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# Signal and Noise in Complex-Valued SENSE MR Image Reconstruction

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# Signal and Noise in Complex-Valued SENSE MR Image Reconstruction

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# OUTLINE

1. Motivation

2. Background

3. Methods

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5. Discussion

## Motivation

In MRI it is not voxel values that are measured.

The actual measurements are spatial frequencies ($k$-space).

The $k$-space measurements are not acquired instantaneously.

In parallel imaging, $k$-space is subsampled and measured in parallel then combined to form a single image.

Image and volume measurement time is decreased at the expense of increased image reconstruction difficulty and time.

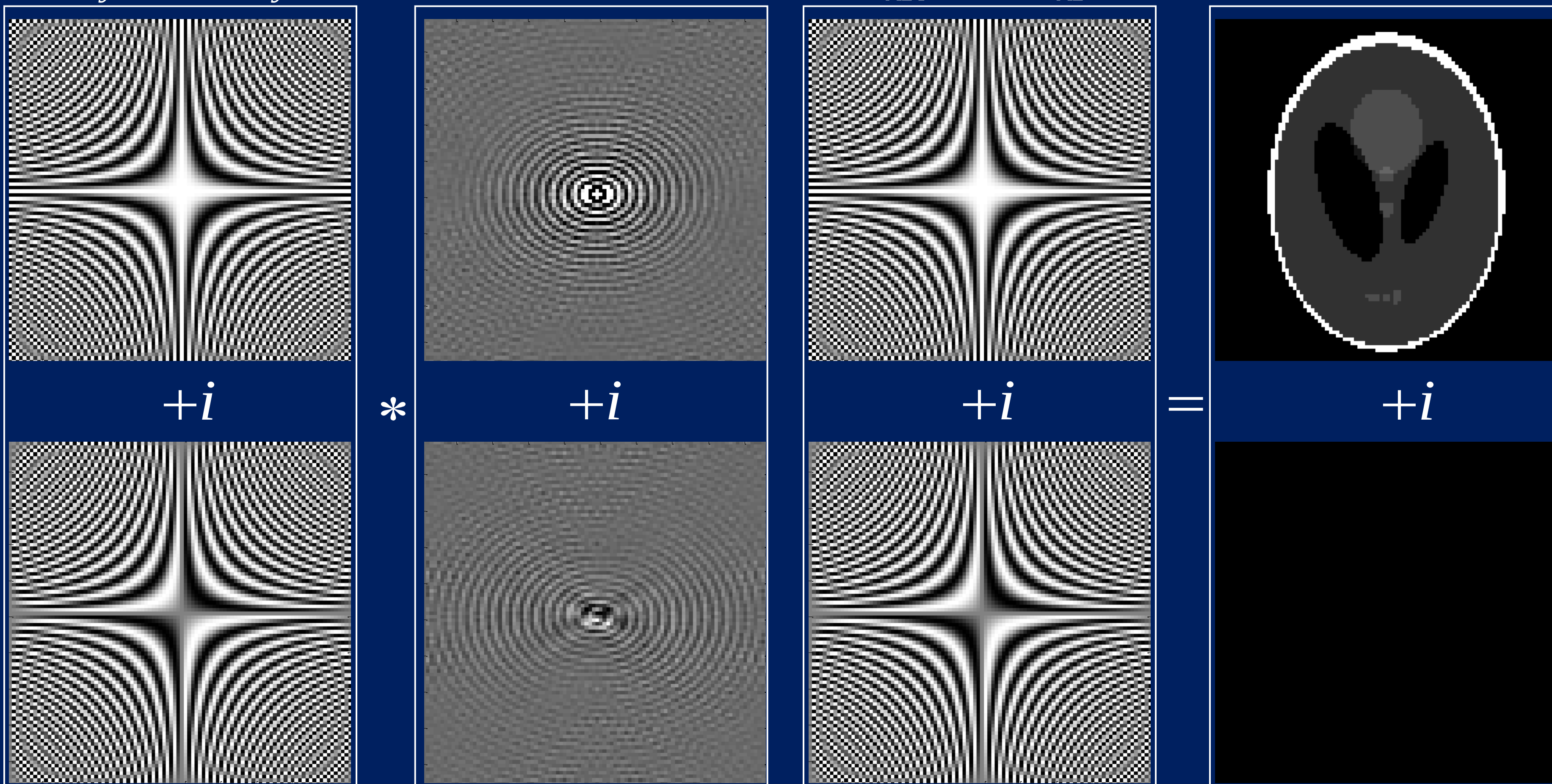
One popular parallel imaging method is SENSE.

Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.

# Background

Image inverse Fourier Reconstruction for single coil.

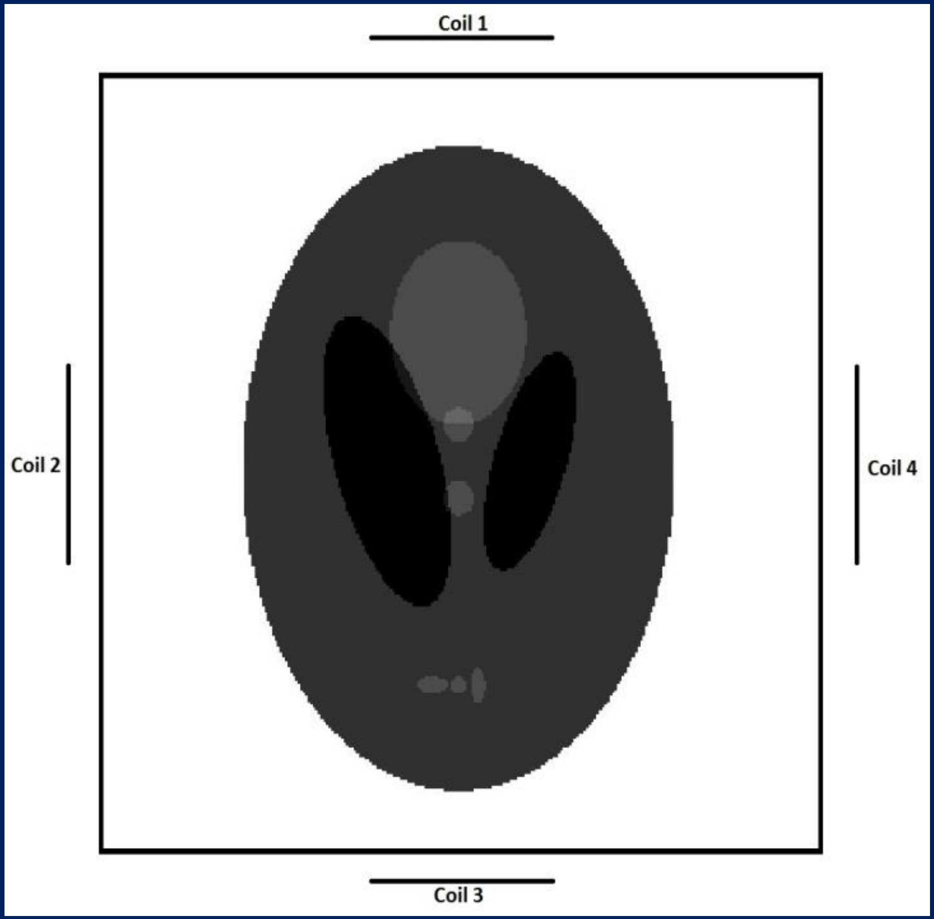
$$(\Omega_{yR} + i\Omega_{yI}) \ast (F_R + iF_I) \ast (\Omega_{xR} + i\Omega_{xI})^T = (V_R + iV_I)$$



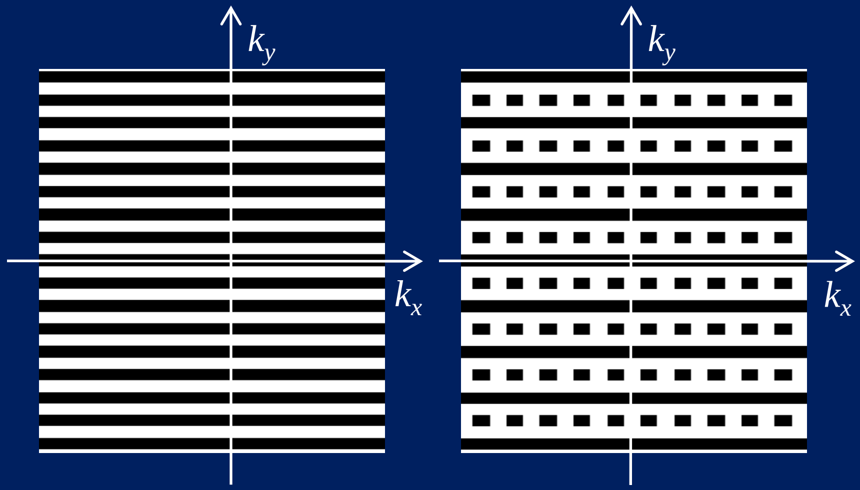

Rowe, Nencka, Hoffmann,Signal and noise of Fourier reconstructed fMRI data. JNSM 159:361-369, 2007.

## Background

In parallel imaging there is more than one receive coil.




Each coil measures a $k$-space array where every $A^{\text{th}}$ line is skipped.

