark after Theorem 5.6 for an alternative derivation. □

Although Tilson's Bonded Component Theorem (Result 3.4) has consequences for the members of the interval $[\mathbf{B}_2, \mathbf{B}]$, it is not at this point clear how they translate in varietal terms.

Turning now to the next 'layer', $[\mathbf{B}_0, \mathbf{B}_0\mathbf{M}]$, recall from Corollary 2.16 its decomposition as the disjoint union of the intervals $[\mathbf{B}_0\mathbf{N}, \mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{N})]$, $\mathbf{N} \in \mathcal{L}(\mathbf{M})$, where $\mathbf{B}_0\mathbf{N}$ is shorthand for $\mathbf{B}_0 \vee \mathbf{N}$. Here the Bonded Component Theorem has significant impact, as follows.

THEOREM 5.5. *For each nontrivial variety* $\mathbf{N}$ *of monoids,* $\mathbf{B}_0\mathbf{N} = \mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{N})$; *that is,* $\mathbf{B}_0\mathbf{N}$ *is the unique variety in the interval* $[\mathbf{B}_0, \mathbf{B}_0\mathbf{M}]$ *whose intersection with* $\mathbf{M}$ *is precisely* $\mathbf{N}$.

Proof. We apply the special case of the Bonded Component Theorem stated in Corollary 3.5. Let $\mathbf{N}$ be a nontrivial variety of monoids and let $\mathbf{V} \in [\mathbf{B}_0\mathbf{N}, \mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{N})]$. Let S be a primitive member of $\mathbf{V}$. Then $C(S)$ is an anticyclic category whose local monoids belong to $\mathbf{N}$. By the cited corollary, $C(S)$ divides a monoid $N \in \mathbf{N}$. Now by Lemma 4.5, $S = R(C(S)) \prec N \times R(C(S)/\tau)$, where since $C(S)$ is anticyclic, $R(C(S)/\tau)$ now belongs to $\mathbf{B}_0$. So $S \in \mathbf{B}_0\mathbf{N}$. □

THEOREM 5.6. *The variety* $\mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{T})$ *covers* $\mathbf{B}_0$.

Proof. Theorem 8.1 of [17] can no longer be applied literally, as in the proof of Proposition 5.3. Instead we apply the result behind that theorem, as stated in Result 3.3. Let $S \in \mathbf{mon}(\mathbf{T}) \cap \mathbf{B}_0\mathbf{M}$, $S \notin \mathbf{B}_0$. Then S has trivial submonoids but a nontrivial $\widetilde{\mathcal{H}}$-class. We may assume that S is completely 0-r-simple. Now $C(S)$ is a locally trivial, but not trivial, category. By Result 3.3, $C(S) \prec A_2^I$, for some set I. By Lemma 4.5, $S = R(C(S)) \prec R(A_2^I) \times R(C(S)/\tau)$ where, since $C(S)$ is anticyclic, $R(C(S)/\tau) \in \mathbf{B}_0$. Now applying Lemma 4.8, $R(A_2^I) \prec R(A_2)^I$. Thus S belongs to the variety generated by $R(A_2)$.

The proof is completed by noting that any subvariety of $\mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{T})$ that properly contains $\mathbf{B}_0$ contains $R(A_2)$. □

Remark. Tilson's proof (Theorem 7.1) that every locally trivial category divides a power of A_2 proceeds in two steps that translate in terms of restriction semigroups as follows. He first shows that the statement holds for locally trivial anticyclic categories. As noted in the proof of our Theorem 5.6, this shows that the semigroup $R(A_2)$ generates $\mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{T})$.

His Corollary 7.6, again in conjunction with our Lemma 3.3, shows that any variety $\mathbf{V}$ in the interval $(\mathbf{B}_2, \mathbf{B} \cap \mathbf{mon}(\mathbf{T})]$ is the join of $\mathbf{V} \cap (\mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{T}))$ with $\mathbf{B}_2$ and thus also contains $R(A_2)$, yielding the covering in Theorem 5.4.

COROLLARY 5.7. *The interval* $[\mathbf{B}_0, \mathbf{B}_0\mathbf{M}]$ *is isomorphic with the lattice obtained by adjoining a new zero to the lattice of varieties of monoids.*

Proof. See the above and Figure 1. □

6 Pseudovarieties of restriction semigroups and of categories

In fact, it is pseudovarieties that are of most interest in the study of finite semigroups and their applications to formal languages. However, little modification is needed to translate our varietal results into their pseudovarietal analogues. Essentially, it must be verified that the appropriate finiteness properties are preserved. Since we are not here considering the equational theory, we need not delve into issues of profiniteness (or of sequences of equations, as in [17]).

