4-2006

Spatial Admittance Selection Conditions for Frictionless Force-guided Assembly of Polyhedral Parts in Single Principal Contact

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IEEE Transactions on Robotics, Vol. 22, No. 2 (April 2006): 225-239. DOI.

Spatial Admittance Selection Conditions for Frictionless Force-Guided Assembly of Polyhedral Parts in Single Principal Contact

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Abstract:
By judiciously selecting the admittance of a manipulator, the forces of contact that occur during assembly can be used to guide the parts to proper positioning. This paper identifies conditions for selecting the appropriate spatial admittance to achieve reliable force-guided assembly of polyhedral parts for cases in which a single feature (vertex, edge, or face) of one part contacts a single feature of the other, i.e., all single principal contact

cases. These conditions ensure that the motion that results from frictionless contact always instantaneously reduces part misalignment. We show that, for bounded misalignments, if an admittance satisfies the misalignment-reducing conditions at a finite number of contact configurations, then the admittance will also satisfy the conditions at all intermediate configurations.

SECTION I. Introduction

Assembly involves contact between the mating parts. For effective use in assembly, robots should regulate the force of contact and comply with that force in such a way to improve part relative positioning. Without force regulation, part positional misalignment may yield excessive contact forces. Without the ability to improve relative positioning, proper assembly cannot be achieved.

A robot's force regulation and motion response behaviors are characterized by its mechanical admittance. The appropriate admittance for assembly is one for which a misalignment-reducing motion is generated as a direct result of contact. Ideally, a single admittance (a single operator mapping input forces to output motions) provides misalignment reduction for all misalignments that may occur during a given assembly task. As such, this single admittance would ensure proper assembly using contact forces alone.

The appropriate admittance for assembly should satisfy the error-reduction conditions for all configurations in the range considered. However, since there are an infinite number of configurations, it is not realistic to impose the error-reduction conditions on the admittance at all configurations. Thus, it is necessary to develop a set of sufficient conditions on the admittance at a finite number of configurations to ensure error reduction for all configurations. Once established, the conditions can be used as testable conditions useful in the search for an appropriate admittance matrix. One way to accomplish this is to use optimization with the conditions used as constraints. Previous work for planar parts with friction [1] showed the success of this strategy.

This paper presents conditions used to select the appropriate spatial manipulator admittance for force-guided assembly of two polyhedral objects when contact is frictionless and is restricted to cases in which a single feature (e.g., vertex, edge, or face) of one part contacts a single feature of the other.

Here, a simple, general linear admittance control law [2] is used. For spatial applications, this type of admittance has the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{A}\mathbf{w}$$

(1)

where $\mathbf{v}_0$ is the nominal twist (a 6-vector), $\mathbf{w}$ is the contact wrench (force and torque) measured in the body frame (a 6-vector), $\mathbf{A}$ is the admittance matrix (a 6×6 matrix), and $\mathbf{v}$ is the motion of the body.

In this paper, a single admittance control law in the form of (1) is used for all contact states considered.

A. Related Work

Other researchers have addressed the design and use of admittance for force guidance. Whitney [3], [4] initially proposed that the linear compliance (i.e., force-deflection relationship and inverse stiffness) of a manipulator be structured so that contact forces lead to decreasing errors. Peshkin [5] addressed the synthesis of the linear accommodation (force-velocity relationship; inverse damping) of a manipulator by specifying the desired force–motion relation at a sampled set of positional errors for a planar assembly task. An unconstrained optimization was then used to obtain an accommodation matrix that does not necessarily provide force guidance.
Asada [6] used a similar unconstrained optimization procedure for the design of an accommodation neural network rather than an accommodation matrix. Others [7], [8] provided synthesis procedures based on spatial

intuitive reasoning. None of the general approaches, however, provides a proof that the admittance selected will, in fact, be reliable for all possible configurations.

A reliable admittance selection approach is to design the control law so that, at each possible part misalignment, the contact force always leads to a motion that instantaneously reduces the existing misalignment. The approach is referred to as *force assembly.* The success and robustness of the approach were initially demonstrated in the workpart into fixture insertion problems in which only *infinitesimal* misalignments were considered [2], [9], [10].

By the definition of force assembly [2], the motion resulting from contact must instantaneously reduce misalignment. How-ever, because the configuration space of a rigid body is non-Euclidian, there is no “natural metric” for *finite* spatial error. As such, several “body-specific metrics” have been established [11]. One of these metrics is based on the Euclidean distance between a single point on the body and its location when properly positioned. The specific point on the body corresponds to the location having the maximum distance from its properly mated position. This point on the body is configuration-dependent.

In this paper, sufficient conditions for admittance selection are presented. We show that, once these conditions are satisfied at a finite number of configurations, error-reducing motion is ensured for *all* configurations and contact states. Thus, these conditions can be used as constraints in a *constrained* optimization procedure, from which the obtained optimal admittance will ensure successful assembly.

B. Approach

Similar to related work addressing planar assembly [1], [12], here we consider a *measure* of error based on the Euclidean distance between an arbitrarily chosen single (fixed) point on the held body and its location when properly positioned. Use of a single measure of this type does not conform with any established metric. As such, multiple measures each based on a single fixed reference point are used to: 1) further restrict the body motion and 2) conform with the established metric (if one of these points is the one that is furthest from its properly mated position).

Since error reduction of the body is described by the error measure, different sets of reference points will yield different error-reduction requirements on the motion of the body. In general, the selection of the reference points is part - and task-specific. One meaningful choice would be the vertexes on the convex hull of the held part. If so selected, since the furthest point of a polyhedral part is one of its vertexes, at least one of the measures becomes the established metric.

Using this point-based measure of misalignment, misalignment reduction can be expressed mathematically if we let d (a 6-vector for spatial motion) be the line vector from the selected point at its properly mated position to its current position. Then, for error reducing motion, the condition is

$$\mathbf{d}^T\mathbf{v} = \mathbf{d}^T(\mathbf{v}_0 + \mathbf{A}\mathbf{w}) < 0.$$

(2)

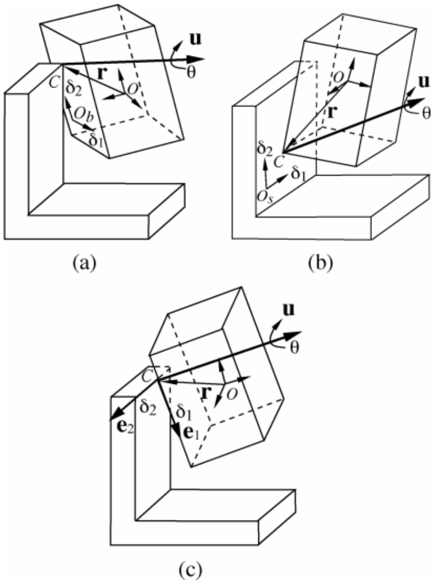

Fig. 1. Configuration variables for single-point pcs. (a) face-vertex contact. (b) vertex—face contact. (c) edge-edge cross contact.

Since force assembly requires that misalignment is reduced at each possible misalignment, this condition must be satisfied for all possible misalignments.

This paper considers polyhedral rigid-body assembly involving spatial motion constrained by frictionless contact. The contact states studied here are the nondegenerate *principal contacts* (PCs) [13] obtained for polyhedral parts.

Because the line vector **d** depends on the rigid-body configuration and because the number of configurations is infinite, it is impossible to impose the error-reduction condition separately for all misalignments. In application, however, the misalignments of the rigid body are bounded by: 1) the extremes within a contact state or 2) the possible inaccuracy of the robotic manipulator. Those misalignments on the "boundary" are of particular interest.

In [1] and [12], sufficient conditions for an admittance to ensure force-guided assembly for *planar* polygonal parts have been identified. In this paper, sufficient conditions for an admittance to ensure force guidance for *spatial* polyhedral parts are identified. We show that, by identifying an admittance matrix that satisfies the error-reduction conditions at a *finite* number of configurations on the boundary of each contact state, the error-reduction requirements are also satisfied for *all* configurations within the bounded area.

Polyhedral bodies in single-point contact have three types of stable principal contacts: "face—vertex" ($\{f-v\}$) contact, "vertex-face" ($\{v-f\}$) contact, and "edge-edge cross" ($\{e-e\}_c$) contact. In "face-vertex" contact, one face of the held body is in contact with one vertex of the mating fixtured part [see Fig. 1(a)]. In "vertex—face" contact, one vertex of the held body is in contact with one face of its mating part. Each of the single-point principal contacts is illustrated in Fig. 1.

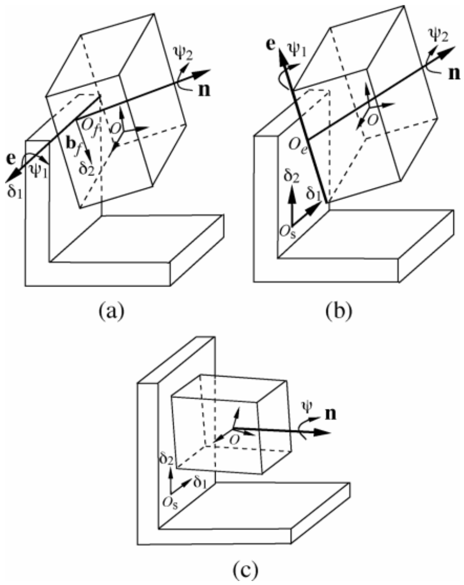

Fig. 2. Configuration variables for multipoint pcs. (a) face-edge contact. (b) edge-face contact state. (c) face-face contact state.

For multipoint contact, there are three PCs: face-edge ($\{f - v\}$, edge-face ($\{e - f\}$) and face-face ($\{f - f\}$) contacts, as shown in Fig. 2.

C. Overview

In this paper, sufficient conditions for an admittance to ensure force-guided assembly are established for each of the six PCs described above. Section II identifies the coordinates used to describe the configuration variation for each contact state. In Section III, means of calculating the motion of a constrained body and an error-reduction function are derived for each type of contact state. Finally, sufficient conditions for error reduction for each PC are derived in Sections IV–IX. These conditions show that an admittance matrix that satisfies the error-reduction conditions at the boundaries of a set of contact configurations also satisfies the error-reduction conditions at all intermediate configurations. A discussion and a brief summary are presented in Sections X and XI.

SECTION II. Configuration Description

In this section, the sets of coordinates used to describe configuration variation for each contact state are presented. For each of the different contact states, the relative configuration of the constrained rigid bodies is described using a different set of generalized coordinates $\mathbf{q}$.

Each PC is characterized by two degrees of freedom (DOFs) in translation. The number of DOFs in rotation, however, is different for different types of PCs. Those PCs associated with single-point contact have three rotational DOFs; PCs associated with line contact have two rotational DOFs, and those associated with plane contact have one rotational DOF.

Below, the variables used to describe the configuration variation within each PC are presented for each of these three classes of PC (based on DOF).

A. Single-Point Contact States

Single-point contact PCs include face-vertex, vertex-face, and edge-edge cross contact cases as shown in Fig. 1. The body can translate in the plane of contact and rotate about the contact point in any direction. As such, five variables describe the relative configuration of the bodies (the relative position of the contact point using two translational variables and the relative orientation using three rotational variables).

1) Orientational Variation

The relative orientation of the rigid body can be described by a 3×3 orthogonal matrix $\mathbf{R}$.

Consider two configurations C_0 and C_1 with the same point of contact. By Euler's theorem, there exists an axis such that configuration C_1 can be achieved from configuration C_0 by a rotation about this single axis. For any given C_0 and C_1, the direction of the axis u and rotation angle θ are unique $(0 \leq \theta \leq \pi)$.

