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Kings and Heirs: A Characterization of the (2,2)-domination Graphs of Tournaments

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Abstract: In 1980, Maurer coined the phrase king when describing any vertex of a tournament that could reach every other vertex in two or fewer steps. A (2,2)-domination graph of a digraph D, $dom_{2,2}(D)$, has vertex set $V(D)$, the vertices of D, and edge uv whenever u and v each reach all other vertices of D in two or fewer steps. In this special case of the (i,j)-domination graph, we see that Maurer's theorem plays an important role in establishing which vertices form the kings that create some of the edges in $dom_{2,2}(D)$. But of even more interest is that we are able to use the theorem to determine which other vertices, when paired with a king, form an edge in $dom_{2,2}(D)$. These vertices are

referred to as heirs. Using kings and heirs, we are able to completely characterize the (2,2)-domination graphs of tournaments.

Keywords: Tournaments; Domination; Kings

1. Introduction

Domination in digraphs has been the focus of research for decades within a variety of areas in mathematics. The current branch of research has evolved from studying dominance in animal societies in the 1950s, led by mathematical sociologist H. Landau.[6;7;8] Further results involving what would later be called a king in a tournament were supplied by Moon[10] in his monograph. Yet it was Maurer in 1980[9] who coined the phrase *king* in a tournament to refer to any vertex that could beat every other vertex in at most two steps. It is that term we will use throughout this paper to refer to such a vertex, as it describes precisely the dominance we wish to explore.

Here, we are interested in a tournament, T, which is a set of n vertices where there is an arc between every pair of vertices. We say that u beats v, $u \rightarrow v$, if arc (u,v) is in T. The set of players that u beats is the outset of u, $O^{+}(u)$, and the set of players that beat u is the inset of u, $O^{-}(u)$. The distance between vertices u and v, $dist(u,v)$ or $dist_T(u,v)$, is the minimum number of arcs in a directed path from u to v.

The authors previously took the concept of (i,j) dominating sets defined by Hedetniemi et al. in their original works[5;3;4] and created (i,j)-domination graphs.[1;2] Here, we look specifically at the (2,2)-domination graphs of tournaments. Given a digraph D, $G=dom_{2,2}(D)$ is the (2,2)-domination graph of D where $V(G)=V(D)$ with edge uv if vertices u and v can each reach all of the remaining vertices in one or two steps. We call u and v a (2,2)-dominating pair. The definition of $dom_{2,2}(D)$ should bring to mind the definition of a king, as any pair of kings is a (2,2)-dominating pair. However, pairs of kings are not the only vertices to create edges in $dom_{2,2}(D)$.

For simplicity of notation we will write $T-\{x\}$ to mean the induced subtournament obtained when x is removed from the vertex set of T. Consider any (2,2)-dominating pair, $\{u,v\}$, that creates an edge in $dom_{2,2}$(D). Say that u beats v. Since u can reach all vertices, including v in one or two steps, u is a king. We know that v can reach all vertices except possibly u in one or two steps, so v must be a king in $T-\{u\}$. If v is not a king in T, then v cannot reach u in two steps, and consequently v fails to form a (2,2)-dominating pair with any vertex other than u. Call such a vertex an *heir*. In other words, an heir is a vertex who is not a king, but when a particular king is removed, it becomes a king.

Lemma 1.1. *If h is an heir of king k, then h is not an heir of any other king.*

Proof. Suppose h is an heir of k_i and k_j. Then h is a king in $T-\{k_i\}$, so must beat vertex k_j in at most two steps. Thus, h is not an heir of k_j.

In a tournament T, on n vertices with kings labeled $x_1, x_2, \ldots, x_k$ define the royal sequence as follows $[k; h_1, h_2, \ldots, h_k; r]$ with $r = n - k - \sum_{i=1}^{k} h_i$, h_i representing the number of heirs of king x_i, and r representing the number of vertices in T which are neither kings nor heirs. Note that it is not strictly necessary to provide k and r in the sequence but it is convenient to do so. Note also that we may label the kings arbitrarily so we may permute the sequence of h_i freely. In Sections[2;3] of this paper we will completely characterize royal sequences, and as a consequence present a complete characterization of (2,2)-domination graphs of tournaments.

To create the environment in which we are working, both within the realms of kings and those of domination graphs, foundational results must be examined. First, we examine three results for kings.

Lemma 1.2 Landau.[8] *Any vertex with highest out degree in a tournament is a king.*

A regular tournament is one where the outdegree of every vertex is the same. Thus, every vertex in a regular tournament is a king. The next two lemmas add more information on how kings interact with vertices in the tournament.