A pseudovariety of *restriction semigroups* is a collection of finite such semigroups that is closed under finite products and division.

A pseudovariety of *categories* (termed a **C**-variety by Tilson) is a collection of finite categories that is closed under finite products and such that if a finite category divides a member of the collection, then it too belongs to the collection. There is a considerable literature on this topic and on that of pseudovarieties of semigroupoids (see [15] and [13]).

Pseudovarieties of *monoids* (termed **M**-varieties by Tilson) may be regarded in either of the above fashions, as appropriate. Although the relevant materials in the monographs by [12], [1] and [13] focus on pseudovarieties of semigroups, the general arguments transfer to the monoid situation and, moreover, many of the examples may be considered in either context.

The respective pseudovarieties form lattices, which we again denote $\mathcal{L}(\mathbf{R})$ and $\mathcal{L}(\mathbf{Cat})$, respectively, using similar notation for the lattice of subpseudovarieties of any given pseudovariety.

The finite members of any variety form a pseudovariety. (In that case, we shall in general use the same notation for the pseudovariety.) In particular, given a finite object, the finite members of the variety that it generates also constitute the pseudovariety that it generates. (Of course many of the interesting examples are not found in this fashion – see the works cited above.)

We refer the reader to [17] for demonstrations that the key results of Section 3 respect finiteness. In particular (cf Result 3.3) every finite locally trivial category divides a *finite* power of A_2; and (cf Result 3.4) every finite category that is not locally trivial divides a *finite* product of its strongly connected components.

Since the author is unaware of any literature on pseudovarieties of restriction semigroups, we outline the appropriate modifications to the general results.

The first statement of Result 2.6 relied on the existence of a proper cover for any restriction semigroup, as defined at the end of Section 2. As remarked there, every finite restriction semigroup has a finite proper cover. Thus the analog of the first statement holds and then it is clear that the map $\mathbf{V} \longrightarrow \mathbf{V} \vee \mathbf{M}$ is again a complete lattice homomorphism on the lattice of pseudovarieties of restriction semigroups. That the same is true for the map $\mathbf{V} \longrightarrow \mathbf{V} \cap \mathbf{M}$, when restricted subpseudovarieties of $\mathbf{B}$, follows from a routine verification that finiteness is preserved throughout the proof of Proposition 2.8.

The decomposition of $\mathcal{L}(\mathbf{SM})$ in Result 2.9 follows easily from the strongness of the semilattice decomposition stated therein.

As in Corollary 2.10, $\mathcal{L}(\mathbf{B})$ is the disjoint union of the intervals $[\mathbf{N}, \mathbf{B} \cap \mathbf{mon}(\mathbf{N})]$, $\mathbf{N} \in \mathcal{L}(\mathbf{M})$.

The descriptions of the finite members of the varieties $\mathbf{B}_2$ and $\mathbf{B}_0$ in Result 2.12 of course remain true. The coverings in (3) of that result need to be shown to again hold in the pseudovariety case. Although they follow the same arguments as in [8], we spell them out because of their fundamental nature in the sequel.

That $\mathbf{T} \prec \mathbf{S}$ follows from the decomposition of $\mathcal{L}(\mathbf{SM})$ cited above. That $\mathbf{S} \prec \mathbf{B}_0$ follows from the same arguments as for varieties, since they relied on local finiteness. To elaborate, it was shown in [7, Lemma 3.12] that for any restriction semigroup S and $a \in S$, there are four possibilities for the restriction subsemigroup generated by a, corresponding to the options (i) $a^+ > a^*$, (ii) $a^* < a^+$, (iii) a^+ and a^* are incomparable, and (iv) $a^+ = a^*$. In the first two cases it is infinite, in the third case B_0 divides it, and in the fourth case it generates a submonoid of S. Since the outcome is actually stronger

than the covering stated above, it is worth a separate statement.

PROPOSITION 6.1. *If a pseudovariety of restriction semigroups does not consist of semilattices of monoids, then it contains* $\mathbf{B}_0$. *That is,* $\mathbf{B}_0$ *is the unique cover of* $\mathbf{SM}$ *in the lattice* $\mathcal{L}(\mathbf{R})$ *of pseudovarieties.*

PROPOSITION 6.2. $\mathbf{B}_0 \prec \mathbf{B}_2$.

