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Analysing Student Work Involving Geometric Concepts

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Analysing Student Work Involving Geometric Concepts

Hyunyi Jung reflects on why students struggle to understand trigonometry

When I taught three high school geometry classes, I was unaware that understanding trigonometric ratios involving special right-angled triangles is often a struggle for high school students (Calzada & Scariano, 2006). I was not sure how to help students recognise that if a triangle has angles that are 30, 60, and 90 degrees, the ratio of the corresponding sides is always $1:\sqrt{3}:2$. As a result of my struggles teaching this the first time, I learned that I need to start by analysing my students' mathematical reasoning before jumping into teaching a topic.

Later, while teaching a university methods course for pre-service mathematics teachers, I wondered if college students, who need to know this concept for advanced study, have the same struggle when they use trigonometric ratios to solve mathematics problems. In order to explore how university students understand this concept, I asked a group of them to solve a mathematics problem involving special right-angled triangles with trigonometric ratios, see Figure 1.

The *Interstate Teacher Assessment and Support Consortium* (INTASC) Math standards state that *being able to assess students' understanding is one of the common core teaching skills that all new teachers should have* (INTASC, 1995). I decided to explore the ways students think about mathematical ideas as a way of analysing their thinking (NCATE, 2003).

Through this analysis, I would like to work towards answering the questions,

"What are students' understanding and struggles with trigonometric ratios involving right triangles?"

and

"What are some suggestions for instruction to help developing learners' understanding of this concept?"

The Process of Analysing Student Work

Fifteen students agreed to solve the problem shown in Figure 1. Four students were pre-service mathematics teachers; eleven students were engineering and food science majors at a University in the U.S., the participants drew pictures and discussed their reasoning while solving the problem. After that, I interviewed the participants to ask them to further explain their solutions, and inquired into their experiences learning the concepts.

I decided to analyse the participants' work and interviews according to three categories: **(1)** prior knowledge and connections, **(2)** definitions, and **(3)** generalisation. These categories were chosen because the participants solved the problem by connecting prior knowledge, using definitions, and creating a generalisation of the specific situation using variables. Furthermore, I found related literature (e.g., Common Core State Standards [CCSS], 2010 & Lappan, 1996) to support these strategies. These sources address the importance of students connecting previous knowledge to new ideas, mastering definitions related to the concepts, and finding patterns among various contexts.

What Students Show

Based on the analysis, three cases of student work are shown in Figure 2. Case 1 illustrates the method that four students used to solve the problem. They marked on their triangle that the lengths of the other two sides were 3 inches and 4 inches When I asked a student how she determined her answers, she replied that she used one of the Pythagorean Triples, 3:4:5. Since the length

Minji and Andrew are talking about how to construct a building model using specially designed equipment.

Minji: In order to construct this building model, we can use the special right triangular equipment that has certain angles that have already been set up. If we set two angles of the triangular equipment, can we know the lengths of the sides?

Andrew: Well, since one of the angles of the right triangle is 60 degrees, we can figure out the other angles. Also the longest side of the triangular equipment is 5 in. because the equipment must fit in the building model.

Minji: Ok, I'll try to draw the triangle and figure out the other two sides.

1) Draw the described triangle. Please mark all the given information on the triangle.
2) Using your picture, find the two sides of the triangle. Show all your work, including all your calculations.
3) What if the longest side of the triangle is unknown? How would you express the two other sides of the triangle? (Hint: You can name the unknown value using any variables of your choice.)

Figure 1: The Story Problem

of hypotenuse of the right triangle was 5 inches, she was quite sure that the other two sides were 3 inches and 4 inches Three students used the approach shown in Case 2. The students were confused with setting up a correct equation using the trigonometric ratio. For example, a student wrote the equation cos60° = $^{\text{opposite}}/_{\text{hypotenuse}}$. Five students used the approach shown in Case 3. Students struggled to express the two other sides of the triangle using variable(s) when the longest side of the triangle is unknown. A student used variables *x, y*, and *z*, but did not indicate the relationships among them.