Consider a rotation about an arbitrary axis $\mathbf{u}$ with angle θ. The rotation matrix associated with this configuration change can be obtained by Rodrigues' formula [14]

$$\mathbf{R}(\mathrm{u}, \theta) = \cos\theta \mathbf{I} + (1 - \cos\theta)\mathbf{u}\mathbf{u}^T + \sin\theta[\mathbf{u} \times]$$

(3)

where $\mathbf{I}$ is the 3×3 identity matrix and $[\mathbf{u} \times]$ denotes the antisymmetric matrix associated with the cross-product operation involving u given by

$$[\mathbf{u} \times] = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

Finite variation from an initial configuration (considered later in establishing sufficient conditions) can be described by placing bounds on the maximum angular magnitude $0 \leq \theta \leq \theta_M$ and with no bounds on the direction of the rotation axis $\mathbf{u}$.

Since the orientational error is bounded by manipulator inaccuracy, only small angular variation $\leq 10°$ is considered. Because u is arbitrary, for a centered coordinate frame with maximum angular variation $\triangle\,\theta$, the bound for the angular magnitude $\theta_M = \left(\frac{1}{2}\right)\triangle\,\theta$. For example, if the maximum angular variation considered is 10°, then $\theta_M = 5°$.

2) Translational Variation

For bodies in contact at a single point, the location of the contact point can be described by two parameters $\delta = (\delta_1, \delta_2)$. The meaning of these variables changes for the different principal contacts.

For face-vertex ($\{f - v\}$) contact, a two-dimensional (2-D) coordinate frame O_b is established on the held body in the plane of the contact face. Two orthogonal coordinates δ_1, δ_2 are used to describe translational variation of the rigid body within this contact state, as shown in Fig. 1(a).

For vertex-face ($\{v - f\}.$) contact, a 2-D coordinate frame O_s is established on the stationary part in the plane of the contact face. Again, two orthogonal coordinates δ_1, δ_2 are used to describe the translational variation of the rigid body within this contact state. as shown in Fig. 1(b).

For edge-edge cross ($\{e - e\}_c$) contact, two translational nonorthogonal coordinates δ_1, δ_2 are chosen to describe translational variation along edges $\mathbf{e}_1$ and $\mathbf{e}_2$, as shown in Fig. 1(c).

Since finite configuration variation is considered, for each contact state, the variation of each δ_i is bounded. By appropriately choosing the coordinate origin (at a central location of contact), the bounds for δ_i can be written as

$$-\delta_{M_i} \leq \delta_i \leq \delta_{M_i}.$$

In summary, the configuration variation for each single-point contact state is given by $\mathbf{q} = (\delta_1, \delta_2, \mathbf{u}, \theta)$.

B. Line Contact

When an edge of a body is in contact with a face of its mating part, the body has four DOFs when maintaining this contact: two in translation and two in rotation. PCs of this type include two cases: edge-face ($\{e-f\}$) and face-edge ($\{f-e\}$) contacts.

1) Face-Edge Contact

To describe the relative configuration variation of the bodies, a 2-D coordinate frame is established on the held body's contact face such that, at an initial configuration, the origin O_f is on the contact edge [see Fig. 2(a)]. Let δ_1 describe the translational variation along edge e and δ_2 describe the translational variation along the direction $\mathbf{b}_f$ in the face plane (where $\mathbf{b}_f \perp \mathbf{e}$). Then, the relative configuration in translation of the body is determined by the two parameters δ_1, δ_2.

To maintain contact, the body can only rotate about edge e and the face normal n. If we denote ψ_1 and ψ_2 as the rotation angles about e and n, respectively, then the configuration of the body can be determined by the four parameters ($\delta_1, \delta_2, \psi_1, \psi_2$).

2) Edge-Face Contact

For this type of contact state, once a reference point O_e is chosen on the contact edge of the body, the translational variation of the body is described by the coordinates δ_1, δ_2 indicating the location of O_e relative to the coordinate frame O_s fixed in the stationary face [see Fig. 2(b)].

To maintain contact, only rotational variation about the contact edge e or/and the face normal II are allowed. Let ψ_1 and ψ_2 be rotation angles about the edge e and axis n, respectively. Then, the body's configuration **q** can be determined by four parameters ($\delta_1, \delta_2, \psi_1, \psi_2$).

For both $\{e-f\}$ and $\{f-e\}$ cases, the four parameters are bounded by

$$-\delta_{M_i} \leq \delta_i \leq \delta_{M_i},$$
$$-\psi_{M_i} \leq \psi_i \leq \psi_{M_i}$$

where the rotation axis lies in the e-n plane.

C. Plane Contact

When one face of the held body is in contact with a face of its mating part, the motion of the body is constrained to the plane of the contact face (if the contact is maintained). Thus, the body has 3 DOFs. The configuration of the body can be characterized by three parameters (δ_1, δ_2, ψ), where δ_1, δ_2 describe the translational variation in the contact plane and ψ describes the rotational variation about the axis in the direction of the plane normal.

Since finite configuration variation is considered, the three parameters are bounded by

$$-\delta_{M_i} \leq \delta_i \leq \delta_{M_i}$$
$$-\psi_M \leq \psi \leq \psi_M$$

where the rotation axis is **n**.

SECTION III. Error-Reducing Motion of a Constrained Rigid Body

In this section, the motion of a partially constrained body is investigated. For each contact state, the frictionless contact force is first discussed and the error-reduction function is then obtained.

A. Single-Point Contact

For single-point contact states, the contact force is imposed at the point of contact and is along the face normal (for $\{v - f\}$ and $\{f - v\}$ contact states) or along the normal determined by the two contact edges (for $\{e - e\}_c$ contact). Let **n** be a unit three-vector indicating the direction of the normal contact force applied to the held body. The unit wrench associated with the normal force has the form

$$\mathbf{w}_n = \begin{bmatrix} \mathbf{n} \\ \mathbf{r} \times \mathbf{n} \end{bmatrix}$$

(4)

where **r** is the position vector from the origin of the held body coordinate frame to the point of contact C, as shown in Fig. 1.

Let ϕ be the magnitude of the normal contact force. The contact wrench is

$$\mathbf{w} = \mathbf{w}_n \phi.$$

(5)

By the control law (1), the motion of the body is

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{A}\mathbf{w}_n \phi.$$

(6)

Because the motion of the rigid body cannot penetrate the surface, the reciprocal condition [15] must be satisfied as follows:

$$\mathbf{w}_n^T \mathbf{v} = \mathbf{w}_n^T \mathbf{v}_0 + \mathbf{w}_n^T \mathbf{A}\mathbf{w}_n \phi = 0.$$

The magnitude ϕ is determined from

$$\phi = \frac{-\mathbf{v}_0^T \mathbf{W}_n}{\mathbf{w}_n^T \mathbf{A}\mathbf{w}_n}.$$

(7)

Substituting (7) into (6) yields

$$\mathbf{v} = \frac{(\mathbf{v}_0 \mathbf{w}_n^T - \mathbf{v}_0^T \mathbf{w}_n \mathbf{I})\mathbf{A}\mathbf{w}_n}{\mathbf{w}_n^T \mathbf{A}\mathbf{w}_n}.$$

(8)

Note that, since $\mathbf{n}$ and/or r can vary in the body frame for the same contact state, $\mathbf{w}_n$ is, in general, a function of configuration for each of the single-point contact PCs.

For the compliant motion to be error-reducing, condition (2) must be satisfied for a given point.
Thus, (2) becomes

$$E = \frac{\mathbf{d}^T (\mathbf{v}_0 \mathbf{w}_n^T - \mathbf{v}_0^T \mathbf{w}_n \mathbf{I})\mathbf{A}\mathbf{w}_n}{\mathbf{w}_n^T \mathbf{A}\mathbf{w}_n} < 0$$

(9)

where $\mathbf{A}$, $\mathbf{d}$, and $\mathbf{w}_n$ are expressed in the held body frame.

Since $\mathbf{A}$ is positive definite, $\mathbf{w}_n^T\mathbf{A}\mathbf{w}_n > 0$, and the denominator of (9) is positive. Therefore, the error-reduction function can be expressed as

$$F_{1p} = \mathbf{d}^T(\mathbf{v}_0\mathbf{w}_n^T - \mathbf{v}_0^T\mathbf{w}_n\mathbf{I})\mathbf{A}\mathbf{w}_n.$$

(10)

Since $\mathbf{d}$ and $\mathbf{w}_n$ are functions of configuration $\mathbf{q}, F_{1p}$ is a function of $\mathbf{q}$. To obtain error reduction, $F_{1p}(\mathbf{q})$ must be negative for all $\mathbf{q}$ considered within the specified principal contact.

B. Line Contact

Next, consider edge-face contact. Let $\mathbf{n}$ be a unit vector along the face normal (pointing toward the held body). The contact force must be in the direction of $\mathbf{n}$ and must pass somewhere through the contact edge.

Let $\mathbf{w}_r$ be the resultant contact wrench with magnitude ϕ and unit wrench $\mathbf{w}_n$ having the form

$$\mathbf{w}_n = \begin{bmatrix} \mathbf{n} \\ \mathbf{r}_r \times \mathbf{n} \end{bmatrix}$$

where $\mathbf{r}_r$ is the position vector indicating the line of action (from the origin of the body frame to an undetermined point on the contact edge). Since the resultant force must pass through the edge, the vector $\mathbf{r}_r$. can be expressed as a linear combination of any two different position vectors terminating on the edge.

Let $p_i(i = 1,2)$ be two arbitrarily chosen points on the contact edge and let $\mathbf{r}_i$ be the position vector associated with p_i (from the body frame origin to p_i). Let $\mathbf{w}_{ni}$ be the unit normal wrench associated with the corresponding p_i. Below, we show that the choice of these points does not influence the calculated reaction force. First, we prove that any wrench in direction n that passes through the edge is a linear combination of the two unit normal wrenches $\mathbf{w}_{n1}$ and $\mathbf{w}_{n2}$. To prove this, we prove that, if p_0 is an arbitrary point on the edge and $\mathbf{w}_{n0}$ is the unit normal wrench associated with p_0, then $\mathbf{w}_{n0}$ is a linear combination of $\mathbf{w}_{n1}$ and $\mathbf{w}_{n2}$.

Let $\mathbf{r}_0$ be the vector from the body frame origin to p_0. Since p_0 is on the edge, $\mathbf{r}_0$ can be expressed as

$$\mathbf{r}_0 = \alpha\mathbf{r}_1 + \beta\mathbf{r}_2$$

where α and β are scalars satisfying $\alpha + \beta = 1$. Thus

$$\begin{aligned} \mathbf{w}_{n0} &= \begin{bmatrix} \mathbf{n} \\ \mathbf{r}_0 \times \mathbf{n} \end{bmatrix} \\ &= \alpha \begin{bmatrix} \mathbf{n} \\ \mathbf{r}_1 \times \mathbf{n} \end{bmatrix} + \beta \begin{bmatrix} \mathbf{n} \\ \mathbf{r}_2 \times \mathbf{n} \end{bmatrix} \\ &= \alpha\mathbf{w}_{n1} + \beta\mathbf{w}_{n2}. \end{aligned}$$

For a wrench $\mathbf{w}_0$ with unit wrench $\mathbf{w}_{n0}$ and magnitude (ϕ, if we denote

$$\phi_1 = \alpha\phi, \phi_2 = \beta\phi$$

then

$$\mathbf{w}_0 = \phi_1\mathbf{w}_{n1} + \phi_2\mathbf{w}_{n2}.$$

Therefore, the two unit normal wrenches $\mathbf{w}_{n1}$ and $\mathbf{w}_{n2}$ establish a basis for all wrenches passing through the edge in the direction **n**.