Lemma 1.3 Maurer.[9] *If vertex u has nonempty inset, then u is beaten by a king.*

Corollary 1.4. If a tournament contains exactly three kings, those kings form a three cycle.

Now we look at the insets of u, $O^{-}(u)$, and outsets of u, $O^{+}(u)$, in relationship to subsets, which ultimately help determine which vertices are or are not kings or heirs.

Lemma 1.5 Factor, Langley.[2] *$u \in V(T)$ is a king if and only if for any $v \in V(T) - \{u\}, O^{-}(v) \not\subseteq O^{-}(u)$.*

The contrapositive to this is the following.

Corollary 1.6. There exists a vertex $v \in V(T) - \{u\}$ where $O^{-}(v) \subseteq O^{-}(u)$ if and only if u is not a king of T.

Since no vertex is in its own outset, we remove the equality in the subset notation and rewrite the contrapositive using the definition of a king so that it is most useful to the approach in this paper.

Corollary 1.7. The vertex u cannot reach vertex v in two or fewer steps if and only if $O^{-}(v) \subset O^{-}(u)$ or equivalently, $O^{+}(u) \subset O^{+}(v)$.

The next sections use some constructions requiring the union of graphs. Given two tournaments $T_1=(V_1,A_1)$ and $T_2=(V_2,A_2)$, then $T_1 \cup T_2$ is a directed graph with vertex set $V_1 \cup V_2$ and arcs $A_1 \cup A_2$. Since we are studying tournaments we will subsequently define arcs between all pairs of vertices $v_1 \in V_1$ and $v_2 \in V_2$ to create a tournament on $V_1 \cup V_2$.

2. Royal sequences and Maurer's theorem

In this section, we begin to examine the existence of royal sequences. We delve into the role that heirs play in ascertaining the existence of the sequences, and observe how Maurer's theorem can be used to constrain the heirs by viewing them as future kings (kings with their associated king removed). With the exception of tournaments with $k=3$ or $k=4$ kings, the application of Maurer's theorem and constructive lemmas allows us to determine all possible royal sequences. The cases of $k=3$ or $k=4$ kings require particular approaches and are reserved for Section 3.

Theorem 2.1 Maurer.[9] *There exists a tournament T with n vertices and k kings,* $n \geq k \geq 1$*, unless* $k=2$ *or* $k=n=4$*.*

Maurer's theorem includes all the tournaments where every vertex is a king, and thus we have the following corollaries:

Corollary 2.2. There exists a tournament T with royal sequence $[k;0,\ldots,0;0]$ *if* $k=1$*,* $k=3$ *or* $k\geq 5$*.*

Corollary 2.3. There exists no tournament T with royal sequence $[2;h_1,h_2;r]$ *or* $[4;0,0,0,0;0]$*.*

The effect of Maurer's theorem on the existence of heirs in the royal sequence is what gives additional depth and interest to this study of (2,2)-domination. Since heir h of king k is king in the tournament $T-\{k\}$, Maurer's result must be extended to the heirs to ascertain which royal sequences are possible.

First we may directly address the case where there is exactly one king. Maurer observed that if a tournament has exactly one king, it must be a *transmitter*. That is, it has empty inset and is directed toward all other vertices in the tournament.

Proposition 2.4. *There exists a tournament T with royal sequence* $[1;h;r]$ *if and only if* $h\neq 2$*, if* $h=0$ *then* $r=0$*, and if* $h=4, r>0$*.*

Proof. If T has one vertex, then we have royal sequence $[1;0;0]$. Otherwise if the unique king is removed from T then the resulting tournament must have $h>0$ kings. Consequently the only case where $h=0$ is the case where T has exactly one vertex. The other values of h and r are restricted only by Maurer's theorem, hence $h\neq 2$ and if $h=4$ then $r>0$. Conversely, attaching a transmitter to tournament with h kings and r additional vertices creates a tournament with royal sequence $[1;h;r]$, consequently existence is assured.

In order to construct royal sequences we will constructively add $h_i>0$ heirs to their corresponding kings. With the exception of $h_i=2$ or $h_i=4$ heirs we may use Maurer's theorem directly, first by attaching a tournament with h_i kings, transforming them into the desired heirs.

Lemma 2.5. *Let T_1 be any tournament with king x, where x has no heirs. Let T_2 be any tournament with exactly k kings. Construct T as follows: Begin with $T_1 \cup T_2$. Then, for any pair of vertices u from T_1 and v from T_2 include arc (u,v) if and only if either (u,x) is an arc or $u=x$. Then, the kings of T are precisely the kings of T_1. The heirs of x are precisely the kings of T_2. The heirs of the other kings of T_1 are unchanged and provide the only other heirs of T.*

Proof. First, x is a king of T_1 and $(x,v) \in E(T)$ for any $v \in V(T_2)$, so x is a king of T.