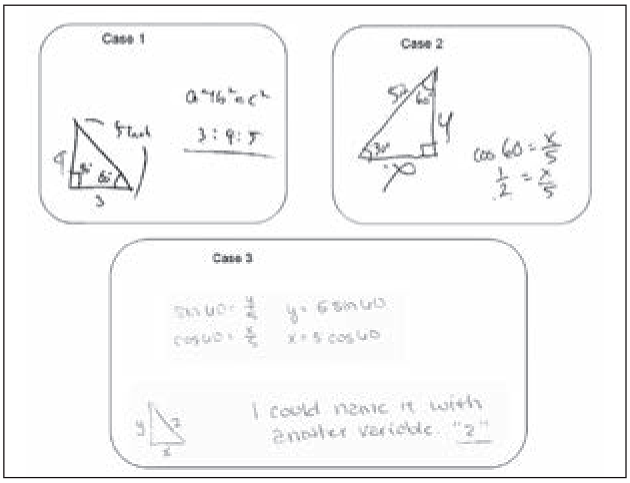

Figure 2: Three cases from student work

Cognitive Challenges That Students Face

These analyses helped me identify cognitive challenges commonly faced by students. The *CCSS* (2010) and the *NCTM Principles and Standards for School Mathematics* (2000) state *the importance of making sense of mathematics, making use of appropriate definitions, connecting related ideas, and finding patterns to generalise concepts.* However, such knowledge was lacking in the student work as described below.

Lacking Prior Knowledge and Connection.

I wondered how the students represented by Case 1 understood prior knowledge of the Pythagorean Triples since that was what they used for solving the problem. The dialogue below shows conversations with Mary. She is a student majoring in food science and said that she has not been confident with mathematics.

IN (Interviewer): *Would you tell me why you used the Pythagorean Triple to solve this problem?*

Mary: *I used it because it's a right triangle and the hypotenuse is 5 inches.*

IN: *I see.. what does the Pythagorean Triples mean?*

Mary: *Well.. Look at this* (she is writing down $a^2 + b^2 = c^2$), *since the hypotenuse is 5 inches, c is 5 and the Pythagorean Triple is 3:4:5, when I plug a = 3, b = 4, it exactly matches with* $3^2 + 4^2 = 5^2$.

IN: *What about the angles?*
Would it be different if the two angles of the triangle are 40 and 90, instead of 60 and 90?

Mary: *Um.. I haven't thought about that.. maybe the sides should be different, but I'm not sure..*

This dialog shows that Mary knew the formula of the Pythagorean Theorem and one of the Pythagorean Triples, 3:4:5. However, she was not sure about the relationship between the lengths of sides and their corresponding angles in the case when the angles are 30, 60, and 90 degrees.

Lacking Conceptual Understanding of Definitions

As shown in Case 2, some students struggled to use the appropriate definition. One of the students, called Eaton, is a pre-service mathematics teacher. He described himself as good at algebra, but not geometry. I asked him,

"How did you learn to solve these kinds of problems in your high school geometry class?"

He said, *"I was taught to memorise formulae to solve such problems."* He continued, *"I am confused with how to use SOH CAH TOA. It's been a while since I memorised it in my high school."*

Another student said that her high school teacher introduced the definition and explained some examples. Then she solved several similar problems for homework and studied for an exam. This instructional strategy is what Mitchelmore (2000) called the ABC method, *where concrete examples are explained after abstract definitions are introduced.*

I recalled that I had introduced the SOH CAH TOA method when I was teaching trigonometric ratios to my high school geometry students. I wondered if my students also had been left with the impression that mathematics was nothing more than memorising rules that lacked meaning, or tedious drills. Would my students be able to justify why they used SOH CAH TOA to solve a problem? Reflecting on my teaching led me to think about how to teach the concept better, which I will discuss later in this article.

