Reliability-Based Design Optimisation Methods in Large Scale Systems

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Abstract

Structural optimisation is an important field of applied mathematics, which has proved useful in engineering projects. Reliability-based design optimisation (RBDO) can be considered a branch of structural optimisation. Different RBDO approaches have been applied in real world problems (e.g. vehicle side impact model, short column design, etc.).

Double-loop, single-loop, and decoupled approaches are three categories in RBDO. This research focuses on double-loop approaches, which consider reliability analysis problems in their inner loops and design optimisation calculations in their outer loops.

In recent decades, double-loop approaches have been studied and modified in order to improve their stability and efficiency, but many shortcomings still remain, particularly regarding reliability analysis methods.

This thesis will concentrate on development of new reliability analysis methods that can be applied to solve RBDO problems. As a local optimisation algorithm, the conjugate gradient method will be adopted. Furthermore, a new method will be introduced to solve a reliability analysis problem in the polar space. The reliability analysis problem must be transformed into an unconstrained optimisation problem before solving in the polar space. Two methods will be introduced here and their stability and efficiency will be compared with the existing methods via numerical experiments.

Next, we consider applications of RBDO models to electricity networks. Most of the current optimisation models of these networks are categorised as deterministic design optimisation models. A probabilistic constraint is introduced in this thesis for electricity networks.

For this purpose, a performance function must be defined for a network in order to define safety and failure conditions. Then, new non-deterministic design optimisation models will be formulated for electricity networks by using the mentioned probabilistic constraint. These models are designed to keep failure probability of the network below a predetermined and accepted safety level.

Dedication

In dedication to my wife, lovely Parisa, who is the most valuable blessing from God for me. If I did not have you, I could never progress in any step of my life. Love you forever and will never forget your helps and supports.

I Love You - Dooset Daram - Sani Seviram
Soorena

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Statement of Authorship

Except where explicit reference is made in the text of the thesis, this PhD thesis contains no material published elsewhere or extracted in whole or in part from a thesis by which I have qualified for or been awarded another degree or diploma. No other persons work has been relied upon or used without due acknowledgment in the main text and bibliography of the thesis.

Signature:	Signature: ....
Dated: 17/11/2015	Dated: 17/11/2015
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List of Abbreviations

AC	Alternating Current
AC	Ant Colony
AMA	Approximate Moment Approach
AMV	Advanced Mean Value
BD	Benders Decomposition
BE	Branch Exchange
CC	Chance Constrained
CDF	Cumulative Distribution Function
CGA	Conjugate Gradient Analysis
CMV	Conjugate Mean Value
DC	Direct Current
DDO	Deterministic Design Optimisation
DOE	Degree Of Freedom
EBDO	Evidence Based Design Optimisation
EV	Electric Vehicle
FORM	First Order Reliability Method
GA	Genetic Algorithms
HCC	Hybrid Chaos Control
HL - RF	Hasofer and Lind - Rackwitz and Fiessler
HMV	Hybrid Mean Value

JPDF	Joint Probability Density Function
KKT	Karush Kuhn Tucker
LAS	Local Adaptive Sampling
LS	Least Square
MCC	Modified Chaos Control
MCS	Monte Carlo Simulation
MILP	Mixed Integer Linear Programming
MLS	Moving Least Square
MORBDO	Multi Objective Reliability Based Design Optimisation
MPFP	Most Probable Failure Point
MPTP	Minimum Performance Target Point
MV	Mean Value
NERS	Nested Extreme Response Surface
PBDO	Possibility Based Design Optimisation
PMA	Performance Measure Approach
PP	Probable Point
QP	Quadratic Programming
RBDO	Reliability Based Design Optimization
RBFN	Radial Basis Function Neural
RDO	Robust Design Optimisation
RIA	Reliability Index Approach
RSM	Response Surface Methodology
SA	Simulated Annealing
SAP	Sequential Approximate Programming
SDP	Semi Definite Programming
SLP	Sequential Linear Programming
SORA	Sequential Optimisation and Reliability Assessment

SQP	Sequential Quadratic Programming
SORM	Second Order Reliability Method
SLSV	Single Loop Single Vector
STTM	Spin Transfer Torque Magnetic
SVM	Support Vector Machine
TS	Tabu Search
3D - IC	Three Dimensional Integrated Circuits
UPRA	Unconstrained Polar Reliability Analysis
WLS	Weighted Least Square

List of Publications and Presentations

1. Ghasem Ezzati, On Probabilistic Constraints in Optimization Models of Electricity Power Networks, *Far East Journal of Applied Mathematics*, Vol. 94, No. 3, pp. 231-246 (2016).

2. Ghasem Ezzati, David Yost, "A Modification on Recently Proposed Reliability Analysis Method", *Australian and New Zealand Industrial and Applied Mathematics (ANZIAM) Conference*, UNSW Canberra, Canberra, February 2016.

3. Ghasem Ezzati, David Yost, A New Approach to Solve First-Order Reliability Analysis Problems, *12th Engineering Mathematics and Applications Conference (EMAC2015)*, University of South Australia, Adelaide, December 2015.

4. Ghasem Ezzati, David Yost, An Improvement of the Conjugate Gradient Analysis Method, *59th Annual Meeting of the Australian Mathematical Society*, Flinders University, Adelaide, September 2015.

5. Ghasem Ezzati, A Reliability-Based Design Optimization Model for Electricity Power Networks, *Dynamics of Continuous, Discrete and Impulsive Systems; Series B: Applications and Algorithms*, Vol. 22, pp. 339-357 (2015).

6. Ghasem Ezzati, An Optimization Model for Electricity Networks using Random Vari-

ables, *The 8th International Congress on Industrial and Applied Mathematics (ICIAM2015)*, Beijing, China, August 2015.

7. Ghasem Ezzati, Musa Mammadov, Siddhivinayak Kulkarni, An Investigation into Probabilistic Constraint in Optimal Power Flow Model, *Australian and New Zealand Industrial and Applied Mathematics (ANZIAM) Conference*, Queensland University of Technology, Gold Coast, February 2015.

8. Ghasem Ezzati, Abbas Rasouli, Evaluating System Reliability using Linear-Exponential Distribution Function, *International Journal of Advanced Statistics and Probability*, Vol. 3, No. 1, pp. 15-24 (2015).

9. Ghasem Ezzati, Musa Mammadov, Sid Kulkarni, A New Reliability Analysis Method Based on the Conjugate Gradient Direction, *Structural and Multi-disciplinary Optimization*, Vol. 51, pp. 89-98 (2015).

10. Ghasem Ezzati, Musa Mammadov, Siddhivinayak Kulkarni, A Novel Non-deterministic Optimisation Model of Electricity Networks, *8th Australia New Zealand Mathematics Convention (ANZMC)*, The University of Melbourne, Melbourne, December 2014.

11. Ghasem Ezzati, Musa Mammadov, Siddhivinayak Kulkarni, Re-optimising Optimal Power Flow Model using Reliability-Based Design Optimisation, *Annual Research Conference*, Federation University Australia, Ballarat, November 2014.

12. Ghasem Ezzati, Musa Mammadov, Sid Kulkarni, Solving Reliability Analysis Problems in the Polar Space, *International Journal of Applied Mathematical Research*, Vol. 3, No. 4, pp. 353-365 (2014).

13. Ghasem Ezzati, Musa Mammadov, Siddhivinayak Kulkarni, Unconstrained Polar Reliability Analysis Method for Reliability-Based Design Optimisation, *11th Engineering Mathematics and Applications Conference (EMAC2013)*, Queensland University of Technology, Brisbane, December 2013.

14. Ghasem Ezzati, Applying Conjugate Gradient Method in Reliability Analysis Problems, *Annual Research Conference*, University of Ballarat, Ballarat, November 2013.

15. Ghasem Ezzati, Musa Mammadov, Siddhivinayak Kulkarni, Conjugate Gradient Analysis Method for Reliability-Based Design Optimisation, *57th Annual Meeting of the Australian Mathematical Society*, The University of Sydney, Sydney, October 2013.

Chapter 1

Introduction

Optimisation is an important and broad part of applied mathematics playing a prominent role in the current industrial world. Many optimisation models are now available for various real world problems delivering promising results.

Non-deterministic design optimisation, as a part of optimisation, has been used in different projects. This kind of optimisation, which is considered in this research project, is related to safety level of engineering systems and tools.

However, the main objective in most of the non-deterministic design optimisation models is still to minimise total cost. Typically, construction, material, operation and maintenance costs are assumed as various parts of total cost.

In this chapter, a background of the intended topic of this research along with its application in large scale systems are provided. Basic concepts of electricity networks as well as motivations of this research are also given in the next sections. Finally, the main aims of this research will be explained.

1.1 Background

In a non-deterministic design optimisation problem, total cost is often minimised, while existing uncertainties are also taken into account. One widely used non-deterministic optimisation model is reliability-based design optimisation (RBDO) that is considered in this thesis.

The main features of an RBDO problem can be seen in its variables and constraints. Variables of an RBDO problem are random and also constraints are probabilistic. Reliability-based design optimisation (RBDO) and large scale systems will briefly be explained in the following subsections.

1.1.1 Reliability-Based Design Optimisation

Reliability-based design optimisation (RBDO) is an optimisation model that is formulated using random variables. The main constraint of an RBDO problem is probabilistic. Deterministic constraints (if they exist in a model) do not have significant roles in an RBDO problem.

RBDO is a class within non-deterministic design optimisation defining a connection between reliability and optimisation. Existing RBDO approaches can be classified into three groups; single-loop (or mono level) approaches, double-loop (or two level) approaches and decoupled approaches. Although many widely-used approaches are available for RBDO problems, there are still many challenges in this area, particularly in regards to stability, robustness and efficiency of these approaches.

The main concentration of this project is on double-loop RBDO approaches. Reliability index approach (RIA) and performance measure approach (PMA) are two commonly used double-loop RBDO approaches. The main concern of this research is about inner loop of performance measure approach (PMA). It must be mentioned that inner loop of PMA is related to a reliability analysis problem.

RBDO model and its applications in large scale systems are considered in this research. It is intended to extend RBDO applications into electricity networks, while stability and efficiency of RBDO are enhanced simultaneously.

1.1.2 Large Scale Systems

Non-deterministic design optimisation has been applied into different engineering projects to develop optimal systems. Uncertainties in material properties, manufacturing conditions, external loading conditions and analytical and/or numerical modeling can be considered in a non-deterministic design optimisation problem.

RBDO, as a non-deterministic design optimisation model, has also been applied in different problems, such as durability model of a road-arm of a military tracked vehicle, a truss with multiple failure modes, climate change, a two-bar steel frame, crash-worthiness vehicle side impact model, etc.

A new area will be used in this thesis for RBDO application. For this purpose, electricity power networks are considered as large scale system in which RBDO models will be applied.

Many optimisation models are available for electricity power networks, but there are still many challenges in these problems. Although reliability related issues of these networks are considered by using different uncertainty indices, there is no comprehensive optimisation model for them so that their cost and reliability are considered at the same time.

The next section is dedicated to illustrate several fundamental concepts and definitions of electricity power networks.

1.2 Basic Concepts in Electricity Networks

In this section, a number of concepts that play prominent roles in electrical engineering and electricity networks are illustrated. These concepts can also be found with more details in [73, 121, 130, 146]. This will be needed in Sections 2.6 to 2.8 and also Chapter 4.

Elements of an electricity power network are generally categorised as below:

1. Resistors that dissipate energy;

2. Inductors that store energy in a magnetic field;

3. Capacitors that store energy in an electric field.

Every electricity network is modelled by using various quantities related to different notions. Voltage and current magnitudes are two fundamental concepts in any network.

Generally, charge density difference between two points is considered as a voltage magnitude. This amount, which is denoted by V, has volt (V) as a unit of measure.

Also, a current flow is caused by an existing force that moves electrons in a conductor. It is common to denote current flow by I and it must be mentioned that amperes (A) is often used as a unit of current flow.

However, voltage and current quantities, themselves, are seldom used in formulating electricity networks involving alternating currents. Instead, their effective values are often used. These values are called root mean square (RMS) values and calculated as below:

$$V_{RMS} = \frac{V_{max}}{\sqrt{2}}$$

$$I_{RMS} = \frac{I_{max}}{\sqrt{2}}$$

where V_{max} and I_{max} are peak values of voltage magnitudes and current flow, respectively.

Another item that is very important in electricity networks and can be calculated by using voltage magnitude and current flow is resistance. Resistance is denoted by R and is measured in ohms (Ω). Resistance is obtained as below:

$$R = \frac{V}{I}$$

Immediate current is equal to immediate voltage divided by resistance. In other words, I_{max} is formulated as follows:

$$I_{max} = \frac{V_{max}}{R}$$

Further, current flow in a circuit consumes energy that is expended not only in consumers' devices as loads, but also in conductors as losses or heat. This energy (power) is found as below:

$$P = I^2 R$$

where I is current and R is resistance.

Irrespective of the direction of current and due to squaring of current in the expression for energy, energy is expended as heat in resistance of conductors even when energy is returned from a magnetic or an electric field to source.

Power, which is a multiplication of voltage and current (i.e. $P = VI$), is always positive. Energy spent in a second is the time integral or the area under the power curve over a duration of one second.

In a time interval (period), there are generally two different tasks in a circuit: for part of the time, energy is delivered from source to fields (electric and magnetic); for the remaining part of the time, energy is returned. These powers are called active and reactive powers.

Active power is the average power in a circuit that is delivered to fields. This power is measured in watts and denoted by P. Active power is formulated as below:

$$P = VIcos(\delta) \tag{1.1}$$

where V and I are the effective values of voltage and current and δ is the phase angle of current in relation to voltage by which the current lags the voltage.