Now, consider the resultant contact wrench $\mathbf{W}_r$ expressed in terms of the two unit normal wrenches $\mathbf{w}_{n1}$ and $\mathbf{w}_{n2}$ as follows:

$$\mathbf{w}_r = \phi_1 \mathbf{w}_{n1} + \phi_2 \mathbf{w}_{n2}.$$

If we denote

$$\begin{aligned} \mathbf{W} &= [\mathbf{w}_{n1}, \mathbf{w}_{n2}] \times \mathbb{R}^{6\times 2} \\ \phi &= [\phi_1, \phi_2]^T \in \mathbb{R}^2 \end{aligned}$$

then the total contact wrench is

$$\mathbf{w}_r = \mathbf{W}\phi.$$

By the reciprocal condition [15], we have

$$\mathbf{W}^T(\mathbf{v}_0 + \mathbf{A}\mathbf{W}\phi) = 0.$$

Solving the above equation for ϕ yields

$$\phi = -[\mathbf{W}^T\mathbf{A}\mathbf{W}]^{-1}\mathbf{W}^T\mathbf{v}_0.$$

Thus, the resultant contact wrench is

$$\mathbf{w}_r = -\mathbf{W}[\mathbf{W}^T\mathbf{A}\mathbf{W}]^{-1}\mathbf{W}^T\mathbf{v}_0.$$

Note that to maintain contact, the reciprocal condition must be satisfied. In doing so, the reaction force can be determined without knowing beforehand the line of action of this force.

The error-reduction function (2) can be expressed as

$$\begin{aligned} F_{\text{er}} &= \mathbf{d}^T(\mathbf{v}_0 + \mathbf{A}\mathbf{w}_r) \\ &= \mathbf{d}^T(\mathbf{v}_0 - \mathbf{A}\mathbf{W}[\mathbf{W}^T\mathbf{A}\mathbf{W}]^{-1}\mathbf{W}^T\mathbf{v}_0). \end{aligned}$$

Let $[\mathbf{W}^T\mathbf{A}\mathbf{W}]^*$ be the adjugate matrix of $[\mathbf{W}^T\mathbf{A}\mathbf{W}]$ (the transpose of the cofactor matrix of $[\mathbf{W}^T\mathbf{A}\mathbf{W}]$. Then, the error-reduction function can be written as

$$F_{\text{er}} = \frac{(\mathbf{d}^T\mathbf{v}_0)\det(\mathbf{W}^T\mathbf{A}\mathbf{W}) - \mathbf{d}^T\mathbf{A}\mathbf{W}[\mathbf{W}^T\mathbf{A}\mathbf{W}]^*\mathbf{W}^T\mathbf{v}_0}{\det(\mathbf{W}^T\mathbf{A}\mathbf{W})}.$$

Since $\det(\mathbf{W}^T\mathbf{A}\mathbf{W}) > 0$, the error-reduction function can be characterized by the numerator of the above equation.

An equivalent analysis also applies to edge-face contact. Thus, for line-contact cases, the error-reduction function is

$$F_{lc} = (\mathbf{d}^T\mathbf{v}_0)\det(\mathbf{W}^T\mathbf{A}\mathbf{W}) - \mathbf{d}^T\mathbf{A}\mathbf{W}[\mathbf{W}^T\mathbf{A}\mathbf{W}]^*\mathbf{W}^T\mathbf{v}_0.$$

(11)

Note that the values of the error-reduction functions F_{lc} are independent of the choice of the two representative points along the edge. Although they can be chosen arbitrarily, since the representative wrenches are functions of configuration, it is convenient to choose them at two fixed locations on the held body or on the stationary body based on the type of contact state. For example, for face-edge contact, the two wrenches can be chosen at the vertices bounding the edge of the fixtured body. For edge-face contact, the two wrenches can be chosen at the vertices bounding the contact edge of the held body. Since the error measure vector **d** and the two selected normal wrenches $\mathbf{W}_{ni}$ depend on the body's configuration $\mathbf{q}, F_{lc}$ for either face-edge or edge-face contact can be described by a known function of $\mathbf{q}$.

C. Plane Contact

Consider face-face contact. If the normal of the faces is $\mathbf{n}$, then the contact force at each contact point in the contact face is in the direction of $\mathbf{n}$. Thus, the resultant contact force must be in the direction of $\mathbf{n}$.

Let w be the resultant contact wrench with magnitude ϕ and unit wrench $\mathbf{w}_n$ having the form

$$\mathbf{w}_n = \begin{bmatrix} \mathbf{n} \\ \mathbf{r} \times \mathbf{n} \end{bmatrix}$$

(12)

where $\mathbf{r}$ indicates the unknown line of action of the force. To ensure its uniqueness, we can suppose that $\mathbf{r}$ is perpendicular to $\mathbf{n}$, i.e.,

$$\mathbf{r}^T \mathbf{n} = 0.$$

(13)

To maintain contact, the reciprocal condition must be satisfied for all contact points in the contact plane. The vector $\mathbf{r}$ indicating the line of action and the magnitude ϕ of the contact force can be determined by these conditions. Let $\mathbf{r}_i (i = 1,2,3)$ be three arbitrarily chosen contact points on the contact face, and let $\mathbf{w}_{ni}$ be the unit wrenches associated with these three locations which have the form of (12). Then

$$\mathbf{w}_{ni}^T (\mathbf{v}_0 + \mathbf{A}\mathbf{w}_n \phi) = 0, i = 1,2,3.$$

(14)

Equations (13) and (14) provide four independent equations. Thus. $\mathbf{r}$ and ϕ can be uniquely determined by satisfying the four equations. Again, because the reciprocal condition at any three noncollinear locations on a plane ensures the same condition for all contact points of the plane, the three contact locations can be chosen arbitrarily.

For the compliant motion to be error-reducing, condition (2) must be satisfied for a given point. Thus

$$F_{ff} = \mathbf{d}^T (\mathbf{v}_0 + \mathbf{A}\mathbf{w}_n \phi) < 0.$$

(15)

For convenience, wrenches associated with three vertices on the contact face of the held body can be selected. Since the contact wrench $\mathbf{w}_n \phi$ is obtained by solving (13) and (14) (independent of the configuration), only the $\mathbf{d}$ vector is a function of configuration.

Conditions for error-reducing motion for each of the different types of contact have now been identified. Each is a function of configuration **q**. Next, we consider conditions imposed on a finite number of configurations such that, when satisfied, error reduction is satisfied for the entire set of possible configurations within the contact state.

SECTION IV. Sufficient Conditions for Face-Vertex Contact

As shown in Section II-A, the relative configuration of the bodies for face-vertex contact is described by the translation variables δ_1, δ_2 and orientational variables $(\mathbf{u}, \theta)$. We prove that, if an admittance matrix $\mathbf{A}$ satisfies a set of conditions at the "boundary" points, then the A matrix ensures error-reducing motion for all intermediate configurations $\delta_i \in \left[-\delta_{M_i}, \delta_{M_i}\right]$ and $\theta \in [0, \theta_M]$ (regardless of the direction of rotation).

A. Error-Reduction Function

In order to obtain the error-reduction function in terms of configuration $\mathbf{q}$, we first express the contact wrench and the error-measure vector **d** as functions of $(\delta_1, \delta_2, \mathbf{u}, \theta)$.

For a face-vertex contact state as shown in Fig. 3(a), when the held body rotates relative to the fixtured body about the contact point O, the description of the contact wrench does not change in a body-based coordinate frame. When the held body translates relative to the fixtured body, the description of the contact wrench changes in a body-based coordinate frame because the contact point changes (although its direction is constant). Thus, the contact wrench is a function of only the translational variables δ_1, δ_2.

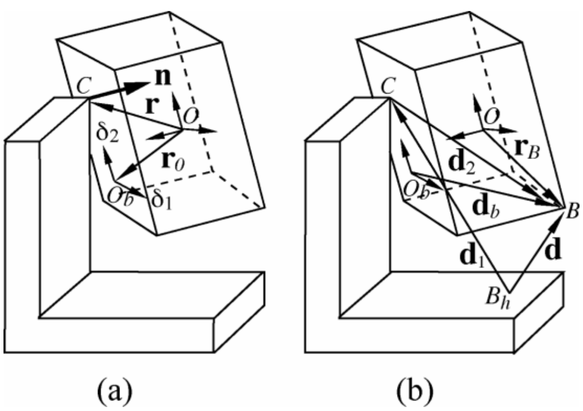

Fig. 3. Face-vertex contact. (a) contact force in the body frame. (b) error-measure vector **d** in the body frame.

For all face-vertex cases, the direction of the surface normal is constant in the body frame while the position vector of the contact point r varies. For arbitrary (δ_1, δ_2), $\mathbf{r}$ can be expressed as

$$\mathbf{r} = \mathbf{r}_0 + \delta_1 \mathbf{b}_1 + \delta_2 \mathbf{b}_2$$

where $\mathbf{r}_0$ is the position vector from the body frame's origin O to the origin of the centrally located coordinate frame O_b, and $\mathbf{b}_1$ and $\mathbf{b}_2$ are unit vectors along the two axes of coordinate frame O_b (constant in body frame). By (4), the unit wrench corresponding to the surface normal is

$$\mathbf{w}_n = \begin{bmatrix} \mathbf{n} \\ \mathbf{r} \times \mathbf{n} \end{bmatrix}$$

(16)

Note that in the body frame, the direction of $\mathbf{w}_n$ is constant while the last component (the moment term) is a linear function of δ_i.

Let B_h be the home position of B (the location where the parts are properly mated) and $\mathbf{d}'$ be the three-vector from B_h to B. As shown in Fig. 3(b), the line vector d associated with error reduction is also a function of configuration. Let $\mathbf{d}_i{}'$ be the three-vector shown in Fig. 3(b) and di be the line vectors (six-vectors) associated with $\mathbf{d}_i{}'$. Namely, let $\mathbf{d}_1{}'$ be the position vector from B_h to the contact point C and $\mathbf{d}_2{}'$ be the position vector from C to B. Then, $\mathbf{d}_1{}'$ is constant in the global frame and $\mathbf{d}_2{}'$, can be expressed as

$$\mathbf{d}_2{}' = \mathbf{d}_b{}' - \delta_1\mathbf{b}_1 - \delta_2\mathbf{b}_2$$

where $\mathrm{d}_b{}'$ is the position vector from the frame origin O_b to point B (constant in body frame). For arbitrary δ_1, δ_2 with $\theta = 0$, the error-measure three-vector d$'$ is

$$\mathbf{d}'(\delta) = \mathbf{d}_1{}' + \mathbf{d}_b{}' - \delta_1\mathbf{b}_1 - \delta_2\mathbf{b}_2, \delta_i \in [-\delta_M, \delta_M].$$

If we denote

$$\delta' = \delta_1\mathbf{b}_1 + \delta_2\mathbf{b}_2$$

then $\mathbf{d}'$ can be expressed as

$$\mathbf{d}'(\delta) = \mathbf{d}_1{}' + \mathbf{d}_b{}' - \delta'.$$

Again, note that $\mathbf{d}_1{}'$ is constant in the global coordinate frame while $\mathbf{b}_2$ and $\mathbf{d}_b{}'$ are constant in the body frame. Thus, for an arbitrary orientation (u, θ) and $\delta_i \in \left[-\delta_{M_i}, \delta_{M_i}\right]$, the error measure three-vector $\mathbf{d}'$ is a function of $u, \theta)$ and δ_i having the form

$$\mathbf{d}'(\mathrm{u}, \theta, \delta) = \mathbf{R}\mathbf{d}_1{}' + \mathbf{d}_b{}' - \delta'$$

where $\mathbf{R}$. is the rotation matrix having the form of (3).