Proof. Suppose $S \in \mathbf{B}_2$ but $S \notin \mathbf{B}_0$. According to Result 2.12, S is $\widetilde{\mathcal{H}}$-combinatorial but does not satisfy the identity $xyx = x^2y^2$ and therefore has an r-principal factor with the same property. Now [8, Lemma 10.1] asserts that for an $\widetilde{\mathcal{H}}$-combinatorial completely 0-r-simple semigroup, failure to satisfy that identity is equivalent to containing a copy of B_2. Thus the variety generated by S is $\mathbf{B}_2$, as required. □

COROLLARY 6.3. *The lattice of subpseudovarieties of* $\mathbf{B}_2$ *comprises the chain of coverings* $\mathbf{T} \prec \mathbf{S} \prec \mathbf{B}_0 \prec \mathbf{B}_2$.

Thus, by the same arguments as for varieties, $\mathcal{L}(\mathbf{B})$ again decomposes in the general fashion of Figure 1, now interpreted as the lattice of subpseudovarieties of $\mathbf{B}$.

Naturally, since any finite strict restriction semigroup is a subdirect product of its r-principal factors, $\mathbf{B}$ is again generated by its completely 0-r-simple members.

Let us now turn to correspondences elicited in Sections 4 and 5. The constructions C(S) and R(C) clearly respect finiteness. Thus the analogs of Propositions 4.1 and 4.2 remain true. Again, finiteness is respected in Lemma 4.5 and the final two lemmas of that section apply directly.

Thus for any pseudovariety $\mathbf{W}$ of categories, let $\mathcal{R}(\mathbf{W})$ be the pseudovariety of strict restriction semigroups generated by $\{\mathrm{R}(C) : C \in \mathbf{W}\}$. Conversely, for any pseudovariety $\mathbf{V}$ of strict restriction semigroups that contains $\mathbf{B}_2$, let $\mathcal{C}(\mathbf{V}) = \{\mathrm{C}(S) : S \text{ is a primitive member of } \mathbf{V}\}$. By the analogue of Proposition 5.1, $\mathcal{C}(\mathbf{V})$ is a pseudovariety of categories. Again, the arguments in the proof of Theorem 5.2 respect finiteness.

THEOREM 6.4. *The mappings* $\mathcal{R}$ *and* $\mathcal{C}$ *are mutually inverse isomorphisms between the lattice* $\mathcal{L}(\mathbf{Cat})$ *of pseudovarieties of categories and the interval* $[\mathbf{B}_2, \mathbf{B}]$ *in the lattice of pseudovarieties of strict restriction semigroups.*

Since, as remarked early in this section, all the relevant category-theoretic theorems from [17] respect finiteness, we may simply state that each result from Section 5 carries over directly to the lattice of pseudovarieties, yielding a diagram completely analogous to Figure 1. In particular, $\mathbf{B} \cap \mathbf{mon}(\mathbf{T}) \succ \mathbf{B}_2$ and $\mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{T}) \succ \mathbf{B}_0$; the interval $[\mathbf{B}_0, \mathbf{B}_0\mathbf{M}]$ consists of $\mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{T})$ and the pseudovarieties $\mathbf{B}_0\mathbf{N}$, for pseudovarieties $\mathbf{N}$ of monoids.

7 Varieties having proper covers

We hasten to say that the term 'cover' is here used as in Section 2, not in the order-theoretic sense. A restriction semigroup is *proper* if $\widetilde{\mathcal{L}} \cap \sigma = \widetilde{\mathcal{R}} \cap \sigma = \iota$, where σ is its least monoid congruence, that is, the least congruence that identifies all the projections. If S is any restriction semigroup and M is a (restriction) monoid, then a *proper cover over* M for S consists of a proper restriction semigroup T such that $T/\sigma \cong M$, and a projection-separating homomorphism upon S. If $\mathbf{N}$ is a variety of monoids, S has a proper cover *over* $\mathbf{N}$ if it has a proper cover over some monoid from $\mathbf{N}$.

Cornock [3] studied varieties of restriction semigroups *having proper covers*: a variety $\mathbf{V}$ has this property if each of its members has a proper cover in $\mathbf{V}$ (necessarily over a monoid from $\mathbf{V} \cap \mathbf{M}$). Cornock [3, Theorem 10.1.6] showed that $\mathbf{V}$ has this property if and only if its free members are proper.