Furthermore, reactive power is the power supplying the stored energy in reactive elements. Since the returned energy is in reaction to cyclic establishment of fields, it is called reactive power. This power is denoted by Q and represented as VAR (volt-amperes reactive). Reactive power flow is obtained as following:

$$Q = VIsin(\delta) \tag{1.2}$$

It must be noted that both P and Q have the same dimensions, $joules/s$.

Phase angle, which is also called power factor, can be calculated by using the following equalities:

$$\delta = arcsin(\frac{Q}{VI}) = arccos(\frac{P}{VI})$$

Phase angle has positive and negative components. A positive angle represents power consumption in the resistive elements. Also, a negative angle indicates stored energy in magnetic field that is returned to the source periodically.

Energy is not delivered into the circuit during the time when the stored energy in electric and magnetic fields is returned to the source. When energy is being returned to source, immediate direction of current is not necessarily negative (i.e. from circuit to source). In this case, immediate direction of current is opposite to that of voltage. This concept is shown in the Figure (1.1).

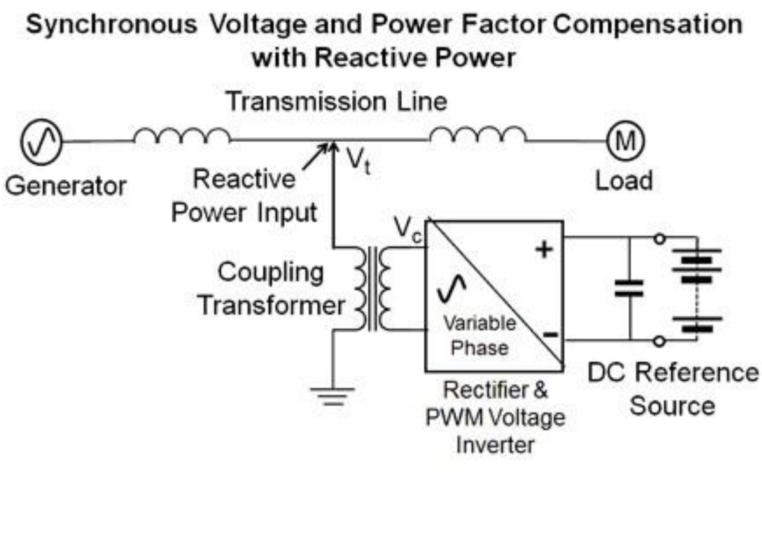

Figure 1.1: Reactive Power Flow

A phasor diagram, including various relationships in an electricity network, is shown in the Figure (1.2).

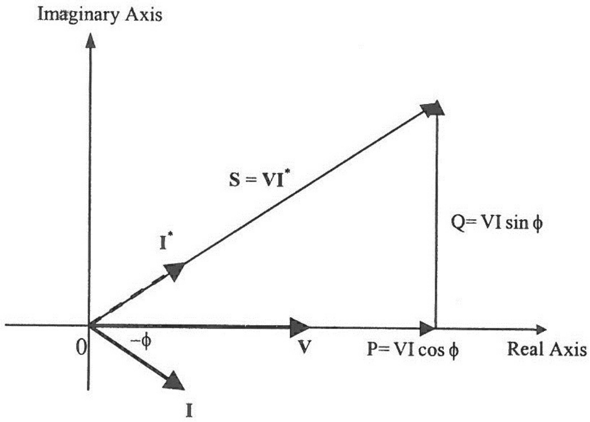

Figure 1.2: Phasor Diagram for Complex Power Relationships

As shown in the Figure (1.2), there is another power quantity in electricity networks in which P and Q are orthogonal components. This is apparent power or complex power and is denoted by S. Apparent power is often defined as follows:

$$S = P + jQ \tag{1.3}$$

Apparent power, which is also formulated as $|S| = \sqrt{P^2 + Q^2}$, can be rewritten as below:

$$S = VI(cos\delta + jsin\delta) = VIe^{j\delta} = VI\angle\delta$$

Apparent power is measured in VA (volt-amperes) since it is nothing but a product of effective values of voltage and current; i.e. $S = V * I$.

Power at i^{th} bus (injected into a network) is called the bus power and is defined as follows:

$$S_i = S_{G_i} - S_{D_i}$$

where S_{G_i} and S_{D_i} are supplied or generated power (incoming power) and drawn or load power (outgoing power), respectively.

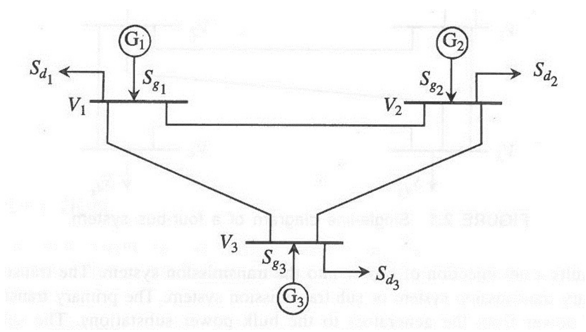

Figure 1.3: Three Bus System

Since $S_{G_i} = P_{G_i} + jQ_{G_i}$ and $S_{D_i} = P_{D_i} + jQ_{D_i}$, we get:

$$S_i = (P_{G_i} - P_{D_i}) + j(Q_{G_i} - Q_{D_i})$$

The last equality can also be written as below:

$$S_i = P_i + jQ_i$$

which is indeed the Equality (1.3).

A simple three bus example system is displayed in the Figure (1.3). Apparent powers as well as supplied and drawn powers are shown in this figure.

All nodes in an electricity network, which are also called buses, are categorised into three groups: generation nodes, load nodes and slack nodes. At each node, two quantities (out of the four following quantities) are specified:

1. Magnitude of the voltage, $|V|$;

2. Phase angle of the voltage, δ;

3. Active or real power, P;

4. Reactive power, Q.

Active power (P) and voltage magnitudes ($|V|$) indicate generation nodes. This type of node is also called a voltage-controlled node and denoted as a $P-V$ node.

A load node is denoted as a $P-Q$ node, because active and reactive powers are specified in this bus. A slack bus, which is also called a swing bus or a reference bus, is denoted as a $V-\delta$ node where voltage magnitude and phase angle are specified.

A slack bus is often considered to provide additional active and reactive power to supply transmission losses. If a slack bus is not specified, then a generation node, usually with the maximum active power, is taken as the slack bus. There can be more than one slack bus in a system.

In a network having n buses, it is obvious that there can be at most $n-1$ outgoing lines at each bus. Each line has different associated quantities such as admittance and impedance. These two quantities are the reciprocal of each other and denoted by Y and Z, respectively; i.e. $Y * Z = 1$.

If a line between two buses does not exist, then its admittance is simply set to zero. Admittance can be decomposed into real and imaginary parts as:

$$Y = g + ib$$

where g and b are conductance and susceptance, respectively.

Further, impedance of a line in an electricity network is calculated as below:

$$Z = r + ix$$

where r and x are real and imaginary impedances, respectively.

Since $Y * Z = 1$, we have:

$$Y = \frac{1}{Z} = \frac{1}{r + ix} = \frac{1}{r + ix} * \frac{r - ix}{r - ix} = \frac{r - ix}{r^2 + x^2}$$

Therefore, as $Y = g + ib$, it can be concluded that conductance and susceptance are computed, respectively, as follows:

$$g = \frac{r}{r^2 + x^2}$$

$$b = \frac{-x}{r^2 + x^2}$$

Suppose that all admittances of different parts of a network are written in a matrix. This matrix is called admittance matrix. In this case, angle of an element of the mentioned matrix is calculated as follows:

$$\theta_{ij} = arctan(\frac{b_{ij}}{g_{ij}})$$

where i and j indicate row and column of an admittance, respectively.

Value of impedance (as a complex number) is used to find the current flow, because division of voltage by impedance gives not only the magnitude of the current but also its phase relationships.

Current flow in a line is given by difference in voltage (in phasor form) at the two ends of the line divided by the impedance of the line.

If both resistance and inductance exist in a circuit, the current lags the voltage by an angle less than $\frac{\pi}{2}$. This angle is called phase angle and calculated as follows:

$$\delta = arctan(\frac{\omega L}{R})$$

where ω is angular frequency and L and R are the inductance and resistance of the circuit, respectively.

The power balance law is another important rule in electricity networks. This law is based on Kirchhoff's Laws. Kirchhoff's Laws arise from the fact that charges cannot be destroyed. In other words, the summation of currents at any junction (node) is zero. In other words, currents flowing into a node and out of it must be the same.

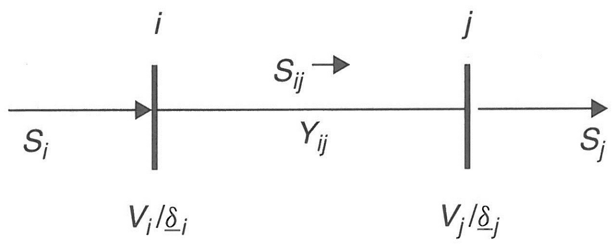

Figure 1.4: Complex Power Flow in a Branch

Considering the Figure (1.4) and based on the Kirchhoff's laws, the current I_{ij} (complex, RMS value) from i to j is given by:

$$I_{ij} = \frac{V_i - V_j}{Z_{ij}} = Y_{ij}(V_i - V_j)$$

where V_i and V_j represent complex voltages at nodes i and j, respectively. Also, Z_{ij} and Y_{ij} are, respectively, impedance and admittance between these nodes.

Complex power, S_{ij}, flowing in the branch from i to j (which is the same as the injected power at node i) is given by the product of the voltage conjugate and current as below:

$$S_{ij} = S_i = V_i^* I_i (= V_i I_i^*) = V_i^* Y_{ij}(V_i - V_j)$$

$$\Rightarrow S_i = S_{ij} = Y_{ij}|V_i|^2 - Y_{ij}V_i^* V_j$$

$$\Rightarrow S_i = S_{ij} = |Y_{ij}||V_i|^2 \angle\theta_{ij} - |Y_{ij}||V_i||V_j| \angle(\theta_{ij} + \delta_j - \delta_i)$$

where θ_{ij} is the angle of the admittance Y_{ij}, and δ_i and δ_j are the angles of voltages at nodes i and j.

$$S_j = -S_{ij} = |Y_{ij}||V_j|^2\angle\theta_{ij} - |Y_{ij}|V_i||V_j|\angle(\theta_{ij} + \delta_i - \delta_j)$$

Moreover, electrical waves can be described as periodic mathematical functions. In this case, the most general mathematical formula for a cosine signal is as below:

$$x(t) = Acos(\omega_0 t + \delta)$$

where x is a function of time (t) and A, ω_0 and δ are fixed numbers. A is called amplitude, ω_0 is radian frequency and δ is phase angle.

Obviously, $x(t)$ is an oscillating function between $-A$ and A with a period of $\frac{2\pi}{\omega_0}$. Also, the unit of δ must be radians. If t has units of seconds, ω_0 must have units of $\frac{radians}{second}$.

The above function of a signal is sometimes written in terms of sine function. In other words, it may be given as $x(t) = Asin(\omega_0 t + \delta')$. In this case, this function can simply be rewritten in terms of cosine as below:

$$x(t) = Asin(\omega_0 t + \delta') = Acos(\omega_0 t + \delta' - \frac{\pi}{2})$$

Radian frequency (ω_0) can be replaced by cyclic frequency (f_0). Hence, we will have:

$$x(t) = Acos(2\pi f_0 t + \delta)$$

where f_0 must have units of sec^{-1}.

There are generally two kinds of electrical flows: direct current (DC) and alternating current (AC). The main difference between an AC and a DC power flows is that a true formulation of steady-state power flow is often used for an AC power flow, while a DC power flow is often approximated by only considering active power, and ignoring reactive power and bus voltage magnitudes.

A solution to any direct current (DC) electrical circuit can be obtained by using two parameters: voltage at nodes and current in branches. On the other hand, generators in an alternating current (AC) electrical circuit generate a sinusoidally time varying voltage expressed as following:

$$v(t) = V_{max} sin(\omega t)$$

where ω and V_{max} are radian frequency and immediate maximum value of the voltage, respectively.

Moreover, if n and l indicate all the nodes and existing lines, respectively, it can easily be shown that:

$$l \leq \frac{n(n-1)}{2}$$

All these basic electrical concepts and corresponding mathematical functions are used in a wide range of research projects in this area. It is intended in this thesis to introduce new optimisation models for electricity power networks based on these concepts.

1.3 Motivation of This Research

Many efforts have been expended to introduce new and improve efficiency of the existing reliability-based design optimisation (RBDO) models. However, there are still many difficulties in these models that have not been solved yet.

Requiring high computational efforts, limitations to dealing with a large number of design variables, sensitivity to initial design points and effects of probability distributions on problems are the most common drawbacks of the existing RBDO approaches mentioned in the literature.

Some of these drawbacks and other properties of RBDO approaches will be investigated in this thesis. High computational cost and instability and inefficient behaviours of double-loop RBDO approaches as well as effects of probabilistic constraints on various problems are discussed in the next chapters.

One of the main aims of this research is to improve stability of reliability analysis problems used in inner loop of the double-loop RBDO approaches. High importance of a reliability analysis problem comes from the fact that if this problem diverges, it is impossible to solve the associated RBDO problem.

Another issue is to reformulate reliability analysis problems. As discussed in this thesis, a reliability analysis problem in the inner loop of an RBDO problem can be reformulated using polar coordinates system. This reformulation changes a constrained optimisation problem to an unconstrained optimisation problem resulting in more robustness.

Moreover, application of RBDO in electricity power networks is another significant motivation of this research. Although many optimisation models are available for these networks, no probabilistic constraint has yet been introduced in the literature for them.

Most of the existing optimisation models developed for electricity networks are mono-objective and just try to find the best (lowest) cost amount. Reliability relevant issues are often considered in the existing models by merely using different safety indices.

Further, a number of multi-objective optimisation models have been developed for electricity networks, but there is a lack of RBDO formulation in the existing literature. Applying RBDO into electricity power networks is a quite difficult task and it is one of the principal motivations of this thesis.