Let $k \neq x$ be a king of T_1. For any $v \in V(T_2)$, either (k,v) and (k,x) or (v,k) and (x,k) are arcs of T. In the latter case, there exists a vertex u of T_1 such that (k,u) and (u,x) are arcs in T_1, so (u,v) is an arc of T, and k reaches v in two steps. Thus, k is a king of T.

Let h be an heir of king $k \neq x$ in T_1. Vertex k is distance 3 from h in T_1. If there exists a path of length 2 in T from h to k, then there is a vertex $v \in V(T_2)$ so that (h,v) and (v,k) are arcs in T. But then (h,x) and (x,k) are arcs in T_1, which is a contradiction. Thus, h is not a king of T. If (h,x) is an arc in T_1, then (h,v) is an arc in T for all $v \in V(T_2)$. If (x,h) is an arc in T, then there is a vertex u in T_1 such that (h,u) and (u,x) are arcs since h is an heir in T_1. Thus, (u,v) is an arc in T for all $v \in V(T_2)$, and h is an heir of k in T.

Let u be a vertex of T_1 that is not a king or heir. As seen previously, no vertex in T_2 exists that creates a path of length 2 between u and w in T_1 where $dist_{T1}(u,w)>2$. Thus, u is not a king or heir of T.

Let k be a king of T_2. (x,k) is an arc of T and for any u in T_1 such that (k,u) is an arc, (x,u) is also an arc of T so $O^+(k) \subset O^+(x)$. Therefore, by Corollary 1.7, k is not a king of T. However, for all $u \in V(T_1)$ where $u \neq x$, either (k,u) is an arc or (u,x) and thus (u,k) are arcs in T. In the latter case, there is an xu-path of length 2 in T_1 so there is a vertex w in T_1 such that (x,w) and (w,u) are arcs. Thus, (k,w) is an arc of T and k reaches u in 2 steps, making k an heir of x.

For any other vertex $v \in V(T_2)$, there is a vertex y in T_2 where $dist_{T2}(v,y) \geq 3$, and there is no path less than 2 in T. For $u \in V(T_1)$ where

(v,u) is an arc indicates all vertices in T_2 beat u, so (u,y) cannot exist. Thus, v is neither a king nor heir in T. So, the kings of T are precisely the kings of T_1, the heirs of the kings of T_1 are still heirs of the same kings in T, and the kings of T_2 are the heirs of x in T.

Corollary 2.6. *If x is a king in a tournament T_1 with zero heirs, there exists a tournament T with exactly the same royal sequence, with the exception that x has $h_x>0$ heirs, for $h_x\neq 2$, $h_x\neq 4$.*

Proof. This follows from the previous lemma and Maurer's theorem by appending to T_1 a tournament T_2 with precisely $h>0, h\neq 2,4$ vertices all of which are kings.

As we see from Maurer's theorem, the challenging cases must involve two or four heirs, since we cannot simply append two or four kings. The following lemmas will be critical in managing two or four kings. The first follows closely the inductive step of Maurer's proof, and will allow for kings with exactly two heirs and transform them into kings with exactly four heirs. Then, the next lemma will provide certain conditions under which we may guarantee a king has exactly two heirs (and consequently may have four heirs instead).

Lemma 2.7. *Let T be a tournament with king x, where x has $h>0$ heirs, then there exists a tournament T' with precisely the same royal sequence as T, except x has $h+2$ heirs.*

Proof. Let v be an heir of x. Construct T' by replacing v with a directed three cycle of vertices v_1,v_2,v_3 and all arcs (v_i,y) if and only if T has arc (v,y), and all other arcs of T preserved. Observe first that for any pair of vertices in $V(T')-\{v_1,v_2,v_3\}$ their distance is unchanged in T' or in $T'-\{x\}$, and that the distance from $y\in T$ to v_i in T' or $T'-\{x\}$ is the same as the distance from $y\in T$ to v in T or $T-\{x\}$ respectively. Furthermore v_i can reach v_j in two or fewer steps around the cycle. Because all distances are preserved, kings of T remain kings in T' and no new kings are formed, heirs of T other than v remain heirs of T and of the same kings, and v_1,v_2,v_3 are heirs of x.