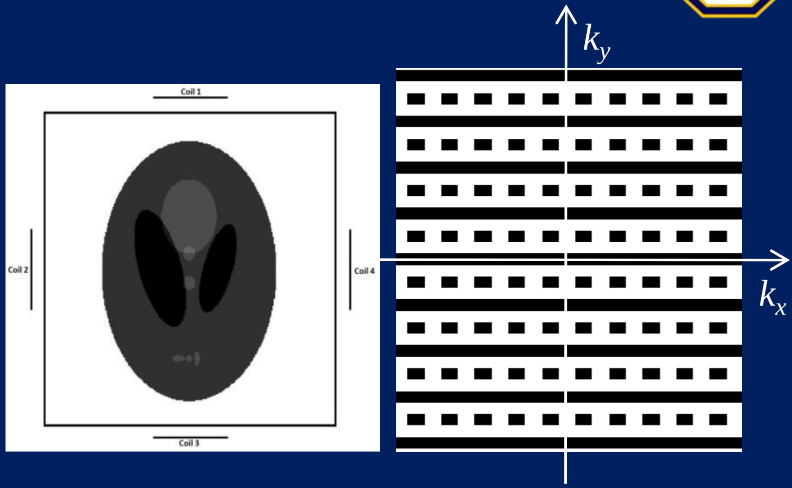



Full $k$-space. Skipped $k$-space.

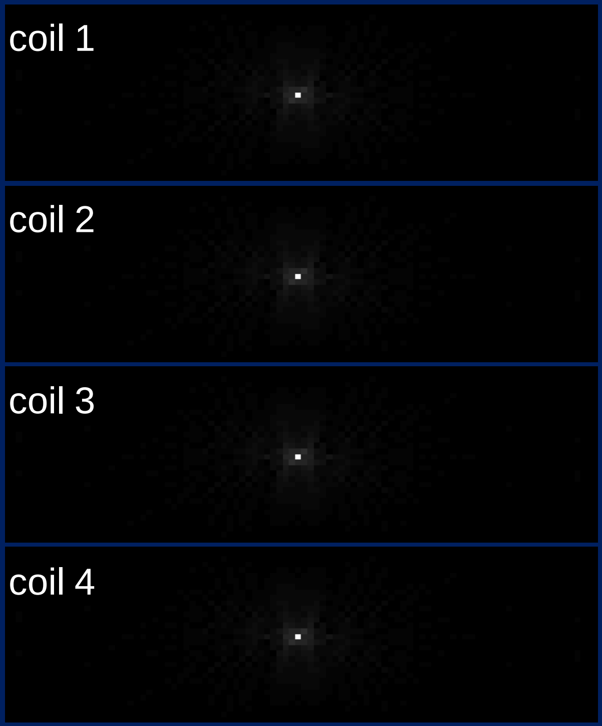
Bruce, Karaman, and Rowe: In Submission, 2011.

# Background

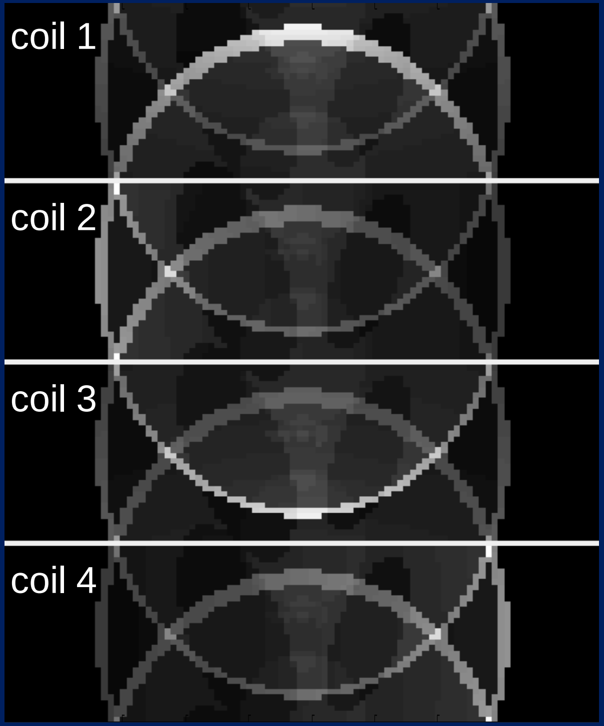
The $k$-space arrays where every $A^{th}$ line is skipped are reconstructed into an aliased image to be combined to form a single image.




Skipped $k$-space.




Aliased images.

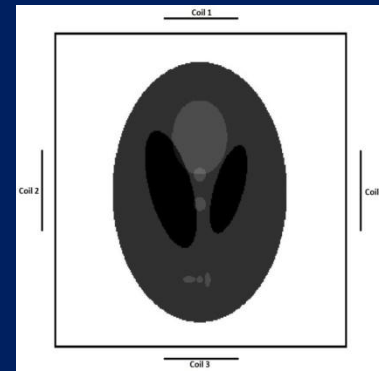


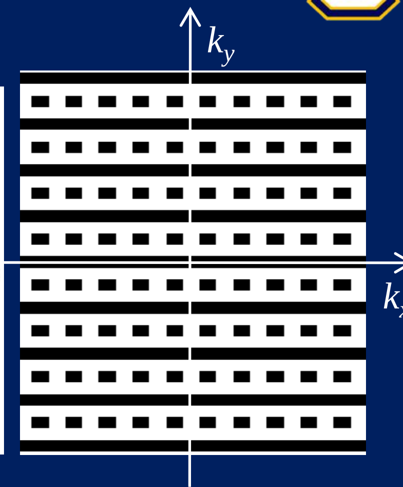

Combined image.

Bruce, Karaman, and Rowe: In Submission, 2011.

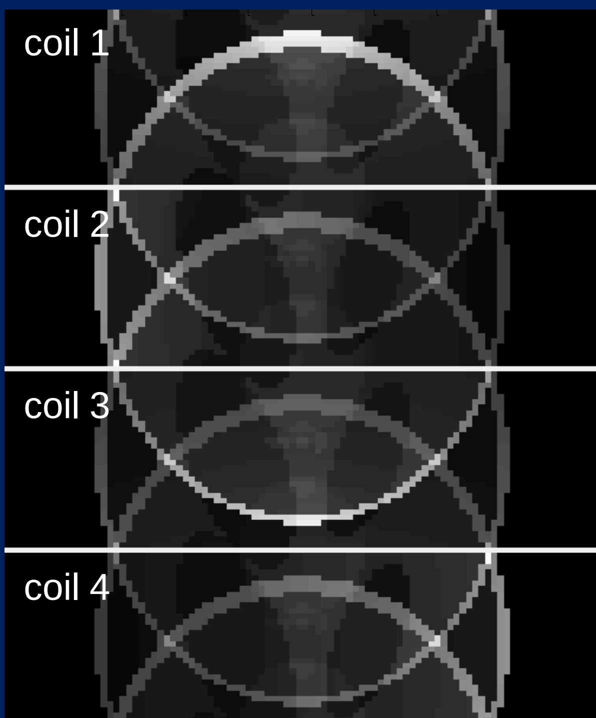
# Background

The combination of aliased images to form a single image utilizes coil sensitivities.

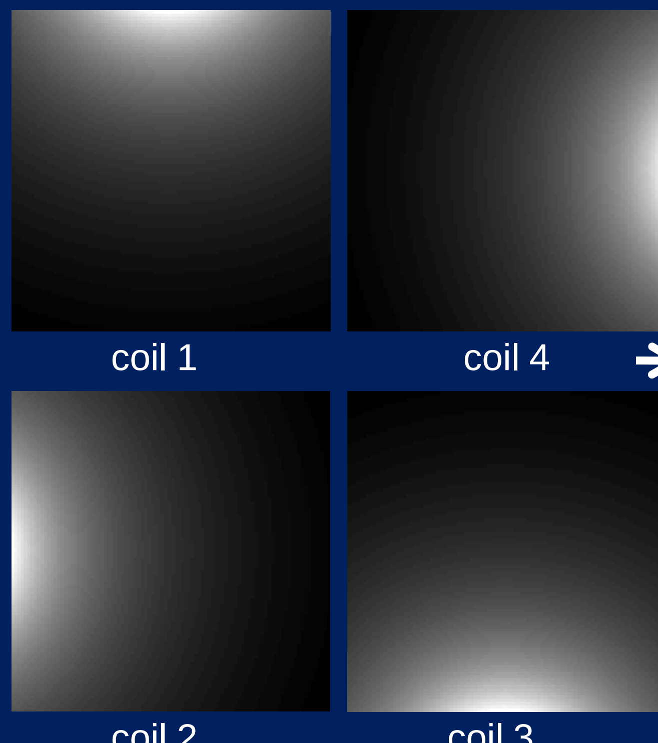


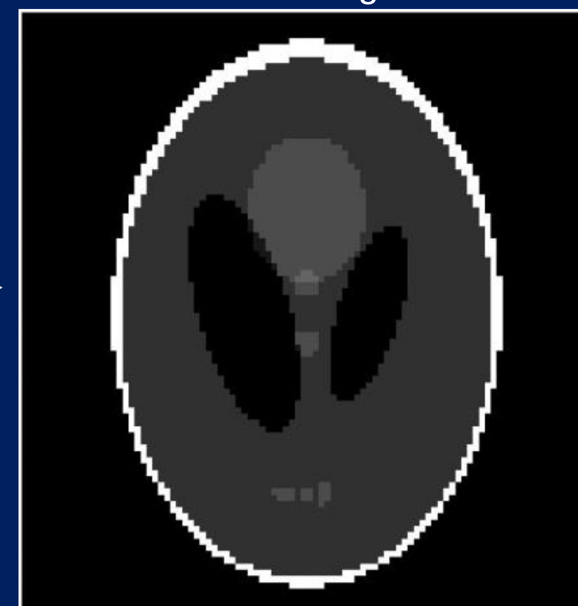

Aliased images. $a_C$




Coil sensitivities. $S_C$




Combined image. $v_C$



Bruce, Karaman, and Rowe: In Submission, 2011.