All above mentioned motivations for this research project will be summarized as various research aims in the next section.

1.4 Research Aims

In this section, main aims of the current research are briefly illustrated. As discussed in the previous section, improvement of reliability-based design optimisation (RBDO) models are considered in this thesis as well as their application in large scale systems.

There are many drawbacks in the existing RBDO models that have made them hard to solve. The main shortcomings are stability and efficiency issues. Further, as there is no RBDO model for electricity power networks, it is intended to introduce such a model for these networks.

Therefore, the main goals of this research can be summarised as following:

1. Improve stability of reliability analysis methods (in the inner loop of a double-loop RBDO approach) to prevent solutions of RBDO problems from diverging.

2. Efficiency enhancement of reliability analysis problems. Efficiency can be improved by reducing the required time or the number of required iterations for solving a problem as well as obtaining a better solution for problem.

3. Formulate a probabilistic constraint for electricity power network as a large scale system in order to use in a non-deterministic design optimisation problem.

4. Introduce non-deterministic design optimisation models for electricity power networks.

1.5 Organisation of the Thesis

This document is organised as follows. The existing literature will be reviewed in the next chapter. The first five sections of literature review are dedicated to the existing literature about reliability-based design optimisation (RBDO) and the last three sections are about electricity power networks.

Then, two new reliability analysis methods will be introduced and illustrated in Chapter 3. Stability and efficiency of these methods will be compared with the existing reliability analysis methods.

Chapter 4 includes an idea of how to formulate a probabilistic constraint for an optimisation model of electricity power networks. Also, a new reliability-based design optimisation (RBDO) model will be introduced in this chapter for these large scale systems.

A conclusion of this thesis will be provided in Chapter 5. In this chapter, existing methods/models will be discussed and compared with the new methods/models introduced in this thesis. Further, this chapter consists of a summary of future works in this area.

Chapter 2

Literature Review

2.1 Design Optimisation

Classical optimisation algorithms try to find the best amount for an objective function such that a set of possible constraints is satisfied. These algorithms are not concerned about probabilistic features of a system whereas these properties (such as probability of failure) are very important factors in all systems.

It has been reported in the existing literature that such algorithms, which ignore uncertainty as a probabilistic feature of a structure, cannot ensure required safety levels, because they do not explicitly consider failure probability of components and systems. Therefore, optimum design obtained without considering uncertainties may result in unreliable or even catastrophic design [26, 61, 83, 84, 115, 133, 155, 172, 173, 195, 203].

Design optimisation methods can be classified into two main groups, deterministic design optimisation (DDO) and non-deterministic design optimisation. The latter category takes into account system uncertainties and includes four subclasses, reliability-based design optimisation (RBDO), possibility-based design optimisation (PBDO), evidence-based design optimisation (EBDO) and robust design optimisation (RDO) [13, 14].

Existence of uncertainties in the physical quantities requires a probability-based approach to design optimisation. Thus, evaluation of a probabilistic constraint in a non-deterministic design optimisation problem is an essential step [43, 45, 173].

2.1.1 Deterministic Design Optimisation

In a deterministic design optimisation problem, the designer seeks the best values of design variables for which an objective function is optimum and deterministic constraints are satisfied. A typical deterministic design optimisation problem in terms of design variable (d) can be formulated as [24]:

$$
\begin{aligned}
Min \quad & f(d) \\
s.t. \quad & g(d) \leq 0 \\
& h(d) = 0 \\
& d^L \leq d \leq d^U
\end{aligned}
\tag{2.1}
$$

where f is an objective function, g is deterministic inequality constraint and h is deterministic equality constraint. In this model, the design space is bounded by d^L and d^U. All the objective and the constraint functions are explicit deterministic functions of the design variable (d).

It must be noted that in deterministic design optimisation, system failure, which is originated from existence of uncertainties in the system, is not taken into account. Therefore, it is required to extend optimisation algorithms so that uncertainties can also be considered. It means that going from deterministic design optimisation to non-deterministic design optimisation is unavoidable [14, 17, 24, 61].

Optimum designs obtained from deterministic design optimisation do not ensure target reliability levels with the most economical solutions [13].

2.1.2 Non-Deterministic Design Optimisation

The second category in the design optimisation is non-deterministic design optimisation that considers probabilistic features of a system. A general non-deterministic design optimisation problem is written as follows [14]:

$$\begin{aligned} &Min \quad f(x) \\ &s.t.\ P_f(x) \leq \tilde{P} \end{aligned} \tag{2.2}$$

where the probabilistic constraint includes failure probability of a system that should be non-greater than an intended level.

The non-deterministic design optimisation methods can be categorised in three classes as parallel-loop methods, serial-loop methods and single-loop methods [13]. The single-loop methods merge double-loops for reliability analysis and design optimisation in a single design loop. Also, the parallel- and serial-loop methods are more stable but less efficient than the single-loop method [61].

In a general non-deterministic design optimisation problem, total cost often depends on a design variable (d) and a random variable (X). Design variables are deterministic control parameters and should be optimised for cost reduction. They are usually probabilistic parameters, such as mean of random variables. Random variables represent uncertainty and possible fluctuations in a system. Each random variable is defined by a probability distribution [13].

1948	Reduction of total cost should include not only construction cost, but also expected failure cost
1950s	Development of the reliability theory
1970s	Solution procedure becomes available
1980-85	Improvement of the solution procedure
1985-90	RBDO was born

Table 2.1: History of Non-Deterministic Design Optimisation

A general non-deterministic design optimisation problem has three goals as low total cost, high reliability level and good structural performance. A brief history of efforts in the non-deterministic design optimisation area is shown in the Table (2.1) [61].

An adaptive-loop method has been proposed for the non-deterministic design optimisation [191]. This method, which has been developed by using an integrated framework, involves different phases including deterministic design optimisation, parallel-loop method and single-loop method.

Deterministic design optimisation speeds up the non-deterministic design optimisation process. The parallel-loop method maintains the numerical stability of the non-deterministic design optimisation. Also, single-loop method completes the non-deterministic design optimisation after confirming numerical convergence [191].

In non-deterministic design optimisation, a performance function is often formulated to define a probabilistic constraint. Suppose that x_i and $G(x_1, x_2, ..., x_n)$ are a random variable and the corresponding system performance function, respectively. It is assumed that system fails if $G(x_1, x_2, ..., x_n) < 0$ and remains safe if $G(x_1, x_2, ..., x_n) > 0$. In this case, $G(x_1, x_2, ..., x_n) = 0$ indicates failure surface that is also called limit-state function [13, 14, 17, 47, 61].

Thus, a performance function $G(x_1, x_2, ..., x_n)$ divides the entire space into three subspaces as below:

1. $G(x_1, x_2, ..., x_n) > 0$ shows safety region;

2. $G(x_1, x_2, ..., x_n) < 0$ displays failure region;

3. $G(x_1, x_2, ..., x_n) = 0$ indicates failure surface or limit-state function;

One question arises here as why do we need to define another random variable? The confusing point, which may result in the mentioned question, is that since $(x_1, x_2, ..., x_n)$ is a random variable, $G(x_1, x_2, ..., x_n)$ (as a performance function) is also a random variable (because it is a function of a random variable).

However, the reason to define a performance function for every system is that we have to determine safety and failure regions for every system (based on the above numeric points). In this case, if we do not define system performance function, and determine safety and failure regions by only the original random variable, then we will have $(x_1, x_2, ..., x_n) < 0$ and $(x_1, x_2, ..., x_n) > 0$ as failure and safety conditions, respectively, for all systems. But it is obvious that these conditions cannot be the same for all systems. Hence, we need to formulate a performance function for every system.

If x_i was a random variable, then probability of failure of a system (P_f) can generally be formulated in a non-deterministic design optimisation problem as follows [14, 17, 61]:

$$\begin{aligned} P_f &= P[G(x_1, x_2, ..., x_n) < 0] = F_G(0, 0, ..., 0) \\ \Rightarrow P_f &= \int \int ... \int_{G(x_1, x_2, ..., x_n) < 0} f(x_1, x_2, ..., x_n) d(x_1, x_2, ..., x_n) \\ x_i^L &\leq x_i \leq x_i^U \qquad\qquad i = 1, 2, ..., n \end{aligned} \tag{2.3}$$

where P_f is system failure probability, $G(x_1, x_2, ..., x_n)$ is the defined performance function, F is the cumulative distribution function (CDF) and $f(x_1, x_2, ..., x_n)$ is the joint probability density function (JPDF). Also, the random space is bounded by lower and upper bounds ($(x_1, x_2, ..., x_n)^L$ and $(x_1, x_2, ..., x_n)^U$, respectively).

This equation can rarely be used because the performance function $G(x_1, x_2, ..., x_n)$ cannot be written as a simple linear function of normally distributed variables. These difficulties require the use of some approximate integration methods, like first order reliability method (FORM), second order reliability method (SORM) and Monte Carlo simulation (MCS) [61, 110, 193].

Furthermore, the least square (LS) method, the moving least square (MLS) method and the weighted least square (WLS) method are very useful tools for reconstructing responses (functions) using some sets of points [7, 132, 134, 157].

Two well-known reliability analysis problems are explained in the next section. These problems are often used to evaluate a probabilistic constraint.

2.2 Reliability Analysis Problems

Evaluating a probabilistic constraint in a non-deterministic design optimisation problem consists of calculating an integration, often a multiple integral. This integration is very difficult and sometimes impossible to solve [61]. Thus, some approximate probability integration methods have been proposed to evaluate probabilistic constraints. These methods include moment methods (such as first-order reliability method (FORM) and second-order reliability method (SORM)) and also sampling-based methods (like Monte Carlo simulation (MCS)) [119, 170, 200].

In sampling-based methods, which usually use MCS, a probabilistic constraint can be approximated by the size of MCS, the realisation of design variables and the accepted failure probability [181, 199].

The sampling-based methods need a very large computational effort and are expensive [193], but the moment methods (especially FORM) are widely accepted for application into the non-deterministic design optimisation problems due to their simplicity. Also, they often provide adequate accuracy because of their efficiency [106, 173, 192, 195].

In the first order reliability method (FORM), a transformation is used to transform problem from original (non-normal) random space (X-space) to the independent and standard normal random space (U-space) [78, 119]. This transformation is generally written as below:

$$T : X \longrightarrow U$$

Also, the transformation can be formulated as following:

$$\Phi_U(u_i) = F_X(x_i) \quad i = 1, 2, ..., n$$

where F is the cumulative distribution function (CDF) in the original random space (X-space) and Φ is the standard normalised cumulative distribution function (CDF) in the standard normal random space (U-space). For example, if random variable x_i was normally distributed with mean value μ and standard deviation σ as statistical parameters ($i.e. x_i \sim N(\mu, \sigma)$), then the transformation is defined as follows:

$$T(x_i) = u_i = \frac{x_i - \mu(x_i)}{\sigma_i} \tag{2.4}$$

where u_i is the projection of x_i in the standard normal random space. In addition, performance function must be transformed from the X-space to the U-space ($i.e. T : G_X(x_1, x_2, ..., x_n) \longrightarrow G_U(u_1, u_2, ..., u_n)$).

The reason for this transformation is to obtain a simpler performance function and then a reliability analysis problem. Different statistical parameters for various random variables may result in complicated constraints for the original reliability analysis problem in X-space.

Failure probability of a system can also be formulated as a function of reliability index (β) by using function Φ (or the standard normal cumulative distribution function (CDF)). This relationship can be formulated as follows [61]:

$$P_f = P[G(x_1, x_2, ..., x_n) < 0] \approx \Phi(-\beta)$$

A reliability index can generally be formulated as below:

$$\beta = \frac{\mu_R - \mu_F}{\sqrt{(\sigma_R^2 + \sigma_F^2)}}$$

where μ_R and μ_F are mean values of reliability and failure, respectively, and also σ_R and σ_F are their standard deviations, respectively.

In some cases and due to difficulties originating from non-linear nature of an objective function, different surfaces (functions) must be approximated. a sensitivity analysis of this process can be obtained by using objective function evaluations of some special samples [167]. This evaluation can be assessed in two ways: either using Hessian approximation [117] or using finite perturbations in the parameter [129].

Reliability analysis problems and probabilistic constraint evaluations have been applied in different research fields. For instance, a reliability analysis tool and design optimisation methodology have been proposed in the literature to reduce mechanical reliability issues in three dimensional integrated circuits (3D-IC) [90].

Reliability analysis is also studied as a part of theory of mathematical uncertainty in the literature [114]. Another novel reliability analysis method is introduced based on a hybrid uncertain model. In this method, several important parameters of probability distribution functions are given as variation intervals (not as precise values) [86].

Spin transfer torque magnetic RAM (STT-MRAM) is another application of reliability analysis in which all possible failures are categorised as soft errors and hard errors. Impacts of these errors on the memory reliability are analysed and then several design solutions are found in order to address these errors and enhance STT-MRAM's reliability [202].

In the coming subsections, two reliability analysis problems that have widely been used in the literature are illustrated.

2.2.1 First Order Reliability Analysis Problem

A first order reliability analysis problem tries to find a point on a hyper-surface $G_U(u_1, u_2, ..., u_n) = g_a$ (g_a is a constant, e.g. zero) in the U-space that has the minimum distance from the origin. The point, which has the maximum joint probability density, is named the most probable point (MPP), $u^*_{g=g_a}$ [13, 173].