Lemma 2.8. *Let T be a tournament, x a king with no heirs and y a king in the outset of x such that x is a king in $T-\{y\}$. Then there exists* *a tournament T' with king x and x has exactly 2 or 4 heirs, but otherwise the heir sequence is unchanged.*

Proof. We will start with 2 heirs. The construction and proof are similar to Lemma 2.5. Let the vertices of T' be the vertices of T together with two new vertices u_1 and u_2. Maintain arcs for pairs vertices within T. Choose arcs (x,u_i) for $1\leq i\leq 2$, (u_2,y), (y,u_1), (u_1,u_2). Finally, for all remaining vertices w if (x,w) is an arc, then (u_i,w) is an arc and if (w,x) is an arc then so is (w,u_i) for $1\leq i\leq 2$. Note that the outset of u_i is strictly contained in the outset of x.

Claim: For any vertices v,w in T, where $v\neq x,y$, if there is no path of length shorter than three from v to w in T, then there is no path of length shorter than three in T'. Proof of claim: Since the arc between v and w are unchanged in T' there is clearly no path of length one from v to w. Suppose there is a path of length two in T', (v,z), (z,w). Then, since there is no path of length two in T, z must be u_i for some i. Since v is not equal to x or y, by construction, (v,x) is an arc in T. Also, since the outset of u_i is strictly contained in the outset of x, then (x,w) is an arc in T. Consequently there is a path of length 2 from v to w entirely contained in T.

Since x is a king in T and reaches both new vertices in one step, x is a king in T'. Since the outset of u_i is strictly contained in the outset of x, no u_i is a king due to Corollary 1.7. Observe that $u_i,1\leq i\leq 2$ can reach $u_j,1\leq j\leq 2$ and $j\neq i$, or y in at most two steps. Consider $v\notin\{x,y,u_i\}$. If (x,v) is an arc then so is (u_i,v), so u_i can reach v in one step. On the other hand if (x,v) is not an arc, since x is a king of $T-\{y\}$, x can reach v in two steps, via $w\notin\{x,y,u_i\}$. However, this means u_i can likewise reach v in two steps via w. Consequently u_i is a king in $T-\{x\}$, so is an heir of x.

Suppose z (possibly equal to y) is a king of T. Since the arcs of T are unchanged within T', z can reach all vertices of T in at most two steps. If (z,x) is an arc then so is (z,u_i). On the other hand if (z,v),(v,x) are arcs, then, since y is in the outset of x, $y\neq v$, so (v,u_i) is an arc of T' and thus z can reach u_i in two steps. Consequently, z is a king of T'.

By the claim above we know that any vertex of T that is not a king of T does not become a king of T'. We need to show that heirs are preserved: That is that any heir of T remains an heir of T' and no new vertex of T becomes an heir in T'.

Suppose z is an heir of T. We have shown z is not a king of T' but we need to confirm z remains an heir. We know z can reach all vertices of T except for its king in two or fewer steps. Since z is not an heir of x, z can reach x in two or fewer steps, (and since (y,x) is not an arc, this path will avoid y). Consequently z can reach u_i in two or fewer steps. By the earlier claim, z cannot reach its own king in fewer than three steps, so remains an heir.

Suppose z is neither a king nor an heir of T. We know that z is not a king of T'. Suppose z were transformed into an heir of a king w. Then, z would be a king of $T'-\{w\}$. This means z can reach every vertex of $T'-\{w\}$ in two or fewer steps. However adding $\{w\}$ back into the tournament cannot lengthen the distance between vertices, so z can reach all vertices in T' except w in two or fewer steps. By the original claim z can reach all vertices of T except w in two or fewer steps, and consequently is an heir of w.

To extend this result to four heirs we may apply Lemma 2.7 to either u_1 or u_2. It will be convenient to apply this lemma and replace u_1 with three heirs.

Note for future reference u_2 has the property that $O^+(u_2)\cap(T-\{x,u_1,u_2\})=O^+(x)\cap(T-\{x,u_1,u_2\})$.

The preceding lemma uses a fairly strict requirement on x and y in order to add exactly 2 or 4 heirs. However it turns out that if y is a king with a sufficiently robust heir, we can meet the conditions of the lemma.

Lemma 2.9. *Suppose x and y are kings of a tournament T with arc (x,y), u is an heir of y. Let V' consist of the vertices of T with y and all heirs of y removed. If $O^+(y)\cap V'=O^+(u)\cap V'$, then x is a king of $T-\{y\}$.*

Proof. For any heir of y, the inset of u must contain the inset of y, otherwise u could reach y in two steps. Consequently, x is directed toward all heirs of y. For any other vertex v of T if x can reach v in two steps via y then x can reach v in two steps via u.

