Out-Tournament Adjacency Matrices with Equal Ranks

Zachary Buelow
Marquette University

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by

Zachary Buelow

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ABSTRACT
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Zachary Buelow

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Much work has been done in analyzing various classes of tournaments, giving a partial characterization of tournaments with adjacency matrices having equal and full real, nonnegative integer, Boolean, and term ranks. Relatively little is known about the corresponding adjacency matrix ranks of local out-tournaments, a larger family of digraphs containing the class of tournaments. Based on each of several structural theorems from Bang-Jensen, Huang, and Prisner, we will identify several classes of out-tournaments which have the desired adjacency matrix rank properties. First we will consider matrix ranks of out-tournament matrices from the perspective of the structural composition of the strong component layout of the adjacency matrix. Following that, we will consider adjacency matrix ranks of an out-tournament based on the cycles that the out-tournament contains. Most of the remaining chapters consider the adjacency matrix ranks of several classes of out-tournaments based on the form of their underlying graphs. In the case of the strong out-tournaments discussed in the final chapter, we examine the underlying graph of a representation that has the strong out-tournament as its catch digraph.

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Contents

CHAPTER 1
Introduction, Brief Survey and Overview

1 Introduction

Many graph theory and linear algebra papers have addressed the topic of matrix ranks of adjacency matrices for graphs and digraphs. In particular, there have been numerous tournament matrix classes analyzed for matrix rank; perhaps most notably, upset tournaments have been well studied. We will take a brief look at some of these, because some of the results found in this paper follow in parallel to existing tournament results, and use similar techniques. This paper will consider matrix ranks of adjacency matrices of *out-tournaments*, a class of digraphs in which each outset induces a tournament.

The $\{0,1\}$-matrix ranks are important for graph theory because adjacency matrix ranks reflect properties of the graph or digraph that produced them. We consider the real rank (over $\Re$), term rank, nonnegative integer rank (over $\{x \in \mathbb{Z} | x \geq 0\}$, denoted here as Z^+ to follow previous literature), and Boolean rank $(1+1=1)$ of adjacency matrices of out-tournaments. We are interested in classes of out-tournaments that have all four of these ranks full and equal, or equal and less than full. These goals arise from previous work in $\{0,1\}$-matrix ranks, work in tournament matrix ranks, and specifically the fairly well explored class of upset tournaments and their adjacency matrix ranks.

2 Recent Work on Related {0,1}-Matrix Ranks

Papers on {0,1}-matrix ranks go back to time immemorial, so we will only mention some of the more recent papers which are relevant to the current work and relevant to previous work in tournament ranks and adjacency matrix ranks in general. In addition to the usual linear algebra techniques, two of the most important matrix rank tools in this paper are independent sets of 1s and isolated sets of 1s. An *independent set of 1s* in a $\{0,1\}$-matrix has no two elements in the same row or column. An *isolated set of 1s* is an independent set of 1s in which no two elements are in a 2×2 submatrix of all 1s. The maximum size of a set of isolated 1s in a $\{0,1\}$-matrix is the *isolation number* of the matrix, and the maximum size of an independent set of 1s is the *independence number* of the matrix. It is well known that the independence number of a matrix equals its term rank. For example in Lundgren's presentation [**28**], independence number as well as its different characterizations are discussed.

The term rank of A can be found by identifying a set of independent 1s and showing that it is of a maximum size.

$$A = \begin{bmatrix} 0 & 0 & 1_t & 1 & 0 \\ 0 & 0 & 1 & 1_t & 1 \\ 0 & 0 & 0 & 1 & 1_t \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.1}$$

In matrix A given by (1.1), the set of three 1s with subscript t indicate a set of independent 1s. Since 5×5 matrix A has two columns of 0s, we know that this set of independent 1s is of maximum size, and thus, we know that the term rank of A is 3.

The isolation number is a well known lower bound for Boolean rank of $\{0,1\}$-matrices. Lines (1.2), (1.3) and (1.4) show how isolated 1s are used to give a lower bound on Boolean rank. Line (1.2) gives an example of a $\{0,1\}$-matrix with Boolean rank 2. The 1s with subscript s indicate a maximum set of isolated 1s in matrix A. We can see that the two 1s cannot be together in a single rank 1 matrix. Each must be in a distinct rank 1 submatrix. In that respect, isolated 1s are the Boolean rank analog of pivot 1s, used in Gaussian elimination to find the real rank of matrices.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1_s & 1 & 1 \\ 0 & 0 & 0 & 1_s & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.2}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{1.3}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.4}$$

A minimum Boolean factorization of A can be seen in (1.3), and (1.4) expresses the same decomposition as the factorization, but in a different way. Matrix A is represented in (1.4) as the Boolean sum of two rank 1 matrices. In practice, we are usually looking for sets of isolated 1s of a maximum size in the original matrix, but the factorization or Boolean sum can be produced when needed. In this simple rank 2 matrix, A, the reader will note that the choice of a maximum set of isolated 1s indicated is not unique.

The isolation number of a $\{0,1\}$-matrix has been analyzed in various contexts. From a strictly linear algebra perspective, papers such as Beaseley [**7**] from 2011 considered the isolation number and Boolean rank. Brown and Roy [**11**], also from 2011, explored isolation number in the context of tournament matrices, while Brown, Lundgren, Roy, and Siewert [**12**] investigated isolation number and intersection number as they relate to upset tournament matrices. Beasley also has other work on {0,1}-matrix ranks with various associates: e.g. Kirkland and Shader [**8**] on rank comparisons, Guterman [**9**] on rank inequalities over semirings, and Norm Pullman [**10**] comparing column ranks to factor ranks over semirings. Also noteworthy, Hefner (K. Factor) and Lundgren [**24**] in 1990, explore minimum ranks of $k - regular$ $\{0,1\}$-matrices. De Caen with Gregory and Pullman [**16**] also considers Boolean rank of matrices in general. Gregory and Pullman [**21**] compare Boolean and nonnegative integer ranks of $\{0,1\}$-matrices, which are both of particular interest in connection with adjacency matrices.

Many of the articles just mentioned were published in *Linear Algebra and its Applications* and other similar journals. They are noteworthy because their authors have also published papers in graph theory journals on the same topics as they have related to tournament matrices in general, as well as tournament subclasses with structures that allow more complete characterization of adjacency matrix ranks in terms of natural graph theoretic properties. In the papers listed, these authors have laid the foundation of proof techniques using independent and isolated sets of 1s to give term rank and Boolean rank of adjacency matrices, as well as $\{0,1\}$-matrices in general. In particular, the investigations in Chapters 3 and 4 analyze the $\{0,1\}$-matrix structure to find ways to describe out-tournament matrices with equal and full ranks. Chapter 3 of the current paper looks at reduction of out-tournament matrices suggested by out-tournament structure operations by Bang-Jensen [**5**] that preserve full ranks. Chapter 4 examines the submatrices that allow the Boolean rank and nonnegative integer rank to be different and attempts to find out-tournament subdigraphs that produce such structures. These follow in the path of the papers cited above in the use of isolated 1s and independent 1s, as well as similar proof methods.

3 Coverings and Partitions of Digraphs

A *covering* of the edges of a graph is a family of subsets of edges whose union is the entire edge set of the graph. A *partition* of the edges of a graph is a covering in which the constituting subsets are pairwise disjoint. Coverings and partitions are analogously defined for digraphs, but of course, the family consists of subsets of arcs instead of edges.

Coverings and partitions usually require a particular form of subsets of edges. For example, a *clique* is a complete subgraph, and the *clique cover number* of a graph is the minimum number of cliques required to cover all the edges of a graph.

The idea of covering or partitioning the edges of a graph or the arcs of a digraph with complete subgraphs or directed bicliques (respectively) goes back at least to 1977, with Orlin's paper [**35**]. That paper considers covering the edges of a graph with *cliques*, and the minimum number of cliques required to do so for various classes of graph.

Following in a progression from clique coverings, many authors have looked at biclique coverings of graphs. A *bipartite graph* or *bigraph*, G, has two disjoint sets of vertices, X and Y, and every edge in the graph connects a vertex in X to a vertex in Y. A *biclique* in a graph is a complete bipartite subgraph in which every vertex in $X' \subseteq X$ is adjacent to every vertex in $Y' \subseteq Y$. Hence, it is natural to consider the minimum number of bicliques required to cover all the edges of a bigraph. In 1991 an important paper by Gregory, Jones, Pullman and Lundgren

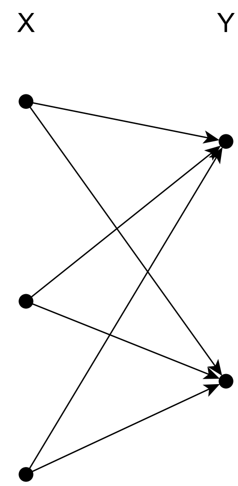

Figure 1. A *directed biclique*, with X represented by the left column of vertices, and Y on the right.

[**22**] gave the connection between biclique coverings of bigraphs and their adjacency matrix ranks. The *biclique cover number* of a bigraph is the minimum number of bicliques needed to cover all the edges of the graph. The *biclique partition number* is the minimum number of bicliques needed to partition the edges of the graph. Gregory et al. [**22**] showed that the biclique partition number of a bigraph equals the nonnegative integer rank of its adjacency matrix, and the biclique cover number of the bigraph equals the biclique cover number of its adjacency matrix. A nice consequence of this is the correspondence of a minimum biclique cover to a minimum Boolean factorization of the adjacency matrix, and likewise, a minimum biclique partition corresponds to a minimum nonnegative integer factorization. Due to the work of Jones, Lundgren and Maybee [**26**], followed by a 1986 paper by Barefoot, Hefner, Jones and Lundgren [**6**], the results with undirected bicliques were brought into the realm of directed biclique coverings and partitions. The paper by Barefoot et al. [**6**] looked at directed biclique coverings of the complements of cycles and paths, which forms a vital part of the foundation for the current research, as most of the Boolean rank proofs in the current paper rely on directed biclique cover numbers of the classes of out-tournaments we are examining.

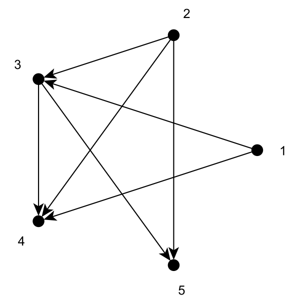

FIGURE 2

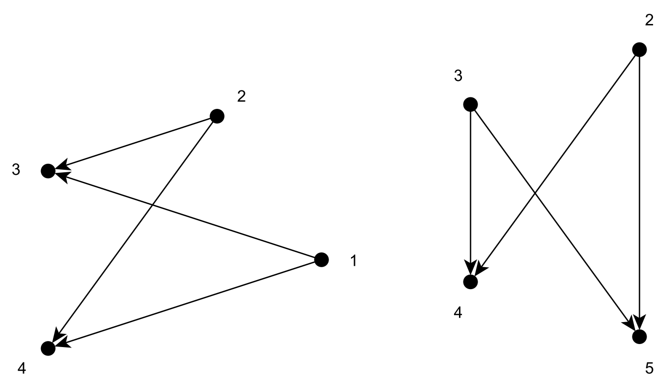

FIGURE 3. Bicliques B_1 and B_2 of our digraph in Figure 2.

A *directed biclique* consists of two disjoint sets of vertices, X and Y, with $x \longrightarrow y$ for each $x \in X$ and $y \in Y$ (see Figure 1). Figure 2 shows a digraph corresponding to adjacency matrix A in (1.2), and Figure 3 shows two bicliques of the graph in Figure 2, which together form a minimum biclique covering of the arcs of the graph in Figure 2. Note that the adjacency matrices of B_1 and B_2 are the two rank 1 matrices in (1.5).

$$
(1.5) \qquad A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{B_1} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{B_2}
$$

The rank 1 matrices in the Boolean sum of (1.5) are the adjacency matrices for the subdigraphs B_1 and B_2, as the submatrices with all the nonzero entries.

The work done by Doherty, Lundgren and Siewert [**17**], for example, carried out an investigation of undirected biclique covers and partitions of graphs and directed biclique covers of digraphs, as well as considerations for the corresponding ranks of their adjacency matrices. Monson, Pullman and Rees [**34**] gave a good summary of clique and biclique coverings, biclique partitions and the corresponding adjacency matrix ranks in 1995.

Henceforth in this paper, *biclique* will be used to mean *directed biclique* and *undirected* will be used to differentiate a biclique in an undirected graph, since it is only directed bicliques that are used in this research.

4 Upset Tournaments and Their Matrix Ranks

Upset tournaments are a particularly well studied class of tournaments with respect to both their structure as well as their adjacency matrix ranks. Because much is known about their properties, upset tournaments form a class of digraphs that is desirable for use in studying adjacency matrix ranks. The *score list* of a digraph is the multi-set of out-degrees of its vertices. An upset tournament on n vertices has the score list $(1, 1, 2, 3, ..., n-2, n-1, n-1)$. Brualdi and Li [**13**] first explored these tournaments in 1983. Later, a series of papers by Lundgren and

Siewert ([**30**],[**31**],[**32**]), as well as Siewert's papers [**40**] and [**41**] studied (directed) biclique covers of upset tournaments, biclique partitions of upset tournaments, and the matrix ranks of their adjacency matrices, giving a complete characterization of the form of upset tournaments with singular adjacency matrices. Together with work from de Caen [**15**], Poet and Shader [**38**], Shader [**39**], and Katzenberger and Shader [**27**], Siewert gave a concise summary of previous matrix rank results on upset tournaments as well as examples of upset tournaments with different relationships between the four matrix ranks [**41**]. Bryan Shader's dissertation [**39**] is remarkable in this context because he showed that for any upset tournament matrix, the nonnegative integer rank and real rank are equal. As we will discuss in more detail, finding classes of digraphs with equal matrix ranks is difficult. Because of this difficulty, Shader's discovery regarding equal real and nonnegative integer ranks for upset tournaments is highly significant.

5 Out-Tournament Properties

Out-tournaments were developed and explored by Jørgen Bang-Jensen in 1990 [**2**], giving some important structural properties of out-tournaments. The central results of this paper are refined and brought into the later 1993 paper [**5**].

These two Bang-Jensen papers are actually centered around in-tournaments, while our research revolves around out-tournaments. Thus, the theorems presented in this paper are actually the out-tournament dual versions of the theorems given for in-tournaments.

Additionally appearing in 1990, Bang-Jensen, Huang and Pavol Hell [**4**], explored chordal proper circular arc graphs. These directly relate to the last few chapters of the current work, in that they present open questions. One in particular is: *What are the catch digraph out-tournaments of representations in the form of chordal proper circular arc graphs having full equal matrix ranks?* This question spurred the classes of out-tournaments found in Chapters 6, 7, and 8 of the current paper.

The following year, Bang-Jensen published [**3**] on digraphs with the path merging property. This is important for our purposes, because out-tournaments are the predominant digraph class with the out-path merging property. An *xy-path* in a digraph is a subdigraph consisting of

$$x \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_k \longrightarrow y, \tag{1.6}$$

where no vertex appears more than once, except for the possibility that $x = y$, in which case the path is called a *cycle*. A *Hamiltonian cycle* contains all the vertices of the digraph.

A digraph has the *out-path merging* property, or *out-path mergeability* if for any internally disjoint xy-path and xz-path, the digraph has a path starting at x, which incorporates all the vertices of both paths, and preserves the relative order of those vertices in the merged path. Every out-tournament has this property, and it will be instrumental to the construction of our out-tournament class in Chapter 8 of this paper.

Some of the results of [**3**] were incorporated into Bang-Jensen, Huang and Erich Prisner's 1993 paper [**5**]. Likewise, the theorems we will be using from Prisner's dissertation [**36**], as well as a subsequent publication [**37**], which are used in this paper were also included in the 1993 paper [**5**]. Specifically, Prisner gives a characterization of in-tournaments as the catch digraph of a family of pointed sets, such that the intersection graph of the family of pointed sets gives the underlying graph of the in-tournament.

A *pointed set* is a set of vertices with one member of the set designated as the point. The *catch digraph* of a family of pointed sets is a digraph with a vertex representing each set in the family, and an arc $x \longrightarrow y$ if x is in the set having y as its point.

The *intersection graph* Γ of a family of sets F consists of a vertex for each set and an edge between two vertices if the intersection of the sets they represent is nonempty. A representation of graph G is a graph H such that there exists a family $\mathcal{F}$ of subsets of vertices where each subset induces a connected subgraph in H and the intersection graph of the family of subsets, denoted $\Gamma(\mathcal{F})$, is G. The idea of representations of graphs by set intersection goes back at least as far as Erdös et al. from 1966 [**18**]. This idea was important to the Bang-Jensen, Huang and Prisner 1993 paper because they went into some detail to characterize the orientability of a graph as an in-tournament based on the form of its possible representations. These possible forms of representations give the last few chapters of the current work a starting point. We consider classes of out-tournaments based on the representations given in [**5**] and build from there; namely, unicyclic representations (Chapter 6), and cactus representations, with varying levels of complexity (Chapters 7, 8). This gives a different perspective on out-tournaments than what had been done before in the relatively unexplored area of out-tournament matrix ranks.

Jing Huang [**25**] went on to explore *local tournaments*. These are digraphs with each outset and each inset inducing a tournament. These are obviously also out-tournaments, but with some additional structure. Local tournament matrix ranks do not play a role in the current paper, but leave open the possibility of matrix rank results analogous to those reached here.

6 Out-Tournament Matrices

The summary of tournament matrix work above is certainly not comprehensive, but gives a reasonable idea of the form that matrix rank results have taken. A complete characterization of the matrix ranks of a class of tournaments is sometimes feasible, but the more general the class, the less likely the possibility of a characterization of matrix ranks of tournaments in the class.

With tournaments, we have the lower bound of $n-1$ on the real rank of the adjacency matrix, and hence on nonnegative integer rank as well as term rank. As soon as we move to out-tournaments, this nice lower bound on three of the ranks is gone, as is much of the structure. Boolean rank, however, can be below $n-1$, even for tournament matrices. For example, Lundgren and Siewert [**30**] collect many examples of tournaments where the different matrix ranks vary. Despite the fact that there is less structure in an out-tournament, some characterization of out-tournament adjacency matrix ranks is possible.

Factor et al. explored matrix ranks of out-tournaments with upset tournament strong components ([**19**],[**20**]). These papers take advantage of the wealth of knowledge available from previous work in upset tournament matrix ranks. Thus, the papers by Factor et al. form the takeoff point for this research. Specifically, they were the first papers examining out-tournament matrix ranks.

Papers [**19**] and [**20**] make use of theorems establishing the form of the strong component digraph of any non-strong out-tournament. A *strong component* is a maximal strongly connected subdigraph. The *strong component digraph* of any digraph has a vertex for each strong component, and $x \longrightarrow y$ if any vertex in strong component X beats any vertex in strong component Y in the original digraph. Because of the acycylic nature of the strong component digraph, some nice results are available for the original out-tournament. The results in [**19**] and [**20**] are generalized in Chapter 2 of the current paper by removing the requirement that strong components are upset tournaments.

As mentioned above, then, this paper proceeds from Factor et al.([**19**],[**20**]), as well as Bang-Jensen, Huang and Prisner [**5**] for out-tournament structure. It gives a partial characterization of the very large class of out-tournaments by considering smaller classes with nice properties, and generalizing whenever possible.

CHAPTER 2
Strong Components and Full Equal Ranks

1 Introduction

Recall from Chapter 1 the four $\{0,1\}$-matrix ranks that we are considering.

- The *real rank* of $A_{n\times n}$ is the usual matrix rank. Real rank, $r(A)$, is the minimum k such that there are matrices $X_{n\times k}$ and $Y_{k\times n}$ over $\Re$ with $A = XY$. Matrix ranks defined in this way are called *factor ranks*.
- The *nonnegative integer rank* of matrix A, $r_{Z^+}(A)$, is defined in the same way as real rank, but the factor matrices are over $Z^+ = \{x \in \mathbb{Z} | x \geq 0\}$.
- The Boolean rank of A, $r_B(A)$, is also a factor rank, but the factor matrices are over the Boolean semiring {0,1}, and Boolean algebra is used for the matrix product.
- The term rank of A, $r_t(A)$, is the maximum size of a set of independent 1s in A.

By the definitions of the matrix ranks that we are interested in, the following chain of inequalities holds for all {0,1}-matrices.

$$r(A) \leq r_{Z^+}(A) \leq r_t(A) \leq n. \tag{2.1}$$

However, there is no standard relationship between $r_B(A)$ and $r(A)$; the Boolean rank of a {0,1}-matrix can be higher, lower, or equal to the rank of A.

Siewert [41] in 2007 gives examples of each case. Following from their definitions, though, we know that for $\{0,1\}$-matrix A, $r_B(A) \leq r_{Z^+}(A)$ necessarily.

In Chapter 1, we referred to Gregory et al. [22], which gave a connection between the undirected biclique cover number of a bigraph, $bc(B)$, and the Boolean rank of the adjacency matrix of the graph, and also gave the connection between the biclique partition number, $bp(B)$, and the nonnegative integer rank.

