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Deterministic and Stochastic Resilience Analysis of Minimum-Time-Controlled Discrete-Time Systems

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Abstract:
The resilience of discrete-time systems subject to minimum-time control is analysed for both deterministic and stochastic control gain perturbations. Lyapunov analysis is used to determine a tight upper bound on the control gain perturbations to maintain asymptotic stability.

Introduction

Minimum-time response of an *n*th-order single-input discrete-time system in which the system state reaches its desired value in *n*-steps is achieved by designing a controller that places the eigenvalues of the closed-loop system exactly at zero in the complex plane [1, 2]. For digitally implemented controllers, the control gains are vulnerable to perturbations such as finite word length round-off errors. When this controller is implemented in hardware, component tolerances may change the effective controller gains. These uncertainties can change the closed-loop system eigenvalues and potentially cause the closed-loop system to become unstable. It is critical to analyse the resilience/non-fragility [3] of the controller design to determine the allowable range of deviations from the designed gains while maintaining stability of the system. Techniques introduced in [3, 4] are used in this Letter to analyse the resilience of minimum-time controllers.

The following notation is used: $x \in \Re^n$ is an *n*-dimensional vector with real elements; x^T represents the transpose of vector $\boldsymbol{x}$; norm $\| x \|_i$, where $\| x \|_1 = \sum_{i=1}^{n} |xi|$ and $\| x \|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{(1/2)}$; $E\{x\} = \bar{x}$ denotes the expected value of *x*; $\boldsymbol{A} \in \Re^{m\times n}$ is an $m \times n$ matrix with real elements; $\boldsymbol{A} > 0$ means $\boldsymbol{A}$ is a positive definite matrix and $\boldsymbol{I}_m$ is the $m \times m$ identity matrix.

System definition

Consider a controllable single-input discrete-time system in (1), where $x_k \in \Re^n$ is the state and $u_k \in \Re$ is the control input; without loss of generality, the system is assumed to be in controllable canonical form (2), when the system is described by the characteristic equation in (3)

$$x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}u_k \quad (1)$$

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} and\ B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (2)$$

$$a(z) = \det(zI_n - A) = z^n + a_1 z^{n-1} + \cdots + a_n \quad (3)$$

Any minimum-time state feedback control design technique for a discrete-time controller can be used for this single-input system because the control gains that place the eigenvalues at zero in the complex plane are unique, $K = [a_1 a_2 \cdots a_n]$. The resulting closed-loop system is

$$x_{k+1} = (\boldsymbol{A} + \mathrm{B}K)_{x_k} = \boldsymbol{A}_{\mathrm{CL}} x_k \quad (4)$$

where the closed-loop system matrix is

$$\boldsymbol{A}_{\mathrm{CL}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad (5)$$

When there are perturbations on the control gains, the closed-loop system becomes

$$x_{k+1} = \left(\boldsymbol{A} + \boldsymbol{B}\left(K + \boldsymbol{\Delta}^{\mathrm{T}}\right)\right) x_k = \left(\boldsymbol{A}_{\mathrm{CL}} + \boldsymbol{B}\boldsymbol{\Delta}^{\mathrm{T}}\right) x_k \quad (6)$$

where $\Delta \in \Re^n$ is the vector of the control gain perturbations.

Resilience analysis

Using the Lyapunov energy function, $V_k = x_k^{\mathrm{T}} \boldsymbol{P} x_k$, and the condition for asymptotic stability, $V_{k+1} - V_k < 0$, together with the system of (6), the following condition is obtained:

$$\boldsymbol{P} - \boldsymbol{A}_{\mathrm{CL}}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{A}_{\mathrm{CL}} - \boldsymbol{A}_{\mathrm{CL}}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B} \boldsymbol{\Delta}^{\mathrm{T}} - \boldsymbol{\Delta} \boldsymbol{B}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{A}_{\mathrm{CL}} - \Delta \boldsymbol{B}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B} \boldsymbol{\Delta}^{\mathrm{T}} > 0 \quad (7)$$

By choosing $Q = I_n$ for the Lyapunov equation, $\boldsymbol{P} - \boldsymbol{A}_{\mathrm{CL}}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{A}_{\mathrm{CL}} = Q$, an analytic solution for $\boldsymbol{P}$ for an *n*-dimensional system is found as

$$P = \begin{bmatrix} n & 0 & \cdots & 0 \\ 0 & n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad (8)$$