## Methods

The SENSE model for aliased voxel values from $n$ coils is

$$\underset{n\times 1}{a_C} = \underset{n\times A}{S_C} \; \underset{A\times 1}{v_C} + \underset{n\times 1}{\varepsilon_C} \quad , \quad \varepsilon_C \sim CN(0, \Psi_C)$$

$$\Psi_C = \Psi_R + i\Psi_I$$

where for each voxel

$a_C$ is a vector of the $n$ complex-valued aliased voxel values

$$a_C = a_R + i a_I$$

$v_C$ is a vector of the $A$ unaliased voxel value

$$v_C = v_R + i v_I$$

$S_C$ is an $n$x$A$ matrix of complex-valued coil sensitivities

$$S_C = S_R + i S_I$$

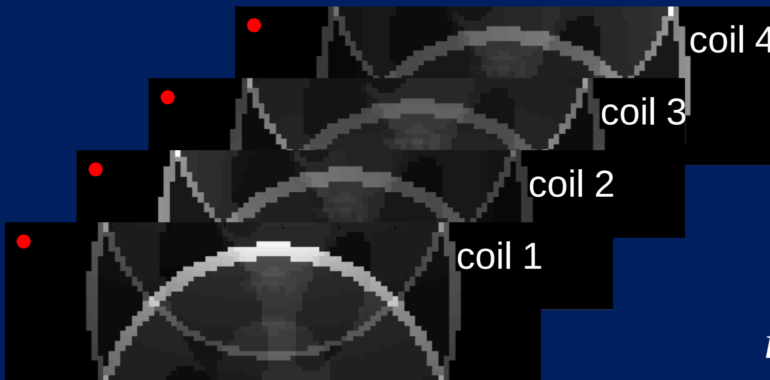
$\varepsilon_C$ is a vector of the $n$ complex-valued error values

$$\varepsilon_C = \varepsilon_R + i\varepsilon_I$$

Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.
Bruce, Karaman, and Rowe: In Submission, 2011.

# Methods

$n \times 1$

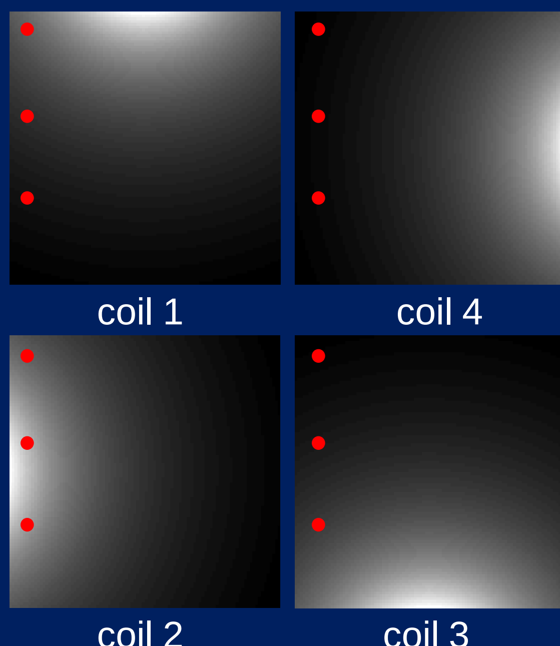



$$\underset{n\times 1}{a_C} = \underset{n\times A}{S_C} \; \underset{A\times 1}{v_C} + \underset{n\times 1}{\varepsilon_C}$$

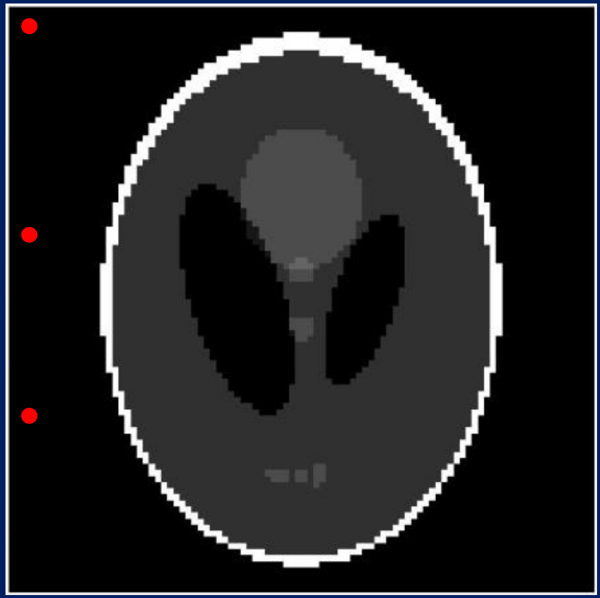
$n \times A$

$A \times 1$

$=$




$*$

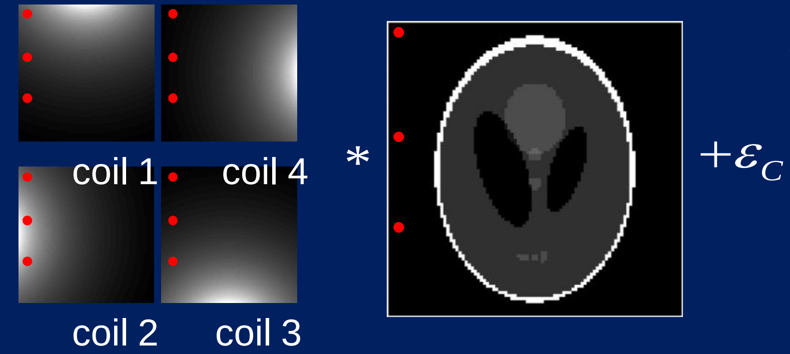


$n \times 1$

$+\varepsilon_C$

Bruce, Karaman, and Rowe: In Submission, 2011.

# Methods


The SENSE process

$$\underset{n\times 1}{a_C} = \underset{n\times A}{S_C} \; \underset{A\times 1}{v_C} + \underset{n\times 1}{\varepsilon_C} \;, \qquad \varepsilon_C \sim CN(0,\Psi_C)$$

$$\Psi_C = \Psi_R + i\Psi_I$$

uses the complex-valued normal distribution

$H$ is the conjugate transpose (Hermetian)

$$f(\varepsilon_C) = (2\pi)^{-n} \left|\Psi_C\right|^{-1} e^{-1/2\varepsilon_C^H \Psi_C^{-1} \varepsilon_C},$$

and for $n$ coil measurements

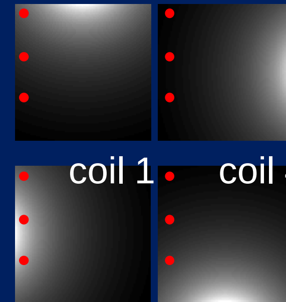
$$f(a_C) = (2\pi)^{-n} \left|\Psi_C\right|^{-1} e^{-1/2(a_C - S_C v_C)^H \Psi_C^{-1} (a_C - S_C v_C)}$$

Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.
Wooding The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.
Bruce, Karaman, and Rowe: In Submission, 2011.

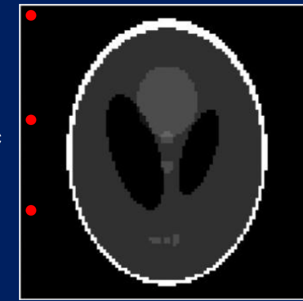
Rowe

# Methods


$=$




$*$



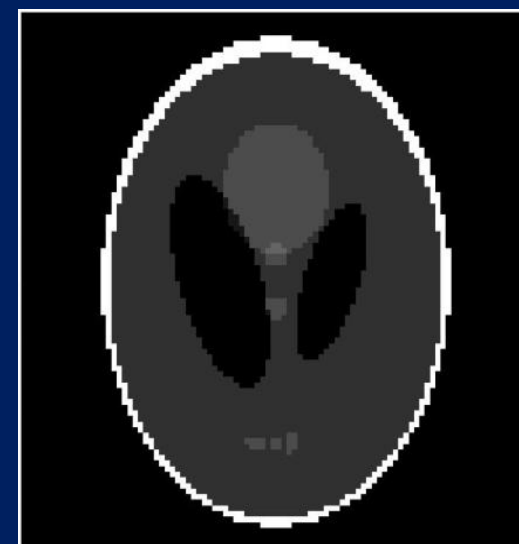
$+\varepsilon_C$

From the distribution for the $n$ coil measurements

$$f(a_C) = (2\pi)^{-n} \left|\Psi_C\right|^{-1} e^{-1/2(a_C - S_C v_C)^H \Psi_C^{-1}(a_C - S_C v_C)}$$

the voxel values can be estimated as

$$v_C = (S_C^H \Psi_C^{-1} S_C)^{-1} S_C^H \Psi_C^{-1} a_C$$



with knowledge of $S_C$ and $\Psi_C$.