Theorem 2.1. [22] *Let B be any bigraph and let A be its adjacency matrix. Then*

$$bc(B) = r_B(A) \text{ and } bp(B) = r_{Z^+}(A).$$

This result translates directly to directed biclique coverings and partitions of arcs in a digraph.

Corollary 2.2. *Let D be any digraph and let A be its adjacency matrix. Then*

$$\vec{bc}(D) = r_B(A) \text{ and } \vec{bp}(D) = r_{Z^+}(A).$$

The corollary follows from the theorem because every $n \times n$ digraph adjacency matrix is also the adjacency matrix of a bigraph on $2n$ vertices. Given digraph D with adjacency matrix A, if the rows are assigned numbers $1, 2, ..., n$ and columns are numbered with $(n+1), (n+2), ..., 2n$, then A represents the adjacency matrix of bigraph B with each edge connecting a vertex in $X = \{1, 2, ..., n\}$ to a vertex of $Y = \{(n+1), (n+2), ..., 2n\}$. A biclique in B is represented by a submatrix of all 1s of A, which also represents a directed biclique of D. Therefore, $bc(B) = \vec{bc}(D) = r_B(A)$ and $bp(B) = \vec{bp}(D) = r_{Z^+}(A)$. In fact, the class of digraph matrices on n vertices is a subclass of bigraph matrices on $2n$ vertices. Given digraph D, define $f : A(D) \longrightarrow E(B)$ by $f(i \longrightarrow j) = \{i, n+j\}$ then f is a 1-1 function.

In this chapter, we will consider out-tournament matrices from the perspective of the strong component structure of the out-tournaments. From this we are able to give the vertices an enumeration that creates a block upper triangular matrix, allowing for fairly straightforward verification of its matrix ranks.

2 Strong Component Structure

Beginning with a series of theorems from Bang-Jensen, Huang, and Prisner [**5**], the revealed structure became the basis for matrix rank theorems in [**19**] and [**20**]. The generalization of that work was featured in [**14**].

We will be asking: when are the adjacency matrix ranks of a non-strong out-tournament full and equal? In other words, when do we have:

$$r_B(A) = r(A) = r_{Z^+}(A) = r_t(A) = n?$$

In a digraph, we say vertex y is *reachable* from x if there is a directed path from x to y. A digraph, D (or a subdigraph) is *strongly connected*, or simply *strong*, if each vertex y in D is reachable from each vertex x in D, distinct from y. A *strong component* of digraph D is a maximal, strongly connected subdigraph.

Thinking about the out-tournament in terms of its strong components and grouping the vertices together accordingly reveals the highly structured nature of the adjacency matrix of a digraph of this type. The digraph has several nice properties which are reflected in the equally nice properties of its adjacency matrix.

Suppose that digraph D has strong components $D_1, D_2, ..., D_k$, which have sizes $n_1, n_2, ..., n_k$ and the adjacency matrices of the components are $A_1, A_2, ..., A_k$, respectively.

THEOREM 2.3. [**19**] *Let D_i and D_j be distinct strong components of out-tournament D. If vertex $v \in D_j$ is dominated by some vertex in D_i, then every vertex in D_i dominates vertex v.*

Observe that there are no arcs from any vertex in D_j to any vertex in D_i. To visualize this, recall that the strong components are maximal, so there can only be arcs going in one direction, otherwise the components are not distinct. Since every vertex in D_i is reachable from any $u \in D_i$ and the parent digraph is an out-tournament, then every vertex in D_i must dominate v.

The notation $D_i \longrightarrow v$ will be used to denote that every vertex in D_i beats vertex v. The *strong component digraph* of a digraph D, $SC(D)$, has a vertex for each strong component, and an arc $u \longrightarrow v$ if any vertex in the strong component represented by u dominates any vertex in the component represented by v. Let the notation $D_i \Longrightarrow D_j$ mean that there is at least one arc from D_i to D_j and there are no arcs going from any vertex in D_j to any vertex in D_i. Now we are able to say that if vertex v in strong component D_j is dominated by any vertex in distinct strong component D_i in out-tournament D, then $D_i \longrightarrow v$ and $D_i \Longrightarrow D_j$. Moreover, if d_i and d_j represent strong components D_i and D_j respectively in $SC(D)$, then $d_i \longrightarrow d_j$.

If D is an out-tournament, then $SC(D)$ is an out-tournament [**5**]. Furthermore, $SC(D)$ is an acyclic digraph. As a corollary to Lemma 3.7 in Factor et al. [**19**], we get the following proposition.

Proposition 2.4. *Let D_i, D_j and D_k be strong components of out-tournament D. If $D_i \Longrightarrow D_j$ and $D_i \Longrightarrow D_k$ then $D_j \Longrightarrow D_k$ or $D_k \Longrightarrow D_j$, but not both.*

Proof. Suppose that D_i, D_j and D_k are distinct strong components of out-tournament D and assume that $D_i \Longrightarrow D_j$ and $D_i \Longrightarrow D_k$. Then there is a vertex $u \in D_i$ that dominates some $v \in D_j$ and some $w \in D_k$. Since $\{v, w\} \subseteq N^+(v)$, then v and w are adjacent in D because D is an out-tournament. Suppose that $v \longrightarrow w$ then $D_j \longrightarrow w$ and $D_j \Longrightarrow D_k$. If $w \longrightarrow v$ then $D_k \longrightarrow v$ and $D_k \Longrightarrow D_j$. We cannot have both $D_j \Longrightarrow D_k$ and $D_k \Longrightarrow D_j$ because D_j and D_k are distinct strong components. $\square$

This proposition reveals a great deal of structure in the digraph. The modified condensation digraph of an out-tournament, using '$\Longrightarrow$' as arcs, is acyclic by Proposition 6.20. The result is that the strong components may be numbered in such a way that for any components D_j and D_k, $D_j \Longrightarrow D_k$ implies that $j < k$. Then, we can use this numbering to form A, a block adjacency matrix of the out-tournament with the form given in (2.2). In the block matrix, '[0]' will mean a block of all 0s of the appropriate dimension.

The size of the blocks off the diagonal are determined by their row and column in the block matrix. All the entries below the diagonal are blocks of all 0s. The blocks above the diagonal are blocks composed of columns of 1s and columns of 0s, due to Theorem 2.3.

$$A = \begin{bmatrix} A_1 & * & \cdots & * \\ [0] & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ [0] & \cdots & [0] & A_k \end{bmatrix} \tag{2.2}$$

3 An Out-Tournament Matrix with Full Ranks

If we assume that an out-tournament has the desired matrix ranks full and equal, then the same must be true of the strong components. Note that throughout this section, A will be the adjacency matrix of the specified out-tournament.

3.1 Real Rank of Components

Proposition 2.5. *If A is the adjacency matrix of an out-tournament with strong component matrices $A_1, ..., A_k$ and $r(A) = n$, then for each $j \in \{1, ..., k\}$, $r(A_j) = n_j$.*

Proof. To prove this, we need only consider the structure of A. Suppose that the out-tournament has adjacency matrix A and has strong components $D_1, ..., D_k$, numbered in the standard way. The resulting matrix with corresponding submatrices $A_1, ..., A_k$ is shown in (2.2). Suppose that $r(A) = n$, the number of vertices in D. Next consider the bottom row of the block matrix shown in (2.2). The function $f : R^n \longrightarrow R^{n_k}$ given by $f(0, ..., 0, a_1, ..., a_{n_k}) = (a_1, ..., a_{n_k})$ is clearly a bijection between the row space of $[[0], .., [0], A_k]$ and the row space of A_k. Thus, $r(A_k) = n_k$ and A_k is row equivalent to I_{n_k}. Therefore, we have that

$$A \simeq \begin{bmatrix} A_1 & * & \cdots & * \\ [0] & \ddots & \ddots & \vdots \\ \vdots & \ddots & A_{k-1} & * \\ [0] & \cdots & [0] & I_{n_k} \end{bmatrix}$$

where $\simeq$ represents row equivalence. Hence, by row elimination, we get

$$A \simeq A' = \begin{bmatrix} A_1 & \cdots & * & [0] \\ [0] & \ddots & \ddots & \vdots \\ \vdots & \ddots & A_{k-1} & [0] \\ [0] & \cdots & [0] & I_{n_k} \end{bmatrix}.$$

Any 1s that lay above I_{n_k} have been eliminated in this step. By the same argument, the row second to the bottom of block matrix A_1 indicates that $r(A_{k-1}) = n_{k-1}$. Continuing in this way, we see that for each $j \in \{1, ..., k\}$, we have $r(A_j) = n_j$. $\square$

Now that we know each component has full real rank, by (2.1), we have $r(A) = r_{Z^+}(A) = r_t(A) = n$. That is, the only rank that remains to be calculated is the Boolean rank for each of the components.

3.2 Boolean Rank of Components

In our search for the Boolean ranks of the strong component submatrices of our out-tournament matrix, it will be convenient to introduce a matrix related to A, which will be useful in the results of this section. Define

$$A_0 = \begin{bmatrix} A_1 & [0] & \cdots & [0] \\ [0] & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & [0] \\ [0] & \cdots & [0] & A_k \end{bmatrix}.$$

Since A is a block diagonal matrix, then any pair of 1s in two different component matrices cannot lie in a single rank 1 submatrix. Clearly,

$$r_B(A_0) = \sum_{j=1}^{k} r_B(A_j).$$

LEMMA 2.6. *If for each j, $r(A_j) = n_j$ then $r_B(A) \leq r_B(A_0)$.*

PROOF. Since $r(A_j) = n_j$ for each j, there are no rows or columns of 0s in A. Hence, in each row, there are 1s that are covered in a minimum biclique covering of A_0. Thus, any minimum biclique cover of A_0 extends to a biclique cover of A with the same number of bicliques. This means that $r_B(A) \leq r_B(A_0)$. □

PROPOSITION 2.7. *If $r_B(A) = n$ then for each j, $r_B(A_j) = n_j$.*

PROOF. Since $n = r_B(A) \leq r_B(A_0) = \sum_{j=1}^{k} r_B(A_j) \leq n$, then $r_B(A_j) = n_j$ for each j. □

PROPOSITION 2.8. *If $r(A) = r_{Z^+}(A) = r_B(A) = r_t(A) = n$, then for each j, $r(A_j) = r_{Z^+}(A_j) = r_B(A_j) = r_t(A_j) = n_j$.*

PROOF. Since $r_B(A) = n$ implies that $r_B(A_j) = n_j$ and $r(A) = n$ implies that $r(A_j) = n_j$, we now get that

$$r(A) = r_{Z^+}(A) = r_B(A) = r_t(A) = n$$

implies that for each j,

$$r(A_j) = r_{Z^+}(A_j) = r_B(A_j) = r_t(A_j) = n_j,$$

which completes the proof. $\square$

4 Component Matrices with Equal, Full Ranks

Here we consider the converse of the previous situation and find that if we assume full and equal matrix ranks for each of the strong components, then the same must hold for the entire out-tournament.

PROPOSITION 2.9. *Let D be an out-tournament with k strong components and adjacency matrix A. Let n_j be the size of D_j, the jth component of D. If for each $j \in \{1, 2, ..., k\}$, $r_B(A_j) = n_j$ then $r_B(A) = n$.*

PROOF. Consider a minimum biclique cover of D. If each biclique contains arcs originating from only one component D_j, then there will be no fewer than $\sum r_B(A_j) = n$ bicliques. Suppose that $r_B(A) < r_B(A_0)$. Then there is a biclique in that minimum cover of the form $\{v_i, v_j, ...\} \longrightarrow \{w_1, w_2, ...\}$ where v_i and v_j are not in the same strong component, and at least one of $\{w_1, w_2, ...\}$ must be in D_i and at least one in D_j. We can assume, without loss of generality, that $i < j$ and that w_1 is in D_i and w_2 is in D_j. If this were not the case, then the biclique we are considering would be an extension of a biclique in A_0, which would not be one of the bicliques that could allow $r_B(A) < r_B(A_0)$. Clearly this is not a biclique in A_0, so there must be a column of 1s above A_j, in the column of w_2. More precisely, it must be the case that $D_i \Longrightarrow D_j$. However, since $D_i \Longrightarrow D_j$, there cannot be an arc from v_j to w_1, so the multi-component biclique cannot exist. For an illustration, see the block diagonal matrix, (2.2). Recall that the upper triangular block matrix was possible because of the acyclic enumeration of the strong components, and this is the fact that prevents any bicliques from having arcs in more than one strong component. Thus, $r_B(A) = r_B(A_0) = n$. □

PROPOSITION 2.10. *Let D be an out-tournament with adjacency matrix A, and k strong components with submatrices $A_1, ..., A_k$. Let n_j be the size of D_j, the jth component of D. If for each $j \in \{1, 2, ..., k\}$, we have $r(A_j) = n_j$, then $r(A) = n$.*

PROOF. Assume that $r(A_j) = n_j$ for each j. Then

$$A \simeq \begin{bmatrix} I_{n_1} & [0] & \cdots & [0] \\ [0] & \ddots & \ddots & \vdots \\ \vdots & \ddots & I_{n_{k-1}} & [0] \\ [0] & \cdots & [0] & I_{n_k} \end{bmatrix} \simeq \begin{bmatrix} A_1 & [0] & \cdots & [0] \\ [0] & \ddots & \ddots & \vdots \\ \vdots & \ddots & A_{k-1} & [0] \\ [0] & \cdots & [0] & A_k \end{bmatrix} = A_0, \tag{2.3}$$

so $r(A) = r(A_0) = \sum r(A_j) = n$. □

5 Conclusion and Future Work

Assembling the preceding propositions, we arrive at a concise statement of exactly when an out-tournament has equal, full ranks, in terms of the ranks of its strong components' adjacency matrices.

THEOREM 2.11. *Let D be an out-tournament with k strong components and matrix A. Let n_j be the size of D_j, the jth component of D, and A_j its matrix. Then,*

$$r_B(A) = r(A) = r_{Z^+}(A) = r_t(A) = n$$

if and only if

$$r_B(A_j) = r(A_j) = r_{Z^+}(A_j) = r_t(A_j) = n_j$$

for each j.

Proof. ($\Rightarrow$) This is Proposition 2.8.

($\Leftarrow$) Follows immediately from Propositions 2.9 and 2.10. □

Along with this result, the analog for in-tournaments comes in a dual fashion. The circumstances under which the preceding matrix ranks are equal and less than full remains to be investigated.

5.1 Related Open Questions, Future Work

This section has dealt with out-tournament matrices in terms of strong component matrices. However, it does not address:

(1) Characterization of adjacency matrices of strong out-tournaments:

When are these equal and full?

When are these equal, and less than full?

(2) Adjacency matrix ranks of non-strong out-tournaments where D has one or more single-vertex strong components.

These are big questions that give the most basic subdivisions of the equal rank problem for out-tournament adjacency matrices in light of the current chapter. Each of the following chapters in this paper fits into one of those categories. As discussed in Chapter 1, partial characterizations will take a subclass from one of these categories and analyze its matrix ranks to partially characterize and add to the previous knowledge of equal rank adjacency matrices for out-tournaments.

CHAPTER 3
Reduction and Substitution by a Tournament

1 Substitution and Reduction of an Out-Tournament

Consider an out-tournament D, with a tournament subdigraph T, such that if $x \in V(D-T)$ beats vertex $v \in V(T)$ then for each $t \in V(T)$, $x \longrightarrow t$. Likewise, if there is a vertex $t \in V(T)$ with $t \longrightarrow y$ for $y \in V(D-T)$, then for each $v \in V(T)$, $v \longrightarrow y$. If these conditions hold, we say D is *reducible* by tournament T. See Figure 4 below. Vertex x is a representative vertex in the X, the set of vertices that beat T. Vertex y is a representative of the set of vertices dominated by T.

Definition 3.1. *Let D be a digraph reducible by tournament T. The reduction of digraph D by tournament T is $D' = [V']_D$ where $V' = V(D-T) \cup \{v\}$, v is any vertex in T, and any arcs to or from removed vertices are also removed.*

In, 1993 Bang-Jensen [**2**] introduced the idea of replacing a tournament with a single vertex. It lends itself well to the discussion of the properties of out-tournaments.

Informally, one vertex $v \in D'$ in the reduced digraph represents the tournament that was in D. In a similar fashion, given any digraph D' we can form a 'larger' digraph D by substituting a tournament T for any vertex $v \in D'$. Since we will be talking about two related digraphs, D and D', we will use a subscript on arrows that represent arcs, in order to indicate which digraph the arc is in. E.g., '$x \longrightarrow_D y$' refers to arc (x, y) in the arc set of digraph D. When we refer to an outset of a vertex, it is also necessary to state which digraph we mean. Similar to the arc notation, the expression $N_D^+(v)$ is used to indicate the outset of vertex v in digraph D, and $N_D^-(v)$ the inset of v in D. The adjacency matrix of digraph D on n vertices is denoted A, and A' is the adjacency matrix of D' on n' vertices.

DEFINITION 3.2. *Given digraph $D' = (V', A')$, digraph $D = (V' \cup V(T), A)$, where $[V(T)]_D = T$ and $x \longrightarrow_{D'} v$ implies $x \longrightarrow_D t$ for each $t \in T$. Likewise, if $v \longrightarrow_{D'} y$ then $t \longrightarrow y$ for each $t \in T$. Then D is formed from D' by the substitution of tournament T for vertex v in V'.*

As noted in [**2**], both substitution and reduction by a tournament are local tournament preserving operations. Digraph D is a local tournament if and only if D' is a local tournament. As a corollary, both substitution by a tournament and reduction by a tournament are out-tournament preserving. D is an out-tournament if and only if D' is an out-tournament.

Given these facts, perhaps a natural question is: *do the operations of substitution and reduction by a tournament preserve the full matrix ranks of the out-tournament matrix?* We will address this question for two matrix ranks that are

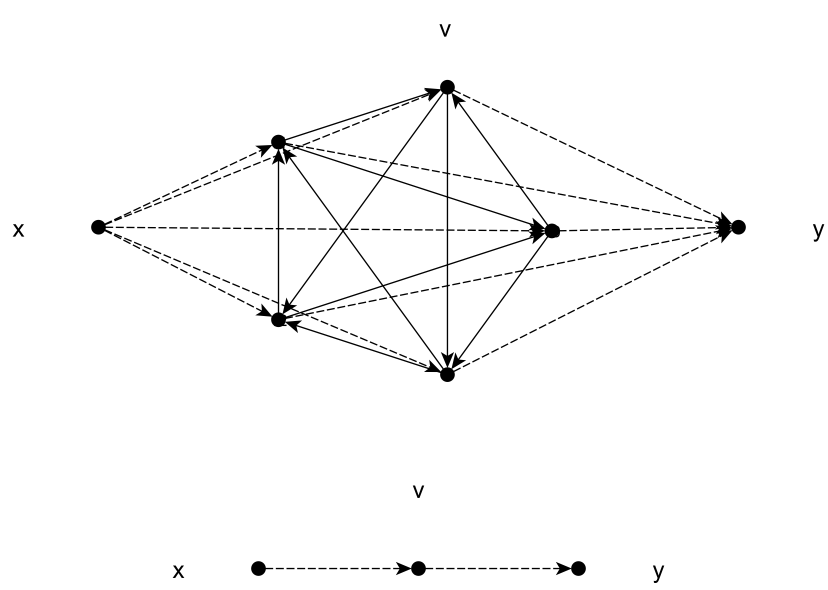

FIGURE 4. Subdigraph reducible by tournament on five vertices, and the corresponding reduced subdigraph below. Arcs with solid lines are within the reduced tournament.

of particular interest for adjacency matrices of digraphs, namely term rank and Boolean rank.

2 Substitution, Reduction and Full Term Rank

Let D be an out-tournament and D' the reduction of D by tournament T.

2.1 Term Rank and Connectedness

PROPOSITION 3.3. *Out-tournament D is strong if and only if D' is strong.*

PROOF. ($\Rightarrow$) Suppose that out-tournament D is strong. Let v be the vertex in D' to which a tournament T in D is reduced. For each vertex $t \in T$ and $x \in D'$, there is a path from t to x and a path from x to t in D. Therefore, in D' there is a path from t to v and a path from v to t. Thus, D' is strong.

($\Leftarrow$) Conversely, suppose that D' is strong. Then for each $x \in D'$ there is an xv-path and a vx-path. By the definition of substitution of tournament T for vertex $v \in D'$, we have $N_D^-(t) \cap V(D') = N_{D'}^-(v)$ and $N_D^+(t) \cap V(D') = N_{D'}^+(v)$ for each $t \in T$. There is an xt-path and a tx-path in D for each $t \in T$ and each $x \in V(D') - \{v\}$. Any xy-path in D' where neither x nor y is v, remains unaffected by the substitution of the tournament T. $\square$

Now, we recall that by out-path mergeability we know that an out-tournament is strong if and only if it has a Hamiltonian cycle [**5**]. Thus, following immediately from Proposition 3.3 we get the desired result.