The line vector associated with $\mathbf{d}'$ can be calculated as

$$\mathbf{d}(\delta, \theta) = \begin{bmatrix} \mathbf{R}\mathbf{d}_1{}' \\ \mathbf{r}_B \times \mathbf{R}\mathbf{d}_1{}' \end{bmatrix} + \begin{bmatrix} \mathbf{d}_b{}' \\ \mathbf{r}_B \times \mathbf{d}_b{}' \end{bmatrix} - \begin{bmatrix} \delta' \\ \mathbf{r}_B \times \delta' \end{bmatrix}$$

(17)

where $\mathbf{r}_B$ is the position vector from the body frame origin O to the error measure point B (constant in body frame).

Thus, for any intermediate configuration $(\delta_1, \delta_2, \theta)$, using (16) and (17), the error-reduction function F_{1p}, in (10) can be expressed as a function of $(\delta_1, \delta_2, \mathbf{u}, \theta)$.

Since only small orientational variation is considered, the angular magnitude θ is small $\leq 5^{\mathrm{o}}$. Thus, the rotation matrix $\mathbf{R}$ in (3) can be accurately approximated by

$$\mathbf{R}(\mathbf{u}, \theta) = \mathbf{I} + \sin\theta[\mathbf{u}\times].$$

(18)

In the following, for an arbitrary wrench 6-D line vector $\mathbf{s}$, we denote $\mathbf{s}_{\mathrm{u}}$ as the cross-product operation of $\mathbf{u}$ on $\mathbf{s}$, i.e., if $\mathbf{s}$ has the form

$$\mathbf{s} = \begin{bmatrix} \mathbf{a} \\ \mathbf{r} \times \mathbf{a} \end{bmatrix}$$

then

$$\mathbf{s_u} = \begin{bmatrix} \mathbf{u} \times \mathbf{a} \\ \mathbf{r} \times (\mathbf{u} \times \mathbf{a}). \end{bmatrix}$$

(19)

If we denote the three-vector

$$\mathbf{d}_0{}' = \mathbf{d}_1{}' + \mathbf{d}_b{}'$$

and denote the 6-D line vectors

$$\mathbf{d}_0 = \begin{bmatrix} \mathbf{d}_0{}' \\ \mathbf{r}_B \times \mathbf{d}_0{}' \end{bmatrix}, \delta = \begin{bmatrix} \delta' \\ \mathbf{r}_B \times \delta' \end{bmatrix}$$

then, using (18), the error-reduction function can be accurately approximated by

$$\begin{aligned} F_{1p}(\delta, \theta) \quad &= (\mathbf{d}_0 - \delta)^T (\mathbf{v}_0 \mathbf{w}_n^T - \mathbf{v}_0^T \mathbf{w}_n \mathbf{I}) \mathbf{A} \mathbf{w}_n \\ &+ [\mathbf{d}_{1\mathbf{u}}^T (\mathbf{v}_0 \mathbf{w}_n^T - \mathbf{v}_0^T \mathbf{w}_n \mathbf{I}) \mathbf{A} \mathbf{w}_n] \sin \theta \end{aligned}$$

(20)

where the subscript $\mathbf{u}$ of a line vector indicates the cross-product operation of $\mathbf{u}$ on the vector [as defined in (19)].

Now consider the matrix norm of the six-vector $\mathbf{d}_{1\mathrm{u}}$. Since $\mathbf{1}$ is a unit vector

$$\|\mathbf{d}_{1\mathbf{u}}\| = \left\| \begin{bmatrix} \mathbf{u} \times \mathbf{d}_1{}' \\ \mathbf{r}_B \times (\mathbf{u} \times \mathbf{d}_1') \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \mathbf{d}_1{}' \\ \mathbf{r}_B \times \mathbf{d}_1{}' \end{bmatrix} \right\| = \|\mathbf{d}_1\|.$$

Thus, the second term in (20) is given as

$$\begin{aligned} &[\mathbf{d}_{1\mathbf{u}}^T (\mathbf{v}_0 \mathbf{w}_n^T - \mathbf{v}_0^T \mathbf{w}_n \mathbf{I}) \mathbf{A} \mathbf{w}_n] \sin \theta \\ &\leq \|\mathbf{d}_1\| \cdot \|(\mathbf{v}_0 \mathbf{w}_n^T - \mathbf{v}_0^T \mathbf{w}_n \mathbf{I}) \mathbf{A} \mathbf{w}_n\| \sin \theta \\ &\leq \mathrm{M} \sin \theta_M \end{aligned}$$

where $M = \|\mathbf{d}_1\| \cdot \left\|(\mathbf{v}_0 \mathbf{w}_n^T - \mathbf{v}_0^T \mathbf{w}_n \mathbf{I}) \mathbf{A} \mathbf{w}_n\right\|$ and the norms used are the conventional matrix norms. Note that, in a specified coordinate frame, M is constant.

Now consider the first term in (20)

$$M = \|\mathbf{d}_1\| \cdot \|(\mathbf{v}_0 \mathbf{w}_n^T - \mathbf{v}_0^T \mathbf{w}_n \mathbf{I}) \mathbf{A} \mathbf{w}_n\|$$

Since $\mathbf{w}_n$ only contains linear terms in δ_i, f is a third-order polynomial in δ_1 and δ_2. If we construct a new function

$$F(\delta_1, \delta_2) = f + \mathrm{M} \sin \theta_M$$

(21)

then F is a third-order polynomial in δ_1 and δ_2 and, for all intermediate configurations, we have

$$F_{1p} \leq F(\delta_1, \delta_2).$$

B. Sufficient Conditions for Error Reduction

The error-reduction condition requires that the error-reduction function in (20) must be negative in the range of configurations considered. In order to obtain sufficient conditions, we consider the "more positive" function defined in (21). The third-order polynomial can be written in the form

$$\begin{aligned} F(\delta_1, \delta_2) \quad &= f_1\delta_1^3 + f_2\delta_1^2\delta_2 + f_3\delta_1\delta_2^2 + f_4\delta_2^3 + f_5\delta_1^2 \\ &+f_6\delta_1\delta_2 + f_7\delta_2^2 + f_8\delta_1 + f_9\delta_2 + f_0. \end{aligned}$$

(22)

Consider a single-variable function of δ_2 defined by

$$f_{\delta_2} = F(0, \delta_2) = f_4\delta_2^3 + f_7\delta_2^2 + f_9\delta_2 + f_0.$$

Let

$$f_{M\delta_2} = \max\{|f_4|, |f_7|, |f_9|\}.$$

(23)

Then, as shown in [12], a root of f_{δ_2}, ξ_2, must satisfy

$$|\xi_2| \geq \frac{|f_0|}{f_{M\delta_2} + |f_0|}.$$

(24)

Thus, if

$$\frac{|f_0|}{f_{M\delta_2} + |f_0|} \geq \delta_{M_2}$$

(25)

then f_{δ_2} has no root in $[-\delta_{M_2}, \delta_{M_2}]$.

Denote

$$\begin{aligned} f_m \quad &= \min_{|\delta_2| \leq \delta_{M_2}} \{|f_{\delta_2}|\} \\ c_M \quad &= \max_{|\delta_2| \leq \delta_{M_2}} \{|f_1|, |f_2\delta_2 + f_5|, |f_3\delta_2^2 + f_6\delta_2 + f_8|\}. \end{aligned}$$

(26)(27)

We prove that if

$$\frac{f_m}{c_M + f_m} \geq \delta_{M_1}$$

(28)

then F_{1p} has no root for all $\delta_1 \in [-\delta_{M_1}, \delta_{M_1}]$ and $\delta_2 \in [-\delta_{M_2}, \delta_{M_2}]$.

To prove this, consider the function F in (22). For an arbitrary $\delta_{2_0} \in [-\delta_M, \delta_M], F(\delta_1, \delta_{2_0})$ is a third-order polynomial in a single-variable δ_1 as follows:

$$F_{\delta_1} = f_1\delta_1^3 + (f_2\delta_{2_0} + f_5)\delta_1^2 + (f_3\delta_{2_0}^2 + f_6\delta_{2_0} + f_8)\delta_1 + f_m.$$

Since $\delta_{2_0} \in [-\delta_{M_2}, \delta_{M_2}]$., for any root of $F(\delta_1), \xi_1$, by (26) and (27), we have

$$|\xi_1| \geq \frac{f_m}{c_M + f_m} > \delta_{M_1}.$$

Thus, F_{δ_1} has no root in $[-\delta_{M_1}, \delta_{M_1}]$ for all $\delta_2 \in [-\delta_{M_2}, \delta_{M_2}]$. Since f_m in (26) and c_M in (27) are functions of the admittance $\mathbf{A}$, (28) imposes a constraint on $\mathbf{A}$. In summary, we have the following.

Proposition 1

For a face-vertex contact state, if: 1) at the configuration $\delta_1, \delta_2, \theta) = (0,0,0)$, the admittance satisfies the error reduction condition (2) and 2) condition (28) is satisfied for the polynomial (22), then the admittance will satisfy the error-reduction conditions for all configurations bounded by $\delta_i \epsilon [-\delta_{M_i}, \delta_{M_i}]$ and $\theta \in [0, \theta_M]$, where $\mathbf{u}$ is arbitrary.

Note that, since the functions in (26) and (27) are all polynomials in δ_2 with order no higher than three, the maximum and minimum values of these functions can be obtained analytically by evaluating the function at the boundary points $\pm\delta_{M_2}$ and the stationary points. Thus, to ensure that contact yields error-reducing motion for the body for a face-vertex contact state, only two conditions [(2) and (28)] need to be satisfied.

SECTION V. Sufficient Conditions for Vertex-Face Contact State

In this section, vertex-face contact is considered. As shown in Fig. 2(b), the configuration of the body can be determined by the orientation of the body $(\mathbf{u}, b)$ and the location of the contact point δ_1, δ_2.

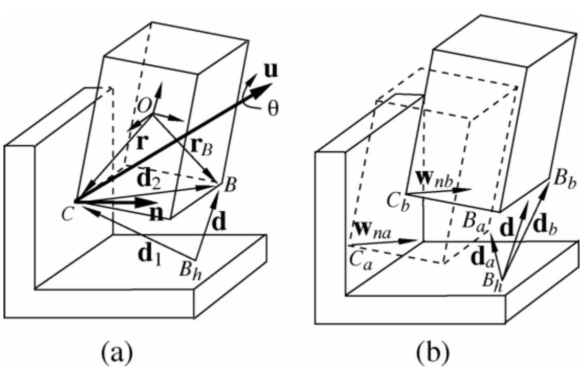

Fig. 4. Vertex—face contact state. (a) orientational variation. (b) translational variation.

Suppose that θ varies within the range of $[0, \theta_M]$ and δ_i varies within the range of $[-\delta_{M_i}, \delta_{M_i}]$. We prove that, if an admittance matrix $\mathbf{A}$ satisfies a set of conditions determined at the "boundary" configurations, then the same admittance will ensure that the motion is error-reducing for any intermediate configuration $\theta \in [0, \theta_M], \delta_i \in [-\delta_M, \delta_M]$.

To prove the results, we first consider configuration variation in orientation and translation separately. Then, by combining the two cases, general results are obtained.

A. Configuration Variation in Orientation

Consider only orientational variation of the contact configuration as illustrated in Fig. 4(a). In this case, the location of the contact vertex of the held body is constant in the face plane, and both the direction of the error-reduction vector $\mathbf{d}$ and the direction of the contact force are changed by changing the orientation. We prove that, for $\theta_M \leq 5^\circ$, if $\mathbf{A}$ satisfies a set of conditions at $\theta = 0$ (defined at a central orientation), then an error-reducing motion is ensured for all configurations obtained by rotating about an arbitrary axis u with angle $\theta < \theta_M$.

