Performance-Robust Dynamic Feedback Control of Lipschitz Nonlinear Systems

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PERFORMANCE-ROBUST DYNAMIC FEEDBACK CONTROL OF LIPSCHITZ NONLINEAR SYSTEMS

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ABSTRACT
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This dissertation addresses the dynamic control of nonlinear systems with finite energy noise in the state and measurement equations. Regional eigenvalue assignment (REA) is used to ensure that the state estimate error is driven to zero significantly faster than the state itself. Moreover, the controller is designed for the resulting closed loop system to achieve any one of a set of general performance criteria (GPC).

The nonlinear model is assumed to have a Lipschitz nonlinearity both in the state and measurement equations. By using the norm bound of the nonlinearity, the controller is designed to be robust against all nonlinearities satisfying the norm-bound. A Luenberger-type nonlinear observer is used to estimate the system state, which is not directly measurable.

The choice of the eigenvalue locations for the linear part of the system is based on the transient response specifications and the separation of the controller dynamics from the observer dynamics. Furthermore, the GPC are incorporated to achieve performance requirements such as H_2, H_∞, etc. The advantage of using GPC is it allows the designer flexibility in choosing a performance objective to tune the system.

The design problem introduced in this dissertation uses various mathematical techniques to derive LMI conditions for the controller and observer design using REA, GPC, and the bounds on the Lipschitz nonlinearities. All work will be demonstrated in both continuous- and discrete-time. Illustrative examples in both time domains are given to demonstrate the proposed design procedure. Multiple numerical approaches are also presented and compared in simulations for ease of use, applicability, and conservatism.

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Chapter 1

Introduction

Modern control theory can broadly be summarized as the use of state-space models to control individual states of a system. The methods for achieving stabilization and control of the modeled system are output feedback and state feedback. When learning modern control theory, the emphasis is on controlling noiseless linear systems with measurable states. In the real world, however, most systems are nonlinear, have states which are not known or measurable, and have some level of external noise acting on the system. Therefore, the goal of controller design is to stabilize the system while also accommodating the effects of the noise and nonlinearities. Given that all the states may not be measurable, it is also important to be able to design an estimator or an observer that can quickly estimate the state while also maintaining the desired performance for the closed loop system. Designers have used many techniques to achieve these design goals while maintaining a specific set of performance objectives.

The goal of this dissertation is to improve on existing techniques by developing a set of feasibility conditions which are necessary to design a state estimate feedback controller for systems with analytic nonlinearities that satisfy the Lipschitz condition. The controller-observer system will meet specific performance criteria such as H_2 or strict passivity, as well as achieve a desired transient response. This is done by combining known control techniques using linear matrix inequalities (LMIs). To understand the path of this research, it is useful to look back at where, in the body of research, various aspects of this research have been developed prior to this dissertation.

1.1 Historical Context

In this section, a brief examination at the origin of specific control techniques will be given. From the first linear matrix inequality problem in control theory to the development of the regional eigenvalue assignment (REA) inequality definition, this dissertation follows in the path of many great minds and builds upon the solid foundation of brilliant mathematicians and engineers.

1.1.1 Linear Matrix Inequalities (LMIs)

The LMI has been used in control system theory for over 120 years. In 1890, Aleksandr Lyapunov published what is now known as the Lyapunov Stability Theorem. It stated that a continuous-time system, represented as a first order differential equation, is stable if and only if there exists a positive definite matrix P such that

$$A^T P + PA < 0$$

where A represents the closed loop system. At the time, solving such LMIs was computationally challenging, so Lyapunov developed a method of turning the inequality into an equation by solving for a positive definite, symmetric matrix, Q, to solve for P explicitly. During the Cold War, while the United States and the West were developing the Dynamic Programming and Pontryagin Maximum Principle, the Soviet Union started applying Lyapunov's inequality to solving real world control problems. Back then, the inequalities that resulted from the design process had to be solved by hand. In the 1960s, noted scholars such as Yanukovych, Popov, and Kalman all described methods of simplifying the LMIs down to various criteria. Yakubovich [1] [2]was instrumental in solving the inequality constraints of automatic control problems. This directly led to the observation that certain LMIs could also be solved using an Algebraic Riccati Equation (ARE). Soon after, certain optimal control problems were being solved using LMIs. In the 1980s, computing had developed to the point where LMIs could be solved using a computer instead of by hand [3]. Since then, LMI techniques have been used for a variety of controller design techniques.

1.1.2 Performance Criteria

In designing controllers, it is desirable to achieve certain performance criteria. This dissertation will describe LMI methods for achieving various performance criteria. To that end, a brief history of the study of each performance criteria will be given.

Asymptotic stability is the most basic performance criterion. The idea that the energy of the system must uniformly converge to zero over time comes directly from the previously mentioned Lyapunov stability theory. As it is the most basic of the performance criteria, all control systems are expected to at least meet this performance criterion.

The H_2 performance has been a staple of modern control design since the Cold War era. Over time, three distinct methods of designing H_2 controllers have been developed: a transfer function solution based on solving Diophantine equations (linear polynomial matrix equation), a Wiener-Hopf analytic solution (partial differential equations), and a state-space solution [4]. Originally used for the Linear Quadratic Gaussian (LQG) problem, the goal of H_2 control is to minimize the cost of the control based on the parameter deemed most important. In the state-space formulation, this takes the form of specified states in the form of the performance output.

Study of H_∞ performance began in the early 1980s as a method of reducing the sensitivity of systems via feedback control [5]. The H_∞ performance criteria optimizes the systems in such a way that for a given performance output or target, the ratio of the norm of the output to the noise is minimized, thereby minimizing the effect of the noise.

The concept of passivity was introduced by Youla [6] in 1959. At the time, the definition was subject to some mathematical interpretation, as seen in Newcomb [7]. However, the definition of passivity was formally clarified in 1972 [8], reconciling the differences between the Youla definition and the Newcomb definition. By 1981, there were three different interpretations of what passivity meant. The first interpretation came from a thermodynamic point of view. The second interpretation was from a transfer function viewpoint that was applied to state-space. The third point of view is the view most commonly used in control application, the internal energy point of view. This interpretation says that a system is passive if the only energy in the system was supplied to the system from the outside. In other words, the supplied energy must be positive semidefinite. It was determined that all 3 interpretations, while different, were in fact equivalent [9].

1.1.3 Regional Eigenvalue Assignment

LMI techniques have been used in assigning eigenvalues of a closed loop linear system to specific regions of the complex plane. Regional eigenvalue assignment uses modified Lyapunov equations and other mathematical equations which are associated with regions in the complex plane. In 1989, a paper by Furuta and Kim used a modified discrete-time Lyapunov Equation to define a circular boundary. Based on the assumption that Q is positive definite, they showed that a valid placement of the eigenvalues would be within the circular region [10]. Other regional shapes, such as vertical strips and cones can also be defined using LMIs. In general, regional eigenvalue assignment is D-space formulation, meaning the region is a closed space, D. If all the eigenvalues of a closed loop system are all located within region D, then the system is considered D-stable. The region is independent of a specific time domain. The LMI formulation for REA has been shown to be valid in the design of controllers in both continuous-time and discrete-time using state-feedback [11] [12] and output feedback control [13]. However, unlike most conventional methods of controller design, D-stability does not guarantee asymptotic stability.

REA has also been applied to the design of observers for linear systems where the states are not measurable [14]. REA has also been used specifically for robust nonlinear control [15] [16]. Controllers designed using REA have been tuned to have specific performance criteria, which will be discussed in more detail in the next section.

Over time, many of the concepts introduced in this section have been integrated collectively into controller designs. Many researchers have examined methods of using REA to design robust controllers via output feedback [17] and state feedback [18]. REA has also been applied to H_2 [19] and H_∞ control [13]. Systems with vanishing nonlinearities have applied REA to the linear component of the system in both continuous-time [15] and discrete-time [7].

1.2 The General Performance Criteria Framework

A general performance criteria (GPC) framework was developed [20] by taking advantage of the similarities of the forms of the various performance criteria. The general performance criteria are built into a basic Lyapunov inequality with additional terms that are tuned to satisfy one of the previously mentioned performance criterion when integrated in continuous-time or summed over time in discrete-time. The GPC has been used to design resilient observers for linear continuous-time [21] and discrete-time [20] systems. The GPC has also been used for nonlinear systems in continuous-time [22] and discrete-time [23]. Resilient controllers [24] and robust feedback controllers [25] have also been achieved using the GPC. Recently, the GPC has been applied to the analysis of resilient dynamic feedback control for linear systems [26] and the design of resilient dynamic feedback control for nonlinear systems [27].

1.3 Outline of Dissertation

Why is this control design procedure being made? In a word, flexibility. The first goal is to have the flexibility to accommodate a variety of different nonlinearities that satisfy a specific Lipschitz bound. The second goal is to have the flexibility to choose a desirable performance criteria. The third goal is to have the flexibility to determine specific transient properties for the closed loop system. The last goal is to have the flexibility to achieve the previously mentioned goals within a single LMI framework without knowledge of the system state.

This dissertation will be formatted in the following way. In Chapter 2, a more in-depth explanation of the general performance criteria will be given. This will include a detailed derivation of each performance criterion within the GPC framework. The application of the GPC, based on previous work, will be given for both controller design and observer design. This will be demonstrated in both continuous-time and discrete-time. Lastly, the application of regional eigenvalue assignment in the design of controllers and observers will be examined.

In Chapter 3, the main results of this dissertation will be presented. This work includes the design of dynamic feedback controllers for both linear and nonlinear systems using the GPC framework together with REA in both continuous-time and discrete-time. Specific LMI techniques will be used to formulate the main results, presented as matrix inequalities, on which the analysis and design will be based.

Chapter 4 will show four methods of applying the main results to design a dynamic feedback controller. Again, LMIs are the cornerstone of each design procedure. Simulation data will be used to illustrate each method. The results produced by each method will be compared and analyzed.

The dissertation will then conclude in Chapter 5 with a summary of the work presented and will briefly describe potential areas of future work that can be explored. This chapter will explore the limitations of the design procedures and suggest paths forward to address these limitations.

1.4 Notation and Lemmas

The following notations will be used throughout the dissertation.

A > 0: A is a positive definite matrix

$x \in R^n$: *n*-dimensional vector with real elements

x^T: Transpose of a matrix x

$\dot{x}$: the first derivative, with respect to time, of x

||x||: The Euclidean norm of x = $(x^T x)^{1/2}$

|a|: the absolute value of scalar a.

$A \in R^{m \times n}$: m x n matrix with real elements.

A^{-1}: Inverse of matrix A,

I_m: Identity matrix of dimension m

$*$: represents the element or submatrix that need to be added to make the matrix symmetric.

Lemma 1a: The Schur Complement [28] (For English, see [29])

For a matrix inequality in the form

$$A - BC^{-1}B^T > 0$$

where

$$C > 0$$

Then the matrix inequality is equivalent to

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0$$

Lemma 1b: The Schur Complement alternate form

For a matrix inequality in the form

$$C - B^T A^{-1} B > 0$$

where

$$A > 0$$

Then the matrix inequality is equivalent to

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0$$

Lemma 2: The S-Procedure [30]

There exists a $\tau \geq 0$ such that

$$\xi^T (A - \tau B) \xi > 0$$

if and only if for a nonzero ξ, A and B satisfy the following inequalities,

$$\xi^T A \xi > 0 \text{ and } \xi^T B \xi \geq 0$$

Chapter 2

Theoretical and Mathematical Backgrounds

Before the main results of the research are presented, the more relevant works in literature that form the basis for the derivation and design procedure in this dissertation will be examined in closer detail. The use of the general performance criteria for linear continuous-time and discrete-time systems will be reviewed. The goal of using general performance criteria (GPC) is to have a framework for which various stability-based criteria can be achieved in the design and analysis of control systems. Some special cases of the general performance criteria will be discussed in detail and examples of designs for controllers and observers using the general performance criteria will be shown.

In addition, Regional Eigenvalue Assignment (REA) for linear systems will be examined. REA is a way to localize the eigenvalues of a closed loop system within an enclosed region of the complex plane. In both continuous-time and discrete-time, REA is useful in designing systems to have certain properties unique to specific regions of the complex plane, such as natural frequency, percent overshoot, and settling time. This Chapter will illustrate the use of REA in controller design and observer design.

The GPC and REA can both be expressed in the form of a Linear Matrix Inequality (LMI). However, most systems are nonlinear. Many of the nonlinearities in systems satisfy the Lipschitz property, which can also be expressed in LMI form. Therefore, these nonlinearities have norm bounds related to the Lipschitz property. This chapter will discuss the technique of applying the norm bound of the system to an LMI based design of controllers satisfying certain performance criteria. The main results of this dissertation, presented in Chapter 3, will be derived from the three techniques highlighted in this chapter.

2.1 The general performance criteria for continuous-time systems

Let us consider the following linear continuous-time system, represented with state, measurement, and performance output equations

$$\dot{x} = Ax + Bu + Fw \tag{2.1}$$

$$y = Cx + Du + Gw \tag{2.2}$$

$$z = C_z x + D_z w \tag{2.3}$$

where $x \in \mathbb{R}^n$ is the state of the system, $y \in \mathbb{R}^p$ is the output, and $w \in \mathbb{R}^w$ is finite-energy (L_2) disturbance. The system is assumed to be controllable and observable. In order to satisfy any of the performance criteria that can come from the general performance criteria framework, there must exist a symmetric positive definite matrix, P, such that for the Lyapunov energy function,

$$V = x^T Px \tag{2.4}$$

the general performance objective,

$$-\dot{V} - \delta z^T z - \varepsilon w^T w + \gamma z^T w > 0 \tag{2.5}$$

is satisfied [25]. This inequality guarantees the energy of the system decays over time while the performance satisfies the following inequality

$$\gamma \int_0^{T_f} z^T w dt > \delta \int_0^{T_f} \|z\|^2 dt + \varepsilon \int_0^{T_f} \|w\|^2 dt \tag{2.6}$$

The general performance criteria can be applied to achieve any of a set of performance objectives. This framework is used to give the designer the flexibility to alter performance goals without having to completely redesign the controller. In the next few sections, the performance objectives for systems with and without noise will be shown.

2.1.1 Non-noisy Cases

Assuming the system has no noise (w=0), inequality (2.5) reduces to

$$-\dot{V} - \delta z^T z > 0 \tag{2.7}$$

Through the GPC framework, a controller for a linear system modeled without noise can be designed specifically for two performance objectives: simple asymptotic stability and the H_2 performance.

2.1.1.1 Asymptotic Stability

The case where δ=0 for a system without noise is known as asymptotic stability. The definition of asymptotic stability is that for any initial condition, the system energy will continuously go to zero. Inequality (2.7) is further reduced to

$$-\dot{V} > 0 \tag{2.8}$$

In other words

$$\lim_{t \to \infty} \|x(t)\| = 0 \tag{2.9}$$

The LMI that provides a feasible asymptotically stable result is

$$-A^T P - PA > 0 \tag{2.10}$$

This LMI can be used to analyze control systems. If the closed loop system satisfies this LMI, it will be asymptotically stable. However, this LMI can also be used to design asymptotically stable controllers.

Asymptotic stability represents the most basic performance criterion to be satisfied in control systems. The use of the LMI for analysis and design also works for the more advance performance criteria discussed throughout the rest of this chapter.

2.1.1.2 The H_2 Performance Criterion

A system that exhibits H_2 performance satisfies the inequality [31]

$$\int_0^t \|z(\tau)\|^2 \, d\tau \le \tfrac{1}{\delta}\lambda_{\max}(P)\|x(0)\|^2 \tag{2.11}$$

where the value for δ is a positive scalar. For the system defined in equations (2.1) -(2.3), where w=0, inequality (2.7) can be expanded

$$-A^T P - PA - \delta C_z^{\,T} C_z > 0 \tag{2.12}$$

Similar to the asymptotic stability case, the analysis and design of continuous-time H_2 controllers or observers can be achieved using (2.12).

2.1.2 Noisy Cases

Given the system described in (2.1) -(2.3) with non-zero noise (w≠0), it follows that (2.5) can be expanded

$$\begin{aligned} -(Ax+Fw)^T Px - x^T P(Ax+Fw) - \delta(C_z x + D_z w)^T (C_z x + D_z w) \\ -\varepsilon w^T w + \gamma(C_z x + D_z w)^T w > 0 \end{aligned} \tag{2.13}$$

This inequality can be expressed in a block matrix form

$$\begin{bmatrix} x^T & w^T \end{bmatrix}\begin{bmatrix} -A^T P - PA - \delta C_z^T C_z & -PF - \delta C_z^T D_z + \tfrac{\gamma}{2} C_z^T \\ -F^T P - \delta D_z^T C_z + \tfrac{\gamma}{2} C_z & -\delta D_z^T D_z - \varepsilon I + \tfrac{\gamma}{2}(D_z^T + D_z) \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \qquad (2.14)$$

This matrix inequality is linear in P and therefore can be used to design and analyze control systems to satisfy various performance criteria.

2.1.2.1 The H∞ Performance Criterion

The H∞ performance criterion is achieved when the output to noise ratio is bounded. Another way of expressing this is with the inequality

$$\int_0^t \|z(\tau)\|^2 \, d\tau < -\varepsilon \int_0^t \|w(\tau)\|^2 \, d\tau \qquad (2.15)$$

In terms of the general performance criteria framework, inequality (2.15) implies that δ=1, γ=0, and ε<0. From (2.13),

$$-(Ax + Fw)^T Px - x^T P(Ax + Fw) - (C_z x + D_z w)^T (C_z x + D_z w) - \varepsilon w^T w > 0 \qquad (2.16)$$

Similar to the way (2.10) can be used to design and analyze asymptotically stable control systems, (2.16) can be used for the analysis and design of H∞ controllers or observers.

2.1.2.2 Strict Passivity

The strict passivity performance criterion [8] is achieved when

$$\int_0^t z(\tau)^T w(\tau) d\tau > 0 \qquad (2.17)$$

In terms of the general performance criteria framework, inequality (2.17) implies that $\delta=0$, $\gamma=1$, and $\varepsilon=0$. From (2.13),

$$-(Ax+Fw)^T Px - x^T P(Ax+Fw) + (C_z x + D_z w)^T w > 0 \tag{2.18}$$

This inequality is used for the analysis and design of strict passivity controllers.

2.1.2.3 Input Strict Passivity

The input strict passivity performance criterion is achieved when

$$\int_0^t z(\tau)^T w(\tau) d\tau > \varepsilon \int_0^t \|w(\tau)\|^2 d\tau \tag{2.19}$$

In terms of the general performance criteria framework, inequality (2.19) implies that $\delta=0$, $\gamma=1$, and $\varepsilon>0$. From (2.13),

$$-(Ax+Fw)^T Px - x^T P(Ax+Fw) - \varepsilon w^T w + (C_z x + D_z w)^T w > 0 \tag{2.20}$$

This inequality is used for the analysis and design of input strict passivity controllers or observers.

2.1.2.4 Output Strict Passivity

The output strict passivity performance criterion is achieved when

$$\int_0^t z(\tau)^T w(\tau) d\tau > \delta \int_0^t \|z(\tau)\|^2 d\tau \tag{2.21}$$

In terms of the general performance criteria framework, inequality (2.21) implies that $\delta>0$, $\gamma=1$, and $\varepsilon=0$. From (2.13),

$$-(Ax+Fw)^T Px - x^T P(Ax+Fw) - \delta(C_z x + D_z w)^T (C_z x + D_z w) + (C_z x + D_z w)^T w > 0 \quad (2.22)$$

This inequality is used for the analysis and design of output strict passivity controllers or observers.

2.1.2.5 Very Strict Passivity

The very strict passivity is the most general of the general performance criteria. It is achieved when

$$\int_0^t z(\tau)^T w(\tau) d\tau > \varepsilon \int_0^t \|w(\tau)\|^2 d\tau + \delta \int_0^t \|z(\tau)\|^2 d\tau \quad (2.23)$$

In terms of the general performance criteria framework, inequality (2.23) implies that $\delta>0$, $\gamma=1$, and $\varepsilon>0$. From (2.13),

$$\begin{aligned} -(Ax+Fw)^T Px - x^T P(Ax+Fw) - \delta(C_z x + D_z w)^T (C_z x + D_z w) \\ -\varepsilon w^T w + (C_z x + D_z w)^T w > 0 \end{aligned} \quad (2.24)$$

This inequality is used for the analysis and design of very strict passivity controllers or observers. It should be noted that in general, γ is set equal to zero or one because any other value can be normalized by dividing both sides of inequality (2.13) by the value of γ. As previously stated, these performance criteria can be used in the design of controllers and observers.

2.1.3 Design Application

The goal of the design procedure to determine the controller and/or observer gains necessary to achieve the desired performance for a closed loop system using LMI techniques. In this section, the process of using the general performance criteria to find the gains for the design of continuous-time observers and controllers will be discussed.

2.1.3.1 Observer Design [22]

In order to design the observer to have any of the general performance criteria, a Luenberger-type observer is designed. The state estimate dynamics are

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x} - Du) \tag{2.25}$$

where $\hat{x}$ represents the estimate of the state. Since the estimation error is defined as

$$e = x - \hat{x} \tag{2.26}$$

the state estimation error dynamics are defined as

$$\dot{e} = A_o e + (F - LG)w \tag{2.27}$$

where the closed loop observer matrix is

$$A_o = A - LC \tag{2.28}$$

with the observer gain defined by L and A is the open-loop system matrix. The GPC can be applied to the observer design by defining

$$V_e = e^T Pe \tag{2.29}$$

Therefore

$$\dot{V}_e = ((A - LC)e + (F - LG)w)^T Pe + e^T P((A - LC)e + (F - LG)w) \tag{2.30}$$

The GPC for the observer design can therefore be defined as

$$-\dot{V}_e - \delta z^T z - \varepsilon w^T w + \gamma z^T w > 0 \tag{2.31}$$

where z is defined as

$$z = Cz \cdot e + Dz \cdot w \tag{2.32}$$

Inequality (2.31) can thus be expanded to

$$\begin{aligned} &-((A-LC)e+(F-LG)w)^T Pe - e^T P((A-LC)e+(F-LG)w) - \delta(C_z e + D_z w)^T (C_z e + D_z w) \\ &\qquad -\varepsilon w^T w + \gamma (C_z e + D_z w)^T w > 0 \end{aligned} \tag{2.33}$$

which can be further expanded into the inequality

$$\begin{aligned} &e^T(-A^T P + C^T L^T P - PA + PLC - \delta C_z{}^T C_z)e + w^T(-(F-LG)^T P - D_z^T C_z + \tfrac{\gamma}{2} C_z)e \\ &+ e^T(-P(F-LG) - C_z^T D_z + \tfrac{\gamma}{2} C_z^T)w + w^T(-\varepsilon I - \delta D_z^T D_z + \tfrac{\gamma}{2}(D_z^T + D_z))w > 0 \end{aligned} \tag{2.34}$$

or in vector matrix form,

$$\begin{bmatrix} e^T & w^T \end{bmatrix} \begin{bmatrix} -A^T P + C^T L^T P - PA + PLC - \delta C_z{}^T C_z & -P(F-LG) - C_z^T D_z + \frac{\gamma}{2} C_z^T \\ -(F-LG)^T P - D_z^T C_z + \frac{\gamma}{2} C_z & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} > 0 \tag{2.35}$$

This matrix inequality is not linear due to the multiplication of unknowns P and L. This is easily remedied by defining a new variable,

$$Y_o = PL \tag{2.36}$$

Simplifying (2.35) yields a linear matrix inequality.

$$\begin{bmatrix} -A^T P + Y_o{}^T P - PA + Y_o C - \delta C_z{}^T C_z & -PF + Y_o G - C_z^T D_z + \frac{\gamma}{2} C_z^T \\ -F^T P + G^T Y_o^T - D_z^T C_z + \frac{\gamma}{2} C_z & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} > 0 \tag{2.37}$$

Since P is a positive definite symmetric matrix, it is invertible. Therefore, when P and Y_o are found, L can be calculated as

$$L = P^{-1} Y_o \tag{2.38}$$

The observer design will guarantee that the state estimator achieves the desired performance criteria.

2.1.3.2 Controller Design

The same principle for designing the observer applies to the controller design. To find the controller gain, the control input is defined as

$$u = Kx \tag{2.39}$$

The closed loop system matrix is defined as

$$A_c = A + BK \tag{2.40}$$

where K is the controller gain. By doing this, the general performance criteria inequality (2.13) becomes

$$\begin{aligned} -((A+BK)x + Fw)^T Px - x^T P((A+BK)x + Fw) - \delta(C_z x + D_z w)^T (C_z x + D_z w) \\ -\varepsilon w^T w + \gamma (C_z x + D_z w)^T w > 0 \end{aligned} \tag{2.41}$$

or in vector matrix form

$$\begin{bmatrix} x^T & w^T \end{bmatrix} \begin{bmatrix} -A^T P - K^T B^T P - PA - PBK - \delta C_z{}^T C_z & -PF - C_z^T D_z + \frac{\gamma}{2} C_z^T \\ -F^T P - D_z^T C_z + \frac{\gamma}{2} C_z & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} > 0$$

(2.42)

This matrix inequality is not linear due to the multiplication of the unknown matrices K and P with B. To address this, the matrix is pre-multiplied and post-multiplied by

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{2.43}$$

The resulting matrix inequality is

$$\begin{bmatrix} -P^{-1} A^T - P^{-1} K^T B^T - AP^{-1} - BKP^{-1} - \delta P^{-1} C_z{}^T C_z P^{-1} & -F - P^{-1} C_z^T D_z + \frac{\gamma}{2} P^{-1} C_z^T \\ -F^T - D_z^T C_z P^{-1} + \frac{\gamma}{2} C_z P^{-1} & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} > 0$$

(2.44)

By using Lemma 1 to deal with the quadratic term in the (1,1) block and defining Y_c as

$$Y_c = KP^{-1} \tag{2.45}$$

an LMI in P and Y_c is obtained

$$\begin{bmatrix} -P^{-1} A^T - Y_c{}^T B^T - AP^{-1} - BY_c & -F - P^{-1} C_z^T D_z + \frac{\gamma}{2} P^{-1} C_z^T & \sqrt{\delta} P^{-1} C_z{}^T \\ -F^T - D_z^T C_z P^{-1} + \frac{\gamma}{2} C_z P^{-1} & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) & 0 \\ \sqrt{\delta} C_z P^{-1} & 0 & I \end{bmatrix} > 0 \tag{2.46}$$

Therefore P^{-1} and Y_c can be found so that LMI (2.46) is satisfied. The controller gains can be calculated based on (2.45)

$$K = Y_c P \tag{2.47}$$

The continuous-time closed loop system designed through this method will satisfy any one of the general performance criteria illustrated in this chapter. However, for discrete-time systems, the matrices will be different. This is explored in detail in the next section.

2.2 The general performance criteria for discrete-time systems

Let us now consider the following linear discrete-time system with state, measurement, and performance output equations of the form

$$x_{k+1} = Ax_k + Bu_k + Fw_k \tag{2.48}$$

$$y_k = Cx_k + Du_k + Gw_k \tag{2.49}$$

$$z_k = C_z x_k + D_z w_k \tag{2.50}$$

where $x_k \in \mathbb{R}^n$ is the state of the system, $y_k \in \mathbb{R}^p$ is the output, and $w_k \in \mathbb{R}^w$ is a finite-energy (l_2) disturbance. The system is assumed to be controllable and observable. In order to satisfy any of the performance criteria that can come from the general performance criteria framework, there must exist a symmetric positive definite matrix, P, such that for the Lyapunov energy function,

$$V_k = x_k{}^T P x_k; \tag{2.51}$$

the general performance objective,

$$-V_{k+1} + V_k - \delta z_k^T z_k - \varepsilon w_k^T w_k + \gamma z_k^T w_k > 0 \tag{2.52}$$

is satisfied. This inequality guarantees the system's energy decays over time in such a way that

$$\gamma \sum_{j=0}^{T_f} z_j^T w_j > \delta \sum_{j=0}^{T_f} \|z_i\|^2 + \varepsilon \sum_{j=0}^{T_f} \|w_j\|^2 \tag{2.53}$$

The general performance criteria can be applied to many different performance objectives. In the following sections, the objectives for systems with and without noise will be examined.

2.2.1 Non-noisy Cases

Similar to the continuous-time case, when it is assumed that the system has no noise (w_k=0), inequality (2.52) can be reduced to

$$-V_{k+1} + V_k - \delta z_k^T z_k > 0 \tag{2.54}$$

Just like with the continuous-time, the two performance criteria that can be designed for a noiseless system are asymptotic stability and the H_2 performance.

2.2.1.1 Asymptotic Stability

The definition of asymptotic stability is that for any initial condition, the system energy will go to zero. In other words,

$$\lim_{k \to \infty} \|x_k\| = 0 \tag{2.55}$$

Mirroring the case continuous-time case, when δ=0 for a system without noise, the energy relationship can be expressed as the following inequality,

$$-V_{k+1}+V_k>0 \tag{2.56}$$

The inequality can be expressed in the following form

$$-A^T PA+P>0 \tag{2.57}$$

However, unlike the continuous-time, this matrix inequality is not linear. In order to put the matrix inequality into a linear form, Lemma 1 is used.

$$\begin{bmatrix} P & A^T P \\ PA & P \end{bmatrix} > 0 \tag{2.58}$$

This LMI is used in the analysis and design of asymptotically stable discrete-time controllers. It should be noted that in order to use LMI techniques to analyze or design discrete-time systems using the general performance criteria, the Schur complement will be used for every case due the nonlinearity in the Lyapunov inequality.

2.2.1.2 The H_2 Performance Criteria

A system that exhibits H_2 performance satisfies the inequality

$$\sum_i \|z_i\|^2 < \delta^{-1}\lambda_{\max}(P)\|x_0\|^2 \tag{2.59}$$

Since the norm of z summed over time, the norm of the initial state, and the max eigenvalue of P must be greater than or equal to zero, the value for δ is a positive scalar. For the system defined in equations (2.48) - (2.50), where w_k=0, inequality (2.52) can be expanded

$$-A^T PA+P-\delta C_z{}^T C_z>0 \tag{2.60}$$

Using Lemma 1 yields the LMI in P

$$\begin{bmatrix} P-\delta C_z{}^T C_z & A^T P \\ PA & P \end{bmatrix} > 0 \tag{2.61}$$

This LMI will be used in the analysis and design of discrete-time H_2 controllers and/or observers.

2.2.2 Noisy Cases

Given the system described in (2.48) -(2.50) where $w_k \neq 0$, it follows that (2.52) can be expanded

$$\begin{aligned} -(Ax_k + Fw_k)^T P(Ax_k + Fw_k) + x_k{}^T Px_k - \delta(C_z x_k + D_z w_k)^T (C_z x_k + D_z w_k) \\ -\varepsilon w_k{}^T w_k + \gamma(C_z x_k + D_z w_k)^T w_k > 0 \end{aligned} \tag{2.62}$$

Furthermore, this inequality can be expressed in a block matrix form

$$\begin{bmatrix} x_k{}^T & w_k{}^T \end{bmatrix} \begin{bmatrix} P - A^T PA - \delta C_z^T C_z & -A^T PF - \delta C_z^T D_z - \frac{\gamma}{2} C_z{}^T \\ -F^T PA - \delta D_z^T C_z - \frac{\gamma}{2} C_z & -F^T PF - \delta D_z^T D_z - \varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \geq 0 \tag{2.63}$$

This matrix inequality forms the basis of the design and analysis of various performance criteria for noisy systems.

2.2.2.1 The H_∞ Performance Criteria

The H_∞ performance criteria is achieved when

$$\sum_i \left\| z_i \right\|^2 < -\varepsilon \sum_i \left\| w_i \right\|^2 \tag{2.64}$$

In terms of the general performance criteria framework, inequality (2.64) implies that $\delta=1$, $\gamma=0$, and $\varepsilon<0$. From (2.62),

$$-(Ax_k + Fw_k)^T P(Ax_k + Fw_k) + x_k{}^T Px_k - (C_z x_k + D_z w_k)^T (C_z x_k + D_z w_k) - \varepsilon w_k{}^T w_k > 0 \tag{2.65}$$

This matrix inequality is used for the analysis and design of H_∞ controllers or observers.

2.2.2.2 Strict Passivity

The strict passivity performance criterion is achieved when

$$\sum_i z_i{}^T w_i > 0 \tag{2.66}$$

In terms of the general performance criteria framework, inequality (2.66) implies that $\delta=0$, $\gamma=1$, and $\varepsilon=0$. From (2.62),

$$-(Ax_k + Fw_k)^T P(Ax_k + Fw_k) + x_k{}^T Px_k + (C_z x_k + D_z w_k)^T w_k > 0 \tag{2.67}$$

This matrix inequality is used for the analysis and design of strict passivity controllers or observers.

2.2.2.3 Input Strict Passivity

The input strict passivity performance criterion is achieved when

$$\sum_i z_i{}^T w_i > \varepsilon \sum_i \|w_i\|^2 \tag{2.68}$$

In terms of the general performance criteria framework, inequality (2.68) implies that δ=0, γ=1, and ε>0. From (2.62),

$$-(Ax_k+Fw_k)^T P(Ax_k+Fw_k)+x_k{}^T Px_k-\varepsilon w_k{}^T w_k+(C_z x_k+D_z w_k)^T w_k>0 \tag{2.69}$$

This inequality is used for the analysis and design of input strict passivity controllers or observers.