THEOREM 3.4. *Let D be a strong out-tournament reducible by tournament T and D' the reduction of D by T. We have $r_t(A) = n$ if and only if $r_t(A') = n'$.*

PROOF. Since D is strong, D has a Hamiltonian cycle. Likewise, by Proposition 3.3 on page 25, D' also has a Hamiltonian cycle. The 1s of a Hamiltonian cycle in the adjacency matrix represent a set of independent 1s of size n, so $r_t(A) = n$ and $r_t(A') = n'$. □

3 Boolean Rank and Substitution

As it turns out, a result similar to Theorem 6.13 for term rank holds for Boolean rank as well. However, as has been noted in previous literature, it is often a bit more difficult to find than the term rank. As before, let D be an out-tournament reducible by T, and let D' denote the reduction of D by T. Recall that this also means D is the result of substituting tournament T for vertex $v \in D'$.

3.1 Reduction Preserves Full Boolean Rank

First, let us consider an out-tournament adjacency matrix A of a digraph D reducible by tournament T. Our goal in this section is to show that if A has full Boolean rank, then A' also has full Boolean rank.

From the Chapter 1, recall that the arcs represented by any rank 1 submatrix form a biclique in the digraph, so we will refer to the rank 1 submatrix of A as a *biclique matrix*. Observe that if biclique B_i is given as $B_i = (X_i, Y_i)$ where $X_i \longrightarrow Y_i$, then X_i gives the row labels and Y_i gives the column labels of the nonzero entries of the rank 1 matrix that represents B_i in matrix A.

Given a particular biclique covering or partition, $B = \{(X_i, Y_i)\}_{i=1}^{k}$, it may be possible for a biclique $B_i \in B$ to be *extended horizontally* to $B_i' = (X_i, Y_i')$ where $Y_i' \supseteq Y_i$ or B_i may be *reduced horizontally* if it can be replaced in collection B by $B_i' = (X_i, Y_i')$ where $Y_i' \subseteq Y_i$.

Similarly, it may be possible for a biclique $B_i \in B$ in collection B to be *extended vertically* to $B_i' = (X_i', Y_i)$ with $X_i' \supseteq X_i$ or *reduced vertically* to $B_i' = (X_i', Y_i)$ with $X_i' \subseteq X_i$. In any of these cases of reduction or extension, the presumption is that the modified collection B is still a biclique covering or partition, respectively, of D.

Proposition 3.5. *Let D be an out-tournament reducible by a tournament T. If $r_B(A) = n$, then $r_B(A_T) = n_T$, where $n_T = |V(T)|$.*

Proof. Suppose not. If $r_B(A_T) < n_T$, then there is a set S_T of rows of A_T that may be covered by fewer than $|S_T|$ rank 1 matrices. Let $|S| = j$, and define the i^{th} biclique as $B_i' = (X_i, Y_i)$, whose matrix covers these rows of A_T. Now, we extend these biclique matrices to cover the corresponding set of rows in A. Let $Y = N^-(T)$ (note that we can refer to this set without ambiguity) then define $B_i = (X_i, Y_i \cup Y)$. Then the set of rows S of A corresponding to the set S_T of rows in A_T is covered by $j < |S|$ biclique matrices, which then implies that $r_B(A) < n$. $\square$

Now, in a similar fashion, we can say that a result analogous to Proposition 3.5 holds for A', the reduction of A by tournament matrix A_T.

Theorem 3.6. *Let A be a connected out-tournament matrix reducible by tournament matrix A_T, and matrix A' its reduction. Then $r_B(A) = n$ implies that $r_B(A') = n'$.*

Proof. Suppose that $r_B(A') < n'$. Then there exists a set with minimum size, of j rows of A' coverable with $j - q$ biclique matrices $B_1', B_2', \ldots, B_{j-q}'$ for some $q > 0$. These may be extended to biclique matrices of A as follows. If $B_i' = (X_i', Y_i')$ then define $B_i = (X_i', Y_i' \cup V(T))$ if $v \in Y_i'$ and $B_i = B_i'$ otherwise. This extends the biclique matrices in A' horizontally. Thus, the corresponding $j - q$ rows of A are covered by $B_1, B_2, \ldots, B_{j-q}$, showing that $r_B(A) < n$. $\square$

3.2 Substitution and Full Boolean Rank

Now, we may question whether the converse of Theorem 3.6 also holds. Suppose that we have out-tournament matrix A' and substitute tournament matrix

A_T. As we have seen in the proofs of Theorem 3.6 and Proposition 3.5, if $r_B(A') < n'$ or $r_B(A_T) < n_T$, then $r_B(A) < n$. If we assume that we are substituting a tournament with a full Boolean rank matrix, does $r_B(A') = n'$ imply $r_B(A) = n$? The answer is: *not in general*.

The matrix A in (3.1) serves as a counterexample. For ease of visualization, we have vertex v as 6, at the end of the enumeration of vertices of D'. Note that D' is a strong out-tournament, with a cyclic enumeration of its six vertices. Then $D' = [\{1, 2 \ldots, 6\}]_D$, with A' consisting of the 6×6 matrix in the upper left hand corner of A. See (3.2) for matrix A' and (3.3) for A_T. Tournament $T = [\{5, 6, \ldots, 11\}]_D$ consists of the induced subdigraph of D on the last six vertices.

$$A = \left[\begin{array}{cccccc|ccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \tag{3.1}$$

$$A' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.2}$$

$$A_T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{3.3}$$

In (3.3) note that the set $\{(6,1),(1,2),(2,3),(3,4),(4,5),(5,6)\}$ forms an isolated set of 1s. Since this set has size 6, it is therefore a maximum set for a 6×6 matrix. Therefore, $r_B(A_T) = n_T = 6$. Likewise, the same set of locations $\{(6,1),(1,2),(2,3),(3,4),(4,5),(5,6)\}$ forms an isolated set of 1s in matrix A'. Thus, $r_B(A') = n' = 6$ as well.

Now, despite the fact that A' and A_T each have full Boolean rank, the matrix A, which is the adjacency matrix resulting from the substitution of tournament T for vertex 6 in D', does not have full Boolean rank. Indeed, consider rows 5 through 11 of matrix A, together with the following six bicliques, given as $B_i = (X_i, Y_i)$, where $X_i \longrightarrow_D Y_i$. We use the insets $N^-(v)$ as X_i because it is a convenient way to refer to the set of row indices of all the nonzero entries in column v.

- $B_1 = (N^-(6), \{1,6\})$
- $B_2 = (N^-(7), \{1,7\})$

- $B_3 = (N^-(8), \{1, 8\})$
- $B_4 = (N^-(9), \{1, 9\})$
- $B_5 = (N^-(10), \{1, 10\})$
- $B_6 = (N^-(11), \{1, 11\})$

In matrix A, column 1 is the Boolean sum of columns 6, 7, 8, 9, 10 and 11. So, we have seven rows (or seven columns) of A coverable in fewer than seven biclique matrices.

3.3 A Case in which Full Boolean Rank is Preserved through Substitution.

In Section 3.2, we considered when it might be possible to subsitute a full Boolean rank tournament T into a full Boolean rank digraph D' and obtain a full Boolean rank digraph. The counterexample, (3.1), was constructed in a specific way in order to allow a minimum covering of A in which there was at least one multi-row, multi-column biclique covering 1s of both A' and 1s of A_T. In order for this to happen, we substituted T for a vertex with $N^+(N^-(v)) \cup N^+(v) \neq \emptyset$. That is, $5 \in N^+(N^-(v)) \cup N^+(v)$ allowed a horizontal extension of a column biclique cover of the 1s of columns 6 through 11 in the submatrix of A consisting of the rows 5 through 11.

There may certainly be other cases in which a substitution of this kind preserves full Boolean rank, but we will consider a particular case, which was perhaps hinted at in the previous counterexample.

This method revolves around an appropriate choice of vertex v in D' for which we will substitute tournament T. Our assumptions are as follows:

Criterion 3.7. *Matrix A' has $r_B(A') = n'$, digraph D is formed by substituting tournament T for vertex v, and $r_B(A_T) = n_T$.*

We want to give a condition such that these three items guarantee full Boolean rank will be preserved by the substitution. Lundgren and Stewart [**32**] give us the following useful result.

LEMMA 3.8. *If $r_B(A) < n$, then there exists a set, S, of rows of A such that these rows of A may be covered by less than $|S|$ bicliques, and S is of a minimum size.*

Thus, if we assume Criterion 3.7 holds, and assuming that $r_B(A) < n$, there must exist such a set S. It will be convenient for us to be able to refer to another related digraph, D^*, which we will define as $D' - \{v\}$ or $[V - \{v\}]_{D'}$, and A^* will be its adjacency matrix. Let M be a minimum biclique covering of the rows of S. Note that there are no single row biclique matrices in M. If there were, then S was not of a minimum size.

LEMMA 3.9. *The set S defined above may not consist only of rows corresponding to A^**

PROOF. Since $r_B(A') = n'$, no collection of rows S in A corresponding to A^* may be covered by less than S bicliques. If there were such a collection of rows S in A, then the corresponding rows of A' could be covered in that same number of bicliques, by horizontal reduction of any minimum cover of the corresponding rows in A, which would make $r_B(A') < n'$. By assumption, this is not so. $\square$

Likewise, such a set S cannot consist only of rows corresponding to A_T, for precisely the same reason: we have assumed that $r_B(A_T) = n_T$.

Now, because of the ranks of A^* and A_T, a similar result holds for any minimum biclique cover, M.

LEMMA 3.10. *In any minimum biclique matrix cover of the 1s in rows of S, there must exist a biclique B_0 whose matrix involves rows of both A^* and A_T, as well as involving columns of both A^* and A_T.*

Proof. Suppose that every biclique matrix of minimum biclique covering M involving rows of A^* does not involve rows of A_T. Let S^* denote the part of S in the rows of A corresponding to A^*, and analogously for S_T. Note that S^* alone requires $|S^*|$ bicliques, since $r_B(A') = n'$. Likewise, S_T requires $|S_T|$ bicliques since $r_B(A_T) = n_T$. But, S was coverable by fewer than $|S|$ bicliques, by assumption. Now let M' be the subset of the covering consisting only of the bicliques involving rows of both A_T and A^*. Suppose that all of the elements of M' involving columns of A^* did not involve columns of A_T. Then M itself is nothing more than a vertical extension of the union of a minimum covering for each of S^* and S_T, which as we have noted would make $|M| > |S|$. Thus, such a biclique matrix with rows of both A^* and A_T, and columns of both A^* and A_T must exist, given our assumptions above. $\square$

We have been talking about a multi-row, multi-column biclique matrix that has at least one column in common with each of two submatrices that we are looking at, as well as having at least one row in common with each of the two submatrices. If biclique B has that property, we will call it a *crossover* biclique. Finalizing the above argument, we get the following proposition.

Proposition 3.11. *Let D be an out-tournament formed by substitution, following Criterion 3.7. Then $r_B(A) < n$ implies that crossover biclique B_0 must exist in any minimum biclique cover.*

Proof. Let $Z \subseteq V(D')$ be the vertices in D' not adjacent to v. Define set X to be the set of vertices $\{x \in D^* | x \longrightarrow T\}$, and define set Y as $\{y \in D^* | T \longrightarrow y\}$. To illustrate the block form of A using this partition of vertices, rows corresponding to X, Y, Z, and T are labelled accordingly in (3.4). We give the matrix A in a way that allows easier visualization of the above concepts. The vertices are grouped by categories above, and D is given an appropriate enumeration to make the block form of A appear as

$$
A = \begin{array}{c} \\ Z \\ X \\ Y \\ T \end{array}
\begin{array}{c}
\begin{array}{cccc} Z & X & Y & T \end{array} \\
\begin{bmatrix}
 & & & [0] \\
 & & & [1] \\
 & & & [0] \\
[0] & [0] & [1] & A_T
\end{bmatrix}
\end{array} . \tag{3.4}
$$

Block entries not relevant are left blank. Note that the submatrix consisting of rows and columns X, Y, Z is exactly matrix A^*. A [0] block is a submatrix of A consisting of all 0s, and a [1] entry in the block is a submatrix of A consisting of all 1s. Notice that we cannot use J_k notation here, since the blocks are not necessarily square. Now, keeping this block form in mind, consider the locations of the 1s that would be involved in any crossover biclique, B_0.

$$
A = \begin{array}{c} \\ Z \\ X \\ Y \\ T \end{array}
\begin{array}{c}
\begin{array}{cccc} Z & X & Y & T \end{array} \\
\begin{bmatrix}
 & & & [0] \\
 & & A_1 & A_2 \\
 & & & [0] \\
[0] & [0] & A_3 & A_4
\end{bmatrix}
\end{array} \tag{3.5}
$$

Since B_0 involves 1s in A_T, there must be at least one in A_4, see (3.5). Now, recall that B_0 also must involve at least one row of A^*, and thus B_0 must cover a 1 in block A_2, the only other nonzero block in column T. Since B_0 also involves columns of A^*, those must be contained in column Y in the block matrix, (3.5), since column Y contains the only nonzero blocks in row T and the columns of A^*. Therefore, B_0 must also cover the appropriate 1s in block A_1. □

Of course, it could happen that the configuration of 1s in A_1 and A_4 does not, in fact, allow such a biclique. In such a case, we would have preservation of full

Boolean rank, but in an *ad hoc* manner. If D is formed from D' by substitution of a tournament, T, with adjacency matrices A, A' and A_T, respectively, then $r_B(A') = n'$ and $r_B(A_T) = n_T$ are together not sufficient to guarantee that $r_B(A) = n$. One universal way to assure preservation of full Boolean rank with substitution is to not allow any 1s in submatrix A_1. Or, to phrase it differently, to substitute T only for a vertex that does not let this happen. The following theorem covers that case. Theorem 3.12 gives a sufficient condition such that full Boolean rank is preserved under tournament substitution. All of the outset and inset operators in this theorem are with respect to digraph D'. Leaving off the subscripts will make the conditions easier to read.

THEOREM 3.12. *Let D' be an out-tournament with $r_B(A') = n'$, T be a tournament with $r_B(A_T) = n_T$, and D the digraph formed by substituting tournament T for vertex $v \in D'$. Let A be the adjacency matrix of D. If $N^+(N^-(v)) \cap N^+(v) = \varnothing$, then $r_B(A) = n$.*

PROOF. Assume that the conditions are satisfied. The requirement that $N^+(N^-(v)) \cap N^+(v) = \varnothing$ precludes the possibility that there may be any multi-row, multi-column biclique matrix with at least one arc in tournament T and at least one arc outside of tournament T. Consider any $a_{ij} = 1$ in submatrix A_T. For any $a_{kj} = 1$ in the same column, but not in the rows representing T, $k \longrightarrow j$ means $k \in N^-(v)$. Any other 1 in row k not in the columns representing T, say in column ℓ, means that $k \longrightarrow v$ in D'. Our assumption, $N^+(N^-(v)) \cap N^+(v) = \varnothing$, guarantees that any row t (including row i) of the rows of A_T has entry 0 in column ℓ. Thus, there can be no reduction of Boolean rank from full when tournament T is substituted for vertex v. Furthermore, $r_B(A) = r_B(A') - 1 + r_B(A_T) = n' - 1 + n_T = n$. $\square$

$$A = \begin{array}{cc} & \begin{array}{ccc}\ell & \quad & j\end{array} \\ \begin{array}{c}k \\ \\ i\end{array} & \left\{\begin{array}{ccc}1 & \leftarrow & 1 \\ \downarrow & & \uparrow \\ 0 & & 1\end{array}\right. \end{array} \tag{3.6}$$

In (3.6), we see an illustration of the progression of the proof of Theorem 3.12, where all labels are those used in the proof. Only important row and column labels are shown, while most of the matrix entries are left out. Arrows indicate the order of the 1s appearance in the proof. The entry a_{ij} is in A_4 of block matrix (3.5), a_{kj} is in A_2, and $a_{k\ell}$ cannot be in A_1, which is all 0s. Finally, any $a_{t\ell}$ must be 0 since columns Y and T of the block matrix contain the only nonzero entries.

The condition this theorem relies on, $N^+(N^-(v)) \cap N^+(v) = \varnothing$, assures us that A_1 in (3.5) is a block of 0s. Hence, there can be no biclique matrix covering part of $A*$ and part of A_T that allows $r_B(A)$ to be less than $r_B(A*) + r_B(A_T)$.

4 Conclusion and Future Work

Although the condition $N^+(N^-(v)) \cap N^+(v) = \varnothing$ is a narrow constraint, it does achieve the desired effect, albeit in a brute force manner. It appears that if we wanted to weaken the conditions on the theorem, the results would be a series of classes of out-tournaments that do not necessarily lend themselves to a more natural characterization. That does not mean that there isn't such a characterization, only that it seems unlikely to reveal itself in this context, based on previous and current research results and limitations. Thus, we must settle for the reduction result, Theorem 3.6, which should be much easier to apply to other theorems and investigations of out-tournament matrix ranks.

The effect of the above results may assist in characterizing full term rank adjacency matrices and their out-tournaments as well as full Boolean rank adjacency matrices and their out-tournaments. Theorem 3.6 allows us to assume that the digraphs are fully reduced, which gives a potentially simpler form with which to work.

CHAPTER 4
Digraph Matrix Classes with Equal Boolean and Nonnegative Integer Ranks

1 Introduction

As discussed previously, papers [**6**] and [**22**] found that for any adjacency matrix A arising from digraph D, $\vec{bc}(D) = r_B(A)$ and $\vec{bp}(D) = r_{Z^+}(A)$; for illustration see Figures 2 and 3, and the Boolean sum in (1.5). Recall that Boolean rank and nonnegative integer ranks of $\{0, 1\}$-matrices are defined as factor ranks, whose factorizations may alternately be thought of as sums of rank 1 matrices. The central idea with regard to the matrices is that in both cases, a rank 1 matrix has all 0s, except for a single maximal submatrix of all 1s.

A biclique covering of the arcs of digraph D appears in the adjacency matrix A as a collection of submatrices of all 1s. Since this is a covering, 1s may be used in more than one rank 1 submatrix. A biclique partition of the arcs of D appears in the matrix in a similar way, except, as the name suggests, each 1 in the matrix is in exactly one rank 1 submatrix.

The motivation for this chapter lies in the differences between covers of arcs and partitions of the arcs in the digraph. When are the biclique cover and biclique partition numbers equal? That is the question we will explore here. The question is not an easy one to answer. It is relevant to the current paper, as a whole, because as we see in Siewert [**41**], there are examples where the Boolean rank of a digraph matrix is above real rank as well as examples with Boolean rank below. Here we look for conditions such that Boolean and nonnegative integer ranks are equal. If we find out-tournaments that fit these requirements, then for that class of out-tournaments, the question of finding equal full ranks reduces to that of finding out-tournaments with full real rank. So, if satisfactory conditions can be found, then this will be an important step toward characterizing all out-tournament matrices with equal full ranks.

2 On Boolean Rank

For any {0,1}-matrix A, Boolean rank is more difficult to analyze than real rank because there is a simple process to find a basis over $\Re$ of the row space of A. Although in some ways we can treat the Boolean rank similarly, we don't have subtraction over Boolean semiring, β. Hence, finding Boolean rank becomes much more difficult than finding real rank.

2.1 Reduction and Extension of Bicliques and Biclique Matrices

In this chapter, we use 'reduction' exclusively in the sense of reduction of bicliques given in Section 3.1. Before proceeding to the main result of this section, we need to develop some language to clarify the successive discussion. Whether we are talking about a biclique partition or a biclique cover, the bicliques in digraph D correspond to rank 1 submatrices of A, the adjacency matrix of D. Also recall the introduction of the biclique operations of extension and reduction, and their matrix equivalents, which was presented in Section 3.1. Observe that a covering of a minimum size is not necessarily fully reduced. There are times when maximum bicliques are required in a minimum covering. In order to simplify the following explanations, we will assume that the minimum biclique coverings are also minimal in the sense that any reduction of any biclique in the cover results in a modified collection B' that is no longer a covering. That is, any possible reduction leaves at least one arc uncovered.

Given any minimum cover of 1s in a matrix A, we will call the covering *fully reduced* if any single row or single column elements of a cover are extended to cover all 1s in their row/column (respectively) and all other elements of the cover are reduced accordingly.

One may observe that for any given biclique B_i in collection B, a maximal extension vertically followed by maximal extension horizontally may well produce a different B_i' than that produced by doing the extensions in the other order. A similar statement holds for reductions. The order of multi-stage extensions or reductions will be mentioned explicitly when it makes a difference to a particular proof.

2.2 A Matrix with Full Nonnegative Integer Rank and Singular Boolean Rank

Recall that term rank, $r_t(A)$, of a matrix A is equal to the independence number of A, the size of a maximum set of independent 1s in the matrix. Alternatively, this has been referred to as the minumum size of a line cover of the 1s of the matrix, or in [**39**] as the minimum size of a claw cover of the arcs of digraph D, where D is digraph with adjacency matrix A. A *claw cover* of the arcs of D is a collection of bicliques that we can write as $B = \{(X_i, Y_i)\}_{i=1}^{k}$ in which for each i, $|X_i| = 1$ or $|Y_i| = 1$. Recall (2.1), and the fact that $r_B(A) \leq r_{Z^+}(A)$ for any $\{0,1\}$-matrix, the inequalities $r_B(A) \leq r_{Z^+}(A) \leq r_t(A)$ always hold. This should also seem plausible since a claw cover is more restrictive than a more general biclique cover.