It is essential at this point to emphasize that adding heirs via Corollary 2.6 or Lemma 2.8 creates an heir with precisely such a relationship as described in the previous lemma. In Theorem 2.11, Theorem 3.5 and Lemma 3.9 we construct examples by sequentially adding heirs, which is allowed by this property. As our final constructive lemma let us observe that it is not difficult to append vertices which are neither kings nor heirs.

Lemma 2.10. *Suppose T is a tournament on at least two vertices with royal sequence $[k;h_1,\dots,h_k;r]$. Then there exists a tournament T' with royal sequence $[k;h_1,\dots,h_k;r+s]$ for any positive integer s.*

Proof. Let S be any tournament on s vertices. Construct T' from T and S by directing every vertex in T toward every vertex is S. We need only show that any vertex of S is neither a king nor an heir. Since there is no path from any vertex of S to any vertex of T, no vertex of S is a king, and since even removing any vertex of T leaves at least one vertex of T remaining, no vertex of S is an heir.

Theorem 2.11. *Let $k\geq 5$. Then $[k;h_1,\dots,h_k;r]$ is a royal sequence.*

Proof. Corollary 2.2 takes care of the case in which all $h_i=0$, so we may assume otherwise.

We will construct examples, beginning with the first case, k is odd. Consider the rotational tournament R_k on $k=2t+1\geq 5$ vertices, labeled $v_0,\dots,v_{2t}$ with v_i directed toward $v_{i+1}, v_{i+2},\dots,v_{i+t}$ using subscript addition mod $2t+1$. Since R_k is a regular tournament, every vertex of R_k is a king. This tournament will form the subtournament of kings of our tournament T. For each of these kings, we want to be able to add as many heirs as desired by adding vertices and arcs outside of R_k, to create T.

Next observe that, v_{2t} is a king of R_k-v_0, since v_{2t} can reach $v_1,\dots,v_{t-1}$ in one step and $v_t,\dots,v_{2t-1}$ via v_{t-1} in two steps, even if v_0 is removed. We will construct our royal sequence as follows: Sort h_i so h_1 through h_l are not zero and h_{l+1} through h_k equal zero (although l might equal k, if no h_i is zero). We may assign h_l heirs to v_l by Corollary 2.6 or Lemma 2.8 depending on whether h_l=2 or 4 or not. Then by Lemma 2.9 we may assign h_{l-1} heirs to v_{l-1}, and then to v_{l-2} and so on through v_1. Finally we may add r vertices which are neither kings nor heirs by Lemma 2.10.

In the second case, let k be even with $k=2t+2\geq 6$, construct a tournament R'_k as follows: We begin with the rotational tournament R_{2t+1} and append one vertex as follows: Add vertex u, arcs (v_{2t},u), (v_t,u) and for $i\neq 2t, i\neq t$, (u,v_i). The vertex uu has highest out degree in R' so is a king. Observe that each of $v_0,\ldots,v_{2t}$ is a king of R' and $R'-\{u\}=R$. We construct a tournament in a similar fashion as before. We append h_1 heirs to v_l by Corollary 2.6 or Lemma 2.8 and proceed, via Lemma 2.9 to add heirs as before until appending heirs to l vertices, including finally vertex u if $l=k$.

3. Three or four kings

The constructions of the previous section require that we begin with exactly k vertices, all kings. Maurer's theorem demonstrates that this is impossible if k=4. Although some of the constructions work for 3 kings, Lemma 2.8 requires that kings have a sufficiently robust relationship, in terms of arcs, which does not necessarily hold when k=3. Therefore we must try different approaches for tournaments with 3 or 4 kings.

Suppose T is a tournament with three kings. These must form a three cycle, so we will name them x_1, x_2, x_3 with arcs (x_1,x_2), (x_2,x_3), (x_3,x_1). Fortunately, in most cases the lemmas from the preceding section do apply.

Lemma 3.1. *If h_1,h_2,h_3 are nonnegative integers such that at least one of h_i is greater than zero and not equal to 2 or 4, then $[3;h_1,h_2,h_3;r]$ is a royal sequence.*

Proof. Assume without loss of generality that h_1 is greater than zero, $h_1\neq 2$, $h_1\neq 4$. Also assume that if exactly one h_i=0, then h_3=0. To construct T we use Corollary 2.6 to append h_1 heirs to x_1. If h_2=0 we are finished. If $h_2=2$ or $h_2=4$ then we use Lemma 2.8 ; Lemma 2.9 to append 2 or 4 heirs to x_3, otherwise we add h_2 heirs to x_3 by Corollary 2.6. We repeat this process by adding h_3 heirs to x_2.