2.2.2.4 Output Strict Passivity

The output strict passivity performance criterion is achieved when

$$\sum_i z_i{}^T w_i > \delta \sum_i \|z_i\|^2 \tag{2.70}$$

In terms of the general performance criteria framework, inequality (2.70) implies that δ>0, γ=1, and ε=0. From (2.62),

$$\begin{aligned}-(Ax_k+Fw_k)^T P(Ax_k+Fw_k)+x_k{}^T Px_k-\delta(C_z x_k+D_z w_k)^T(C_z x_k+D_z w_k)\\ +(C_z x_k+D_z w_k)^T w_k>0\end{aligned} \tag{2.71}$$

This inequality is used for the analysis and design of output strict passivity controllers or observers.

2.2.2.5 Very Strict Passivity

The very strict passivity performance criterion is achieved when

$$\sum_i z_i{}^T w_i > \delta \sum_i \|z_i\|^2 + \varepsilon \sum_i \|w_i\|^2 \tag{2.72}$$

In terms of the general performance criteria framework, inequality (2.72) implies that δ>0, γ=1, and ε>0. From (2.62),

$$\begin{aligned} -(Ax_k + Fw_k)^T P(Ax_k + Fw_k) + x_k{}^T Px_k - \delta(C_z x_k + D_z w_k)^T (C_z x_k + D_z w_k) \\ -\varepsilon w_k{}^T w_k + (C_z x_k + D_z w_k)^T w_k > 0 \end{aligned} \tag{2.73}$$

This inequality is used for the analysis and design of very strict passivity controllers or observers. The inequalities used in this section are not linear and during the design or analysis, they will need to be put into a linear form. As previously stated, these performance criteria can be used in the design of controllers and observers. In the next section, a closer look at the design procedure will be given.

2.2.3 Design Application

Similar to the continuous-time case, in discrete-time the design of the controller and/or observer is achieved by finding the appropriate gains to achieve the desired performance criterion. This section highlights how the LMI techniques are used to put the matrix inequalities into a linear form in such a way that the gains can be determined.

2.2.3.1 Observer Design [23]

In order to design the observer to have any of the general performance criteria, a Luenberger-type observer is designed. The state estimate update equation is

$$\hat{x}_{k+1} = Ax_k + Bu_k + L(y_k - C\hat{x}_k - Du_k) \tag{2.74}$$

where $\hat{x}_k$ represents the estimate of the state. Since the estimation error is defined as

$$e_k = x_k - \hat{x}_k \tag{2.75}$$

the state estimation error update equation is

$$e_{k+1} = (A-LC)e_k + (F-LG)w_k \tag{2.76}$$

By applying the general performance criteria inequality to the error dynamics, (2.62) becomes

$$\begin{aligned}&-((A-LC)e_k + (F-LG)w_k)^T P((A-LC)e_k + (F-LG)w_k) + e_k{}^T P e_k \\ &-\delta(C_z e_k + D_z w_k)^T (C_z e_k + D_z w_k) - \varepsilon w_k{}^T w_k + \gamma(C_z e_k + D_z w_k)^T w_k > 0\end{aligned} \tag{2.77}$$

which can be further expanded into the inequality

$$\begin{aligned}e_k^T(P-(A-LC)^T P(A-LC) - \delta C_z^T C_z)e_k - e_k^T((A-LC)^T P(F-LG) + \delta C_z^T D_z - \tfrac{\gamma}{2} C_z^T)w_k \\ -w_k^T((F-LG)^T P(A-LC) + \delta D_z^T C_z - \tfrac{\gamma}{2} C_z)e_k - w_k^T((F-LG)^T P(F-LG) + \delta D_z^T D_z \\ +\varepsilon I - \tfrac{\gamma}{2}(D_z^T + D_z))w_k > 0\end{aligned}$$

(2.78)

or in vector matrix form,

$$\begin{bmatrix} e_k \\ w_k \end{bmatrix}^T \begin{bmatrix} P-(A-LC)^T P(A-LC) - \delta C_z^T C_z & \boxed{\begin{array}{r}-(A-LC)^T P(F-LG) \\ -\delta C_z^T D_z + \frac{\gamma}{2} C_z^T\end{array}} \\ \boxed{\begin{array}{r}-(F-LG)^T P(A-LC) \\ -\delta D_z^T C_z + \frac{\gamma}{2} C_z\end{array}} & \boxed{\begin{array}{r}-(F-LG)^T P(F-LG) - \delta D_z^T D_z \\ -\varepsilon I + \frac{\gamma}{2}(D_z^T + D_z)\end{array}} \end{bmatrix} \begin{bmatrix} e_k \\ w_k \end{bmatrix} > 0$$

(2.79)

The inequality is not linear. The first step in putting this into a linear form is to make use of the Schur complement.

$$\begin{bmatrix} P-\delta C_z^T C_z & -\delta C_z^T D_z + \frac{\gamma}{2} C_z^T & (A-LC)^T P \\ -\delta D_z^T C_z + \frac{\gamma}{2} C_z & -\delta D_z^T D_z - \varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) & (F-LG)^T P \\ P(A-LC) & P(F-LG) & P \end{bmatrix} > 0 \tag{2.80}$$

This matrix inequality is not linear due to the multiplication of unknowns P and L. This is easily remedied by defining a new variable,

$$Y_o = PL \tag{2.81}$$

This makes (2.80) a linear matrix inequality.

$$\begin{bmatrix} P-\delta C_z^T C_z & -\delta C_z^T D_z + \frac{\gamma}{2} C_z^T & A^T P - C^T Y_o^T \\ -\delta D_z^T C_z + \frac{\gamma}{2} C_z & -\delta D_z^T D_z - \varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) & F^T P - G^T Y_o^T \\ PA - Y_o C & PF - Y_o G & P \end{bmatrix} > 0 \tag{2.82}$$

Since P is a positive definite symmetric matrix, it is invertible. Therefore, when P and Y_o are found in inequality (2.82), L can be calculated as

$$L = P^{-1} Y_o \tag{2.83}$$

The observer design will guarantee that the state estimate error achieves the desired performance criteria.

2.2.3.2 Controller Design

The same principle for designing the observer applies to the controller design. To find the controller gain, the control input is defined as

$$u_k = K x_k \tag{2.84}$$

The closed loop system matrix becomes

$$A_c = A + BK \tag{2.85}$$

where K is the controller gain. By doing this, the general performance criteria inequality (2.62) becomes

$$\begin{aligned} -((A+BK)x_k + Fw_k)^T P((A+BK)x_k + Fw_k) + x_k{}^T P x_k - \delta(C_z x_k + D_z w_k)^T (C_z x_k + D_z w_k) \\ -\varepsilon w_k{}^T w_k + \gamma(C_z x_k + D_z w_k)^T w_k > 0 \end{aligned} \tag{2.86}$$

which can be further expanded into the inequality

$$\begin{aligned} x_k^T (P - (A+BK)^T P(A+BK) - \delta C_z^T C_z) x_k - x_k^T ((A+BK)^T PF + \delta C_z^T D_z - \tfrac{\gamma}{2} C_z^T) w_k \\ -w_k^T (F^T P(A+BK) + \delta D_z^T C_z - \tfrac{\gamma}{2} C_z) x_k - w_k^T (F^T PF + \delta D_z^T D_z + \varepsilon I - \tfrac{\gamma}{2}(D_z^T + D_z)) w_k > 0 \end{aligned} \tag{2.87}$$

or in vector matrix form

$$\begin{bmatrix} x_k{}^T & w_k{}^T \end{bmatrix} \begin{bmatrix} \boxed{\begin{array}{l} P - (A+BK)^T P(A+BK) \\ -\delta C_z^T C_z \end{array}} & \begin{array}{l} -(A+BK)^T PF \\ -\delta C_z^T D_z + \tfrac{\gamma}{2} C_z^T \end{array} \\ \begin{array}{l} -F^T P(A+BK) \\ -\delta D_z^T C_z + \tfrac{\gamma}{2} C_z \end{array} & \boxed{\begin{array}{l} -F^T PF - \delta D_z^T D_z \\ -\varepsilon I + \tfrac{\gamma}{2}(D_z^T + D_z) \end{array}} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} > 0 \tag{2.88}$$

The inequality is not linear. The first step in putting this into a linear form is to make use of Lemma 1.

$$\begin{bmatrix} P - \delta C_z^T C_z & -\delta C_z^T D_z + \tfrac{\gamma}{2} C_z^T & A^T P + K^T B^T P \\ -\delta D_z^T C_z + \tfrac{\gamma}{2} C_z & -\delta D_z^T D_z - \varepsilon I + \tfrac{\gamma}{2}(D_z^T + D_z) & F^T P \\ PA + PBK & PF & P \end{bmatrix} > 0 \tag{2.89}$$

Since this matrix inequality is not linear due to the multiplication of the unknown matrices K and P with B. To address this, the matrix is pre-multiplied and post-multiplied by

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P^{-1} \end{bmatrix} \tag{2.90}$$

The resulting matrix inequality is

$$\begin{bmatrix} P^{-1}-\delta P^{-1}C_z^T C_z P^{-1} & -\delta P^{-1}C_z^T D_z + \frac{\gamma}{2}P^{-1}C_z^T & P^{-1}A^T + P^{-1}K^T B^T \\ -\delta D_z^T C_z P^{-1} + \frac{\gamma}{2}C_z P^{-1} & -\delta D_z^T D_z - \varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) & F^T \\ AP^{-1} + BKP^{-1} & F & P^{-1} \end{bmatrix} > 0 \tag{2.91}$$

By using Lemma 1 to deal with the quadratic term in the (1,1) block and defining Y_c as

$$Y_c = KP^{-1} \tag{2.92}$$

yields an LMI in P and Y_c

$$\begin{bmatrix} P^{-1} & \frac{\gamma}{2}P^{-1}C_z^T & P^{-1}A^T + Y_c^T B^T & \sqrt{\delta}P^{-1}C_z^{\ T} \\ \frac{\gamma}{2}C_z P^{-1} & -\varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) & F^T & \sqrt{\delta}D_z^T \\ AP^{-1} + BY_c & F & P^{-1} & 0 \\ \sqrt{\delta}C_z P^{-1} & \sqrt{\delta}D_z & 0 & I \end{bmatrix} > 0 \tag{2.93}$$

Therefore, when P^{-1} and Y_c are found so that LMI (2.93) is feasible, the controller gain can be calculated based on (2.92)

$$K = Y_c P \tag{2.94}$$

The discrete-time closed loop system designed through this method will satisfy any one of the general performance criteria illustrated in this chapter.

The general performance criteria can be used for both continuous-time and discrete-time systems to define various stability performance requirements within a single framework. The analysis allows for closed loop systems to be tested against the various performance criteria. This analysis has been expanded to also test the resilience and robustness of those closed loop systems to certain types of perturbations in the system and input. This framework is also useful in finding the gains that will allow a controller to give an open loop system the desired characteristics of the performance criteria. Table 2.1 provides a summary of the GPC for both continuous-time systems and discrete-time systems.

Table 2.1: General Performance Criteria

Design Criteria	Design Parameters	Continuous-time	Discrete-time
Non-noisy cases (w=0)			
Asymptotic Stability	$\delta = 0, \gamma = 0, \varepsilon = 0$	$\lim_{t\to\infty} \Vert x(t) \Vert = 0$	$\lim_{k\to\infty} \Vert x_k \Vert = 0$
H_2 Controller	$\delta = 1, \gamma = 0, \varepsilon = 0$	$\int_0^t \Vert z(\tau) \Vert^2 \, d\tau \le \delta^{-1} \lambda_{\max}(P) \Vert x(0) \Vert^2$	$\sum_i \Vert z_i \Vert^2 < \delta^{-1} \lambda_{\max}(P) \Vert x_0 \Vert^2$
Noisy Cases (w as a non-zero l_2 disturbance)			
H_∞ Controller	$\delta = 1, \gamma = 0, \varepsilon < 0$	$\int_0^t \Vert z(\tau) \Vert^2 \, d\tau < -\varepsilon \int_0^t \Vert w(\tau) \Vert^2 \, d\tau$	$\sum_i \Vert z_i \Vert^2 < -\varepsilon \sum_i \Vert w_i \Vert^2$
Strict Passivity	$\delta = 0, \gamma = 1, \varepsilon = 0$	$\int_0^t z(\tau)^T w(\tau) d\tau > 0$	$\sum_i z_i^T w_i > 0$
Input Strict Passivity	$\delta = 0, \gamma = 1, \varepsilon > 0$	$\int_0^t z(\tau)^T w(\tau) d\tau > \varepsilon \int_0^t \Vert w(\tau) \Vert^2 \, d\tau$	$\sum_i z_i^T w_i > \varepsilon \sum_i \Vert w_i \Vert^2$
Output Strict Passivity	$\delta > 0, \gamma = 1, \varepsilon = 0$	$\int_0^t z(\tau)^T w(\tau) d\tau > \delta \int_0^t \Vert z(\tau) \Vert^2 \, d\tau$	$\sum_i z_i^T w_i > \delta \sum_i \Vert z_i \Vert^2$
Very Strict Passivity	$\delta > 0, \gamma = 1, \varepsilon > 0$	$\int_0^t z(\tau)^T w(\tau) d\tau > \varepsilon \int_0^t \Vert w(\tau) \Vert^2 \, d\tau + \delta \int_0^t \Vert z(\tau) \Vert^2 \, d\tau$	$\sum_i z_i^T w_i > \delta \sum_i \Vert z_i \Vert^2 + \varepsilon \sum_i \Vert w_i \Vert^2$

2.3 Regional eigenvalue assignment for linear systems

Regional eigenvalue assignment (REA) is the technique of placing the eigenvalues of a linear system within specific regions of the complex plane. This placement contrasts with exact pole placement which is also used in control system design. The size and shape of the regions can vary from vertical strips to cones to circles and many more. These regions can be used to provide flexibility in the location of the eigenvalues while maintaining certain transient properties.

The regions can be represented using one or more matrix inequalities. Also, the regions are independent of the domain. This means the same formula for placing eigenvalues in a circle in the continuous-time domain can also be used in the discrete-time domain. This independent space is known as a D-space. The eigenvalues within the defined D-space are considered D-stable. For the purposes of the research presented in this thesis, the region defined will be circular. This is because with only a single LMI, an upper and lower boundary to the region is established, both in terms of the real part and the imaginary components of the eigenvalues.

To define the REA inequality for a circular region, the Lyapunov equality for discrete-time systems is modified. Furuta and Kim were able to shift and scale the mathematical definition of the unit circle into a D-space centered at (a,0) with radius r, or D (r, a). This matrix equation for a circular region is,

$$\frac{(A-aI)^T}{r} P \frac{(A-aI)}{r} - P = \frac{-Q}{r^2} \tag{2.95}$$

where A is the closed loop system matrix, Q is a positive definite matrix, r is a positive definite scalar, and a is a scalar. Since Q is positive definite and r is positive, equation (2.95) can be rewritten and expressed as a matrix inequality

$$r^2 P - (A - aI)^T P (A - aI) > 0 \tag{2.96}$$

This inequality may be used when calculating the gains of the observer or the controller.

2.3.1 Linear Observer Design

The observer design uses (2.96) with the A replaced with A_o.

$$r_o^{\,2} P - (A - LC - a_o I)^T P (A - LC - a_o I) > 0 \tag{2.97}$$

To determine the gains, Lemma 1 is used to linearize the matrix inequality and the product of P and L are defined as in (2.36). The resulting LMI is

$$\begin{bmatrix} r_o^2 P & A^T P - C^T Y_o^{\,T} - a_o P \\ PA - Y_o C - a_o P & P \end{bmatrix} > 0 \tag{2.98}$$

Once a P and Y_o are found that results in a feasible LMI (2.98), the observer gain L is calculated. The observer system, using the calculated L, will place the eigenvalues of the observer within a circular region of radius r_o and centered at a_o.

2.3.2 Linear Controller Design

Similarly, the controller design uses (2.96) with A_c substituted for A.

$$r_c^{\,2} P - (A + BK - a_c I)^T P (A + BK - a_c I) > 0 \tag{2.99}$$

Lemma 1 is used to obtain an equivalent positive definite matrix

$$\begin{bmatrix} r_c^2 P & A^T P + K^T B^T P - a_c P \\ PA + PBK - a_c P & P \end{bmatrix} > 0 \tag{2.100}$$

Like in the previous sections, the multiplication of K, B and P make the matrix inequality nonlinear. To obtain a solvable LMI, inequality (2.100) is pre-and post-multiplied by

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix} \tag{2.101}$$

Using the previous definition for Y_c, the resulting LMI is

$$\begin{bmatrix} r_c^2 P^{-1} & P^{-1} A^T + Y_c^T B^T - a_c P^{-1} \\ AP^{-1} + BY_c - a_c P^{-1} & P^{-1} \end{bmatrix} > 0 \tag{2.102}$$

Once P^{-1} and Y_c are found that results in a feasible solution, the controller gain K can be calculated. The closed loop system, using the calculated K, will place the eigenvalues of the controller within a circular region of radius r_c and centered at a_c.

2.4 Controller Design for Analytic Nonlinear Systems Via LMI Techniques [15]

The linear model is very useful; however, most real world systems are nonlinear. In this section, the design of controllers using LMI techniques will be explored for systems with analytic nonlinearities in the system model.

For continuous-time systems, let’s consider a nonlinear system with state and output equations

$$\dot{x} = \Omega(x,u) + Fw \tag{2.103}$$

$$y = \Psi(x,u) + Gw \tag{2.104}$$

where $x \in \mathbb{R}^n$ is the unknown state of the system, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, and $w \in \mathbb{R}^w$ is finite-energy (L_2) disturbance [13]. It is assumed that the nonlinear system is analytic and therefore, the matrices Ω and Ψ have linear parts which can be expressed separately from the higher order nonlinear terms. The state and measurement equations can thus be represented as,

$$\dot{x} = Ax + Bu + f(x) + Fw \tag{2.105}$$

$$y = Cx + Du + g(x) + Gw \tag{2.106}$$

where f and g are the differences between the nonlinear system matrix and the extracted linear component, also known as the higher order nonlinear terms. The linear components are assumed to be (A, B) controllable and (A, C) observable. These nonlinear terms can come in many forms. Some nonlinearities are unbounded and therefore the use of LMI techniques are ill suited for designing controllers with these types of nonlinearities. Other types can be bounded with respect to a constant or with respect to the state. In this dissertation, a specific type of nonlinearity is considered, Lipschitz-type nonlinearities.

2.4.1 Lipschitz-type nonlinearity

It is assumed that in the continuous-time and discrete-time domains, the nonlinearities, f, and g, are Lipschitz and therefore obey the following conditions:

$$f(0) = 0 \tag{2.107}$$

$$\left\| f(x_1) - f(x_2) \right\| \le \sqrt{\alpha} \left\| x_1 - x_2 \right\| \tag{2.108}$$

$$\|g(x_1)-g(x_2)\| \le \sqrt{\beta}\,\|x_1-x_2\| \tag{2.109}$$

for any x_1 and $x_2 \epsilon\ \mathbb{R}^n$. Note that (2.108) and (2.109) are Lipschitz conditions on f(x) and g(x), respectively. The constants α and β are linear growth bounds on the nonlinearity, also known as Lipschitz constants. From (2.107) and (2.108), it follows that

$$\|f(x)\| \le \sqrt{\alpha}\,\|x\| \tag{2.110}$$

Examples of nonlinearities that satisfy this condition are shown in Figure 2.1.

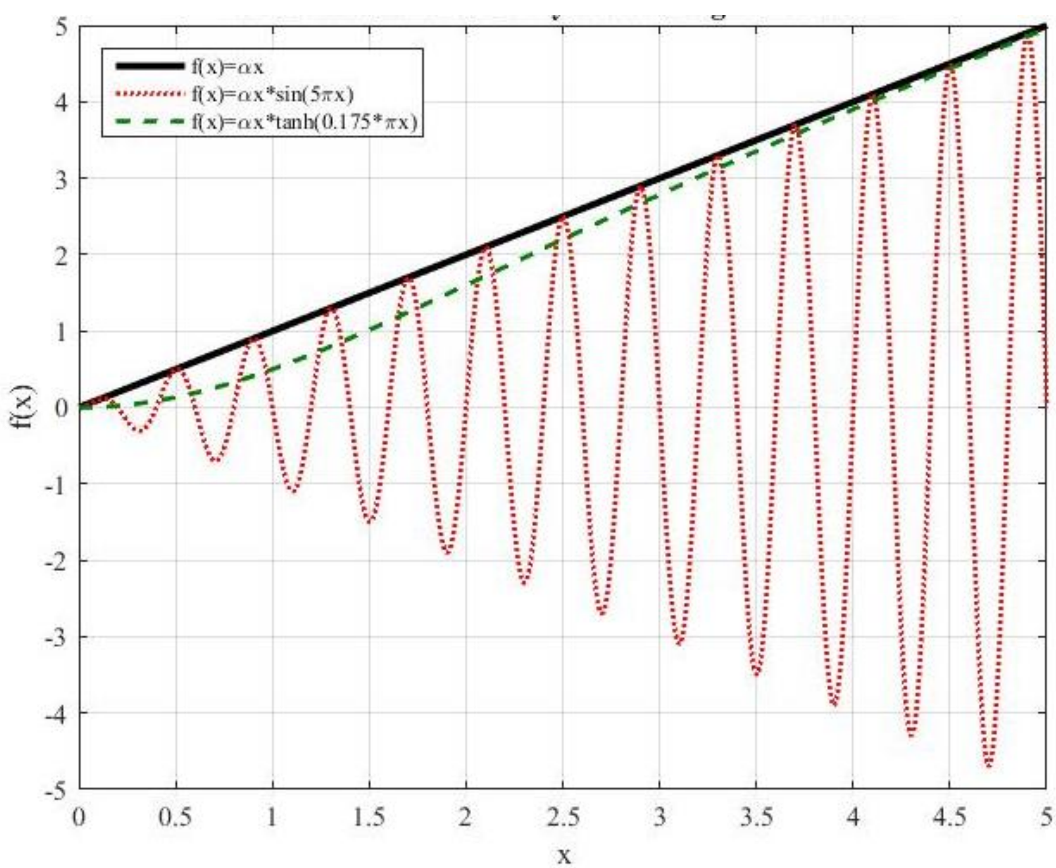

Figure 2.1: Examples of Lipschitz nonlinearities

Figure 2.1 shows nonlinearities that are bounded by the linear function α|x| where α>0. Nonlinear functions that satisfy (2.110) will converge to zero over time. This can therefore be considered the region of attraction. While the nonlinearities in Figure 2.1 are globally bounded, the results of this dissertation are also valid for locally bounded Lipschitz nonlinearities as long at the nonlinearity remain within the region of attraction.

2.4.2 Continuous-time Controller Design [15]

To design the controller for the nonlinear system described in (2.104) and (2.105), where w is assumed to be zero, the input is defined as

$$u = Kx \tag{2.111}$$

By doing this, the asymptotic stability condition becomes

$$-((A+BK)x+f(x))^T Px - x^T P((A+BK)x+f(x)) > 0 \tag{2.112}$$

this can be expanded into vector matrix form

$$\begin{bmatrix} x^T & f(x)^T \end{bmatrix} \begin{bmatrix} -A^T P - K^T B^T P - PA - PBK & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ f(x) \end{bmatrix} > 0 \tag{2.113}$$

The inequality is not linear. The first step in putting this into a linear form is to make use of the bounding conditions on the nonlinearity

$$\| f(x)\|^2 \leq \alpha \|x\|^2 \tag{2.114}$$

or

$$\begin{bmatrix} x^T & f(x)^T \end{bmatrix} \begin{bmatrix} \alpha I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ f(x) \end{bmatrix} > 0 \tag{2.115}$$

Since the vectors multiplying the matrices of (2.113) and (2.115) are the same, Lemma 2 can be used to combine the two expressions into the matrix inequality

$$\begin{bmatrix} -A^T P - K^T B^T P - PA - PBK - \tau\alpha I & P \\ P & \tau I \end{bmatrix} > 0 \tag{2.116}$$

As in Section 2.1.3.2, the matrix inequality needs to be pre-and post-multiplied by

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{2.117}$$

To obtain

$$\begin{bmatrix} -P^{-1}A^T - P^{-1}K^T B^T - AP^{-1} - BKP^{-1} - \tau\alpha P^{-2} & I \\ I & \tau I \end{bmatrix} > 0 \tag{2.118}$$

This results in a quadratic term so the Schur complement is used. The resulting LMI is

$$\begin{bmatrix} -P^{-1}A^T - P^{-1}K^T B^T - AP^{-1} - BKP^{-1} & I & P^{-1} \\ I & \tau I & 0 \\ P^{-1} & 0 & \alpha_\tau^{-1} I \end{bmatrix} > 0 \tag{2.119}$$

where

$$\alpha_\tau = \tau\alpha \tag{2.120}$$

The continuous-time closed loop system designed through this method will be asymptotically stable. It will also be able to accommodate the nonlinearity while maintaining the stability criterion.

2.4.2.1 Discrete-time Controller Design [16]

Let's consider a nonlinear discrete-time system with state and output equations,

$$x_{k+1} = \Omega(x_k, u_k) + Fw_k \tag{2.121}$$

$$y_k = \Psi(x_k, u_k) + Gw_k \tag{2.122}$$

where $x_k \in \mathbb{R}^n$ is the unknown state of the system, $u_k \in \mathbb{R}^m$ is the input, $y_k \in \mathbb{R}^p$ is the output, and $w_k \in \mathbb{R}^w$ is finite-energy (l_2) disturbance. It is again assumed that Ω and Ψ are analytic nonlinearities and can be expressed in such a way that the state space model is represented as,

$$x_{k+1} = Ax_k + Bu_k + f(x_k) + Fw_k \tag{2.123}$$

$$y_k = Cx_k + Du_k + g(x_k) + Gw_k \tag{2.124}$$

The pair (A, B) is assumed to be controllable and the pair (A, C) is assumed to be observable. It is also assumed that the nonlinearities, f and g, follow the Lipschitz conditions similar to those for continuous- time systems.

In order to design an asymptotically stable controller for the nonlinear system described in (2.123) and (2.124) with no noise, the input is defined as

$$u_k = Kx_k \tag{2.125}$$

By doing this, the general performance criteria inequality becomes

$$-((A+BK)x_k + f(x_k))^T P((A+BK)x_k + f(x_k)) + x_k^T P x_k > 0 \tag{2.126}$$

this can be expanded into vector matrix form

$$\begin{bmatrix} x_k \\ f(x_k) \end{bmatrix}^T \begin{bmatrix} P-(A+BK)^T P(A+BK) & -(A+BK)^T P \\ -P(A+BK) & -P \end{bmatrix} \begin{bmatrix} x_k \\ f(x_k) \end{bmatrix} > 0 \tag{2.127}$$

The inequality is not linear. The first step in putting this into a linear form is to make use of the bounding conditions on the Lipschitz nonlinearity

$$\| f(x_k) \|^2 \le \alpha \| x_k \|^2 \tag{2.128}$$

Or

$$\begin{bmatrix} x_k^T & f(x_k)^T \end{bmatrix} \begin{bmatrix} \alpha I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x_k \\ f(x_k) \end{bmatrix} > 0 \tag{2.129}$$

Using Lemma 2 yields the matrix inequality

$$\begin{bmatrix} P-(A+BK)^T P(A+BK) - \alpha_\tau I & -(A+BK)^T P \\ -P(A+BK) & \tau I - P \end{bmatrix} > 0 \tag{2.130}$$

where $\alpha_\tau = \tau\alpha$. To put this inequality into linear form, Lemma 1 is used. The resulting matrix inequality is

$$\begin{bmatrix} P-\alpha_\tau I & 0 & (A+BK)^T P \\ 0 & \tau I & P \\ P(A+BK) & P & P \end{bmatrix} > 0 \tag{2.131}$$

As in Section 2.2.3.2, the matrix inequality needs to be pre-and post-multiplied by

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P^{-1} \end{bmatrix} \tag{2.132}$$

To obtain

$$\begin{bmatrix} P^{-1}-\alpha_\tau P^{-2} & 0 & P^{-1}A+Y^T B^T \\ 0 & \tau I & I \\ AP^{-1}+BY & I & P^{-1} \end{bmatrix} > 0 \tag{2.133}$$

This results in a quadratic term so Lemma 1 is used. The resulting LMI is

$$\begin{bmatrix} P^{-1} & 0 & P^{-1}A+Y^T B^T & P^{-1} \\ 0 & \tau I & I & 0 \\ AP^{-1}+BY & I & P^{-1} & 0 \\ P^{-1} & 0 & 0 & \alpha_\tau^{-1} \end{bmatrix} > 0 \tag{2.134}$$

The discrete-time closed loop system designed through this method is asymptotically stable and accommodates the nonlinearity of the system.

2.5 Summary

In this chapter, the general performance criteria framework was described for linear systems. Its uses in the controller and observer design process were explained. Table 2.1

summarizes the various performance objectives that can come from the general performance criteria framework.

The regional eigenvalue assignment LMIs were derived as well. For linear system, regional eigenvalue assignment is a method to provide flexibility of eigenvalue locations without losing specific transient properties. While many different shaped regions could be used, for this dissertation, a circular region is used.

The use of LMI techniques in the design of controllers for nonlinear systems was also examined. By making use of the Lipschitz property of the nonlinearity, a linear upper bound can be incorporated into the design of the controller in such a way that all nonlinearities that stay within the bound will be accommodated. This was shown to work in both continuous-time and discrete-time.

In the next chapter, the general performance criteria will be used in conjunction with the regional eigenvalue assignment to design a state-estimate feedback controller for systems with Lipschitz nonlinearities in the system model. This will be derived for systems in continuous-time and discrete-time. The dynamic feedback controller will be designed to satisfy any of the general performance criteria in the presence of any nonlinearity that satisfies the Lipschitz bound. The regional eigenvalue assignment will separate eigenvalues of the closed loop linear component of the system from those of the linear component of the observer system. This separation will be reflected in the overall performance of the dynamic state feedback controller for the whole nonlinear system.

Chapter 3

Dynamic Feedback Controller Design Via The General Performance Criteria Framework With Regional Eigenvalue Assignment Constraints

The previous chapter laid the intellectual foundation for using general performance criteria (GPC), regional eigenvalue assignment (REA), and Lipschitz nonlinearities within a linear matrix inequality (LMI) framework. Moving forward, LMI techniques will be used to derive feasibility conditions for a dynamic state-feedback controller. The controller will consist of an observer to estimate the unknown states and a controller to stabilize the system, which can meet any one of the GPC. In the design of this controller, REA will be used to separate the eigenvalues of the observer from the eigenvalues of the controller in such a way that the state estimation error will be driven to zero much faster than the state.

To build the derivation up to the goal of applying this technique to nonlinear systems, it is advisable to first look at the technique applied to a linear system. This chapter will start by deriving the dynamic GPC controller with REA constraints for a continuous-time linear system. The GPC will be applied to nonlinear systems and derive the LMI conditions for the design of a controller. The derivation for a feasible nonlinear observer that satisfies the GPC also will be developed using LMI techniques. Lastly, the main result of this dissertation for continuous-time systems, a GPC LMI for a nonlinear dynamic feedback controller, will be derived. In addition to the GPC LMI constraints, the REA constraints will also be included to drive the estimation error to zero relatively quickly.

After the continuous-time main result has been derived, the work will then be duplicated for discrete-time systems. While the REA LMIs will not change, the GPC LMI will be derived again. To that end, the linear dynamic feedback controller with GPC will be derived for discrete-time systems. Afterwards, the derivation for the controller, the observer, and the dynamic feedback controller will be derived. The GPC LMI for nonlinear dynamic feedback controllers, the discrete-time main result of this dissertation, will include the REA constraints to insure the separation of the controller and observer eigenvalues.

3.1 Continuous-time Dynamic Feedback Controller Design

The GPC controller design that was derived in the previous chapter can also be applied to the design of dynamic state-feedback controllers. This type of controller can estimate the state and control the system based on the estimate. Individually, the general performance criteria can be applied to the observer and controller. But for the purposes of time and design efficiency, applying the GPC to both at the same time is preferable.

The goal is to derive an LMI that can produce a feasible gain for both the controller and observer that satisfies a GPC while accommodating Lipschitz type nonlinearities. But first, the LMI that can produce feasible gains while satisfying a GPC for linear systems should be derived as a proof of concept for the design process.

3.1.1 Dynamic Feedback Controller Design for Continuous-time Linear Systems

Let us consider the following linear continuous-time system,

$$\dot{x} = Ax + Bu + Fw \tag{3.1}$$

$$y = Cx + Du + Gw \tag{3.2}$$

where x ϵ $\mathbb{R}^n$ is the state of the system, u ϵ $\mathbb{R}^m$ is the input, y ϵ $\mathbb{R}^p$ is the output, and w ϵ $\mathbb{R}^w$ is finite-energy (L_2) system noise.

The nth-order Luenberger observer used in Chapter 2 is used to calculate a state estimate, $\hat{x}$.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x} - Du) \tag{2.25}$$

The feedback control, based on the state estimate, is given by

$$u = K\hat{x} \tag{3.3}$$

Theorem 3.1: Given the model of a continuous-time linear system described in (3.1) and (3.2) and the dynamic feedback control law given by (3.3), the closed loop system satisfies the GPC if the matrix inequality

$$\begin{bmatrix} -A_c^T P_c - P_c A_c - \delta C_{z1}^T C_{z1} & P_c BK - \delta C_{z1}{}^T C_{z2} & -P_c F - \delta C_{z1}^T D_z + \frac{\gamma}{2} C_{z1}^T \\ K^T B^T P_c - \delta C_{z2}{}^T C_{z1} & -A_o^T P_o - P_o A_o - \delta C_{z2}^T C_{z2} & -P_o F + P_o L_o G - \delta C_{z2}^T D_z + \frac{\gamma}{2} C^T{}_{z2} \\ -F^T P_c - \delta D_z^T C_{z1} + \frac{\gamma}{2} C_{z1} & -F^T P_o + G^T L^T P_o - \delta D_z^T C_{z2} + \frac{\gamma}{2} C_{z2} & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} > 0 \tag{3.4}$$

is feasible.