1) Error-Reduction Function

Let $\mathbf{w}_0$ be the wrench and $\mathbf{d}_0$ be the error measure line vector associated with $\theta = 0$. Suppose that, at $\theta = 0$, an error-reducing motion is obtained, i.e.,

$$\mathbf{d}_0^T \mathbf{v}_0 + \mathbf{d}_0^T \mathbf{A} \mathbf{w}_0 < 0.$$

(29)

Consider a rotation given by an angle change $\theta \in [0, \theta_M]$ about an axis u. If we denote $\mathbf{n}_0$ as the surface normal associated with $\theta = 0$, then, in the body coordination frame, the surface normal associated with varying $(\mathbf{u}, \theta)$ is

$$\mathbf{n}_\theta = \mathbf{R}(\theta)\mathbf{n}_0$$

(30)

where $\mathbf{R}$. is the rotation matrix having the form of (18).

Since contact is frictionless, the contact force is along the surface normal at the contact point. Thus, the unit contact wrench is

$$\mathbf{w}_n(\theta) = \begin{bmatrix} \mathbf{n}_\theta \\ \mathbf{r} \times \mathbf{n}_\theta \end{bmatrix} = \begin{bmatrix} \mathbf{R}\mathbf{n}_0 \\ \mathbf{r} \times \mathbf{R}\mathbf{n}_0 \end{bmatrix}$$

(31)

where $\mathbf{r}$ is the position vector from the origin of the body frame to the contact point (constant in body frame).

Since the orientational variation considered corresponds to pure rotation about the contact point, the error-measure three-vector d' for an intermediate configuration can be expressed in the body frame as

$$\mathbf{d}' = \mathbf{R}{\mathbf{d}_1}' + {\mathbf{d}_2}'$$

where ${\mathbf{d}_1}'$ is the position three-vector from B_h to the contact point C and ${\mathbf{d}_2}'$ is the position three-vector from C to point B. Note that ${\mathbf{d}_1}'$ is a constant in the global frame and ${\mathbf{d}_2}'$ is constant in the body frame. Then, in the body frame, the line vector associated with $\mathbf{d}'$ is obtained as

$$\mathbf{d}(\theta) = \begin{bmatrix} \mathbf{d}' \\ \mathbf{r}_B \times \mathbf{d}' \end{bmatrix} = \begin{bmatrix} \mathbf{R}{\mathbf{d}_1}' \\ \mathbf{r}_B \times \mathbf{R}{\mathbf{d}_1}' \end{bmatrix} + \begin{bmatrix} {\mathbf{d}_2}' \\ \mathbf{r}_B \times {\mathbf{d}_2}' \end{bmatrix}$$

(32)

where $\mathbf{r}_B$ is the position vector from the body frame origin to point B.

By (10), the error-reduction function can be written as

$$F_{1p}(\theta) = \mathbf{d}^T \mathbf{v}_0 (\mathbf{w}_n^T \mathbf{A} \mathbf{w}_n) - \mathbf{d}^T \mathbf{A} \mathbf{w}_n (\mathbf{w}_n^T \mathbf{v}_0).$$

(33)

From (31) and (32), it can be seen that $\mathbf{W}_n$ and $d(\theta)$ involve the rotation matrix $\mathbf{R}$.
Substituting (31) and (32) into (33) and using (18), the error-reduction function can be expressed as a function of $(\mathbf{u}, \theta)$ in the form

$$F_{1p}(\theta) = F_{1p}(0) + F_1 \sin\theta + F_2 \sin^2\theta + F_3 \sin^3\theta$$

(34)

where

$$\begin{aligned}
F_1 \quad &= \mathbf{d}_{1\mathbf{u}}^T (\mathbf{v}_0 \mathbf{w}_0^T - \mathbf{v}_0^T \mathbf{w}_0 \mathbf{I}) \mathbf{A} \mathbf{w}_0 \\
&+ \mathbf{d}_0^T (\mathbf{v}_0 \mathbf{w}_0^T - \mathbf{v}_0^T \mathbf{w}_0 \mathbf{I}) \mathbf{A} \mathbf{w}_{0\mathbf{u}} \\
&+ \mathbf{d}_0^T (\mathbf{v}_0 \mathbf{w}_{0\mathbf{u}}^T - \mathbf{v}_0^T \mathbf{w}_{0\mathbf{u}} \mathbf{I}) \mathbf{A} \mathbf{w}_0 \\
F_2 \quad &= \mathbf{d}_{1\mathbf{u}}^T (\mathbf{v}_0 \mathbf{w}_0^T - \mathbf{v}_0^T \mathbf{w}_0 \mathbf{I}) \mathbf{A} \mathbf{w}_{0\mathbf{u}} \\
&+ \mathbf{d}_0^T (\mathbf{v}_0 \mathbf{w}_{0\mathbf{u}}^T - \mathbf{v}_0^T \mathbf{w}_{0\mathbf{u}} \mathbf{I}) \mathbf{A} \mathbf{w}_{0\mathbf{u}} \\
&+ \mathbf{d}_{1\mathbf{u}}^T (\mathbf{v}_0 \mathbf{w}_{0\mathbf{u}}^T - \mathbf{v}_0^T \mathbf{w}_{0\mathbf{u}} \mathbf{I}) \mathbf{A} \mathbf{w}_0 \\
F_3 \quad &= \mathbf{d}_{1\mathbf{u}}^T (\mathbf{v}_0 \mathbf{w}_{0\mathbf{u}}^T - \mathbf{v}_0^T \mathbf{w}_{0\mathbf{u}} \mathbf{I}) \mathbf{A} \mathbf{w}_{0\mathbf{u}}
\end{aligned}$$

where $\mathbf{w}_0$ and $\mathbf{d}_0$ are the wrench and the error measure line vector when $\theta = 0$, respectively, and where the subscript u of a line vector indicates the cross-product operation of **u** on the vector as defined in (19).

2) Error-Reduction Conditions

To achieve error reduction at all other orientations considered, $F_{1p}(\theta)$ must be negative for $\theta \in [0, \theta_M]$ and an arbitrary rotation axis u. Since **u** is a unit vector, the bounds for Fi in (34) can be obtained.

If we denote

$$M = \|\mathbf{d}_0\| \cdot \|(\mathbf{v}_0 \mathbf{w}_0^T - \mathbf{v}_0^T \mathbf{w}_0 \mathbf{I}) \mathbf{A}\| \cdot \|\mathbf{w}_0\|$$

where the norm used is the conventional matrix norm, then

$$|F_1| \leq 3M, |F_2| \leq 3M, |F_3| \leq M.$$

Consider the new function constructed by

$$F = F_{1p}(0) + 3M \sin\theta_M + 3M \sin^2\theta_M + M \sin^3\theta_M.$$

Then, for $\theta \in [0, \theta_M]$ with an arbitrary rotation axis, we have

$$F_{1p}(\mathbf{u}, \theta) \leq F.$$

Thus, if

$$F = F_{1p}(0) + 3M \sin\theta_M + 3M \sin^2\theta_M + M \sin^3\theta_M < 0$$

(35)

then $F_{1p}(\mathbf{u},\theta) < 0$ for all orientational variations considered.

B. Configuration Variation in Translation

Now consider the translational variation of the contact configuration illustrated in Fig. 4(b). In this case, only translation in the contact face is allowed, and the contact force does not change in the body frame. For a given orientation, the configuration of the body can be determined by the location (δ_1, δ_2) of the vertex C.

Suppose that, at the two configurations described by d_α, and d_b, the error-reduction conditions are satisfied as follows:

$$\mathbf{d}_a^T\mathbf{v}_0 + \mathbf{d}_a^T\mathbf{A}\mathbf{w}_{na} < 0$$
$$\mathbf{d}_b^T\mathbf{v}_0 + \mathbf{d}_b^T\mathbf{A}\mathbf{w}_{nb} < 0$$

(36)(37)

where $\mathbf{w}_{na}$ and $\mathbf{w}_{nb}$ are the contact wrenches at $\mathbf{d}_a$ and $\mathbf{d}_f$, respectively. Since the contact wrench $\mathbf{W}_n$ is the same in the body frame for all contact configurations, $\mathbf{w}_n = \mathbf{w}_{na} = \mathbf{w}_{nb}$. Thus, for any $\alpha, \beta \geq 0$, we have

$$(\alpha\mathbf{d}_a + \beta\mathbf{d}_b)^T\mathbf{v}_0 + (\alpha\mathbf{d}_a + \beta\mathbf{d}_b)^T\mathbf{A}\mathbf{w}_n < 0.$$

(38)

Consider $\mathrm{d}_a(\delta_1, \delta_2)$ and $\mathbf{d}_b(\delta_1{}', \delta_2)$ at two configurations with the same δ_2. Let $\mathrm{d}(\delta_{1_0}, \delta_2)$ be an arbitrary line vector with the same δ_2 but different $\delta_{1_0} \in [\delta_1, \delta_1{}']$. Since the ends of these three vectors must be on a straight line. $\mathbf{d}$ is a convex combination of the vectors $\mathbf{d}_a$ and $\mathbf{d}_f$, i.e.,

$$\mathbf{d} = \alpha\mathbf{d}_a + \beta\mathbf{d}_b$$

(39)

where $\alpha, \beta \geq 0$, and $\alpha + \beta = 1$.

Substituting (39) into (38) yields

$$\mathbf{d}^T\mathbf{v}_0 + \mathbf{d}^T\mathbf{A}\mathbf{w}_n < 0.$$

Thus, if at two configurations $-\delta_{M_1}, \delta_2$ and $(\delta_{M_1}, \delta_2$:) the error-reduction condition is satisfied, then the error-reduction condition must be satisfied for all intermediate configurations δ_1, δ_2 with $\delta_1 \in [-\delta_{M_1}, \delta_{M_1}]$. The same result holds true for variation in δ_2 while δ_1 is constant.

C. General Case

The results presented in Sections V-A and B can be generalized to intermediate vertex-face contact configurations involving both translational and orientational variations from configurations at which the conditions were imposed.

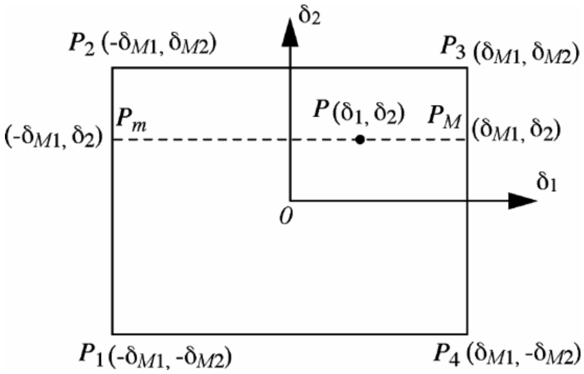

Fig. 5. Error-reduction condition for general vertex-face contact state. By satisfying the orientational variation conditions at four translational boundary configurations, the error-reducing motion for all intermediate configurations is ensured.

In the $\delta_1 - \delta_2$ plane, consider the rectangular region defined by the four extremal points $P_i(i = 1, \dots ,4)$ as shown in Fig. 5. Suppose that, at these four boundary points, condition (35) is satisfied. Then, at these four locations, the error-reduction condition must be satisfied for all orientations $(\mathrm{u}\ \theta)$ with $\theta \in [0, \theta_M]$.

Let $P(\delta_1, \delta_2, \mathbf{u}, \theta)$ be an arbitrary configuration with $\delta_i \in \left[-\delta_{M_i}, \delta_{M_i}\right]$. and $\theta \in [0, \theta_M]$.

Consider first, the two configurations P_m and P_M determined by $(-\delta_{M_1}, \delta_2$:) and (δ_{M_1}, δ_2), respectively. Since at configurations P_1 and P_2 the error-reduction condition (2) and inequality (35) are satisfied, by the results presented in Section V-B, the error-reduction condition must be satisfied at configuration P_m for all orientations considered. By the same reasoning, the error-reduction condition is also satisfied at the configuration P_M. Then, because the error-reduction condition is satisfied at P_m and P_M ", by the results presented in Section V -B, the error-reduction condition must also be satisfied for any $\delta_1 \in \left[-\delta_{M_1}, \delta_{M_1}\right]$. Thus, we have the following proposition.