The minimum distance, named the first order reliability index $\beta_{a,FORM}$, is an approximation of the generalized reliability index corresponding to g_a. This approximation can be formulated as below:

$$\beta_{a,FORM} = \|u^*_{g=g_a}\| \approx \beta_a = \beta_G(g_a)$$

In traditional first order reliability analysis problems, the first order reliability index $\beta_{a,FORM}$ is obtained by solving the following non-linear optimisation problem [119, 172].

$$\begin{aligned} Min \quad & \|(u_1, u_2, ..., u_n)\| \\ s.t. \quad & G_U(u_1, u_2, ..., u_n) = g_a \end{aligned} \tag{2.5}$$

where the optimum point is the MPP $u^*_{g=g_a}$ and thus $\beta_{a,FORM} = \|u^*_{g=g_a}\|$.

If $g_a = 0$, then the mentioned hyper-surface ($G_U(u_1, u_2, ..., u_n) = g_a$) is changed to the failure surface (i.e. $G_U(u_1, u_2, ..., u_n) = 0$). In this case, the optimum point is called most probable failure point (MPFP).

Some algorithms are available to solve this problem. The Hasofer and Lind - Rackwitz and Fiessler (HL - RF) method is a particular algorithm that is often applied to solve a reliability analysis problem. However, general optimisation algorithms, such as sequential linear programming (SLP) and sequential quadratic programming (SQP), can be used for this purpose as well [46, 115, 179, 184, 185, 187, 197].

2.2.2 First Order Inverse Reliability Analysis Problem

Optimum solution of a first order inverse reliability analysis problem is often a point on reliability surface ($\beta = \beta_a$) that minimises standard normalised performance function ($G_U(u_1, u_2, ..., u_n)$) in the U-space [195]. This point is named minimum performance point (MPP), $u^*_{\beta=\beta_a}$ [13, 173].

The performance function value at the MPP $u^*_{\beta=\beta_a}$ is an approximation of the probabilistic performance measure corresponding to β_a. This concept has appeared in the literature as below:

$$g_{a,FORM} = G_U(u^*_{\beta=\beta_a}) \approx g_a = g(\beta_a)$$

In this case, probabilistic performance measure $g_{a,FORM}$ is found by solving the following sphere-constrained non-linear optimisation problem [172].

$$\begin{aligned} &Min \quad G_U(u_1, u_2, ..., u_n) \\ &s.t. \quad \|(u_1, u_2, ..., u_n)\| = \beta_a \end{aligned} \tag{2.6}$$

where the optimum point is the MPP $u^*_{\beta=\beta_a}$ and thus $g_{a,FORM}(\beta_a) = G_U(u^*_{\beta=\beta_a})$.

β_t is a commonly used reliability index that is called target reliability index. If $\beta_a = \beta_t$, then the reliability surface ($\beta = \beta_a$) is changed to target reliability surface ($\beta = \beta_t$). In this case, the optimum point is called minimum performance target point (MPTP).

General optimisation algorithms (such as SLP and SQP) can be used to solve this sphere-constrained optimisation problem, which is generally easier to solve than the optimisation problem in the first order reliability analysis due to the regular sphere constraint [17, 45].

Moreover, mean value (MV) based methods have been proposed as particular algorithms to solve the inverse reliability analysis problem. These methods include advanced mean value (AMV) method, conjugate mean value (CMV) method and hybrid mean value (HMV) method [43, 184, 185].

Particular reliability analysis methods, which are often applied to solve a first-order and a first-order inverse reliability analysis problems, will be explained in the next section.

2.3 Reliability Analysis Methods

The main objective of a reliability analysis problem is the assurance of a requested level of reliability for a system. Further, it must be noted that an engineered system has numerous sources of uncertainties and the absolute safety of a system cannot be guaranteed [17].

In this regard, there are many numerical tools for probabilistic constraint evaluation in a non-deterministic design optimisation problem. As mentioned in the previous section, Hasofer and Lind - Rackwitz and Fiessler (HL - RF) method is a preferred tool to solve first order reliability analysis problems [72, 115, 144]. Furthermore, some algorithms that have been established based on the mean value (MV) method are useful tools in order to solve first order inverse reliability analysis problems [61, 159, 172].

In addition, it was first reported in the literature that effect of size of search space on efficiency of the MPP search algorithm is significant [173], but it has recently been found that a MPP search algorithm is not affected by the size of search space considerably [195].

In an RBDO problem, a performance function $G(x_1, x_2, ..., x_n)$ is defined in order to introduce a failure/safety condition for system. In other words, $G(x_1, x_2, ..., x_n) < 0$ results in system failure and $G(x_1, x_2, ..., x_n) > 0$ indicates safety region.

As x_i is a random variable, $G_X(x_1, x_2, ..., x_n)$ is also a random variable. Considering x_i as a normally distributed random variable, we can find its standard normalised random variable (u_i) and its performance function $(G_U(u_1, u_2, ..., u_n))$. However, we do not consider finding density or distribution functions of $G_X(x_1, x_2, ..., x_n)$ even if it is simple to do so.

There is a wide range of real world problems in which reliability analysis problems have been applied. For instance, these methods are extended to investigate reliability of different repairable and non-repairable electrical systems with various cold and warm standby switches [56, 177, 189, 198]. Further, human reliability analysis or human sustainability is another area that has successfully employed reliability analysis methods [20].

2.3.1 Hasofer and Lind - Rackwitz and Fiessler Method

This method was first introduced for second-moment reliability analysis problems [72] and then extended to include distribution information [144]. The Hasofer and Lind - Rackwitz and Fiessler (HL - RF) method, which is a specific iterative scheme, is widely used to solve different optimisation problems in structural reliability [119].

In the first order reliability analysis problems, it is assumed that g_a in the Problem (2.5) is equal to zero. Hence, the reliability analysis problem to find an MPFP $u^*_{g=0}$ is, in fact, calculating the minimum distance from the failure surface $(i.e. G_U(u_1, u_2, ..., u_n) = 0)$ to the origin of the standard normal random space (U-space).

Thus, the reliability analysis problem will be changed to the following non-linear optimisation problem:

$$\begin{aligned} &Min \quad \|(u_1, u_2, ..., u_n)\| \\ &s.t. \quad G_U(u_1, u_2, ..., u_n) = 0 \end{aligned} \tag{2.7}$$

This problem can be solved by the HL - RF method using the steepest descent direction. This vector must be computed by using standard normalised performance function $G_U(u_1, u_2, ..., u_n)$ at $u^{(k)} = (u_1^k, u_2^k, ..., u_n^k)$ as below [72, 115, 144, 193]:

$$u^{(k+1)} = (u^{(k)}.n^{(k)})n^{(k)} + \frac{G_U(u^{(k)})}{\|\nabla_U G(u^{(k)})\|} n^{(k)} \tag{2.8}$$

where $\|.\|$ is the Euclidean norm and $u^{(0)}$ is the origin of the U-space; i.e. $u^{(0)} = 0$. Moreover, the steepest descent direction $(n^{(k)})$ is obtained as:

$$n^{(k)} = \frac{\nabla_U G(u^{(k)})}{\|\nabla_U G(u^{(k)})\|}$$

The HL - RF method is a commonly accepted method to solve the first-order reliability analysis problems (Problem (2.5)). It has been reported that this method is efficient and stable to apply in inner loop of reliability index approach (RIA) [13, 111, 173, 193].

2.3.2 Advanced Mean Value Method

Different approaches have been used to evaluate a probabilistic constraint. One of the most popular methods for this purpose, which is also applied in inner loop of performance measure approach (PMA), is to solve a first-order inverse reliability analysis problem (shown in Problem (2.6)).

An inverse reliability analysis problem minimises a performance function on a target reliability surface. In this process, all probable points have a fixed distance from the origin of the U-space. This distance is target reliability index that is a design parameter (predetermined by an engineer or a designer based on the previous designed systems) and is denoted by β_t. As mentioned earlier, the optimum answer of this problem is often called minimum performance target point (MPTP).

It has been reported that mean value (MV) based methods are powerful tools to find an MPTP in a first-order inverse reliability analysis problem in inner loop of PMA. These methods are based on the steepest descent direction.

Mean value (MV) method is the first method in this category. Optimum point of the MV method is computed in the U-space as below [43, 115, 193]:

$$u^*_{MV} = \beta_t.n(0) \tag{2.9}$$

where $n(0) = -\frac{\nabla_U G(u^{(0)})}{\|\nabla_U G(u^{(0)})\|}$.

In this method, the normalised steepest descent direction $n(0)$ is defined to minimise the standard normalised performance function $G_U(u_1, u_2, ..., u_n)$ at the mean value that is the origin of the U-space; i.e. $u_i^{(0)} = 0$.

The optimum point of the MV method is the initial design point of the advanced mean value (AMV) method; i.e. $u^{(1)}_{AMV} = u^*_{MV}$. Advanced mean value (AMV) method is a useful tool in order to find MPTP of a convex performance function [115, 195].

This method iteratively updates the direction vector of the steepest descent method at the probable point u_{AMV} until stopping criterion is held.

A design point of the AMV method is updated as follows:

$$u^{(k+1)}_{AMV} = \beta_t.n(u^{(k)}_{AMV}) \quad\quad k \geq 1 \tag{2.10}$$

where $n(u^{(k)}_{AMV}) = -\frac{\nabla_U G(u^{(k)}_{AMV})}{\|\nabla_U G(u^{(k)}_{AMV})\|}$.

It is assumed that the AMV method has converged when the distance of two consecutive probable points becomes less than a predetermined acceptable convergence parameter. Thus, if this parameter is denoted by ϵ, then stopping criterion would be as follows:

$$|G_U(u_{AMV}^{(k+1)}) - G_U(u_{AMV}^{(k)})| \leq \varepsilon$$

It has been reported that the AMV method is effective for evaluating convex performance functions, while it diverges or has a slow rate of convergence and also exhibits numerical instability and inefficiency to evaluate concave performance functions. This shortcoming comes from a lack of updated information during the iterative reliability analysis [115, 193, 195].

A modified chaos control (MCC) method has been introduced to apply in the AMV method. This adjustment has been done to improve efficiency of the AMV method when evaluating concave performance functions. Then, due to inefficiency of the MCC method to evaluate convex performance functions, a hybrid chaos control (HCC) method has been introduced by adaptively using the AMV end MCC methods [124].

However, a modification has also been implemented on the AMV method to evaluate concave performance functions. This modified method, which is called conjugate mean value (CMV) method, will be illustrated in the next subsection.

2.3.3 Conjugate Mean Value Method

Another MV based method to apply into inner loop of the PMA is conjugate mean value (CMV) method. As discussed in the previous subsection, the AMV method has some drawbacks when applied for evaluating concave performance functions.

The CMV method has been proposed to overcome these difficulties using information of both the current and previous design points [195].

In this method, the new search direction is obtained by combining $n(u^{(k-2)})$, $n(u^{(k-1)})$ and $n(u^{(k)})$. For evaluating convex performance functions, the conjugate steepest descent direction has a slow rate of convergence, while it has a much better convergence rate as well as stability when the performance function is concave [43].

Execution of the CMV method starts with the AMV method, as below:

$$u_{CMV}^{(0)} = 0, \;\; u_{CMV}^{(1)} = u_{AMV}^{(1)}, \;\; u_{CMV}^{(2)} = u_{AMV}^{(2)}$$

This vector is computed from the fourth iteration as follows:

$$u_{CMV}^{(k+1)} = \beta_t . \frac{n(u_{CMV}^{(k)}) + n(u_{CMV}^{(k-1)}) + n(u_{CMV}^{(k-2)})}{\|n(u_{CMV}^{(k)}) + n(u_{CMV}^{(k-1)}) + n(u_{CMV}^{(k-2)})\|} \qquad k \geq 2 \tag{2.11}$$

where $n(u_{CMV}^{(k)}) = -\frac{\nabla_U G(u_{CMV}^{(k)})}{\|\nabla_U G(u_{CMV}^{(k)})\|}$.

Like in the AMV method, it is assumed that the CMV method has converged when distance of two consecutive probable points becomes less than a stopping criterion parameter. Thus, this criterion would be as below:

$$|G_U(u_{CMV}^{(k+1)}) - G_U(u_{CMV}^{(k)})| \leq \varepsilon$$

Although the CMV method works better than the AMV method for evaluating the concave performance functions, it converges very slowly or even diverges for evaluating convex performance functions.

It has been reported that the CMV method is inefficient to evaluate concave performance functions. Therefore, the type of performance function must first be identified in order to select an appropriate MPTP search algorithm [195]. This idea has led to another MV based method that is explained in the last part of this section.

2.3.4 Hybrid Mean Value Method

As mentioned before, it has been reported that the AMV method behaves well for evaluating convex performance functions, but it exhibits numerical shortcomings, such as slow convergence or even divergence, when applied for evaluating concave performance functions. To overcome these difficulties, the CMV method has been proposed that uses both the current and previous design points information [43].

Therefore, it can be concluded that once the type of a performance function is recognised, a suitable numerical tool can be used to solve the corresponding reliability analysis problem. In other words, to select an appropriate MPTP search algorithm, the type of performance function should be identified first. In this context, hybrid mean value (HMV) method has been introduced [195].

A function type criterion is used in the HMV method by employing the steepest descent directions for three consecutive iterations as follows:

$$\zeta^{(k+1)} = (n^{(k+1)} - n^{(k)}).(n^{(k)} - n^{(k-1)}) \tag{2.12}$$

where $\zeta^{(k+1)}$ is the criterion for the performance function type at the $(k + 1)^{th}$ iteration and $n^{(k)}$ is the steepest descent direction of performance function at k^{th} design point.

Once type of the performance function is determined, one of two numerical algorithms (AMV or CMV) is adaptively selected for the MPTP search. A suitable numerical algorithm can be selected as follows:

1. If $\zeta^{(k+1)}$ is positive, then the performance function is convex at $u_{HMV}^{(k+1)}$ and the AMV method must be selected.

2. If $\zeta^{(k+1)}$ is zero or negative, then the performance function is concave at $u_{HMV}^{(k+1)}$ and the CMV method must be selected.