Proof

The state estimate error defined is the same as (2.26) and therefore has a corresponding state estimate error update equation that matches (2.27). Therefore, the augmented system can be represented as,

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_c & -BK \\ 0 & A_o \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} F \\ F - LG \end{bmatrix} w \tag{3.5}$$

where A_o is as defined in (2.28) and A_c is as defined in (2.40). Equation (3.5) can be compactly expressed as,

$$\dot{\mathrm{X}} = \Lambda \mathrm{X} + Hw \tag{3.6}$$

where

$$\mathrm{X} = \begin{bmatrix} x \\ e \end{bmatrix} \tag{3.7}$$

$$\Lambda = \begin{bmatrix} A_c & -BK \\ 0 & A_o \end{bmatrix} \tag{3.8}$$

$$H = \begin{bmatrix} F \\ F - LG \end{bmatrix} \tag{3.9}$$

The performance output is defined as

$$z = C_z X + D_z w \tag{3.10}$$

where

$$C_z = \begin{bmatrix} C_{z1} & C_{z2} \end{bmatrix} \tag{3.11}$$

The GPC is revised by using the following Lyapunov energy function

$$V = \mathrm{X}^T P \mathrm{X}; \tag{3.12}$$

For mathematical simplicity, it is assumed that P is a block diagonal symmetric positive definite matrix

$$P = \begin{bmatrix} P_c & 0 \\ 0 & P_o \end{bmatrix} \tag{3.13}$$

Therefore, the GPC becomes

$$-\dot{V} - \delta z^T z - \varepsilon w^T w + \gamma z^T w > 0 \tag{3.14}$$

Using the system defined in (3.6), it follows that

$$\begin{aligned} -(\Lambda \mathrm{X} + Hw)^T P X - X^T P(\Lambda \mathrm{X} + Hw) - \delta(C_z X + D_z w)^T (C_z X + D_z w) \\ -\varepsilon w^T w + \gamma (C_z X + D_z w)^T w > 0 \end{aligned} \tag{3.15}$$

This can be expressed in the matrix inequality

$$\begin{bmatrix} X^T & w^T \end{bmatrix} \begin{bmatrix} -\Lambda^T P - P\Lambda - \delta C_z^T C_z & -PH - \delta C_z^T D_z + \frac{\gamma}{2} C_z^T \\ -H^T P - \delta D_z^T C_z + \frac{\gamma}{2} C_z & -\delta D_z^T D_z - \varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} \begin{bmatrix} X \\ w \end{bmatrix} > 0 \tag{3.16}$$

Expanding (3.16) yields the matrix inequality (3.4).

This theorem forms the basis of the general performance criteria design procedure for linear dynamic feedback controllers in continuous-time. In Chapter 4, various methods of solving inequality (3.4) for design purposes will be examined.

3.1.2 Continuous-time Nonlinear GPC Controller Design with REA Constraints

In Chapter 2, the work of Siljak and Stipanovic allowed for certain types of nonlinearities to be accommodated through use of LMI techniques. As a direct expansion of that work, the GPC will be incorporated into the design. This time, the L2 noise will not be assumed to be zero.

Theorem 3.2: Given the model of a continuous-time nonlinear system described in (2.105) and (2.106), the performance output as defined in (2.3) and the input is defined in (2.39), the closed loop system satisfies the GPC for the designed controller if the matrix inequality

$$\begin{bmatrix} -P^{-1}A^T - P^{-1}K^TB^T - AP^{-1} - BKP^{-1} & I & -F - P^{-1}C_z^TD_z + P^{-1}\frac{\gamma}{2}C_z^T & \sqrt{\delta}P^{-1}C_z^T & P^{-1} \\ I & \tau I & 0 & 0 & 0 \\ -F^T - D_z^TC_zP^{-1} + \frac{\gamma}{2}C_zP^{-1} & 0 & -\varepsilon I - \delta D_z^TD_z + \frac{\gamma}{2}(D_z^T + D_z) & 0 & 0 \\ \sqrt{\delta}C_zP^{-1} & 0 & 0 & I & 0 \\ P^{-1} & 0 & 0 & 0 & \alpha_\tau^{-1}I \end{bmatrix} > 0 \tag{3.17}$$

where

$$\alpha_\tau = \tau\alpha \tag{3.18}$$

is feasible.

Proof

In order to design the continuous-time GPC controller for the nonlinear system, the general performance criteria inequality (3.14) is expanded based on the system model

$$-((A+BK)x+f(x)+Fw)^T Px - x^T P((A+BK)x+f(x)+Fw) - \delta(C_z x + D_z w)^T (C_z x + D_z w) - \varepsilon w^T w + \gamma(C_z x + D_z w)^T w > 0$$

(3.19)

this can be expanded into vector matrix form.

$$\begin{bmatrix} x^T & f(x)^T & w^T \end{bmatrix} \begin{bmatrix} -A^T P - K^T B^T P - PA - PBK - \delta C_z{}^T C_z & P & -PF - \delta C_z^T D_z + \frac{\gamma}{2} C_z^T \\ P & 0 & 0 \\ -F^T P - \delta D_z^T C_z + \frac{\gamma}{2} C_z & 0 & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} \begin{bmatrix} x \\ f(x) \\ w \end{bmatrix} > 0$$

(3.20)

The inequality is not linear and therefore LMI techniques cannot be used. The first step in putting this into a linear form is to use the bounding conditions on the nonlinearity

$$\| f(x) \|^2 \leq \alpha \| x \|^2 \qquad (3.21)$$

or

$$\begin{bmatrix} x^T & \mathrm{T}^T & w^T \end{bmatrix} \begin{bmatrix} \alpha I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \mathrm{T} \\ w \end{bmatrix} > 0 \qquad (3.22)$$

Using the S-procedure yields the matrix inequality

$$\begin{bmatrix} -A^T P - K^T B^T P - PA - PBK - \delta C_z{}^T C_z - \tau\alpha I & P & -PF - C_z^T D_z + \frac{\gamma}{2} C_z^T \\ P & \tau I & 0 \\ -F^T P - D_z^T C_z + \frac{\gamma}{2} C_z & 0 & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} > 0$$

(3.23)

Like in the previous chapter, the matrix inequality needs to be pre-and post-multiplied by

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{3.24}$$

To get

$$\begin{bmatrix} \boxed{\begin{matrix} -P^{-1}A^T - P^{-1}K^T B^T - AP^{-1} - BKP^{-1} \\ -\delta P^{-1} C_z{}^T C_z P^{-1} - \tau\alpha P^{-2} \end{matrix}} & I & -F - P^{-1} C_z^T D_z + P^{-1} \frac{\gamma}{2} C_z^T \\ I & \tau I & 0 \\ -F^T - D_z^T C_z P^{-1} + \frac{\gamma}{2} C_z P^{-1} & 0 & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} > 0 \tag{3.25}$$

Expanding (3.25) will produce the result stated in theorem 3.2. The continuous-time closed loop system designed through this method will satisfy any one of the general performance criteria illustrated in this chapter.

If a specific circular region is the desired location of the observer eigenvalues, the regional eigenvalue assignment will need to be solved simultaneously with (3.17). The linear growth bound must be incorporated into the regional eigenvalue assignment LMI to guarantee that the nonlinearity will not cause the performance to deviate beyond the limits set by the regional assignment.

Theorem 3.3: Given the model of a continuous-time nonlinear system described in (2.105) and (2.106), the performance output as defined in (2.3) and the input is defined in (2.39), the closed loop system satisfies the GPC and place the eigenvalues of the linear component of the system within a circular region of radius, r_c, and centered at a_c if the matrix inequalities (3.17) and

$$\begin{bmatrix} r_c^2 P^{-1} & P^{-1}A^T + Y_c^T B^T - a_c P^{-1} & P^{-1} \\ AP^{-1} + BY_c - a_c P^{-1} & P^{-1} & 0 \\ P^{-1} & 0 & \alpha^{-1} \end{bmatrix} > 0 \tag{3.26}$$

are feasible.

Proof

The new regional eigenvalue assignment inequality, which considers the nonlinearity, is

$$r_c^2 P - (A + BK - a_c I)^T P (A + BK - a_c I) - \alpha I > 0 \tag{3.27}$$

After using the Schur Complement, the regional eigenvalue assignment LMI is

$$\begin{bmatrix} r_c^2 P - \alpha I & A^T P + K^T B^T P - a_c P \\ PA + PBK - a_c P & P \end{bmatrix} > 0 \tag{3.28}$$

As was done previously, the multiplication of K, B and P make the matrix inequality nonlinear. To deal with this, inequality (3.28) is pre-and post-multiplied by

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix} \tag{3.29}$$

Using the previous definition for Y_c, the resulting LMI is

$$\begin{bmatrix} r_c^2 P^{-1} - \alpha P^{-2} & P^{-1}A^T + Y_c^T B^T - a_c P^{-1} \\ AP^{-1} + BY_c - a_c P^{-1} & P^{-1} \end{bmatrix} > 0 \tag{3.30}$$

Unlike with the linear case, the (1,1) term that has the linear growth bound is quadratic. Therefore, Lemma 1 is used to put inequality (3.30) into a linear form. The resulting LMI is (3.26).

It should be noted that the additional constraints on the REA have the unintended consequence of making the allowable Lipschitz constant more conservative since instead of being only with respect to the stability of the system, it is with respect to the D-stability of the region. Therefore, if separation of the eigenvalues is all that matters and the transient properties of the state response and the estimation error are deemed irrelevant, it may be advisable to use the simple REA described in Chapter 2.

3.1.3 Continuous-time Nonlinear GPC Observer Design with REA Constraints

Theorem 3.4: Given the model of a continuous-time nonlinear system described in (2.105) and (2.106) and the performance output as defined in (2.3), the closed loop system satisfies the GPC for the observer design if the matrix inequality

$$\begin{bmatrix} -A^T P + C^T Y_o^T - PA + Y_o C - \delta C_z{}^T C_z - \tau(\alpha+\beta) I & P\mathrm{N} & -PF + Y_o G - C_z^T D_z + \frac{\gamma}{2} C_z^T \\ \mathrm{N}^T P & \tau I & 0 \\ -F^T P + G^T Y_o^T - D_z^T C_z + \frac{\gamma}{2} C_z & 0 & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} > 0 \tag{3.31}$$

is feasible.

Proof

In order to design the observer for the nonlinear system to have any of the general performance criteria, a Luenberger-type nonlinear observer is designed with state estimate dynamics

$$\dot{\hat{x}} = A\hat{x} + Bu + f(\hat{x}) + L(y - C\hat{x} - Du - g(\hat{x})) \tag{3.32}$$

The state estimation error differential equation is

$$\dot{e} = A_o e + \Delta f - L\Delta g + Fw - LGw \tag{3.33}$$

where

$$\Delta f = f(x) - f(\hat{x}) \tag{3.34}$$

$$\Delta g = g(x) - g(\hat{x}) \tag{3.35}$$

This error dynamic can be expressed in a more compact form

$$\dot{e} = A_o e + \mathrm{NT} + (F - LG)w \tag{3.36}$$

where

$$\mathrm{N} = \begin{bmatrix} I & -L \end{bmatrix} \tag{3.37}$$

$$\mathrm{T} = \begin{bmatrix} \Delta f \\ \Delta g \end{bmatrix} \tag{3.38}$$

The GPC, defined in (2.29-2.31), is used in the design of the observer. Using the error dynamics described in (3.36), inequality (2.31) is expanded into

$$
\begin{aligned}
-((A-LC)e+\mathrm{NT}+(F-LG)w)^T Pe - e^T P((A-LC)e+\mathrm{NT}+(F-LG)w) \\
-\delta(C_z e + D_z w)^T (C_z e + D_z w) - \varepsilon w^T w + \gamma (C_z e + D_z w)^T w > 0
\end{aligned}
\tag{3.39}
$$

which can be further expanded and put into vector matrix form

$$
\begin{bmatrix} e^T & \mathrm{T}^T & w^T \end{bmatrix}
\begin{bmatrix}
-A^T P + C^T L^T P - PA + PLC - \delta C_z{}^T C_z & P\mathrm{N} & -PF + PLG - C_z^T D_z + \frac{\gamma}{2} C_z^T \\
\mathrm{N}^T P & 0 & 0 \\
-F^T P + G^T L^T P - D_z^T C_z + \frac{\gamma}{2} C_z & 0 & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z)
\end{bmatrix}
\begin{bmatrix} e \\ \mathrm{T} \\ w \end{bmatrix} > 0
$$

(3.40)

This matrix inequality is not linear due to the multiplication of the unknown P and L. This is remedied by defining a new variable,

$$
Y_o = PL \tag{3.41}
$$

The resulting matrix inequality is not a valid LMI since there is a zero on the diagonal. This is due to incomplete information on the nature of the nonlinearity. This information is incorporated into the design by putting the bounds into vector matrix form. Due to the Lipschitz condition on the nonlinearities, the bound on Γ is

$$
\|\mathrm{T}\|^2 = \|\Delta f\|^2 + \|\Delta g\|^2 \le \alpha \|e\|^2 + \beta \|e\|^2 \tag{3.42}
$$

or

$$
\begin{bmatrix} e^T & \mathrm{T}^T & w^T \end{bmatrix}
\begin{bmatrix}
(\alpha+\beta) I & 0 & 0 \\
0 & -I & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} e \\ \mathrm{T} \\ w \end{bmatrix} > 0 \tag{3.43}
$$

Applying Lemma 2 to (3.40) and (3.43) yields the LMI (3.31)

The observer design will guarantee that the state estimate error achieves the desired performance criteria. Also, if it is desired to place the observer eigenvalues in a specific region, a specific circular region is the desired location of the observer eigenvalues, the regional eigenvalue assignment LMI can be use simultaneously with (3.31).

Theorem 3.5: Given the model of a continuous-time nonlinear system described in (2.105) and (2.106) and the performance output as defined in (2.3), the closed loop system satisfies the GPC and place the eigenvalues of the linear component of the observer's eigenvalues within a circular region of radius, r_o, and centered at a_o if the matrix inequalities (3.31) and

$$\begin{bmatrix} r_o^2 P - (\alpha+\beta)I & A^T P - C^T Y_o^T - a_o P \\ PA - Y_o C - a_o P & P \end{bmatrix} > 0 \tag{3.44}$$

are feasible.

Proof

From Chapter 2, the regional eigenvalue assignment inequality is

$$r_o^2 P - (A - LC - a_o I)^T P (A - LC - a_o I) > 0 \tag{2.97}$$

However, in order to maintain the regional stability in the presence of perturbations, the linear growth condition must be considered. Therefore, the regional eigenvalue assignment inequality becomes

$$r_o^2 P - (A - LC - a_o I)^T P (A - LC - a_o I) - (\alpha+\beta)I > 0 \tag{3.45}$$

After using Lemma 1, the regional eigenvalue assignment LMI is (3.44).

The solution to both LMIs (3.41) and (3.44) will not only give the desired closed loop system performance, the eigenvalues of the linear component of the system will be within the circular region of the desired radius and centered where specified.

3.1.4 Dynamic Feedback Design

In order to control a continuous-time nonlinear system with unmeasurable states, a Luenberger-type nth-order nonlinear observer is used to calculate a state estimate, $\hat{x}$.

$$\dot{\hat{x}} = A\hat{x} + Bu + f(\hat{x}) + L(y - C\hat{x} - Du - g(\hat{x})) \tag{3.46}$$

The feedback control, which is based on the state estimate, is

$$u = K\hat{x} \tag{3.47}$$

The state estimate error update equation is

$$\dot{e} = (A - LC)e + f(x) - f(\hat{x}) + Fw - L(g(x) - g(\hat{x}) - Gw) \tag{3.48}$$

The closed loop system can thus be represented as,

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_c & -BK \\ 0 & A_o \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} f(x) \\ \Delta f - L\Delta g \end{bmatrix} + \begin{bmatrix} F \\ F - LG \end{bmatrix} w \tag{3.49}$$

Equation (3.49) can be compactly expressed as,

$$\dot{\mathrm{X}} = \Lambda \mathrm{X} + J\Gamma + Hw \tag{3.50}$$

where X, Λ, and H are as previous defined in (3.7), (3.8), and (3.9) and

$$J = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -L \end{bmatrix} \tag{3.51}$$

$$\Gamma = \begin{bmatrix} f(x) \\ \Delta f \\ \Delta g \end{bmatrix} \tag{3.52}$$

The performance output is defined in (3.10) and (3.11).

Theorem 3.6: Given the model of a continuous-time nonlinear system described in (2.105) and (2.106) and the dynamic feedback control law given by (3.4), the closed loop system satisfies the GPC if the matrix inequality

$$\begin{bmatrix}
-A_c^T P_c - P_c A_c - \frac{\delta}{\tau} C_{z1}^T C_{z1} - \alpha I & P_c BK - \frac{\delta}{\tau} C_{z1}^T C_{z2} & -P_c & 0 & 0 & -P_c F - \frac{\delta}{\tau} C_{z1}^T D_z + \frac{\gamma}{2\tau} C_{z1}^T \\
K^T B^T P_c - \frac{\delta}{\tau} C_{z2}^T C_{z1} & -A_o^T P_o - P_o A_o - \frac{\delta}{\tau} C_{z2}^T C_{z2} - (\alpha+\beta) I & 0 & -P_o & P_o L & -P_o F + P_o LG - \frac{\delta}{\tau} C_{z2}^T D_z + \frac{\gamma}{2\tau} C_{z2}^T \\
-P_c & 0 & I & 0 & 0 & 0 \\
0 & -P_o & 0 & I & 0 & 0 \\
0 & L^T P_o & 0 & 0 & I & 0 \\
-F^T P_c - \frac{\delta}{\tau} D_z^T C_{z1} + \frac{\gamma}{2\tau} C_{z1} & -F^T P_o + G^T L^T P_o - \frac{\delta}{\tau} D_z^T C_{z2} + \frac{\gamma}{2\tau} C_{z2} & 0 & 0 & 0 & -\frac{\varepsilon}{\tau} I - \frac{\delta}{\tau} D_z^T D_z + \frac{\gamma}{2\tau}(D_z^T + D_z)
\end{bmatrix} > 0 \tag{3.53}$$

is feasible.

Proof

A system satisfies a general performance criterion if there exists a symmetric positive definite matrix, P, such that for the Lyapunov energy function,

$$V = \mathrm{X}^T P \mathrm{X}; \tag{3.54}$$

where P is as defined in (3.13), the GPC is

$$-\dot{V} - \delta z^T z - \varepsilon w^T w + \gamma z^T w > 0 \tag{3.55}$$

From this, it follows

$$\begin{aligned} -(\Lambda \mathrm{X} + J\Gamma + Hw)^T PX - X^T P(\Lambda \mathrm{X} + J\Gamma + Hw) - \delta(C_z X + D_z w)^T (C_z X + D_z w) \\ -\varepsilon w^T w + \gamma(C_z X + D_z w)^T w > 0 \end{aligned} \tag{3.56}$$

This can be expressed in the matrix inequality

$$\begin{bmatrix} \mathrm{X} \\ \Gamma \\ w \end{bmatrix}^T \begin{bmatrix} -\Lambda^T P - P\Lambda - \delta C_z^T C_z & -PJ & -PH - \delta C_z^T D_z + \frac{\gamma}{2} C_z^{\ T} \\ -J^T P & 0 & 0 \\ -H^T P - \delta D_z^T C_z + \frac{\gamma}{2} C_z & 0 & -\delta D_z^T D_z - \varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} \begin{bmatrix} \mathrm{X} \\ \Gamma \\ w \end{bmatrix} > 0 \tag{3.57}$$

The bound on f (x_k) is

$$\| f(x_k)\|^2 \le \alpha \|x_k\|^2 \tag{3.58}$$

Therefore, the bound on Γ is

$$\|\Gamma\|^2 = \| f(x)\|^2 + \|\Delta f\|^2 + \|\Delta g\|^2 \le \alpha \|x\|^2 + (\alpha + \beta)\|e\|^2 \tag{3.59}$$

This can be represented in block matrix for as

$$\Gamma^T \Gamma \le \mathrm{X}^T M \mathrm{X} \tag{3.60}$$

where

$$M = \begin{bmatrix} \alpha I & 0 \\ 0 & \alpha I + \beta I \end{bmatrix} \tag{3.61}$$

Inequality (3.60) is expressed in such a way that it can be used with Lemma 2.

$$\begin{bmatrix} \mathrm{X}^T & \Gamma^T & w^T \end{bmatrix} \begin{bmatrix} M & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{X} \\ \Gamma \\ w \end{bmatrix} > 0 \tag{3.62}$$

Lemma 2 is applied to the (3.57) and (3.62) in order to combine the system dynamics with the bounds on the nonlinearities. The resulting matrix inequality is then multiplied by τ^{-1}. Redefining P for P/τ yields

$$\begin{bmatrix} -\Lambda^T P - P\Lambda - \frac{\delta}{\tau} C_z^T C_z - M & -PJ & -PH - \frac{\delta}{\tau} C_z^T D_z + \frac{\gamma}{2\tau} C_z^{\ T} \\ -J^T P & I & 0 \\ -H^T P - \frac{\delta}{\tau} D_z^T C_z + \frac{\gamma}{2\tau} C_z & 0 & -\frac{\delta}{\tau} D_z^T D_z - \frac{\varepsilon}{\tau} I + \frac{\gamma}{2\tau} (D_z^T + D_z) \end{bmatrix} > 0 \tag{3.63}$$

Expanding (3.63), the resulting matrix inequality is inequality (3.53)

This result, when solved using the REA constraints defined in (3.44) and (3.26), will place the eigenvalues of the linear component in the specified regions in such a way that the nonlinearity is accommodated and the desired performance criterion is met.

In dynamic control design, it is important that the error in the estimate of the state go to zero much faster than the state. In continuous-time, this means placing the observer eigenvalues closer to negative infinity. To force this separation, REA is used.

Theorem 3.7: Given the model of a continuous-time nonlinear system described in (2.105) and (2.106), the performance output as defined in (2.3) and the input is defined in (2.39), the closed loop system satisfies the GPC and place the eigenvalues of the linear component of the controller’s eigenvalues within a circular region of radius, r_c, and centered at a_c and the observer's eigenvalues within a circular region of radius, r_o, and centered at a_o if the matrix inequalities (3.53), (3.26), and (3.44) are feasible.

The basic example of this theorem was demonstrated in [31] for noiseless systems.

3.2 Discrete-time

The goal of deriving a matrix inequality result for a GPC dynamic feedback controller with REA constraints for nonlinear continuous-time systems has been met. Using similar methodology, a matrix inequality result in discrete-time will also be done. Like in the first half of this chapter, the dynamic feedback controller design for a linear system will be done first. Then, including the nonlinearity, the design for a controller, an observer, and a dynamic feedback controller with REA constraints will be derived.

3.2.1 Dynamic feedback Controller Design for Linear Systems

Let us consider the following linear discrete-time system,

$$x_{k+1} = Ax_k + Bu_k + Fw_k \tag{3.64}$$

$$y_k = Cx_k + Du_k + Gw_k \tag{3.65}$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, and $w \in \mathbb{R}^w$ is finite-energy (L_2) disturbance

A Luenberger-type nth-order observer is used to calculate a state estimate, $\hat{x}_k$.

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k - Du_k) \tag{3.66}$$

The feedback control, which is based on the state estimate, is

$$u_k = K\hat{x}_k \tag{3.67}$$

Theorem 3.8: Given the model of a discrete-time linear system described in (3.64) and (3.65) and the dynamic feedback control law given by (3.67), the closed loop system satisfies the GPC if the matrix inequality

$$\begin{bmatrix} P_c - A_c^T P_c A_c - \delta C_{z1}^T C_{z1} & A_c^T P_c BK - \delta C_{z1}{}^T C_{z2} & -A_c^T P_c F - \delta C_{z1}^T D_z - \frac{\gamma}{2} C_{z1}^T \\ K^T B^T P_c A_c - \delta C_{z2}{}^T C_{z1} & P_o - A_o^T P_o A_o - K^T B^T P_c BK - \delta C_{z2}^T C_{z2} & K^T B^T P_c F - A_o^T P_o F + A_o^T Y_o G - \delta C_{z2}^T D_z - \frac{\gamma}{2} C^T{}_{z2} \\ -F^T P_c A_c - \delta D_z^T C_{z1} - \frac{\gamma}{2} C_{z1} & F^T P_c BK - F^T P_o A_o + G^T Y_o^T A_o - \delta D_z^T C_{z2} - \frac{\gamma}{2} C_{z2} & \Xi \end{bmatrix} > 0$$

$$\tag{3.68}$$

where

$$\Xi = -F^T P_c F - F^T P_o LG + G^T L^T P_o F - G^T L^T P_o LG - \varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z)$$

$$\tag{3.69}$$

is feasible.

Proof

The state estimate error defined is the same as (2.74) and therefore has a corresponding state estimate error update equation that matches (2.75). Therefore, the augmented system can be represented as,

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} A_c & -BK \\ 0 & A_o \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix} + \begin{bmatrix} F \\ F - LG \end{bmatrix} w_k \tag{3.70}$$

Equation (3.70) can be compactly expressed as,

$$\mathrm{X}_{k+1} = \Lambda \mathrm{X}_k + Hw_k \tag{3.71}$$

where Λ and H are as defined in Section 3.1 and

$$\mathrm{X}_k = \begin{bmatrix} x_k \\ e_k \end{bmatrix} \tag{3.72}$$

The performance output is defined as

$$z_k = C_z X_k + D_z w_k \tag{3.73}$$

where C_z is the same as what was defined for continuous-time systems. Like in continuous time, the GPC is revised by using the following Lyapunov energy function

$$V_k = \mathrm{X}_k^T P \mathrm{X}_k ; \tag{3.74}$$

where P is defined in (3.13). The GPC in discrete-time is

$$V_k - V_{k+1} - \delta z_k^T z_k - \varepsilon w_k{}^T w_k + \gamma z_k^T w_k > 0 \tag{3.75}$$

Given the system described in (3.71), it follows that

$$\begin{aligned} -(\Lambda \mathrm{X}_k + Hw_k)^T P(\Lambda \mathrm{X}_k + Hw_k) + \mathrm{X}_k{}^T P \mathrm{X}_k - \delta(C_z X_k + D_z w_k)^T (C_z X_k + D_z w_k) \\ -\varepsilon w_k{}^T w_k + \gamma (C_z X_k + D_z w_k)^T w_k > 0 \end{aligned} \tag{3.76}$$

This can be expressed in the matrix inequality

$$\begin{bmatrix} X_k{}^T & w_k{}^T \end{bmatrix} \begin{bmatrix} P - \Lambda^T P \Lambda - \delta C_z^T C_z & -\Lambda^T PH - \delta C_z^T D_z - \frac{\gamma}{2} C_z{}^T \\ -H^T P \Lambda - \delta D_z^T C_z - \frac{\gamma}{2} C_z & -H^T PH - \delta D_z^T D_z - \varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} \begin{bmatrix} X_k \\ w_k \end{bmatrix} > 0 \tag{3.77}$$

Expanding (3.78) yields the matrix inequality (3.68)

This inequality forms the basis of the general performance criteria design procedure for linear dynamic feedback controllers for discrete-time systems. In a future chapter, various methods of solving inequality (3.68) for design purposes will be examined.

3.2.2 Discrete-time Nonlinear GPC Controller Design with REA Constraints

In Chapter 2, the work of Siljak and Stipanovic allowed for certain types of nonlinearities to be accommodated through use of LMI techniques. In Section 3.1.2 of this chapter, the work was expanded for continuous-time systems. In this section, the same expansion will be done in discrete-time for systems with non-zero l2 noise.

Theorem 3.9: Given the model of a discrete-time nonlinear system described in (2.123) and (2.124), the performance output as defined in (2.49) and the input is defined in (2.125), the closed loop system satisfies the GPC for the designed controller if the matrix inequality

$$\begin{bmatrix} P^{-1} & 0 & \frac{\gamma}{2}P^{-1}C_z^T & P^{-1}A^T + Y_c^T B^T & \sqrt{\delta}P^{-1}C_z^T & P^{-1} \\ 0 & \tau I & 0 & I & 0 & 0 \\ \frac{\gamma}{2}C_z P^{-1} & 0 & -\varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) & F^T & \sqrt{\delta}D_z^T & 0 \\ AP^{-1} + BY_c & I & F & P^{-1} & 0 & 0 \\ \sqrt{\delta}C_z P^{-1} & 0 & \sqrt{\delta}D_z & 0 & I & 0 \\ P^{-1} & 0 & 0 & 0 & 0 & \alpha_\tau^{-1} \end{bmatrix} > 0 \qquad (3.78)$$

where

$$\alpha_\tau = \tau\alpha \tag{3.79}$$

is feasible.

Proof

To design the controller for the discrete-time nonlinear system described in (2.123) and (2.124) with a feedback control law defined in (2.125), the GPC is expanded, resulting in the matrix inequality

$$\begin{aligned} &-((A+BK)x_k + f(x_k)+Fw_k)^T P((A+BK)x_k + f(x_k)+Fw_k)+x_k^T P x_k \\ &-\delta(C_z x_k + D_z w_k)^T (C_z x_k + D_z w_k) - \varepsilon w_k{}^T w_k + \gamma(C_z x_k + D_z w_k)^T w_k > 0 \end{aligned} \tag{3.80}$$

this can be expanded into vector matrix form

$$\begin{bmatrix} x_k \\ f(x_k) \\ w_k \end{bmatrix}^T \begin{bmatrix} P-(A+BK)^T P(A+BK)-\delta C_z{}^T C_z & -(A+BK)^T P & -(A+BK)^T PF - \delta C_z^T D_z + \frac{\gamma}{2} C_z^T \\ -P(A+BK) & -P & 0 \\ -F^T P(A+BK) - \delta D_z^T C_z + \frac{\gamma}{2} C_z & 0 & -\varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} \begin{bmatrix} x_k \\ f(x_k) \\ w_k \end{bmatrix} > 0$$

(3.81)

The inequality is not linear. The first step in putting this into a linear form is to make use of the bounding conditions on the nonlinearity

$$\| f(x_k)\|^2 \le \alpha \|x_k\|^2 \tag{3.82}$$

Or

$$\begin{bmatrix} x_k{}^T & \mathrm{T}_k{}^T & w_k{}^T \end{bmatrix} \begin{bmatrix} \alpha I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \mathrm{T}_k \\ w_k \end{bmatrix} > 0 \tag{3.83}$$

Using the S-procedure yields the matrix inequality

$$\begin{bmatrix} P-(A+BK)^T P(A+BK)-\delta C_z^T C_z-\alpha_\tau I & -(A+BK)^T P & -(A+BK)^T PF-\delta C_z^T D_z+\frac{\gamma}{2}C_z^T \\ -P(A+BK) & \tau I-P & 0 \\ -F^T P(A+BK)-\delta D_z^T C_z+\frac{\gamma}{2}C_z & 0 & -\varepsilon I-\delta D_z^T D_z+\frac{\gamma}{2}(D_z^T+D_z) \end{bmatrix}>0$$

(3.84)

To put this inequality into linear form, the Schur Complement is used. The resulting matrix inequality is

$$\begin{bmatrix} P-\delta C_z^T C_z-\alpha_\tau & 0 & -\delta C_z^T D_z-\frac{\gamma}{2}C_z^T & (A+BK)^T P \\ 0 & \tau I & 0 & P \\ -\delta D_z^T C_z-\frac{\gamma}{2}C_z & 0 & \varepsilon I-\delta D_z^T D_z+\frac{\gamma}{2}(D_z^T+D_z) & F^T P \\ P(A+BK) & P & PF & P \end{bmatrix}>0 \tag{3.85}$$

Like in the previous chapter, the matrix inequality needs to be pre-and post-multiplied by

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P^{-1} \end{bmatrix} \tag{3.86}$$

To get

$$\begin{bmatrix} P^{-1}-\delta P^{-1}C_z^T C_z P^{-1}-\alpha_\tau P^{-2} & 0 & -\delta P^{-1}C_z^T D_z+\frac{\gamma}{2}P^{-1}C_z^T & P^{-1}A^T+P^{-1}K^T B^T \\ 0 & \tau I & 0 & I \\ -\delta D_z^T C_z P^{-1}+\frac{\gamma}{2}C_z P^{-1} & 0 & -\delta D_z^T D_z-\varepsilon I+\frac{\gamma}{2}(D_z^T+D_z) & F^T \\ AP^{-1}+BKP^{-1} & I & F & P^{-1} \end{bmatrix}>0$$

(3.87)

This results in two quadratic terms so the Schur complement is used twice. The resulting LMI is (3.78).

If a specific circular region is the desired location of the controller eigenvalues, the regional eigenvalue assignment LMI (3.26) can be solve simultaneously with (3.78). The solution to both LMIs will not only give the closed loop system performance, the eigenvalues will be with the region specified.