Proposition 2

For a vertex-face contact state with variation of orientation $[0, \theta_M]$ and variation of translation $\left[-\delta_{M_i}, \delta_{M_i}\right]$, if inequality (35) is satisfied at the four translational boundary points $\left(\pm\delta_{M_1}, \pm\delta_{M_2}\right)$, then the admittance will satisfy the error-reduction condition for all configurations bounded by $\delta_i \in \left[-\delta_{M_i}, \delta_{M_i}\right]$., and $\theta \in [0, \theta_M]$ in any rotation direction.

Thus, for a vertex-face contact state, to ensure that the motion response due to contact is error reducing for all configurations considered, only four conditions need be satisfied.

SECTION VI. Sufficient Conditions for Edge-Edge Cross Contact

Below, for edge-edge cross contact, we identify the set of conditions that, when satisfied for a given admittance matrix $\mathbf{A}$ at the "boundary" points, ensures error-reducing motion for all intermediate configurations $\theta \in [0, \theta_M], \delta_i \in \left[-\delta_{M_i}, \delta_{M_i}\right]$.

A. Error-Reduction Function

In order to obtain the error-reduction function, we first express the contact wrench and the error-measure vector d in terms of δ_i and θ.

For an edge-edge cross contact state as shown in Fig. 6(a), the direction of the contact force is along the common normal of the two edges. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the two unit vectors along the two edges, respectively, then the direction of the force must be along $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2$. Note that $\mathbf{e}_1$ is constant in the body frame while $\mathbf{e}_2$ is constant in the global frame. When the held body rotates relative to the fixtured body about the contact point O, the vector $\mathbf{e}_2$ in the body frame can be expressed as $\mathbf{Re}_2$ where $\mathbf{R}$ is the rotation matrix. When the held body translates relative to the fixtured body along $\mathbf{e}_1$, as shown in Fig. 6(a), the description of the contact wrench changes in a body-based coordinate frame as the contact point changes (although its direction is constant). Thus, the contact wrench is a function involving both the translational and orientational variables $(\delta_1, \delta_2, \theta)$.

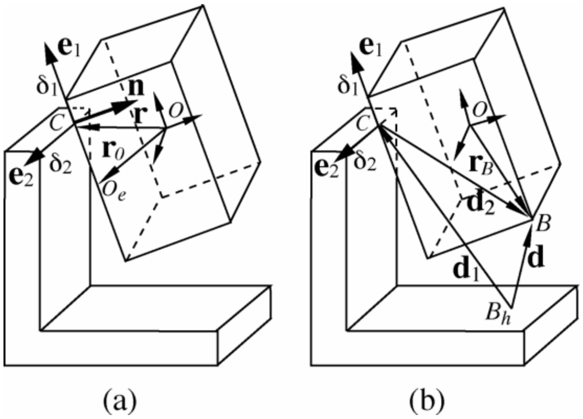

Fig. 6. Edge-edge cross contact. (a) contact force in the body frame. (b) error-measure vector d in the body frame.

For all edge-edge cross contact cases, the direction of the force depends only on the orientational variation while the position vector of the contact point r depends only on the translational variation along the contact edge of the held body $\mathbf{e}_1$. For arbitrary (δ_1, δ_2), $\mathbf{r}$ can be expressed as

$$\mathbf{r} = \mathbf{r}_0 + \delta_1 \mathbf{e}_1$$

where $\mathbf{r}_0$ is a vector from the body frame to a centrally located point on the edge $\mathbf{e}_1$ (constant). By (4), the unit wrench corresponding to the surface normal is

$$\mathbf{w}_n = \begin{bmatrix} \mathbf{n} \\ \mathbf{r} \times \mathbf{n} \end{bmatrix}.$$

(40)

Note that the direction of $\mathbf{w}_n$ is determined by $\mathbf{e}_1$ and $\mathbf{e}_2$ and the last component (the moment term) is a linear function of δ_1.

Let $\mathbf{d}_1{}'$ and $\mathbf{d}_2{}'$ be the two vectors from B_h. to C and from C to B for $(\delta,\ \theta) = (0,0)$, respectively, then, as shown in Fig. 6(b), for arbitrary (δ_1, δ_2) with $\theta = 0$, the error-measure vector $\mathbf{d}'$ is

$$\mathbf{d}(\delta)' = \mathbf{d}_1{}' + \mathbf{d}_2{}' + \delta_1 \mathbf{e}_1 + \delta_2 \mathbf{e}_2, \delta_i \in \left[-\delta_{M_i}, \delta_{M_i}\right].$$

Note that $\mathbf{d}_1{}'$ and e2 are constant in the global coordinate frame while $\mathbf{d}_2{}'$, and $\mathbf{e}_1$ are constant in the body frame. Thus, for an arbitrary orientation $(\mathbf{u}, \theta)$ and contact location $\delta_i \in \left[-\delta_{M_i}, \delta_{M_i}\right]$, the error-measure three-vector **d'** is a function of $(\mathbf{u}, \theta)$ and δ_i having the form

$$\mathbf{d}'(\mathbf{u}, \theta, \delta) = \mathbf{R}(\mathbf{d}_1{}' + \delta_2 \mathbf{e}_2) + \mathbf{d}_2{}' + \delta_1 \mathbf{e}_1$$

where $\mathbf{R}$ is the rotation matrix.

Let $\mathbf{d}_i (i = 1{,}2)$ be the line vectors associated with $\mathbf{d}_i{}'$. If we denote

$$
\begin{aligned}
\delta_1 \quad &= \delta_1 \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{r}_B \times \mathbf{e}_1 \end{bmatrix} \\
\delta_2 \quad &= \delta_2 \begin{bmatrix} \mathbf{e}_2 \\ \mathbf{r}_B \times \mathbf{e}_2 \end{bmatrix} \\
\text{and } \mathbf{d}_{1\mathbf{R}} \quad &= \begin{bmatrix} \mathbf{Rd}_1{}' \\ \mathbf{r}_B \times \mathbf{Rd}_1{}' \end{bmatrix} \\
\delta_{2\mathbf{R}} \quad &= \delta_2 \begin{bmatrix} \mathbf{Re}_2 \\ \mathbf{r}_B \times \mathbf{Re}_2 \end{bmatrix}
\end{aligned}
$$

then the error-measure function $\mathbf{d}$ can be expressed as

$$\mathbf{d} = (\mathbf{d}_{1\mathbf{R}} + \delta_{2\mathbf{R}}) + \mathbf{d}_2 + \delta_1.$$

For rotation $\mathbf{R}$, the direction of the force is

$$\mathbf{n} = \mathbf{e}_1 \times \mathbf{Re}_2.$$

The unit contact wrench can be expressed as

$$\mathbf{w}_n = \begin{bmatrix} \mathrm{e}_1 \times \mathbf{Re}_2 \\ \mathbf{r} \times (\mathbf{e}_1 \times \mathbf{Re}_2) \end{bmatrix}.$$

For small θ, the expression of $\mathbf{R}$ in (18) provides an accurate approximation. Thus, $\mathbf{w}_n$ and $\mathbf{d}$ can be expressed in terms of $\mathbf{u}$ and $\sin\theta$ as

$$
\begin{aligned}
\mathbf{w}_n \quad &= \mathbf{w}_0 - \mathbf{w}_{0\mathbf{u}} \sin\theta \\
\mathbf{d} \quad &= \mathbf{d}_1 + \mathbf{d}_2 + \delta_1 + \delta_2 + (\mathbf{d}_1 + \delta_2)_{\mathbf{u}} \sin\theta
\end{aligned}
$$

where $\mathbf{w}_0$ is the wrench when $\theta = 0$ and the subscript $\mathbf{u}$ of a wrench indicates the cross-product operation of $\mathbf{u}$ on the wrench [as defined in (19)].

Substituting the above $\mathbf{w}_n$ and $\mathbf{d}$ into (10) and sorting the coefficients of $\sin\theta$, the error-reduction function can be expressed as

$$
\begin{aligned}
&F_{1p}(\delta, \theta) = F_0 + F_1 \sin\theta + F_2 \sin^2\theta + F_3 \sin^3\theta \\
\text{where } F_0 \quad &= (\mathbf{d}_1 + \mathbf{d}_2 + \delta_1 + \delta_2)^T (\mathbf{v}_0 \mathbf{w}_0^T - \mathbf{v}_0^T \mathbf{w}_0 \mathbf{I}) \mathbf{A} \mathbf{w}_0 \\
F_1 \quad &= -(\mathbf{d}_1 + \mathbf{d}_2 + \delta_1 + \delta_2)^T (\mathbf{v}_0 \mathbf{w}_{0\mathbf{u}}^T - \mathbf{v}_0^T \mathbf{w}_{0\mathbf{u}} \mathbf{I}) \mathbf{A} \mathbf{w}_0 \\
&\quad -(\mathbf{d}_1 + \mathbf{d}_2 + \delta_1 + \delta_2)^T (\mathbf{v}_0 \mathbf{w}_0^T - \mathbf{v}_0^T \mathbf{w}_0 \mathbf{I}) \mathbf{A} \mathbf{w}_{0\mathbf{u}} \\
&\quad +(\mathbf{d}_1 + \delta_2)_{\mathbf{u}}^T (\mathbf{v}_0 \mathbf{w}_0^T - \mathbf{v}_0^T \mathbf{w}_0 \mathbf{I}) \mathbf{A} \mathbf{w}_0 \\
&\quad -(\mathbf{d}_1 + \delta_2)_{\mathbf{u}}^T (\mathbf{v}_0 \mathbf{w}_{0\mathbf{u}}^T - \mathbf{v}_0^T \mathbf{w}_{0\mathbf{u}} \mathbf{I}) \mathbf{A} \mathbf{w}_0 \\
&\quad -(\mathbf{d}_1 + \delta_2)_{\mathbf{u}}^T (\mathbf{v}_0 \mathbf{w}_0^T + \mathbf{v}_0^T \mathbf{w}_0 \mathbf{I}) \mathbf{A} \mathbf{w}_{0\mathbf{u}} \\
F_3 \quad &= (\mathbf{d}_1 + \delta_2)_{\mathbf{u}}^T (\mathbf{v}_0 \mathbf{w}_{0\mathbf{u}}^T - \mathbf{v}_0^T \mathbf{w}_{0\mathbf{u}} \mathbf{I}) \mathbf{A} \mathbf{w}_{0\mathbf{u}}.
\end{aligned}
$$

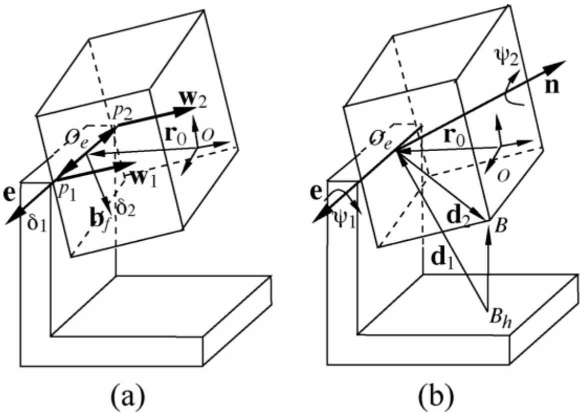

Fig. 7. Face-edge contact state. (a) the representative wrenches are chosen on the edge. (b) the error-measure vector is decomposed into two components.

