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Comments on "The Principal Axes Decomposition of Spatial Stiffness Matrices"

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Abstract—**A significant amount of research has been directed toward developing a more intuitive appreciation of spatial elastic behavior. Results of these analyses have been described in terms of behavior *decompositions* and in terms of behavior *centers*. In a recent paper titled "The Principal Axes Decomposition of Spatial Stiffness Matrices" by Chen, Wang, Lin, and Lai (IEEE Trans. Robot., vol. 31, no. 1, pp. 191-207), a decomposition of spatial stiffness was presented and centers of stiffness and compliance were identified. The results presented in the paper have substantial overlap with previously published results and redefine previously used terms. The objective of this communication is to clarify the contributions of prior work and to standardize the terminology used in describing spatial elastic behavior.**

Index Terms—**Spatial stiffness matrix, force-deflection behavior, center of stiffness/compliance, elastic behavior decomposition.**

I. Introduction

A recent issue of the IEEE Transaction on Robotics contained a paper titled "The Principal Axes Decomposition of Spatial Stiffness Matrices" by Chen, Wang, Lin, and Lai [1]. In it the authors identified three proposed contributions. They were:

1) A new coordinate-invariant decomposition of spatial stiffness matrices;
2) New definitions of the center of stiffness and center of compliance (including a proof of the coincidence of the two locations);
3) A physical appreciation of the inherent structure of spatial elastic behavior.

Interestingly, the work presented in the paper has substantial overlap with previously published results. Most of the core concepts in [1] were developed in work published more than 20 years ago. The objective of this communication is to clarify the contributions of prior work and to standardize the terminology used in describing spatial elastic behavior.

A. Technical Context

For linear spatial elastic behavior, the force-deflection relationship is characterized by $\mathbf{w} = \mathbf{K}\mathbf{t}$, where $\mathbf{t}$ is the body deflection twist (a 6-vector), $\mathbf{w}$ is the applied wrench (a 6-vector), and $\mathbf{K}$ is the stiffness matrix (a 6×6 symmetric PSD matrix).

For consistency with most previous work (but not [1]), the wrench $\mathbf{w} = [\mathbf{f},\ \boldsymbol{\tau}]^T$ is expressed here in Plücker's ray coordinates, the twist $\mathbf{t} = [\boldsymbol{\delta},\ \boldsymbol{\gamma}]^T$ is expressed in Plücker's axis coordinates, and the 6×6 stiffness is partitioned into 3×3 blocks in the form:

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{bmatrix}. \tag{1}$$

B. Overview

In this communication, we clarify the contributions of prior work related to that presented in [1] in order to avoid potential confusion and misunderstanding of concepts related to general spatial elastic behavior. Specifically, existing coordinate-invariant decompositions and existing concepts of elastic behavior centers are reviewed and compared to the content in [1]. Finally, elastic behavior centers are interpreted in terms of unique physical realizations.

II. The Decomposition of a Linear Elastic Behavior

In the analysis of linear spatial elastic behavior, a decomposition of the stiffness or compliance matrix into a simpler form helps characterize and describe the nature of the behavior. If a stiffness is decomposed into a sum of n rank-1 PSD components:

$$\mathbf{K} = k_1\mathbf{w}_1\mathbf{w}_1^T + k_2\mathbf{w}_2\mathbf{w}_2^T + \cdots + k_n\mathbf{w}_n\mathbf{w}_n^T, \tag{2}$$

then the behavior can be physically realized with n springs (including screw springs and simple springs) connected in parallel [2]. The 6-vector (screw) $\mathbf{w}_i$ is called the *spring wrench*.

The decomposition (2) can be expressed as:

$$\mathbf{K} = \mathbf{W}\mathbf{K}_d\mathbf{W}^T, \tag{3}$$

where $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n] \in \mathbb{R}^{6\times n}$ is the wrench matrix and $\mathbf{K}_d = \mathrm{diag}(k_1, k_2, \cdots, k_n) \in \mathbb{R}^{n\times n}$ is the joint-space stiffness matrix.

