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Abstract

In this short article the following inequality called the "Pitman inequality" is proved for the exchangeable random vector $(X_1, X_2, \ldots, X_n)$ without the assumption of continuity and symmetry for each component X_i:

$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \leq \left|\sum_{i=1}^{n} \alpha_i X_i\right|\right) \geq \frac{1}{2}$,

where all $\alpha_i \geq 0$ are special weights with $\sum_{i=1}^{n} \alpha_i = 1$.

Keywords

Exchangeability, Equality in distribution, Pitman inequality, Characteristic function

1. Introduction

Bose et al. (1993) established the following result: if X_1 and X_2 are *i.i.d.* (independent and identically distributed), continuous and symmetric about θ, then $(X_1 + X_2)/2$ is the Pitman-closest estimator of θ within the class of all the estimators of the form $\alpha X_1 + (1-\alpha)X_2$ for $0 \leq \alpha \leq 1$. In other words we have

(1) $P(|(X_1 + X_2)/2 - \theta| \leq |\alpha X_1 + (1-\alpha)X_2 - \theta|) \geq \frac{1}{2}$.

Assume, without loss of generality, that $\theta = 0$ and consider the general form of (1) where $X_1, X_2, \dots, X_n$ are *i.i.d.* continuous and symmetric, i.e.

(2)$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \leq \left|\sum_{i=1}^{n} \alpha_i X_i\right|\right) \geq \frac{1}{2}$,

where all $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. We wish to show that (2) is false for $n > 2$. Let $n = 3, \alpha_1 = \alpha_2 = 0, \alpha_3 = 1$ and let $f(x)$ be a symmetric *pdf* (probability density function) of the *i.i.d.* random variables X_1, X_2, X_3. Let $g = f * f$ be the convolution of f with itself, so g is the *pdf* of $X_1 + X_2$ and is symmetric as well. It is easy to show that

$$1 - P\left(\left|\frac{1}{3}\sum_{i=1}^{3} X_i\right| \leq |X_3|\right) = 2\int_0^\infty \int_{-u/4}^{u/2} f(v)g(u)dvdu = 2\int_0^\infty \left(F^*\left(\frac{u}{2}\right) + F^*\left(\frac{u}{4}\right)\right) g(u)du,$$

where $F^*(u) = \int_0^u f(x)dx$. It appears that (2) may be true for many symmetric *pdf*'s; for example, it is true for the standard normal *pdf* with $n = 3, \alpha_1 = \alpha_2 = 0$ and $\alpha_3 = 1$. However, consider

$$f(x) = \frac{1}{4(1+|x|)^{3/2}}.$$

Then, after some computation, for $x > 0$,

$$g(x) = -\frac{1}{2x^2} + \frac{\sqrt{1+x}(x^2+2x+4)}{2(x+2)^2x^2}.$$

It can be shown that

$$1 - P\left(\left|\frac{1}{3}\sum_{i=1}^{3} X_i\right| \leq |X_3|\right) = \frac{5}{12} - \frac{1}{8}ln3 - \frac{1}{4}\sqrt{2}arctan\left(\frac{1}{4}\sqrt{2}\right) - \frac{1}{16}ln\left(3\sqrt{2}+4\right) + \frac{5}{12}\sqrt{2} = 0.5281\ldots > \frac{1}{2}.$$

The goal here is to prove (2) for an exchangeable random vector $(X_1, X_2, \dots, X_n)$ with special weights α_i via a simple proof and without the assumption of continuity and symmetry. That is, we want to show that for exchangeable random variables $X_1, X_2, \dots, X_n$,

(3)$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \leq \left|\sum_{i=1}^{n} \alpha_i X_i\right|\right) \geq \frac{1}{2}$, where all $\alpha_i \geq 0$ are special weights with $\sum_{i=1}^{n} \alpha_i = 1$.