Now, we are interested in describing matrices that have $r_B(A) = r_{Z^+}(A) = n$. Because of their relationship, if Boolean rank is full, then the nonnegative integer rank is full as well. In this section, we approach the problem by considering matrices with full nonnegative integer rank and identify sufficient conditions to guarantee that Boolean rank is also full.

To accomplish this, first we look at matrices that have $r_B(A) < r_{Z^+}(A) = n$.

THEOREM 4.1. *If a square $\{0,1\}$-matrix A has $r_B(A) < r_{Z^+}(A) = n$, then there is a submatrix of the form*

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{4.1}$$

or some permutation of PCQ of the rows and columns of C, where P and Q are permutation matrices.

PROOF. By assumption, a minimum partition of the 1s in matrix A has size n, but a minimum cover of the 1s of A has size $n-k$, with $k>0$. Note that there cannot be any rows of 0s. Since the Boolean rank is less than n, given any minimum covering $B = \{B_1, B_2, ..., B_{n-k}\}$, there exists a minimal set S of rows of A that are covered by $|S| - k$ rank 1 submatrices. Within this subcover, there exists an $a_{xy} = 1$ that lies in at least two distinct elements of the cover, say $B_i = (X_i, Y_i)$ and $B_j = (X_j, Y_j)$, which is to say that $x \in X_i \cap X_j$ and $y \in Y_i \cap Y_j$. If not, then the 1s in the rows of S are actually partitioned by the subcover lying in these rows. Since the 1s in these rows are partitioned by less than $|S|$ rank 1 matrices, $r_{Z+}(A) < n$, contradicting our assumption.

Without loss of generality, we may suppose that minimum covering B is fully reduced. For ease of discussion, we will refer to only the part of the subcover that lies in the rows of S. Let $b_i = (x_i, Y_i)$, where x_i consists of $X_i \cap S$. The biclique b_i is a sub-biclique matrix of B_i. The collection of $b = \{b_i | B_i \cap S \neq \varnothing\}$ forms a minimum cover of the 1s of rows of S. Suppose not. Then A itself may be covered in less than $n-k$ rank 1 matrices, contradicting our assumption.

Observe that every b_i lies in multiple rows, that is $|x_i| > 1$ for each i. If we were to have a single row b_0, then this rank 1 matrix could be extended to cover all the 1s in that row, and the other b_i reduced accordingly. We have now modified the subcover, but it still has the same number of elements. However, the minimality of S is violated, as the removal of that row from S produces a set of $|S| - 1$ rows covered by $(|S| - 1) - k$ rank 1 matrices.

Now, going back to the 1 that is in b_i and b_j, if b_i is in only column y, then b_i may be extended to cover every 1 in that column, and since they are covered, b_j may be reduced to $b'_j = (x_i, Y_i - y)$. But, we assumed that the covering b was fully reduced.

If r_i and c_i stand for a row vector index of A and a column vector index, respectively, we know that there is $r_1 \in X_i - X_j$ and $r_2 \in X_j - X_i$, and likewise, there is $c_1 \in Y_i - Y_j$ and $c_2 \in Y_j - Y_i$. The submatrix of A consisting of rows r_1, x, r_2 and columns c_1, y, c_2 has two rank 1 matrices, each lying in two rows and two columns, with exactly one nonzero entry in both rank 1 matrices, which we named a_{xy}. Therefore the submatrix of A consisting of rows r_1, x, r_2 and columns c_1, y, c_2 is a permutation of matrix C given by (4.1). □

Corollary 4.2. *If A has $r_{Z^+}(A) = n$ and contains no submatrix PCQ where P and Q are any 3×3 permutation matrices, and C is*

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

then $r_B(A) = r_{Z^+}(A) = n$.

Proof. This follows from the contrapositive of Theorem 4.1. □

Note that we have a sufficient condition that $r_B(A) = r_{Z^+}(A) = n$, but the condition is not necessary. A full nonnegative integer rank matrix may have a submatrix of the given form, and still have full Boolean rank as well.

The essential feature of matrix C is that the 1 at $c_{2,2}$ is in two distinct maximal J_i matrices. As a result, even though $r_B(C) = 2$, clearly $r_{Z^+}(C) = 3$. Up to isomorphism, this is the smallest square $\{0,1\}$-matrix with Boolean rank less than nonnegative integer rank.

We may look at the cause for the difference in the two ranks of matrix C above by giving a proposition extending a remark in [**35**]. Orlin's remark was about decompositions of undirected graphs, whereas ours follows a similar line of reasoning, but for digraphs and bicliques. Although we have not yet talked about C as part of an adjacency matrix, that is where this line of inquiry is leading.

PROPOSITION 4.3. *Let A be the adjacency matrix of digraph D. If every arc in digraph D belongs to a unique maximal biclique, then* $r_{Z^+}(A) = r_B(A) = \vec{bc}(D) = \vec{bp}(D)$ *and this common rank equals the number of distinct maximal bicliques in D as well as the number of distinct maximal J_i in A.*

PROOF. Let $B = \{B_i\}$ be any minimum biclique covering of D. Suppose that $(x, y) \in B_j \cap B_k$. But (x, y) is in a unique maximal biclique, so $B_j = B_k$ otherwise B is not a minimum cover. Thus, the collection B is disjoint, making it a partition. Since $\vec{bc}(D) \leq \vec{bp}(D)$, we have $\vec{bc}(D) = \vec{bp}(D)$. $\square$

Although this is an interesting result in itself, consider that every 1 in matrix C is in a unique maximal J_i, or biclique matrix, with the notable exception of $c_{2,2}$.

3 Digraphs with Matrices Containing C and Their Ranks

Theorem 4.1 gives a submatrix C that must be in any $\{0,1\}$-matrix with differing Boolean and nonnegative integer ranks. We will use that result and apply it to adjacency matrices. The reader will observe that matrix C is not a digraph matrix as it is usually interpreted. It can represent a submatrix of an adjacency matrix, however. The rows and columns of C could be numbered in a way that allow it to be the submatrix of an adjacency matrix. We consider the different ways that this can be done. As noted by Corollary 4.2, if a $\{0, 1\}$-matrix A does not have submatrix C, then $r_B(A) = r_{Z^+}(A)$. But the presence of submatrix C does not guarantee that $r_B(A) < r_{Z^+}(A)$. To explore some possibilities, we will consider classes of tournaments and out-tournaments whose adjacency matrices contain at least one submatrix C and yet has $r_B(A) = r_{Z^+}(A)$. We will do this by considering the vertices involved in that part of the matrix containing submatrix C. That is, we look at the submatrix C consisting of row set X and column set Y, which is exactly the submatrix representing biclique (X, Y). Then, with the restriction that the matrices must be adjacency matrices of a digraph, we consider $[X \cup Y]_D$, the induced subdigraph of parent digraph D on vertices in $X \cup Y$, which generated submatrix C.

First consider that for any digraph adjacency matrix, C arises on a subdigraph consisting of a minimum of five vertices, and no more than six are necessary. Of course, it can appear in matrices of digraphs on more than 6 vertices, but we will only consider adjacency matrices of the induced subdigraph adjacency matrix of the vertices actually involved in the substructure.

3.1 Five-Vertex Subdigraphs

Consider the form of a digraph on five vertices that causes submatrix C to appear in its adjacency matrix.

Refer to digraph D_1 in Figure 5. This five vertex digraph is the basic unit that allows Boolean rank to fall below nonnegative integer rank. As we noted above, if the parent digraph has an adjacency matrix that is free of submatrix C, then $r_B(A) = r_{Z^+}(A)$. Now, we will consider possible configurations of induced subdigraphs on the vertices D_1 in a tournament, and their effects on the matrix ranks of a parent digraph.

Up to isomorphism, the D_1-induced out-tournament digraph adjacency matrices are as follows.

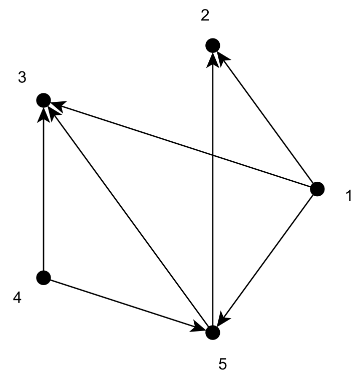

FIGURE 5. Digraph D_1.

$$E_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.2}$$

$$E_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.3}$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

$$E_4 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.5}$$

Now, to make use of these matrices, observe the following, where $D(A)$ represents the digraph D with adjacency matrix A.

(1) $r_B(E_1) = r_{Z^+}(E_1) = 5$, $H_1 = D(E_1)$ is strong and is $\vec{C}_3$-free.

(2) $r_B(E_2) = r_{Z^+}(E_2) = 4$, $H_2 = D(E_2)$ is not strong and contains $\vec{C}_3$.

(3) $r_B(E_3) = r_{Z^+}(E_3) = 4$, $H_3 = D(E_3)$ is not strong and contains $\vec{C}_3$.

(4) $3 = r_B(E_4) \neq r_{Z^+}(E_4) = 4$, $H_4 = D(E_4)$ is not strong and contains $\vec{C}_3$.

There are some direct conclusions that we can draw from this information.

THEOREM 4.4. *Let T be a tournament with matrix A such that every instance of submatrix C arises from subdigraph D_1.*

(1) *If each occurrence of D_1 induces H_1, then $r_B(A) = r_{Z^+}(A)$.*

(2) *If each occurrence of D_1 induces a strong subdigraph, then $r_B(A) = r_{Z^+}(A)$.*

Notice that in (1) above, we cannot include H_2 and H_3 since the fact that their ranks are not full allows the possibility of $r_B(A) < r_{Z^+}(A)$.

For an example that illustrates the previous theorem, consider the following construction.

COROLLARY 4.5. *If T is an out-tournament with matrix A on $5k$ vertices with k strong components consisting of H_1, then $r_B(A) = r_{Z^+}(A) = 5k$.*

PROOF. Let out-tournament T have k strong components, each consisting of H_1. Then there are no other induced copies of D_1 in the digraph, because $SC(D)$ is acyclic. Since every copy of D_1 induces H_1, then $r_B(A) = r_{Z^+}(A)$. Due to the fact that $r_B(A_j) = 5$ and $r_B(A) = \sum_{j=1}^{k} r(A_j)$, where A_j is the adjacency matrix of component j, we have $r_B(A) = r_{Z^+}(A) = 5k$. $\square$

Along the same lines, we may extend a similar result to a class of strong out-tournaments. Consider the rotational tournaments on $n \geq 7$ vertices, defined by $N^+(x) = \{x+1, x+2, x+3\}$ with all entries (mod n)+1. If T is such an out-tournament with adjacency matrix A, there are many submatrices of the form C; each is generated by a subdigraph on five vertices, and each of those subdigraphs induce H_1. For example,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.6}$$

This example is a tournament, but the form of the matrix will be the same for higher n, which will form proper out-tournaments. Note that D is strong in this case, and the 1s that represent the Hamiltonian cycle, $\{x \longrightarrow [(x \bmod n) + 1] | x \in V(D)\}$, form a full set of isolated 1s. Hence, $r_B(A) = r_{Z^+}(A) = n$.

3.2 Some Six-Vertex Subdigraphs

An induced subdigraph D on six vertices with matrix A containing C has many more possible configurations than with five vertices. We can nevertheless make some observations of classes, which despite having such induced digraphs, have full, equal Boolean and nonnegative integer ranks.

Proposition 4.6. *Let D be a tournament with adjacency matrix A having full nonnegative integer rank. If C is a submatrix of A produced by an induced subdigraph on six vertices, and no five vertex subdigraph produces C, then D has at least one $\vec{C}_4$ subdigraph.*

Proof. Consider matrix C again, produced by a subdigraph on six vertices and no five vertex subdigraph produces C. The removal of any of the six vertices causes C to no longer be a submatrix of A. Then by definition the rows and columns represent disjoint sets of vertices. Up to isomorphism, the vertices are as labeled in (4.7).

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \tag{4.7}$$

Since D is a tournament and $a_{1,6} = 0$, then $6 \longrightarrow 1$. Likewise, $4 \longrightarrow 3$. Thus, $1 \longrightarrow 4 \longrightarrow 3 \longrightarrow 6 \longrightarrow 1$. □

Now, we can say the following.

Corollary 4.7. *If D is a tournament with adjacency matrix A having*

(1) *full nonnegative integer rank,*
(2) *every submatrix C generated by a subdigraph on six vertices, and*
(3) *no $\vec{C}_4$ subdigraph,*

then $r_B(A) = n = r_{Z^+}(A)$.

Proof. This follows directly from Proposition 4.6. □

4 Possible Direction of Future Work

It is possible that a more general characterization of out-tournament matrices with equal Boolean and nonnegative integer ranks could arise by considering other matrices with submatrix C, yet having full Boolean rank. What we have done in this chapter gives a partial answer to the difficult question of characterizing out-tournament matrices, and digraph matrices in general, with equal Boolean and nonnegative integer ranks. In the cases where matrix C was generated on five vertices, the possible forms were limited for the induced subdigraph that produced submatrix C. We were able to deal with them on a case by case basis. The work in this chapter leaves open the analogous question for six vertex induced subdigraphs generating C, in which no five vertex subdigraph generates C. There are many more possibilities here, as well as different ways in which the induced subdigraphs may overlap. Because of that, methods used in the five vertex cases may not generalize well to the six vertex cases. There may be a more efficient way to reach a similar characterization of the digraphs containing six vertex generators of matrix C. What was done here laid the foundation for future investigations for the six vertex case, and illustrated the difficulty of identifying digraph structures producing matrices with a particular Boolean rank.

CHAPTER 5
Out-Tournaments with a Relative Minimum Cycle Length

1 Introduction

This chapter will focus first on non-strong out-tournament matrices, identifying several classes which have determinate matrix ranks. The second part turns to a family of strong out-tournaments that build on the properties established in [**5**].

Recall that for any out-tournament, the adjacency matrix has full, equal ranks if and only if the strong component matrices have full and equal ranks, which was Theorem 2.11.

This provides a good deal of information about out-tournaments with multiple strong components. However, if an out-tournament has any single-vertex strong components, then the submatrix for that strong component is simply [0], which of course has rank 0. Likewise, if we are considering a single strong component whether the out-tournament itself or a component of a larger out-tournament, then Theorem 2.11 gives no information whatsoever. It is precisely those two cases that begin the current investigation.

In this section, we need to refer to the size of the inset and outset of a vertex. The *out-degree* of vertex v is the size of its outset, denoted $d^+(v) = |N^+(v)|$. The *in-degree* of v is $d^-(v) = |N^-(v|$. If we talk about the same vertex in the context of different digraphs, a subscript indicates which digraph we mean. For example, $d^+(v)_D$ denotes the out-degree of vertex v in digraph D. If the digraph is clear from the context, the subscript is omitted.

LEMMA 5.1. *Let D be an out-tournament with adjacency matrix A. Then $r_t(A) = n$ if and only if D has no trivial strong components.*

PROOF. ($\Rightarrow$) By definition, $r_t(A) = n$ if and only if there is an independent set of 1s of size n. Suppose that

$$S = \{(i_1, j_1),\ (i_2, j_2),\ ...,\ (i_n, j_n)\}$$

is the set of arcs represented by a set of independent 1s in A of size n. Since each row and each column is represented in the set of 1s, then each of $1, 2, ..., n$ appears in the first coordinate of exactly one ordered pair in S and, likewise, each of $1, 2, ..., n$ appears in the second coordinate of exactly one ordered pair in S. If this set of arcs constitutes a Hamiltonian cycle in D, then we are done. Suppose not. Let D' be the subgraph of D containing only the arcs of S. In D', each vertex v has $d^+(v) = d^-(v) = 1$. Let $s(y)$ be the successor function: $s(y) = z$ if $y \longrightarrow z$. The function is well defined in D'. Consider the list $y, s(y), s^2(y), ..., s^{n-1}(y), s^n(y)$, with $n+1$ entries. With only n possible indices, we know that there is a repeated index in the list, implying that there is a cycle represented in the list. Suppose that v is repeated in the list. There is a yv-path, however, and so working backwards from $v = s^j(y)$, vertex y must be on that cycle. If y is not on the cycle then we have paths $y, w_1, w_2, ..., w_k, v$ and $v, v_1, v_2, ..., v_m, v$, with only vertex v in common. Then w_k and v_m both dominate v in D', which cannot happen, as we noted that all the in-degrees must be 1 in D'. Now, y was an arbitrary vertex, so every vertex in D' lies on a cycle and there are no trivial strong components. Thus, the same holds in D since $D' \subseteq D$.

($\Leftarrow$) Assume that there are no trivial strong components. Then each vertex lies on a cycle. Consider the strong component D_0 containing y. By [**5**], a digraph is strong if and only if it has a Hamiltonian cycle. Thus, D_0 has a Hamiltonian cycle, as does every strong component. The collection of all the arcs on these Hamiltonian cycles forms a set of n independent 1s, so $r_t(A) = n$. $\square$

2 Acyclic Out-Tournaments

Throughout the current work, we are concerned only with connected digraphs. We can assume that the underlying graph is connected without any loss of generality, since any rank of a disconnected digraph is simply the sum of the respective ranks for each of the components of the digraph.

Let us consider out-tournaments that have one or more single-vertex strong-components. The simplest possible case is the digraph in which there are only single-vertex strong components. What does a connected out-tournament without any non-trivial strong-components look like? An important structural theorem from Bang-Jensen et al. below begins to answer the question. An *in-branching* is a spanning tree rooted at vertex r oriented in such a way that every other vertex has exactly one arc out of it.

THEOREM 5.2. [**5**] *Every connected out-tournament has an in-branching.*

Thus, we first consider a digraph D that is an in-branching, i.e., a digraph in which there are only single-vertex strong components. Note that since the underlying graph $UG(D)$ is a tree, D must be acyclic. Often, a particular enumeration of the vertices will reveal patterns in the adjacency matrix of the digraph that would otherwise not be obvious.

Rather than naming vertices $v_1, v_2, ..., v_n$ we will usually refer to vertices by their index alone, 1,2,...,n, which will make the notation slightly less cluttered. Thus, an arc from v_1 to v_2 is written as $1 \longrightarrow 2$ or (1,2) interchangeably. The ordered pair form is particularly useful in this context, since if $i \longrightarrow j$ is an arc in D with adjacency matrix A, then entry $a_{ij} = 1$ and the ordered pair (i, j) may be thought of as the coordinates in the matrix of the 1 that indicates this arc.

An *acyclic enumeration* is an ordering of the vertices with $i \longrightarrow j$ only if $i < j$. If D is an in-branching, it is acyclic. And, since any partial order may be embedded in a linear order, preserving relationships, an acyclic enumeration exists.

A *transitive tournament* is a tournament such that the relation '$\longrightarrow$' is transitive. Observe that for any transitive tournament, there exists an acyclic enumeration. If out-tournament D is acyclic, and $UG(D) = K_n$, where K_n is the complete graph on n vertices, then D is a transitive tournament. The transitive tournament on n vertices is unique, up to isomorphism.

PROPOSITION 5.3. *If out-tournament D is acyclic and $UG(D) \neq K_n$, then D is a subdigraph of a transitive tournament.*

PROOF. An acyclic out-tournament is a partial order of vertices under relation '$\longrightarrow$'. Let D be an acyclic out-tournament on n vertices. Assign the vertices an enumeration that preserves relations so that $i \longrightarrow j$ only if $i < j$. Then D is a subdigraph of the transitive tournament on n vertices with an acyclic enumeration. □

Since our main interest is out-tournaments that are not tournaments, we focus on those that aren't transitive tournaments. The argument below applies in that case as well, however.

For an in-branching D, it will be convenient to refer to the vertices j with $d^-(j) = 0$ as the *leaves* of D. Let L denote the set of all leaves, and $|L|$ the cardinality of L.

As well as in-branchings, the following results apply to all acyclic out-tournaments.

REMARK 5.4. *Let D be an acyclic out-tournament with L the set of leaves of D. Then $r_t(A) \leq n - |L|$.*

This can be seen by observing that for each $l \in L$, column l is all 0s. Thus, there cannot be a set of independent 1s bigger than $r_t(A) \leq n - |L|$ and each of the matrix ranks is now bounded above by $n - |L|$.

Consider the set of S 1s in A consisting of the 'lowest' non-zero entry in each column.

Lemma 5.5. *Let D be an acyclic out-tournament with an acyclic enumeration and adjacency matrix A and set $S = \{a_{ij} = 1 \mid a_{kj} = 0$ for each $k > i\}$. Then S is a maximum set of isolated 1s of A.*

Proof. First, observe that no two elements of S lie in the same column of A, by definition. Second, no two may lie in the same row. Indeed, suppose that $s_1 = a_{ij}$ and $s_2 = a_{ik}$ are both elements of S. Then $i \longrightarrow j$ and $i \longrightarrow k$. Since D is an out-tournament, j and k are adjacent. Without loss of generality, assume that $j < k$. Since D has an acyclic enumeration, it cannot be the case that $k \longrightarrow j$, so we must have that $j \longrightarrow k$ and $a_{jk} = 1$, which therefore implies that a_{ik} was not actually an element of S. Therefore, S is an independent set of 1s.