Pruessmann et al.: SENSE: Sensitivity Encoding for Fast MRI. MRM 42:952–962, 1999.
Wooding The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.
Bruce, Karaman, and Rowe: In Submission, 2011.

## Methods

Instead of writing the model with complex numbers as

$$\underset{n\times 1}{a_C} = \underset{n\times A}{S_C}\ \underset{A\times 1}{v_C} + \underset{n\times 1}{\varepsilon_C}\ ,$$

$$a_C = a_R + i a_I \quad S_C = S_R + i S_I \quad v_C = v_R + i v_I \quad \varepsilon_C = \varepsilon_R + i\varepsilon_I$$

we can write the model using an isomorphism as

$$\underset{2n\text{x}1}{a} = \underset{2n\text{x}2A}{S}\ \underset{2A\text{x}1}{v} + \underset{2n\text{x}1}{\varepsilon}$$

$$a = \begin{pmatrix} a_R \\ a_I \end{pmatrix} \quad S = \begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix} \quad v = \begin{pmatrix} v_R \\ v_I \end{pmatrix} \quad \varepsilon = \begin{pmatrix} \varepsilon_R \\ \varepsilon_I \end{pmatrix}.$$

Wooding The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.
Bruce, Karaman, and Rowe: In Submission, 2011.

## Methods

Then the distribution for $n$ coil measurements is

$$f(a) = (2\pi)^{-n} \left|\Psi\right|^{-1/2} \mathrm{e}^{-1/2(a-Sv)'\Psi^{-1}(a-Sv)} \quad ,$$

with

$$\underset{2n\text{x}1}{a} = \begin{pmatrix} a_R \\ a_I \end{pmatrix} \quad \underset{2n\text{x}2A}{S} = \begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix} \quad \underset{2A\text{x}1}{v} = \begin{pmatrix} v_R \\ v_I \end{pmatrix} \quad \underset{2n\text{x}1}{\varepsilon} = \begin{pmatrix} \varepsilon_R \\ \varepsilon_I \end{pmatrix} \quad ,$$

and the imposed skew-symmetric covariance structure

$$\Psi = \begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix} \quad .$$

Wooding The multivariate distribution of complex normal variables. Biometrika 43:212–215, 1956.
Bruce, Karaman, and Rowe: In Submission, 2011.

# Methods

The SENSE voxel values can be estimated by

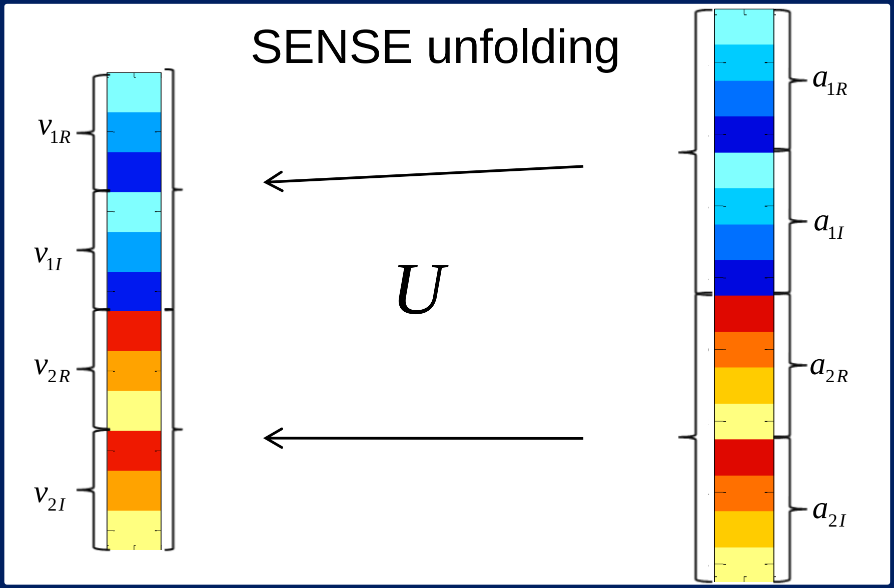
$$v_C = (S_C^H \Psi_C^{-1} S_C)^{-1} S_C^H \Psi_C^{-1} a_C$$

or in terms of an isomorphism

$$\underset{2A\times 1}{\begin{pmatrix} v_R \\ v_I \end{pmatrix}} = \left[ \underset{2A\times 2n}{\begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix}^T} \underset{2n\times 2n}{\begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix}^{-1}} \underset{2n\times 2A}{\begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix}} \right]^{-1} \underset{2A\times 2n}{\begin{pmatrix} S_R & -S_I \\ S_I & S_R \end{pmatrix}^T} \underset{2n\times 2n}{\begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix}^{-1}} \underset{2n\times 1}{\begin{pmatrix} a_R \\ a_I \end{pmatrix}}$$

$$\underset{2A\times 1}{\begin{pmatrix} v_R \\ v_I \end{pmatrix}} = \underset{2A\times 2n}{U} * \underset{2n\times 1}{\begin{pmatrix} a_R \\ a_I \end{pmatrix}}$$

$n$=4 real / $n$=4 imaginary true aliased voxel values

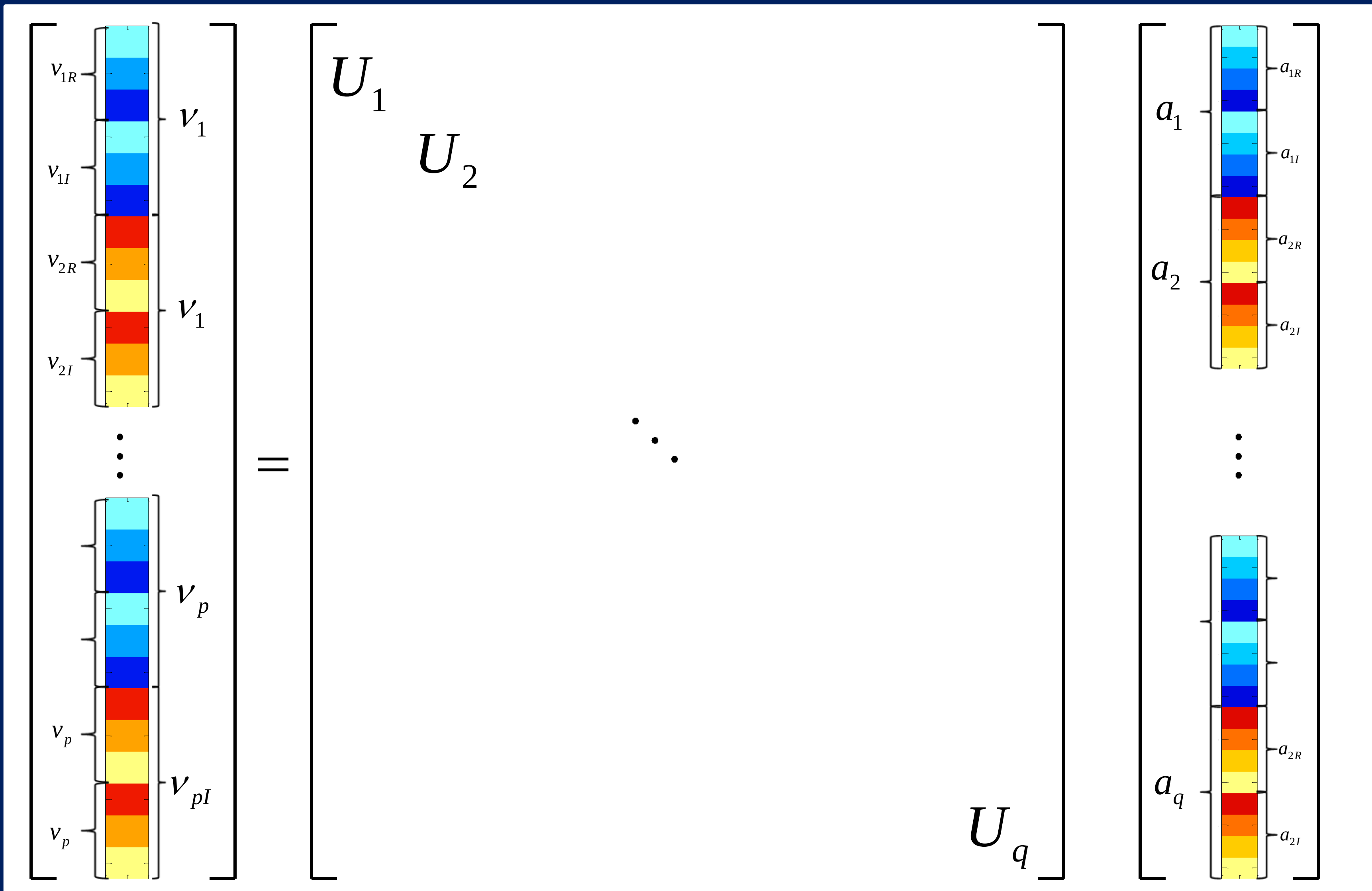



Acceleration $A$ $A$=3 real / $A$=3 imaginary un-aliased fold values

Bruce, Karaman, and Rowe: In Submission, 2011.