THEOREM 7.1. *The varieties in the interval $[\mathbf{B}_2, \mathbf{B}]$ that have proper covers are those of the form $\mathbf{B}_2\mathbf{N}$, where $\mathbf{N}$ consists neither entirely of $\mathcal{R}$-trivial monoids nor entirely of $\mathcal{L}$-trivial monoids.*

These varieties of monoids are precisely those with the property that every category in $g\mathbf{N}$ star-divides a member of $\mathbf{N}$.

With the exception of $\mathbf{B}_0$ and $\mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{T})$, every subvariety of $\mathbf{B}_0\mathbf{M}$ has proper covers.

The following (probably well known) proposition provides an alternative, useful, characterization of the varieties of monoids that are excluded in the statement of the theorem. Recall that B_2^1 is the monoid obtained from B_2 by adjoining an identity element (its *only* projection, when regarded as a restriction monoid); RB_2^1 is the 2×2 rectangular band with adjoined identity. Also recall that a variety of monoids contains no nontrivial groups if and only if it satisfies $x^{n+1} = x^n$ for some $n \geq 1$.

PROPOSITION 7.2. *A monoid variety consists of $\mathcal{R}$-trivial monoids or of $\mathcal{L}$-trivial monoids if and only if it contains neither B_2^1, RB_2^1 nor any nontrivial group.*

Proof. Necessity is obvious. Conversely, let $\mathbf{N}$ be a variety that contains monoids M_1 and M_2 on which $\mathcal{R}$ and $\mathcal{L}$ are respectively nontrivial, but contains no nontrivial group, so that it satisfies some identity $x^{n+1} = x^n$. If M_1 contains a pair of $\mathcal{R}$-related idempotents and M_2 contains a pair of $\mathcal{L}$-related idempotents, then $M_1 \times M_2$ contains a copy of RB_2^1. Suppose then (using duality) that M_1 contains no such pair.

Since $\mathcal{R}$ is nontrivial on M_1 and M_1 is periodic, it contains a pair of distinct *regular* $\mathcal{R}$-related elements (well known, cf [12, Proposition 3.4.1]) and thus, in view of our supposition above, an idempotent e and an $\mathcal{R}$-related nonidempotent a. Since H_e is trivial, $L_e \neq L_a$. Let a' be an inverse of a and $f = a'a$. By the assumption on M_1, a' is also nonidempotent. Consider the submonoid generated by a and a'. By periodicity (more precisely, by 'stability' [13, Appendix A.2]), $\{e, f, a, a'\}$ comprises the $\mathcal{J}$-class immediately below 1, while the remaining elements form an ideal. The Rees quotient is then isomophic to the six-element monoid $\{1, e, f, a, a', 0\} \cong B_2^1$. $\qquad\square$

Further to this proposition it is straightforward to show, along with the dual statement, that a variety of monoids consists of $\mathcal{R}$-trivial monoids if and only if it satisfies $(xy)^n = (xy)^n x$ for some $n \geq 1$. This is not needed in the sequel.

Before proving the theorem, we collect some further preliminaries. As mentioned earlier, in the cited work on restriction semigroups, relational morphisms were called 'subhomomorphisms'. The following result is a slight modification of the characterization of such covers given by Cornock [3, Proposition 10.3.2], extending results for E-unitary covers of inverse semigroups.

RESULT 7.3. *[3, Proposition 10.3.2] Let S be a restriction semigroup and M a monoid. Suppose there is a relational morphism $\phi : S \to M$ such that if $a, b \in S$ and $a\phi \cap b\phi \neq \emptyset$, then $a^+b = b^+a$ and $ab^* = ba^*$. Then $T = \{(s, m) \in S \times M : (s, m) \in \phi\}$ is a proper cover for S over a submonoid of M. Conversely, any proper cover is determined in this way.*

We interpret the last proposition in category-theoretic terms through Proposition 4.3 and the results that follow it.

PROPOSITION 7.4. *Let $\mathbf{N}$ be a variety of monoids and S a completely 0-r-simple restriction semigroup. Then S has a proper cover over $\mathbf{N}$ if and only if there is a monoid $M \in \mathbf{N}$ and a star division $\mathrm{C}(S) \longrightarrow M$ of categories.*

Hence a variety $\mathbf{N}$ of monoids has the property that the variety $\mathbf{B}_2\mathbf{N}$ of strict restriction semigroups has proper covers if and only if every category in $g\mathbf{N}$ star-divides a member of $\mathbf{N}$.