Suppose that two elements of S lie in a 2×2 submatrix of A containing all 1s. Let the elements of S be a_{ij} and $a_{k\ell}$. We know that they must be in different rows and different columns, so, without loss of generality, suppose that $i < k$ and $j < \ell$. However, recall that each element a_{ij} of S was chosen in such a way that there are no non-zero entries below a_{ij} in column j. So, no two elements of S lie in a 2×2 submatrix of all 1s, which means that S is a set of isolated 1s. □

Note that for any out-tournament D with adjacency matrix A, the following are equivalent:

(1) an acyclic enumeration of $V(D)$ exists,

(2) D is acyclic, and

(3) A is upper triangular.

Theorem 5.6. *If D is a connected acyclic digraph with adjacency matrix A, and $L \subseteq V(D)$ is the set of leaves of D, then*

$$r_B(A) = r(A) = r_{Z^+}(A) = r_t(A) = n - |L|.$$

Proof. Note that column j is all 0s if and only if j is a leaf of acyclic digraph D. There is an element of S for each column that is not all 0s, so $|S| = n - |L|$. Since S is an isolated set of 1s, then $S = n - |L| \leq r_B(A)$ and because there are $|L|$

columns of 0s in A, we know $r_t(A) \leq n - |L|$. Thus we have $r_B(A) = r_{Z^+}(A) = r_t(A) = n - |L|$. To find $r(A)$, note that S is a set of leading 1s and that A is in row echelon form, so $|S| = n - |L| \leq r(A)$, and since $r(A) \leq r_{Z^+}(A)$, we get $r(A) = n - |L|$. □

3 Strongly Connected Out-Tournaments

In much the same way as we proceeded in the previous section for out-tournaments with single-vertex strong-components, we now consider strongly connected out-tournaments. Here again, we find a helpful structural theorem in [**5**].

THEOREM 5.7. [**5**] *An out-tournament has a Hamiltonian cycle if and only if it is strongly connected.*

Using Theorem 5.7 as guidance, we begin our study of strong out-tournaments by considering cycles themselves. Consider out-tournament $D = \vec{C_n}$, with $n \geq 3$. Out-tournament D is strongly connected if and only if there is a cyclic enumeration of the vertices. Let D have a cyclic enumeration. Let A be the adjacency matrix of D. Matrix A has 1s on the superdiagonal and in the position $(n, 1)$ only. For any digraph D, let $M(D)$ be the adjacency matrix of D.

REMARK 5.8. *Let* $A = M(\vec{C}_n)$. *Then* $r_B(A) = r(A) = r_{Z^+}(A) = r_t(A) = n$.

This follows from the fact that $A \simeq I_n$, which makes $r(A) = n$ and the fact that the set of all 1s in A forms a set of isolated 1s of size n. Note that unless $n = 3$, $\vec{C}_n$ is not a tournament.

Now, we relax the constraints on D a bit. Suppose we take a strong out-tournament D and put a restriction on the minimum size of the cycles it contains. In this way, we can keep some of the structure of those acyclic out-tournaments. Here, we will cause all subdigraphs of a size smaller than the minimum cycle size to be acyclic.

We will say that A is *nearly upper-triangular* if $A' = PAQ$, and A' is upper-triangular, with permutation matrices P and Q . Defined in this way, for any n, $M(\vec{C_n})$ is nearly upper-triangular.

LEMMA 5.9. *If out-tournament D has a Hamiltonian cycle and the removal of one arc makes it acyclic with an acyclic enumeration, then $A = M(D)$ is nearly upper-triangular.*

PROOF. Consider permutation matrix family P_0 given by

$$P_0 = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & \ddots & & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Let D be a strong out-tournament. Assume that the removal of some arc on that cycle produces an acyclic digraph D'. Give the vertices of D' an acyclic enumeration and use the same enumeration for D. Let $A = M(D)$, then P_0A is upper triangular. □

LEMMA 5.10. *Let out-tournament D have a Hamiltonian cycle such that the removal of one arc makes it acyclic. Let D have a cyclic enumeration that is also an acyclic enumeration under the removal of $(n, 1)$. If $A = M(D)$ then the set of 1s on the superdiagonal and $a_{n,1}$ is a set of isolated 1s.*

PROOF. The 1s representing the Hamiltonian cycle arcs lie on the superdiagonal and $(n, 1)$ under a cyclic enumeration. The removal of arc $(n, 1)$ makes the enumeration acyclic, and hence if $a_{n,1} = 0$, then A would be upper-triangular. Therefore P_0A is upper triangular. Then, the collection of all the 1s on the diagonal of P_0A is a set of isolated 1s because each element a_{ij} has only 0s below it. That is, $a_{kj} = 0$ for each $k > i$. Hence, the corresponding set of 1s in A is also isolated. □

We are going to consider a digraph D with shortest cycle length of $n - k$, which also contains an arc $i \longrightarrow j$, where $j = i + k + 1$. These assumptions will lead to an out-tournament with adjacency matrices having ranks that can be easily evaluated. First, we make the following observation.

LEMMA 5.11. *Let out-tournament D have a cyclic enumeration, a shortest cycle length of $n - k$, where $k \geq 3$, and let $1, k+2, \ldots, n-1, n, 1$ be one such cycle. Let $W \subseteq V(D)$ be the set of vertices $\{2, \ldots, k+1\}$. Take any i and j such that $\{i, j\} \cap W = \varnothing$. Then $i \longrightarrow j$ only if $j = i + 1$.*

PROOF. Take i and j in W^C (the complement of set W). If $j - i > 1$, then $i, j, j+1, \ldots, n-1, n, 1, 1+k+2, \ldots, i$ would be a cycle with length $l < n - k$, which contradicts our assumption. □

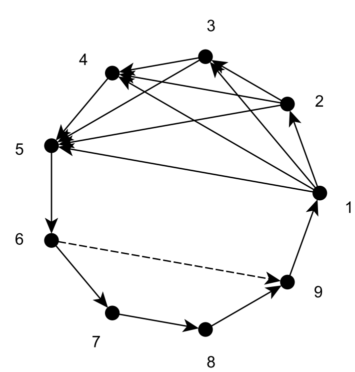

FIGURE 6. With $6 \longrightarrow 9$, cycle $1, 5, 6, 9, 1$ is created that is shorter than the minimum length.

For illustration, where $n = 9$ and $k = 3$, see Figure 6. The i and j referred to in Lemma 5.11 are vertices 6 and 9, respectively. This example shows why in that part of the digraph, a vertex only beats its successor on the Hamiltonian cycle. This condition occurs because of the assumption of the form of the shortest cycle in relation to the Hamiltonian cycle. Specifically, it is due to the assumption of arc $i \longrightarrow j$, which we will consider in more detail. Let strong out-tournament D on n vertices have a cyclic vertex enumeration. Let $n-k$ be the length of the shortest cycle in D. If there is an arc of the form $i \longrightarrow i+k+1$, just as arc $1 \longrightarrow 5$ is in Figure 6, then we refer to that arc as a *skip arc*, with respect to some cycle. Here, the reference cycle will be the Hamiltonian cycle that gave rise to the cyclic enumeration we are using. These arcs create a cycle that skips over some of the other vertices.

LEMMA 5.12. *Let D be a strong out-tournament. If the shortest cycle of D has length $n-k$ and $k < \frac{n-3}{2}$, then $A = M(D)$ is nearly upper-triangular.*

PROOF. There are many possible cases; we will consider the limiting cases first. Let D have a cyclic enumeration and have $n-k$ as the length of its shortest cycle. Further, assume that D has an arc of the form $i \longrightarrow j$ where $j = i+k+1$, see Figure 7.

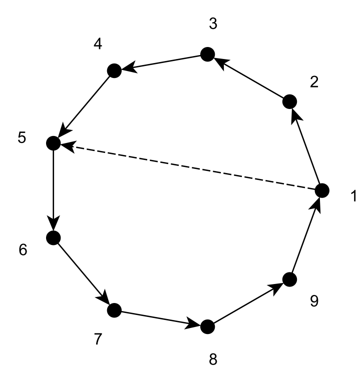

FIGURE 7. The dashed arc indicates $i \longrightarrow j$.

Without loss of generality, we can take $i = 1$. Then $1, k+2, \ldots, n-1, n, 1$ is a cycle of the shortest length in D. By Lemma 5.11, there can be no arcs $i \longrightarrow j$ for $j > i+1$ if $i, j \notin W = \{2, \ldots, k+1\}$, other than the assumed skip arc, $1 \longrightarrow k+2$. However, there may be more arcs of the form $i \longrightarrow i+k+1$ that are incident on set W. Again, without loss of generality, we may assume that if there is any arc

$i \longrightarrow i+k+1$ with at least one of i and $i+k+1$ in W, then $i > 1$. Let i be the vertex with the highest index that has $i \longrightarrow i+k+1$. We claim that $i \leq 2$. To observe this, note the following.

Given a cyclic enumeration, skip arc $i \longrightarrow j$ creates the cycle $i, j, j+1, j+2, \ldots, n, 1, 2, \ldots, i$. The vertices $i+1, i+2, \ldots, j-1$ are left out of this cycle, so its length is $n-(j-i-1) = n-k$.

Each skip arc, $i \longrightarrow i+k+1$ implies that arcs $j \longrightarrow i+k+1$ also must exist,for each $i < j < i+k+1$. This happens because D is an out-tournament and W induces an acyclic digraph. Note that by our construction, subdigraph $[W]_D$ already has an acyclic enumeration. Recall that we assumed skip arc $1 \longrightarrow k+2$ was present. If $i \longrightarrow i+k+1$ also exists, with $i \geq 3$, then cycle $1, i, i+k+1$, ..., $n-1, n, 1$ is shorter than $n-k$, which contradicts the assumption. Therefore, if there are two skip arcs in D, then they are of the form $i \longrightarrow i+k+1$ and $(i+1) \longrightarrow (i+k+2)$.

Suppose that D has two skip arcs, $1 \longrightarrow k+2$ and $2 \longrightarrow k+3$. Since we assumed that $k < \frac{n-3}{2}$, then $2k < n-3$, giving us $k+3 < n-k$. Induced subdigraphs $[\{1, 2, \ldots, k+2\}]$, $[\{2, 3, \ldots, k+3\}]$ as well as $[\{1, 2, \ldots, k+3\}]$ each has fewer than $n-k$ vertices, and so they must be acyclic. Furthermore, the cyclic enumeration we used is an acyclic enumeration for each of these induced subdigraphs. The removal of arc $(n, 1)$ creates an acyclic graph, and the enumeration already assigned is an acyclic enumeration. Therefore, $A = M(D)$ is nearly upper-triangular and P_0A is upper-triangular with all 1s on the main diagonal. The form of the adjacency matrix of D with two skip arcs $i \longrightarrow i+k+1$ is given in (5.1). The two skip arcs are indicated by bold **1**s in the adjacency matrix. $\square$

$$(5.1)\qquad A = \begin{array}{cc} & \begin{array}{cccccccccc} 1 & 2 & 3 & \cdots & k+2 & k+3 & k+4 & \cdots & n-1 & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ k+2 \\ k+3 \\ k+4 \\ \vdots \\ n-1 \\ n \end{array} & \begin{bmatrix} 0 & 1 & 1 & \cdots & \mathbf{1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 1 & \mathbf{1} & 0 & & 0 & \\ & 0 & 0 & \ddots & 1 & 1 & 0 & & & \\ \vdots & & 0 & \ddots & 1 & \vdots & \vdots & & \vdots & \\ & \vdots & & & 0 & 1 & 0 & & & \vdots \\ & & \vdots & & 0 & 0 & 1 & & & \\ & & & & 0 & 0 & 0 & \ddots & 0 & \\ 0 & & \vdots & & \vdots & \vdots & \vdots & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \end{array}$$

The benefit of this careful enumeration shows in the form of the adjacency matrix. It allows us to show that A has full, equal ranks in a straightforward manner.

Corollary 5.13. *Let out-tournament D have a Hamiltonian cycle such that the removal of one arc makes it acyclic. Let D have a cyclic enumeration that is also an acyclic enumeration under the removal of $(n, 1)$. If $A = M(D)$, then $r_B(A) = n$.*

Proof. This corollary follows from Lemma 5.10. Since the 1s in the matrix representing the Hamiltonian cycle form a full set of isolated 1s, we know that the Boolean rank is full. □

Theorem 5.14. *Let D be a strong out-tournament with shortest cycle length of $n - k$ where $k < \frac{n-3}{2}$ and matrix $A = M(D)$. If there is a skip arc of the form $i \longrightarrow i + k + 1$, then adjacency matrix A has equal full ranks:*

$$r_B(A) = r(A) = r_{Z^+}(A) = r_t(A) = n.$$

Proof. By Corollary 5.13 and the basic rank inequalities, we know $r_B(A) = r_{Z^+}(A) = r_t(A) = n$. Recall the proof of Lemma 5.10, in which we observed that P_0A was upper-triangular with all 1s on the main diagonal. Then
$r(P_0A) = r(A) = n$. $\square$

4 Conclusion and Future Work

In the strong out-tournaments we considered, a certain number of vertices are 'skipped' on the shortest cycle. Specifically, all the skipped vertices were in one connected subgraph of D. The other extreme case would be if the sets of vertices skipped on a cycle of the shortest length were in as many small connected subdigraphs as possible. For instance, consider an out-tournament on n vertices with a cyclic enumeration, and arcs defined as $A(D) = \{j \longrightarrow j + \ell \mid \ell = 1, 2, ..., m\}$, where the number of skipped vertices in a shortest cycle dictates the value of m relative to the number of vertices in the digraph. The value of m, in turn, dictates whether or not the real and Boolean ranks are full in the adjacency matrix. For any shortest cycle, the skipped vertices induce multiple connected graphs in the underlying graph, in contrast to the type discussed in Theorem 5.14. Each diagonal of the matrix would be either all 1s or all 0s, which should lend itself to both Boolean rank and real rank calculations. A 'diagonal' of square matrix A means a set of entries $\{(j, (j \bmod n) + k) \mid j = 1, 2, ..., n\}$ for some $0 \leq k \leq n - 1$.

In this chapter, we have identified a class of out-tournament with determinate adjacency matrix ranks, including cycles, in-branchings, and acyclic digraphs. Particularly we looked in-depth at a class with a great deal of structure due to the fairly large minimum cycle length and the connectedness of the set of vertices skipped on a shortest cycle.

CHAPTER 6
Out-Tournament Orientations of Unicyclic Graphs

1 Introduction

In the remainder of the paper we will look at constructing classes of out-tournaments that are based on representation theorems given in Bang-Jensen et al. [**5**], where the authors explored the idea of a local in-tournament, giving structural theorems as well as many propositions on the orientability of graphs with specific structures as in-tournaments. The representations give a perspective on out-tournaments and their matrices that is different from previous work. For example, none of the works cited in the Chapter 1 have investigated tournament or out-tournament matrices based on representations and catch digraphs.

The following definitions are a necessary foundation upon which the results in this chapter are built. A graph $G = (V, E)$ is *orientable* as an out-tournament if there is an assignment of an arc (x, y) or (y, x) to each edge $\{x, y\} \in E$ for which the digraph (V, A) is an out-tournament, where arc set A is the image of E under the assignment. A *pointed set* is an ordered pair (X, a) consisting of a set X and an element $a \in X$, designated as the *point*. Athough there may be other uses of pointed sets, for our purposes, the pointed sets will always represent sets of vertices of a connected subdigraph of H. Let $\mathcal{F} = \{(H_x, p_x)|x \in V\}$ be a family of pointed sets. A *catch digraph* $\Omega^-(\mathcal{F})$ is a digraph with vertex set V and arc (x, y) if $p_x \in H_y$ and x, y are distinct.

The *intersection graph* $\Gamma(\mathcal{F})$ of a family of sets $\mathcal{F} = \{H_x|x \in V\}$ has vertex set V and edge xy whenever $H_x \cap H_y \neq \varnothing$. A graph G is *representable* in graph H if G is isomorphic to the intersection graph of a family of connected subsets of vertices

$\mathcal{F} = \{H_x | x \in V(G)\}$ which induce a connected subgraph in H. The family $\mathcal{F}$ is a *representation* of G in H. So far, we have used Bang-Jensen et al. [**5**] notation exactly. In this paper, we can simplify it somewhat, for better clarity.

Bang-Jensen et al. used the subscript of H_x to indicate the vertex of the catch digraph to which H_x corresponds. This leaves open the possibility that the pointed sets may represent something other than vertices of the underlying graph. Here, the vertices of the representation graph and its catch digraph are the same set. Thus, we will use H_x as the pointed set that has x as its point, where $x \in V(G)$ and $x \in V(\Omega^-(\mathcal{F}))$. That is, the vertex sets of the three objects in question – graph G, representation graph H, and catch digraph $\Omega^-(\mathcal{F}) = D$ – are identical.

This chapter, in particular, will consider a class of catch digraphs produced by unicyclic representations of a graph G. All unicyclic graphs are orientable as out-tournaments and can be used to construct out-tournaments that are more complex than the orientations are. Out-tournament orientations are the simplest out-tournaments obtainable by a catch digraph from the graph H, whatever the form of H. Refer to Figure 8, which shows a unicyclic graph, G, and Figure 9 shows an out-tournament orientation of G.

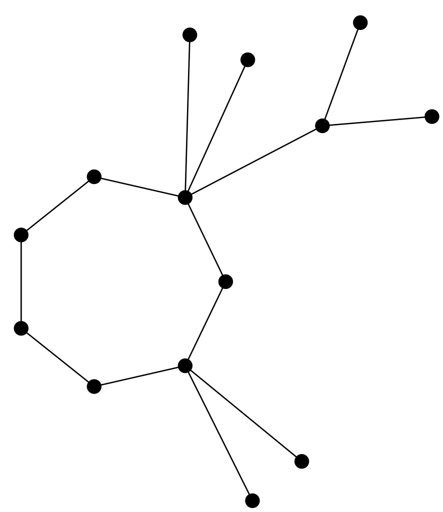

FIGURE 8. A unicyclic graph, G.

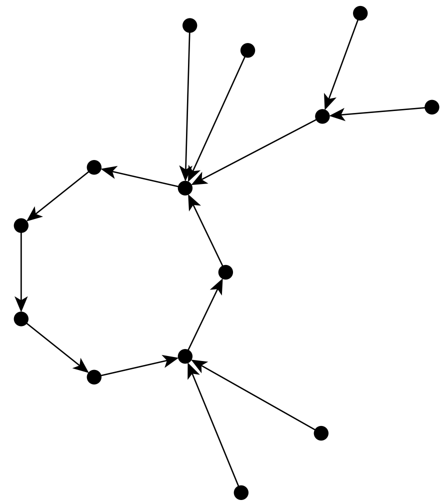

FIGURE 9. Out-tournament orientation of G.

We will also consider the intersection number of a digraph and equivalent notions. The intersection graph is, loosely speaking, the inverse operation of representation. The intersection number of a digraph is related to the Boolean rank of the adjacency matrix, which is a central theme in this paper.

Note that the following lemma is modified for out-tournaments from the cited lemma.

LEMMA 6.1. [**5**] *If D is an out-tournament, then $\Omega^{-}\{(N^{-}[x], x)|x \in V\} = D$ and $\Gamma\{(N^{-}[x], x)|x \in V\} = UG(D)$.*

This lemma tells us that if we start with an out-tournament, take closed insets as the pointed sets and form the catch digraph, we arrive back at the same out-tournament itself. It also tells us that the intersection graph of that family of insets is the underlying graph of the out-tournament we started with. An immediate consequence, which may or may not be obvious, is that any out-tournament is representable in its own underlying graph.

The following theorem puts together all of the pieces. It insures that every out-tournament can be written as a catch digraph of a representation, which opens the door for the representation-based investigation of out-tournament structure as well as orientability of graphs as out-tournaments found in [**5**].

THEOREM 6.2. [**23**] *A digraph $D = (V, E)$ is an out-tournament if and only if it is the catch digraph of a family $\{(S_x, p_x)|x \in V\}$ such that $UG(D)$ is $\Gamma(S_x|x \in V)$.*

We now have a foundation in representations, but before proceeding to the constructions and matrices in this chapter, we must take a look at intersection graphs and the intersection number.

2 Intersection Number of a Digraph

The intersection graph of a family of sets will be an important tool for this paper because, as we have seen, there is an important characterization of out-tournaments based on their representations in families of sets. Closely related is the concept of an intersection digraph. Let $\mathcal{F}$ be a family of ordered pairs (S_i, T_j), with S_i, T_j subsets of some parent set S. Digraph D is an *intersection digraph* of family $\mathcal{F}$ means that $u \longrightarrow v$ if and only if $S_u \cap T_v \neq \varnothing$. Any digraph D is an intersection digraph of a set. The *intersection number* of D, denoted $int(D)$, is the minimum size of a set S such that D is the intersection graph of family $\mathcal{F}$ as defined above. The proof given below is different from that in Brown and Roy [**11**].