Thus, the HMV method can be summarised as below [43]:

1. Set the iteration counter $k = 0$. Select the convergence parameter ε. Compute the steepest descent direction of the performance function at the initial design point in the

U-space. In other words, compute the following vector:

$$n(u_{HMV}^{(0)}) = -\frac{\nabla_U G(u_{HMV}^{(0)})}{\|\nabla_U G(u_{HMV}^{(0)})\|}$$

where $u_{HMV}^{(0)} = 0$.

2. If $k < 3$ or $\zeta^{(k+1)}$ is positive, then use the AMV method to calculate the next design point as below:

$$u_{HMV}^{(k+1)} = \beta_t . n(u_{HMV}^{(k)})$$

If $k \geq 3$ and $\zeta^{(k+1)}$ is zero or negative, then use the CMV method to calculate the next design point as below:

$$u_{HMV}^{(k+1)} = \beta_t . \frac{n(u_{HMV}^{(k)}) + n(u_{HMV}^{(k-1)}) + n(u_{HMV}^{(k-2)})}{\|n(u_{HMV}^{(k)}) + n(u_{HMV}^{(k-1)}) + n(u_{HMV}^{(k-2)})\|}$$

In all cases, we have:

$$n(u_{HMV}^{(k)}) = -\frac{\nabla_U G(u_{HMV}^{(k)})}{\|\nabla_U G(u_{HMV}^{(k)})\|}$$

Also, it must be noted that when $k < 3$, this step of the AMV method is the same as the corresponding step of the CMV method.

3. Calculate the performance function at the new design point; i.e. compute $G_U(u_{HMV}^{(k+1)})$. Then check whether the convergence criterion holds, i.e. $|G_U(u_{HMV}^{(k+1)}) - G_U(u_{HMV}^{(k)})| \leq \varepsilon$. If the convergence criterion is satisfied, then stop; otherwise, go to the next step.

4. Check the function type criterion $\zeta^{(k+1)}$ for determining performance function type and set $k = k + 1$. Then return to the second step.

Therefore, it can be concluded that the hybrid mean value (HMV) method is the most efficient and robust method for finding the MPTPs of performance functions. It has been reported that this method performs quite well for any type of performance functions [13, 45].

Based on the numerical efficiency and robustness when applied to inverse reliability analysis problems, this method is an effective numerical tool for evaluating probabilistic constraints in a reliability-based design optimisation (RBDO) problem.

2.4 Reliability-Based Design Optimisation

In recent decades, many efforts have been made to introduce new and improve the existing non-deterministic design optimisation models in order to apply them into real world problems. Reliability-based design optimisation (RBDO) is one of these models that aims at searching the best compromise between cost reduction and safety assurance and also involves evaluation of probabilistic constraints. In other words, RBDO not only provides a cost-effective manufacturing process, but also a requested confidence level [45].

Difficulties in RBDO originate from the nature of the input data, which is non-deterministic. It can be said that RBDO ensures a minimum total cost without affecting target safety level. The total cost is a summation of initial cost (design and construction costs), failure cost and maintenance cost [13, 61, 173].

RBDO is a probabilistic design model that tries to obtain an optimal design under probabilistic constraints and performance functions. An RBDO problem is a non-linear optimisation problem with inequality probabilistic constraint. The major difficulties arise from the non-deterministic input data [180].

The main difference between an RBDO model and other engineering designs is that system parameters are non-deterministic in an RBDO model. Thus, constraints will be appear in probabilistic form. The probabilistic constraints and the uncertainties play important roles in RBDO. It can be seen that evaluating probabilistic constraints in an RBDO problem is the most important and difficult part to deal with and hence some numerical approaches are needed in this process [45, 193].

A multi-objective reliability-based design optimisation (MORBDO) has also been proposed in the literature in order to explore design of a vehicle door. It is intended in this model to enhance optimisation efficiency [57].

Probabilistic approaches are very popular within all existing RBDO approaches. Various probabilistic approaches have been studied in [42, 91, 169, 203]. However, non-probabilistic RBDO approaches are also investigated in the existing literature [92].

A simplified safety index based on the advanced second-moment method and also a linearised reliability index using linear programming optimisation are two numerical algorithms that have been proposed to solve some special problems [109, 147].

It is commonly accepted that ensuring a high level of system reliability is one of the most significant concerns in practical engineering design. For this purpose, a nested extreme response surface (NERS) approach is available to carry out time-dependent reliability analysis and find an optimum design [181].

Uncertainties in RBDO models are identified by variation of random parameters. Also, the existence of uncertainties in the physical quantities requires a reliability-based approach to design optimisation. In this case, it can be concluded that RBDO problems are rather complicated by nature due to the inherent non-deterministic input data [173, 192, 193, 195].

However, in terms of whether or not to use or not to use the theories of probability and statistics, RBDO can be classified into two categories: methods requiring probability and statistical analysis and methods not requiring these. "Worst Case Analysis", "Corner Space Evaluation" and "Variation Patterns Formulation" are three methods that do not need probability and statistical analyses [27, 70]. It has been reported that the probabilistic formulations are the best methods [173].

An RBDO problem is often solved by search methods for constrained non-linear optimisation, like sequential linear programming (SLP) and sequential quadratic programming (SQP). A search method starts with an initial design and iteratively improves it with the design change, obtained by solving an approximate sub-problem, defined by linearised probabilistic constraints. The linearised probabilistic constraints are not equivalent from different perspectives in predicting a design change [173].

Another issue that is very important in the RBDO process (and also other types of optimisation problems) is non-linearity that can be altered in various cases due to some structural properties and also differences between non-linear functions used in an optimisation problem. Efficiency of non-linear optimisation in the RBDO process significantly depends on complexity of constraints in an optimisation problem [192].

Non-linearity of an RBDO problem can be dramatically increased by non-linearity of reliability analysis and design optimisation problems (in the inner and outer loops of RBDO problem). Thus, as this property can affect efficiency and robustness of RBDO process, it must be noted that a proposed algorithm should have the lowest non-linearity [44].

Since there are some non-linear mappings between X- and U-spaces ($e.g. T : x_i \longrightarrow u_i$) in an RBDO problem and also various probability distributions are used in this process, non-linearity of an RBDO problem depends on type of probability distributions of random parameters. Also, the transformation between X- and U- spaces may introduce additional non-linearity.

In this case, most transformations used in RBDO, except Gaussian distribution, are highly non-linear. Total number of function evaluation is used to measure efficiency of an RBDO process [192].

An RBDO method has been proposed, which employs the response surface methodology (RSM) [43, 44, 71]. The proposed method is based on the moving least squares (MLS) method and a design of experiment (DOE). Also, it has been found that the MLS method is better compared with the least squares (LS) method for obtaining an approximation of implicit responses [43, 101, 149].

Furthermore, the response surface methodology (RSM) has been studied for reliability assessment. In this case, various methods, such as radial basis function neural (RBFN) network and support vector machine (SVM), have been considered and further discussed [168].

Another method has been introduced that changes reliability analysis to deterministic design optimisation. In this method, the probabilistic constraints are converted to deterministic constraints in reliability analysis problem and then an improved design is obtained by a deterministic design optimisation problem [37, 186].

RBDO approaches have been applied into different real world problems. Marine structures, vehicle crash-worthiness, cloud migration, and aero-elasticity problems are only a few examples of these applications [4, 6, 85, 94, 126, 143, 163, 196].

In general, reliability-based design optimisation approaches have been classified in to three categories that will be reviewed in the next subsections. These categories are as below [13, 61, 194]:

1. Mono-level or single-loop approaches;
2. Two-level or double-loop approaches;
3. Decoupled approaches;

As RBDO suffers from high computational cost, Kriging-based model RBDO has been proposed to overcome this difficulty [53]. Further, a local adaptive sampling (LAS) has been introduced to improve Kriging method's efficiency for RBDO approaches [40].

A typical formulation of RBDO problems is reviewed in the next subsection.

2.4.1 General Formulation of RBDO Model

As discussed earlier, in the RBDO process, random variables, which characterise physical quantities under uncertainties, are often modelled by using probability distributions.

Suppose that $G(x_1, x_2, ..., x_n)$ is system performance function that was illustrated in Subsection 2.1.2. Statistical description of a system's failure probability is generally characterised by its cumulative distribution function (CDF) as below [13, 61, 173, 174, 191–193, 195]:

$$P[G(x_1, x_2, ..., x_n) < g] = F_G(g) = \int\int ... \int_{G(x_1,x_2,...,x_n)<g} f_X(x_1, x_2, ..., x_n) d(x_1, x_2, ..., x_n)$$
$$(x_1, x_2, ..., x_n)^L \leq (x_1, x_2, ..., x_n) \leq (x_1, x_2, ..., x_n)^U \tag{2.13}$$

where P, F_G and f_X are probability function, cumulative distribution function (CDF) and joint probability density function (JPDF), respectively. Also, g is a probabilistic performance measure. Random space is bounded in this formulation by lower and upper bounds, $(x_1, x_2, ..., x_n)^L$ and $(x_1, x_2, ..., x_n)^U$, respectively.

The probabilistic constraint in an RBDO problem defines a feasible region by restricting probability of violating limit state $(G(x_1, x_2, ..., x_n))$ to an admissible failure probability $(i.e. \overline{P}_f = \Phi(-\beta_t))$ [13]. In other words, probabilistic constraint of an RBDO problem is evaluated so that failure probability of a system (i.e. $P_f = P[G(x_1, x_2, ..., x_n) \leq 0]$) is kept below than a predetermined level (i.e. $\overline{P}_f$). This accepted level of failure probability is calculated by using target reliability index (β_t).

A basic RBDO formulation consists of minimising the cost function under probabilistic constraints. An RBDO model is generally formulated in terms of design variable $x = (x_1, x_2, ..., x_n)$ as follows [13, 61, 173, 174, 191–193, 195]:

$$\begin{aligned}
&Min \quad Cost(x_1, x_2, ..., x_n) \\
&s.t. \ P_{f_j} \leq \overline{P}_{f_j} \qquad\qquad j = 1, 2, ..., npc \\
&(x_1, x_2, ..., x_n)^L \leq (x_1, x_2, ..., x_n) \leq (x_1, x_2, ..., x_n)^U
\end{aligned} \tag{2.14}$$

where the cost function can be any function in terms of design variable, P_{f_j} is failure probability of the j^{th} performance function and $\overline{P}_{f_j}$ is a given acceptable failure probability limit of the j^{th} performance function that is set on the basis of engineering knowledge and experience with respect to the previous designs. Also, $(x_1, x_2, ..., x_n)^L$ and $(x_1, x_2, ..., x_n)^U$ are the lower and upper boundaries of the design variable, respectively.

Although the design variable $(x_1, x_2, ..., x_n)$ may be an independent deterministic variable, probability distribution parameters (like μ_x) are also often considered as design variables of an RBDO problem. Further, upper and lower boundaries of design variable are typically known as deterministic constraints [13].

However, variables of an RBDO problem are indeed random variables. Their standard deviations are fixed in the entire process, while their expected values are changed in each iteration. In other words, expected values are assumed as design variables in an RBDO problem. Thus, $[x_1, x_2, ..., x_n] = [\mu(x_1), \mu(x_2), ..., \mu(x_n)]$ is design variable.

A general reliability index β_G, which is a function of probabilistic performance measure (g), is introduced as $F_G(g) = \Phi(-\beta_G)$ where Φ is the standard normal cumulative distribution function (CDF) [119]. Hence, the probabilistic performance measure (g) and also the general reliability index (β_G) can be formulated as a function of each other [173].

$$F_G(g) = \Phi(-\beta_G) \Longrightarrow g(\beta_G) = F_G^{-1}[\phi(-\beta_G)] \;\; and \;\; \beta_G(g) = -\phi^{-1}[F_G(g)]$$

Thus, we have [173]:

$$if \quad \beta_G = \beta_t \Longrightarrow \Phi(-\beta_t) = \overline{P}_f \quad \Longrightarrow \quad \beta_t = -\Phi^{-1}(\overline{P}_f) \tag{2.15}$$

$$if \quad \beta_G = \beta_s \Longrightarrow \Phi(-\beta_s) = P_f \quad \Longrightarrow \quad \beta_s = -\Phi^{-1}(P_f) \tag{2.16}$$

$$if \quad g = 0 \Longrightarrow F_G(0) = P_f = \Phi(-\beta_s) \tag{2.17}$$

$$if \quad g = g^* \Longrightarrow F_G(g^*) = \overline{P}_f = \Phi(-\beta_t) \tag{2.18}$$

where β_t is target reliability index. Also, β_s and g^* are a safety reliability index and a target probabilistic performance measure, respectively. A safety reliability index, which is denoted as $\beta_s = \beta_G(0)$, is often used to define a minimum level to assure system safety.

Therefore, the probabilistic constraint of an RBDO model can be expressed by using either of the following inequalities:

$$P_{f_j} \leq \overline{P}_{f_j} \tag{2.19}$$

$$F_{G_i}(0) \leq F_{G_i}(g^*) \tag{2.20}$$

$$\Phi(-\beta_{s_i}) \leq \Phi(-\beta_{t_i}) \tag{2.21}$$

$$F_{G_i}(0) \leq \Phi(-\beta_{t_i}) \tag{2.22}$$

The last expression is the most popular notation of the probabilistic constraint in the RBDO model.

In the following subsections, different categories of RBDO approaches will be reviewed.

2.4.2 Mono-Level RBDO Approaches

Mono-level approaches, also known as single-loop RBDO approaches, solve an RBDO problem in a single loop procedure. In these approaches, reliability analysis problem is avoided. The probabilistic constraints are replaced in a mono-level RBDO approach by the optimality conditions such that the RBDO problem will be reformulated in a single loop optimisation problem [13].

Karush-Kuhn-Tucker (KKT) optimality conditions in the RBDO, single-loop single-vector (SLSV) and approximate moment approach (AMA) are the main single-loop RBDO approaches [45, 118, 193].