Exchangeability plays an important role in forecasting. Suppose one is interested in forecasting a random variable Y based on exchangeable or *i.i.d.* random variables $Z_1, Z_2, \dots, Z_n$. An important problem is to find a best predictor of Y based on a linear function of $Z_1, Z_2, \dots, Z_n$. It is well known that $\overline{Z} = n^{-1} \sum_{i=1}^{n} Z_i$ is the best predictor in the least squares sense. In the sense of Pitman closeness, one may be interested in proving

$$P\left(\left|Y - \overline{Z}\right| \leq \left|Y - \sum_{i=1}^{n} \alpha_i Z_i\right|\right) \geq \frac{1}{2}.$$

This is equivalent to (3) with $X_i = (Y - Z_i)$. Note that in this case $X_1, X_2, \dots, X_n$ are exchangeable, but not necessarily independent. Therefore, it is necessary to consider exchangeable X_i's in (3).

We refer the interested reader to related works by Balakrishnan et al. (2009), Bose et al. (1993) and Ghosh and Sen (1989). We would like to mention here that no results have been reported in the literature regarding (1), (3) for exchangeable random vectors. Over the past six decades many researchers have been working on projects dealing with exchangeability, which is a weaker assumption than that of *i.i.d*. The interested reader is referred, among others, to the book by Chow and Teicher (1997).

In Section 2, we state a useful technique called "the equal in distribution technique (EDT)", which will be employed in this short article. Section 3 is devoted to the proof of (3) for $n = 2$. The final section deals with the proof of (3) for $n > 2$.

2. The equal in distribution technique (EDT)

Two vectors $U = (X_1, X_2, \dots, X_n)$ and $V = (Y_1, Y_2, \dots, Y_n)$ are said to be equal in distribution and denoted by $U \stackrel{d}{=} V$ if they have the same distributions or characteristic functions. It is clear that if $U \stackrel{d}{=} V$ then $g(U) \stackrel{d}{=} g(V)$ for any measurable function $g: \mathbb{R}^n \to \mathbb{R}^k$. The EDT plays a significant role in the proofs of many results in probability theory. For example, using the exchangeability of $(X_1, X_2, \dots, X_n)$, i.e. $(X_1, X_2, \dots, X_n) \stackrel{d}{=} \left(X_{i_1}, X_{i_2}, \dots, X_{i_n}\right)$ for any one of the $n!$ permutations of $(1,2,\dots,n)$, we conclude that

(i) the X_i's are identically distributed;

(ii) $(X_1, X_2, \dots, X_m)$ is exchangeable for $2 \leq m \leq n$;

(iii) $\sum_{j=1}^{m} X_j \stackrel{d}{=} \sum_{j=1}^{m} X_{i_j}$ for $1 \leq m \leq n$.

In particular, $(X_1, X_2, \dots, X_n)$ is exchangeable if and only if

$(X_1 - \theta, X_2 - \theta, \dots, X_n - \theta)$ is exchangeable.

Remark 1

We would like to mention here that although (3) is true for any convex combination of X_i's when n=2 (see Section 3 below), it is not true when $n > 2$, even in the case of symmetry, as the following example shows. Therefore, in proving (3) for $n > 2$ we have to restrict the α_i's to certain "special weights".

Example 1

Let $\alpha_1 = 1$ and $\alpha_i = 0, i = 2,3,\dots,n$ in (3). Then

∑i=jnP(|1n∑i=1nXi|≤|Xj|)≥1,

$$\sum_{i=j}^{n} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \le |X_j|\right) \ge 1,$$

since for every ω there is a j such that $|X_j(\omega)| \ge |X_i(\omega)|$ for all $i = 1,2,\dots,n$. Since the X_i's are exchangeable, we obtain

(4) $P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \le |X_1|\right) \ge \frac{1}{n}$.

To show that $\frac{1}{n}$ in (4) is optimal, we let $e_i, i = 1,2,\dots,n$, denote the standard unit vectors in $\mathbb{R}^n$. Let P be the probability measure with mass $\frac{1}{n}$ at each e_i. Let $X_i: \mathbb{R}^n \to \mathbb{R}$ be the ith projection map. Then, the X_i's are exchangeable and

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \le |X_1|\right) = P(e_1) = \frac{1}{n}.$$

Note that if we take $2ne_i$'s and $-e_i$'s, the random variables X_i will be symmetric as well.