SENSE unfolding $\begin{pmatrix} v_R \\ v_I \end{pmatrix} = U * \begin{pmatrix} a_R \\ a_I \end{pmatrix}$

# Methods



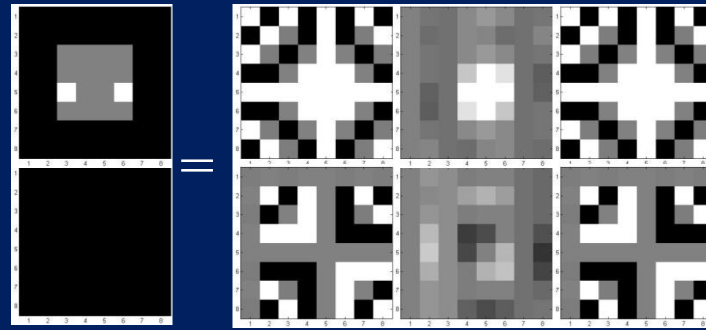

Bruce, Karaman, and Rowe: In Submission, 2011.

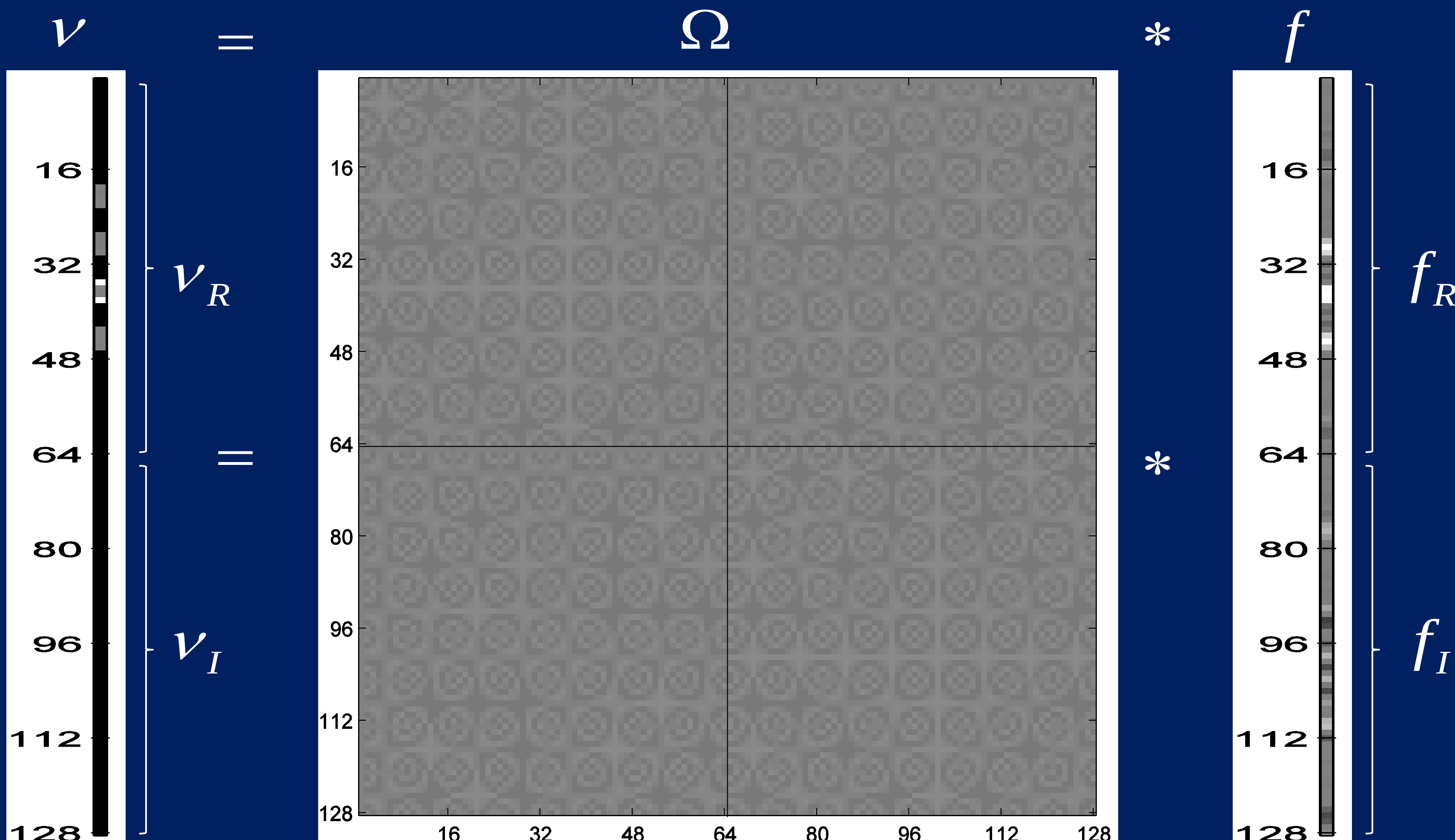
# Methods

Real-valued isomorphism

real

imaginary

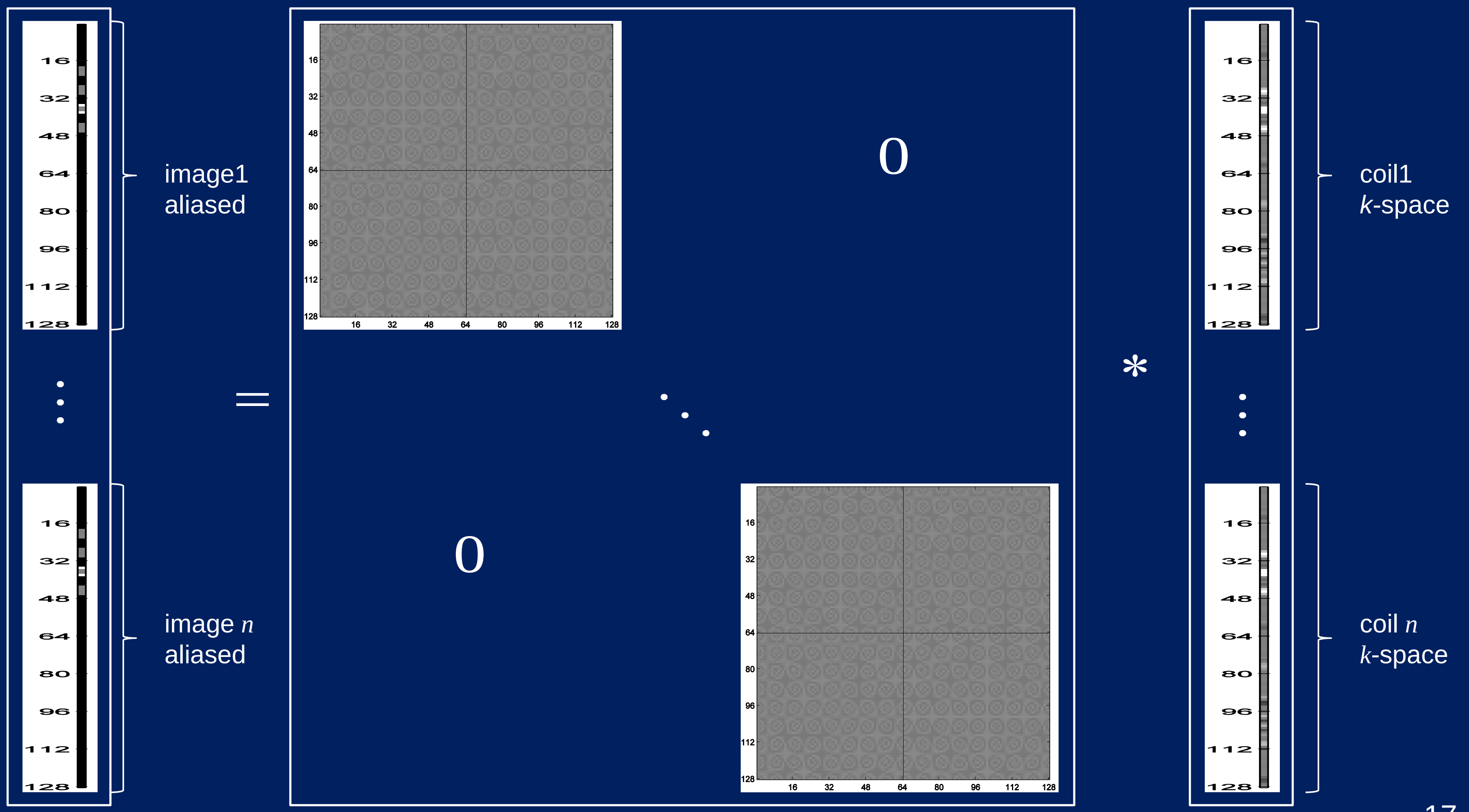





Rowe, Nencka, Hoffmann,Signal and noise of Fourier reconstructed fMRI data. JNSM 159:361-369, 2007.

# Methods

$$v = I_n \otimes \Omega * f$$

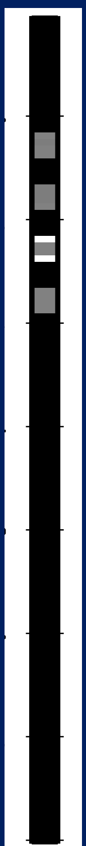



Bruce, Karaman, and Rowe: In Submission, 2011.