THEOREM 6.3. [**11**] *Let D be a digraph and $M = A(D)$. Then $int(D) = r_B(M)$.*

PROOF. Let D be a digraph, A its adjacency matrix and $r_B(A) = k$. Let $XY = A$ be a Boolean factorization of A with X being $n \times k$ and Y being $k \times n$. Using the notation and results from Shader [**39**], let X_j be the jth column of matrix X and Y_j the jth row of matrix Y. If we interpret these as characteristic vectors, then the collection $\{B_i|1 \leq i \leq k\}$ is a biclique cover with $B_i = X_i \longrightarrow Y_i$. To follow the previous notation, let $M_i = X_i Y_i^T$. Conversely, starting with the collection $\{M_i\}_{i=1}^m$ we can construct X and Y. Now, consider the sets $S_{v_i} = \{k | (\exists j)(m_{i,j}^k = 1)\}$ and $T_{v_j} = \{k | (\exists i)(m_{i,j}^k = 1)\}$ as given above. Then,

$$\begin{aligned} S_{v_i} \cap T_{v_i} \neq \varnothing \quad &\Leftrightarrow \{k | i \in X_k\} \cap \{l | j \in Y_l\} \neq \varnothing \\ &\Leftrightarrow (\exists k)(i \in X_k \wedge j \in Y_k) \\ &\Leftrightarrow (i \longrightarrow j) \ in \ D. \end{aligned}$$

The set of matrices $\{M_1, M_2, \ldots, M_m\}$ determines a factorization XY and vice versa. Since $int(D)$ is the minimum such m, then m is also the minimum positive integer with $XY^T = A$ and X is $n \times m$, Y^T is $m \times n$. This is the definition of Boolean rank of A, $int(A) = r_B(A)$. □

2.1 Equivalence of Intersection Number

So far, we have considered the parallel notions of

- a family of ordered pairs $\mathcal{F} = \{(S_{v_i}, T_{v_j})\}$ as defined above,
- a Boolean factorization $A = XY^T$,
- a biclique cover $\{B_i = (X_i \longrightarrow Y_i)\}$ of D, and
- a collection of minimum-size of rank 1 matrices whose Boolean sum is A.

There is a 1-1 correspondence between any two items in the list. The different perspectives given by these alternate characterizations of the same principle can give insight into properties of out-tournaments and their matrix ranks.

3 Orientability of Graphs

In the process of demonstrating the orientability of graphs representable in unicyclic graphs as out-tournaments, the proofs from [**36**] quoted in [**5**] have thereby drawn attention to a class of out-tournament whose matrices have completely determinate ranks, based on the structure of the representation. This class is the focus of the current chapter.

A *unicyclic graph* has only one cycle. Recall, a graph G is *representable in graph H* with family $\mathcal{F} = \{(H_x, p_x) | x \in V\}$ of pointed sets of vertices of connected subgraphs of H if $\Gamma(\mathcal{F}) = G$.

$$\begin{array}{ccc} \mathscr{G} & \longrightarrow & \mathscr{H} \\ & \searrow & \downarrow \\ & & \mathscr{D} \end{array} \tag{6.1}$$

The relationship between graph G, a representation of G in graph H, and D, the catch digraph of the representation, is shown in (6.1). By the definition of a representation of G in H, digraph D is also an orientation of G.

The following theorem forms the foundation upon which the current section builds.

THEOREM 6.4. [**36**] *If graph G is representable in a unicylic graph, then G is orientable as an out-tournament.*

However, note that the converse of this theorem is false. A counterexample is given in [**5**]. There are graphs that are orientable as an out-tournament but are not representable in a unicyclic graph. The authors of that paper put forward a conjecture that *any* graph orientable as an out-tournament is representable in a cactus, which is an interesting subject for further study. Here we focus more on the matrix ranks of these out-tournaments, which result from their structured nature.

We will first consider the simplest representation of this type. Once we determine the structure of our catch digraph, we look for an enumeration to allow computation of the adjacency matrix ranks.

3.1 Orientations of Graphs Representable in a Cycle.

Clearly, a cycle is, itself, unicyclic. Following the notation of Prisner in [**36**] and [**5**], let G be a graph representable in cycle C, with vertices $z_0, z_1, ..., z_{i-1}$ numbered sequentially in a clockwise direction.

Recall that a representation of graph G in H consists of a family of subsets H_x of vertices in H that induce a connected subgraph of H. The representation graph H need not be the same as graph G. However, every graph is representable in some subgraph of itself.

On our cycle, the connected subgraphs are paths on the cycle. If our representation graph H has trees rooted on the cycle, then each connected subgraph is either entirely in a tree, or it contains at least one vertex on the cycle.

Given representation $\mathcal{F} = \{H_x | x \in V\}$ in H, define the point p_x corresponding to H_x as the element farthest *left* (counterclockwise) on the cycle, if it contains any, and if there are none, then define p_x to be the vertex of H_x whose removal would separate the remaining vertices of H_x from the rest of H. This is the vertex of H_x that is closest to the cycle. Note that in both cases, the point is uniquely defined if we make the convention that when H_x contains the entire cycle C, we designate the point as z_0.

Refer to Figure 10, where a sample element H_x of family $\mathcal{F}$ is indicated. The set of white vertices $\{4, 3, 2, 1, 9\}$ induces a connected subgraph. The furthest left element is vertex 4, so this construction assigns vertex 4 as p_x, the point of set H_x. Since we would map vertex 4 in H to vertex 4 in D, we can write $H_4 = \{4, 3, 2, 1, 9\}$. Recall that the catch digraph uses H_x as the inset of the point of H_x. Thus, in the catch digraph D, $N^-(4) = \{3, 2, 1, 9\}$.

LEMMA 6.5. *If graph G is representable by family $\mathcal{F} = H_x$ in cycle C, then the subgraph induced by each H_x induces a path, $[z_j, z_{j+1}, ..., z_{j+k}]_H$.*

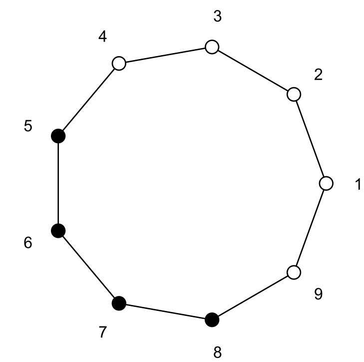

FIGURE 10. Vertex set $\{4, 3, 2, 1, 9\}$ induces a connected subgraph, so this set can be an element, H_x, of family $\mathcal{F}$.

PROOF. This can be seen by recalling that the subgraphs H_x must be connected. $\square$

LEMMA 6.6. *Let graph G be representable in cycle C_n with family $\mathcal{F} = H_x$. At most one of H_x contains all the vertices of C.*

PROOF. If H_1 and H_2 both contain the entire graph H, then the sets of vertices are the same, so H_1=H_2 is actually just one element of the family. $\square$

To facilitate the following theorem and proof, we will need to have the following definition. This definition will only be useful for $H = C_n$. A *predecessor* of vertex $x \in D$ is the vertex $y \in H_x$ such that xy is an edge in H. Recall that H_x includes p_x. It may be convenient to define $H'_x = H_x - p_x$, which is pointed set H_x but without the point. In the catch digraph D, the sets H_x are closed insets of the point, x. That is $H_x = N^-[x]_D$, the closed inset of x in digraph D. Then $H'_x = N^-(x)_D$ is the standard inset of x in D.

THEOREM 6.7. *If G is representable in cycle C_n, such that $1 < |H_x| < n$ for each x, and family $\mathcal{F}$ is formed by assigning the leftmost vertex of each H_x as its point, then $A = M(\Omega^-(\mathcal{F}))$ has full Boolean rank. Furthermore, if C_n is of a minimum size to represent G, then $r_B(A) = |V(C_n)| = |V(D)|$.*

Proof. Observe that since G is representable in C_n, which is clearly unicyclic, if $H_x \cap H_y \neq \varnothing$, then $p_x \in H_y$ or $p_y \in H_x$. By assumption, $H'_x \neq \varnothing$. Take any $y \in H'_x$, then y is a predecessor of p_x in H. Each of these predecessor edges yp_x in H correspond to arcs $y \longrightarrow x$ in D. Define P to be a set of predecessors. Note that this set is well defined for $H = C_n$. Suppose that y is a predecessor of both x and z. Now, $y \in H_x \cap H_z$ implies (WLOG) that $p_x \in H_z$. Thus, in H there is a p_xp_z-path, not going through y. Also, yp_z is in H, which forms a cycle that is completely contained in H_z. But, since we assumed that $1 < |H_x| < n$ for each x, this cannot happen. Let $S = \{(y, x)|y \in P, \{yp_x\} \in E(H), y \in H_x\}$. We claim that the set S of representative predecessor arcs form a set of isolated 1s in the matrix $A = M(D)$. Since there is one arc (y, x) in S for each vertex x in D, the 1s in A are independent, recalling that the set P is well defined. Next, we need to verify that the set of 1s in question is actually an isolated set of 1s. Suppose that $t_x \in H_x$ and $t_y \in H_y$ are elements of S. Now, by way of contradiction assume that (X, Y) is a biclique of D, where $X = \{t_x, t_y\}$ and $Y = \{p_x, p_y\}$. Observe that $t_x \in H_y$, so there is a path from t_x to p_y and a path from t_x to p_x. As noted above $H_x \cap H_y \neq \varnothing$, then $p_x \in H_y$ or $p_y \in H_x$. Because of the two paths above, if $p_x \in H_y$ then the p_xp_y-path completes a cycle, which is impossible given our assumption that $1 < |H_x| < n$ for each x. So, the arcs of D corresponding to the elements of S form a set of isolated 1s in the adjacency matrix A of out-tournament D. Since $|S| = n \leq r_B(A) \leq n$, matrix A has full Boolean rank.

For the second claim of the theorem, note that a representation of a graph is not unique, and that a representation with a minimum number of subgraphs has n elements, the number of vertices of G as well as that for D, which is an orientation of G under the construction given. □

The class of out-tournaments referred to in Theorem 6.7 is strong and has cyclic enumeration by construction. It has arc set $A(D) = \{i \longrightarrow j|j = i + \ell$ for $\ell = 1, 2, ...m_i\}$ where m_i varies for each i, but $m_i < n - 1$ because of the condition $1 < |H_x| < n$, which prevents any source vertices and sink vertices.

Lemma 6.8. *If there is an* $H_x = \{x\}$, *then* $r_B(A) < n$.

Proof. This can be seen by considering the fact that if H_x contains only x, then $d^-(x)_D = 0$ – vertex x is a source vertex in D. Thus, column x in matrix A is all 0s. □

Lemma 6.9. *If there is an* $H_x = V(D)$, *then* $r_B(A) < n$.

Proof. Observe that $H_x = V(D)$ implies that for every vertex y in D, $y \longrightarrow x$, which is to say that $d^+(x) = 0$ (x is a sink) and row x of matrix A is all 0s. □

3.2 Unicyclic Representations with Trees

Next, we allow trees 'growing' out of the cycle. That is, there are tree subgraphs having cycle vertices as a cut vertex. See Figure 9 for an orientation of a unicyclic graph with two trees 'growing' out of the cycle. In other words, there are two maximal tree subgraphs of $H = UG(D)$ with root vertices on the cycle. A *cut vertex* or *articulation vertex* of a connected graph is a vertex whose removal separates the graph into two or more components. As we shall see, constructing family $\mathcal{F}$ in the way we have defined leads to similarly predictable adjacency matrix ranks, as it did for the representation graphs that are cycles.

Proposition 6.10. *Let* H *be a unicyclic representation of graph* G *of minimum size, which has at least one tree growing out of the cycle. Let* L *be the set of all leaves of* H, *with* $|L| = \ell$. *Also, define* $K = \{x \in H \mid H'_x = \varnothing$ *and* x *is not a leaf*$\}$, *with* $|K| = k$. *If* A *is the adjacency matrix of the catch digraph of* $\mathcal{F}$, *then* $r_B(A) \geq n - \ell - k$.

Proof. Similarly to the previous use of predecessors, the only vertices for which we cannot produce a predecessor are those that have none – namely, those vertices x that have $H_x = \{x\}$. These are exactly the vertices of the disjoint union, $L \,\dot\cup\, K$. However, for the remaining vertices of H, we can choose a predecessor as shown in

Section 3.1 of this chapter. Since the 1s of A corresponding to a maximum set of predecessors S form a set of isolated 1s of A, we have $n - \ell - k \leq r_B(A)$, and this completes the proof. $\square$

The result of Proposition 6.10 gives a lower bound for the Boolean rank of the adjacency matrix of the out-tournament. The upper bound that results from the structure of the digraph allows us to give the Boolean rank of the matrix exactly.

THEOREM 6.11. *Let H be a unicyclic representation of graph G of minimum size, which has at least one tree growing out of the cycle. Let L be the set of all leaves of H, with $|L| = \ell$. Also, define $K = \{x \in H | H'_x = \varnothing$ and x is not a leaf$\}$, with $|K| = k$. If A is the adjacency matrix of the catch digraph of $\mathcal{F}$, then $r_B(A) = n - \ell - k$.*

PROOF. By Proposition 6.10, we know that $n - \ell - k \leq r_B(A)$. Observe that for each of the vertices in $L \dot\cup K$, there is a column of 0s in the corresponding column of A. Thus, $r_B(A) \leq n - \ell - k$, and we have $r_B(A) = n - \ell - k$. $\square$

COROLLARY 6.12. *$r_B(A) = n$ if and only if $0 < |H'_x| < n - 1$ for each x.*

PROOF. ($\Leftarrow$) Assume that $0 < |H'_x| < n - 1$ for each x. This guarantees that H has no trees growing out of the cycle, since any tree has a leaf and $|H'_x| = 0$ for any leaf x. It is also necessary to assume that $n - 1 > |H'_x|$ for each vertex, to prevent rows of 0s in A. Further, since every vertex has a predecessor, by Theorem 6.11, we have $r_B(A) = n$.

($\Rightarrow$) Full Boolean rank implies that A has no rows or columns of 0s. Thus, $0 < |H'_x| < n - 1$ for each x. $\square$

3.3 Other Matrix Ranks for Unicyclic Representations

We have found the Boolean rank for adjacency matrices of out-tournament catch digraphs of unicyclic graphs. We now explore the other matrix ranks.

Theorem 6.13. *Let D be the catch digraph of unicyclic graph H, which itself is a representation of graph G. Let $A = M(D)$ Then $r_t(A) = n - \ell - k$.*

Proof. Recall that $r_B(A) = n - \ell - k \leq r_t(A)$. Now referring to the sets L and K, which are disjoint, we have ℓ columns of 0s and k other rows of 0s, so $r_t(A) \leq n - \ell - k$. Therefore $r_t(A) = r_B(A) = n - \ell - k$. □

Corollary 6.14. *Let D be the catch digraph of unicyclic H, which itself is a representation of G. Let $A = M(D)$ Then $r_{Z^+}(A) = n - \ell - k$*

Proof. This follows immediately from (2.1) and Theorem 6.13. □

3.4 Relation of Rank Over $\mathbb{R}$ with the Other Matrix Ranks

Now we will consider the remaining matrix rank, the real rank. Although this is the matrix rank we are (usually) most acquainted with, recall that there is not a standard relationship between the real rank and Boolean rank of a $\{0,1\}$-matrix. However, the following theorem gives a bit of insight to the relationship for this class of matrices.

For real rank of the adjacency matrix of the catch digraph of unicyclic H and family of subgraphs $\mathcal{F}$ constructed as above, we will need a careful enumeration of the vertices. Set S was made in such a way that each vertex v that is not a source was represented by exactly one arc $u \longrightarrow v$ in S, where u, v were adjacent in H. We claim that it is possible to number the vertices so that $x \longrightarrow v$ implies that $x \geq u$, where u is the vertex with $u \longrightarrow v$ in set S. Note that this already occurs on the cycle. Next, number all vertices at a distance of 1 away from the cycle, in any order. Then all vertices at a distance of 2, and so on, until all vertices are numbered.

Theorem 6.15. *For unicyclic graph H and family of subgraphs $\mathcal{F}$ constructed as above, with $D = \Omega^-(H, \mathcal{F})$, define R as the set of all source vertices in D. Then $A = M(D)$ has $r(A) = |S| = n - |R|$.*

Proof. To see this, we need only consider the 1s representing the arcs contained in set S. As we observed, this set S represents a set of isolated arcs of maximum size. There is an element of S represented in the inset of every nonsource vertex. Furthermore, in A, each nonzero entry $a_{ij} = 1$ for $i \longrightarrow j$ in S is the highest nonzero entry in its column. So, the set of 1s in A representing the arcs of S form leading 1s in each of their respective columns, where no two are in the same row, and there is one in every column not equal to $\vec{0}$. Therefore, $r(A) = |S| = n - r$. □

Corollary 6.16. *Let D be the catch digraph of unicyclic H, which itself is a representation of G. Let $A = M(D)$. Then $r(A) = n - \ell - k$.*

Proof. Refer to Proposition 6.10 and Theorem 6.15. We observe that $|R| = \ell + k = |L \dot{\cup} K|$. □

4 Structure and Connectedness of D

The construction used above is very specific. For this class of out-tournament, we can say a great deal about the structure and matrix ranks of an out-tournament oriented unicyclic graph.

Theorem 6.17. *Let D be an out-tournament oriented unicyclic graph. The following are equivalent:*

(1) *D is strong.*

(2) *A has full matrix ranks.*

(3) *$H = UG(D)$ is a cycle.*

(4) *D is a directed cycle.*

Proof. $(1 \Rightarrow 3)$ If D is strong, then it has no sources, so H has no trees, since every tree has at least one source. $(3 \Rightarrow 4)$ As we have seen previously, the only out-tournament orientation of a cycle is a directed cycle. $(4 \Rightarrow 2)$ Recall from Remark 5.8 that the matrix of a directed cycle is a permutation of I_n. $(2 \Rightarrow 1)$ If A has full matrix ranks, then it has no rows of 0s, meaning that D has no sources, and D is a directed cycle, making D strongly connected. □

4.1 Out-Tournament Orientation of a Unicyclic Graph

From Theorem 6.17, we can see a great deal of structure in a strong digraph of this type. This section asks: what are the strong components of an out-tournament orientation of a unicyclic graph? We are not able to answer that question for catch digraphs of unicyclic representations in general, but we can describe the class of oriented unicyclic graphs in terms of connectedness.

LEMMA 6.18. *Let $G = UG(D)$ be a unicyclic graph, and let D be an out-tournament orientation of G. If G has a cycle $v_1, v_2, ..., v_k, v_1$ then the set $\{v_1, v_2, ..., v_k\}$ induces directed cycle $\vec{C}_k$ in D.*

PROOF. Since G is unicyclic, the set $\{v_1, v_2, ..., v_k\}$ induces C_k in G, which is to say that there are no chords between non-consecutive vertices of the cycle. Otherwise, that would form another cycle. Consider three consecutive vertices v_{i-1}, v_i, v_{i+1} on the cycle in D and the possible orientations of the edges between them in out-tournament D. Suppose that $v_i \longrightarrow v_{i-1}$ and $v_i \longrightarrow v_{i+1}$ in D. But, D is an out-tournament, and this assumption leads to the conclusion that v_{i-1}, v_{i+1} are adjacent in D. However, this cannot be the case, since $UG(D) = G$ had no edges between non-consecutive vertices on this cycle. Thus, no vertex on the cycle beats more than one other vertex on the cycle. Since there are k vertices and k edges to orient, each vertex on the cycle must dominate exactly one other vertex on the cycle. Without loss of generality, we can assume that $(v_1, v_2, ..., v_k, v_1)$ is the corresponding directed cycle in D, or we may enumerate the vertices of G so that

$$1 \longrightarrow 2 \longrightarrow ... \longrightarrow (k-1) \longrightarrow k \longrightarrow 1.$$

□

Proposition 6.19. *Let D be an out-tournament orientation of unicylic graph H. Then any edges not on the central cycle are oriented toward the cycle.*

Proof. Suppose that vertex c on the cycle is the root vertex of a tree. As seen in Lemma 6.18, $c \longrightarrow c+1$ in D, where vertex $c+1$ is the next vertex on the cycle. Let x be in the tree rooted at c, and let x be adjacent to c. Edge $\{c, x\}$ in H must be oriented toward c, otherwise $\{c, c+1, x\}$ induce a cycle in H that is not the central cycle, contradicting our assumption that H is unicyclic. If there are more vertices adjacent to x in the tree, then the same argument applies, since x already has a non-empty outset in D and we cannot create any new cycle in $UG(D) = H$. □

Proposition 6.20. *Let H be a unicyclic graph with at least one tree, and D its out-tournament orientation. Each vertex of H not on the cycle is a one-vertex strong component.*

Proof. To show this, observe that any vertex in a tree is not reachable from any vertex on the directed cycle in D. Refer also to Figure 9. □

For an orientation of a unicyclic graph, if there are any trees growing out of the cycle, the central cycle is one strong component and each vertex off the central cycle is its own strong component.

More generally, however, if D is a catch digraph of a family of subsets of vertices in unicyclic graph H, there are many more possibilities for both form of D and relations of its matrix ranks. The following proposition essentially rephrases Theorem 6.7 in terms of a more general construction.