In the KKT based approach, the probabilistic constraint of RBDO is replaced by KKT optimality conditions in the RBDO. Although the computational cost in this method can be reduced by parallel convergence in both design and random spaces, the reduction of the total cost is not very impressive.

Moreover, in another method design and random variables are combined in a hybrid optimisation space by multiplying the structural cost into the objective function of the first-order reliability method (FORM) [95].

Single-loop single-vector (SLSV) is another method based on an approximation of the limit-state function. In this method, the RBDO problem is converted to a deterministic optimisation problem by finding the minimum performance target point (MPTP) in terms of the target reliability index and limit-state derivatives [38].

Approximate moment approach (AMA) is another mono-level approach that is originated from the robust design optimisation concept [193]. In this approach, the first and second statistical moments (mean value and variance, respectively) are approximated to evaluate the probabilistic constraint. This is achieved by approximately matching statistical moments [45].

AMA does not require a reliability analysis problem, but second-order sensitivity analysis is needed resulting a large amount of computational effort. However, an inaccuracy in measuring the failure probability and numerical instability due to this inaccuracy are two major shortcomings of this probabilistic approach [193].

2.4.3 Two-Level RBDO Approaches

Two-level approaches (as direct solutions of RBDO problems) are based on an improvement of the traditional double-loop approaches by increasing efficiency of the reliability analysis loop. These approaches solve the RBDO problem in two nested loops that is a heavy task due to the nested non-linear procedures (reliability analysis and design optimisation).

These approaches consider the probabilistic constraints inside an optimisation loop where the inner loop is concerned with reliability analysis and the outer loop involves design optimisation. This category is also called double-loop approaches and includes reliability index approach (RIA) and performance measure approach (PMA) [13, 61]. The main concentration of this research is on double-loop RBDO approaches, with a especial focus on PMA.

The RIA uses a first-order reliability analysis problem in its inner loop that leads to repeated evaluations of the performance function [54]. In this approach, the limit-state function can also be expanded at a point with the highest probability, known as most probable failure point (MPFP) [135]. Although many methods have been proposed to reduce the cost of this approach, RIA still involves a high computational cost [69, 109, 147].

Moreover, a dual method has been proposed in order to approximate the limit-state function using a response surface. In this method, the failure probability is approximated by interpolation functions in terms of the design variables [66].

Due to shortcomings of the RIA, especially the high computational cost and also some difficulties in the numerical approach, which result in slow convergence or even divergence, performance measure approach (PMA) has been proposed. This approach converts the probability measure to a performance measure. A first-order inverse reliability analysis problem is used in the inner loop of the PMA [173].

The PMA tries to find a point that yields the minimum value of the performance function on the target reliability surface that is called minimum performance target point (MPTP). In fact, the PMA is created because minimising a complicated performance function under simple constraints is easier than minimising a simple cost function under complex constraints [13, 172].

2.4.4 Decoupled RBDO Approaches

Reliability analysis and design optimisation procedures are decoupled in decoupled RBDO approaches. In other words, the reliability analysis is not carried out within the design optimisation in these approaches. The RBDO problem is transformed to a sequence of deterministic design optimisation problems so that deterministic constraints are linked to reliability analysis problem [13].

One of the best decoupled RBDO approaches is sequential optimisation and reliability assessment (SORA), which is based on a transformation from the RBDO problem to a sequence of deterministic design optimisation and reliability cycles [52].

In fact, the probabilistic constraint of the RBDO is replaced by deterministic functions related to probable points in this approach. The SORA method improves a design point from cycle to cycle until convergence. This method is developed to improve efficiency of the probabilistic optimisation.

The SORA method uses a serial-loop strategy with a cycle of deterministic design optimisation and reliability assessment problems. In this method, design optimisation and reliability analysis are decoupled from each other in each cycle. In this approach, reliability analysis is generally done after deterministic design optimisation in order to verify feasibility of constraints under uncertainty [52].

Another decoupled approach is sequential approximate programming (SAP) that formulates an RBDO problem as a sub-programming problem. The probabilistic constraint is replaced by the first order Taylor series at the current design point. It can be seen in the SAP method that the RBDO problem is transformed to a sequence of approximate programming sub-problems [41].

As mentioned before, this PhD thesis mainly focuses on two-level (double-loop) RBDO approaches. These approaches are discussed in more details in the next section.

2.5 Double-Loop RBDO Approaches

Double-loop RBDO approaches include reliability index approach (RIA), which considers the cost reduction under the reliability constraints, and performance measure approach (PMA), which involves an inverse reliability analysis problem as an alternative constraint. The efficiency of these approaches depends on activeness of the probabilistic constraints [61, 173, 174].

It must be noted that in spite of some positive properties of probabilistic design optimisation methods, they often involve high computational cost due to existence of a double-loop procedure for overall optimisation and reliability assessment [34, 69, 122].

A typical double-loop RBDO solution process iteratively carries out a design optimisation in the original random space (X-space) and a reliability analysis of the performance function in the standard normal random space (U-space).

It has been found that PMA is robust and more efficient in evaluating inactive probabilistic constraints, while RIA is more efficient for evaluating violated probabilistic constraints and is also unstable for some problems [43]. Thus, it can be concluded that these approaches are not equivalent when solving various RBDO problems. Also, RBDO often yields a higher rate of convergence using PMA, while RIA yields singularities in some cases [173].

Moreover, a study of non-linearity of double-loop RBDO approaches can be carried out by observing nonlinearity of constraints in two optimisation problems (reliability analysis and design optimisation). Efficiency and robustness of RBDO process depend on non-linearity of reliability analysis and design optimisation problems.

Furthermore, it has been reported that different reliability analysis methods employed in the RIA and the PMA result different behaviours of non-linearity in these approaches [192].

In addition, performance function $G(x_1, x_2, ..., x_n)$ is itself a non-linear function that requires a complex engineering analysis. In general, a small non-linearity has been introduced in the PMA, while a significant non-linearity has been found in the RIA [44].

RIA is originated from reliability analysis and describes a probabilistic constraint as a reliability index, while PMA is originated from reliability-based design concept and converts a performance function to a performance measure [193].

The main difference between reliability analysis problems of the RIA and the PMA is that inner loop of RIA is intended to find the minimum distance of the failure surface $(i.e. G_U(u_1, u_2, ..., u_n) = 0)$ from the origin of the U-space, while inner loop of PMA aims at minimising standard normalised performance function on the target reliability surface $(i.e. \|(u_1, u_2, ..., u_n)\| = \beta_t)$.

Additionally, it has to be noted that in an RBDO model, the mean values of random parameters are often used as design variables and the variances are assumed to be fixed [173].

2.5.1 Probabilistic Constraint Evaluation

The reliability analysis of a system performance function is to evaluate the relationship between a reliability index (β) and the corresponding probabilistic performance measure (g). A generalised reliability index β_G, which is a non-increasing function of g, is often defined as below:

$$F_G(g) = \Phi(-\beta_G) \tag{2.23}$$

Thus, it can easily be shown that $\beta_G(g) = -\Phi^{-1}[F_G(g)]$ and also $g(\beta_G) = F_G^{-1}[\Phi(-\beta_G)]$. Since the system performance function is often not normally distributed, the $\beta_G \sim g$ relationship is generally non-linear [173].

As displayed earlier, probabilistic constraint of the RBDO model ($P[G(x_1, x_2, ..., x_n) \leq 0] \leq \Phi(-\beta_t)$) includes two inequality relationships. This constraint can be represented by a set of three simple constraints, where two inequality constraints are related to each other through an equality constraint [173]. These three constraints are written as follows:

$$\beta_G \geq \beta_t \qquad (i)$$

$$g \geq 0 \qquad (ii)$$

$$F_G(g) = \phi(-\beta_G) \qquad (iii)$$

Inequality constraints (i) and (ii) represent the limit-state of reliability index and probabilistic performance measure, respectively. Also, equality constraint (iii) represents the non-increasing $\beta_G \sim g$ curves.

Thus, as shown in Figure 2.1, $\beta_G - g$ space is divided into four regions, as following:

1. **Active Point:** $\beta_G = \beta_t \;\; and \;\; g = 0$

2. **Feasible Region:** $\beta_G \geq \beta_t \;\; and \;\; g \geq 0$

3. **Infeasible Region:** $\beta_G \leq \beta_t \;\; and \;\; g \leq 0$

4. **Ambiguous Regions:** $(\beta_G - \beta_t).g < 0$

In this regard, a probabilistic constraint can be evaluated by finding any point on the curve that is outside the ambiguous regions [173] as below:

1. **Active Constraint:** The probabilistic constraint is active, if the corresponding $\beta_G \sim g$ curve passes through the active point.

2. **Inactive Constraint:** The probabilistic constraint is inactive, if the corresponding $\beta_G \sim g$ curve passes through the feasible region.

3. **Violated Constraint:** The probabilistic constraint is violated, if the corresponding $\beta_G \sim g$ curve passes through the infeasible region.

In other words, a given design is feasible/infeasible, if the corresponding $\beta_G \sim g$ curve passes through the feasible/infeasible region [173].

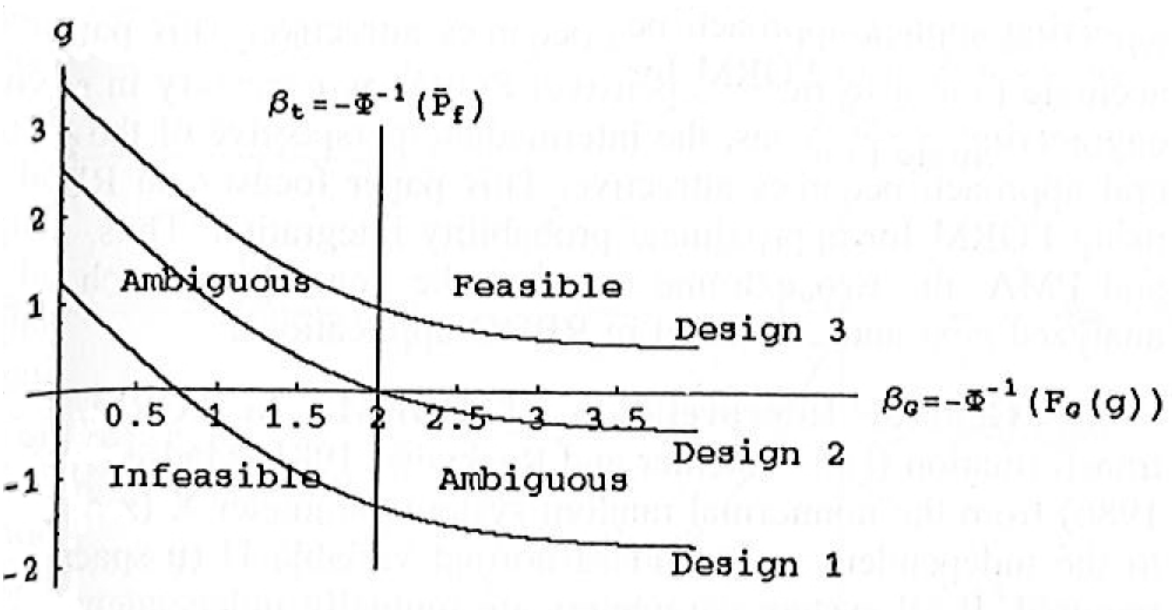

Figure 2.1: General Interpretation of Probabilistic Constraint

A probabilistic constraint may be evaluated by using any point between $(\beta_s, 0)$ and (β_t, g^*). These two points are often used for this purpose as they are known as good points of RIA and PMA, respectively. Suppose that (β_a, g_a) is the intended point between two mentioned good points such that $g_a = \alpha g^*$ and $\beta_a = \alpha\beta_t + (1-\alpha)\beta_s$ where $0 \leq \alpha \leq 1$.

Obviously, if $\alpha = 0$, then $g_a = 0$ and $\beta_a = \beta_s$ that shows RIA. Also, if $\alpha = 1$, then $g_a = g^*$ and $\beta_a = \beta_t$ that shows PMA. In other words, the point (β_a, g_a) shows the MPFP in the RIA and the MPTP in the PMA if α is either zero and one, respectively. These points are shown in the Figure (2.2).

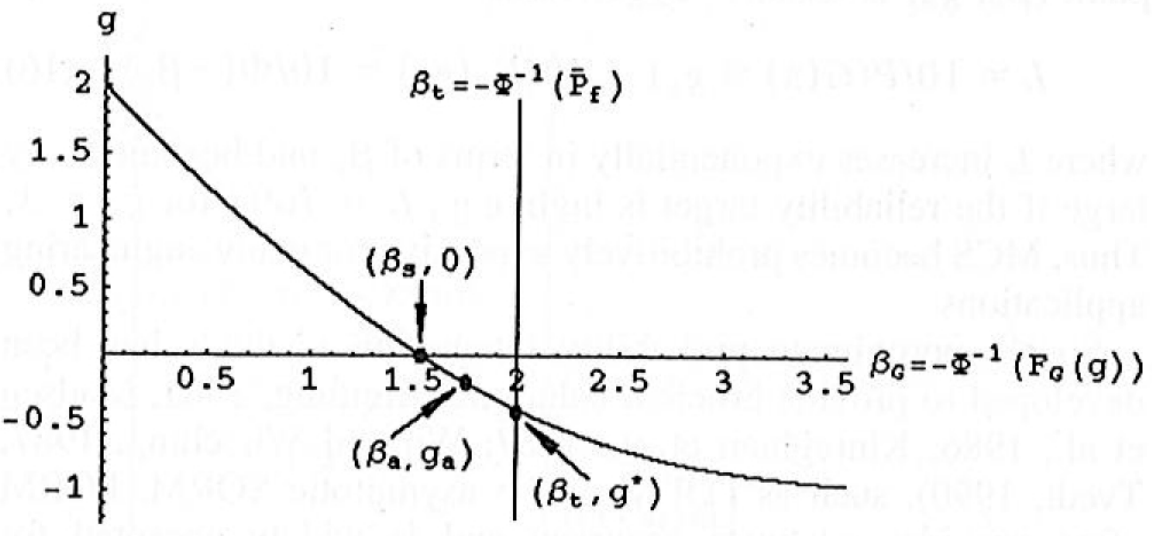

Figure 2.2: Illustration of Probabilistic Constraint Evaluation

In this case, regarding $\beta_G(g) = -\Phi^{-1}[F_G(g)]$ and $g(\beta_G) = F_G^{-1}[\Phi(-\beta_G)]$ and also using the Taylor Series expansion, we will have:

$$\beta_G(g) = \beta_G(g_a) + \sum_{n=1}^{\infty} \frac{d^n \beta_G}{dg^n}(g_a).\frac{(g-g_a)^n}{n!}$$

and

$$g(\beta_G) = g(\beta_a) + \sum_{n=1}^{\infty} \frac{d^n g}{d\beta_G^n}(\beta_a).\frac{(\beta_G-\beta_a)^n}{n!}$$

(Note that $\beta_G(g_a) = \beta_a$ and also $g(\beta_a) = g_a$). In fact, the Taylor Series have been expanded for β_G and g at g_a and β_a, respectively.