This form for $\boldsymbol{P}$ together with the canonical forms for $\boldsymbol{A}_{\mathrm{CL}}$ and $\boldsymbol{B}$, (5) and (2), respectively, leads to a limit on the control gain perturbations as developed in (9)–(11) below

$$\left(\boldsymbol{A}_{\mathrm{CL}}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B}\right) \boldsymbol{\Delta}^{\mathrm{T}} = 0 \text{ and } \boldsymbol{\Delta}\left(\boldsymbol{B}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{A}_{\mathrm{CL}}\right) = 0, \text{since } \boldsymbol{A}_{\mathrm{CL}}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B} = 0_{n \times 1} \quad (9)$$

$$\boldsymbol{\Delta}\left(\boldsymbol{B}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B}\right) \boldsymbol{\Delta}^{\mathrm{T}} = n \boldsymbol{\Delta} \boldsymbol{\Delta}^{\mathrm{T}}, \text{since } \boldsymbol{B}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B} = n \quad (10)$$

$$\text{thus, } I_n - n \boldsymbol{\Delta} \boldsymbol{\Delta}^{\mathrm{T}} > 0 \quad (11)$$

When the trace inequality for a square matrix [5], $\boldsymbol{\Delta} \boldsymbol{\Delta}^{\mathrm{T}} \leq \mathrm{tr}\left(\boldsymbol{\Delta} \boldsymbol{\Delta}^{\mathrm{T}}\right) \mathrm{I}_n = \boldsymbol{\Delta}^{\mathrm{T}} \boldsymbol{\Delta} I_n$, is applied to (11), a bound on the perturbation elements is found as

$$\left(\sqrt{n} \parallel \boldsymbol{\Delta} \parallel_2\right)^2 < 1 \quad (12)$$

The equivalent norms inequalities between the 1- and 2-norms [5], $\parallel \boldsymbol{\Delta} \parallel_2 \leq \parallel \boldsymbol{\Delta} \parallel_1 \leq \sqrt{n} \parallel \boldsymbol{\Delta} \parallel_2$, are used to express the limit in terms of the 1-norm as

$$(\parallel \boldsymbol{\Delta} \parallel_1)^2 \leq \left(\sqrt{n} \parallel \boldsymbol{\Delta} \parallel_2\right)^2 < 1, \text{thus } \parallel \boldsymbol{\Delta} \parallel_1 < 1 \quad (13)$$

The same type of analysis can be done for stochastic perturbations, where the perturbations, $\boldsymbol{\Delta}_i$ for $i = 1, \ldots, n$, are assumed to be white (uncorrelated in time) random sequences with constant means, $\bar{\Delta}_i$, and variances, σ_i^2, [4]

$$E\left\{\boldsymbol{\Delta}_i^2\right\} = \overline{\boldsymbol{\Delta}_i^2} = (\bar{\Delta}_i)^2 + \sigma_i^2 \quad (14)$$

Following a similar analysis procedure as before, we obtain an equivalent form of (7):

$$\boldsymbol{P} - \boldsymbol{A}_{\mathrm{CL}}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{A}_{\mathrm{CL}} - \boldsymbol{A}_{\mathrm{CL}}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B} \bar{\boldsymbol{\Delta}}^{\mathrm{T}} - \bar{\boldsymbol{\Delta}} \boldsymbol{B}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{A}_{\mathrm{CL}} - \overline{\boldsymbol{\Delta} \boldsymbol{B}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B} \boldsymbol{\Delta}^{\mathrm{T}}} > 0 \quad (15)$$

Using $\boldsymbol{P}$ of (8), we have results similar to (9) and (10), with $\boldsymbol{\Delta}$ replaced by $\bar{\Delta}$, which yields

$$I_n - n \overline{\Delta \Delta^{\mathrm{T}}} > 0 \quad (16)$$

By defining $\rho = \left[\left(\overline{\Delta_1^2}\right)^{(1/2)} \left(\overline{\Delta_2^2}\right)^{(1/2)} \cdots \left(\overline{\Delta_n^2}\right)^{(1/2)}\right]^{\mathrm{T}}$ and applying bounding techniques similar to those used above, (16) can be further simplified to (20) as follows:

$$n \sum_{i=1}^{n} \overline{\Delta_i^2} = n \sum_{i=1}^{n} \left(\left(\overline{\Delta_i^2}\right)^{(1/2)}\right)^2 < 1 \quad (17)$$

$$\| \rho \|_1 \leq \sqrt{n} \| \rho \|_2 < 1 (19)$$

$$\| \rho \|_1 = \left\| \left((\bar{\Delta}_i)^2 + \sigma_i^2 \right)^{\left(\frac{1}{2}\right)} \right\|_1 < 1 (20)$$

Simulation

Given any second-order discrete-time system in controllable canonical form, the application of minimum-time control yields a closed-loop system of the form in (4) with *n* equal to 2. An investigation of the magnitude of the perturbation elements in (6) is performed to determine the values of $[\mathbf{\Delta}_1 \ \mathbf{\Delta}_2] = [\delta \cos \theta \ \delta \sin \theta]$ that cause the magnitude of the closed-loop system eigenvalues to exceed one (exit the unit circle). The analysis is performed by increasing the magnitude, δ, for each angle, θ, to determine the stability boundary. The 1-norm of the corresponding perturbation values that represent the stability boundary is shown in Fig. 1 as the solid line, and the bound (13) is shown as the dashed line. The result of this simulation shows that, in quadrants I and II, the bound on the control gain perturbations follow (13) exactly, indicating that the stability boundary result is tight in these two quadrants.