Theorem 3.10: Given the model of a discrete-time nonlinear system described in (2.123) and (2.124), the performance output as defined in (2.49) and the input is defined in (2.125), the closed loop system satisfies the GPC and place the eigenvalues of the linear component of the system within a circular region of radius, r_c, and centered at a_c if the matrix inequalities (3.26) and (3.78) are feasible.

3.2.3 Discrete-time Nonlinear GPC Observer Design with REA Constraints

Theorem 3.11: Given the model of a discrete-time nonlinear system described in (2.123) and (2.124) and the performance output as defined in (2.49), the closed loop system satisfies the GPC for the observer design if the matrix inequality

$$\begin{bmatrix} P-\delta C_z^T C-\tau(\alpha+\beta)I & P\mathrm{N} & -\delta C_z^T D_z+\frac{\gamma}{2}C_z^T & A^T P-C^T Y_{oT} \\ \mathrm{N}^T P & \tau I & 0 & 0 \\ -\delta D_z^T C_z+\frac{\gamma}{2}C_z & 0 & -\delta D_z^T D_z-\varepsilon I+\frac{\gamma}{2}(D_z^T+D_z) & F^T P-G^T Y_o^T \\ PA-Y_o C & 0 & PF-Y_o G & P \end{bmatrix} \tag{3.88}$$

is feasible.

Proof

In order to design the observer to have any of the general performance criteria, it is assumed that the A matrix represents the closed loop system matrix. A Luenberger-type observer is designed. The state estimate dynamics is

$$\hat{x}_{k+1} = Ax_k + Bu_k + f(\hat{x}_k) + L(y_k - C\hat{x}_k - Du_k - g(\hat{x}_k)) \tag{3.89}$$

where $\hat{x}$ represents the estimate of the state. The state estimation error dynamics are

$$e_{k+1} = A_o e_k + \Delta f_k - L\Delta g_k + (F - LG)w_k \tag{3.90}$$

where

$$\Delta f_k = f(x_k) - f(\hat{x}_k) \tag{3.91}$$

$$\Delta g_k = g(x_k) - g(\hat{x}_k) \tag{3.92}$$

The error dynamics can be expressed in a more compact form

$$e_{k+1} = A_o e_k + \mathrm{N}\mathrm{T}_k + (F - LG)w_k \tag{3.93}$$

where

$$\mathrm{N} = \begin{bmatrix} I & -L \end{bmatrix} \tag{3.94}$$

$$\mathrm{T}_k = \begin{bmatrix} \Delta f_k \\ \Delta g_k \end{bmatrix} \tag{3.95}$$

By applying the general performance criteria inequality to the error dynamics, (3.93) becomes

$$
\begin{aligned}
-((A-LC)e_k + \mathrm{NT}_k + (F-LG)w_k)^T P((A-LC)e_k + \mathrm{NT}_k + (F-LG)w_k) + e_k{}^T P e_k \\
-\delta(C_z e_k + D_z w_k)^T (C_z e_k + D_z w_k) - \varepsilon w_k{}^T w_k + \gamma(C_z e_k + D_z w_k)^T w_k > 0
\end{aligned}
\quad (3.96)
$$

which can be further expanded and vector matrix form,

$$
\begin{bmatrix} e_k \\ \mathrm{T}_k \\ w_k \end{bmatrix}^T
\begin{bmatrix}
\boxed{\begin{array}{r} P-(A-LC)^T P(A-LC) \\ -\delta C_z^T C \end{array}} & P\mathrm{N} & \boxed{\begin{array}{r} -(A-LC)^T P(F-LG) \\ -\delta C_z^T D_z + \frac{\gamma}{2} C_z^T \end{array}} \\
\mathrm{N}^T P & 0 & 0 \\
\boxed{\begin{array}{r} -(F-LG)^T P(A-LC) \\ -\delta D_z^T C_z + \frac{\gamma}{2} C_z \end{array}} & 0 & \boxed{\begin{array}{l} -(F-LG)^T P(F-LG) \\ -\delta D_z^T D_z - \varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) \end{array}}
\end{bmatrix}
\begin{bmatrix} e_k \\ \mathrm{T}_k \\ w_k \end{bmatrix} > 0 \quad (3.97)
$$

The resulting matrix inequality is not a valid LMI since there is a zero on the diagonal. This is due to incomplete information on the nature of the nonlinearity. This information is incorporated into the design by putting the bounds into vector matrix form. Due to the Lipschitz condition on the nonlinearities, the bound is

$$
\|\mathrm{T}_k\|^2 = \|\Delta f_k\|^2 + \|\Delta g_k\|^2 \le \alpha \|e\|_k^2 + \beta \|e_k\|^2 \quad (3.98)
$$

or

$$
\begin{bmatrix} e_k{}^T & \mathrm{T}_k{}^T & w_k{}^T \end{bmatrix}
\begin{bmatrix} (\alpha+\beta)I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} e_k \\ \mathrm{T}_k \\ w_k \end{bmatrix} > 0 \quad (3.99)
$$

Applying Lemma 2 to (3.97) and (3.99) yields the LMI

$$\begin{bmatrix} P-(A-LC)^T P(A-LC)-\delta C_z^T C-\tau(\alpha+\beta)I & P\mathrm{N} & -(A-LC)^T P(F-LG)-\delta C_z^T D_z+\frac{\gamma}{2}C_z^T \\ \mathrm{N}^T P & \tau I & 0 \\ -(F-LG)^T P(A-LC)-\delta D_z^T C_z+\frac{\gamma}{2}C_z & 0 & \boxed{\begin{matrix} -(F-LG)^T P(F-LG)-\delta D_z^T D_z \\ -\varepsilon I+\frac{\gamma}{2}(D_z^T+D_z) \end{matrix}} \end{bmatrix} > 0$$

(3.100)

The inequality is not linear. The first step in putting this into a linear form is to make use of Lemma 1 to obtain (3.88). The observer gain, L, can be calculated as

$$L = P^{-1}Y_o \tag{3.101}$$

The observer design will guarantee that the state estimate error achieves the desired performance criteria. If a specific circular region is the desired location of the observer eigenvalues, the regional eigenvalue assignment LMI (3.44) can be solve simultaneously with (3.89). The solution to both LMIs will not only give the closed loop system performance, the eigenvalues will be with the region specified.

Theorem 3.12: Given the model of a continuous-time nonlinear system described in (2.123) and (2.124) and the performance output as defined in (2.49), the closed loop system satisfies the GPC and place the eigenvalues of the linear component of the observer's eigenvalues within a circular region of radius, r_o, and centered at a_o if the matrix inequalities (3.89) and (3.1024) are feasible.

3.2.3.1 Dynamic Feedback Design

To control a discrete-time nonlinear system with states that are not known, a Luenberger-type nth-order nonlinear observer is designed.

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + f(\hat{x}_k) + L(y_k - C\hat{x}_k - Du_k - g(\hat{x}_k)) \tag{3.103}$$

The feedback control law is

$$u_k = K\hat{x}_k \tag{3.104}$$

The state estimate error update equation is

$$e_{k+1} = (A - LC)e_k + f(x_k) - f(\hat{x}_k) + Fw_k - L(g(x_k) - g(\hat{x}_k) - Gw_k) \tag{3.105}$$

The closed loop system is thus represented as,

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} A_c & -BK \\ 0 & A_o \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix} + \begin{bmatrix} f(x_k) \\ \Delta f_k - L\Delta g_k \end{bmatrix} + \begin{bmatrix} F \\ F - LG \end{bmatrix} w_k \tag{3.106}$$

where the differences in nonlinearities and their estimate are defined as,

$$\Delta f_k = f(x_k) - f(\hat{x}_k) \tag{3.107}$$

$$\Delta g_k = g(x_k) - g(\hat{x}_k) \tag{3.108}$$

Equation (3.106) can be expressed as,

$$\mathrm{X}_{k+1} = \Lambda \mathrm{X}_k + J\Gamma_k + Hw_k \tag{3.109}$$

where Λ, H, and J are as previously defined in the continuous-time equations (3.8), (3.9), and (3.51) and X_k is defined in (3.72) and

$$\Gamma_k = \begin{bmatrix} f(x_k) \\ \Delta f_k \\ \Delta g_k \end{bmatrix} \tag{3.110}$$

The performance output is defined in (3.73) and (3.11). The bound on Γ_k is,

$$\|\Gamma_k\|^2 = \|f(x_k)\|^2 + \|\Delta f\|^2 + \|\Delta g\|^2 \le \alpha \|x_k\|^2 + (\alpha+\beta)\|e_k\|^2 \tag{3.111}$$

this can be represented in block matrix form as

$$\begin{bmatrix} \mathrm{X}_k^T & \Gamma_k^T & w_k^T \end{bmatrix} \begin{bmatrix} M & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{X}_k \\ \Gamma_k \\ w_k \end{bmatrix} > 0 \tag{3.112}$$

Theorem 3.13: Given the model of a discrete-time nonlinear system described in (2.123) and (2.124), the performance output as defined in (2.49) and the input is defined in (2.125), the closed loop system satisfies the GPC if the matrix inequality

$$\begin{bmatrix}
P_c - A_c^T P_c A_c - \delta C_{z1}^T C_{z1} - \tau\alpha I & A_c^T P_c BK - \delta C_{z1}^T C_{z2} & -A_c^T P_c & 0 & 0 & -A_c^T P_c F - \delta C_{z1}^T D_z - \frac{\gamma}{2} C_{z1}^T \\
K^T B^T P_c A_c - \delta C_{z2}^T C_{z1} & P_o - A_o^T P_o A_o - K^T B^T P_c BK - \delta C_{z2}^T C_{z2} - \tau(\alpha+\beta) I & K^T B^T P_c & -A_o^T P_o & A_o^T Y_o & K^T B^T P_c F - A_o^T P_o F + A_o^T Y_o G - \delta C_{z2}^T D_z - \frac{\gamma}{2} C^T{}_{z2} \\
-P_c A_c & P_c BK & \tau I - P_c & 0 & 0 & -P_c F \\
0 & -P_o A_o & 0 & \tau I - P_o & P_o L & -P_o F + P_o LG \\
0 & Y_o^T A_o & 0 & L^T P_o & \tau I - L^T P_o L & -L^T P_o F + L^T P_o LG \\
-F^T P_c A_c - \delta D_z^T C_{z1} - \frac{\gamma}{2} C_{z1} & F^T P_c BK - F^T P_o A_o + G^T Y_o^T A_o - \delta D_z^T C_{z2} - \frac{\gamma}{2} C_{z2} & -F^T P_c & -F^T P_o + G^T L^T P_o & F^T P_o L - G^T L^T P_o L & \boxed{\begin{matrix} -F^T P_c F - F^T P_o LG + G^T L^T P_o F \\ -G^T L^T P_o LG - \varepsilon I - \delta D_z^T D_z + \frac{\gamma}{2}(D_z^T + D_z) \end{matrix}}
\end{bmatrix} > 0$$

(3.113)

is feasible.

Proof

Using the augmented system and performance output, the GPC is expanded into the form

$$-(\Lambda \mathrm{X}_k + J\Gamma_k + Hw_k)^T P(\Lambda \mathrm{X}_k + J\Gamma_k + Hw_k) + \mathrm{X}_k{}^T P \mathrm{X}_k - \delta(C_z X_k + D_z w_k)^T (C_z X_k + D_z w_k)$$
$$-\varepsilon w_k{}^T w_k + \gamma(C_z X + D_z w_k)^T w_k > 0$$

(3.114)

where P is as defined in (3.13). This can also be expressed in the matrix inequality

$$\begin{bmatrix} \mathrm{X}_k \\ \Gamma_k \\ w_k \end{bmatrix}^T \begin{bmatrix} P - \Lambda^T P \Lambda - \delta C_z^T C_z & -\Lambda^T PJ & -\Lambda^T PH - \delta C_z^T D_z - \frac{\gamma}{2} C_z{}^T \\ -J^T P^T \Lambda & -J^T PJ & -J^T PH \\ -H^T P \Lambda - \delta D_z^T C_z - \frac{\gamma}{2} C_z & -H^T PJ & -H^T PH - \delta D_z^T D_z - \varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} \begin{bmatrix} \mathrm{X}_k \\ \Gamma_k \\ w_k \end{bmatrix} > 0 \quad (3.115)$$

Lemma 2 is used to combine the system dynamics with the bounds on the nonlinearities.

$$\begin{bmatrix} P - \Lambda^T P \Lambda - \delta C_z^T C_z - \tau M & -\Lambda^T PJ & -\Lambda^T PH - \delta C_z^T D_z - \frac{\gamma}{2} C_z{}^T \\ -J^T P^T \Lambda & \tau I - J^T PJ & -J^T PH \\ -H^T P \Lambda - \delta D_z^T C_z - \frac{\gamma}{2} C_z & -H^T PJ & -H^T PH - \delta D_z^T D_z - \varepsilon I + \frac{\gamma}{2}(D_z^T + D_z) \end{bmatrix} > 0 \quad (3.116)$$

When (3.116) is expanded, the resulting matrix inequality is (3.113).

In dynamic control design, it is important that the error in the estimate of the state go to zero much faster than the state. In discrete-time, the observer eigenvalues are placed closer to the origin. To address this weakness, REA is used. Since the REA formulation is the same in discrete-time as continuous-time, the LMIs for REA, (3.44) and (3.26) are used.

Theorem 3.14: Given the model of a discrete-time nonlinear system described in (2.123) and (2.124), the performance output as defined in (2.49) and the input is defined in (2.125), the closed loop system satisfies the GPC and place the eigenvalues of the linear component of the controller's eigenvalues within a circular region of radius, r_c, and centered at a_c and the observer's eigenvalues within a circular region of radius, r_o, and centered at a_o if the matrix inequalities (3.113), (3.26), and (3.44) are feasible.

The H_∞ controller was designed using this theorem in [32].

3.3 Summary

In this chapter, the dynamic state-feedback controller design was derived in such a way that the system will satisfy a general performance criterion for both linear and nonlinear systems. The nonlinearity was assumed to be analytic and therefore a linear part could be extracted for use in eigenvalue assignment. The remaining nonlinearities were assumed to satisfy a linear growth bound. Since this technique alone could not guarantee separation between the controller eigenvalues and observer eigenvalues, Regional Eigenvalue assignment was used for dynamic state-feedback controllers to separate those eigenvalues.

In the next chapter, the results from this chapter will be applied using various methods of designing the controller. Simulation results will be analyzed and compared to one another. Simulations will also compare the linear and nonlinear results of each method.

Chapter 4

Methods of Implementation and Simulation Results

In Chapter 3, the matrix inequalities were derived for GPC-based controller, observer, and dynamic feedback controller designs in continuous-time and discrete time with REA constraints. By developing the theory for linear systems first, the procedure used in the derivation of the matrix inequality was shown. When including the nonlinearity, use of the S-procedure allowed Lipschitz nonlinearities to be integrated into the matrix inequality framework in such a way that LMI techniques may be used. The resulting matrix inequalities can be used to find gains that satisfy a specified performance criterion while accommodating the nonlinear component of the system and satisfying a given REA constraint.

The design technique is applied in both continuous-time and discrete-time in this chapter for the design of dynamic state-feedback controllers. The design procedure can utilize at least four distinct methods of using the main results from Chapter 3 to obtain the controller and observer gains that satisfy both the performance objective and the regional eigenvalue constraints. The first method utilizes the necessary condition of the (1, 1) block of main result matrix inequality simultaneously with the controller REA to solve for the positive definite matrix, P_c, and the controller gain, K. The second method solves the first 2n x 2n block simultaneously with the controller REA for P_c and K. The third method uses the controller regional eigenvalue assignment LMI to calculate the controller gain K to turn the main result matrix inequalities into an LMIs. These three methods use the P_c and K for the controller in order to then determine the observer gain, L. The fourth method designs the controller based on the necessary condition of the (1,1) block and the controller REA and designs the observer based on the necessary condition of the (2,2) block and the observer REA. In all four cases, the main result matrix inequality becomes solvable using LMI techniques once certain unknowns are calculated, allowing the full LMI to be solved. By solving the LMI, stability, performance, nonlinear accommodation, and REA constraints are satisfied.

The methods are compared in the final section. The comparison will measure which methods work best in terms of the ease of getting a feasible answer and the maximum feasible Lipschitz constant. For each time domain, the control design procedure will be used on the same system, allowing for a fair comparison of the effectiveness of each method.

4.1 Dynamic feedback controller design for nonlinear continuous-time systems

All methods proposed in continuous-time will use the following system model. Consider a simple inverted harmonic system with friction, shown in Figure 4.1.

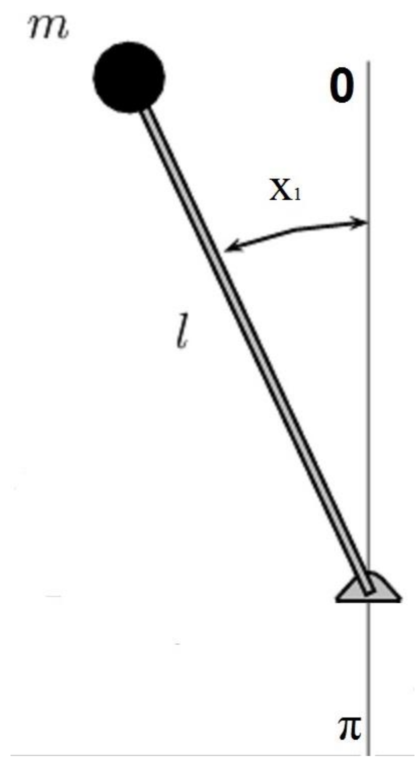

Figure 4. 1: Simple inverted pendulum

This system is expressed in state-space format as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ \alpha \sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \tag{4.1}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + 0.1w \tag{4.2}$$

where $\alpha = 10\times10^{-6}$ and $b = 5\times10^{-3}$. The initial conditions are chosen as

$$x_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \hat{x}_o = \begin{bmatrix} -10\times10^{-6} \\ 0 \end{bmatrix}$$

The initial conditions place the state at the origin, which is an unstable equilibrium point for the system. The finite energy noise generally is unknown; but for the purposes of simulation, it is defined as

$$w_k = \Psi e^{-\chi t}$$

where Ψ=0.95 and χ =5.13. The finite energy noise causes the system to move from its unstable equilibrium point.

A time response plot of the open loop system is shown in Figure 4.2. The plot shows the system settles at x1=π radians and x2=0 rad/sec.

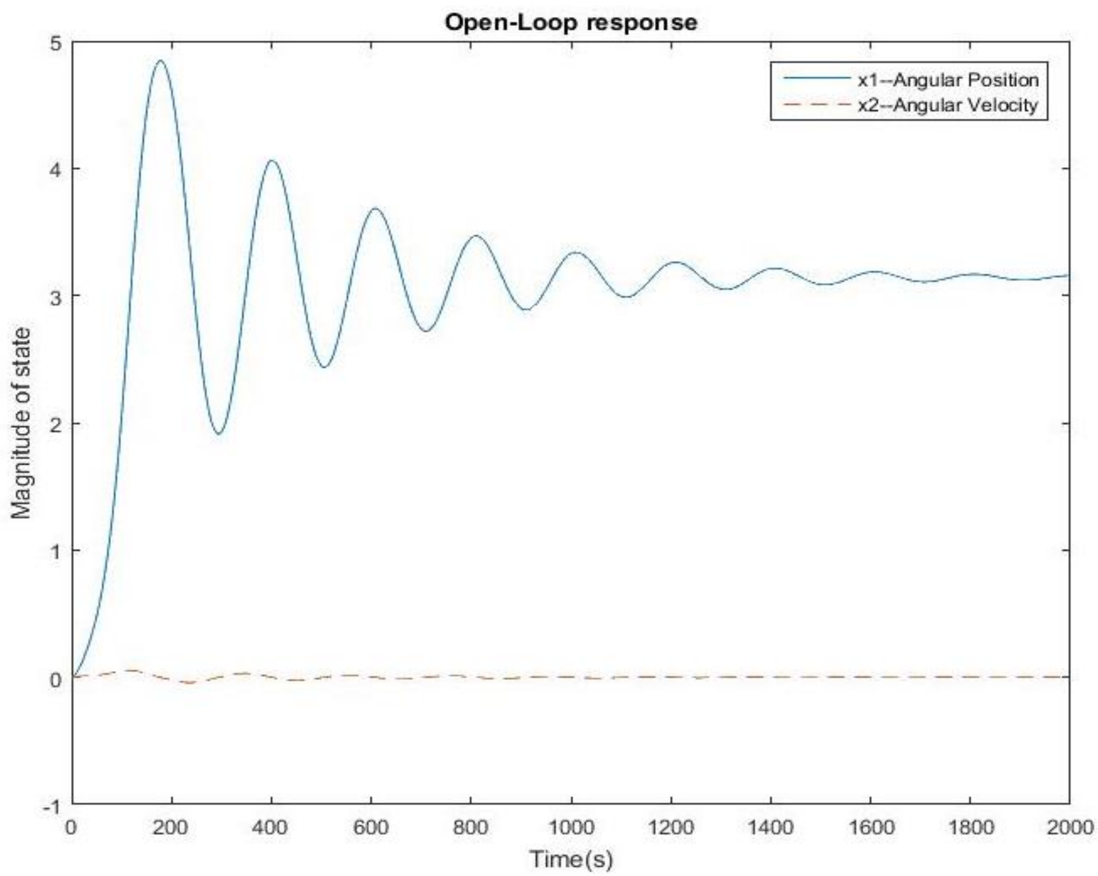

Figure 4.2: Simple inverted pendulum open-loop response

Figure 4.2 shows the pendulum starting at a position pointing upwards. Due to it being slightly off the unstable equilibrium point, it proceeds to fall toward a position π radians, or 180 degrees from its initial position. Over the course of falling, the pendulum swings back and forth across the new stable equilibrium point and as the energy decays, the pendulum settles in a position pointing downward. The goal of the control problem will be to return the pendulum to the unstable equilibrium point despite the imprecise knowledge of the state and despite the noise in the system. Furthermore, a specific performance criteria, either H_∞ or Very Strict Passivity will be achieved in the process. To that end, the performance output is defined as

$$z = [1 \quad 0 \quad 0.1 \quad 0] \begin{bmatrix} x \\ e \end{bmatrix} + 0.1w$$

To satisfy the GPC parameters outlined in Chapter 2, the GPC design parameters for H_∞ and Very Strict Passivity will be set to the values in Table 4.1

Table 4.1: Performance Criteria design parameters

	δ	ε	γ
H_∞	1	-100	0
Very Strict Passivity	0.001	0.001	1

It is assumed that the state cannot be accurately measured. Therefore, a dynamic state feedback controller will be designed. As mentioned in Chapter 3, REA is used to separate the controller eigenvalues from the observer eigenvalues. The design parameters for the desired regions for the controller eigenvalues, Dc (r_c, a_c), and the observer eigenvalues, Do (r_o, a_o), are

$$r_c = 1 \qquad a_c = -1.5$$

$$r_o = 4 \qquad a_o = -8.5$$

For all simulations, Matlab will be used. The LMI solver will be used to calculate the feasible gains for the controller and the observer using one of the four methods discussed in this dissertation. Once a feasible set of gains have been found, Matlab will be used to run simulations based on the closed loop state-space model. Both a pole-zero map and a time response plot will be displayed. When the design is completed using either of the four methods, the dynamic state-feedback controller will have H_∞ or very strict passivity performance, have REA constraints that force the state-estimate error to go to zero much faster than the state, and will accommodate the nonlinearity in the system model.

4.1.1 Method 1: First dimension necessary condition method

4.1.1.1 Design Procedure

For the matrix inequality described in (3.53) to be satisfied, it is necessary that the blocks along the diagonal be positive definite. Using this necessary condition, the (1,1) block of (3.53) is used to derive an LMI system that can find a feasible solution for P_c and K. The matrix inequality system is

$$-(A+BK)^T P_c - P_c(A+BK) - \tfrac{\delta}{\tau} C_{z1}^T C_{z1} - \alpha I > 0 \tag{4.3}$$

In its current form, (4.3) is not linear due to the way the unknown matrices K and P_c multiply with B. This is remedied by pre- and post-multiplying each element of (4.3) by P_c^{-1} to group the unknowns together and then grouping the unknowns together by defining $Y_c = KP_c^{-1}$

$$-P_c^{-1}A^T - Y_c^T B^T - AP_c^{-1} - BY_c - \alpha P_c^{-2} - \tfrac{\delta}{\tau} P_c^{-1} C_{z1}^T C_{z1} P_c^{-1} > 0 \tag{4.4}$$

Matrix inequality (4.4), because of the quadratic term, is not linear. This is addressed by using Lemma 1a twice, resulting in the LMI

$$\begin{bmatrix} -P_c^{-1}A^T - Y_c^T B^T - AP_c^{-1} - BY_c & P_c^{-1} & \sqrt{\delta} P_c^{-1} C_{z1}^T \\ P_c^{-1} & \alpha^{-1} I & 0 \\ \sqrt{\delta} C_{z1} P_c^{-1} & 0 & \tau I \end{bmatrix} > 0 \tag{4.5}$$

The quantities A, B, δ, and C_{z1} are known. If LMI (4.5) is feasible, outputs P_c^{-1} and Y_c are obtained. These results allow the value for the control gain, K, to be calculated. The value of α can also be an input if the Lipschitz bound is known. In that case, a value for τ is found and used in the observer design matrix inequality.

Once the controller has been designed based on the necessary condition used above, Y_c and P_c, are known, allowing for K and A_c=A+BK to be calculated. These values can then be used in the full matrix inequality (3.53). As was done in previous chapters, Y_o=P_oL. This yields the LMI,

$$\begin{bmatrix} \boxed{\begin{matrix} -A_c^T P_c - P_c A_c \\ -\frac{\delta}{\tau} C_{z1}^T C_{z1} - \alpha I \end{matrix}} & P_c BK - \frac{\delta}{\tau} C_{z1}{}^T C_{z2} & -P_c & 0 & 0 & -P_c F - \frac{\delta}{\tau} C_{z1}^T D_z + \frac{\gamma}{2\tau} C_{z1}^T \\ * & \boxed{\begin{matrix} -A^T P_o + C^T Y_o^T - P_o A_o + Y_o C \\ -\frac{\delta}{\tau} C_{z2}^T C_{z2} - \alpha I - \beta I \end{matrix}} & 0 & -P_o & Y_o & -P_o F + Y_o G - \frac{\delta}{\tau} C_{z2}^T D_z + \frac{\gamma}{2\tau} C^T{}_{z2} \\ * & * & I & 0 & 0 & 0 \\ * & * & * & I & 0 & 0 \\ * & * & * & * & I & 0 \\ * & * & * & * & * & -\frac{\varepsilon}{\tau} I - \frac{\delta}{\tau} D_z^T D_z + \frac{\gamma}{2\tau}(D_z^T + D_z) \end{bmatrix} > 0$$

(4.6)

If LMI (4.6) is feasible, the outputs P_o and Y_o are obtained, allowing the observer gain, L, to be calculated. The variable, β, can be a known input quantity or calculated as an output value with this LMI technique. The controller gains and observer gains calculated using this design procedure place the eigenvalues of the linear component of the controller and observer within left half plane and can accommodate nonlinearity in the measurement and state.

However, our goal is to be able to explicitly locate the eigenvalues of the linear component of the controller and observer systems in specific regions of the unit circle while satisfying the performance criteria and accommodating the nonlinearity. To that end, a second set of LMIs is needed. To place the eigenvalues of the linear components in the desired locations, the REA LMI (3.26) is used to place the eigenvalues of the controller within one circular region and the REA LMI (3.44) is used to put the eigenvalues of the observer in a separate circular region in the complex plane.

The advantage to using regional eigenvalue assignment is the LMI region is in D-space, meaning it is not tied directly to continuous-time or discrete-time systems. This means that the LMIs for circular regional eigenvalue assignment can be used in either continuous-time or discrete-time. The disadvantage is that to guarantee stability, a second LMI with the stability constraints in the appropriate time domain is needed. The LMI system consisting of (4.5) and (3.26) places the eigenvalues of (A+BK) within the circular region for the controller. Then the LMI system consisting of (4.6) and (3.44) places the eigenvalues of (A-LC) within the circular region for the observer. The closed loop system is robust with respect to the nonlinear deviation within the system and is asymptotically stable.

4.1.1.2 Method 1 application to the continuous-time system model

Using the system model defined in (4.1) and (4.2), Method 1 is applied. Solving the combination of the (1,1) block of the main result and the controller Regional Eigenvalue Assignment LMI (3.26) yields a feasible value for P_c^{-1} and Y_c, which then allows for the calculation of the controller gains, K.

Table 4.2: Controller gains using Method 1

	H_∞	Very Strict Passivity
Controller Gains, K	[-1.8115 -2.7124]	[-1.8115 -2.7124]

Using the values for P_c^{-1} and K, the feasible observer gains, L are calculated by solving the main result LMI (4.6) for P_o and Y_o.

Table 4.3: Observer gains using Method 1

	H_∞	Very Strict Passivity
Observer Gains, L	$[13.2686 \;\; 43.1780]^T$	$[13.7334 \;\; 46.7140]^T$

When the design parameters for H_∞ control are used in 4.5, the eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.3.

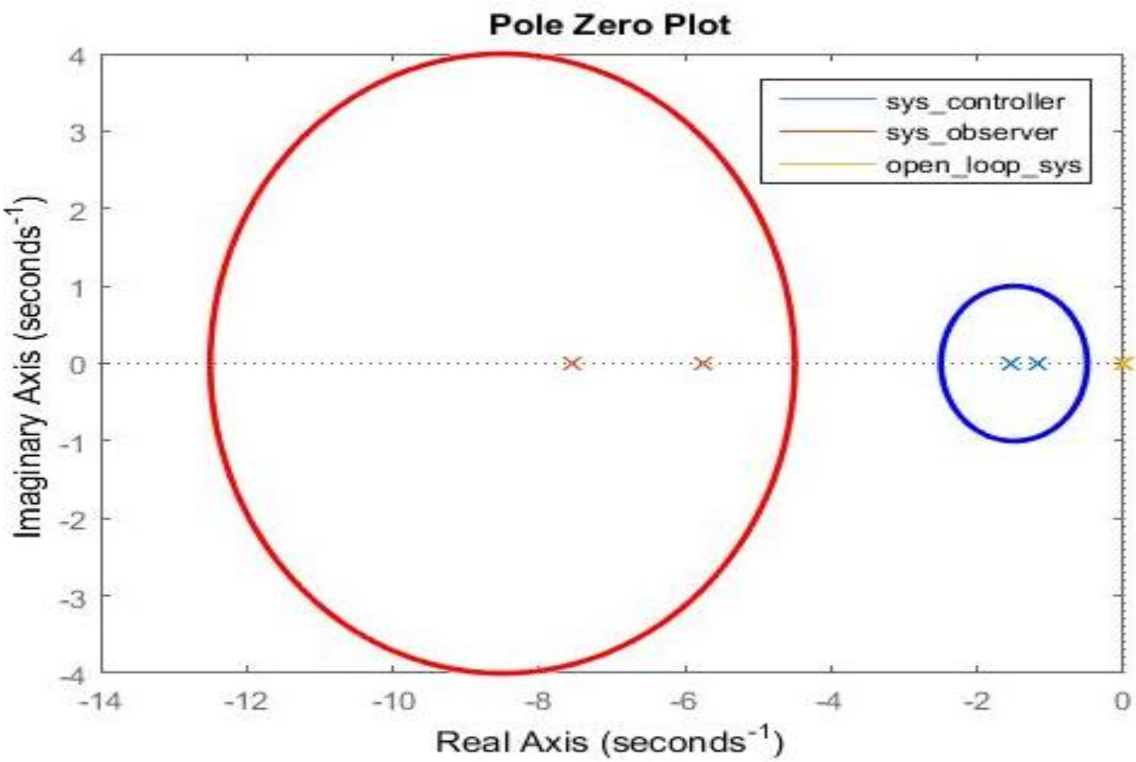

Figure 4.3: Pole-Zero map for H_∞ control using Method 1

When the design parameters for very strict passivity control are used, the eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.4.