Thus, once a stiffness matrix $\mathbf{K}$ is decomposed into the form of (2) or (3), a realization of the behavior with a set of springs is achieved.

By duality [3], a decomposition of a compliance matrix will yield a realization of the behavior with a serial mechanism.

For a given stiffness, the decomposition in (2) or (3) is not unique. There are infinitely many decompositions for the same elastic behavior. Most decomposition methods depend on the coordinate frame selected to describe the elastic behavior. Decompositions that are independent of the coordinate frame are of interest because they characterize the inherent properties of the behavior. Below, we review some coordinate-invariant decompositions of elastic behaviors, then compare these with the decomposition presented in [1].

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A. The Eigenwrench-Eigentwist Decomposition

Dimentberg [4] first discussed the *screw axis* of an elastic behavior, in which a wrench along the axis yields a pure translational (linear) deflection parallel to the direction of the axis. This screw axis was defined as a *wrench-compliant axis* by Patterson and Lipkin [5]. In [5], they described the properties of a wrench-compliant axis and its dual, the *twist-compliant axis* (in which a twist deflection along the axis yields a pure couple parallel to the direction of the axis). The wrench- and twist-compliant axes can be obtained by solving the following eigenvalue problems respectively:

$$\mathbf{K}\boldsymbol{\Gamma}\mathbf{w}_f = k_f\mathbf{w}_f \tag{4}$$

$$\boldsymbol{\Delta}\mathbf{C}\boldsymbol{\Delta}\boldsymbol{\Gamma}\mathbf{t}_\gamma = c_\gamma\mathbf{t}_\gamma \tag{5}$$

where the 6×6 matrix $\boldsymbol{\Gamma}$ is defined as:

$$\boldsymbol{\Gamma} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{6}$$

The screws representing the wrench- and twist-compliant axes were defined in [6] as *eigenwrenches* and *eigentwists* respectively.

It can be seen that the eigenvalue problem (7) depends solely on the first three columns of the stiffness matrix $\mathbf{K}$. Thus, the eigenwrenches $\mathbf{w}_{fi}$ and the corresponding translational stiffness k_{fi} are determined by the two 3×3 block matrices $\mathbf{A}$ and $\mathbf{B}$ of $\mathbf{K}$ in (1). Similarly, the eigenvalue problem (8) depends solely on the last three columns of the compliance matrix $\mathbf{C}$.

It was also shown in [5] that:

1) every full-rank elastic behavior has three orthogonal wrench-compliant axes (eigenwrenches) and three orthogonal twist-compliant axes (eigentwists);
2) the wrench-compliant axes and twist-compliant axes are reciprocal.

Based on these properties, Lipkin and Patterson [6] developed a decomposition of stiffness in the form of (3):

$$\mathbf{K} = [\mathbf{W}_f, \mathbf{W}_\gamma] \begin{bmatrix} \mathbf{K}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_\gamma \end{bmatrix} [\mathbf{W}_f, \mathbf{W}_\gamma]^T, \tag{7}$$

where $\mathbf{W}_f = [\mathbf{w}_{f1}, \mathbf{w}_{f2}, \mathbf{w}_{f3}] \in \mathbb{R}^{6\times3}$ and each $\mathbf{w}_{fi}$ is an eigenwrench; $\mathbf{W}_\gamma = [\mathbf{w}_{\gamma1}, \mathbf{w}_{\gamma2}, \mathbf{w}_{\gamma3}] \in \mathbb{R}^{6\times3}$ and each $\mathbf{w}_{\gamma i} = [\mathbf{0}, \ \boldsymbol{\tau}_{\gamma i}]^T$ is a pure couple in the direction of an eigentwist; and where both $\mathbf{K}_f$ and $\mathbf{K}_\gamma$ are diagonal: $\mathbf{K}_f = \mathrm{diag}(k_{f1}, k_{f2}, k_{f3})$ and $\mathbf{K}_\gamma = \mathrm{diag}(k_{\gamma1}, k_{\gamma2}, k_{\gamma3})$. Thus, (10) can be written as:

$$\begin{aligned}\mathbf{K} &= k_{f1}\mathbf{w}_{f1}\mathbf{w}_{f1}^T + k_{f2}\mathbf{w}_{f2}\mathbf{w}_{f2}^T + k_{f3}\mathbf{w}_{f3}\mathbf{w}_{f3}^T \\ &\quad + k_{\gamma1}\mathbf{w}_{\gamma1}\mathbf{w}_{\gamma1}^T + k_{\gamma2}\mathbf{w}_{\gamma2}\mathbf{w}_{\gamma2}^T + k_{\gamma3}\mathbf{w}_{\gamma3}\mathbf{w}_{\gamma3}^T.\end{aligned} \tag{8}$$

In [6], Lipkin and Patterson did not provide a physical interpretation of this decomposition. However, in [2], Huang and Schimmels showed that any rank-1 PSD component can be realized with a screw spring or a simple spring. Thus, decomposition (11) indicates that $\mathbf{K}$ is realized with three *screw springs* along the three eigenwrenches (wrench-compliant axes) and three *torsional springs* in the directions of the three eigentwists (twist-compliant axes). By the properties of wrench- and twist-compliant axes, the three screw springs and the three torsional springs are each orthogonal. Since the wrench- and twist-compliant axes are coordinate invariant, the decomposition (10) or (11) is unique for the generic case.

B. The Decomposition Described in [1]

In [1], a different decomposition of stiffness is described. First, the stiffness matrix $\mathbf{K}$ is decomposed into two rank-3 components:

$$\begin{aligned}\mathbf{K} &= \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \end{bmatrix} \\ &= \mathbf{K}_S + \mathbf{K}_T.\end{aligned} \tag{9}$$

Then, $\mathbf{K}_S$ is further decomposed into the sum of rank-1 PSD matrices based on the wrench-compliant axes; and $\mathbf{K}_T$ is further decomposed into the sum of rank-1 matrices based on an eigenvalue decomposition. The three components of $\mathbf{K}_S$ are realized with three screw springs associated with the three wrench-compliant axes; and the three components of $\mathbf{K}_T$ are realized with three orthogonal torsional springs.

In the decomposition process, the core step is the decomposition (12) (presented in [1] as equation (13)). The decomposition in (12), however, was previously developed and originally presented in [7] for full-rank stiffness and then in [8] for arbitrary PSD stiffness in order to realize an elastic behavior with a set of concurrent springs. Additionally, the decomposition properties in [1] (described as Lemmas 1-3) were originally provided in [7] and [8]. Also the rank-1 decomposition defined as the "principal axes decomposition" in [1] (equation (16) in [1]) is the same decomposition as (11) that was developed by Lipkin and Patterson [6].

Finally, the term "principal axes decomposition" used in [1] is not consistent with previous usage. In [9], Ball defined the *principal screws* of a 3-system. These three orthogonal principal screws are associated with the 3-system formed by the three wrench- or twist-compliant axes, but are not along the eigenwrenches or eigentwists.

III. Centers of Elastic Behavior

Based on decomposition (12), [1] defined a "stiffness center" to be the center of the cuboid bounded by the three wrench-compliant axes, and defined a "compliance center" to be the center of the cuboid bounded by the three twist-compliant axes. The authors showed that these two centers are coincident. These terms, however, have already been defined by Loncaric [10] and are associated with different concepts. In addition, the centers of the wrench-compliant axes and twist-compliant axes were already defined by Lipkin and Patterson [11], [12] as the *center of elasticity*. The traditional center of stiffness, center of compliance and the center of elasticity are different locations that are used to provide different characterizations of elastic behavior. Below, the different centers of elastic behavior are clarified.

A. Centers of Stiffness and Compliance

Loncaric [10] defined the *center of stiffness* as the location at which the 3×3 off-diagonal block of the stiffness matrix

($\mathbf{B}$ in (1)) is symmetric. Similarly, the center of compliance is defined as the location at which the 3×3 off-diagonal block of the compliance matrix is symmetric. It was proved that for the generic case, the two centers are unique and non-coincident.