This means we need only consider the cases where h_i is 0, 2, or 4. We know that in the case of $[3;0,0,0;r]$ we can construct such a tournament by Corollary 2.2 and by Lemma 2.10, so we may assume at least one of h_i is not zero.

Lemma 3.2. *There exists no tournament T with royal sequence* $[3;2,0,0;0]$ *or* $[3;4,0,0;0]$*.*

Proof. We may assume without loss of generality that x_1 has 2 or 4 heirs while x_2 and x_3 have none. This set of heirs must form a subtournament T'. None of these heirs may be directed toward x_1 or x_3 since they cannot reach x_1 in two or fewer steps. However each vertex must be able to reach x_3 in two or fewer steps when x_1 is removed, consequently they must all be directed toward x_2, since $O^-(x_3)=\{x_2\}$. A consequence of this, is that in order for vertex $v_i \in T'$ to reach $v_j \in T'$ it must follow a path of length 2 entirely in T', and thus must be a king in T'. This means T' has either exactly two kings or exactly 4 kings and 4 vertices, which is a contradiction.

Lemma 3.3. *Let* $h_1=2$ *or 4. There exists a tournament T with royal sequence* $[3;h_1,0,0;r]$*, with* $r>1$*.*

Proof. Begin with the case $h_1=2, r=1$. We construct a tournament on six vertices as follows: x_1,x_2,x_3 form a three cycles as above. In addition we have u_1, u_2, v with the following relationships: (x_1,u_i), (u_i,x_2), (x_3,u_i), (x_i,v), (u_1,u_2),(v,u_1), (u_2,v), $1\le i\le 2$. This tournament is illustrated in the first tournament of Fig. 1. Each x_i reaches v directly and either u_i directly, or in the case of x_2 in two steps via x_3. Observe that the outsets of u_i and v are strictly contained in the outset of x_1, none of the u_i nor v is a king, and if any were an heir it must be an heir of x_1. Now u_i can reach x_2 directly and x_3 via x_2. Since u_1,u_2,v form a directed three cycle each can reach the other in two steps. Consequently u_1 and u_2 are heirs of x_1. On the other hand v cannot reach x_3 in two steps, so is not an heir of x_1, and therefore, not an heir of any king of T. We can change r to any positive integer by

Lemma 2.10 and h_1 to 4 by Lemma 2.7 and replacing u_1 with a three cycle.

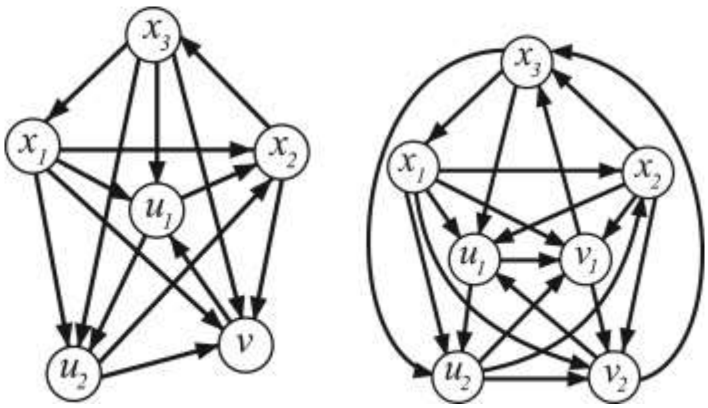

Fig. 1. The tournaments with sequences [3; 2, 0, 0; 1] and [3; 2, 2, 0; 0] respectively.

Lemma 3.4. *Let h_1=2 or 4 and h_2=2 or 4. There exists a tournament T with royal sequence* $[3;h_1,h_2,0;r]$.

Proof. We start with the case that $h_1=h_2=2, r=0$ and construct a tournament on seven vertices as follows: x_1,x_2,x_3 form a three cycles as above. In addition we have u_1, u_2, v_1, v_2 with the following relationships: (x_1,u_i), (x_1,v_i), (x_2,u_1), (u_2,x_2), (x_2,v_i), (x_3,u_i), (v_i,x_3), (u_1,u_2), (v_1,v_2), (u_1v_1), (v_2,u_1), (u_2,v_i). This tournament is the second tournament in Fig. 1. First observe that each of x_i can reach each of u_j and v_k in one or two steps, so each x_i is a king. Next observe that the $O^+(u_i)\subset O^+(x_1)$ and $O^+(v_j)\subset O^+(x_2)$, so none of u_i nor v_j are kings. Finally we observe that u_i is a king when x_1 is deleted from the tournament and v_i is a king when x_2 is deleted from the tournament so each of u_i and v_j is an heir. We can change h_i to 4 by Lemma 2.7 and increase r by Lemma 2.10.