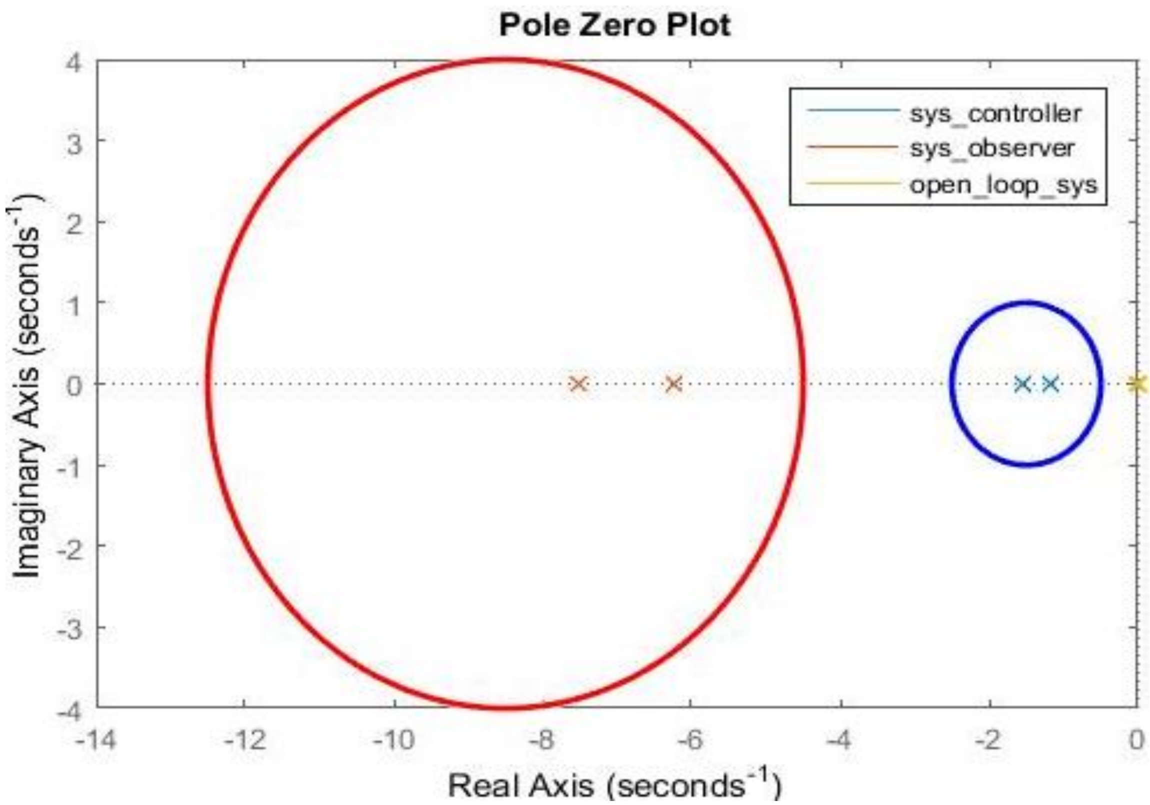

Figure 4.4: Pole-Zero map for very strict passivity control using Method 1

The pole-zero maps in Figure 4.3 and 4.4 show that the eigenvalues are being placed within the regions specified. Table 4.3 also shows that the gains are the same between the 2 performance criteria for the controller design phase, but the observer gains have a small difference. This means that specifying the performance objective does change the eigenvalue location within the region for this method, demonstrating that this design method does indeed distinguish between different performance objectives.

The gains, which have been found though this design procedure, are applied to the augmented system. The closed loop time response is shown in Figure 4.5.

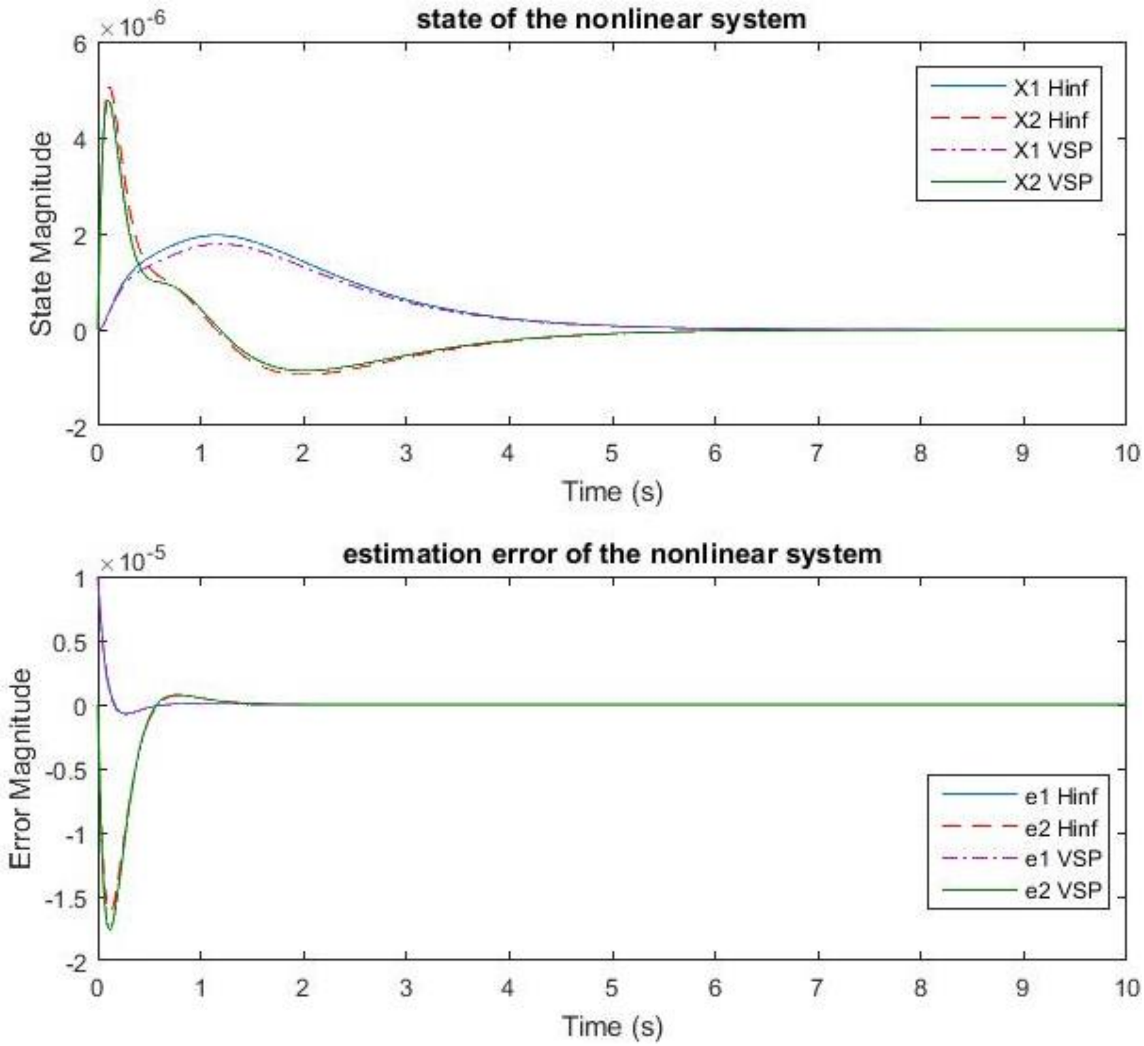

Figure 4.5: Method 1 Time response plot

Figure 4.5 shows that the system response was driven to zero. In physical terms, this mean the inverted pendulum is stabilized about its unstable equilibrium point. Furthermore, despite the state of the system being unknown, the estimator eliminates the estimation error quickly, thereby allowing the state to be accurately driven to zero. These results are demonstrated for both the H_∞ controller and the very strict passivity controller. Figure 4.5 also shows only a slight different in the time response between the H_∞

controller and the very strict passivity controller. In the state plot, the very strict passivity plots stay closer to zero than the H_∞ plot.

4.1.2 Method 2: Second necessary condition method

4.1.2.1 Design Procedure

For the matrix inequality described in (3.129) to be satisfied, all blocks of all dimensions with the matrix along the main diagonal must be positive definite. Using this necessary condition, the composite block consisting of the (1,1), (1,2), (2,1), and (2,2) blocks of (3.129) are used to derive an LMI system that can solve for P_c and K. This differs from the first method by including more information into the controller design in the form of the off diagonal elements and information about the observer in the (2,2) block. The matrix inequality system is

$$\begin{bmatrix} -(A+BK)^T P_c - P_c(A+BK) - \frac{\delta}{\tau} C_{z1}^T C_{z1} - \alpha I & P_c BK - \frac{\delta}{\tau} C_{z1}^T C_{z2} \\ K^T B^T P_c - \frac{\delta}{\tau} C_{z2}^T C_{z1} & -A_o^T P_o - P_o A_o - \frac{\delta}{\tau} C_{z2}^T C_{z2} - (\alpha+\beta) I \end{bmatrix} > 0 \tag{4.7}$$

In order to design the controller, the terms associated with the observer gains are bounded, as defined by the matrix inequality

$$-A_o^T P_o - P_o A_o > R \tag{4.8}$$

Applying (4.8) to (4.7) yields

$$\begin{bmatrix} -(A+BK)^T P_c - P_c(A+BK) - \frac{\delta}{\tau}C_{z1}^T C_{z1} - \alpha I & P_c BK - \frac{\delta}{\tau}C_{z1}^T C_{z2} \\ K^T B^T P_c - \frac{\delta}{\tau}C_{z2}^T C_{z1} & R - \frac{\delta}{\tau}C_{z2}^T C_{z2} - (\alpha+\beta)I \end{bmatrix} > 0 \qquad (4.9)$$

In its current form, (4.9) is not linear due to the way the unknown matrices K and P_c multiply with B. This is remedied by pre- and post-multiplying each element of (4.9) by P_c^{-1} to group the unknowns together and then defining $Y_c = KP_c^{-1}$

$$\begin{bmatrix} -P_c^{-1}A^T - Y_c^T B^T - AP_c^{-1} - BY_c - \frac{\delta}{\tau}P_c^{-1}C_{z1}^T C_{z1}P_c^{-1} - \alpha P_c^{-2} & BY_c - \frac{\delta}{\tau}P_c^{-1}C_{z1}^T C_{z2}P_c^{-1} \\ Y_c^T B^T - \frac{\delta}{\tau}P_c^{-1}C_{z2}^T C_{z1}P_c^{-1} & R_o - \frac{\delta}{\tau}P_c^{-1}C_{z2}^T C_{z2}P_c^{-1} - (\alpha+\beta)P_c^{-2} \end{bmatrix} > 0$$

(4.10)

where

$$R_o = P_c^{-1} R P_c^{-1} \qquad (4.11)$$

This matrix inequality is made linear by using Lemma 1a twice, resulting in the LMI

$$\begin{bmatrix} -P_c^{-1}A^T - Y_c^T B^T - AP_c^{-1} - BY_c & BY_c & \sqrt{\delta}P_c^{-1}C_{z1}^T & \sqrt{\alpha}P_c^{-1} & 0 \\ Y_c^T B^T & R_o & \sqrt{\delta}P_c^{-1}C_{z2}^T & 0 & \sqrt{\alpha+\beta}P_c^{-1} \\ \sqrt{\delta}C_{z1}P_c^{-1} & \sqrt{\delta}C_{z2}P_c^{-1} & I & 0 & 0 \\ \sqrt{\alpha}P_c^{-1} & 0 & 0 & \tau^{-1}I & 0 \\ 0 & \sqrt{\alpha+\beta}P_c^{-1} & 0 & 0 & \tau^{-1}I \end{bmatrix} > 0$$

(4.12)

The known variables are A, B, R_o, δ, C_{z1} and C_{z2}. If LMI (4.12) is feasible, outputs P_c^{-1} and Y_c are obtained. These results allow the value for the control gain, K, to be calculated. The values of α and β can also be a known input or an output of the LMI. The value for R can also be calculated based on the equation (4.11)

Once the controller has been designed based on (4.7) and A_c, Pc, K, R, α and β are known, it can be expanded by setting $Y_o=P_oL$ to get the LMI. If LMI (4.6), (4.8), and (3.44) are feasible, the outputs P_o and Y_o are obtained, allowing the observer gain, L, to be calculated. The gains calculated using this design procedure place the eigenvalues of the linear component of the controller and observer within left half plane and can accommodate nonlinearity in the measurement and state.

The LMI system consisting of (4.11), and (3.26) places the eigenvalues of (A+BK) within the circular region for the controller. Then the LMI system consisting of (4.6), (4.8), and (3.44) places the eigenvalues of (A-LC) within the circular region for the observer. The closed loop system is robust with respect to the nonlinear deviation within the system and is asymptotically stable.

4.1.2.2 Method 2 application to the continuous-time system model

Using the system model defined in (4.1) and (4.2), Method 2 is used. Normally, solving the combination of the 2n x 2n block of the main result and the controller Regional Eigenvalue Assignment LMI would yield P_c^{-1} and Y_c, which would then allow for the calculation of the controller gains, K. However, for the given system parameters, Matlab's LMI solver was unable to find a feasible result for the observer design stage. In order to get a feasible result, the known Lipschitz bound on the nonlinearity, α, is reduced to 10^{-15} and the performance output is modified to

$$z = [0.1\ 0.001\ 0.001\ 0.0001]\mathrm{x} + 0.001\mathrm{w}$$

The resulting controller gains are shown in the following table.

Table 4.4: Controller gains using Method 2

	H_∞	Very Strict Passivity
Controller Gains, K	[-3.4682 -3.4458]	[-2.1607 -3.0168]

Using the values for P_c^{-1} and K, the observer gain, L is calculated by solving the main result LMI for P_o and Y_o.

Table 4.5: Observer gains using Method 2

	H_∞	Very Strict Passivity
Observer Gains, L	$[14.8437 \quad 49.8542]^T$	infeasible

When the design parameters for H∞ control are used, the eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.6.

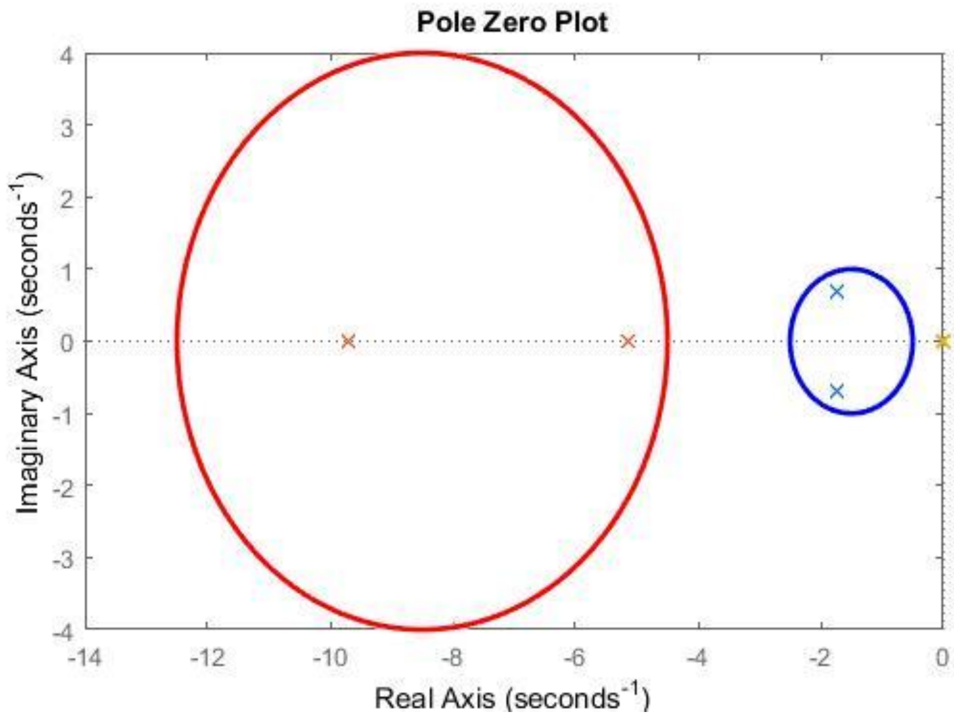

Figure 4.6: Pole-Zero map for H$_\infty$ control using Method 2

When the design parameters for very strict passivity control are used, no feasible answer was achieved for the design of the observer. This shows that given very strict constraints, it is possible to not be able to get a working design using this method. Figure 4.6 shows that like with Method 1, the eigenvalues were placed within the desired regions. It is also notable that the eigenvalue locations are different from those in Method 1. Specifically, the controller region contains eigenvalues that are complex, unlike the real eigenvalues that resulted from using Method 1.

The H$_\infty$ closed loop time response is shown in Figure 4.7.

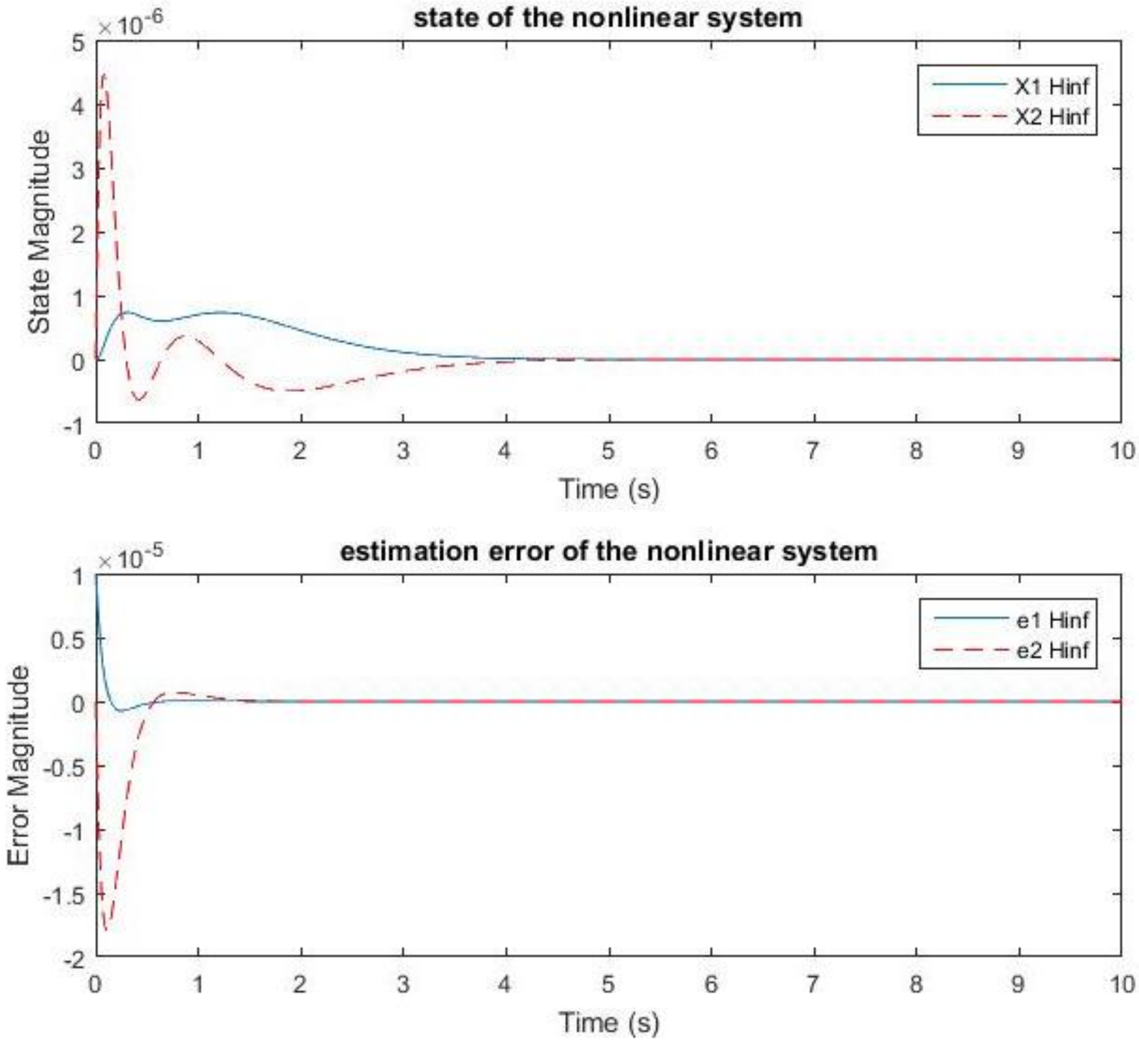

Figure 4.7: Method 2 time response plot

Like Method 1, despite the state of the system being unknown, the estimator eliminates the estimation error quickly. This allows the state to be accurately driven to zero. Unlike Method 1, the use of Method 2 to design a very strict passivity controller failed due to Matlab's inability to find a feasible solution for all unknowns. This demonstrates the limitation of Method 2.

4.1.3 Method 3: The REA controller method

4.1.3.1 Design Procedure

The third method uses the controller regional eigenvalue assignment LMI to solve for a controller gain, K, that will place the eigenvalues of the controller within the prescribed region. Unlike the previous two methods, this method relies only on the REA constraint for the controller design. There is no GPC consideration for the controller, only with the observer. Once the controller has been designed using the controller REA (3.26) and P_c and K are known, (3.129) can be used as an LMI. Setting Yo=PoL yields the LMI,

$$\begin{bmatrix} -A_c^T P_c - P_c A_c - \frac{\delta}{\tau} C_{z1}^T C_{z1} - \alpha I & P_c BK - \frac{\delta}{\tau} C_{z1}{}^T C_{z2} & -P_C & 0 & 0 & -P_c F - \frac{\delta}{\tau} C_{z1}^T D_z + \frac{\gamma}{2\tau} C_{z1}^T \\ K^T B^T P_c - \frac{\delta}{\tau} C_{z2}{}^T C_{z1} & -A^T P_o + C^T Y_o^T - P_o A_o + Y_o C - \frac{\delta}{\tau} C_{z2}^T C_{z2} - \alpha I - \beta I & 0 & -P_O & Y_O & -P_o F + Y_o G - \frac{\delta}{\tau} C_{z2}^T D_z + \frac{\gamma}{2\tau} C^T{}_{z2} \\ -P_C & 0 & I & 0 & 0 & 0 \\ 0 & -P_O & 0 & I & 0 & 0 \\ 0 & Y_O^T & 0 & 0 & I & 0 \\ -F^T P_c - \frac{\delta}{\tau} D_z^T C_{z1} + \frac{\gamma}{2\tau} C_{z1} & -F^T P_o + G^T Y_o^T - \frac{\delta}{\tau} D_z^T C_{z2} + \frac{\gamma}{2\tau} C_{z2} & 0 & 0 & 0 & -\frac{\varepsilon}{\tau} I - \frac{\delta}{\tau} D_z^T D_z + \frac{\gamma}{2\tau}(D_z^T + D_z) \end{bmatrix} > 0 \tag{4.13}$$

If LMI (4.13) and the observer regional eigenvalue assignment LMI produce a feasible result, the outputs P_o and Y_o are obtained, allowing the observer gain, L, to be calculated. The variable β can be used as a known input quantity or calculated as an output value with this LMI technique. The gains calculated using this design procedure place the eigenvalues of the linear component of the controller and observer within specified circular regions in the left half plane and can accommodate nonlinearity in the measurement and state.

4.1.3.2 Method 3 application to the continuous-time system model

Using the system model defined in (4.1) and (4.2), Method 3 is used. Solving the controller regional eigenvalue assignment LMI yields P_c^{-1} and Y_c, from which the controller gains, K, can be calculated.

Table 4.6: Controller gains using Method 3

	H_∞	Very Strict Passivity
Controller Gains, K	[-1.8276 -2.7212]	[-1.8276 -2.7212]

Using the values for P_c^{-1} and K, the observer gains, L are calculated by solving the main result LMI for P_o and Y_o.

Table 4.7: Observer gains using Method 3

	H_∞	Very Strict Passivity
Observer Gains, L	$[16.6876\ \ 58.7608]^T$	$[16.9641\ \ 56.4191]^T$

When the design parameters for $H\infty$ control are used, the eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.8.

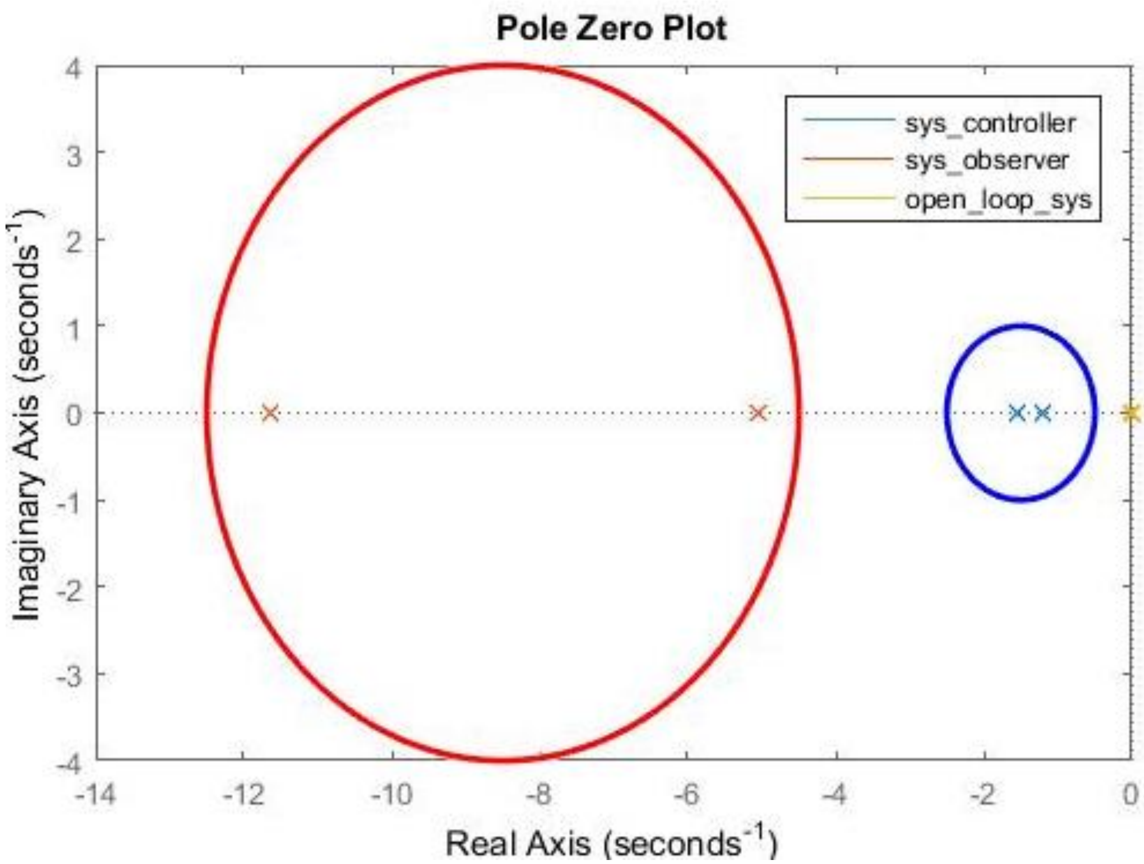

Figure 4.8: Pole-Zero map for H_∞ control using Method 3

When the design parameters for very strict passivity control are used, the eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.9

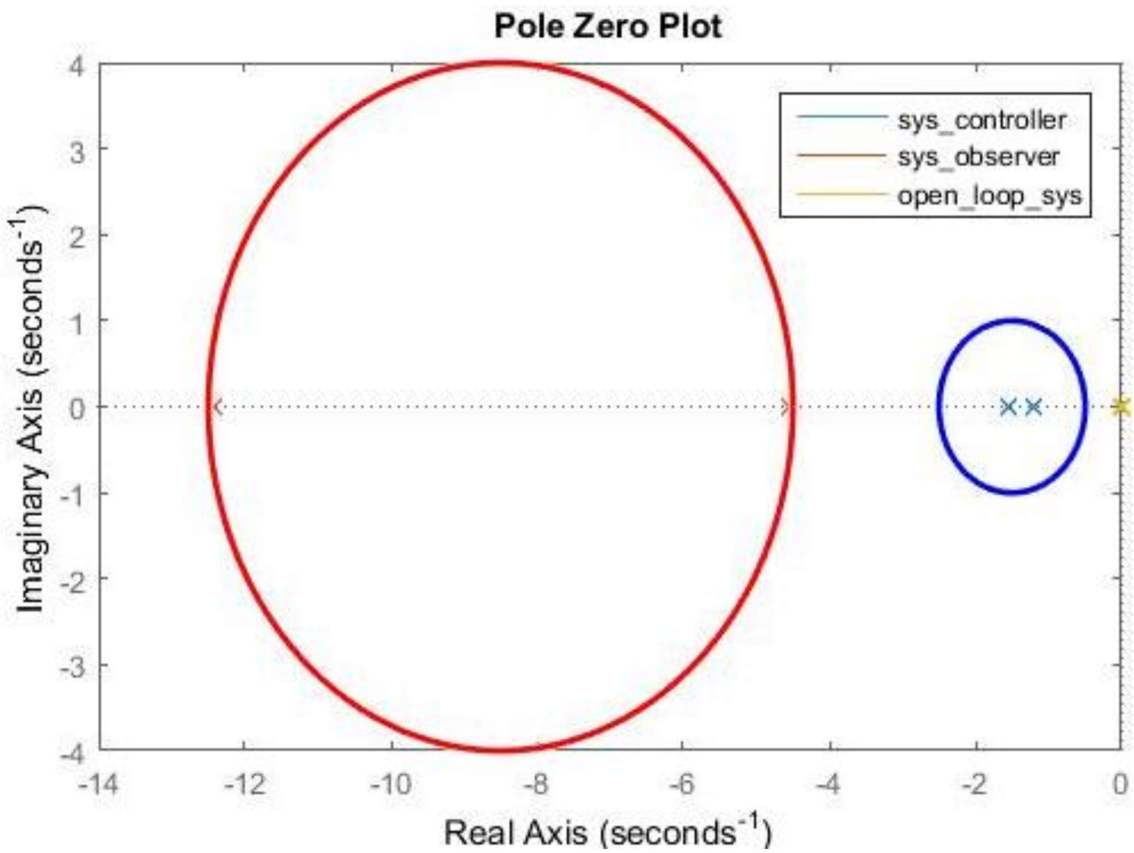

Figure 4.9: Pole-Zero map for very strict passivity control using Method 3

The controller eigenvalues were placed in the same location. This is not surprising since without the GPC consideration, the design is the same for both the H_∞ and very strict passivity controller.

The closed loop time response is shown in Figure 4.10.

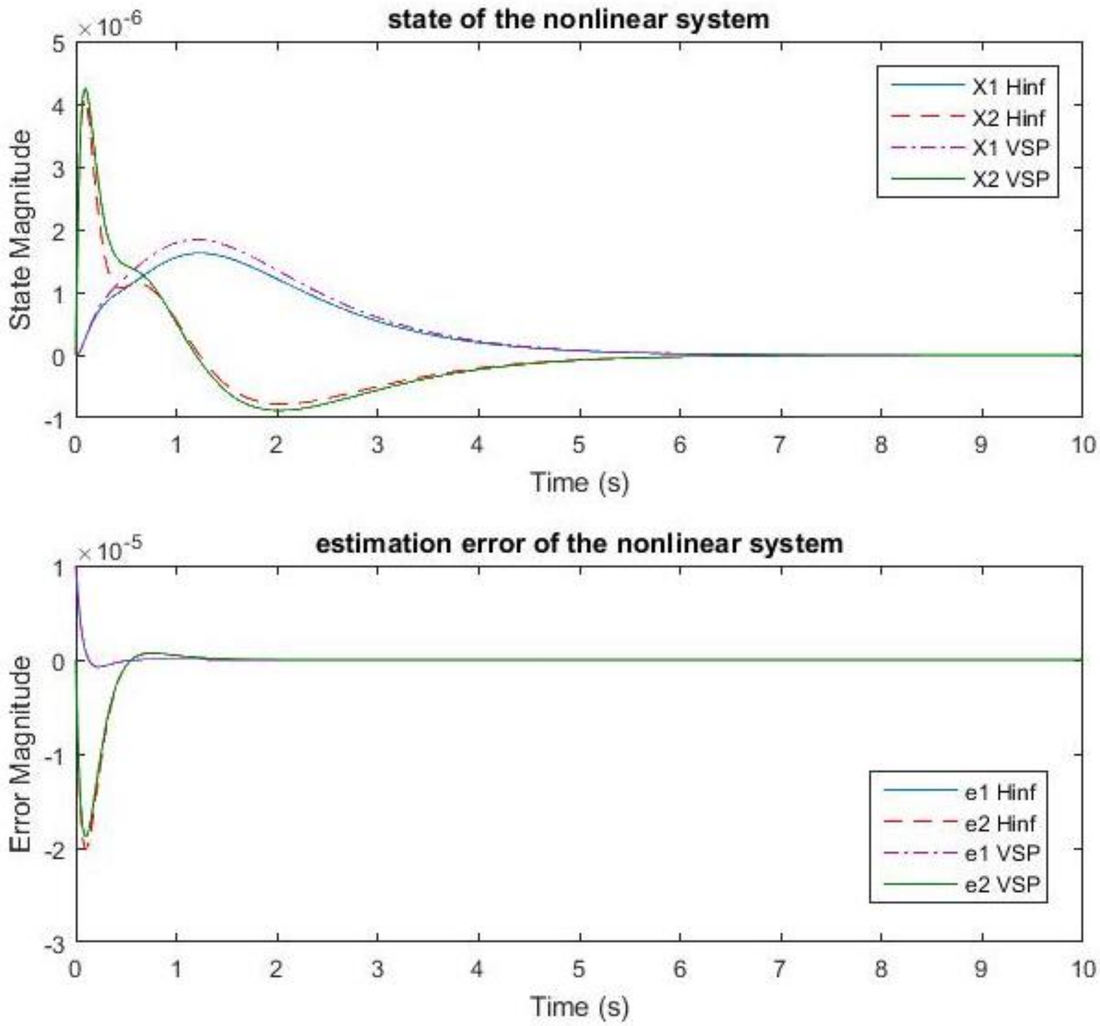

Figure 4.10: Method 3 time response plot

Figure 4.10 is very similar to Figure 4.5 in terms of transient response. The primary differences are that the estimation error drops have a larger magnitude initial drop when using Method 3.

4.1.4 Method 4: The design and check method

4.1.4.1 Design Procedure

The design procedure for this method starts off similarly to Method 1. Using this necessary condition, the (1,1) block of (3.129) is used to derive an LMI system that can solve for P_c and K. The matrix inequality system is (4.3) LMI techniques are used, resulting in LMI (4.5). Like Method 1, the input variables are A, B, δ, and C_{z1}. If LMI (3.129) and REA LMI (3.26) are feasible, outputs P_c^{-1} and Y_c are obtained. These results allow the value for the control gain, K, to be calculated.