Similar to the results presented in Section V-A.2, because u is a unit vector, each F_i in the above equation is bounded. If we denote

$$F_{M_i} = max\{|F_i|\}, i = 1,2,3$$

and consider the function defined by

$$F = F_0 + F_{M_1} \sin\theta_M + F_{M_2} \sin^2\theta_M + F_{M_3} \sin^3\theta_M$$

(41)

then F is a linear function in δ_1 and δ_2. Then, for all $\delta_i \in [-\delta_{M_i}, \delta_{M_i}]$ and $\theta \in [0, \theta_M]$

$$F_{1p} \leq F.$$

Thus, if F is negative for $\delta_i \in [-\delta_{M_i}, \delta_{M_i}]$, then F_{1p} must be negative for all $\delta_i \in [-\delta_{M_i}, \delta_{M_i}]$ and for all rotations with $\theta \leq \theta_M$ in any direction. Since F is a linear function in δ_1 and $\delta_2, F < 0$ for all δ_i's in the bounded area if and only if, at the four extremal points $(\pm\delta_{M_1}, \pm\delta_{M_2}), F < 0$. Thus, we have the following proposition.

Proposition 3

For an edge-edge cross contact state with variation of orientation $[0, \theta_M]$ and variation of translation $[-\delta_{M_i}, \delta_{M_i}]$, if, at the four translational boundary points $(\pm\delta_{M_1}, \pm\delta_{M_2})$ the function F defined in (41) is negative, then the admittance will satisfy the error-reduction condition for all configurations bounded by $\delta_i \in [-\delta_{M_i}, \delta_{M_i}]$ and rotation in an arbitrary direction with angle $\theta \leq \theta_M$.

SECTION VII. Sufficient Conditions for Face-Edge Contact

As shown in Fig. 7, four parameters $(\delta_1, \delta_2, \psi_1, \psi_2)$ are chosen to describe the relative configuration variation of the bodies for face-edge contact. The parameter δ_1 describes translation along the edge $\mathbf{e}$, δ_2 describes translation along the direction perpendicular to the edge in the face plane $\mathbf{b}_f$, while ψ_1 and ψ_2 describe rotations about the edge e and the face normal $\mathbf{n}$, respectively.

First, we consider the case for which δ_1 is constant while δ_2 varies. For this case, the body has no translation along the edge e. As shown in Section III-B, the resultant contact wrench can be represented by two representative wrenches chosen on the edge. Here, two representative wrenches are chosen on the edge at fixed locations $p_i (i = 1,2)$ as illustrated in Fig. 7(a).

Suppose that the position of p_i relative to a reference point O_e on the edge is $\mathbf{r}_{ei}$. Then, the two wrenches have the form

$$\mathbf{w}_i = \begin{bmatrix} \mathbf{n} \\ (\mathbf{r}_0 + \mathbf{r}_{ei}) \times \mathbf{n} \end{bmatrix}$$

where $\mathbf{r}_0$ is the position vector from the origin of the body frame to point O_e. Note that $\mathbf{r}_{ei}$ is constant in the global frame and a rotation about the contact edge e does not influence the expressions of $\mathbf{r}_{ei}$ and $\mathbf{b}_f$ in the body frame. Then, for translational variation δ_2 and orientational variation (ψ_1, ψ_2), the wrenches have the form

$$\mathbf{w}_i = \begin{bmatrix} \mathbf{n} \\ \left(\mathbf{r}_0 + \mathbf{R}\mathbf{r}_{ei} - \delta_2 \mathbf{R}\mathbf{b}_f\right) \times \mathbf{n} \end{bmatrix}$$

where $\mathbf{b}_f$ is the unit vector in the direction perpendicular to the edge in the contact face of the held body.

Consider the error-measure vector $\mathbf{d}'$. As illustrated in Fig. 7(b), $\mathbf{d}'$ can be expressed as

$$\mathbf{d}' = \mathbf{d}_1{}' + \mathbf{d}_2{}'$$

where $\mathbf{d}_1{}'$ is the position vector from the home point B_h to point O_e and $\mathbf{d}_2{}'$ is the position vector from O_e to the error-measure point B. Note that $\mathbf{d}_1{}'$ is constant in the global frame. For translational variation δ_2 and orientational variation (ψ_1, ψ_2) the error-reduction vector has the form

$$\mathbf{d}' = \mathbf{R}\left(\mathbf{d}_1{}' + \delta_2 \mathbf{b}_f\right) + \mathbf{d}_2{}'.$$

The line vector associated with $\mathbf{d}$ is calculated as

$$\mathbf{d} = \begin{bmatrix} \mathbf{R}\left(\mathbf{d}_1{}' + \delta_2 \mathbf{b}_f\right) + \mathbf{d}_2{}' \\ \mathbf{r}_B \times \left[\mathbf{R}\left(\mathbf{d}_1 + \delta_2 \mathbf{b}_f\right) + \mathbf{d}_2\right] \end{bmatrix}.$$

Let $\mathbf{R}_\psi$ and $\mathbf{R}_{\psi_2}$ be the rotation matrices associated with the two rotations about the edge $\mathbf{e}$ and the face normal $\mathbf{n}$, respectively. For small ψ_i, $\mathbf{R}_{\psi_i}$ has the form of (18). The total rotation matrix $\mathbf{R}$ is

$$\mathbf{R} = \mathbf{I} + \sin\psi_1 [\mathbf{e}\,\times] + \sin\psi_2 [\mathbf{n}\,\times]$$

(42)

where $[\mathbf{e}\,\times]$ and $[\mathbf{n}\,\times]$ are antisymmetric matrices associated with the cross-product operation of $\mathbf{e}$ and $\mathbf{n}$, respectively.

Substituting the above $\mathbf{w}_i$, $\mathbf{d}$, and $\mathbf{R}$ into the error-reduction function (11) and neglecting the second-order and higher order terms involving sin ψ_1 and sin ψ_2, we have

$$F_{lc} = f_4 \delta_2^4 + \cdots + f_1 \delta_2 + f_0$$

(43)

where $f_i{}'$s have the form

$$f_i = a_i \sin\psi_1 + b_i \sin\psi_2 + c_i$$

and a_i, b_i and c_i are functions of the admittance $\mathbf{A}$.

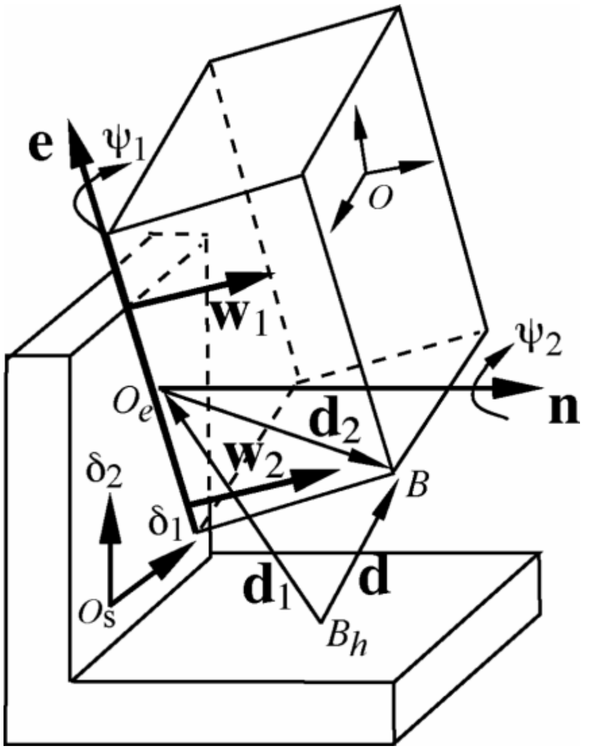

Fig. 8. Edge-face contact state. The two representative wrenches are chosen on contact edge of the held body.

If we denote

$$f_M = \max\{|\alpha_i| \sin\psi_{M_1} + |b_i| \sin\psi_{M_2} + |c_i|, i = 1,2,3,4\}$$
$$c_m = \min\{|a_0 \sin\psi_1 + b_0 \sin\psi_2 + c_0|, |\psi_i| \leq \psi_{M_i}\}$$

then the condition

$$\frac{c_m}{f_M + c_m} > \delta_{M_2}$$

(44)

guarantees that, for all $\psi_i \in [-\psi_{M_i}, \psi_{M_i}]$, F_{lc} has no root over $[-\delta_{M_2}, \delta_{M_2}]$.

Now consider the body's translation along the edge e. Note that, for any given orientation and δ_2, a variation on. δ_1 (a translation along the edge) does not change the contact force. Thus, the same procedure used in Section V-B applies. Therefore, we have the following proposition.

Proposition 4

For a face-edge contact state with variation of orientations $[-\psi_{M_1}, \psi_{M_1}]$ about the edge and $[-\psi_{M_2}, \psi_{M_2}]$ about the face normal, and variation of translation $[-\delta_{M_i}\delta_{M_i}]$, if, at the four configurations with different contact boundary locations $[(\delta_1, \delta_2) = (\pm\delta_{M_1}, \pm\delta_{M_2})]$: 1) the admittance satisfies the error-reduction conditions and 2) inequality (44) is satisfied for $\pm\psi_{M_1}$ and $\pm\psi_{M_2}$, then the admittance will satisfy the error-reduction condition for all configurations bounded by the configurations $\delta_i \in [-\delta_{M_i}, -\delta_{M_i}]$ and $\psi_i \in [-\psi_{M_i}, \psi_{M_i}]$

SECTION VIII. Sufficient Conditions for Edge-Face Contact

In this section, edge-face contact is considered. As shown in Fig. 8, a reference point O_e is chosen on the held body edge. The translation of the body can be described by the location of $O_e(\delta_1, \delta_2)$ in the plane of the contact face. The orientation can be described by a rotation ψ_1 about the edge $\mathbf{e}$ and a rotation ψ_2 about the axis $\mathbf{n}$ along the normal of the face. Note that $\mathbf{e}$ is constant in the body frame and $\mathbf{n}$. is constant in the global frame. Since a translation does not change the contact force, the same procedure used in Section V can be applied to this case in which the orientational and translational variations can be analyzed separately.

First, we consider orientational variation only. Let $\mathbf{w}^1$ and $\mathbf{w}^2$ be the two representative wrenches fixed on the contact edge of the held body with position $\mathbf{r}_i$ relative to the body frame O. At a given configuration, $\mathbf{w}_i$ has the form

$$\mathbf{w}_i = \begin{bmatrix} \mathbf{n} \\ \mathbf{r}_i \times \mathbf{n} \end{bmatrix}.$$

Let $\mathbf{R}_{\psi_1}$ and $\mathbf{R}_{\psi_2}$ be the rotation matrices associated with the two rotations. Since a rotation about $\mathbf{n}$ does not influence $\mathbf{w}_i$ in the body frame, for an arbitrary orientation variation, the wrench has the form

$$\mathbf{w}_i = \begin{bmatrix} \mathbf{R}_{\psi_1}\mathbf{n} \\ \mathbf{r}_i \times \mathbf{R}_{\psi_1}\mathbf{n} \end{bmatrix}.$$

(45)

The error-measure vector $\mathbf{d}'$ can be expressed as

$$\mathbf{d}' = \mathbf{d}_1{}' + \mathbf{d}_2{}'$$

where $\mathbf{d}_1{}'$ is the position three-vector from the home point B_h to O_e and $\mathbf{d}_2{}'$ is the position three-vector from O_e to B. Note that $\mathbf{d}_1'$ is constant in the global frame while $\mathbf{d}_2{}'$ is constant in the body frame. Thus, for an arbitrary orientation variation, $\mathbf{d}'$ has the form

$$\mathbf{d}' = \mathbf{R}\mathbf{d}_1{}' + \mathbf{d}_2{}'.$$

(46)

For small ψ_1 and ψ_2, (42) can be used for the rotation matrix $\mathbf{R}$ associated with ψ_3 and ψ_2. Thus, (46) can be written as

$$\mathbf{d}' = \mathbf{d}_1{}' + \sin\psi_1(\mathbf{e} \times \mathbf{d}_1') + \sin\psi_2(\mathbf{n} \times \mathbf{d}_1') + \mathbf{d}_2{}'.$$

If we denote

$$\mathbf{d}_{\psi_1\psi_2}{}' = \sin\psi_1(\mathbf{e} \times \mathbf{d}_1') + \sin\psi_2(\mathbf{n} \times \mathbf{d}_1')$$

then the line vector associated with $\mathbf{d}'$ is

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_1' + \mathbf{d}_2' \\ \mathbf{r}_B \times (\mathbf{d}_1' + \mathbf{d}_2') \end{bmatrix} + \begin{bmatrix} \mathbf{d}_{\psi_1\psi_2}' \\ \mathbf{r}_B \times \mathbf{d}_{\psi_1\psi_2}' \end{bmatrix}.$$

(47)

By (11), the error-reduction function is

$$F_{lc} = \det\left(\mathbf{W}^T\mathbf{A}\mathbf{W}\right)\mathbf{d}^T\mathbf{v}_0 - \mathbf{d}^T\mathbf{A}\mathbf{W}\left[\mathbf{W}^T\mathbf{A}\mathbf{W}\right]^*\mathbf{W}^T\mathbf{v}_0.$$

Since ψ_1 and ψ_2 are small, neglecting all second-order or higher order terms involving $\sin\psi_1$ and $\sin\psi_2$, we have

$$F_{lc} = f_0 + f_1\sin\psi_1 + f_2\sin\psi_2$$

(48)

where $f_i{}'$s are functions of the admittance $\mathbf{A}$.