Theorem 3.5. *There exists a tournament T with royal sequence* $[3;h_1,h_2,h_3;r]$, *with the exceptions of* [3;2,0,0;0] *and* [3;4,0,0;0].

Proof. We have covered almost all cases in the previous lemmas. The only remaining cases have all h_i=2 or 4 for i=1,2,3. However, in the previous lemma observe that x_1 and u_2 satisfy the conditions of Lemma 2.9, so we may add 2 or 4 heirs to x_3 by Lemma 2.8.

Our final case is four kings. We know that it is impossible for a tournament to have exactly four kings and four vertices, so we must have at least one extra vertex. We will first prove that we must have at least one heir.

Lemma 3.6. [4;0,0,0,0;r] *fails to be a royal sequence for any tournament T.*

Proof. Let T be a tournament with exactly four kings. Let T' be the induced subgraph of T on exactly those four kings. T' will have either one or three kings. In the first case, if T' has exactly one king, say x, since T has four kings, $O^{-}(x)$ is not empty in T. No other vertex of T' may be in $O^{-}(x)$ since x is the sole king in T'. However, since x has nonempty inset and therefore, by Lemma 1.3 must contain a king, T has a king which is not in T', contradicting our construction. In the second case, let v be the sole vertex of T' which is not a king of T', let x be a king of T' which v cannot reach in one or two steps within T', and let y be a king in the inset of x. By examining both tournaments of size 4 with three kings (see Fig. 2), we observe that y is unique to each x. Consider now the tournament $T-\{y\}$. Since v must reach x in T via some vertex not in T', the inset of x is not empty in $T-\{y\}$ and so contains a king. That king is not in T' by our choice of y and so is an heir of y, consequently T must have at least one heir.

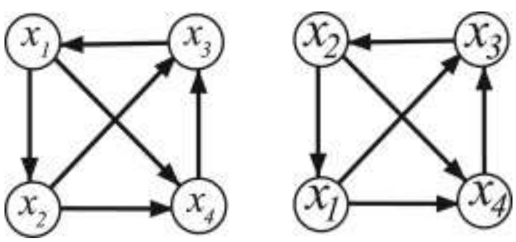

Fig. 2. Two tournaments with four vertices and three kings.

Lemma 3.7. [4;m,0,0,0;0]*,* $m>0$*,* $m\neq 2,4$ *is a royal sequence.*

Proof. Let T_1 be the unique strongly connected tournament on four vertices, labeled as in the first tournament in Fig. 2, and T_2 be any tournament on m vertices with exactly m kings. Create a new tournament T consisting of the vertices and arcs of T_1 and T_2 with arcs as follows: Let u be any vertex of T_2. Set arcs (u,x_2) and (x_i,u) for all vertices x_i, $i\neq 2$. Observe that x_2 can reach all of T_2 in two steps via x_3 or x_4 and that each of x_i can reach x_j in one or two steps, so all of x_i

are kings of T. Observe that for any u in T_2, the outset of u is strictly contained in the outset of x_1, so u is not a king, however u is a king in T_2 and can reach x_2 directly or x_3 or x_4 in two steps via x_2, so u is a king in $T-\{x_1\}$ and hence is an heir of x_1.

Lemma 3.8. $[4;m,0,0,0;0]$*, $m=2$, $m=4$ is a royal sequence.*

Proof. We will start with two heirs and then extend this to four vertices via Lemma 2.7. Let T_1 be the first tournament from Fig. 2. Add two vertices u_1, and u_2 with arcs as follows: (u_1,u_2), (u_i,x_2), (x_1,u_i), (x_3,u_i), (x_4,u_1), (u_2,x_4) to create the tournament in Fig. 3. We observe that u_i is an heir of x_1, thus we have royal sequence $[4;2,0,0,0;0]$. Furthermore u_2 and x_1 satisfy the conditions of Lemma 2.9 so we may extend this tournament to one with royal sequence $[4;4,0,0,0,0]$.

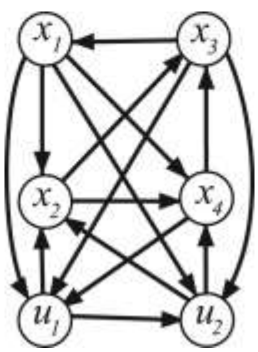

Fig. 3. A tournament with four kings, x_1 having exactly two heirs.