Ciblak and Lipkin [13] investigated the relations between Loncaric's centers and the wrench- and twist-compliant axes. They showed [13] that if $\mathbf{r}_{fi}$ is the perpendicular vector from the center of stiffness P_s to the wrench-compliant axis $\mathbf{w}_{fi}$, $(i = 1, 2, 3)$, then

$$k_{f1}\mathbf{r}_{f1} + k_{f2}\mathbf{r}_{f2} + k_{f3}\mathbf{r}_{f3} = \mathbf{0}, \tag{10}$$

where k_{fi} is the value of translational stiffness (or the eigenvalue of the corresponding eigenvalue problem (7)) associated with wrench-compliant axis $\mathbf{w}_{fi}$.

Similarly, if $\mathbf{r}_{\gamma i}$ is the perpendicular vector from the center of compliance P_c to the twist-compliant axis $\mathbf{t}_{\gamma i}$ $(i = 1, 2, 3)$, then

$$c_{\gamma 1}\mathbf{r}_{\gamma 1} + c_{\gamma 2}\mathbf{r}_{\gamma 2} + c_{\gamma 3}\mathbf{r}_{\gamma 3} = \mathbf{0}, \tag{11}$$

where c_i is the value of rotational compliance (or the eigenvalue of the corresponding eigenvalue problem (8)) associated with twist-compliant axis $\mathbf{t}_{\gamma i}$.

Thus, for the generic case, the center of stiffness can be uniquely determined by the wrench-compliant axes and the corresponding values of translational stiffness k_{fi}. Likewise, the center of compliance can be uniquely determined by the twist-compliant axes and the corresponding values of rotational compliance $c_{\gamma i}$.

B. Centers Defined by Wrench-Compliant Axes and Twist-Compliant Axes

For an elastic behavior, the three wrench-compliant axes (eigenwrenches) form Ball's 3-system S_f [9]. The *principal screws* of S_f are the three screws that are reciprocal, orthogonal and concurrent [9]. The intersection point of the principal screws is the center of the three wrench-compliant axes. For its dual, the three twist-compliant axes define a 3-system S_γ and the center of the principal screws of S_γ can be similarly defined. The two 3-systems S_f and S_γ are reciprocal.

Lipkin and Patterson [11] proved that the two centers associated with the wrench-compliant axes and the twist-compliant axes are coincident. This unique point was defined as the *center of elasticity* [11].

It was proved that:

1) In an arbitrary coordinate frame, the location of the center of elasticity, P_e, can be calculated [11], [13] using

$$\mathbf{r}_e = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3), \tag{12}$$

where $\mathbf{r}_i$ is perpendicular vector from the coordinate origin to the wrench- or twist-compliant axes. If the coordinate frame is at the center of elasticity, then

$$\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = \mathbf{0}. \tag{13}$$

2) If the coordinate frame origin is at P_e, the block matrix $\mathbf{B}^T\mathbf{A}^{-1}$ (or $\mathbf{A}^{-1}\mathbf{B}$) is symmetric [14].

It is obvious from (15) or (16) that the center of elasticity is the center of the cuboid bounded by the three wrench-compliant axes (eigenwrenches) or by the three twist-compliant axes (eigentwists).

C. Centers Incorrectly Redefined in [1]

Bellow, the centers defined by Loncaric [10] and by Lipkin and Patterson [11] are compared with those from [1].

Based on the decomposition in [1], a stiffness is realized with three orthogonal screw springs and three orthogonal torsional springs. The center of the cuboid formed by the three screw springs is incorrectly redefined (relative to that from Loncaric [10]) as the "center of stiffness." Using the same process, performing the decomposition on a compliance matrix yields three orthogonal screw joint twists and three orthogonal prismatic joint twists in a serial mechanism that realizes the behavior. In [1], the center of the cuboid formed by the three screw joint twists is incorrectly redefined as the "center of compliance" also relative to that from Loncaric [10]. Then, it was shown that the two centers are coincident and that the block matrix $\mathbf{B}^T\mathbf{A}^{-1}$ is symmetric when the coordinate frame is at the "center."