Struggling with Generalising Concepts

Generalising has not only been a "*driving force*" of mathematics and science historically, but also a powerful strategy for students to have (Driscoll, DiMatteo, Nikula, & Egan, 2007, p.7). When students needed to generalise an idea, some of the students said that the problem was complicated. An example is shown in Gary's work in Case 3. Gary was a freshman majoring in economics. He said that he panicked when he realised he must figure out two sides of the right triangles given only angles and a side. He expressed the two sides using the given 5 inches. He also said that he could visualise the story problem and solve for the unknown variables *x* and *y*. However, he did not show that sides *x* and *y* could be represented using the longest side *z*.

What Do Students Understand? How Can We Improve Their Understanding?

Analysing students' understanding through observations and interviews helped me reflect on my teaching of trigonometric ratios of special right-angled triangles. This also led me to reconsider the curriculum that I had previously accepted without thinking deeply about their implications. If I had analysed my students' understanding of a concept before I taught it, I would have selected an approach that was appropriate for my students' current knowledge, rather than following my predetermined plan.

When I was teaching, I did not fully understand why students were struggling with the tasks from a so-called "traditional" textbook that I was encouraged to use in my class. After analysing students' understanding, reading several articles, and reflecting on my teaching, I realised that several strategies, such as definitions, explorations, connections, and geometric proofs would be beneficial for students learning (CCSS, 2010; NCTM, 2000). I will describe these strategies and ways to use them to teach special triangles.

Understanding Definitions

Mary understood that Pythagorean Triples are related to the Pythagorean Theorem, which applies to right-angled triangles. However, she did not realise that the Pythagorean Triple only applies to right-angled triangles with specific interior angles. In order to avoid such confusion, students should explore not only the ratio of sides in Pythagorean Triples, but also the angle measures. For example, if students knew that the two interior angles of the right-angled triangle with the side ratio 3:4:5 are always approximately 37° and 53°, they might be less likely to assume the ratio 3:4:5 for all right-angled triangles with a 5 inch-long hypotenuse.

Eaton understood that he could use trigonometric ratios rather than the Pythagorean Triple to solve the problem; however, he did not remember the definition of trigonometric ratios. According to Kendal and Stacey (1997), many students do not know that trigonometry is the study of ratios that represent the lengths of the sides in right-angled triangles. Students should understand that *sine, cosine,* and *tangent* are ratios for a right-angled triangle, which changes depending on the lengths of the sides of the triangles.

Teaching definitions does not mean beginning a lecture with a definition. It can be meaningless for students if they start learning definitions without reasoning deeply about the concepts. One way of introducing definition is to provide visual examples and ask students to differentiate those which fall within a definition and those which do not.

Learning through Exploration.

"Do you think the SOH CAH TOA method was helpful for you to understand the trigonometric ratios?" When I posed this question to the students, some answered that the method was good for rapid recall of what ratios needed to be used. Others said that the method was not very useful for them because it does not help them understand what trigonometric ratios are. According to Cavangh (2008), students ceased trying to learn the meaning behind the rule when they had the simple formula that they could apply.

A suggestion for preventing the ill-advised use of mnemonic is to encourage students to explore the concepts behind the shortcut. Quinlan (2004) suggests that students should explore concrete examples of a mathematical idea, which helps them to develop a sound understanding of the basic concept. For instance, students in groups can draw several triangles with the information that was originally given, such as a right-angled triangle that has an angle to be 60 degrees and the longest side to be 2, 4, and 6 inches. Then they can measure to find the other two unknown sides of each triangle and compare the lengths of the corresponding sides of the similar triangles. Finally, they can be asked to find the relationships between the three sides of each triangle. This kind of exploration helps students make sense of the rules and develops them into successful problem solvers (Driscoll, DiMatteo, Nikula, & Egan, 2007). Another way of helping students have deeper knowledge is to encourage them to understand proofs, a strategy discussed in the following section.