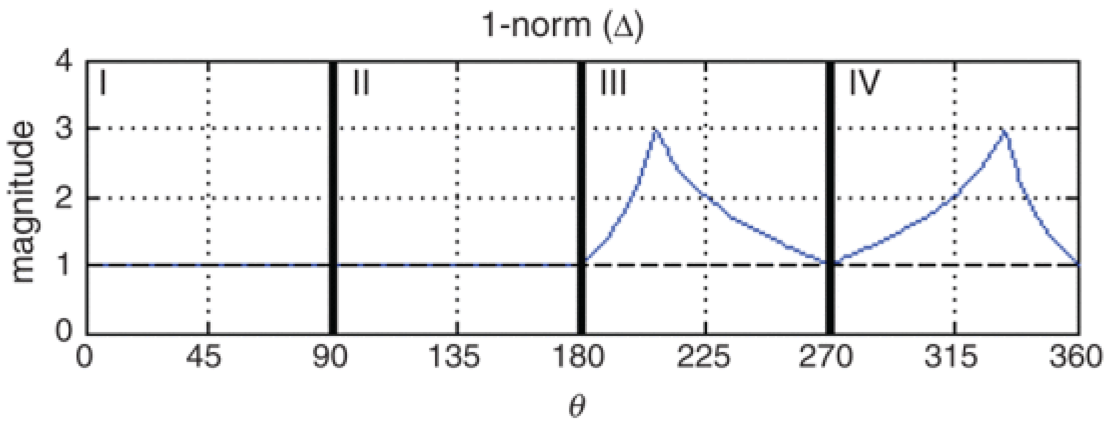

Fig. 1 Stability boundary for second-order system with perturbations; quadrants are identified

Additionally, a third-order minimum-time-controlled discrete-time system is put into the form of (6) with the perturbations, $[\mathbf{\Delta}_1 \ \mathbf{\Delta}_2 \ \mathbf{\Delta}_3] = [\delta \cos \theta \ \delta \sin \theta \ z]$. The investigation of the 1-norm of the perturbation elements is done by taking slices in the *z*-axis and increasing the magnitude, δ, for each angle, θ, to determine the stability boundary. Fig. 2 shows the bound (13) as the dashed line and the stability boundary of the third-order system is the solid line. For the third-order system, the stability bound is again tight in two regions: octants I and VI. Similarly, simulations for higher-order systems show that the stability boundary has exactly two orthants (in *n*-dimensions, the intersection of *n* mutual orthogonal half-spaces) with a tight result.

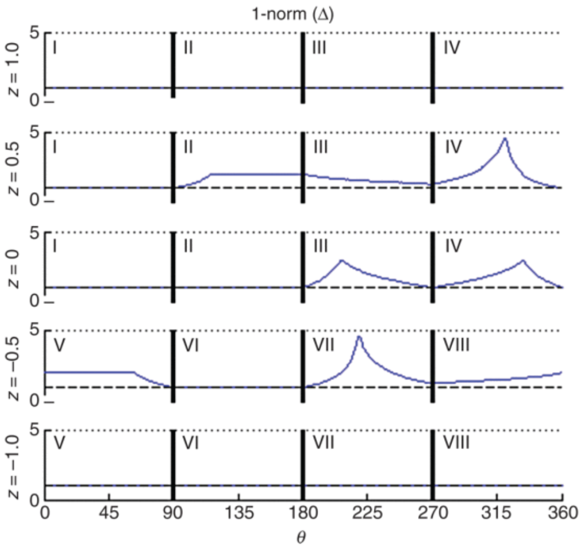

Fig. 2 Stability boundary for third-order system with perturbations

Conclusion

This Letter has presented the resilience analysis results for both deterministic and stochastic perturbations in minimum-time-controlled discrete-time systems. The analysis demonstrates that for a closed-loop system of any order, the sum of the absolute values of the deterministic perturbation elements must be less than one to guarantee system stability. Similarly, the sum of the absolute values of the square root of the stochastic perturbations' second moments must be less than one to guarantee stability. Simulations show that this result with deterministic perturbations on a second- and third-order system is exact in two orthants and therefore tight.

Citation Map

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