Method 4 differs from Method 1 in the following steps. Method 4 uses the necessary condition of the (2,2) block, as well as the observer REA to do the observer design. The necessary condition for the observer design is

$$-(A-LC)^T P_o - P_o(A-LC) - \tfrac{\delta}{\tau} C_{z2}^T C_{z2} - (\alpha+\beta) I > 0 \tag{4.14}$$

This matrix inequality can be put in linear form using the previously defined variable, Y_o.

$$-A^T P_o - C^T Y_o - P_o A - Y_o C - \tfrac{\delta}{\tau} C_{z2}^T C_{z2} - \alpha I - \beta I > 0 \tag{4.15}$$

This matrix inequality is linear and does not require additional LMI techniques. If LMIs (4.16) and observer REA LMI (3.44) are feasible, the outputs P_o and Y_o are obtained, allowing the observer gain, L, to be calculated. The variable β can be used as a known input quantity or calculated as an output value with this LMI technique. The gains calculated using this design procedure place the eigenvalues of the linear component of the controller and observer within left half place, but it does not necessarily guarantee the satisfaction of a general performance criterion.

Using the values of K, L, α, and β, the main result LMI is solved for P_c and P_o. If the resulting LMI is feasible, then the closed loop system will accommodate the nonlinearity and satisfy the general performance criterion while satisfying REA constraints.

4.1.4.2 Method 4 application to the continuous-time system model

Using the system model defined in (4.1) and (4.2), Method 4 is used. Solving the combination of the (1,1) block of the main result and the controller Regional Eigenvalue Assignment LMI yield P_c^{-1} and Y_c, which then allows for the calculation of the controller gains, K.

Table 4.8: Controller gains using Method 4

	H_∞	Very Strict Passivity
Controller Gains, K	[-1.8115 -2.7124]	[-1.8115 -2.7124]

Using the values for P_c^{-1} and K, the observer gains, L are calculated by solving the main result LMI for P_o and Y_o.

Table 4.9: Observer gains using Method 4

	H_∞	Very Strict Passivity
Observer Gains, L	$[15.5880\ \ 54.2390]^T$	$[15.5880\ \ 54.2390]^T$

Note that the controller and observer gains are the same. This is because the relevant design parameter, the δ parameter, is the same for both performance criteria.

Though the magnitudes of the delta terms differ between the performance criteria, they are still relatively small when compared to other elements of the LMIs. Therefore, they have the same gains to a significant figure and will therefore have the same pole-zero and time response plots. This differs from previous methods by excluding the information from the off diagonal terms and not including the information about the system noise.

Using Method 4, the eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.11.

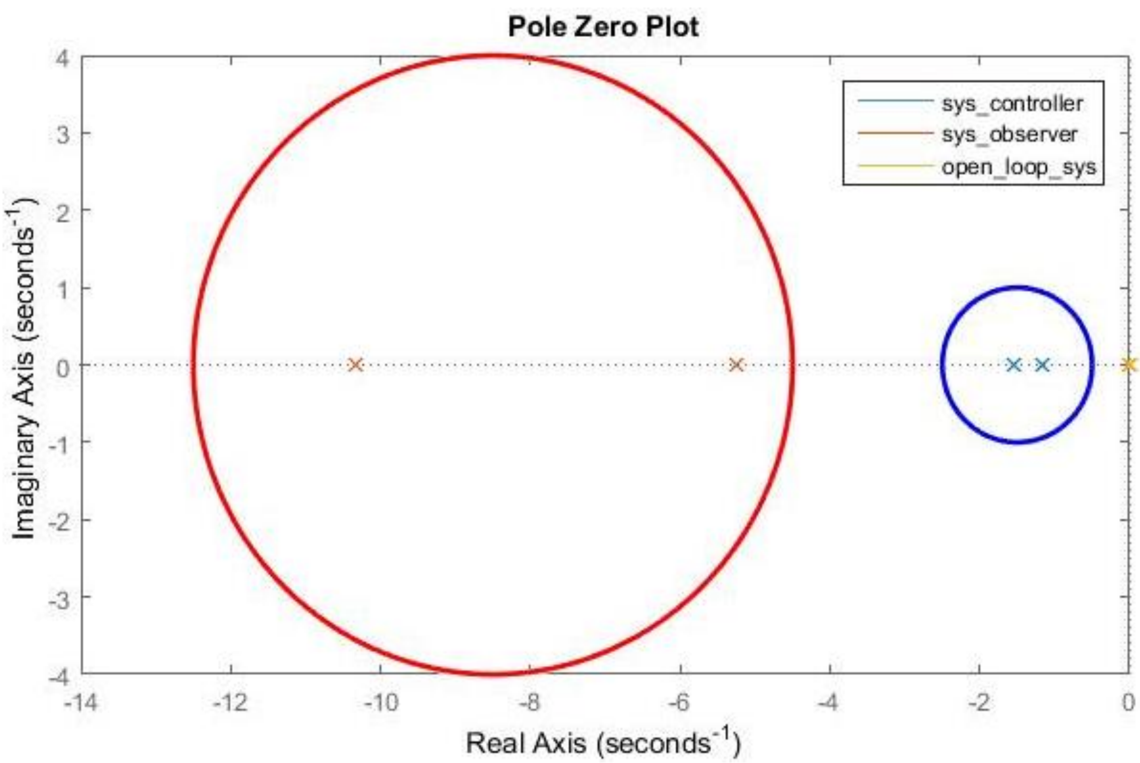

Figure 4.11: Pole-Zero map using Method 4

The closed loop time response is shown in Figure 4.12.

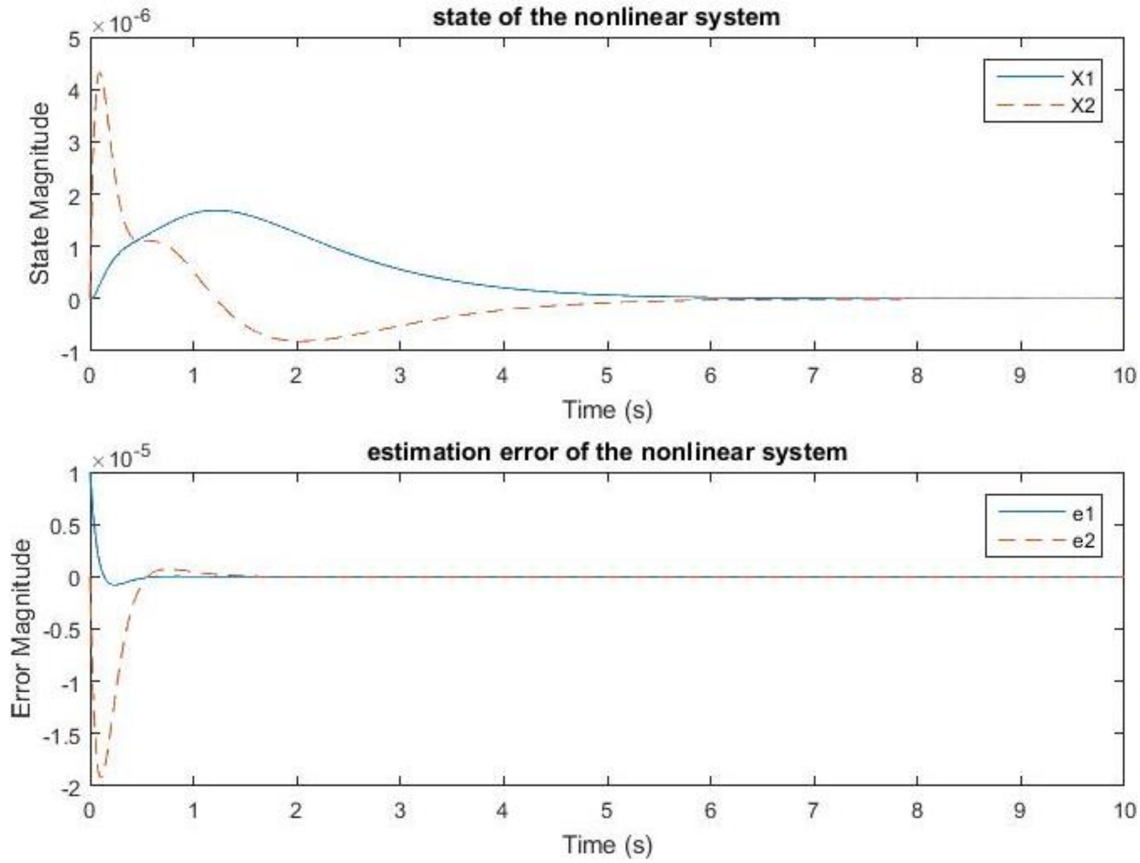

Figure 4.12: Method 4 time response plot

The transient response seen in Figure 4.12 is like those seen when methods 1 and 3 were used. The goal of separating the eigenvalues of the controller and observer have had the desired result, the state estimation error goes to zero faster than the state.

The 4 methods demonstrated show that given a feasible solution, the controller can be designed to specifications. The variety of methods allows the designer flexibility in using the design procedure. However, certain limitations can influence which methods are used and when. In the next section, the differences in the methods will be explored in more detail.

4.1.5 Comparison

The first metric of comparison is the maximum Lipschitz constant each of the methods allows. This value, α, is determined by using a consistent system model and increasing α until the LMI solver no longer provides a K and L that result in a strictly feasible result for the system of LMIs. Using this methodology, the results in are tabulated in Table 4.5.

Table 4.10: Comparison of maximum alpha values between the 4 methods

	max α for H_∞	Rank
Method 1	0.792×10^{-3}	2
Method 2	28×10^{-15}	4
Method 3	0.104×10^{-3}	3
Method 4	4.693×10^{-3}	1

The results show that Method 4 has the highest value of alpha that produces a feasible result. Method 2 is the most restrictive method and this shows in the small value for alpha it needs to achiever feasibility.

A look at the pole zero map for an H_∞ controller using α=10e-16 shows the relative pole locations within the regions the eigenvalues are placed for all the design methods.

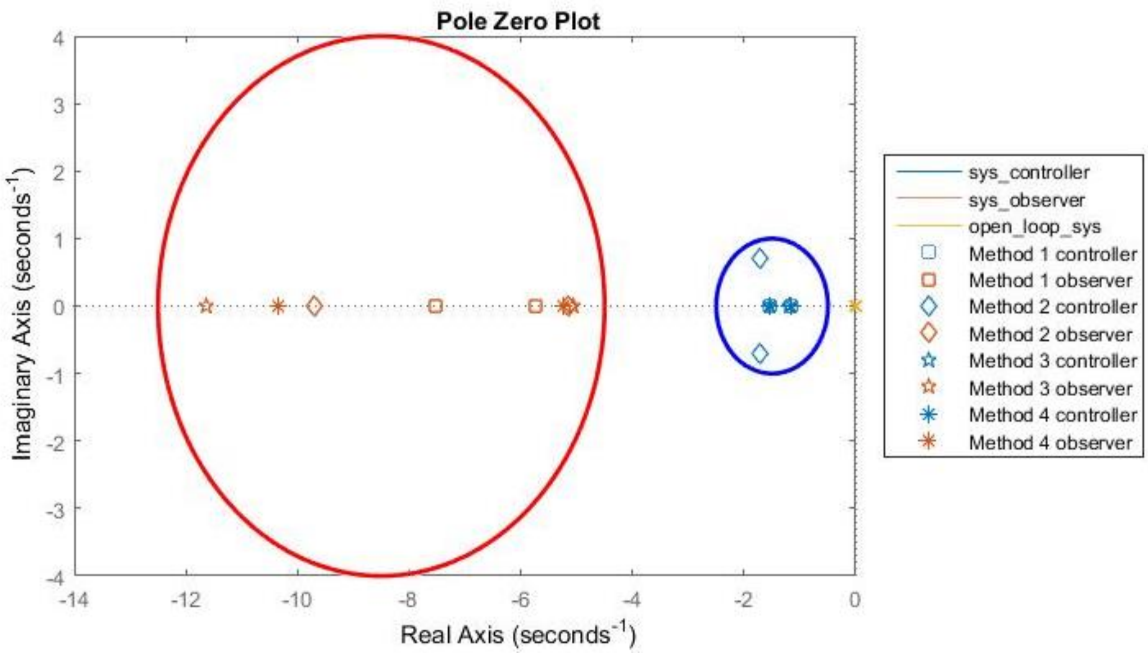

Figure 4.13: Pole-Zero comparison of Methods 1, 3, and 4

From Figure 4.13, the observer eigenvalues are indeed being placed differently within their regions. However, the controller eigenvalues do not appear to have much difference in their eigenvalue locations, except for Method 2. To get a better idea of where within those regions the eigenvalues lie, a zoomed in version of Figure 4.13 for each region is done.

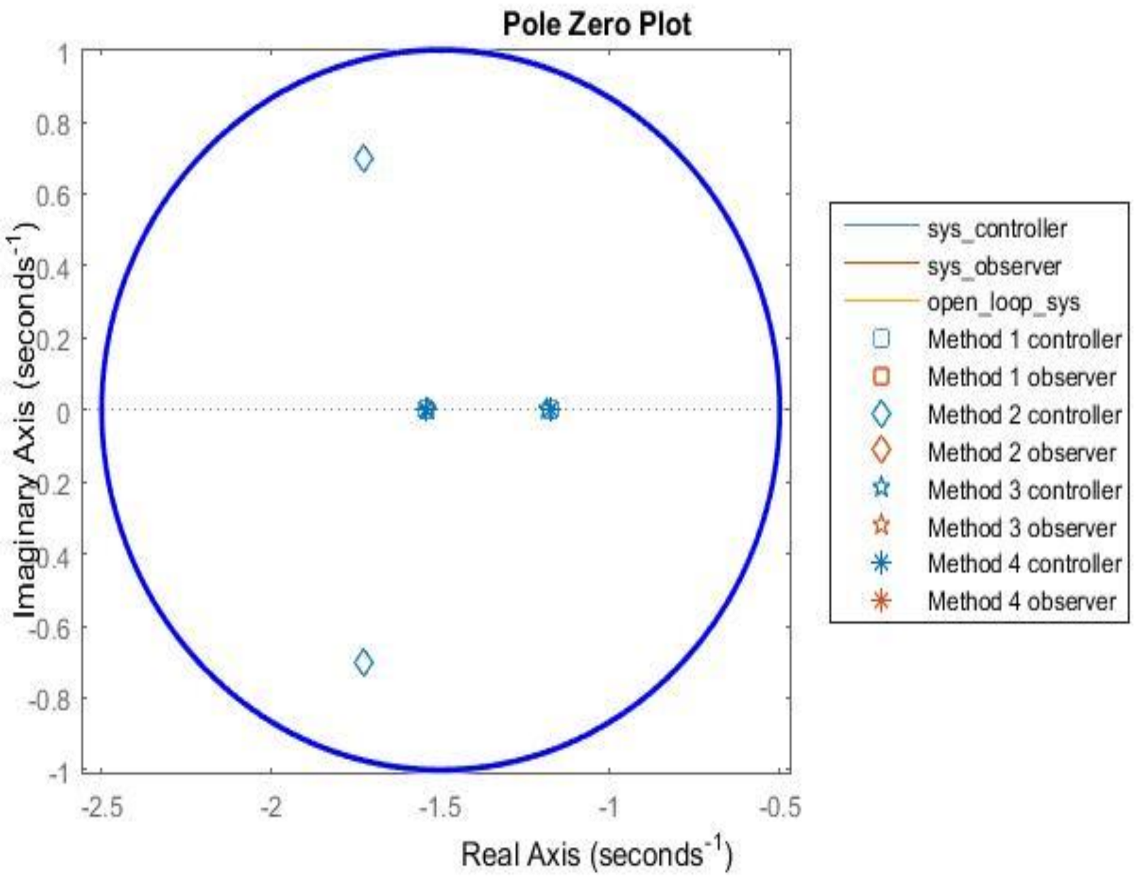

Figure 4.14: Pole-Zero map for the controller region

Figure 4.14 shows the very close eigenvalue locations between methods 1, 3, and 4. However, it should be noted that only methods 1 and 4 have the same eigenvalue location. Method 3 places the eigenvalues slightly closer together.

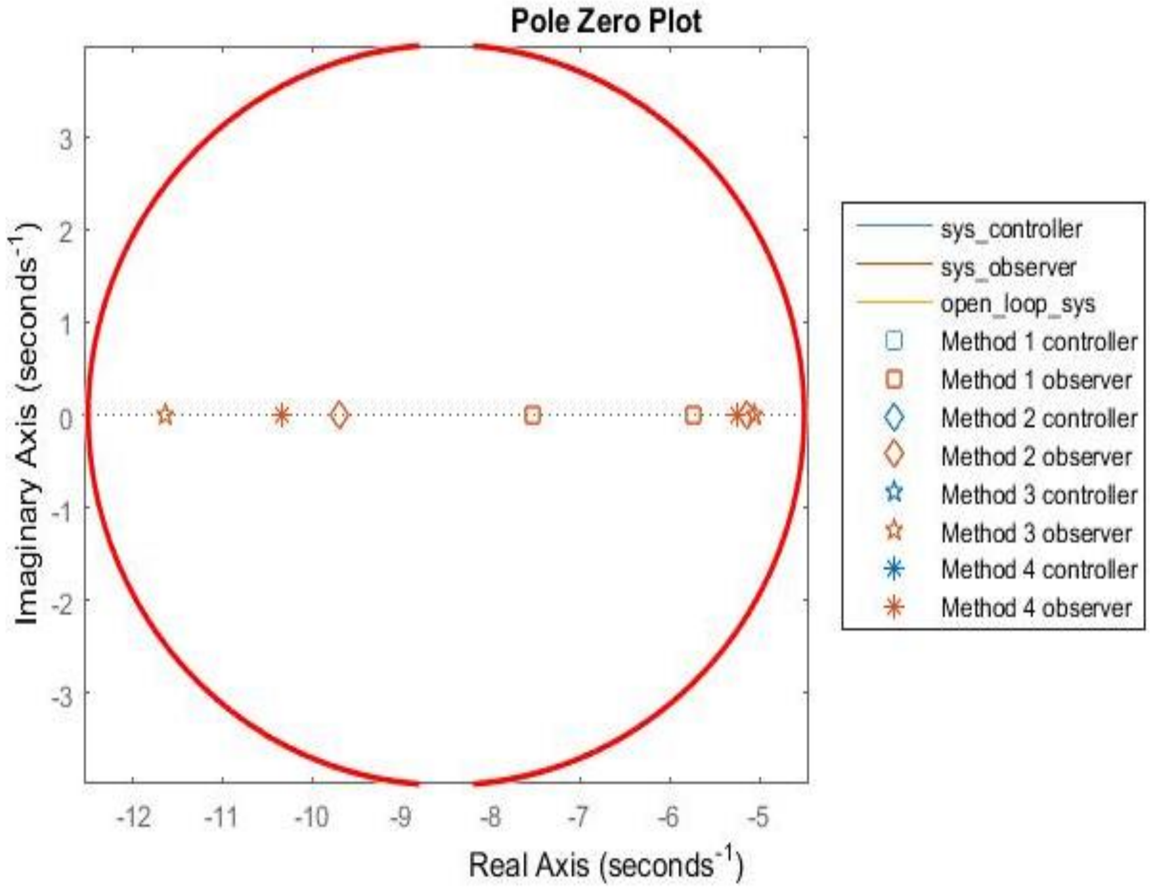

Figure 4.15: Pole-Zero comparison in the observer region

Figure 4.15 shows that in terms of the observer design, Method 3 places the eigenvalues farthest apart and closest to the edge of the observer region. Method 1 places it's eigenvalues closest to the center of the region and to each other.

These results show that the matrix inequality (3.129) can be used to achieve design goals using multiple methods. Some methods are computationally easier and provide a broader range of feasible solutions. Other methods are computationally more intense and provide a narrower range of feasible results. These results are only for continuous-time systems; to get a broader understanding of the effectiveness of each method, they are also being tested in discrete-time

4.2 Discrete-time

All methods proposed in discrete-time will use the following system model. Consider a simple harmonic system with friction, like the system pictured in Figure 4.1, expressed in discrete-time state-space format as

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ -\dfrac{(2-T\cdot b)}{(2+T\cdot b)} & \dfrac{4}{(2+T\cdot b)} \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ \sqrt{\alpha}\sin(x_{k,2}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} w_k \tag{4.16}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + \sqrt{\beta}\tan^{-1}(x_{k,1}) + 10^{-6} w_k \tag{4.17}$$

where $\alpha = 10\times10^{-12}$, $\beta=10^{-3}$, $b = 3$ is the friction coefficient, and T=0.001 is the sample rate. The initial conditions are

$$x_{k,o} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \qquad \hat{x}_{k,o} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The initial condition is estimated to be at the equilibrium point, which is unstable. However, the actual position is perturbed from its equilibrium. The finite energy noise generally is unknown but for the purposes of simulation, it is defined as

$$w_k = \Psi^k \cos(k)$$

where Ψ=0.99. From the system model, it is seen that the finite energy noise also contributes to the system's deviation from its equilibrium point.

A time response plot of the open loop system is shown in Figure 4.16. The plot shows the system settles at x1=π for two different values of alpha, the one used in the system model, α= 10×10^{-12}, and a much larger alpha, $\alpha = 10\times10^{-5}$. The output with the larger alpha reaches steady state in less time than the system with the smaller alpha due to the dominance of the friction in the system model.

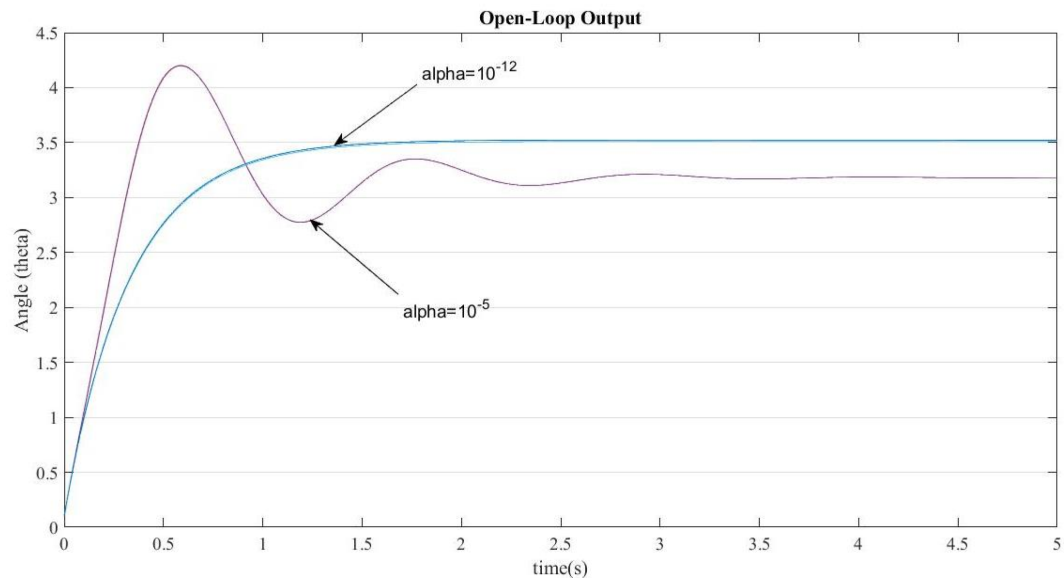

Figure 4.16:Simple discrete-time inverted pendulum open-loop response

The performance output is

$$z_k = [1 \cdot 10^{-6} \quad 0.01 \cdot 10^{-3} \quad 0.1 \quad 0.1]x_k + 0.1w_k$$

This performance output signifies that the transient of the state is less important than the estimation error and the noise in terms of how the system performance is measured. The GPC design parameters for H_∞ and Very Strict Passivity are

Table 4.11: Performance Criteria design parameters --Discrete-time

	δ	ε	γ
H_∞	1	-10	0
Very Strict Passivity	1	0.001	1

It is assumed that the state cannot be accurately measured. Therefore, a dynamic state feedback controller will be designed. As mentioned in Chapter 3, REA is used to separate the controller eigenvalues from the observer eigenvalues. The design parameters for the desired regions for the controller eigenvalues, Dc (rc, ac), and the observer eigenvalues, Do (ro, ao), are

$$r_c = 0.1 \qquad a_c = 0.75$$

$$r_o = 0.2 \qquad a_o = 0.25$$

The control goal is to keep the system at (0,0) though the use of state estimate feedback control. This is achieved through one of the four methods described in the previous section. However, this time, it will be applied toward the design of a discrete-time controller. When the following methods are completed, a dynamic state-feedback controller is designed with either H_∞ or very strict passivity performance, has REA constraints that force the state-estimate error to go to zero much faster than the state, and accommodates the nonlinearity in the system model.

4.2.1 Method 1: First dimension necessary condition method

4.2.1.1 Design Procedure

For the matrix inequality described in (3.113) to be satisfied, it is necessary that the blocks along the diagonal be positive definite. Unlike in continuous-time, the necessary conditions used are both the (1,1) and the (3,3) blocks of (3.113), to derive an LMI system that can solve for P_c and K. The matrix inequality system is

$$P_c - (A+BK)^T P_c (A+BK) - \delta C_{z1}^T C_{z1} - \tau\alpha I > 0 \tag{4.18}$$

with the constraint

$$\tau I - P_c > 0 \tag{4.19}$$

However, in order to convert (4.17) into a linear form, Lemma 1a is used to give

$$\begin{bmatrix} P_c - \tau\alpha I - \delta C_{z1}^T C_{z1} & A^T P_c + K^T B^T P_c \\ P_c A + P_c BK & P_c \end{bmatrix} > 0 \tag{4.20}$$

In its current form, (4.20) is not linear due to the way the unknown matrices K and P_c multiply with B. This a remedied by pre- and post-multiplying each element of (4.20) by P_c^{-1} in order to group the unknowns together

$$\begin{bmatrix} P_c^{-1} - P_c^{-1}(\tau\alpha + \delta C_{z1}^T C_{z1}) P_c^{-1} & P_c^{-1} A^T + Y_c^T B^T \\ AP_c^{-1} + BY_c & P_c^{-1} \end{bmatrix} > 0 \tag{4.21}$$

This matrix inequality is still not linear due to the quadratic term in the (1, 1) block of (4.21). This is addressed by using Lemma 1a twice, resulting in the LMI

$$\begin{bmatrix} P_c^{-1} & P_c^{-1} A^T + Y_c{}^T B^T & P_c^{-1} & \sqrt{\delta} C_{z1}^T P_c^{-1} \\ AP_c^{-1} + BY_c & P_c^{-1} & 0 & 0 \\ P_c^{-1} & 0 & (\tau\alpha I)^{-1} & 0 \\ \sqrt{\delta} C_{z1} P_c^{-1} & 0 & 0 & 1 \end{bmatrix} > 0 \tag{4.22}$$

The second necessary condition is a constraint on P_c. To be consistent with (4.22), (4.19) is also pre- and post-multiplied by P_c^{-1}. The new constraint on P_c^{-1} is

$$P_c^{-1} - \tau^{-1} I > 0 \tag{4.23}$$

The known quantities are A, B, δ, and C_{z1}. If the LMI system consisting of (4.19), (4.22), and (3.26) is feasible, outputs P_c^{-1} and Y_c are obtained. These results allow the value for the control gain, K, to be calculated. The resulting eigenvalue locations for the linear component of the controller eigenvalues are assigned to the region specified by the REA constraints. The value of α can also be an input if the Lipschitz bound is known. In that case, τ is found and used in the observer design matrix inequality.

The controller has been designed based on the necessary conditions of (3.113), A_c, Pc, K, and α are known. In order to design the observer, Lemma 1 is used on (3.116). The resulting matrix inequality is

$$\begin{bmatrix} P-\delta C_z^T C_z-\tau M & 0 & -\delta C_z^T D_z-\frac{\gamma}{2}C_z^T & \Lambda^T P \\ 0 & \tau I & 0 & J^T P \\ -\delta D_z^T C_z-\frac{\gamma}{2}C_z & 0 & -\delta D_z^T D_z-\varepsilon I+\frac{\gamma}{2}(D_z^T+D_z) & H^T P \\ P\Lambda & PJ & PH & P \end{bmatrix} > 0 \tag{4.24}$$

Inequality (4.24) is expanded based on the system model (3.106), P is as defined in (3.13) and Y_o is defined in (2.36). This yields the LMI,

$$\begin{bmatrix} P_c-\tau\alpha I-\delta C_{z1}^T C_{z1} & -\delta C_{z2}^T C_{z1} & 0 & 0 & 0 & -\delta C_{z1}^T D_z-\frac{\gamma}{2}C_{z1}^T & A^T P_c+K^T B^T P_c & 0 \\ -\delta C_{z1}^T C_{z2} & P_o-\delta C_{z2}^T C_{z2}-\tau\alpha I-\tau\beta I & 0 & 0 & 0 & -\delta C_{z2}^T D_z-\frac{\gamma}{2}C_{z2}^T & -K^T B^T P_c & A^T P_o-C^T Y_o^T{}_o \\ 0 & 0 & \tau I & 0 & 0 & 0 & P_c & 0 \\ 0 & 0 & 0 & \tau I & 0 & 0 & 0 & P_o \\ 0 & 0 & 0 & 0 & \tau I & 0 & 0 & -Y_o^T \\ -\frac{\gamma}{2}C_{z1}-\delta D_z^T C_{z1} & -\frac{\gamma}{2}C_{z2}-\delta D_z^T C_{z2} & 0 & 0 & 0 & -\varepsilon I-\delta D_z^T D_z+\frac{\gamma}{2}(D_z^T+D_z) & F^T P_c & F^T P_o-G^T Y_o^T \\ P_c A+P_c BK & -P_c BK & P_c & 0 & 0 & P_c F & P_c & 0 \\ 0 & P_o A-Y_o C & 0 & P_o & -Y_o & P_o F-Y_o G & 0 & P_o \end{bmatrix} > 0 \tag{4.25}$$

If LMI (4.25) and (3.44) are feasible, the outputs P_o and Y_o are obtained, allowing the observer gain, L, to be calculated. The variable β can be used as a known input quantity or calculated as an output value with this LMI technique. Then the LMI system consisting of (4.25) and (3.44) place the eigenvalues of (A-LC) within the circular region for the observer. The gains calculated using this design procedure place the eigenvalues of the linear component of the controller and observer within the unit circle and can accommodate nonlinearity in the measurement and state. The general performance criterion chosen guarantees the eigenvalues are within the unit circle. The closed loop system is robust with respect to the nonlinear deviation within the system and is asymptotically stable.

4.2.1.2 Example of discrete-time design using Method 1

Using the system model defined in (4.16) and (4.17), Method 1 is used with initial conditions and noise defined at the beginning of Section 4.2. Solving the combination of the (1,1) block of the main result, the (3,3) block, and the controller REA LMI yield P_c^{-1} and Y_c, which then allows for the calculation of the controller gains, K.

Table 4.12: Controller gains using method 1--Discrete-time

	H_∞	Very Strict Passivity
Controller Gains	[0.4172 -0.4834]	[0.4172 -0.4834]

Using the values for P_c^{-1} and K, the observer gains, L are calculated by solving the main result LMI for P_o and Y_o.