Since the three screw-spring wrenches (screw joint twists) in the decomposition (12) are along the three wrench-compliant (twist-compliant) axes, the center defined in [1] is exactly the *center of elasticity* defined by Lipkin and Patterson [11], [12]. The coincidence of the two centers and the symmetry of block matrix $\mathbf{B}^T\mathbf{A}^{-1}$ at the center are known results [6], [11], [14].

D. Clarification and Physical Appreciation of Elastic Behavior Centers

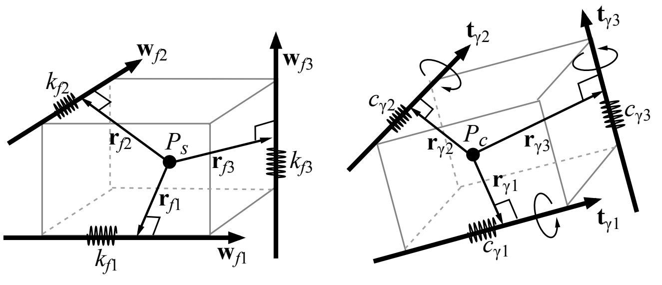
The center of stiffness, the center of compliance, and the center of elasticity are three points used to characterize an elastic behavior. These three distinct points reflect the nature of elastic behavior in different aspects.

Figure 1 shows the geometric relations of the *centers of stiffness and compliance* with respect to the three wrench-compliant axes and twist-compliant axes identified by Ciblak and Lipkin [13]. The center of stiffness can be determined by (13), and the center of compliance can be determined by (14). Since the locations of the wrench- and twist-compliant axes, and the coefficients k_{fi} and $c_{\gamma i}$ in (13) and (14) are not the same, the two centers are not coincident in general.

It can be seen that the center of stiffness depends on both the wrench-compliant axes and the values of the translational stiffness k_{fi}. Thus, changing k_{fi} in (11) changes the location of the center of stiffness.

Figure 2 shows the geometric significance of the *center of elasticity*, which is located at the center of the cuboid bounded by the three eigenwrenches or the three eigentwists. The center is also the intersection point of the three *principal screws* associated with the 3-system defined by the three wrench- or twist-compliant axes [9].

Unlike the centers of stiffness/compliance, the center of elasticity depends solely on the locations of the three wrench- or twist-compliant axes and is independent of the values of



(a) (b)

Fig. 1. The center of stiffness/compliance and the eigenwrenches/eigentwists. (a) At the center of stiffness, $k_{f1}\mathbf{r}_{f1}+k_{f2}\mathbf{r}_{f2}+k_{f3}\mathbf{r}_{f3}=\mathbf{0}$. (b) At the center of compliance, $c_{\gamma 1}\mathbf{r}_{\gamma 1}+c_{\gamma 2}\mathbf{r}_{\gamma 2}+c_{\gamma 3}\mathbf{r}_{\gamma 3}=\mathbf{0}$.

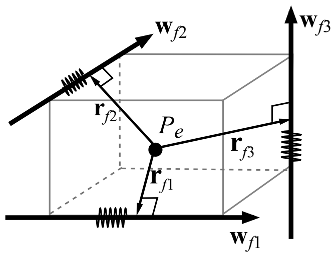


Fig. 2. The center of elasticity defined to be the center of the three wrench- or twist-compliant axes. At the center, $\mathbf{r}_{f1}+\mathbf{r}_{f2}+\mathbf{r}_{f3}=\mathbf{0}$.

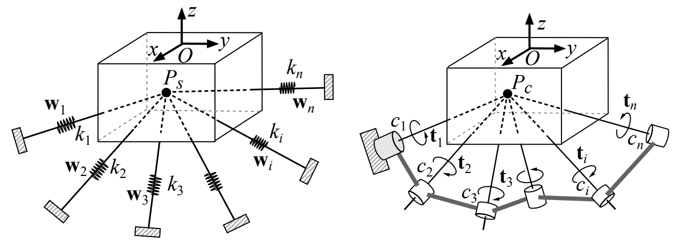
k_{fi} or $c_{\gamma i}$. Thus, changing the value of k_{fi} in (11) will not change the location of the center of elasticity.