Because $\sin\psi_1$ and $\sin\psi_2$ are monotonic functions for small ψ_1 and ψ_2 [e.g., $\psi_i \le (\pi/10)$], F_{lc} is negative for all $\psi_i \in \left[-\psi_{M_i}, \psi_{M_i}\right]$ if and only if, at the four boundary points $\left(\pm\psi_{M_i}, \pm\psi_{M_2}\right), F_{lc} < 0$.

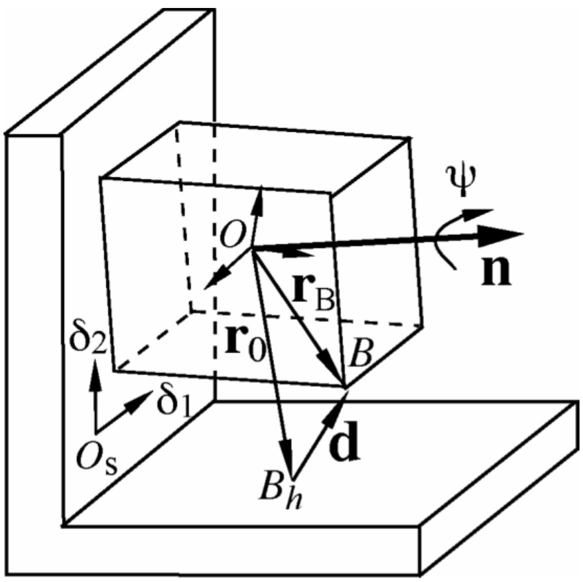

Fig. 9. Face-face contact state.

For a translational variation, similar to the case in Section V-B, it can be proved that, for a given orientation, if, at four translational locations the condition $F_{lc} < 0$ is satisfied, then, for any intermediate location bounded by these four points, the same condition must be satisfied for the given orientation. Thus, we have the following proposition.

Proposition 5

For an edge-face contact state with variation of orientations $\left[-\psi_{M_1}, \psi_{M_1}\right]$ about the edge and $\left[-\psi_{M_2}, \psi_{M_2}\right]$ about the normal direction and variation of translation $\left[-\delta_{M_i}\delta_{M_i}\right]$, if, at the four translational boundary locations $\left(\pm\delta_{M_1}, \pm\delta_{M_2}\right)$ the function F_{lc} in (48) is negative for each $\pm\psi_{M_i}$, then the admittance will satisfy the error-reduction condition for all configurations bounded by the configurations $\delta_i \in \left[-\delta_{M_i}, -\delta_{M_i}\right]$and $\psi_i \in \left[-\psi_{M_i}, \psi_{M_i}\right]$.

SECTION IX. Face-Face Contact State

Consider face-face contact as shown in Fig. 9. If the contact is maintained, the motion of the body occurs in the plane containing the two faces. Thus, the configuration of the body can be described with three parameters $(\delta_1, \delta_2, \psi)$.

Let wn be the unit wrench associated with the resultant contact force, then, as shown in Section III-C, $\mathbf{w}_n$ is constant in the body frame. In a centered configuration with $(\delta_1, \delta_2, \psi)$ being zeros, the error-measure vector can be expressed as

$$\mathbf{d}' = \mathbf{r}_B - \mathbf{r}_0$$

where $\mathbf{r}_0$ is the position vector from the body frame origin at a centrally located configuration to the home point of B_h and $\mathbf{r}_B$ is the position vector from the body frame origin to point B. For arbitrary $(\delta_1, \delta_2, \psi)$, the error-measure vector is

$$\mathbf{d}' = \mathbf{r}_B - \mathbf{R}(\mathbf{r}_0 + \delta_1\mathbf{s}_1 + \delta_2\mathbf{s}_2)$$

where $\mathbf{s}_i$'s are unit vectors along the two coordinate axes on the stationary face (constant in global frame) and $\mathbf{R}$ is the rotation matrix associated with ψ in the direction $\mathbf{n}$.

Let

$$\delta = \delta_1\mathbf{s}_1 + \delta_2\mathrm{s}_2.$$

The line vector associated with $\mathbf{d}'$ is

$$\mathbf{d} = \begin{bmatrix} \mathbf{r}_B - \mathbf{R}(\mathbf{r}_0 + \delta) \\ \mathbf{r}_B \times [\mathbf{r}_B - \mathbf{R}(\mathbf{r}_0 + \delta)] \end{bmatrix} = \begin{bmatrix} \mathbf{r}_B - \mathbf{R}(\mathbf{r}_0 + \delta) \\ -\mathbf{r}_B \times \mathbf{R}(\mathbf{r}_0 + \delta) \end{bmatrix}.$$

The error-reduction function F_{ff} (15) is

$$F_{ff} = \mathbf{d}^T(\mathbf{v}_0 + \mathbf{A}\mathbf{w}).$$

Note that, in F_{ff}, only $\mathbf{d}$ contains the orientation matrix $\mathbf{R}$. Using (3) for $\mathbf{R}$ with $\mathbf{u}$ replaced by $\mathbf{n}$, the error-reduction function can be expressed in the form

$$\begin{aligned} F_{ff} \quad &= (a_1\delta_1 + a_2\delta_2 + a_0) + (b_1\delta_1 + b_2\delta_2 + b_0)\sin\psi \\ &+(c_1\delta_1 + c_2\delta_2 + c_0)\cos\psi \end{aligned}$$

(49)

where a_i, b_i, and c_i are functions of the admittance $\mathbf{A}$.

The error-reduction condition requires that the error-reduction function in (49) must be negative in the range of configurations considered. In order to obtain sufficient conditions, we construct two functions F_0 and F_M by replacing the $\cos\psi$ terms in (49) with 1 and $\cos\psi_M$, respectively, to obtain

$$\begin{aligned} F_0(\delta,\psi) \quad &= (a_1\delta_1 + a_2\delta_2 + a_0) + (b_1\delta_1 + b_2\delta_2 + b_0)\sin\psi \\ &+(c_1\delta_1 + c_2\delta_2 + c_0) \\ F_M(\delta,\psi) \quad &= (a_1\delta_1 + a_2\delta_2 + \alpha_0) + (b_1\delta_1 + b_2\delta_2 + b_0)\sin\psi \\ &+(c_1\delta_1 + c_2\delta_2 + c_0)\cos\psi_M\,. \end{aligned}$$

(50)(51)

For small ψ (e.g., $\psi \leq 5^{\circ}$), F_0 and F_M are close approximations of F_{ff}, and, for any $(\delta,\ \psi)$ in the range considered, we have

$$\min\{F_0, F_M\} \le F_{ff} \le \max\{F_0, F_M\}.$$

Thus, if both F_0 and F_M are negative over the range $\delta \in [-\delta_M, \delta_M]$ and $\psi \in [-\psi_M, \psi_M]$, error-reducing motion is ensured.

Now consider the function F_0. Note that F_0 contains only linear terms in δ_1 and δ_2 and, for small ψ (e.g., $\psi \le \left(\frac{\pi}{10}\right)$), sinψ is a monotonic function in ψ. Thus, for $|\psi| \le \psi_M$, if, at the four boundary points $(\pm\delta_1, \pm\delta_2), F_0$ is negative, then, for all $\delta_i \in \left[-\delta_{M_i}, \delta_{M_i}\right], F_0$ is negative. The same reasoning applies to F_M. Therefore, we have the following proposition.

Proposition 6

For a face-face contact state with variation of orientation [−ψM, ψM] and variation of translation $\left[-\delta_{M_i}, \delta_{M_i}\right]$, if, at the four boundary points $\left(\pm\delta_{M_1}, \pm\delta_{M_2}\right)$, the functions F_0 and F_M defined in (50) and (51) are negative for $\psi = 0$ and ψ_M, respectively, then the admittance will satisfy the error-reduction condition for all configurations bounded by $\delta_i \in \left[-\delta_{M_i}, \delta_{M_t}\right]$ and rotation $\psi \in [-\psi_M, \psi_M]$.

SECTION X. Discussion

In this paper, error reduction of a single point on the held body is considered when evaluating error reduction of the held body. If the point selected corresponds to that which is maximally displaced from its proper position, an established metric [11] is used as a measure of error reduction. Alternately, the results could be applied to a finite set of points to further restrict the description of error reduction. If, for example, n points on a body are selected as the reference points, then the error-reduction conditions must be satisfied for all of the n error measures. Therefore, the associated conditions (Propositions 1–6) must be applied to all of the n points.

The polyhedral body discussed is not necessarily the entire held body. It could be any portion of the held part of interest. As a consequence, the set of reference points can be selected based only on the chosen subpart.

The conditions for each PC ensure error-reducing motion only within the same contact state. In order to achieve reliable assembly in tasks that involve multiple PCs, conditions for each of the PCs that may occur in the assembly must be imposed on the admittance simultaneously.

In robotic application, the orientational misalignment due to the manipulator's inaccuracy is small. Thus, the orientational variation considered is small (approximately $\pm 5^{\mathrm{o}}$). For this range, the simplification of the rotation matrix in (18) is an accurate approximation of that in (3). Also, to obtain sufficient conditions for each contact state, conservative bounds on functions for translational and orientational variations are used. Thus, the sufficient conditions obtained are conservative for all contact states.

Once the sufficient conditions are established, an optimization procedure can be used to find a desired admittance. In this optimization, the sufficient conditions can be imposed on the admittance as constraints. Our previous work for planar assembly problems [1] showed the success of this strategy.

In this paper, only frictionless, single PC contact is considered. In practical assembly problems, friction and multi-PC contact must be considered. In spatial cases with friction, since the body motion and the friction are coupled in more complicated nonlinear equations, it is difficult to determine the direction of the contact force, which is needed in determining the motion of the held body. In extension of this study to frictional cases, a way to characterize the friction force when the motion of the body is not known is needed.

SECTION XI. Summary

We have presented a set of conditions for admittance selection for force-guided assembly of two polyhedral rigid bodies. We have shown that, for single-PC contact states, the admittance control law can be selected based on imposed behavior at a *finite* number of configurations. If the error-reduction conditions are satisfied at these configurations, the error-reduction conditions will be satisfied for all intermediate configurations.

In future work, more general admittance selection problems involving multi-PC contact states and contact forces including friction will be investigated.

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