Table 4.13: Observer gains using method 1 --Discrete-time

	H_∞	Very Strict Passivity
Observer Gains	$[1.4532 \quad 1.9607]^T$	$[1.4533 \quad 1.9610]^T$

When the design parameters for H_∞ control are used, the eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.17

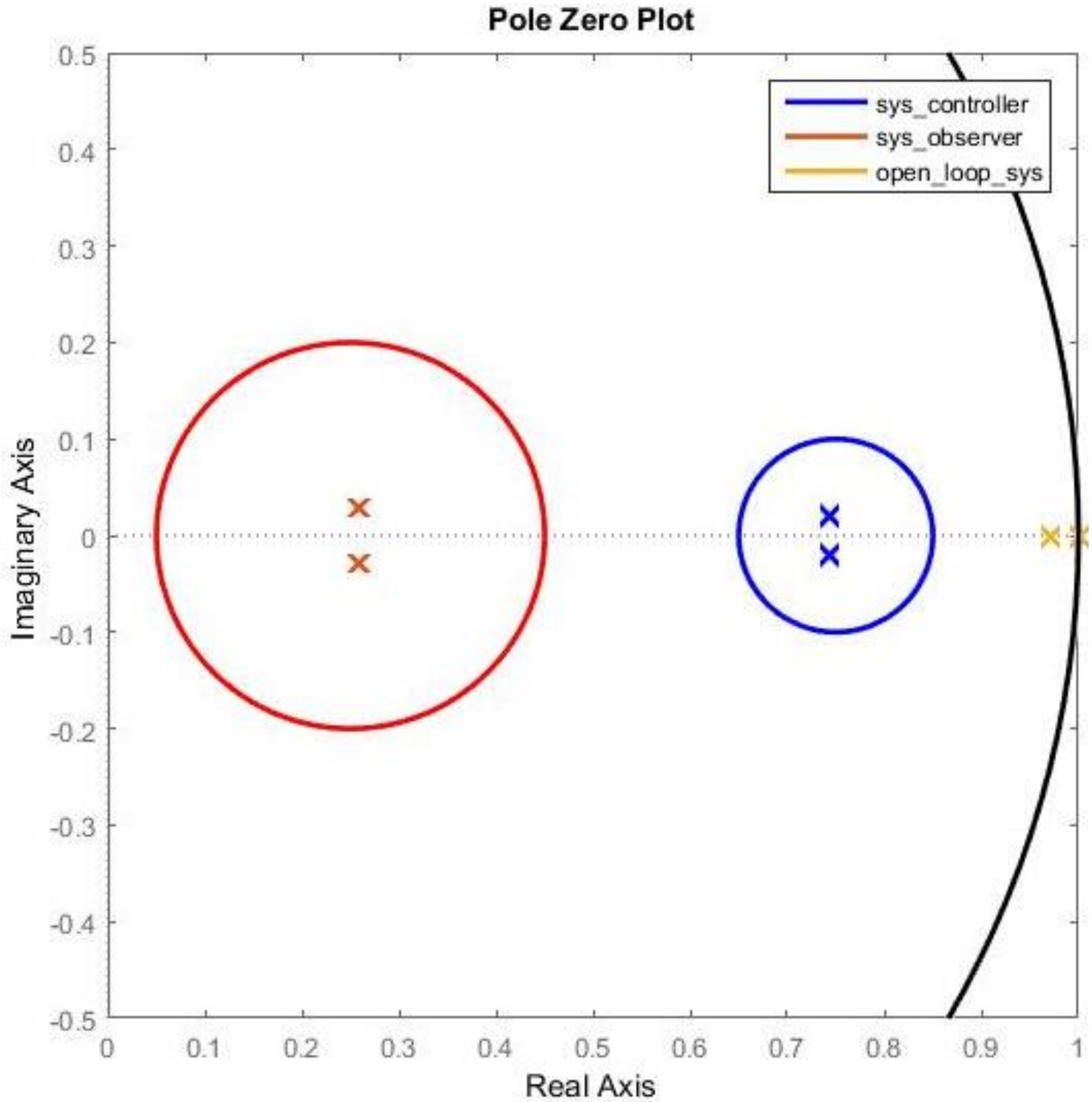

Figure 4.17: Pole-Zero Map H_∞ control using method 1--Discrete-time

When the design parameters for very strict passivity control are used, the eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.18

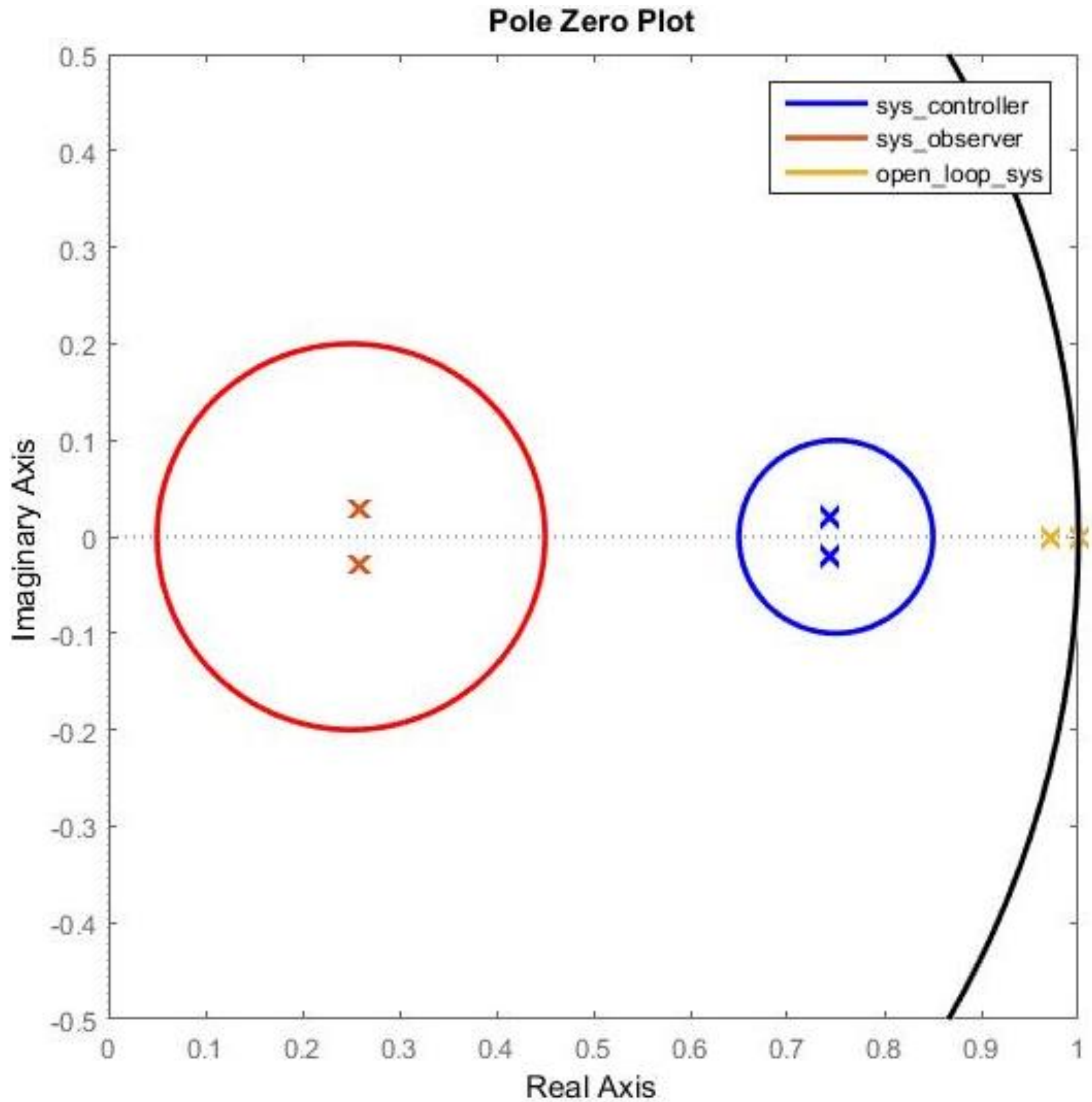

Figure 4.18: Pole-Zero map for very strict passivity using method 1 --Discrete-time

The pole-zero maps in Figure 4.17 and 4.18 show that the eigenvalues are being placed within the regions specified. Table 4.12 also shows that the gains are the same between the two performance criteria for the controller design phase, but the observer gains have a small difference. This means that specifying the performance objective does change the eigenvalue location within the region.

The closed loop time response is shown in Figure 4.19.

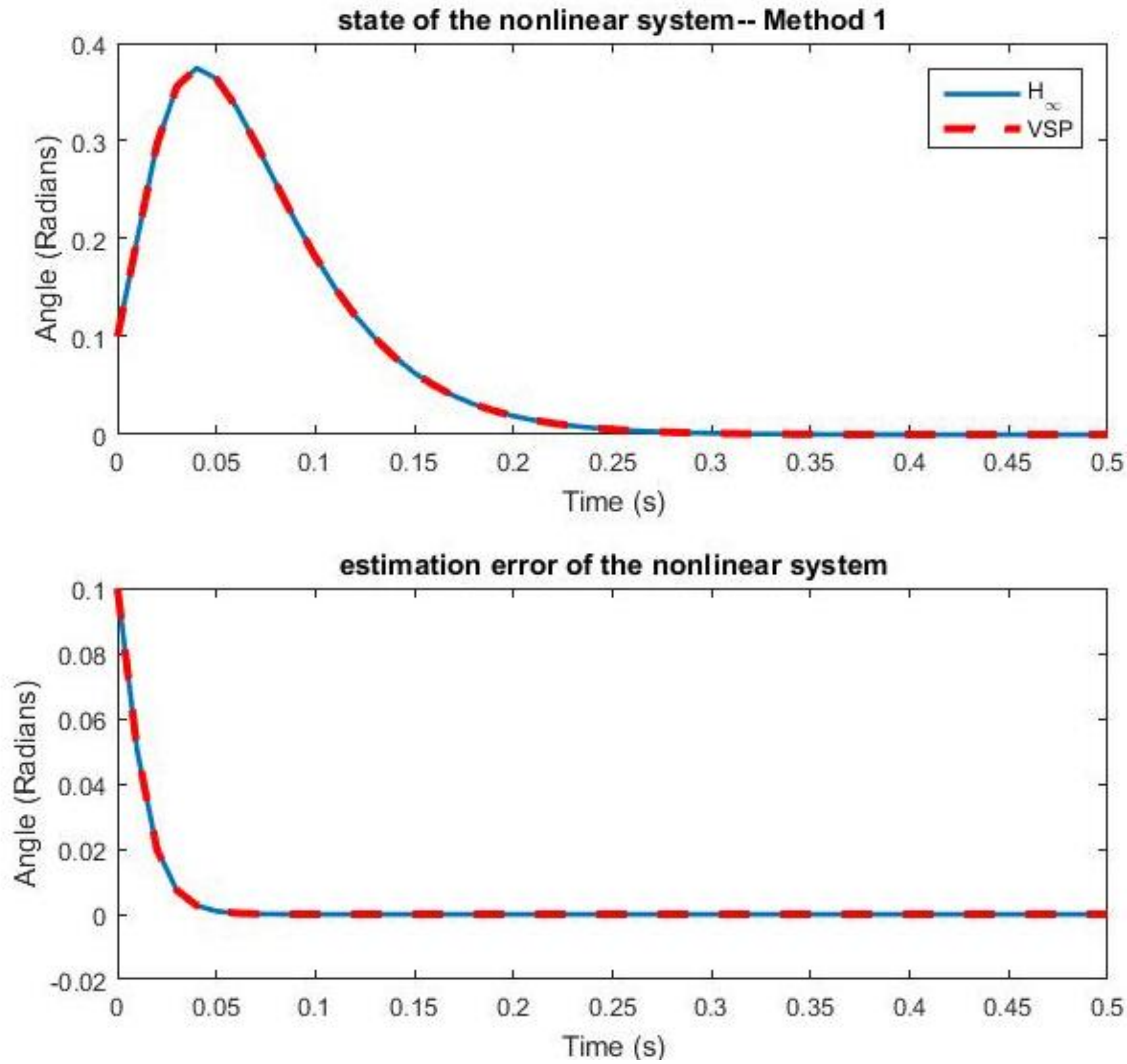

Figure 4.19: Method 1 Time response plot-- Discrete-time

The very slight difference in the eigenvalue locations in the observer have made it so that the time response is virtually identical for the H_∞ controller and the very strict passivity controller. Both controllers drive the estimation error to zero faster than the state reaches zero and both controllers succeed in keeping the pendulum balanced at its equilibrium point.

4.2.2 Method 2: Second dimension necessary condition method

4.2.2.1 Design Procedure

For the matrix inequality described in (3.113) to be satisfied, it is necessary that the blocks along the diagonal be positive definite. Using this necessary condition, the composite block consisting of the (1,1), (1,2), (2,1), and (2,2) blocks of (3.113) are used to derive an LMI system that can solve for P_c and K. The matrix inequality system is

$$\begin{bmatrix} P_c-(A+BK)^T P_c(A+BK)-\delta C_{z1}^T C_{z1}-\tau\alpha I & A_c^T P_c BK-\delta C_{z1}^T C_{z2} \\ K^T B^T P_c A_c-\delta C_{z2}{}^T C_{z1} & \boxed{\begin{matrix} P_o-A_o^T P_o A_o-K^T B^T P_c BK \\ -\delta C_{z2}^T C_{z2}-\tau(\alpha+\beta)I \end{matrix}} \end{bmatrix}>0 \qquad (4.26)$$

Since the observer information at this point in the design is not needed. Therefore, the observer information is bounded, as defined by the matrix inequality

$$P_o-A_o^T P_o A_o>R \qquad (4.27)$$

Or expressed in LMI form

$$\begin{bmatrix} P_o-R & A^T P_o-C^T Y_o^T \\ P_o A-Y_o C & P_o \end{bmatrix}>0 \qquad (4.28)$$

Applying (4.8) to (4.7) yields

$$\begin{bmatrix} P_c-(A+BK)^T P_c(A+BK)-\delta C_{z1}^T C_{z1}-\tau\alpha I & A_c^T P_c BK-\delta C_{z1}^T C_{z2} \\ K^T B^T P_c A_c-\delta C_{z2}{}^T C_{z1} & R-K^T B^T P_c BK-\delta C_{z2}^T C_{z2}-\tau(\alpha+\beta)I \end{bmatrix}>0$$

(4.29)

Lemma 1a is applied to (4.28)

$$\begin{bmatrix} P_c - \tau\alpha I & 0 & (A+BK)^T P_c & \delta C_{z1}^T \\ 0 & R - \tau(\alpha+\beta)I & K^T B^T P_c & \delta C_{z2}^T \\ P_c(A+BK) & P_c BK & P_c & 0 \\ \delta C_{z1} & \delta C_{z2} & 0 & \delta I \end{bmatrix} > 0 \tag{4.30}$$

In its current form, (4.29) is not linear due to the way the unknown matrices K and P_c multiply with B. This is remedied by pre- and post-multiplying each element of (4.30) by

$$\begin{bmatrix} P_c^{-1} & 0 & 0 & 0 \\ 0 & P_c^{-1} & 0 & 0 \\ 0 & 0 & P_c^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{4.31}$$

Y_c is substituted in the matrix inequality for the product KP_c^{-1}.

$$\begin{bmatrix} P_c^{-1} - \tau\alpha P_c^{-2} & 0 & P_c^{-1}A^T + Y_c^T B^T & \delta P_c^{-1} C_{z1}^{\ T} \\ 0 & R_o - \tau(\alpha+\beta)P_c^{-2} & Y_c^T B^T & \delta P_c^{-1} C_{z2}^{\ T} \\ AP_c^{-1} + BY_c & BY_c & P_c^{-1} & 0 \\ \delta C_{z1} P_c^{-1} & \delta C_{z2} P_c^{-1} & 0 & \delta \end{bmatrix} > 0 \tag{4.32}$$

where

$$R_o = P_c^{-1} R P_c^{-1} \tag{4.33}$$

This matrix inequality is still not linear due to the quadratic term in the (1,1) and (2,2) blocks. This is addressed by using Lemma 1a twice, resulting in the LMI

$$\begin{bmatrix} P_c^{-1} & 0 & P_c^{-1}A^T + Y_c^T B^T & \delta P_c^{-1} C_{z1}{}^T & P_c^{-1} & 0 \\ 0 & R_o & Y_c^T B^T & \delta P_c^{-1} C_{z2}{}^T & 0 & P_c^{-1} \\ AP_c^{-1} + BY_c & BY_c & P_c^{-1} & 0 & 0 & 0 \\ \delta C_{z1} P_c^{-1} & \delta C_{z2} P_c^{-1} & 0 & \delta I & 0 & 0 \\ P_c^{-1} & 0 & 0 & 0 & (\tau\alpha)^{-1} I & 0 \\ 0 & P_c^{-1} & 0 & 0 & 0 & \tau^{-1}(\alpha+\beta)^{-1} I \end{bmatrix} > 0 \quad (4.34)$$

The input variables are A, B, R_o, δ, C_{z1} and C_{z2}. If LMI (4.34) is feasible, outputs P_c^{-1} and Y_c are obtained. These results allow the value for the control gain, K, to be calculated. The values of α and β can be an input and τ^{-1} can be found immediately or α and (α+β) solved and β can be calculated. The value for R can also be calculated based on the equation (4.10)

Once the controller has been designed based on the necessary condition of (3.113), A_c, Pc, K, R, α and β are known, it can be expanded. Setting Yo=PoL yields the LMI (4.25). If LMI (4.25), LMI (3.44), and LMI (4.27) are feasible, the outputs P_o and Y_o are obtained, allowing the observer gain, L, to be calculated. The gains calculated using this design procedure place the eigenvalues of the linear component of the controller and observer within left half plane and can accommodate nonlinearity in the measurement and state.

The LMI system consisting of (4.32), and (3.26) places the eigenvalues of (A+BK) within the circular region for the controller. Then the LMI system consisting of (4.25), (4.27), and (3.44) places the eigenvalues of (A-LC) within the circular region for the observer. The closed loop system is robust with respect to the nonlinear deviation within the system and is asymptotically stable.

4.2.2.2 Example of discrete-time design using Method 1.

Using the system model defined in (4.16) and (4.17), Method 2 is attempted for the same system and performance parameters. However, in this particular case, the method described above failed to yield a feasible LMI solution for all LMIs.

The LMIs for the controller design could find a feasible solution, however, when the values of P_c and K were substituted into the discrete-time main result LMI, a feasible solution could not be found. This either means this method could not find a feasible LMI solution for the observer or the resulting closed loop solution could not match the performance criteria. When the main result LMI was removed, a feasible result was found, but it lacked the performance criteria information needed to guarantee the desired performance.

The reasons for the failure of this method on this system could be that the LMI solver is not advanced enough to find the feasible solution. However, it is also important to remember that in continuous-time, this method was the worst at generating a feasible solution and only when system parameters are drastically changed was a feasible solution for the continuous-time H_∞ controller found. However, like with the continuous-time very strict passivity case, a feasible solution could not be obtained via Matlab.

4.2.3 Method 3: The REA only controller method

4.2.3.1 Design Procedure

The third method uses the controller regional eigenvalue assignment LMI to solve for a controller gain, K, that will place the eigenvalues of the controller within the desired region. Once the controller has been designed and A_c, K, and α are known, (3.113) can be used as an LMI. A feasible solution for LMI (4.25) can then be found. If LMI (4.25) and the observer regional eigenvalue assignment LMI produce a feasible result, the outputs P_o and Y_o are obtained, allowing the observer gain, L, to be calculated. The variable β can be used as a known input quantity or calculated as an output value with this LMI technique. The gains calculated using this design procedure place the eigenvalues of the linear component of the controller and observer within left half plane and can accommodate nonlinearity in the measurement and state.

4.2.3.2 Example of the design procedure for method 3

Using the system model defined in (4.15) and (4.16), method 3 is used. Solving the controller regional eigenvalue assignment LMI yields P_c^{-1} and Y_c, from which the controller gains, K, can be calculated.

Table 4.14: Controller gains using method 3--Discrete-time

	H_∞	Very Strict Passivity
Controller Gains	[0.4117 -0.4755]	[0.4117 -0.4755]

Using the values for P_c^{-1} and K, the observer gains, L are calculated by solving the main result LMI for P_o and Y_o.

Table 4.15: Observer gains using method 3-- discrete-time

	H_∞	Very Strict Passivity
Observer Gains	$[1.4536 \quad 1.9710]^T$	$[1.4589 \quad 1.9635]^T$

When the design parameters for H∞ control are used, the eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.20.

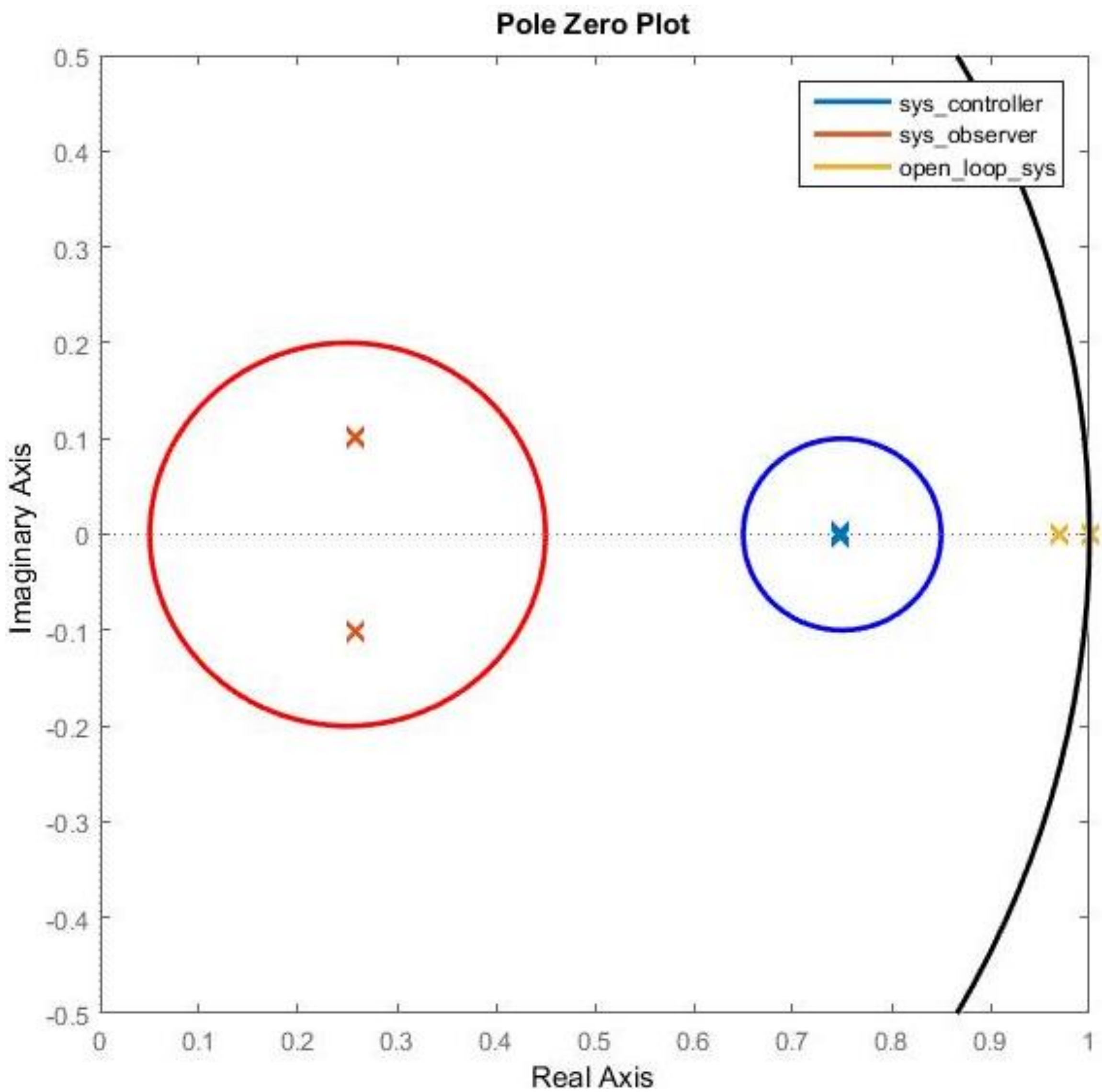

Figure 4.20: Pole-Zero map for H_∞ control using Method 3-- Discrete-time

When the design parameters for very strict passivity control are used, the eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.21.

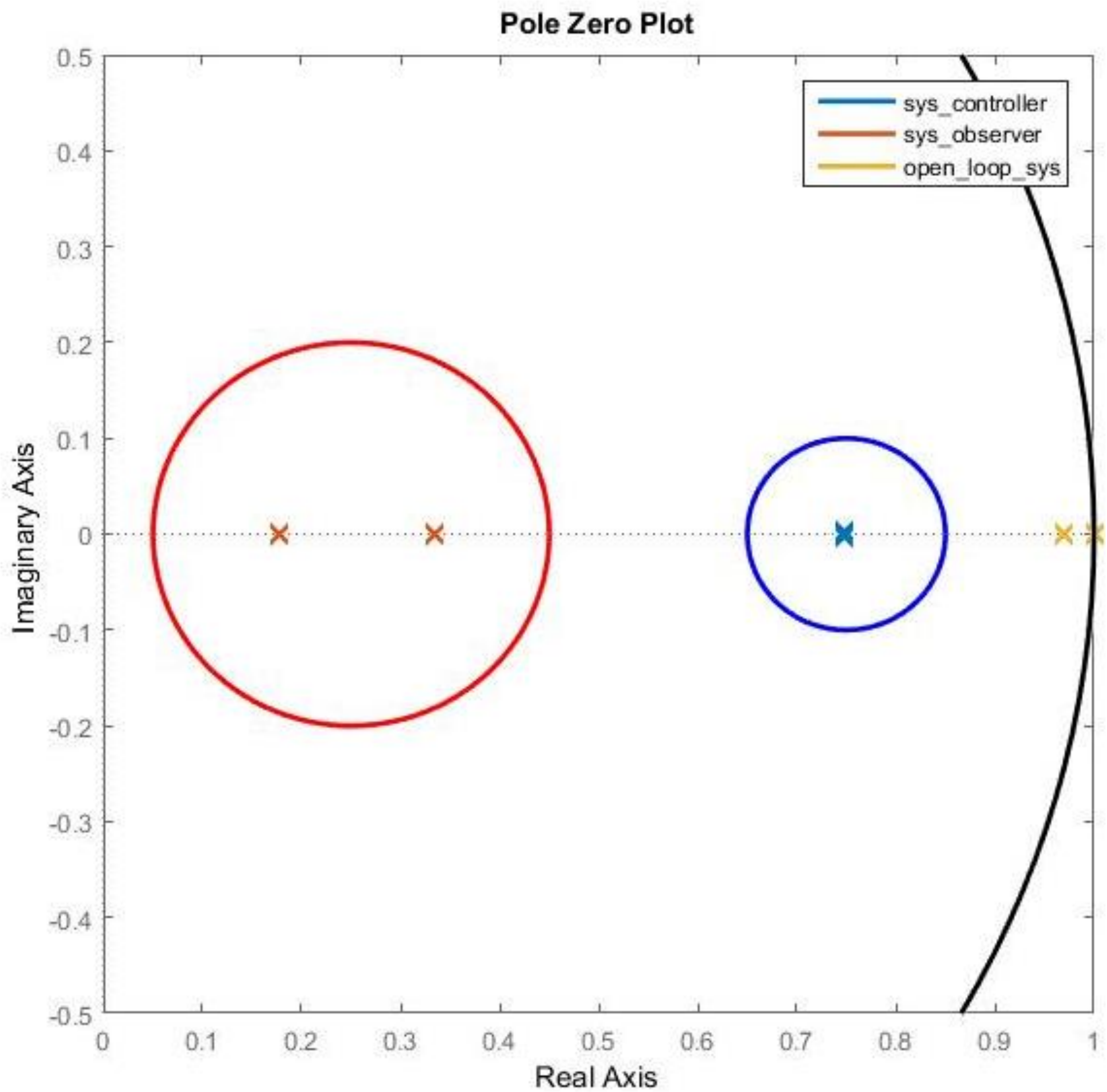

Figure 4.21: Pole-Zero map for very strict passivity control using method 3--Discrete-time

Note that the H_∞ controller has complex eigenvalues in the observer region while Very Strict Passivity controller has real eigenvalues in the observer region. The closed loop time response is shown in Figure 4.22.

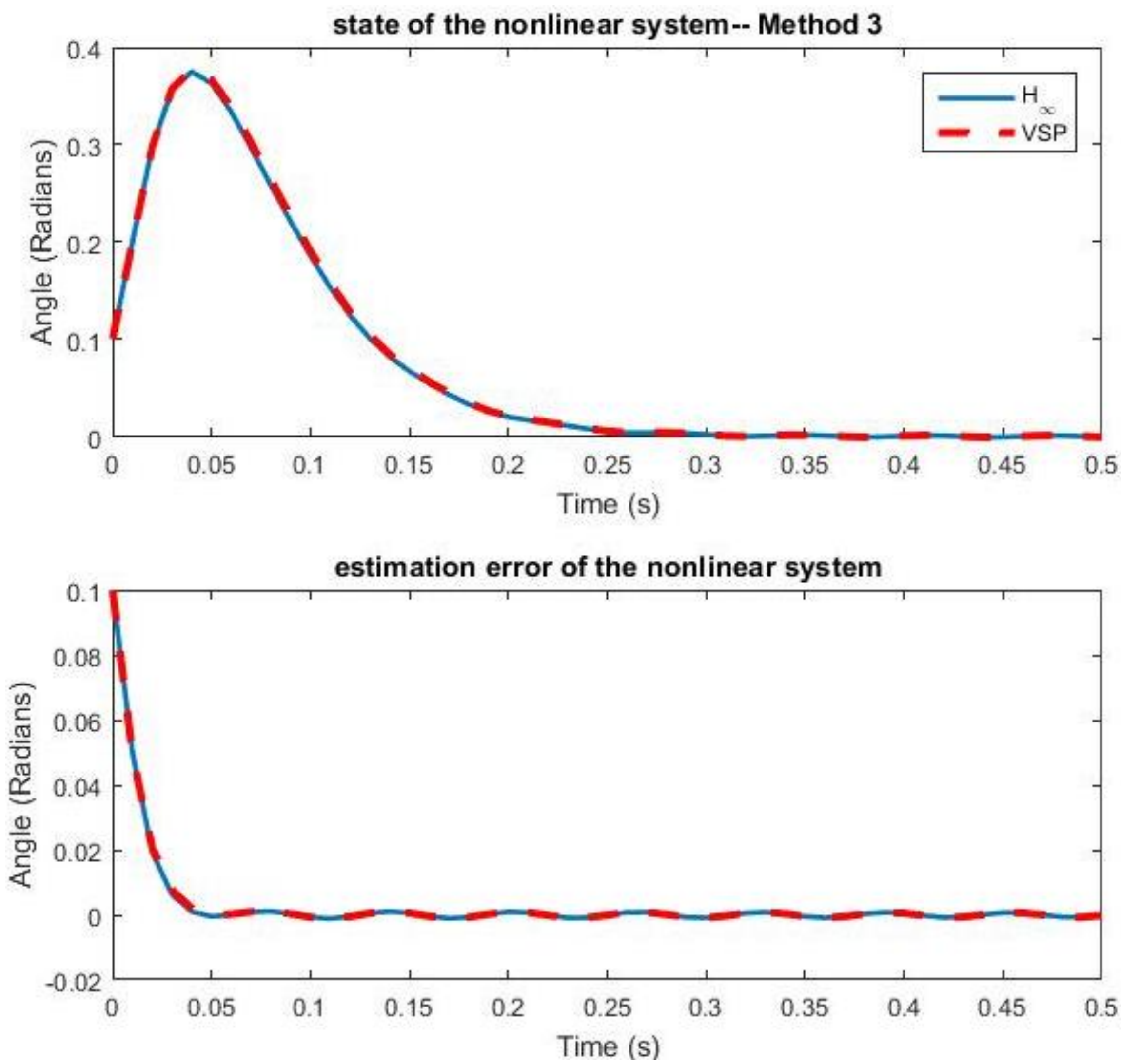

Figure 4.22: Method 3 time response plot-- Discrete-time

Figure 4.22 is very like Figure 4.19 in terms of transient response. The difference between performance criteria is seen in the error where the effects of the noise are still visible. This slight decrease in noise accommodation differs from the smooth error estimation error steady-state seen in Method 1. But like the design using Method 1, the design objectives are achieved.

4.2.4 Method 4: Design and Check

4.2.4.1 Design Procedure

The design procedure for this method starts off similarly to method 1. Using this necessary condition, the (1,1) block of (3.113) is used to derive an LMI system that can solve for P_c and K. The matrix inequality system is

$$P_c-(A+BK)^T P_c(A+BK)-\delta C_{z1}^T C_{z1}-\tau\alpha I>0 \tag{4.35}$$

with the constraint from the (3,3) block

$$\tau I-P_c>0 \tag{4.36}$$

LMI techniques are used, resulting in the LMIs

$$\begin{bmatrix} P_c^{-1} & P_c^{-1}A^T+Y_c^T B^T & P_c^{-1} & \sqrt{\delta}C_{z1}^T P_c^{-1} \\ AP_c^{-1}+BY_c & P_c^{-1} & 0 & 0 \\ P_c^{-1} & 0 & (\tau\alpha I)^{-1} & 0 \\ \sqrt{\delta}C_{z1}P_c^{-1} & 0 & 0 & 1 \end{bmatrix}>0 \tag{4.37}$$

and

$$P_c^{-1}-\tau^{-1}I>0 \tag{4.38}$$

The input variables are A, B, δ, and C_{z1}. If LMI (4.37), (4.38), and REA LMI (3.26) are feasible, outputs P_c^{-1} and Y_c are obtained. These results allow the value for the control gain, K, to be calculated.

Method 4 then uses the necessary conditions from the (2,2), (4,4), and (5,5) block, as well as the observer REA to do the observer design. The first necessary condition for the observer design is

$$P_o - A_o^T P_o A_o - K^T B^T P_c BK - \delta C_{z2}^T C_{z2} - \tau(\alpha+\beta)I > 0 \tag{4.39}$$

which can be expressed in a linear form as

$$\begin{bmatrix} P_o - K^T B^T P_c BK - \delta C_{z2}^T C_{z2} - \tau(\alpha+\beta)I & A^T P_o - C^T Y_o^T \\ P_o A - Y_o C & P_o \end{bmatrix} > 0 \tag{4.40}$$

The second necessary condition is

$$\tau I - P_o > 0 \tag{4.41}$$

and the third necessary condition is

$$\tau I - L^T P_o L > 0 \tag{4.42}$$

which is expressed in linear form as

$$\begin{bmatrix} \tau I & Y_o^T \\ Y_o & P_o \end{bmatrix} > 0 \tag{4.43}$$

If LMIs (4.40), (4.41), (4.43) and observer REA LMI (3.44) are feasible, the outputs P_o and Y_o are obtained, allowing the observer gain, L, to be calculated. The variable β can be used as a known input quantity or calculated as an output value with this LMI technique. The gains calculated using this design procedure place the eigenvalues of the linear component of the controller and observer within the unit circle, but it does not necessarily guarantee the satisfaction of a general performance criterion.

Using the values of K, L, α, and β, LMI (4.25) is solved for a new P_c and a new P_o. If the resulting LMI is feasible, then the closed loop system will accommodate the nonlinearity and satisfy the chosen general performance criterion.

4.2.5 Example of discrete-time design using method 4

Using the system model defined in (4.16) and (4.17), Method 4 is used. In simulation, the controller gains found from the first set of LMIs, LMIs (3.26), (4.36), and (4.37), place the eigenvalues within the desired circular region for the controller design.