Proposition 6.21. *Let D be the catch digraph of unicyclic representation H with some family of connected subgraphs. Then the induced subdigraph on C, $[C]_D$, need not be strong. Let the points of the sets H_x be assigned as the leftmost cycle vertex in H_x, and if there is no cycle vertex in H_x, then the point is the vertex closest to the cycle. Then $1 < |H_z| < n$ for each z on the cycle in H implies that the image of C in D is strong.*

PROOF. Follows from Theorem 6.7. □

Proposition 6.21 tells us that as long as there are no source or sink vertices on C_k, then the induced digraph on C_k must be strong in the catch digraph D.

5 Going Forward From Here

The result of Proposition 6.20 is that D may have many single vertex strong components. Because of that, the results from Chapter 2 do not apply. But further exploration could prove fruitful in finding more general classes of out-tournaments that are catch digraphs of unicyclic graphs. Relaxing the restrictions on the family of connected subgraphs is the natural direction in which to take this line of investigation. Generalization may begin with allowing in-degree $d^-(v)$ to be restricted to two or three. In this chapter, out-degree was limited to two, in effect. That restriction wasn't stated explicitly, since it was implied by the construction itself. However, despite the fact that the class of catch digraphs from unicyclic representations is also very large, a partial characterization may be possible in terms of matrix ranks.

CHAPTER 7
Out-Tournament Orientation of a Cactus

1 Introduction

Chapter 6 started with a representation graph and carefully assigned pointed sets such that the adjacency matrix of the resulting catch digraph had some very nice properties, and its ranks were then characterized in terms of the representation graph properties.

Here, we will start at a different point and consider different objects, so the proof techniques differ from previous methods. Note that we will bypass the representation family of subsets entirely, since it's unnecessary in this case. Thus, starting with an underlying graph of the appropriate form, we give the graph an out-tournament orientation and consider the form and ranks of its resulting adjacency matrix.

2 An Oriented Cactus

In Chapter 6, we considered a class of digraphs based on its representation in a unicyclic graph. Now, we take a different approach and look at a known class of underlying graph that lends itself well to out-tournament orientation. A *cactus* is a graph in which no two cycles share an edge.

From Theorem 6.4, we know that any graph representable in a unicyclic graph is orientable as an out-tournament. The class of cactus graphs contains the class of unicyclic graphs.

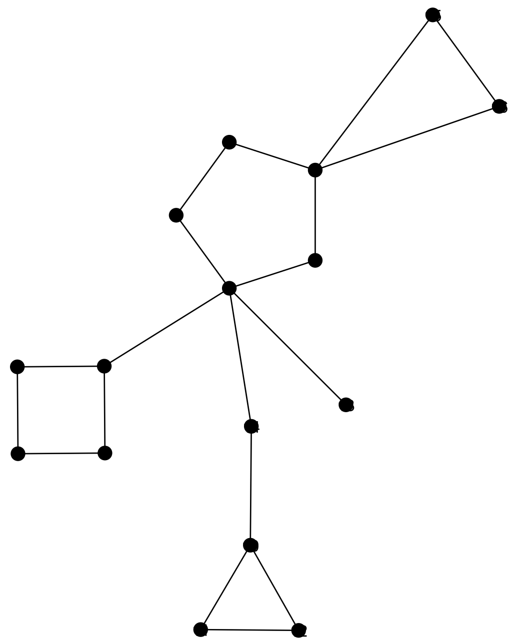

Figure 11. A cactus that is not orientable as an out-tournament.

The basis for this approach is the conjecture in [**5**] that every graph orientable as an out-tournament is representable in a cactus. The converse, however, is false.

Remark 7.1. [**5**] *Any cactus with more than one cycle of length* $k \geq 4$ *is not orientable as an out-tournament.*

Note that every cactus is representable in a subgraph of itself, which must, therefore, also be a cactus. The class of representations of graphs orientable as out-tournaments is only a subset of the class of all cactus graphs. For example, the graph in Figure 11 is a cactus, but it is not orientable as an out-tournament. However, the graph in Figure 12 is orientable as an out-tournament, as we will see. Likewise, any graph that may be represented in that graph is also orientable as an out-tournament. This accounts for a rather large class of graphs. To see that, one must only consider all the possible families $\mathcal{F}$ of connected subgraphs of the cactus in Figure 12.

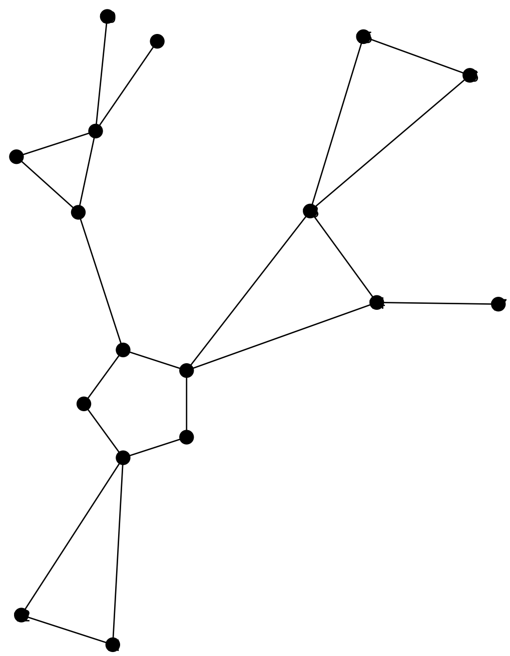

FIGURE 12. A cactus graph orientable as an out-tournament

Also observe that the cactus representation $(H, \mathcal{F})$ of any graph G resembles the unicyclic graph of Chapter 6, which had only one cycle. Our cactus may have only one cycle C_k with $k \geq 4$. Any other cycles in the cactus are triangles, in the branches off of any C_k, $k \geq 4$, there may be.

If any cactus is the underlying graph of an out-tournament, it must have no more than one cycle of length four or more, and it is representable in a cactus, since it is representable in some subgraph of itself. We consider the form of an out-tournament oriented cactus, then consider the ranks of its adjacency matrix.

2.1 Structure of an Out-Tournament Orientation of a Cactus

Let D be an out-tournament with $UG(D) = G$ and let G be a cactus. Much like the class of oriented unicyclic graphs in Chapter 6, the form of out-tournament D is almost completely determined by the assumption of the form of its underlying graph. The following results characterize the form that D must take.

COROLLARY 7.2. *Let $G = UG(D)$ be a cactus, and let D be an out-tournament. If G has a cycle $v_1, v_2, ..., v_k, v_1$ with $k \geq 4$, then the set $\{v_1, v_2, ..., v_k\}$ induces directed cycle $\vec{C}_k$ in D.*

PROOF. This follows from Lemma 6.18. Although that lemma was dealing with the out-tournament orientation of a cycle in a unicyclic graph, the result applies to all out-tournament oriented cycles. The only out-tournament orientation of C_k is $\vec{C}_k$. $\square$

Now that the structure of what we will refer to as 'the big cycle,' and alternately 'the main cycle,' has been established, we consider the form of the remaining structures in D. The other cycles may only be triangles, and none may share an edge, by definition. Thus, the remainder of G, outside of the big cycle, if one exists, resembles the trees growing out of the central cycle in the unicyclic representations of Section 4.1. To facilitate the following discussion, define distance $d'(v)$ to be the distance in G of vertex v to the nearest vertex on C_k. We can define $d'(v) = 0$ for any vertex v on the cycle, and refer to the distance classes of vertices, which will prove useful in upcoming results.

First, consider all those vertices that have $d'(v) = 1$, that is, those adjacent to a vertex of C_k.

LEMMA 7.3. *Let $G = UG(D)$ be a cactus with cycle C_k, $k \geq 4$, and D an out-tournament. Let v not on C_k be adjacent to vertex j on C_k in H. Then $v \longrightarrow j$ in D.*

PROOF. Suppose that $j \longrightarrow v$ in D. Then $(j+1) \longrightarrow v$ in D and $j, (j+1)$ and v form a triangle in G sharing edge $\{j, (j+1)\}$ with C_k, which is impossible. $\square$

Now, observe the following about cactus G.

LEMMA 7.4. *If u, v, w lie in the same triangle in G, then at least two of these vertices are not on C_k. Those two vertices must have the same distance d' from C_k. Furthermore, the shortest path to the cycle from each of these two must pass through the third vertex.*

PROOF. The first part of the lemma follows directly from the assumption that G is a cactus, as no two cycles share an edge. Thus, at least two of the vertices of the triangle are not on C_k. For the second part of the lemma, consider that one vertex of the triangle is a cut vertex, with its removal separating G into a component with C_k, and a second component consisting of one edge of the triangle with anything that is adjacent to that edge. If not, then G is not a cactus. Thus, the shortest path to C_k from each of the two other vertices of the triangle is through that cut vertex and the two vertices are the same distance from C_k. $\square$

COROLLARY 7.5. *In any triangle of cactus G, two vertices of the triangle have $d'(v) = d'(u) = \ell + 1$ where the third vertex has $d'(w) = \ell$.*

We have defined distance $d'(v)$ from C_k. It is also critical to refer to edges $\{x, y\}$ of cactus H with $d'(x) = d'(y)$ as *level edges* and their orientations as *level arcs*. Similarly, if $d'(x) \neq d'(y)$ then $\{x, y\}$ is a *trans-level edge* and its orientation is a *trans-level arc*. The set of all vertices with a particular distance from C_k is a *level*. Now, by the same argument as Lemma 7.4, we can state the following.

LEMMA 7.6. *If u, v are adjacent in G with $d'(u) = d'(v) - 1$ then $v \longrightarrow u$ in D.*

PROOF. Assume that $\{u, v\} \in E(G)$, and $d'(u) = d'(v) + 1$. By Lemma 7.3 , we know that if $d'(u) = 1$, then the conclusion holds. Suppose that the conclusion holds for all levels up to level $\ell \geq 1$. Then assume $d'(u) = \ell + 1$, $d'(v) = \ell$, with edge $\{u, v\}$ in G. Suppose $v \longrightarrow u$ in D. But, there must exist w adjacent to v in G with $d'(w) = \ell - 1$ and by the induction hypothesis, $v \longrightarrow w$ in D. Then u, w are adjacent in G since $\{u, w\} \subseteq N^+(v)$ and D is an out-tournament. Vertex w is adjacent to another vertex, x, either on the same level or one level lower. If $w \longrightarrow u$, then $\{u, x\} \in E(G)$, creating a cycle on $\{u, w, x\}$ that shares an edge with the triangle on $\{u, v, w\}$. On the other hand, if $u \longrightarrow w$, then $d'(u) = d'(v)$ contradicting our assumption. Therefore, $u \longrightarrow v$ and the lemma holds for all levels ℓ. □

To paraphrase: except for triangles, any trans-level edges in G are all oriented toward $\vec{C}_k$ in D. That is to say that other than the orientation of the level edges in G, the form of the out-tournament orientation of G is completely determined by our assumption that D has a cactus as an underlying graph. Furthermore, the orientations of the level edges do not matter, until we are assigning an enumeration.

Remember that the motivation to look at this class of digraph is to analyze its adjacency matrix ranks. Now that we understand the structure of the digraph, we are well equipped to do just that.

2.2 Adjacency Matrix Ranks of Out-tournament Oriented Cacti

The underlying graph $UG(D) = G$ resembles the unicyclic representations of Section 4.1: there is potentially only one big cycle and something like trees growing out of the cycle. The difference here is that triangles may exist in the trees. If each of those triangles were condensed to a single vertex, then the branches of G off the cycle would indeed be trees. If we call a cactus having no cycle larger than C_3 a *triangle cactus*, then we can say that D consists of a directed cycle, $\vec{C}_k$, with a number of directed triangle cacti, each rooted at a cycle vertex. Recall that an in-branching is an oriented spanning tree subdigraph with exactly one vertex having out-degree zero.

Remark 7.7. *Let U be the induced subdigraph on a triangle cactus branch of an out-tournament oriented cactus. If each triangle is condensed to a single point then U becomes an in-branching.*

Each triangle cactus in G may have leaves, or it may have triangles at the end of branches. In this case, we will refer to such a triangle as a *terminal triangle*.

In the same way that we have begun previous matrix rank investigations, consider the source vertices in D. Each leaf vertex in G represents a source in D, by Lemma 7.3. Let $\{u, v, w\}$ induce a terminal triangle in G, with $\{u, v\}$ being its same level edge. Now, either $u \longrightarrow v$ or $v \longrightarrow u$ in D, but either way, the terminal triangle contains exactly one source. Let the set of all leaves have size ℓ, and the set of all terminal triangles have size t.

Lemma 7.8. *Let D be an out-tournament orientation of cactus G. Let T be the set of all terminal triangles in G, L be the set of all leaves in G and $A = M(D)$. Then $r_t(A) \leq |V(D)| - |L| - |T|$.*

Proof. In each terminal triangle of G there is a vertex representing a source vertex in D, and each leaf of G is also a source in D. For each of these vertices, the column representing its inset in A is $\vec{0}$. Thus, term rank of A cannot exceed $n - \ell - t$. □

Next we make an observation important for the term rank and Boolean rank, the ranks that are analogous to line cover number and biclique cover number of the corresponding digraph, respectively. A biclique $X \longrightarrow Y$ is *substantial* if $|X| > 1$ and $|Y| > 1$; in other words, a substantial biclique is a non-claw biclique.

Lemma 7.9. *Let D be an out-tournament oriented cactus. Then D has no substantial bicliques.*

Proof. For each vertex v, $d^+(v) \leq 2$. And, for each u, v, $|N^+(u) \cap N^+(v)| \leq 1$. If the intersection of any two outsets were 2 or greater, then $UG(D) = G$ is not a cactus, contradicting our assumption. □

Following immediately from Lemma 7.9, we have a corollary relating the term and Boolean ranks.

COROLLARY 7.10. *If D is an out-tournament oriented cactus, then any minimum biclique cover of the arcs of D corresponds to a line cover of the 1s of $A = M(D)$. Thus $r_t(A) = r_B(A)$.*

PROOF. Since there are no substantial bicliques in D, then the arcs of any biclique in a minimum cover represent the non-zero entries in a single row or single column of A. Therefore $r_t(A) \leq r_B(A)$. By the matrix rank inequalities given in (2.1), $r_B(A) \leq r_t(A)$. Therefore, $r_t(A) = r_B(A)$. $\square$

So far, we have an upper bound for term rank, and we know that term rank and Boolean rank are the same. The next proposition addresses the Boolean rank problem using independent/isolated sets of 1s in A, thus giving a lower bound.

In order to accomplish this, we will build up a set sequentially. At the same time, we will give the vertices an enumeration that will lend itself to the clarity of the proof of a theorem below.

ALGORITHM 7.11. ***Stage 0*** *Assuming that the vertices are given a cyclic enumeration for the central cycle $\vec{C}_k$, put each arc $j+1 \longrightarrow j$ $(j = 2, 3, ..., k)$ and arc $1 \longrightarrow k$ into set S, which is the opposite of the usual convention, but the reason will become clear in the next section.*

Stage $\ell \geq 1$

(1) *If there is a level $\ell - 1$ vertex y reachable from level ℓ in D without an arc $x \longrightarrow y$ in S so far, then add one such arc for each such vertex y.*
(2) *Add every arc $u \longrightarrow v$ where $d'(u) = d'(v) = \ell$.*

As each arc $u \longrightarrow v$ is put into S, if it becomes the i^{th} element of S, then v is given the label i.

The steps are repeated until the last stage z is completed, where z is the level of the tallest triangle cactus growing out of the cycle.

Note that at the finish of the construction of set S, not every vertex will have been assigned an index. The remaining vertices are numbered in any satisfactory manner. For our purposes, it is only important that they are numbered last. As we will see below, it is exactly the set of sources of D that are not indexed until the end. So, if there are n vertices in all, and s sources, the sources will have labels $n-s+1, n-s+2, ..., n$.

PROPOSITION 7.12. *The set S represents a maximum set of independent 1s in $A = M(D)$, and $|S| = |\{v \in V | N^-(v) \neq \varnothing\}|$. Then $r_t(A) = |S'|$.*

PROOF. The set S is independent if no two arcs have the same starting point, and no two arcs have the same ending point. Because of the way we have chosen arcs and the order in which we have chosen them, neither of these things happen.

For the size of S, we will use a function $f : S \longrightarrow V$ given by $f(x, y) \longrightarrow y$. Function f maps each arc in S to its ending point. Since we noted above that no two arcs in S have the same starting point, f is well-defined. Likewise, f is 1-1 since no two arcs in S have the same ending point.

For convenience, define $S' = \{v \in V | N^-(v) \neq \varnothing\}$, which is the set of all vertices of D except for the sources. Our goal is to show that $f(S) = S'$.

Take $y \in f(S)$. There is $x \in V$ such that $(x, y) \in S$. Hence, y is not a source, and we have that $f(S) \subseteq S'$. Choose any $y \in S'$. By definition, $N^-(y) \neq \varnothing$. Suppose that $d^+(y) = \ell$. At stage ℓ in the construction of S, if y were dominated by another level ℓ vertex, x, then that arc would have been added to set S at this stage. Suppose that there were no such x on level ℓ. By assumption, y is dominated by some vertex and by Lemma 7.3, $x \longrightarrow y$ implies that $0 \leq d'(x) - d'(y) \leq 1$. So there must be an x on level $\ell + 1$ with $x \longrightarrow y$, which would be added to S at stage $\ell + 1$. So, $S' \subseteq f(S)$, which gives the desired result.

The only thing yet to demonstrate is that S is a maximum set of independent arcs in D. We know that $r_t(A) = |S|$ for any set S of independent 1s of maximum size. Earlier, we showed that $r_t(A) \leq |S'|$ with Lemma 7.8 , and since the size of any independent set of 1s forms a lower bound for the term rank, we have $|S'| = |S| \leq r_t(A) \leq |S'|$. Therefore, we know that S is of a maximum size, and $r_t(A) = |S'|$. □

Therefore, we can make the following statement immediately, combining the results of Corollary 7.10 and Proposition 7.12.

THEOREM 7.13. *Let D be an out-tournament oriented cactus, with $A = M(D)$ and r source vertices. Then $r_B(A) = r_{Z^+}(A) = r_t(A) = n - r$.*

PROOF. Consider the set S, represented in A by a maximum set of independent 1s. By Corollary 7.10, there are no substantial bicliques in D. Therefore, any independent set of 1s is automatically an isolated set of 1s. Now $s \leq r_B(A) \leq s$, thus $r_B(A) = s = n - r$. By (2.1), $r_B(A) \leq_{Z^+} (A) \leq r_t(A)$ for any $\{0, 1\}$-matrix, therefore the nonnegative integer rank of A is also $n - r$. □

As we have observed in the past, real rank is a bit different from the others, as the graph property corresponding to a singular matrix is frequently less obvious than that for other ranks. Thus, to find this common rank for a given digraph class frequently requires different techniques than the other ranks.

However, as often is the case, a strategic enumeration of the vertices can place the adjacency matrix in a form that makes identification of its real rank a very simple matter. In our case, all the hard labor has already been done. At this point, just sit back and let the enumeration do all the work!

2.3 Real Rank of Adjacency Matrix of Out-tournament Oriented Cacti

Given the enumeration in Algorithm 7.11, the form of A can be visualized by first observing the following:

Lemma 7.14. *Let D be any out-tournament orientation of a cactus, with vertex enumeration given by Algorithm 7.11. If $i \longrightarrow j$ in D, then $i > j$, with the exception of $1 \longrightarrow k$ on $\vec{C}_k$, the central cycle.*

Proof. Clearly the result holds for every j on $\vec{C}_k$ except for $1 \longrightarrow k$. Take any j with $N^-(j) \neq \varnothing$. As we noted in Lemma 7.3, for any out-tournament oriented cactus, all trans-level arcs are directed toward the central cycle, $\vec{C}_k$. As the set S was constructed in Algorithm 7.11, every level ℓ vertex has a lower label than any level $\ell + 1$ vertex. Suppose that $i \longrightarrow j$ and $i \leq j$. Then $d'(i) = d'(j) = \ell$. If i was labeled first, then there was an arc $x \longrightarrow i$ added to S before $i \longrightarrow j$. However, (x, i) cannot be a level arc, because G is a cactus, and that would cause two cycles to share an edge. But x cannot be a level $\ell + 1$ vertex since in that case, i would not have been labeled until stage $\ell + 1$, after j. Thus, the conclusion holds for all arcs in D. □

We will define set L_j as the set of all level j vertices with nonempty insets, and R the set of all source vertices. Let x be any vertex with $d'(x) = z$ a maximum in H. Then there are z stages in the construction of set S, and the vertices V of D are partitioned into $L_0, L_1, ..., L_z, R$. Let $|L_j| = \ell_j$ for each $0 \leq j \leq z$, and define R to be the set of source vertices in D, with $|R| = r$.