# Methods

here is $a$ for each voxel

$$y = P_u \quad U \quad a \quad P_S P_C \quad v$$



$$= P_U \begin{bmatrix} U_1 & & 0 \\ & \ddots & \\ 0 & & U_p \end{bmatrix} \qquad P_S P_C$$

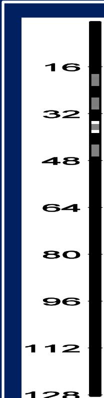



image1 aliased

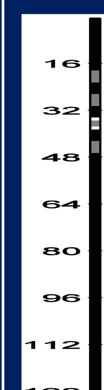



image $n$ aliased

Single image vector

permute

unfold matrix $U$'s have $S$ and $\Psi$

permute to by folded voxel

Bruce, Karaman, and Rowe: In Submission, 2011.

# Methods

processing on unfolded image vector

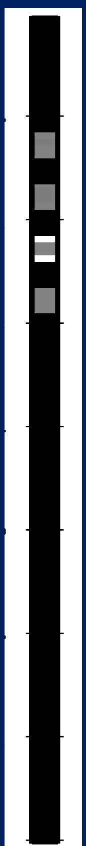
here is $a$ for each voxel

$a$

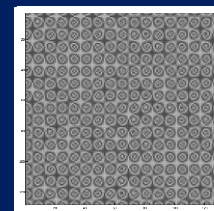
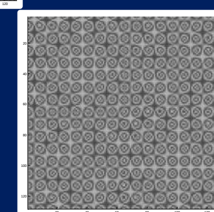
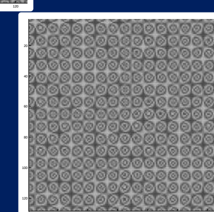
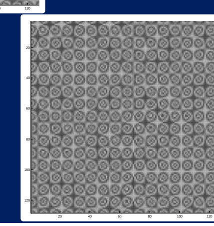
processing on each coil image vector

insert processing on each coil $k$-space vector

$$y = P_u \quad U \quad P_S P_C \quad (I_n \otimes \Omega) \quad f$$

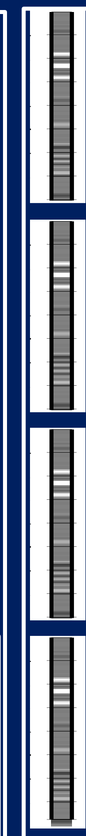


$$= P_U \begin{pmatrix} U_1 & & 0 \\ & \ddots & \\ 0 & & U_p \end{pmatrix} P_S P_C$$

0

0



permute

Single image vector

unfold matrix $U$'s have $S$ and $\Psi$

permute to by folded voxel

reconstruct $n$=4 images

$k$-space vector of $n$ images

Bruce, Karaman, and Rowe: In Submission, 2011.

## Methods

$$y = \underbrace{O_I \; P_u \; U \; P_S P_C \; (I_n \otimes \Omega_a O_k)}_{O} \; f$$

where

$f = (f_1, \ldots, f_n)'$ are coil $k$-space

$O_k$ is $k$-space preprocessing

$\Omega_a$ is adj. inverse Fourier matrix

$P_u$ , $P_S$ , $P_C$ , permutation matrices

$U$ SENSE unfolding matrix

$O_I$ is image space preprocessing

$$f = P_C \mathcal{R} C \mathcal{F}$$

k-space vector
censor blip points
row reverse
permute

$$O_k = \mathcal{A} \, \mathcal{Z} \, \mathcal{H} \, \underbrace{P_R^{-1} \Omega_{row}^{-1} \Phi \Omega_{row} P_R}$$

$\Omega_a = \Omega$ adjusted for $\Delta B$ and for $T_2^*$

$O_I = S_m$ ← Image smoothing

Bruce, Karaman, and Rowe: In Submission, 2011.

## Methods

Statistical Expectation and Covariance.

If E($f$)=$f_0$, then for $Mf$, E($Mf$)=$Mf_0$.

If cov($f$)=$\Gamma$, then for $Mf$, cov($Mf$)=$M\Gamma M'$.

This means that with $y = Of$,

$$E(y) = Of_0 \quad \text{and} \quad \text{cov}(y) = O\Gamma O' = \underset{2p\times 2p}{\Sigma}$$

$\rightarrow$ $\text{cor}(\nu) = D_{\Sigma}^{-1/2}\Sigma D_{\Sigma}^{-1/2}$

So even if $\Gamma = \sigma^2 I$, it is not necessarily true that $\Sigma = \sigma^2 I$ !

This has $H_0$ fMRI noise and fcMRI connectivity implications!

Nencka, Hahn, Rowe: JNSM, 181:268-282, 2009.

Nencka, Rowe: OHBM, 2007.

## Results

Since

$$y = Of\,,$$

we inverted and made the $n$ coil spatial frequencies from

$$(O^T O)^{-1} O^T v = f$$

where $O$ and $v$ are known

$v$ is true/noiseless Shepp-Logan phantom (scaled by 50)

$$O = S_m P_U U P_S P_C (I_n \otimes \Omega)$$

The number of coils, $n$, and the reduction factor, $A$, are specified in the dimensions of operators, $O$.

Bruce, Karaman, and Rowe: In Submission, 2011.

$$y = Of$$

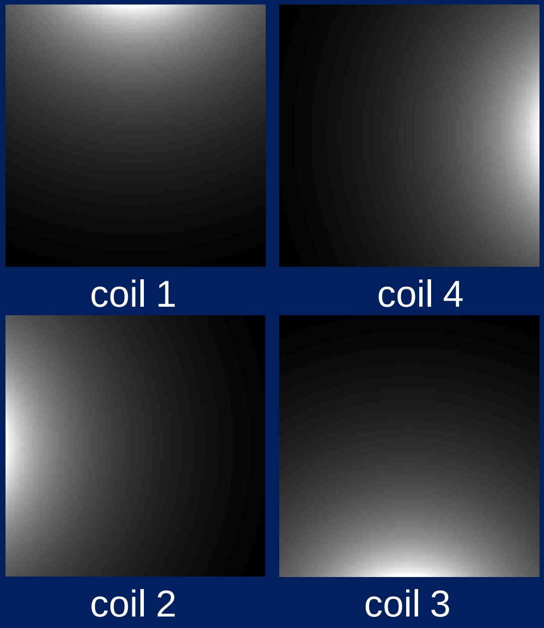
## Results

Noiseless data $f = (O^T O)^{-1} O^T v$ generated for $N_X$=$N_Y$=96, $n$=4, $A$=3

$$O = S_m P_U U P_S P_C (I_n \otimes \Omega)$$

$O$ had diagonal blocks $U_j = (S_j^T \Psi^{-1} S_j)^{-1} S_j^T \Psi^{-1}$

Sensitivities, $S$




Markovian coil covariance, $\Psi$

$$\Psi = \begin{pmatrix} \Psi_R & \Psi_{RI} \\ \Psi'_{RI} & \Psi_I \end{pmatrix} \qquad \Psi_R = \begin{pmatrix} 1 & .33 & .11 & .33 \\ .33 & 1 & .33 & .11 \\ .11 & .33 & 1 & .33 \\ .33 & .11 & .33 & 1 \end{pmatrix}$$

Not skew-symmetric

$$\Psi_I = \Psi_R \qquad \Psi_{RI} = \begin{pmatrix} 0 & -.11 & -.07 & -.11 \\ .26 & 0 & -.11 & -.07 \\ .42 & .26 & 0 & -.11 \\ .26 & .42 & .26 & 0 \end{pmatrix}$$

Bruce, Karaman, and Rowe: In Submission, 2011.

## Results

Gaussian Smoothing applied in image-space

- FWHM = 3 voxels,

-Normalized to leave variance unaffected (Scales mean by 4.516)

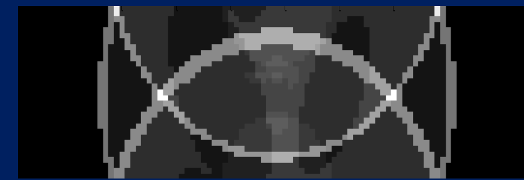
By definition, smoothing induces a covariance and correlation between voxels and their neighbors.

This effect is in turn transferred to the correlated voxels from each fold in SENSE.