Proof. Suppose S has a proper cover over $\mathbf{N}$, let $\phi : S \longrightarrow M \in \mathbf{N}$ be a relational morphism as described in Result 7.3 and let ϕ^* be the category-theoretic relational morphism $\mathrm{C}(S) \longrightarrow M$ induced according to Proposition 4.3. If a, b are arrows in $\mathrm{C}(S)$ with the same initial vertex such that $a\phi \cap b\phi \neq \emptyset$ then, regarded as members of S, $a^+ = b^+$ and so $a = b$. Necessarily, ϕ is a division. A dual argument applies if they have the same terminal vertices.

Conversely, if there is a star division $\theta : \mathrm{C}(S) \longrightarrow M$ of categories, put $C = \mathrm{C}(S)$, so that $S = \mathrm{R}(C)$. Let $\overline{\theta} : S \longrightarrow M$ be the relational morphism induced according to Proposition 4.3. Now if $a, b \in S$ and $a\overline{\theta} \cap b\overline{\theta} \neq \emptyset$, then the conclusion $a^+b = b^+a$ certainly holds when either a or b is zero or when $a^+ \neq b^+$, by Result 2.1. But when $a^+ = b^+$, the associated arrows in $\mathrm{C}(S)$ have the same initial vertex and so $a = b$. By duality, we may apply Result 7.3.

The final statement now follows from the equation $\mathcal{R}(g\mathbf{N}) = \mathbf{B}_2\mathbf{N}$ in Theorem 5.3, noting that we need only consider connected categories and completely 0-r-simple restriction semigroups. $\qquad\square$

RESULT 7.5. *[3, Theorem 10.3.5] For any variety $\mathbf{N}$ of monoids, the restriction semigroups that have proper covers over $\mathbf{N}$ form a variety, defined by the the restriction semigroup identities $v^+u = u^+v$ and $uv^* = vu^*$, over all the identities $u = v$ satisfied in $\mathbf{N}$.*

It follows that if the members of a collection of restriction semigroups all have proper covers over a given variety $\mathbf{N}$ of monoids, then the variety they generate has this property. Moreover, a non-monoidal variety of *strict* restriction semigroups has this property if and only if each of its completely 0-r-simple semigroups has this property. We can now elucidate the corresponding category-theoretic property, using Result 7.3.

COROLLARY 7.6. *If a variety $\mathbf{V}$ of strict restriction semigroups has proper covers (over $\mathbf{N} = \mathbf{V} \cap \mathbf{M}$), then $\mathbf{V} = \mathbf{N}$, $\mathbf{SN}$, $\mathbf{B}_0\mathbf{N}$ or $\mathbf{B}_2\mathbf{N}$ (see Figure 1). The first two of these always have proper covers. The last two have proper covers if and only if the respective semigroups B_0 and B_2 have proper covers over $\mathbf{N}$.*

Proof. Suppose $\mathbf{V}$ has proper covers. Then for any completely 0-r-simple member S of $\mathbf{V}$, $\mathrm{C}(S)$ divides a monoid in $\mathbf{N}$, by Proposition 7.4, so by Theorem 5.3, $S \in \mathcal{R}(g\mathbf{N}) = \mathbf{B}_2\mathbf{N}$. (This may also be proved directly.) According to the results of Section 5, the listed varieties constitute the interval $[\mathbf{N}, \mathbf{B}_2\mathbf{N}]$.

Since $\mathbf{SN}$ is generated by semilattices (which are proper) and monoids, it has proper covers, by the remarks following Result 7.5. Since $\mathbf{B}_0\mathbf{N}$ and $\mathbf{B}_2\mathbf{N}$ are generated by B_0 and B_2, together with the monoids of $\mathbf{N}$, the last statement follows likewise. $\qquad\square$

The next step of the proof of Theorem 7.1 could be accomplished directly by using Result 7.5 and the identities cited following Proposition 7.2, but we prefer a direct verification (in part because that allows a direct translation to the pseudovarietal situation).