Since high order derivatives in the above Taylor Series expansions are difficult to obtain, the m^{th} order approximation is often used (if $m = 1$, then first-order approximation refers to first-order reliability method (FORM)). Hence, the point (β_a, g_a) can sufficiently identify the limit-state of a probabilistic constraint [173].

Assuming $g = 0$ (in RIA) and $\beta_G = \beta_t$ (in PMA) in the Taylor Series expansion, for the first order reliability method (FORM), we will obtain (respectively):

$$\beta_G(0) = \beta_a + (0 - g_a).\nabla_g \beta_a \geq \beta_t \Rightarrow \beta_a - g_a.\nabla_g \beta_a \geq \beta_t \tag{2.24}$$

$$g(\beta_t) = g_a + (\beta_t - \beta_a).\nabla_\beta g_a \geq 0 \tag{2.25}$$

On the other hand, the computational efforts associated with RIA (using first order reliability analysis) and PMA (using first order inverse reliability analysis) cannot be easily quantified, since RIA and PMA are searching different points.

The computational difference between RIA and PMA becomes significant if $u^*_{g=0}$ and $u^*_{\beta=\beta_t}$ are far apart in the U-space, while the exact status of probabilistic constraint is unknown until $u^*_{g=0}$ or $u^*_{\beta=\beta_t}$ is finally found.

Generally, it is easier to find a point that is closer to the origin of the U-space (searching in a more restrictive solution space) [173]. Thus, the estimations of the computational efforts associated with the RIA and the PMA can be established as following:

1. If $\beta_s < \beta_t$, then $u^*_{g=0}$ (RIA) is closer to the origin. In this case, the probabilistic constraint passes through the infeasible region and is violated.

2. If $\beta_s > \beta_t$, then $u^*_{\beta=\beta_t}$ (PMA) is closer to the origin. In this case, the probabilistic constraint passes through the feasible region and is inactive.

3. If $\beta_s = \beta_t$, then RIA and PMA search the same point ($u^*_{g=0} = u^*_{\beta=\beta_t}$). In this case, the probabilistic constraint passes through the active point and is active.

However, it is often assumed that g and β_G are fixed numbers in the RIA and the PMA, respectively. Thus, the RBDO problem should be solved (using FORM) so that β_G and g are obtained. In this case and for the RIA and the PMA we have:

$$RIA : g = 0 \Rightarrow \beta_G(0) = \|u^*_{g=0}\| = \beta_s$$

$$PMA : \beta_G = \beta_t \Rightarrow g(\beta_G) = G(u^*_{\beta=\beta_t}) = g^*$$

The next subsections explain further details inside the RIA and the PMA, respectively.

2.5.2 Reliability Index Approach

Reliability index approach (RIA) tries to find a point on the failure surface ($G_U(u_1, u_2, ..., u_n) = 0$) in the reliability analysis loop in order to obtain the required change to update the current design point. RIA looks at the probabilistic constraint as a reliability index and originates from the reliability analysis concept [45, 54, 135].

In a general RIA formulation, we have:

$$F_G(0) = \phi(-\beta_G(0)) = \phi(-\beta_s) \leq \phi(-\beta_t) \Longleftrightarrow \beta_t \leq \beta_s \tag{2.26}$$

The RIA yields a singularity in two cases:

1. When the $\beta_G \sim g$ curve is always positive (the curve is completely above the β_G axis), g can never be zero. In this case, the failure probability of the system is zero ($g > 0 \Rightarrow P_f = 0$).

2. When the $\beta_G \sim g$ curve is always negative (the curve is completely below the β_G axis), g can never be zero. In this case, the failure probability of the system is one ($g < 0 \Rightarrow P_f = 1$).

These occur because the reliability index (β_s) tends to positive and negative infinity, respectively, and hence the point $(\beta_s, 0)$ does not exist.

In other words, reliability analysis problem in RIA may fail to have a solution whenever the corresponding failure surface in the U-space ($G_U(u_1, u_2, ..., u_n) = 0$) is outside the infinite probability integration domain; i.e. $\beta = \|(u_1, u_2, ..., u_n)\| = \infty$ [173].

RIA uses the first order reliability method (FORM) approximation to perform reliability analysis where the probabilistic constraints are replaced by reliability index constraints.

Hence, the probabilistic constraint of RIA can generally be written as below [13, 135]:

$$\beta_G(0) = -\Phi^{-1}[F_G(0)] \geq \beta_t \Rightarrow \beta_s \geq \beta_t \tag{2.27}$$

Due to sequential changes of optimal point and the MPFP, RIA leads to a slow convergence scheme and also zigzagging [61]. RBDO method using the conventional RIA is known as a good approach that requires a large computational time [173].

Moreover, it must be mentioned that a modified version of RIA has been proposed in the literature to improve stability and efficiency of the existing traditional RIA. In the modified RIA, reliability index is redefined and then drawbacks related to convergence of RIA are found and further studied in order to consider non-normally distributed design variables [111].

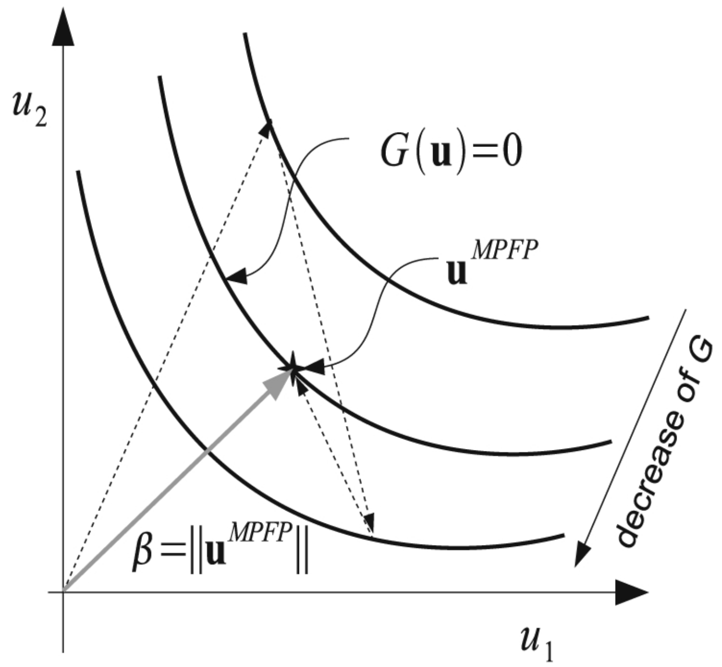

Figure 2.3: Reliability Analysis in RIA

However, if a quadratic programming problem was used in the design optimisation loop of RIA in order to calculate design change, then RBDO problem is actually solved by two nested sub-problems as below:

1. Inner loop; reliability analysis problem: This problem aims at calculating the minimum distance of the limit-state function $G_U(u_1, u_2, ..., u_n) = 0$ from the origin of the U-space that is called safety reliability index. This concept is displayed in the Figure (2.3).

   Reliability analysis problem of RIA is often formulated as follows:

$$Min \quad \|(u_1, u_2, ..., u_n)\|$$

$$s.t. \quad G_U(u_1, u_2, ..., u_n) = 0$$

2. Outer loop, design optimisation problem: This problem finds a search direction for updating the current design point. It has commonly been accepted that a constrained minimisation problem is used in outer loop of RIA.

   This problem often includes a quadratic objective function. Constraint of the mentioned quadratic programming problem is $\beta_s \geq \beta_t$.

Safety reliability index β_s is used as the RIA probabilistic constraint in the design optimisation loop. β_s is a simple n-dimensional quadratic function in the U-space and should inversely be transformed into the X-space to perform a design optimisation problem. Also, the objective function of the reliability analysis problem in RIA does not involve a non-linear transformation of probability distributions.

Thus, the inverse transformation in the probabilistic constraint evaluation introduces additional non-linearity for all probability distributions, except the Gaussian distribution that requires a linear transformation [192].

Generally, it has been reported that RIA fails to converge for probability distributions with bounds (such as Uniform distribution) and extreme type distributions (like Gumbel distribution) in which the infinite integration domain may not include a failure surface. Therefore, reliability index approach (RIA) depends so much on the non-linear transformation that does not yield a good RBDO tool [44, 135].

A preferred method for evaluating the probabilistic constraint of RIA is the Hasofer and Lind - Rackwitz and Fiessler (HL - RF) method, whereas any general optimisation algorithm, like sequential linear programming (SLP) and sequential quadratic programming (SQP) can be used as well [135, 193].

2.5.3 Performance Measure Approach

The performance measure approach (PMA) has been established on this fact that it is easier to minimise a complex cost function subject to a simple constraint function than to minimise a simple cost function subject to a complicated constraint function.

In other words, the PMA with a spherical equality constraint is easier to solve than RIA with a complicated equality constraint when evaluating the probabilistic constraint of an RBDO problem [45, 172, 195]. The PMA converts the performance function into a performance measure and is originated from the reliability-based design concept [45].

In the inverse reliability analysis problem of PMA, the probabilistic constraint must be replaced by a new optimisation problem, which minimises the standard normalised performance function $G_U(u_1, u_2, ..., u_n)$ as a cost function. The optimum solution has to satisfy the spherical equality constraint $\|(u_1, u_2, ..., u_n)\| = \beta_t$ so that β_t is a target reliability index.

Further, an enhanced PMA has been introduced in the literature in order to improve PMA's computational efficiency. In this approach, which is very useful when applied in large-scale system problems, numerical efficiency has been improved by a reduction in the number of required iterations in an RBDO problem.

Probabilistic constraints are efficiently evaluated in the enriched PMA, which is also called PMA+, by reusing some information obtained in previous RBDO iterations [194].

However, the probabilistic constraint of RBDO using PMA can generally be written as below:

$$g(\beta_t) = F_G^{-1}[\Phi(-\beta_t)] \geq 0 \Rightarrow g^* \geq 0 \tag{2.28}$$

where g^* is target probabilistic performance measure. This constraint may be linearised in order to use in the design optimisation loop for estimating a new search direction as follows:

$$G_i(x_i^*) + \nabla G_i(x_i^*)^T.D \geq 0$$

where D is the design change.

Therefore, supposing that a quadratic programming problem has been used in design optimisation loop, two nested sub-problems in the PMA can be summarised as follows:

1. Inner loop; reliability analysis problem: This problem aims at calculating the minimum amount of the standard normalised performance function $G_U(u_1, u_2, ..., u_n)$ on the target reliability surface. This idea is displayed in the Figure (2.4).

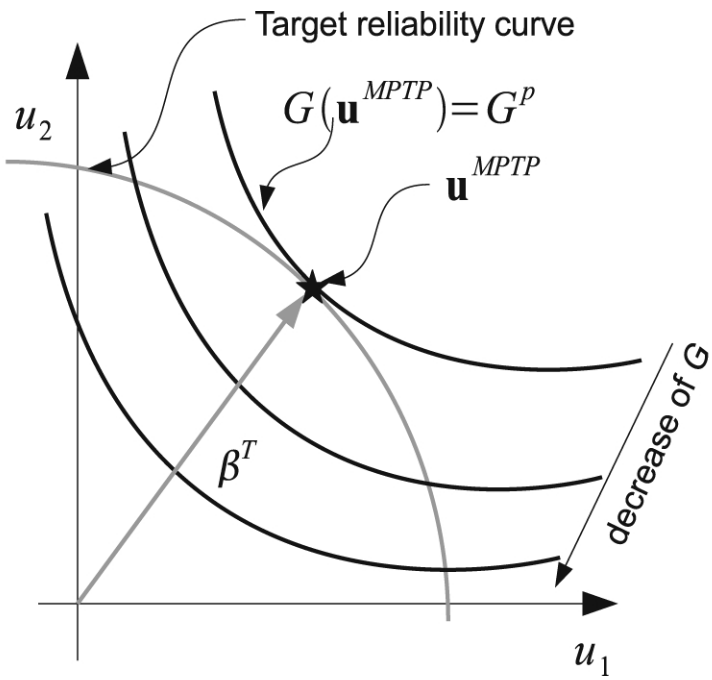

Figure 2.4: Reliability Analysis in PMA

Reliability analysis problem of inner loop of PMA is often formulated as follows:

$$Min \quad G_U(u_1, u_2, ..., u_n)$$

$$s.t. \quad \|(u_1, u_2, ..., u_n)\| = \beta_t$$

2. Outer loop; design optimisation problem: This problem estimates a design change for updating the current design point. This goal is often obtained by solving a constrained minimisation problem in which objective function is quadratic. Constraint of this problem is written as $G_i(x_i^*) + \nabla G_i(x_i^*)^T.D \geq 0$.