Comparing the different types of centers, the center of stiffness by Loncaric [10] depends on both the eigenvalues and eigenvectors of the eigenvalue problem (7); whereas the center of elasticity by Lipkin and Patterson [11] only depends on the eigenvectors of (7). In this sense, the center of stiffness better represents the elastic behavior. For example, if the translational stiffness k_{fi} is increased, then the center of stiffness will move closer to the wrench-compliant axis $\mathbf{w}_{fi}$.

Since the center of elasticity depends only on the two block matrices $\mathbf{A}$ and $\mathbf{B}$ in (1), changing in the lower-right block matrix **D** in the stiffness matrix **K** will not change the location of the center of elasticity. Similarly, changing in the upper-left block matrix in the compliance matrix will not change the location of the center of elasticity.

For Loncaric's centers, although changing the lower-right block matrix of the stiffness matrix does not change the location of the stiffness center, it changes the location of the center of compliance. Similarly, changing the upper-left block matrix of the compliance matrix does not change the location of the compliance center, but changes the location of the center of stiffness. Thus, the non-coincidence of the two centers better describes the nature of an elastic behavior.

Huang and Schimmels [7] provided a different physical appreciation of the centers of stiffness and compliance. It was shown that, if a stiffness matrix is realized with a set of springs having concurrent axes, the intersection point must be the center of stiffness; and that any full-rank stiffness matrix can be realized with a set of springs intersecting at the center of stiffness (Fig. 3a). By duality, it was shown that, if a compliance matrix is realized with a serial mechanism having concurrent joint axes, the intersection point must be the center of compliance; and that any compliance matrix can be realized with a serial mechanism having all joint axes intersecting at the center of compliance (Fig. 3b).



(a) (b)

Fig. 3. A physical appreciation of stiffness center and compliance center. (a) The stiffness center is the intersection of all spring wrenches in a parallel mechanism with concurrent axes. (b) The compliance center is the intersection of all joint twists in a serial mechanism with concurrent axes.

Although for a general elastic behavior the three centers are distinct in space, these three locations are related in some cases. An obvious case is when the translational and rotational components in an elastic behavior can be completely decoupled, i.e., in a coordinate frame, the off-diagonal block **B** in (1) vanishes. For this case, the three centers are coincident.

Ciblak and Lipkin [13] investigated the relations of the three centers when the elastic behavior has at least one *compliant axis*. A compliant axis is defined to be a screw axis for which a force in the direction of the axis produces a parallel translational deformation, and a rotational deformation about the axis produces a parallel couple. It was shown [13] that if one compliant axis exists then all three centers P_s, P_c and P_e must be on the axis; if two compliant axes exist, then all three centers must be coincident. Means of physically realizing elastic behaviors having compliant axes were presented in [15].

IV. Summary

The three proposed contributions of [1] listed in Section I have significant overlap with previously published results. In the development of the coordinate-invariant decomposition in [1], the rank-3 decomposition used the methods presented in [7], [8] and the rank-1 decomposition was presented in [6]. The authors of [1], however, did provide an alternate way to obtain the same results. The proofs of properties of the decomposition provided in the paper confirmed the previously published results contained in [7], [8], [6], [14]. The center "redefined" in [1] is not a new location that better characterizes an elastic behavior, but the "center of elasticity" identified and defined in [11], [12]. The center of stiffness, the center of compliance, and the center of elasticity are different locations in space. The center defined by the wrench- or twist-compliant axes should not replace the existing definitions of centers of stiffness/compliance by Loncaric [10].

In this communication, the contributions of previous works were identified relative to those presented in [1] and clarified to avoid potential confusion. Discussions about the coordinate-invariant decompositions of spatial stiffness, and the centers of an elastic behavior were presented to avoid potential misunderstanding of concepts and misuse of existing terms.

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