3. Proof of (3) for $n = 2$ and $\alpha_1 + \alpha_2 = 1$

Applying the two-variable function

$$g(t_1, t_2) = (\alpha_1 t_1 + \alpha_2 t_2, \alpha_2 t_1 + \alpha_1 t_2)$$

on both sides of $(X_1, X_2) \stackrel{d}{=} (X_2, X_1)$, by the EDT we obtain

$$(\alpha_1 X_1 + \alpha_2 X_2, \alpha_2 X_1 + \alpha_1 X_2) \stackrel{d}{=} (\alpha_1 X_2 + \alpha_2 X_1, \alpha_2 X_2 + \alpha_1 X_1).$$

Now, setting $U_1 = \alpha_1 X_1 + \alpha_2 X_2$ and $U_2 = \alpha_2 X_1 + \alpha_1 X_2$ we have $(U_1, U_2) \stackrel{d}{=} (U_2, U_1)$. Applying the function $h(t_1, t_2) = |t_1| - |t_2|$ on both sides of $(U_1, U_2) \stackrel{d}{=} (U_2, U_1)$, by the EDT we obtain $|U_1| - |U_2| \stackrel{d}{=} |U_2| - |U_1|$, i.e. $|U_1| - |U_2|$ is symmetric. Using the fact that $U_1 + U_2 = X_1 + X_2$, we have

$$P(|(X_1 + X_2)/2| \le |\alpha_1 X_1 + \alpha_2 X_2|) = P(|U_1 + U_2| \le 2|U_1|) \ge P(|U_1| + |U_2| \le 2|U_1|) = P(|U_1| - |U_2| \ge 0) = \frac{1}{2}.$$

Thus, (3) is proved for $n = 2$ with $\alpha_1 + \alpha_2 = 1$.

4. Proof of (3) for n>2

We will prove (3) for $n = 3$ and $n = 4$ for some specific weights α_i. For n=3, we prove (3) when one of the weights is the average of the other two weights.

Theorem A

If (X_1, X_2, X_3) is exchangeable, then

(5) $P\left(\left|\frac{1}{3}\sum_{i=1}^{3} X_i\right| \le \left|\frac{2}{3}\alpha X_1 + \frac{1}{3}X_2 + \frac{2}{3}\beta X_3\right|\right) \ge \frac{1}{2}$,

where $0 \le \alpha \le 1$ and $\alpha + \beta = 1$.

Proof

Let

$$T_1 = \frac{2}{3}X_1 + \frac{1}{3}X_2, T_2 = \frac{2}{3}X_3 + \frac{1}{3}X_2,$$

Then

$$\frac{1}{3}\sum_{i=1}^{3} X_i = \frac{1}{2}\sum_{j=1}^{2} T_j .$$

Clearly

$$\alpha T_1 + \beta T_2 = \frac{2}{3}\alpha X_1 + \frac{1}{3}X_2 + \frac{2}{3}\beta X_3.$$

The pair (T_1, T_2) is exchangeable. To show this, we observe that by exchangeability

$$(X_1, X_2, X_3) \stackrel{d}{=} (X_3, X_2, X_1).$$

Define the function $g: \mathbb{R}^3 \to \mathbb{R}^2$ as follows:

$$g(u_1, u_2, u_3) = \left(\frac{2}{3}u_1 + \frac{1}{3}u_2, \frac{2}{3}u_3 + \frac{1}{3}u_2\right).$$

Then EDT implies that (T_1, T_2) is exchangeable.

Now, inserting T_1 and T_2 in the left hand side of (5), by the case $n = 2$, we have

$$P\left(\left|\frac{T_1 + T_2}{2}\right| \leq |\alpha T_1 + \beta T_2|\right) \geq \frac{1}{2}.$$

Remark 2

Theorem A can easily be generalized for any odd integer $2n + 1 > 3$.