Table 4.16: Controller gains using method 4-- Discrete-time

	H_∞	Very Strict Passivity
Controller Gains	[0.4612 -0.5469]	[0.4612 -0.5469]

Using the values for P_c^{-1} and K, the observer gains, L is calculated by solving the second set of LMIs (3.44), (4.40), (4.41), and (4.43) for P_o and Y_o.

Table 4.17: Observer gains using method 4-- Discrete-time

	H_∞	Very Strict Passivity
Observer Gains	$[1.5127 \quad 2.0497^T$	$[1.5127 \quad 2.0497]^T$

Note that the gains are the same. This occurs because the design process itself only taking delta into account, which is equal to 1 for both design objectives. Therefore, they have the same gains and will therefore have the same pole-zero and time response plots. The eigenvalues for the linear component are placed within the prescribed circular regions, as shown in Figure 4.23.

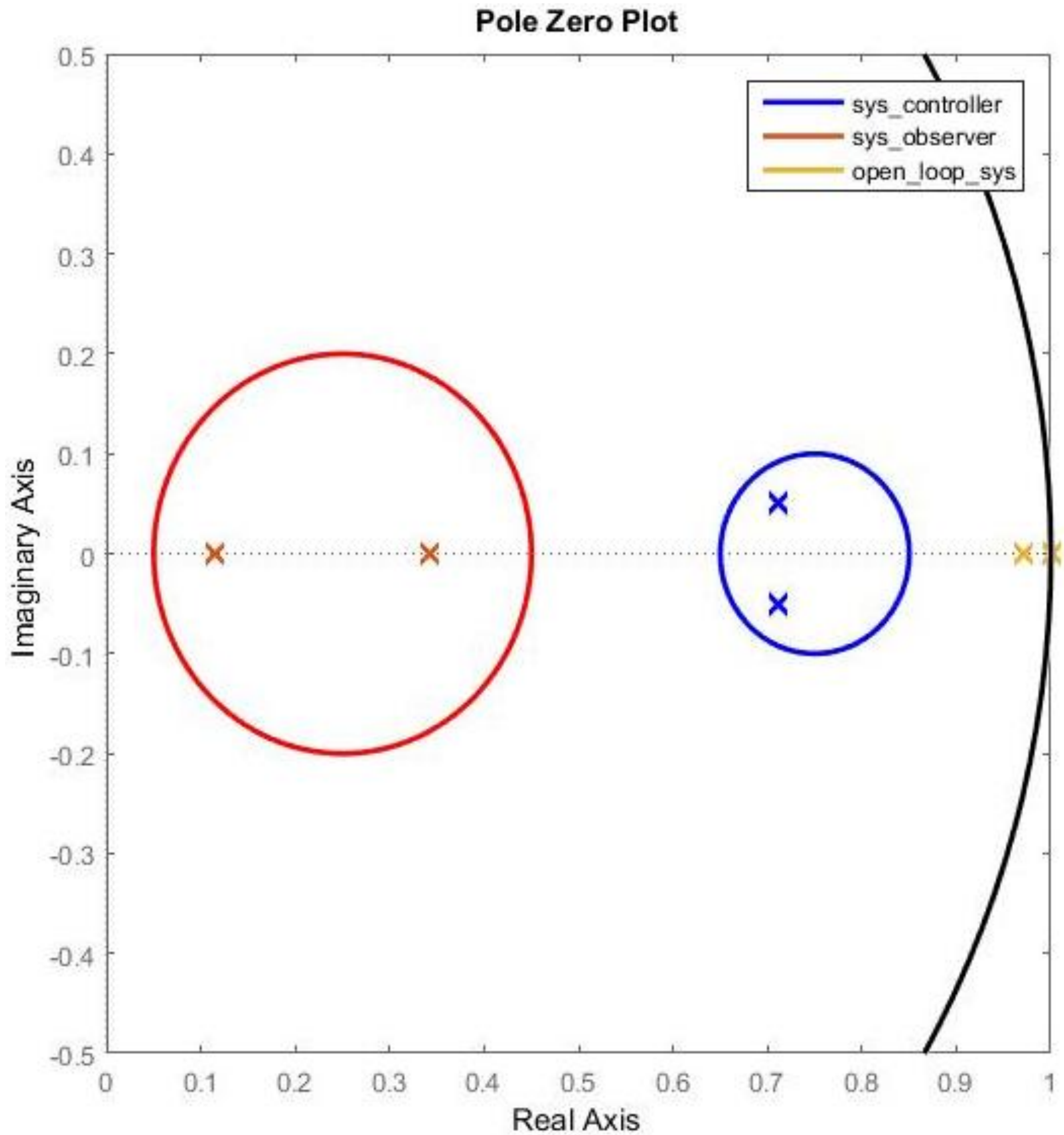

Figure 4.23: Pole-Zero map for method 4--Discrete-time

The third LMI (4.25) is feasible for both the H_∞ controller and the very strict passivity controller for the found controller and observer gains. This means that the closed loop system meets the desired performance criteria. The closed loop time response is shown in Figure 4.24.

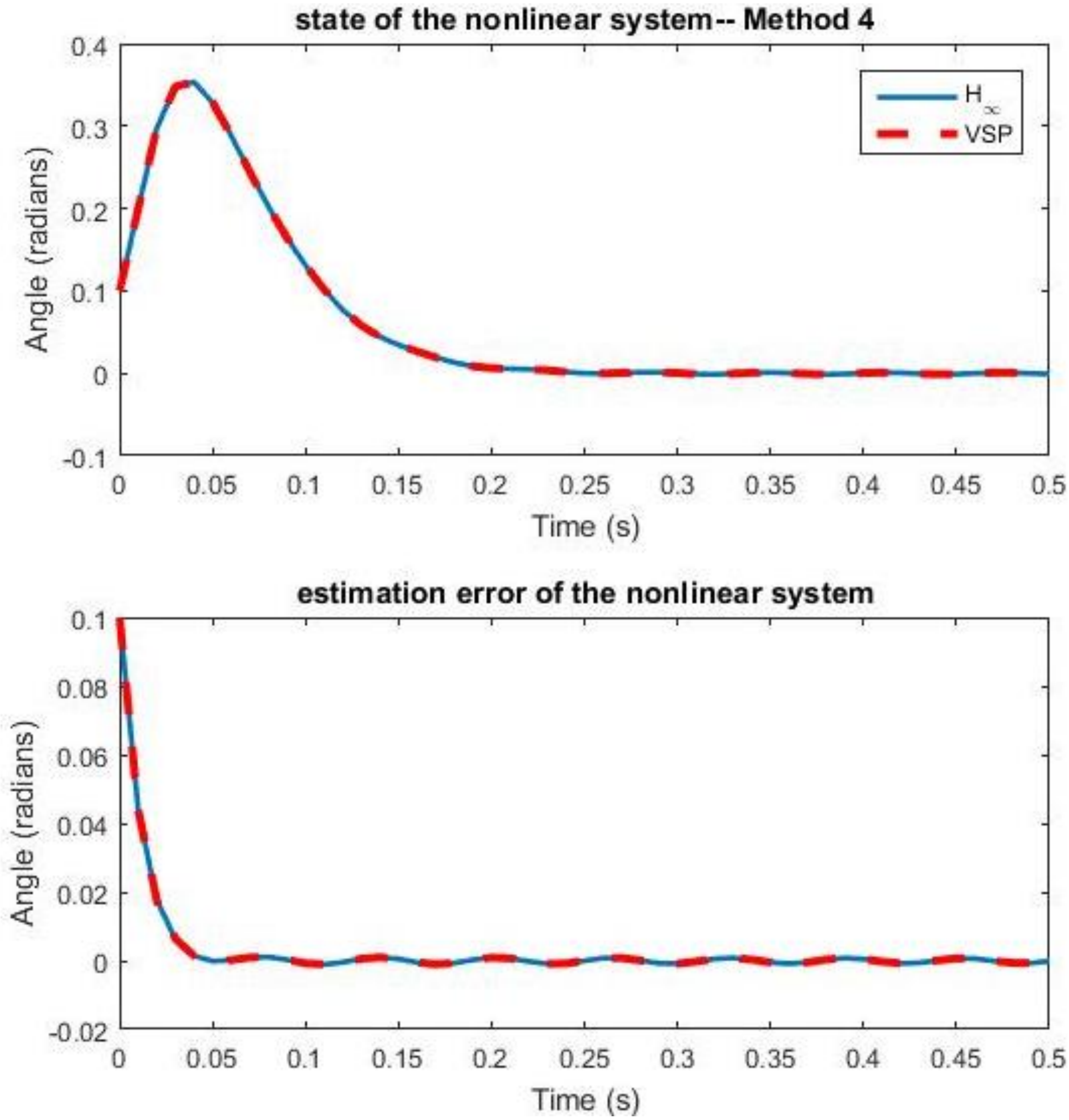

Figure 4.24: Method 4 time response plot-- Discrete-time

The time response of the controller designed using method 4, seen in Figure 4.24, is very like the time response from the controller designed using method 3, seen in Figure 4.22. The noise is still influencing the estimation error. However, because the gains satisfy the main result matrix inequality and the verification via the time response plot, it is concluded that method 4, like methods 1 and 3, work.

Three of the four methods demonstrated show that given a feasible solution, the controller can be designed to the desired specifications. In the next section, the differences will be explored in more detail.

4.2.6 Comparison

Just as in the continuous-time case, the first metric of comparison will examine the maximum Lipschitz constant. Table 4.5 shows that Method 4 has the highest values of α and β that produce feasible results. This matches the result in continuous-time.

Table 4.18:Comparison of maximum alpha and beta values between the 4 methods--Discrete-time

	Maximum α H_∞	**Maximum α VSP**	**Maximum β H_∞**	**Maximum β VSP**
Method 1	1.4189×10^{-12}	1.0049×10^{-12}	19.680×10^{-4}	11.939×10^{-4}
Method 3	0.4714×10^{-4}	0.4698×10^{-4}	20.660×10^{-4}	20.545×10^{-4}
Method 4	3.1687×10^{-4}	3.1687×10^{-4}	188.44×10^{-4}	189.24×10^{-4}

From Table 4.18, it is uniformly seen that the smallest maximum Lipschitz bounds are found using Method 1 while the largest are found using Method 4. This shows that the most conservative method that works is Method 1. In Figure 4.25, an examination of the pole zero map for an H_∞ controller shows the relative pole locations within the regions the eigenvalues are placed.

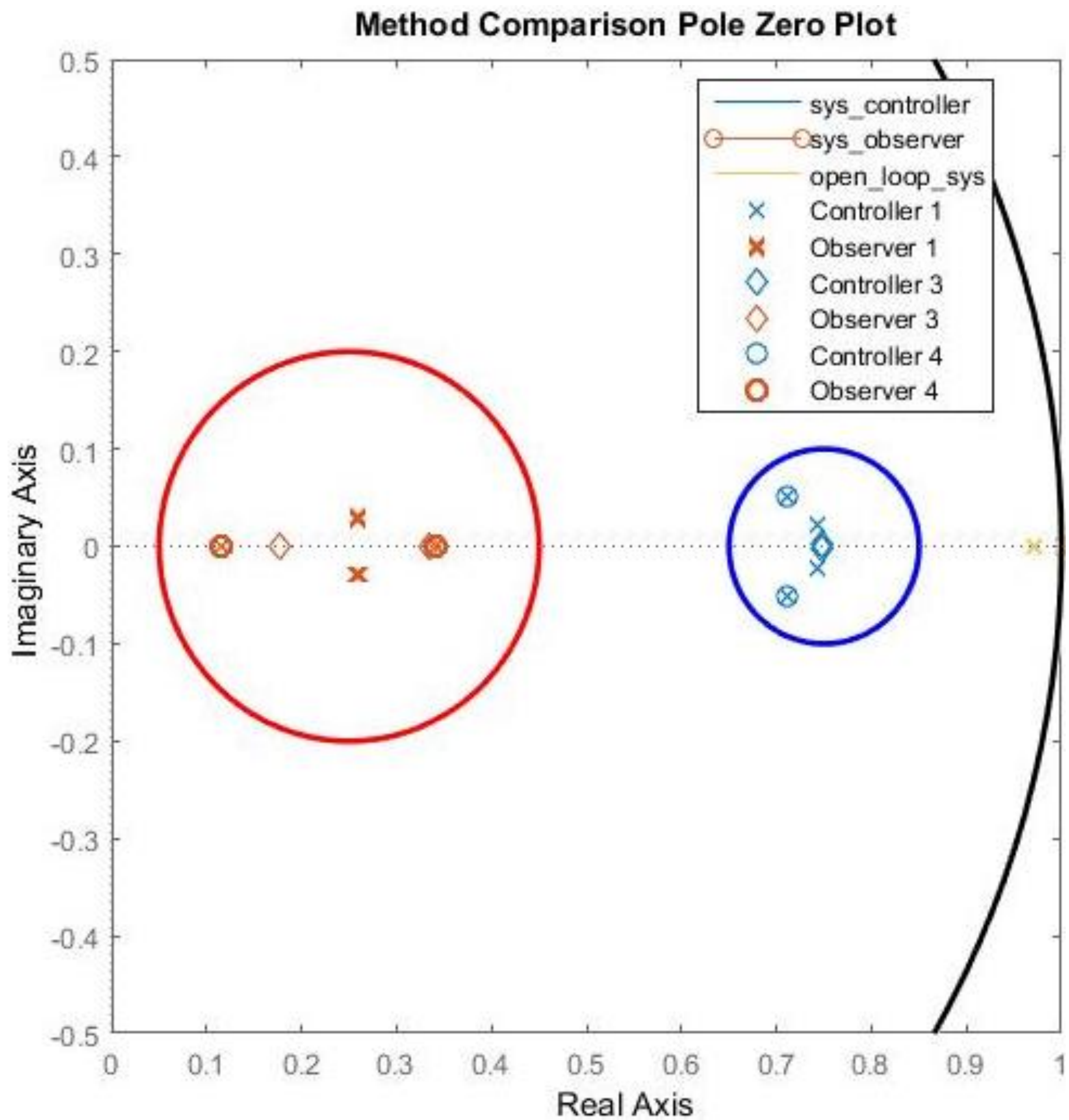

Figure 4.25: Pole zero comparison-- Discrete-time

Unlike in continuous-time, the eigenvalues of the observer region are observed to have imaginary components. There also appears to be less overlap in terms of the eigenvalues within the controller region. It should be noted that unlike in continuous-time, method 1 and method 4 do not have the same eigenvalue locations for the controller despite the similarity in design methods. Like in continuous-time, Method 3 places the eigenvalues closer together than the other methods.

The observer region shows that the eigenvalues calculated using method 1 are complex while the other two methods produced real eigenvalues. Unlike in continuous-time, method 4 places an eigenvalue closest to the edge of the region and the eigenvalues found using method 1 are closer to the center.

These discrete-time results show that the main result can be used to achieve design goals using three of the four methods that were used successfully for continuous-time. The design method is an important factor to consider when implementing this design procedure. Some methods are computationally easier and provide a broader range of feasible solutions. Other methods are computationally more intense and provide a narrower range of feasible results. But how does this controller compare to a linear controller for the same model with the same parameters?

4.3 Linear vs. Nonlinear

In continuous- and discrete-time, the differences in the norms of the states between the linear system and the nonlinear system are small. This is due to the small magnitudes of the system response, the relatively large size of the noise, and the small value for α. More analysis of this controller design procedure with different systems will need to be done in future work.

4.4 Discussion

In this chapter, four methods of applying the main results from Chapter 3 are demonstrated. Method 2 is the least workable method due to the difficulty of obtaining a feasible solution in Matlab. This method is included in this dissertation because in the future, LMI software will improve to better calculate feasible solutions and it is the author's opinion that this method will be useful for control systems designers.

In contrast, the fourth method discussed in this chapter was the easiest method for which a feasible solution was found. This is due to the design being based solely on REA and the necessary conditions of the main results from Chapter 3. These relatively lax conditions allowed the LMI solver to calculate larger Lipschitz bounds on the nonlinearity. The downside of this method is that the performance criteria are treated as an afterthought. This may be what leads to a slower convergence of the system. Method 4 is best used as an initial design to test for feasibility. If method 4 fails to obtain a feasible solution, it is very likely that the other methods will also not be able to find a feasible solution.

Method 3 and Method 1 both worked well for the given system and performance parameters. While Method 3 found a higher Lipschitz constant, Method 1 did a better job accommodating the noise, as evidenced by the lack of oscillation in the estimation error.

Overall, apart from method 2, all three methods succeeded in designing a state-estimate feedback controller that achieved a desired performance and accommodated the nonlinearity in the system. The successful application of this dissertation's main results, using multiple methods, show the viability and flexibility of this design technique.

Chapter 5

Conclusion and Future Work

In this dissertation, a dynamic feedback controller design procedure is proposed. The goal of this dissertation is to develop a design procedure which uses LMI constraints, the GPC, and REA to estimate and control certain types of nonlinear systems. This goal was achieved in this dissertation.

In Chapter 2, there was an in-depth look at the previous work in the various areas of control theory that contribute to the controller design procedure introduced in this dissertation. The various performance criteria that are encompassed by the GPC framework briefly examined in both continuous-time and discrete-time. The GPC was then applied to the design of observers for state estimation and controllers for stabilization using LMI techniques. The LMI derivation for REA was used to demonstrate how a combination of LMI techniques and REA could be used to design observers and controllers for linear systems. Chapter 2 concluded by examining the method of using the Lipschitz property to bound certain types of nonlinearities. This allowed for LMI techniques to be applied to the analysis and design of controllers for nonlinear systems.

In Chapter 3, the main theorems were derived in both continuous-time and discrete-time. The matrix inequality conditions necessary to design a linear dynamic feedback controller which satisfied the GPC was proven first. The theorem was then expanded to include Lipschitz nonlinearities in the system model for the design of observers, controllers, and dynamic state-feedback controllers. The REA constraints were added to the design of the observers and controllers. In this dissertation, the main result was the application of the REA constraints to the system of matrix inequalities that, when satisfied, designed a GPC dynamic feedback controllers for systems with Lipschitz nonlinearities.

Chapter 4 applied the main results derived in Chapter 3 to a simple real world system, the simple inverted pendulum, in continuous-time and discrete-time. By using any one of four methods, the matrix inequalities were manipulated to design the controller first. Once the controller gains had been calculated, the observer was designed. When the design process was complete, the closed loop system would exhibit the desired performance in the presence of the nonlinearity and noise. Chapter 4 also compared the different methods since all the methods gave slightly different designs.

The design technique is a combination of previously established mathematical and control design techniques. Using the LMI constraints allows for flexibility in the design procedure; this is illustrated using 4 methods to design the dynamic feedback controller. The GPC provides flexibility in terms of the performance criteria, as well as building in extra noise accommodation. The REA is used to guarantee that the state estimation goes to zero faster than the state itself. Using the properties of Lipschitz nonlinearities, bounds on the nonlinearity are incorporated into the LMI formulation to accommodate the nonlinearity in the system. By bringing these various techniques together in a single design procedure, an innovative approach to control design has been conceived and tested. Furthermore, this design technique works in both continuous-time and discrete time.

The design technique is not without limitations. The biggest limiting factor is being able to find the feasible solution to the main results of this dissertation. The LMI software algorithm is not always able to find a strictly feasible result. This limits the effectiveness of this technique for more complex systems. The simulated maximum Lipschitz bound is conservative for both continuous-time and discrete-time systems. A small Lipschitz bound limits the potential use of this design technique on real world systems. These computational shortcomings are outside the purview of this dissertation. However, new LMI software is being developed and used. As a direct extension of this work, testing out other LMI solvers on a different system may provide additional insight into the problem.

This research can be further expanded and explored in many other ways. While this dissertation based the dynamic feedback controller design on a full order observer, the work could be expanded to address systems where a few states are unknown and the rest are known using a reduced order observer. A reduced-order observer would reduce the computational cost to the controller while providing a more accurate estimate of the state.

This research also used the GPC to guarantee a single performance criterion is achieved. For future work, mixed-criteria design can be explored. The simplest way of implementing a mixed-criteria design would be to incorporate additional GPC-based LMI constraints [33]. However, this could make the design process more computationally complex than it needs to be. Other methods of addressing the mixed criteria objective should also be investigated.

Other areas of potential future work include:

- Use of this design procedure for stochastic systems
- Incorporating the resilience property into the design
- Analysis of how the controller design affects the observer design/ GPC constraint
- Exploration of additional methods of using the main result
- Optimal Control via LMI optimization

Future graduate students can explore these areas.

BIBLIOGRAPHY

[1] V. A. Yakubovich, "The method of matrix inequalities in the stability theory of nonlinear control systems. II. Absolute stability in a class of nonlinearities with a condition on the derivative.," *Avtomat. i Telemeh*, vol. 26, p. 577–592, 1965.

[2] V. A. Yakubovich, "The method of matrix inequalities in the theory of stability of non-linear controlled systems. III. Absolute stability of systems with hysteresis non-linearities.," *Avtomat. i Telemeh.,* vol. 26, p. 753–763, 1965.

[3] S. Boyd, L. ElGhaoui, E. Feron and V. Balakrishnan, "Linear Matrix Inequalities In System and Control Theory", Philadelphia: SIAM, 1994.

[4] V. Kučera, "A Tutorial on H2 Control Theory: The Continuous Time Case," in *Polynomial Methods for Control Systems Design*, London, Springer London, 1996, pp. 1-55.

[5] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Transactions on Automatic Control,* Vols. AC-26, no. 2, pp. 301-320, 1981.

[6] D. C. Youla, L. J. Castriota and H. J. Carlin, "Bounded real scattering matrices and the foundations of linear passive network theory,," *IRE Transactions on Circuit Theory,* vol. 6, no. 1, pp. 102-124, 1959.

[7] R. W. Newcomb, "On causality, passivity and single-valuedness," *IRE Transaction on Circuit Theory,* vol. 9, no. 1, pp. 87-89, 1962.

[8] R. Nedunuri, "On the Definition of Passivity," *IEEE Transactions on Circuit Theory,* vol. 19, no. 1, p. 72, 1972.

[9] J. L. Wyatt Jr., L. O. Chua, J. W. Gannett, I. C. Goknar and D. N. Green, "Energy Concepts in the State-space Theory of Nonlinear n-Ports: Part I--Passivity," *IEEE Transactions On Circuits and Systems*, Vols. CAS-28, no. 1, pp. 48-61, 1981.

[10] K. Furuta and S. B. Kim, "Pole Assignment in a Specified Disk," *IEEE Transactions on Automatic Control,* Vols. AC-32, no. 5, pp. 423-427, 1987.

[11] B. Ling, "State-feedback Regional Pole Placement Via LMI Optimization," in *Proceedings of the Amercan Control Conference,* Arlington, pp. 3475-3480, 2001.

[12] B. S. Kim, H. S. Han and J. G. Lee, "Pole Placement of Uncertain Discrete Systems in the Smallest Disk by State Feedback," in *Proceedings of the 35th Conference on Decision and Control*, Kobe, Japan, pp. 4558 - 4563, 1996.

[13] M. Chilali, P. Gahinet and P. Apkarian, "Robust pole placement in LMI regions," *IEEE Transactions on Automatic Control*, vol. 44, no. 1, pp. 2257-2270, 1999.

[14] S. Chaib, D. Boutat and A. Benali, "Hysteresis Switched Observer with Regional Pole Placements for General Manipulation Systems," in *Proceedings of the 2006 IEEE International Conference on Control applications*, Munich, pp. 1487 - 1492, 2006.

[15] D. Siljak and D.M.Stipanovic, "Robust Stabilization of Nonlinear Systems: The LMI Approach," *Mathematical Problems in Engineering*, vol. 6, pp. 461-493, 2000.

[16] D.M.Stipanovic and D.D.Siljak, "Robust Stability and Stabilization of Discrete-time Non-linear Systems: the LMI Approach," *International Journal of Control*, vol. 74, no. 9, pp. 873-879, 2001.

[17] W. Haddad and D. Bernstein, "Controller Design with Regional Pole Constraints," *IEEE Transactions on Automatic Control*, vol. 37, no. 1, pp. 54-69, 1992.

[18] G. Garcia and J. Bernussou, "Pole Assignment for Uncertain Systems in a Specified Disk by State Feedback," *IEEE Transactions on Automatic Control*, vol. 40, no. 1, pp. 184-190, 1995.

[19] J. Xu and M. Li, "Robust Control of Norm-bounded uncertain linear system with Regional Pole Assignment," in *Proceedings of International Conference on Mechatronics and Automation*, Changchun, pp. 772- 777, 2009.

[20] C. S. Jeong, E. E. Yaz and Y. I. Yaz, "Resilient design of discrete-time observers with general criteria using LMIs," in *Proceedings of 2003 IEEE Conference on Control Applications, 2003.*, Istanbul, pp. 1416 - 1419, 2003.

[21] C. S. Jeong, E. E. Yaz, A. Bahakeem and Y. I. Yaz, "Resilient design of observers with general criteria using LMIs," in *Proceedings of the American Control Conference*, Minneapolis, pp. 111-116, 2006.

[22] E. E. Yaz, C. S. Jeong, A. Bahakeem and Y. I. Yaz, "Nonlinear Observer Design with General Criteria," in *Proceedings of the American Control Conference*, Portland, pp. 3636 - 3640, 2005.

[23] E. E. Yaz, C. S. Jeong and A. Bahakeem, "Discrete-time nonlinear observer design with general criteria," in *Proceedings of the Electro/Information Technology Conference,* Milwaukee, pp. 281 - 285, 2004.

[24] C. S. Jeong, F. Feng, E. E. Yaz and Y. I. Yaz, "Robust and resilient optimal controller design for a class of nonlinear system with general criteria," in *Proceedings of the American Control Conference*, Baltimore, pp. 6363 - 6368, 2010.

[25] F. Feng, C. S. Jeong, E. E. Yaz, S. C. Schneider and Y. I. Yaz, "Robust controller design with general criteria for uncertain conic nonlinear systems with disturbances," in *Proceedings of the America Control Conference*, Washington DC, pp. 5869 - 5874, 2013.

[26] F. Feng, S. C. Schneider and E. E. Yaz, "Performance analysis of resilient dynamic feedback H_2 controllers for discrete-time systems," in *Proceedings of the European Control Conference*, Linz, pp. 2003 - 2008, 2015.

[27] F. Feng, W. A. Baker Jr., E. E. Yaz and S. C. Schneider, "Design of Resilient General Criteria controllers for Nonlinear Systems with Uncertainties," in *Proc. of the IASTED International Conference on Intelligent Systems and Control*, Marina del Rey, pp. 238-245, 2013.

[28] J. Schur, "Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind.," *Journal für die reine und angewandte Mathematik*, vol. 147, no. 1, pp. 205-232, 1917.

[29] G.-R. Duan and H.-H. Yu, LMIs in Control Systems, Boca Raton: CRC Press, 2013.

[30] V. A. Yakubovich, "The s-procedure in nonlinear control theory," *Vestnik Leningrad Univ. Math.*, vol. 4, pp. 73-93, 1977.

[31] W. A. Baker , S. C. Schneider and E. E. Yaz, "Robust regional eigenvalue assignment by dynamic state-feedback control for nonlinear continuous-time systems," in *Proceedings of the IEEE Conference on Decision and Control*, Osaka, pp. 6903-6908, 2015.

[32] W. A. Baker, S. C. Schneider and E. E. Yaz, "Robust H_∞ Dynamic State-Feedback Control for Nonlinear Discrete-Time Systems via LMI-Based Regional Eigenvalue Assignment," in *Proceedings of the ASME Dynamic Systems and Control Conference*, Minneapolis, 2016.

[33] X. Wang, E. E. Yaz and C. S. Jeong, "Robust nonlinear feedback control of discrete-time nonlinear systems with mixed performance criteria," in *Proceedings of the American Control Conference*, Baltimore, pp. 6357 - 6362, 2010.

[34] R. W. Newcomb, Linear Multiport Synthesis, McGraw-Hill, 1966.

[35] L. Knockaert, "A note on strict passivity," *Systems and Control Letters*, vol. 54, no. 9, pp. 865-869, 2005.

[36] V. Balakrishnan and R. L. Kashyap, "Robust stability and performance analysis of uncertain systems using linear matrix inequalities," *J. Optim. Theory Appl*, vol. 100, no. 3, pp. 457-478, 1999.

[37] W. Assawinchaichote, "Further results on robust fuzzy dynamic systems with LMI D-stability constraints.," . *Int. J. Appl. Math. Comput. Sci.* , vol. 24, no. 4, p. 785–794, 2014.

[38] Z. M. Fu, Q. T. Wu and S. M. Fei, "H_∞ dynamic output feedback stabilization for a class of switched linear systems: the LMIs approach.," *Math. Practice Theory,* vol. 38, no. 2, pp. 70-77, 2008.

[39] W. N. Zhou, H. Y. Su and J. Chu, "Condition of reliable D-stability for a class of dynamic systems—an LMI approach.," *Acta Math. Sci. Ser. A Chin. Ed.* , vol. 28, no. 1, pp. 144-154, 2008.

[40] S. K. Nguang and P. Shi, "Robust H_∞ output feedback control design for fuzzy dynamic systems with quadratic D stability constraints: an LMI approach. ," *Inform. Sci.,* vol. 176, no. 15, pp. 2161-2191, 2006.

[41] B. Song and J. K. Hedrick, "Observer-based dynamic surface control for a class of nonlinear systems: an LMI approach," *IEEE Trans. Automat.* , vol. 49, no. 11, pp. 1995-2001, 2004.

[42] S. Gupta, "Robust stability analysis using LMIs: beyond small gain and passivity. Linear matrix inequalities in control theory and applications. ," *Internat. J. Robust Nonlinear Control* , vol. 6, no. 9-10, pp. 953-968, 1996.

[43] S. Sayyaddelshad and T. Gustafsson, " H_∞ observer design for uncertain nonlinear discrete-time systems with time-delay: LMI optimization approach. ," *Internat. J. Robust Nonlinear Control* , vol. 25, no. 10, pp. 1514-1527, 2015.

[44] A. Trofino and T. J. M. Dezuo, "LMI stability conditions for uncertain rational nonlinear systems. ," *Internat. J. Robust Nonlinear Control,* vol. 24, no. 18, pp. 3124-3169, 2014.

[45] B. Grandvallet, A. Zemouche, H. Souley-Ali and M. Boutayeb, "New LMI condition for observer-based H_∞ stabilization of a class of nonlinear discrete-time systems. ," *SIAM J. Control Optim.,* vol. 51, no. 1, pp. 784-800, 2013.

[46] A. Zemouche and M. Boutayeb, "On LMI conditions to design observers for Lipschitz nonlinear systems," *Automatica J. IFAC* , vol. 49, no. 2, pp. 585-591, 2013.

[47] S. Sathananthan, M. J. Knap, A. Strong and L. H. Keel, "Robust stability and stabilization of a class ofnonlinear discrete time stochastic systems: an LMI approach. ," *Appl. Math. Comput.,* vol. 219, no. 4, pp. 1988-1997, 2012.

[48] M. Rajchakit and G. Rajchakit, "LMI approach to robust stability and stabilization of nonlinear uncertain discrete-time systems with convex polytopic uncertainties.," *Adv. Difference Equ.,* p. 14, 2012.

[49] A. Zemouche and M. Boutayeb, "Observer synthesis method for Lipschitz nonlinear discrete-time systems with time-delay: An LMI approach. ," *Appl. Math. Comput.*, vol. 218, no. 2, pp. 419-429, 2011.

[50] K. Mukdasai and P. Niamsup, "An LMI approach to stability for linear time-varying system withnonlinear perturbation on time scales.," *Abstr. Appl. Anal.*, vol. 2011, no. 1, p. 15, 2011.

[51] M. Abbaszadeh and H. J. Marquez, "Dynamical robust H_∞ filtering for nonlinear uncertain systems: an LMI approach," *J. Franklin Inst*, vol. 347, no. 7, pp. 1227-1241, 2010.

[52] D. F. Coutinho, C. E. de Souza, K. A. Barbosa and A. Trofino, "Robust linear H_∞ filter design for a class of uncertain nonlinear systems: an LMI approach.," *SIAM J. Control Optim.*, vol. 48, no. 3, pp. 1452-1472, 2009.

[53] S. Ibrir and S. Diopt, "Novel LMI conditions for observer-based stabilization of Lipschitzian nonlinearsystems and uncertain linear systems in discrete-time. ," *Appl. Math. Comput.*, vol. 206, no. 2, pp. 579-588, 2008.

[54] C.-H. Lien, K.-W. Yu, Y.-F. Lin, Y.-J. Chung and L.-Y. Chung, "Robust reliable H_∞ control for uncertain nonlinear systems via LMI approach.," *Appl. Math. Comput.*, vol. 198, no. 1, pp. 453-462, 2008.

[55] Y. Yang and L. Huang, "An LMI-based criterion for robust absolute stability of uncertain nonlinear systems.," *Acta Automat. Sinica*, vol. 29, no. 6, pp. 932-936, 2003.

[56] C. H. Fang, "Regional eigenvalue-assignment robustness for uncertain systems. (English summary)," *Control Theory Adv. Tech.*, vol. 10, no. 1, pp. 81-87, 1994.

[57] P. J. Moylan, "Implications of Passivity in a Class of Nonlinear Systems," *IEEE Transactions on Automatic Control*, vol. 18, no. 2, pp. 373-381, 1974.

[58] H. Ouyang, "Robust stabilization for a class of uncertain nonlinear systems via an LMI approach.," *Acta Scientiarum Naturalium Universitatis Pekinensis,* vol. 40, no. 5, pp. 722-728, 2004.