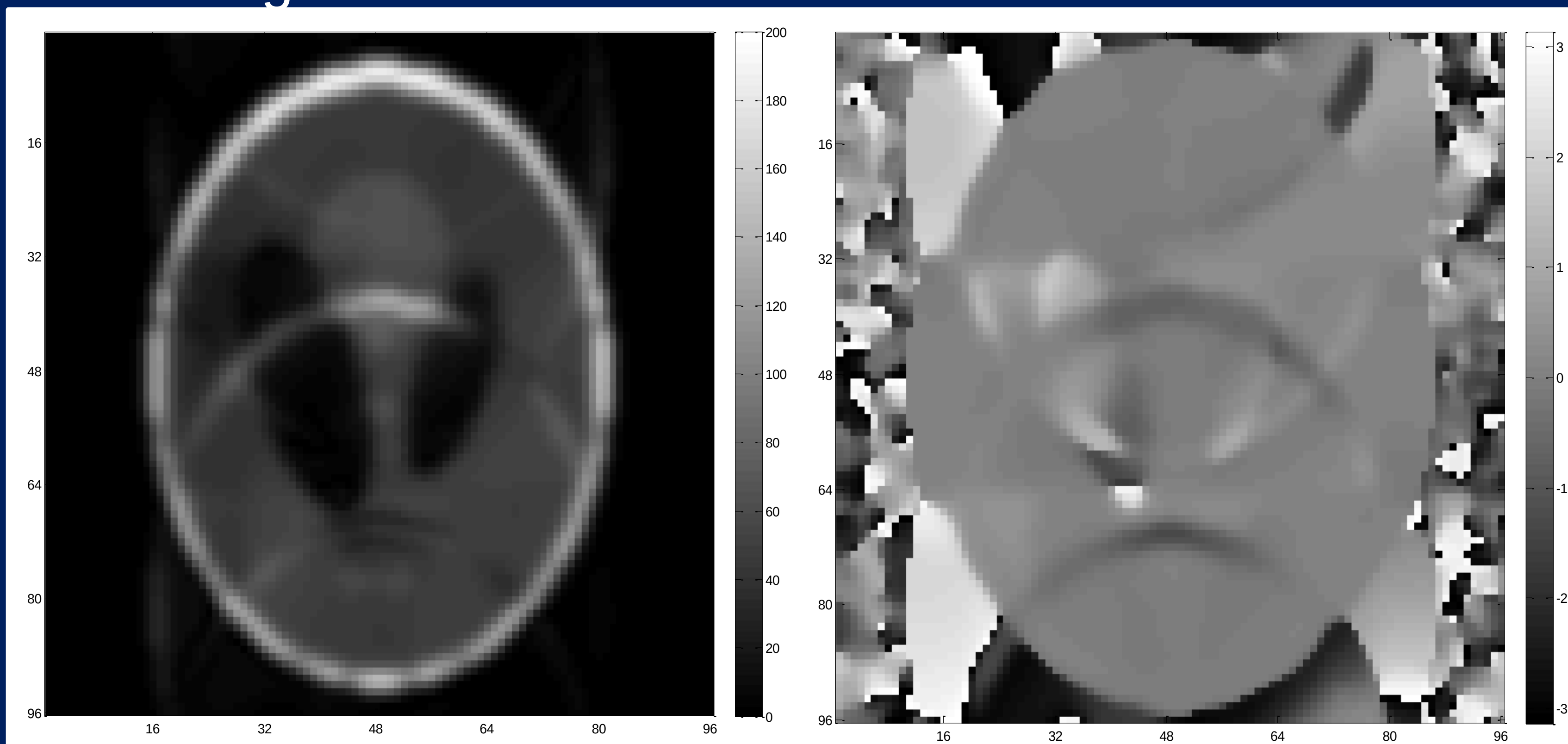
Gaussian smoothing kernel, $S_m$, was applied in image-space to reconstructed images.

Bruce, Karaman, and Rowe: In Submission, 2011.

Nencka et al.: A Mathematical Model for Understanding the STatistical effects of k-space (AMMUST-k) Preprocessing Operators on Observed Voxel Measurements in fcMRI and fMRI. Journal of Neuroscience Methods 181:268-282, 2009.



## Results




Ghosting because symmetric coil cov $\Psi$ used $\rightarrow \Psi = \begin{pmatrix} \Psi_R & -\Psi_I \\ \Psi_I & \Psi_R \end{pmatrix}$

Alternatice symmetric coil cov $\Psi$ proposed.

$$\Psi = \begin{pmatrix} \Psi_R & \Psi_{RI} \\ \Psi'_{RI} & \Psi_I \end{pmatrix}$$

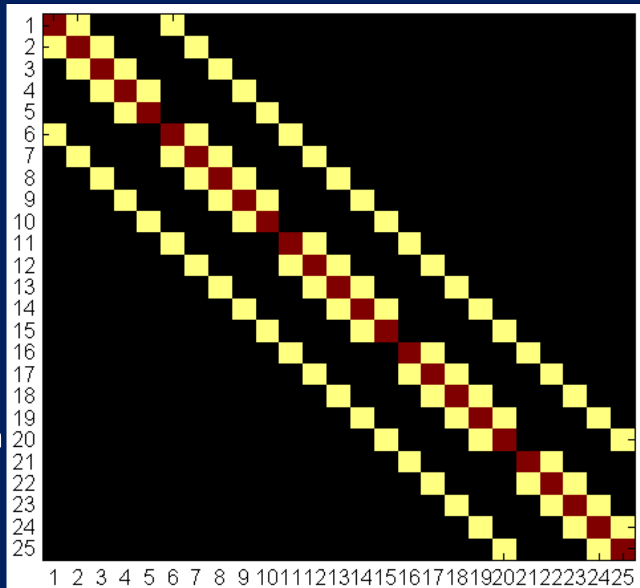
Phase is important in complex-valued fMRI!

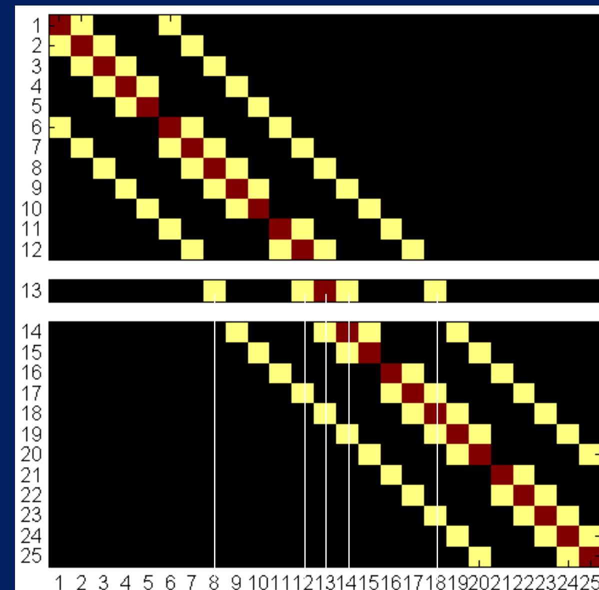
# Results

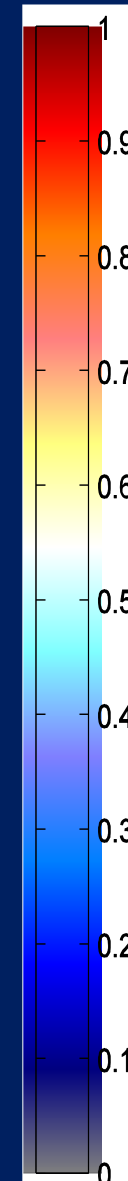
| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 |
| 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 |