For $n = 4$, we prove (3) when any pair of weights is the same as the weights of the other pairs. Due to exchangeability, it suffices to consider $\alpha_1 = \alpha_2 = \frac{\alpha}{2}, \alpha_3 = \alpha_4 = \frac{\beta}{2}$ and $\alpha \neq \beta$ with $\alpha + \beta = 1$.

Theorem B

If $(X_1, X_2, X_3, X4)$ is exchangeable, then

(6) $P\left(\left|\frac{1}{4}\sum_{i=1}^{4} X_i\right| \leq \left|\frac{\alpha}{2}X_1 + \frac{\alpha}{2}X_2 + \frac{\beta}{2}X_3 + \frac{\beta}{2}X_4\right|\right) \geq \frac{1}{2},$

where $0 \leq \alpha \leq 1, \alpha \neq \beta$ and $\alpha + \beta = 1$.

Proof

Let

$$T_1 = \frac{X_1 + X_2}{2}, T_2 = \frac{X_3 + X_4}{2},$$

Then

$$\frac{1}{4}\sum_{i=1}^{4} X_i = \frac{1}{2}\sum_{j=1}^{2} T_j\,.$$

Clearly

$$\alpha T_1 + \beta T_2 = \frac{\alpha}{2}X_1 + \frac{\alpha}{2}X_2 + \frac{\beta}{2}X_3 + \frac{\beta}{2}X_4.$$

The pair (T_1, T_2) is exchangeable. To show this, we observe that by exchangeability

$$(X_1, X_2, X_3, X_4) \stackrel{d}{=} (X_3, X_4, X_2, X_1).$$

Define the function $g: \mathbb{R}^4 \to \mathbb{R}^2$ as follows:

$$g(u_1, u_2, u_3, u_4) = \left(\frac{u_1 + u_2}{2}, \frac{u_3 + u_4}{2}\right).$$

Then EDT implies that (T_1, T_2) is exchangeable.

Now, inserting T_1 and T_2 in the left hand side of (6), by the case $n = 2$, we have

$$P\left(\left|\frac{T_1 + T_2}{2}\right| \leq |\alpha T_1 + \beta T_2|\right) \geq \frac{1}{2}.$$

Remark 3

Theorem B can easily be generalized for any even integer $2n > 2$.

Remarks 4

(i) Let X_1, X_2, X_3 be *i.i.d.* positive continuous random variables, then it is easy to see that for $(\alpha = 1, \beta = 0)$ or $\left(\alpha = \frac{1}{3}, \beta = \frac{2}{3}\right)$, (5) becomes equality and hence Theorem A is optimal with $\frac{1}{2}$. (ii) The same can be said for Theorem B with four *i.i.d.* random variables. (iii) Let X_1, X_2, X_3 be *i.i.d.* with an exponential distribution function F with parameter $\frac{1}{2}$ $(F(x) = 1 - e^{-x/2}, x > 0)$.

Observe that

$$\frac{1}{3}\sum_{i=1}^{3} X_i \leq \frac{1}{6}(X_1 + 4X_2 + X_3)$$

is the same as $X_1 + X_3 \leq 2X_2$ and

$$P(X_1 + X_3 \leq 2X_2) = P(E) = \iiint_E \frac{1}{8} e^{-\frac{1}{2}(x_1 + x_2 + x_3)} dx_1 dx_2 dx_3 = \frac{4}{9}.$$

This shows that Theorem A is not true if the conditions on the coefficients do not hold. (iv) If we take four *i.i.d.* random variables in (iii) we observe that

$$\frac{1}{4}\sum_{i=1}^{4} X_i \leq \frac{1}{8}(X_1 + X_2 + 2X_3 + 4X_4)$$

is the same as $X_1 + X_2 \leq 2X_4$ and $P(X_1 + X_2 \leq 2X_4) = \frac{4}{9}$. Therefore, Theorem B is not true if the conditions on the coefficients do not hold.

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