PROPOSITION 7.7. *The restriction semigroup B_2 does not have a strict proper cover over any variety of monoids consisting entirely of $\mathcal{R}$-trivial monoids or entirely of $\mathcal{L}$-trivial monoids.*
The restriction semigroup B_0 does not have a proper cover over the variety of trivial monoids.

Proof. Suppose $\theta : T \longrightarrow B_2$ is a projection-separating homomorphism of a proper restriction semigroup upon B_2, where $M = T/\sigma$ is $\mathcal{R}$-trivial, the other case being dual. Let $c\theta = a, d\theta = b, g = c^+, h = d^+$ and $k = gh$, using our usual notation for B_2. Thus $g\theta = e$, $h\theta = f$ and $k\theta = 0$. The monoid $\widetilde{\mathcal{H}}_k = 0\theta^{-1}$ is therefore an r-ideal of T. By properness, it embeds in M and so satisfies $x^{n+1} = x^n$ for some $n \geq 1$, whereby $(kcd)^{n+1} = (kcd)^n$. As a result, $(kcd)^n \mathcal{R}(kcd)^n(kc)$ in $\widetilde{\mathcal{H}}_k$, whence $(kcd)^n = (kcd)^nkc$. But $k \,\sigma\, c^+$, so $(cd)^n \,\sigma\, (cd)^nc$ in T. However, since cd belongs to the submonoid $\widetilde{\mathcal{H}}_g$, the same is true of

$(cd)^n$, while $(cd)^n c \in \widetilde{\mathcal{H}}_c$. Now the relation $(cd)^n \mathcal{R}(cd)^n c$ in T contradicts properness, since $(cd)^n \theta = e$ while $((cd)^n c)\theta = a$.

The second statement follows from the fact that the only proper restriction semigroups over trivial monoids are semilattices (since, in that event, $(a, a^+) \in \widetilde{\mathcal{R}} \cap \sigma$ for all a). $\qquad\square$

Finally, the following proof could be treated entirely semigroup-theoretically, using Result 7.3, but the category-theoretic version better illustrates the prior connections that we have established.

PROPOSITION 7.8. *The semigroup B_2 has a strict proper cover over any nontrivial group and over the monoids B_2^1 and $(RB_2)^1$. It therefore has such a cover over any variety of monoids consisting neither entirely of $\mathcal{R}$-trivial monoids nor entirely of $\mathcal{L}$-trivial monoids.*

The semigroup B_0 has a strict proper cover over any nontrivial monoid and therefore over any nontrivial monoid variety.

Proof. For the first statements, it suffices by Corollary 7.4 to show that there are star divisions from $C_2 = C(B_2)$ to each of the stated monoids. First, if G is a nontrivial group, let $g \in G \backslash 1$. It is straightforward to verify that ϕ, given by $e\phi = f\phi = 1$, $a\phi = g$ and $b\phi = g^{-1}$, has the requisite properties. The last two cases follow from Corollary 4.6 and Proposition 4.7, for C_2 divides the trivial monoid.

The second statement follows from Proposition 7.2. For the last statement, it suffices to show that there is a star division from $C_0 = C(B_0)$ to any nontrivial monoid M. Let $m \in M \backslash 1$ and define ϕ by $e\phi = f\phi = 1$ and $a\phi = m$. Again it is straightforward to verify that ϕ has the requisite properties. $\qquad\square$

This completes the proof of Theorem 7.1.

We conclude this section with the pseudovarietal version of the theorem. It is well known [12, 1] that the finite $\mathcal{R}$-trivial monoids and the finite $\mathcal{L}$-trivial monoids respectively form the pseudovarieties **L** and **R** (not to be confused with our notation **R** for strict restriction semigroups). In view of the results of Section 6, it remains only to verify that finiteness is preserved during the proof of the theorem, a task that is routinely accomplished.

THEOREM 7.9. *The pseudovarieties in the interval $[\mathbf{B}_2, \mathbf{B}]$ that have proper covers are those of the form $\mathbf{B}_2\mathbf{N}$, where $\mathbf{N} \nsubseteq \mathbf{R}$ and $\mathbf{N} \nsubseteq \mathbf{L}$. These pseudovarieties of monoids are precisely those with the property that every category in $g\mathbf{N}$ star-divides a member of $\mathbf{N}$. With the exception of $\mathbf{B}_0$ and $\mathbf{B}_0\mathbf{M} \cap \mathbf{mon}(\mathbf{T})$, every subpseudovariety of $\mathbf{B}_0\mathbf{M}$ has proper covers.*

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