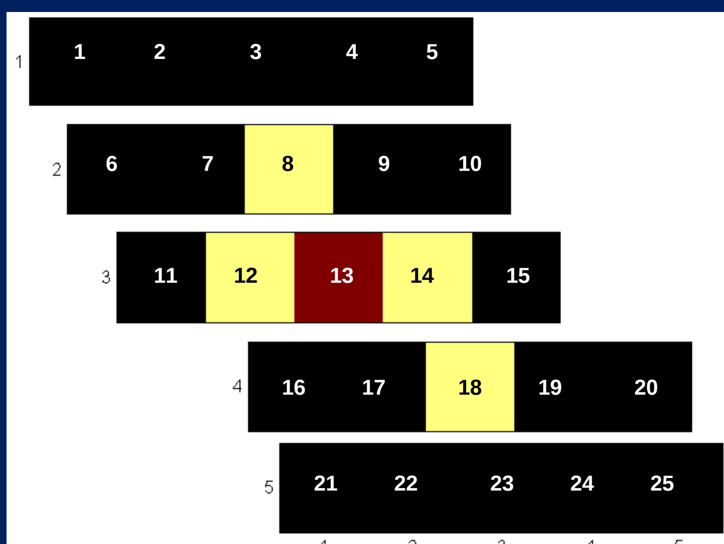
5x5 image

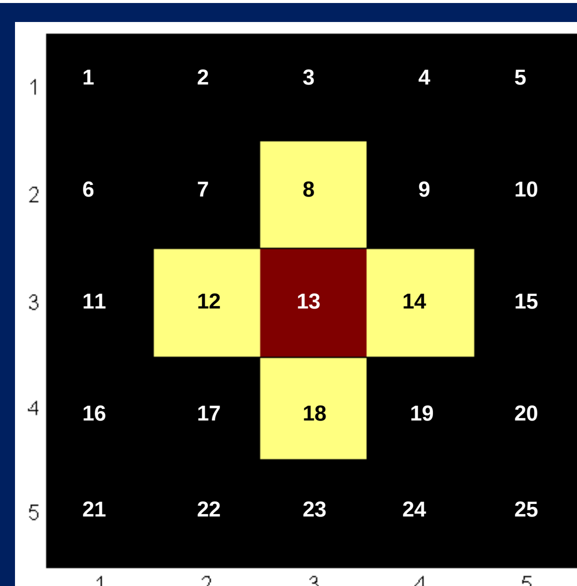
cor =
25x25
correlation
matrix



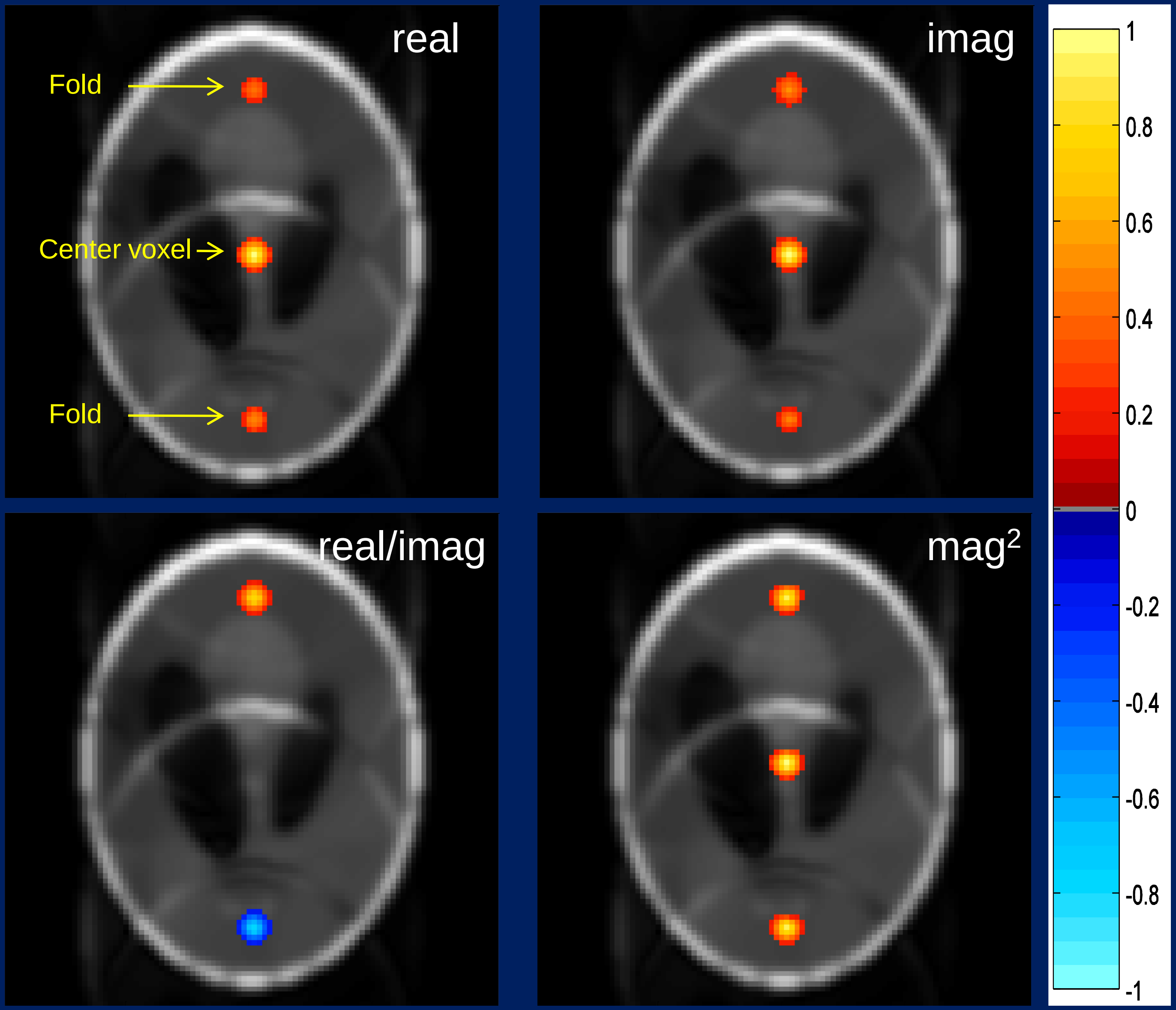




5x5 correlation image

# Results

Correlations induced about the center voxel.

$N_X$=96
$N_Y$=96
$n$ = 4
$A$ = 3



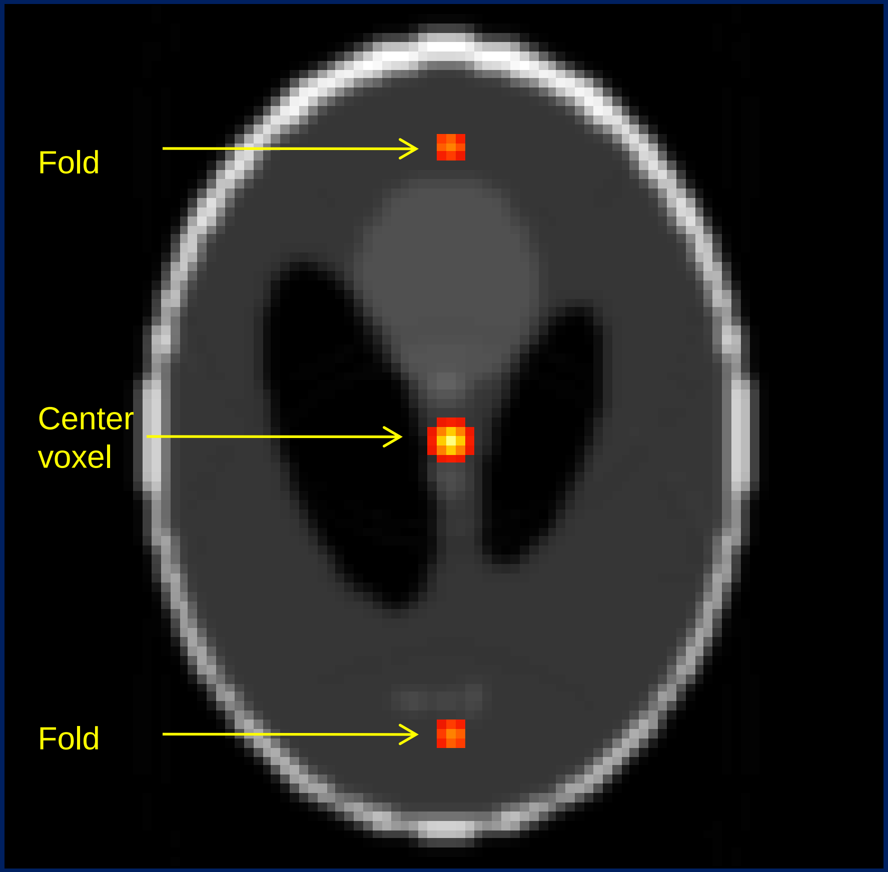

TH=0.01

Functional connectivity implications

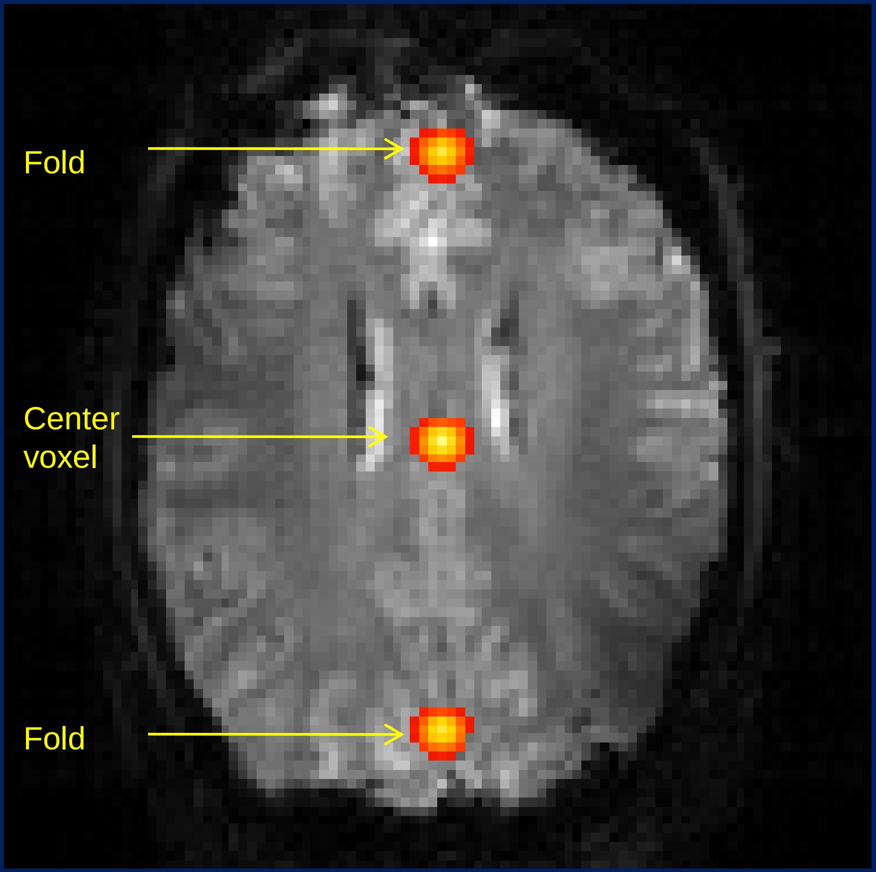
Bruce, Karaman, and Rowe: In Submission, 2011.

# Results

Phantom




Human




underlay artificially expanded

Extrapolate to human, mistakenly conclude regions correlated!

## Discussion

The SENSE image reconstruction method was described.

Wrote SENSE reconstruction with an isomorphism

$$y = O_I P_U U P_S P_C (I_n \otimes \Omega O_k) f = Of \, .$$

The new mean $E(y) = Of_0$ and covariance $\Sigma = O \Gamma O'$

of complex-valued SENSE described.

Theoretical results of SENSE reconstruction presented.

Ghosting present in SENSE magnitude and phase images.

Induced correlation between folds of no biological origin.

# Thank You

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**Sweet 16!**


**vs.**


**Go Marquette!**