Familiarisation with Geometric Proof

Among all the participants, Paula was the only one to show the effective use of geometric proof to solve the problem, see Figure 3. She divided the right-angled triangle into two triangles (a right-angled triangle and an isosceles triangle), and then found the sides of those two triangles. She said that the small triangle in her figure would be equilateral with a side length of half the hypotenuse. Since she first drew a 60-degree angle at the vertex of the right triangle, it can be proved using the transitive property of equality that this will bisect the hypotenuse. By doing this, she proved that the length of the base is the half of the length of the hypotenuse and the length of the height is $1:\sqrt{3}/2$ of that hypotenuse.

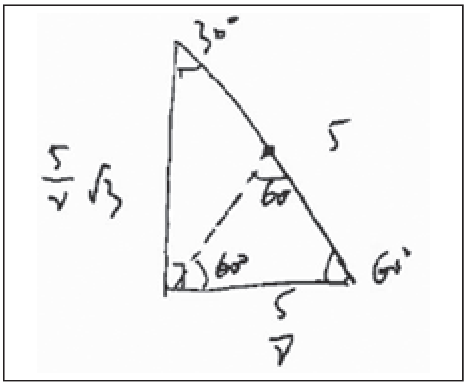

Figure 3: Paula's answer using geometric proof

There are several advantages of using geometric proofs to solve the problem. First, this way of solving the problem does not require students to memorise a complex rule or formulae. If they know the basic concepts, such as the characteristics of an equilateral triangle in terms of sides and angles, they would not need to memorise the ratio

1:√3: 2 without context. Furthermore, students would have the opportunity to recall prior knowledge while they solve the problem using geometric proof. The importance of connecting new concepts with previous knowledge follows in the next section.

Connections between Subjects

Calzada and Scariano (2006) claimed that there should be a smooth transition into trigonometry by introducing related topics from algebra and geometry. Some students struggled when they lacked basic, prior knowledge, such as proportional reasoning and irrational numbers.
On the other hand, Cameron used proportional reasoning to solve the problem as he set the equation, $\sqrt{3}/2 = x/5$, see Figure 4. He also knew how to multiply and divide an irrational number by a rational number. Such knowledge helped him go through the process to arrive at accurate answers.

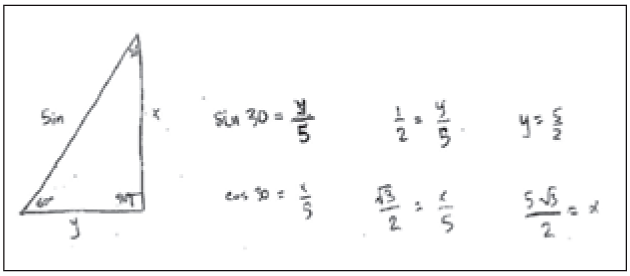

Figure 4: Cameron's answer using proportional reasoning

What if a student does not have prior knowledge? Recognising which basic concepts she or he does not understand would be the first step, and then helping them understand the specific concepts would be next. I recall that I was overwhelmed by the many new topics that I should teach and could not help all the students review the prior concepts they should have already learned. Once I realised that continually introducing new concepts confused and confounded students who had not mastered the basic knowledge, I put more effort into connecting new topics with prior concepts.

One suggestion is to include algebraic contexts when students learn geometry. For instance, a teacher could introduce a right-angled triangle whose lengths of sides are irrational numbers and revisit the topic of irrational numbers. Similarly, students can learn proportional reasoning using several similar geometric figures. Such connection between geometry and algebra would help students perceive the scope of trigonometric ratios.

Conclusion

In this article, I discussed students' understanding and misunderstanding of trigonometric ratios of special right-angled triangles, as well as some ideas for assisting students with understanding concepts related to this topic. Admittedly, there are additional ideas for helping students visualise trigonometric ratios which are not addressed in this article, for example, trigonometric ratios in the context of unit circles or graphing (Thompson, Carlson & Silverman, 2007). In the context of right-angled triangles, it is important for students to connect prior knowledge to new concepts, understand definitions, and generalise mathematical ideas. In order to develop such knowledge, students need to have opportunities to gain better comprehension of definitions, learn through exploration, build knowledge with geometric proofs, and perceive the interconnection among different mathematics topics.

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