Lemma 3.9. $[4;h_1,h_2,h_3,h_4;r]$ *is a royal sequence for any tournament T provided at least one of* $h_i>0$*.*

Proof. Without loss of generality we may assume h_1 is not zero. First suppose none of h_i are equal to two or four. Then we use the construction from Lemma 3.7 to set h_1 heirs and Corollary 2.6 for nonzero values of h_2, h_3, and h_4. Finally we add r vertices which are neither kings nor heirs by Lemma 2.10. Suppose at least one of h_i equals 2 or 4. We may assume, without loss of generality that h_1 is one such number. Furthermore we will assume that if $h_i=0$ and $h_j\neq0$ that $j<i$. We use Lemma 3.8 to attach heirs to x_2, then by either Lemma 2.8 or 2.9 add h_2 heirs to x_3, x_4 until $h_j=0$. Finally we add r remaining vertices using Lemma 2.10.

We collect the previous theorems.

Theorem 3.10. *There exists a tournament with royal sequence* $[k;h_1,\ldots,h_k;r]$, $k\geq 1$, $h_i\geq 0$, $r\geq 0$, *with the following restrictions (up to permutations of* h_i*):* $[1;0;r]$ *if* $r>0$, $[1;2;r]$, $[1;4;0]$, $[2;h_1,h_2;r]$, $[3;2,0,0;0]$, $[3;4,0,0;0]$, $[4;0,0,0,0;r]$.

4. (2,2)-domination graphs

In the (2,2)-domination graph of a tournament T, the edges are formed by any two vertices each of whom beat all of the remaining vertices in one or two steps. In the previous sections, we formed the restrictions on what kings and heirs can exist in a tournament. From that information, we characterize the structure of (2,2)-domination graphs of tournaments. First we list the only pairs of vertices that can form an edge in $dom_{2,2}(T)$.

Lemma 4.1. *Let* T *be a tournament. Then* uv *is an edge in* $dom_{2,2}(T)$ *if and only if*

(a) *u and v are kings in T, or*
(b) *u is a king in T and v is an heir of u.*

Proof. (⇒) Since uv is an edge in $dom_{2,2}(T)$, u beats every vertex except possibly v in at most two steps. Similarly for v. Say that u beats v. If v beats u in at most two steps, they are both kings. If not, v must still beat all other vertices in at most two steps, so u is a king and v is an heir of u. (⇐) In (a) and (b), both u and v beat all other vertices in $T-\{u,v\}$ in at most two steps, so form the edge uv in $dom_{2,2}(T)$.

Further, we formulate the structures within the (2,2)-domination graphs with the following lemma.

Lemma 4.2. *The* (2,2)*-domination graph of a tournament is formed as follows:*

(a) *The kings of T form a complete subgraph of $dom_{2,2}(T)$.*
(b) *An heir forms a pendant vertex with its associated king.*
(c) *All vertices that are neither kings nor heirs are isolated vertices.*

Proof. (a) All kings beat all other vertices in at most two steps, so form a clique. (b) An heir does not beat its king in 2 or fewer steps, but does beat all others, so can only form an edge with that one vertex. (c) All other vertices have at least two vertices they cannot reach in one or two steps, so cannot form an edge with any other vertex.

Finally, we combine the structure of the (2,2)-domination graph with Theorem 3.10 to obtain the characterization of (2,2)-domination graphs of tournaments.

Theorem 4.3. *G is the* (2,2)*-domination graph of a tournament T on* $n\geq 1$ *vertices with royal sequence* $[k;h_1,\ldots,h_k;r]$ *if and only if G is a complete graph with or without pendant vertices and with or without isolated vertices except for the following:*

1. K_2 *with or without pendant vertices and with or without isolated vertices;*
2. K_4 *with or without isolated vertices;*
3. *The graph of* $n\geq 2$ *isolated vertices;*
4. $K_{1,4}$*;*
5. K_3 *with 2 or 4 vertices pendant to exactly one of its vertices.*

Proof. Theorem 3.10 provides all sequences which are not royal sequences, so lays the groundwork for the structure of kings and heirs in a tournament. Lemma 4.1 ; Lemma 4.2 give us the only way the $dom_{2,2}(T)$ edges can be formed and the structures that are created. Using Theorem 3.10, we have the following. $[1;0;r],r>0$ disallows graph 3 in the theorem. $[2;h_1,h_2;r]$ disallows any copy of K_2 with or without pendant vertices and with or without isolated vertices, which is graph 1. Sequence $[1;2;r]$ is a subset of the previous constraint, as it is K_2 with a pendant vertex and possible isolated vertices. We obtain graph 2 with sequence $[4;0,0,0,0;r]$ and graph 4 with sequence [1;4;0]. Finally, royal sequences [3;2,0,0;0] and [3;4,0,0;0] disallow graph 5.

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