Define $A_j = M[L_j]_D$, the adjacency matrix of the induced subdigraph on vertices L_j in digraph D. Then A has the block form shown below. Also, $A_{(i+1)i}$ contains all of the arcs from a level $i+1$ vertex to a level i vertex. All the blocks below the first subdiagonal and those above the main diagonal are blocks of all 0s. The bottom right corner block, labeled A_R, represents the matrix of the induced subdigraph on the set of all sources, which were numbered last. This is, of course, a subdigraph with no arcs. So A_R is an $r \times r$ block of all 0s.

$$A = \begin{bmatrix} A_0 & [0] & \dots & \dots & [0] \\ A_{1,0} & A_1 & \ddots & & \vdots \\ [0] & A_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_z & [0] \\ [0] & \dots & [0] & A_{r,z} & A_R \end{bmatrix}$$

Each block A_j contains relatively few nonzero entries. Each 1 in submatrix A_j represents a level arc in level j. Each of these arcs are represented in the set S, constructed in Algorithm 7.11.

THEOREM 7.15. *Let $A = M(D)$ be the adjacency matrix of an out-tournament oriented cactus D, with r sources. Then $r(A) = n - r$.*

PROOF. Consider the 1s in matrix A that represent the arcs in set S. As we noted earlier, no two are in the same row, and no two are in the same column. By our enumeration, each of these 1s is also the highest nonzero entry in its column, which we can think of as leading 1s in their respective columns. Then the column rank is greater than or equal to $n - r$, the number of linearly independent columns with leading 1s, and less than or equal to $n - r$, since A has r columns of 0s on the far right. Therefore $r(A) = n - r$. $\square$

THEOREM 7.16. *Let $A = M(D)$ be the adjacency matrix of an out-tournament oriented cactus D, with r sources. Then*

$$r(A) = r_B(A) = r_{Z^+}(A) = r_t(A) = n - r.$$

PROOF. Follows from Theorems 7.13 and 7.15. $\square$

3 Conclusion and Future Work

The work in this chapter goes together with that in Chapter 6. The next investigation in both unicyclic and in cactus representations would be some determinate way of assigning the family of pointed sets such that the catch digraph produces an out-tournament with predictable adjacency matrix ranks. A second possible direction is the following.

CONJECTURE 1. [**5**] *Every graph orientable as an out-tournament has a cactus representation.*

This remains to be seen, while this chapter produced particular out-tournament cactus orientations, it was not designed to verify the conjecture. Note that Prisner [**36**] in his dissertation did extensive work with representations, and together with Bang-Jensen and Jing Huang [**5**], produced results but were unable to verify the conjecture. Another direction might be to consider, in light of the current chapter, families of subgraphs of a cactus that have strong out-tournaments as their catch digraphs. Perhaps some similar adjacency matrix rank properties will hold such as those for oriented cactus graphs. For example, several classes given in Bang-Jensen, Huang and Prisner [**5**] seem promising. The immediate results of Theorem 7.17 were given in the same paper.

THEOREM 7.17. [**5**] *Any graph representable in a cactus with no more than one cycle of length 4 or greater is orientable as an out-tournament.*

Each of the following is orientable as an out-tournament:

(1) Chordal graphs;

(2) Circular arc graphs; and

(3) Graphs with exactly one induced cycle length of 4 or greater.

Thus, these classes all merit a closer look because they may well lend themselves to an analysis of the matrix ranks of out-tournament catch digraphs from their representations.

CHAPTER 8
A Strong Catch Digraph of a Cactus Representation

1 Introduction

Following in a progression through the last two chapters, we have considered underlying graphs of increasing levels of complexity to give classes of out-tournaments having adjacency matrices with full ranks. In each of those chapters, because of our constructions, there has not been the possibility of a strongly connected out-tournament. So this chapter attempts to find a class of out-tournaments as simple as possible that can be shown to have full, equal matrix ranks, and is strongly connected. Recall that:

Proposition 8.1. [**5**] *An out-tournament is strong if and only if it has a Hamiltonian cycle.*

Since our primary goals here are strong connectedness and equal matrix ranks, we must necessarily be looking for cases in which the four matrix ranks are full. If our out-tournament is strong, the term rank of its matrix is automatically full, due to the fact that the matrix must have a full set of independent 1s, representing the Hamiltonian cycle that is guaranteed to exist. Therefore, equal ranks means equal and full ranks in this case.

As it is usually desirable to start with simpler cases and generalize from there, our goals in this chapter will be to:

(1) add the fewest arcs necessary to an oriented cactus to make a strong out-tournament, and
(2) avoid creating any substantial bicliques.

If we can achieve items 1 and 2, assigning the vertices a cyclic enumeration based on a Hamiltonian cycle, then Boolean rank of the matrix will automatically be full. As we shall see, the cyclic enumeration reveals a well-organized pattern in the adjacency matrix, which will allow easy evaluation of the real rank of the matrix.

2 Cactus-Based Strong Out-Tournament

Let H be a cactus with at most one cycle of length 4 or greater. We will assume that the cactus has this central cycle of length at least 4. If not, then one of the triangles serves as the central cycle. Recall that a triangle cactus is a cactus with no cycle larger than C_3. If there are no triangles, then the cactus is a tree, and we will need a different construction in that case, which we will not address at this time. Suppose that C_k is the big cycle, and $T_1, T_2, ..., T_k$ are the triangle cacti whose roots are the cycle vertices. Cactus T_j has cycle vertex c_j as its root. The root vertex c_j is a cut vertex, separating T_j from the remainder of graph H (see Figure 13).

The following construction differs from those in the previous chapters. Because we want a strong out-tournament as the catch digraph, we can no longer simply orient a cactus. The graph H will be the basis of a representation of $UG(D)$, where D will be our out-tournament. In fact, the only cactus orientable as a strong out-tournament is a cycle C_k, which we have already addressed in a Chapter 6, Lemma 6.18.

In this construction, we will orient C_k as a directed cycle $\vec{C_k}$ in D, because that allows $UG[C_k]_D = C_k$.

Recall our distance measure $d'(v)$, which gives distance to the nearest cycle vertex in graph H. Along with that, if $d'(v) = \ell$, then we may say that vertex v is in a level ℓ. Since this is a cactus, it is completely possible that H has two vertices adjacent that are on the same level. For example, see Figure 13. Vertices u and v are both on level 1, so $\{u, v\}$ is a level edge.

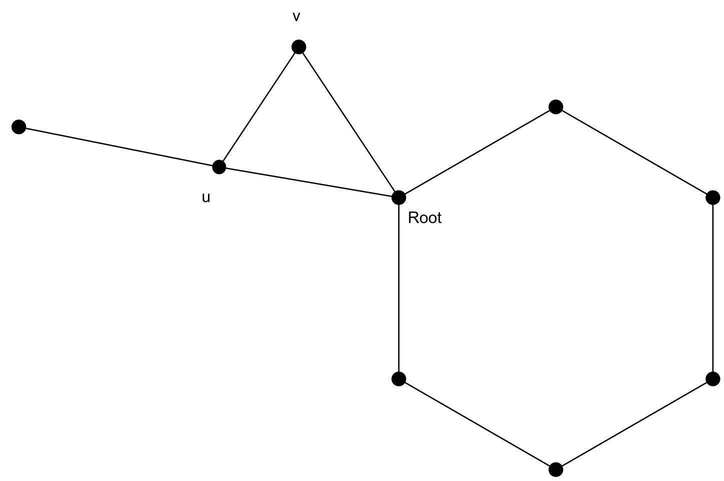

FIGURE 13. A cactus orientable as an out-tournament, with c_j labeled *Root*.

Recalling the oriented cactus in the previous chapter, note that since all of the trans-level edges were oriented toward the central cycle by Lemma 7.3, we ended up with a source vertex at each end of a branch. Rather than adding arcs from the cycle out to those sources, which would be one way to make the resulting digraph strong, here we will take a different approach.

This construction will orient all trans-level arcs *away* from the central cycle. That is, any trans-level arc is oriented from a lower level vertex to a higher level vertex. Now every vertex is at least reachable from somewhere else in the digraph and there are no more source vertices in the digraph. If possible, we would like $UG[T_j]_D = T_j$, which is to say that T_j induces an orientation of itself in D. This limits the form of T_j in that it must be a path, with the possibility that some of the edges are replaced with triangles (refer to Figure 14).

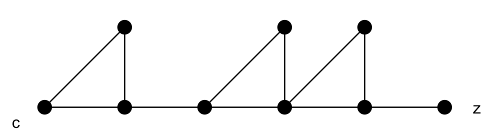

FIGURE 14. Example of T_j satisfying our principles.

Lemma 8.2. *Let H be a cactus that is orientable as an out-tournament, with T_j the triangle cactus having cycle vertex c as its root. If $UG[T_j]_D = T_j$ and the trans-level edges of T_j are oriented outward, then T_j is a path, possibly with some edges replaced with triangles.*

Proof. Suppose that vertex c is adjacent to 3 or more vertices in T_j. Then the set of vertices dominated by c in T_j, represented by $N^+[c_j] \cap T_j$, induces a tournament in D.

As shown in Figure 15, the underlying graph is not a cactus. By our assumption, we wanted $UG[T_j]_D = T_j$, where T_j is a cactus. □

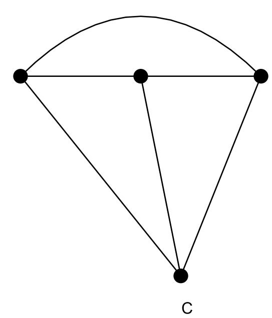

Figure 15. Underlying graph of $[(N^+[c_j] \cap T_j)]_D$

At this point, the forms of $[C_k]_D$ and $[T_j]_D$ have been determined, but D is not yet an out-tournament. To see this, consider the outset of any cycle vertex that is a root vertex of a cactus in H.

Before adding arcs to create a strong out-tournament, we make an observation about the triangle cactus, T_j, rooted on the central cycle.

Lemma 8.3. *Let D_j be an orientation of triangle cactus T_j rooted at vertex c_j with all trans-level arcs directed away from c_j. The orientation D_j is an out-tournament. Furthermore, there is a directed path in D_j containing all the vertices of T_j.*

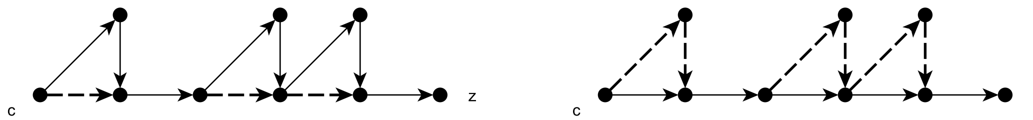

FIGURE 16. The long cz-path, with all vertices (left), and the shortest cz-path (right). Dashed arcs indicate exclusion from the path.

PROOF. To prove this, refer to Figure 14 to see the form that T_j must take: a path with some edges replaced by triangles. The distance of any vertex $v \in T_j$ from root vertex c_j is exactly the $d'(v)$ distance measure in the larger digraph D. With all trans-level arcs oriented away from c_j, any vertex that dominates more than one vertex in the orientation automatically induces a tournament on its outset. Observe that every vertex in D_j is reachable from c_j. By out-path mergeability, D_j has a directed path that includes all of the vertices of D_j. $\square$

Note that in each branch D_j off of the central cycle, there will be exactly one sink vertex. We will denote that vertex z. As noted in Lemma 8.3, each D_j has a path that includes all of its vertices. This path, therefore, ends at z. There is also a shortest path from c to z that skips a vertex at each triangle, if there is one. For an example of the shortest path and the Hamiltonian path of a sample D_j, see Figure 16.

We now have the necessary understanding of our orientation of H to add a relatively small number of arcs to complete the digraph D as a strong out-tournament. Recall that each triangle cactus T_j grows out of the central cycle at vertex c_j. We need a path for the j^{th} sink vertex, z_j, to reach the vertices on the central cycle. So, we add arc $(z_j, s(c_j))$ where $s(c)$ represents the *successor* of a cycle vertex on the orientation of the central cycle. But, observe that the pointed set for $s(c_j)$ does not represent a connected subgraph of H. Currently, it consists of c_j and z_j. To achieve connectedness in the pointed sets (closed insets in D), we need arcs from each vertex on the shortest $c_j z_j$-path to $s(c_j)$. Adding those arcs makes D strongly connected, but the presence of these arcs causes the digraph to cease to be an out-tournament. Specifically, consider the outset of any vertex x containing a vertex of T_j, say y, that was not included in the shortest $c_j z_j$-path. That outset contains y and $s(c_j)$, which are not adjacent in T_j or in D_j. The solution is to add an arc from every vertex in T_j to cycle vertex $s(c_j)$. This creates a strong out-tournament without adding any unneccessary arcs to our oriented cactus.

3 Connectedness and Enumeration of Vertices

At this point, we have a strong out-tournament that has a minimal number of arcs that are not oriented edges from the representation H. The goal is to find the cyclic enumeration to assist in matrix rank evaluations. In this construction, the Hamiltonian cycle is unique, so we can refer to 'the cyclic enumeration' without ambiguity. The following proposition verifies that D is strong by giving the cyclic enumeration.

Proposition 8.4. *Let H be a cactus orientable as an out-tournament. Let C_k be the central cycle, and T_j any cactus subgraph growing out of the cycle at vertex c_j. Let each T_j induce an out-tournament orientation of itself in D, with all trans-level edges of each T_j oriented away from the central cycle. Let C_k induce $\vec{C_k}$ in D. Add arcs $x \longrightarrow s(c_j)$ for each $x \in T_j$. Then D is a strong out-tournament.*

Proof. Consider outsets of the different types of vertices in D. For any cycle vertex, c_j, we have $N^+(c_j)$ containing $s(c_j)$ and there are at most two level 1 vertices in T_j. By the construction, this outset must induce a tournament. The level 1 vertices must be adjacent, because of the form of T_j and any level 1 vertices of T_j must beat $s(c_j)$, so $[N^+(c_j)]_D$ is a tournament for each $1 \leq j \leq k$. For any vertex z_j which is the sink vertex for D_j, the outset is the single vertex $s(c_j)$. Take any vertex v in T_j other than c_j and z_j. Observe that the form of D_j dictates that $N^+(v) \cap V(T_j)$ has either one or two vertices. If two, then those vertices are adjacent in D, as well as both beating $s(c_j)$, thus $N^+(v)$ induces a tournament. Since every outset induces a tournament in D, digraph D is an out-tournament.

If we can provide a Hamiltonian cycle, then clearly D is strong. For use in the next section, addressing the ranks of the adjacency matrix, we will assign a cyclic enumeration to the vertices at the same time as identifying the Hamiltonian cycle. Take any cycle vertex c_j as vertex 1. If c_j has a cactus growing out of it, we number those vertices next, in order, following the long path through T_j to z_j. From z_j, $z_j \longrightarrow s(c_j)$ takes us back to the cycle at the next cycle vertex $s(c_j)$. If c_j did not have a cactus, then we proceed directly to $s(c_j)$. Repeat for vertex $s(c_j)$ and so on, until arriving back at starting point cycle vertex, c_j. Then

$$1 \longrightarrow 2 \longrightarrow ... \longrightarrow n \longrightarrow 1$$

represents a Hamiltonian cycle, so D is strong. □

4 Adjacency Matrix and Matrix Ranks

The cyclic enumeration provided in Proposition 8.4 gives an adjacency matrix with characteristics that allow us to identify the matrix ranks. Let D be the strong out-tournament constructed from cactus H as discussed in the last section. Recall that D_j denotes the induced digraph on cactus T_j. Following previous convention, let A be the adjacency matrix of D and A_j the adjacency matrix of subdigraph D_j.

Figure 17 gives an example of the construction we have been discussing. The dashed arcs are the added arcs, in the sense that those are the only ones that are not orientations of the edges in H.

It was critical that we started the enumeration at a cycle vertex to enable us to represent the block form of matrix A as:

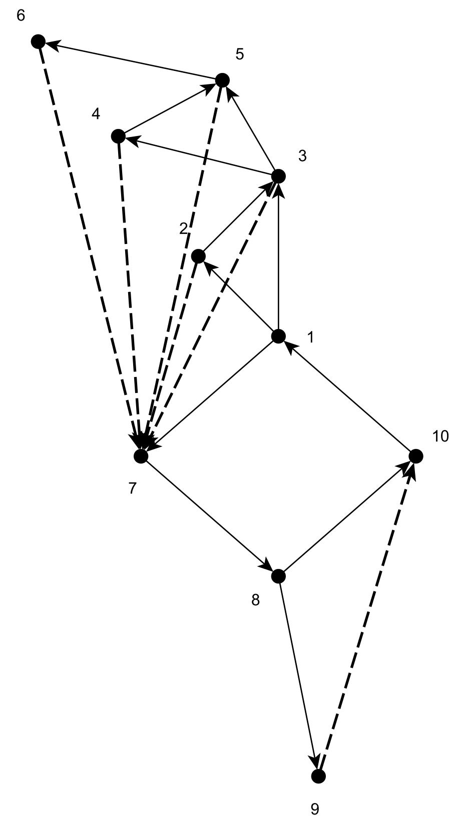

FIGURE 17. Example of construction, with enumeration.

$$A = \begin{bmatrix} A_1 & X_1 & [0] & \dots & [0] \\ [0] & A_2 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & [0] \\ [0] & \ddots & \ddots & A_{k-1} & X_{k-1} \\ X_k & [0] & \dots & [0] & A_k \end{bmatrix}.$$

Since n usually refers to the number of vertices in digraph D, let n_j represent the number of vertices in subdigraph D_j. In the block matrix above, the matrices A_j are upper triangular. In fact, in each A_j, the superdiagonal, excluding $a_{n_j,1}$, is all 1s. The superdiagonal of A is all 1s, including $a_{n,1}$. Those 1s represent the Hamiltonian cycle. In each A_j, the second superdiagonal may have some 1s, representing the arcs that were left out of the long path through T_j, as in Figure 16. All other entries are 0s.

The matrices X_j represent the added arcs originating in the j^{th} triangle cactus, D_j, and ending at the next cycle vertex. Thus, each X_j has its first column consiting of all 1s, and the rest of the matrix all 0s.

PROPOSITION 8.5. *For each A_j, $1 \leq j \leq k$,*

$$r(A_j) = r_B(A_j) = r_{Z^+}(A_j) = r_t(A_j) = n_j - 1.$$

PROOF. Consider the subdigraph D_j with the path we used for the enumeration. No vertex in D_j beats a vertex with a lower index. Thus, A_j has all 0s below the first superdiagonal. The set of $n_j - 1$ 1s on the superdiagonal form a maximum isolated set of 1s, a maximum independent set of 1s and a maximum set of pivot 1s (for real rank). Thus, the four matrix ranks are all equal to $n_j - 1$. $\square$

Recall that any out-tournament oriented cactus has no substantial bicliques, and so we automatically get

$$r_B(A) = r_{Z^+}(A) = r_t(A)$$

if A is the adjacency matrix of any out-tournament oriented cactus. With the current strong out-tournament construction, we made note of the fact that the resulting digraph is no longer an orientation of a cactus, however it nearly is. The only arcs not that do not appear as edges in the graph H are the added arcs $T_j \longrightarrow c_{j+1}$. Adding these arcs does not create any substantial bicliques, and so the Boolean, nonnegative integer, and term ranks of our adjaency matrices also must be equal to each other.

THEOREM 8.6. *Let H be a cactus orientable as an out-tournament. Let C_k be the central cycle, and T_j any cactus subgraph growing out of the cycle at vertex c_j. Let each T_j induce an out-tournament orientation of itself in D, with all trans-level edges of each T_j oriented away from the central cycle. Let C_k induce $\vec{C_k}$ in D. Add* *arcs* $x \longrightarrow s(c_j)$ *for each* $x \in T_j$. *The adjacency matrix* A *of strong out-tournament* D *has full, equal matrix ranks.*

Proof. Observe that the 1s corresponding to the arcs of the Hamiltonian cycle represent a full set of n isolated 1s in adjacency matrix A. These 1s are in each of the positions $a_{j,j+1}$ for $j = 1, 2, ..., n-1$ and $a_{n,1}$. Matrix A is nearly upper triangular: relabeling columns by cyclic permutation $(n, n-1, ..., 2, 1)$ produces an upper triangular matrix with all 1s on the main diagonal. So $r(A) = n$. □

As an illustration, consider matrix A, in (8.1), which corresponds to our example digraph shown in Figure 17.

$$A = \left[\begin{array}{ccccc|ccccc}
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{8.1}$$

5 Conclusion and Future Work

In this chapter, we have found a class of strong out-tournaments with full and equal ranks of their adjacency matrices. Consider that for each vertex in our finished digraph, with the exception of cycle vertices, the insets (whose closures form the pointed sets for the catch digraph) have size 1 or 2. Each vertex off the big cycle can only be dominated by a vertex on the same level or one level higher, and cannot be dominated by a vertex in another branch, T_j. Each of those conditions can be eased, one at a time, to see what generalizations can be made. There undoubtedly are limitations to the approach of starting with an out-tournament orientable cactus, but as we observed before, it offers a new perspective on the problem of classifying out-tournament matrices by their ranks. Ultimately, the limitations lie in the fact that the class of out-tournaments is huge, even for fairly small n. Any foothold we can find, however, gets us closer to the complete characterization of all out-tournament matrices by the four matrix ranks.

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