The PMA has an inverse reliability analysis problem where the probabilistic constraint is transformed to a performance measure corresponding to the target reliability level.

This approach goes first to the hyper-sphere with a radius equal to the target reliability index (β_t), then iterations are carried out on this hyper-sphere. PMA is increasingly used for the large-scale systems [13, 61].

Although a general optimisation algorithm, such as sequential linear programming (SLP) and sequential quadratic programming (SQP), can be used for evaluating the probabilistic constraint in PMA, there are many efficient particular algorithms for this purpose, like advanced mean value (AMV), conjugate mean value (CMV) and hybrid mean value (HMV) methods.

It has been reported that PMA using the HMV method provides the best result in the RBDO problem [193]. Also, it can be concluded that PMA is robust because the point (β_t, g^*) always exists [173].

The constraint of reliability analysis problem in PMA is a simple n-dimensional quadratic function without any non-linear transformation, while the cost function involves a non-linear transformation. Also, since the probabilistic constraints of the PMA are the original performance function evaluated at $x^*_{\beta=\beta_t}$, there is no non-linear transformation in design optimisation loop of PMA [192].

PMA is much less dependent on the non-linear transformation and thus can handle a variety of probability distributions without significantly increasing number of function evaluations [192]. Also, it can be seen that at the solution point u^*, the limit-state function is tangent to the hyper-sphere with radius β_t [13].

2.6 Optimisation in Electricity Power Networks

Mathematical optimisation is increasingly applied in a wide range of real world problems such as engineering, economics, the health sciences, etc. In this regard, electricity power networks utilize optimisation approaches in order to find optimum amounts for various objectives in economical aspects (cost) and/or electrical engineering aspects (power, voltage, etc.).

Electricity networks are one of the most complex systems ever known. There are many optimisation models available for these networks. These models include mono- and multi-objective optimisation problems [33, 58, 64, 65, 81, 93, 98, 125, 176].

An electricity power network generates, controls, transmits and finally consumes electrical power. Electricity is produced in generators, transformed to an appropriate voltage level in transformers and then distributed via buses on transmission lines for final distribution to customers. Figure (2.5) displays a common view of electricity networks.

Power is injected into a bus from generators, while loads are tapped from it. Some buses may have no generation facility. The surplus power at a bus is transported via transmission lines to buses which have deficit in power [99].

A mathematical model must be formulated for power networks in order to predict the flow picture. This model is power flow equation (PFE). Substations and loads on feeders are considered as sources and demands in an optimisation problem of electricity networks, respectively [130, 175].

In general, nodes and line loads are treated as sources and demands in an effective model, respectively. There are four major approaches for planning an electricity power network. These approaches can briefly be explained as below [175]:

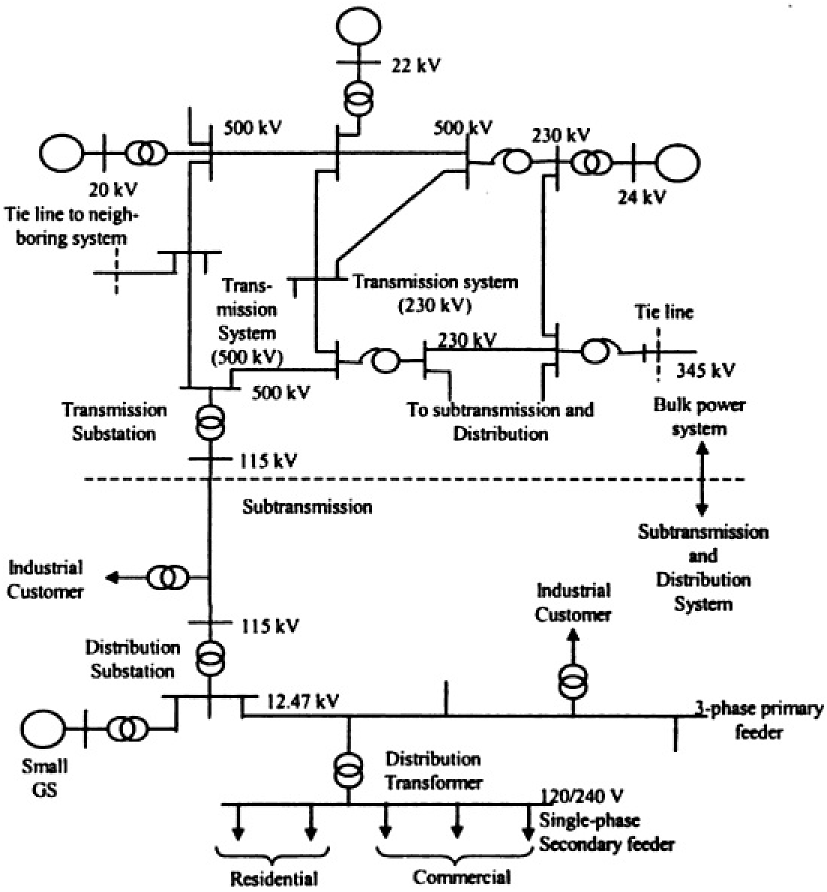

Figure 2.5: A General Electricity Power Network

1. The alternative policy method that compares a number of policies and selects the best.

2. The decomposition approach in which a large optimisation problem is divided into several smaller subproblems.

3. The linear programming and the integer programming methods.

4. The dynamic programming method.

Constraints of a problem are also constructed based on the assumptions below [175]:

1. The required load (by consumers) should be supplied all the time.

2. The transmitted power in each line should not be more than its thermal power capacity.

3. Substations installed in a given area should be less than the accepted number of substations in that area.

4. The power flow in the lines should be unidirectional.

5. Total undelivered energy must be kept less than an accepted rate.

6. The power flow cannot be negative; i.e. there cannot be a flow from the demand node to the supply node.

Reducing electricity losses in networks is another important issue in electrical energy management. Cost of electricity is reduced and efficiency of a network is improved by reducing electricity losses [55, 60].

These losses are typically categorised as technical losses (such as losses due to physical processes) and non-technical losses (such as unauthorized line tapping or meter by passing) [51]. Technical losses are often less than non-technical losses [39, 97].

In the following subsections, more details of an electricity network and its optimisation model will be provided.

2.6.1 Components and Subsystems of an Electricity Network

There are different components and subsystems in a typical electricity power network. An electricity network is often divided into three subsystems while each sub-system is able to affect the overall system's behaviour. These subsystems are as below:

1. Generation subsystem;

2. Transmission subsystem;

3. Distribution subsystem.

A common structure for an electricity network includes the following properties [130]:

1. Electric power is generated in generation subsystems.

2. Generated power is transmitted from generators to load centers through transmission subsystems.

3. Low voltage networks that deliver the generated electricity power to consumers are distribution subsystems.

4. Tight tolerance levels of voltage and frequency are used to ensure a high quality product.

Generator voltages are usually in the range of 11 to 35 kV. These are stepped up to transmission voltage level. A transmission system connects all major generating stations (GSs) and main load centers in the system. Transmission system is often regarded as the power system's spine and operates at the highest voltage levels (+230 kV) [130].

Transmission subsystem consists of transmission lines, transformers and switching devices. When generated power is transmitted to transmission substations, voltages are stepped down to a range from 69 to 138 kV, which is called sub-transmission level.

Sub-transmission systems often supply large industrial customers directly. This system transmits power from transmission substations to distribution substations at a lower voltage and in smaller quantities. Further, a set of generation and transmission subsystems is often called a bulk power system.

Distribution system performs the last step of power transformation by delivering electricity power to the customers described below [130]:

1. Primary feeders supplying small industrial customers (4 to 34.5 kV).

2. Secondary feeders supplying commercial and residential customers (120/240 V).

Generally, buses in a power flow problem are often shown using PQ, PV and S notations. PQ buses enforce active and reactive power equations and PV buses enforce active power and voltage magnitude equations. S buses, which are also called slack buses, enforce specified values of V_{d_k} and V_{q_k}.

All equations of power flows and voltage magnitude must be satisfied at all buses of an electricity power network. However, within solution procedure of a power flow problem, only two equations (within the set of all possible equations) are directly enforced at each bus [107].

Moreover, a controlled electric vehicle (EV) has been considered as a possible utility to eliminate or reduce voltage disturbances. This can be modelled as a multi-period, unbalanced load flow and rolling optimisation method. It is intended in this method to focus on different rates and times with minimum cost subject to certain constraints [136].

A strategy for interactions between commercial buildings and a smart grid is proposed based on building power demand management [190].

2.6.2 General Optimisation Model

An optimisation model of electricity power networks based on a radial network model is, in general, a non-linear programming problem (non-linear objective function with non-linear constraints) which should be solved with consideration to optimality conditions. Difficulties in optimal planning of electricity networks are originated from two different sources: non-linear nature of the models and large number of scenarios [2, 16].

Although various objectives, such as total cost, network area and voltage drops, have often been considered for optimisation in electricity power networks [55, 60, 138, 161, 165, 183], there are two major goals in a typical optimisation model of electricity networks [98, 128]:

1. Determining the optimum number and locations of distribution substations;
2. Finding an optimum way of connecting load nodes to these substations through interconnection of feeders.

A general optimisation model of electricity networks minimises total cost. Manufacturing cost, instalment cost and maintenance cost are often considered as different components of total cost [60, 98]. Further, total cost is sometimes defined as a summation of fixed and variable costs where fixed costs include costs of construction of nodes and lines (manufacturing cost) and instalment cost. Also, operation, maintenance and energy losses costs are often considered as variable costs [96].

Further, technical constraints in an electricity network optimisation problem are summarized as follows:

1. The first Kirchhoff's law in the existing nodes of a power network;

2. Restrictions of power transport capacity for each line;

3. Restrictions of power supply capacity associated with substations;

Many optimisation algorithms have been applied into electricity power networks. A short list of these algorithms is as below [98, 128, 138]:

1. Mixed Integer Linear Programming (MILP)

2. Ant Colony (AC)

3. Genetic Algorithms (GA)

4. Tabu Search (TS)

5. Branch Exchange (BE)

6. Simulated Annealing (SA)

7. Bender's Decomposition (BD)

8. Particle Swarm Optimisation (PSO)

Mono- and multi-objective optimisation models are applied in electricity networks. Multi-objective models are intended to consider various objectives simultaneously. For instance, total cost and network reliability are taken into account in these models. These models will be discussed in more details in the next section.

However, a mono-objective optimisation model of electricity power networks is concerned with cost only. A typical optimisation model for these networks is formulated as follows [2, 33, 113, 120]:

$$
\begin{aligned}
& Min \quad Cost(d) \\
& s.t. \quad f_i(d) = 0 \qquad i = 1, 2, ..., m \\
& \qquad\quad h_j(d) \leq 0 \qquad j = 1, 2, ..., p
\end{aligned}
\tag{2.29}
$$

where $d = [d_1, d_2, ..., d_n]$ are the design variables and $f(d)$ and $h(d)$ are deterministic constraints.

2.6.3 Other Optimisation Problems

Different objectives have been considered for power networks optimisation in the existing literature. These objectives and their associated optimisation problems are discussed in this subsection. A number of well-known minimisation problems in electricity networks can be found in [15, 77, 80, 105, 158].

Static or dynamic planning can be used to formulate an optimisation problem for an electricity power network. A static approach is based on one-step planning, and a dynamic approach plans a network with load growth at the existing nodes [128].

As previously mentioned, there are two kinds of costs in a network, namely fixed cost and variable cost. In this case, distance between start and end nodes of a link is often considered as fixed cost of the link while value of transmitted power is known as variable cost. It should be noted that each source has a maximum limit of the power supply [3, 128].

One of the existing approaches for generation cost reduction in electricity networks is topology control. This approach is also called optimal transmissions switching (OTS) [59, 63, 137, 141]. It has been reported in the literature that network congestion, which is often created by line thermal limits or nominal voltage requirements, may be reduced or even eliminated by changing topology of a power network [148, 156].

A usual difficulty in electricity networks is power quality disturbance, such as voltage sag, voltage swell and harmonic distortion. An accepted method to protect conventional and sensitive loads against these disturbances is to use custom power devices (CPD) [58].

The proper placement of CPDs has an important effect on quality improvement and ensures that total cost is minimal in accordance with maximum efficiency. The CPDs can be installed for an individual customer or a group of customers. Further, it's accepted that the best solution to improve quality, reliability and availability (QRA) is to fit a network with proper types of CPDs.

Central improvement and distributed improvement are two widely used methods for locating CPDs. It has been reported in the literature that it is preferred to apply a distributed improvement configuration and a central improvement should be used at the same time [58].

Determining the optimal location and size of CPDs is the main goal of many radial electricity networks. These networks are formulated so that power quality improvement is maximised while minimising total cost.

A function of attributed cost of low power quality delivered to customers (C_{LPQ}), cost of CPDs including investment, operation and maintenance costs (C_{CPD}), utility revenue due to installation of CPDs (C_{UTI}) and penalty cost charged to the utility due to the lack of the regulatory targets (P_{UTI}) is often considered as objective function of an optimisation model in which it is intended to locate CPDs optimally [58].

Constraints of the mentioned optimisation model must be well-defined so that all variables are considered based on their real applications. These constraints are often defined as follows:

1. Bus voltage limit: $V_{min} \leq |V_i| \leq V_{max}$ where V_i is voltage magnitude of i^{th} bus.

2. Frequency of voltage sag: $VS_i \leq VS_{i-Limit}$ where VS_i is i^{th} bus voltage sag frequency.

3. CPD rating limit: $S_{CPD} \leq S_{CPD-max}$ where S_{CPD} is the maximum power rating of CPDs.

4. Total harmonic distortion (THD) limits: $THD_i \leq THD_{max}$ where THD_i is i^{th} bus total harmonic distortion.

When a suitable objective function and a set of constraints are determined, a stopping criteria should be defined based on improvement of the QRA level. If the obtained solution cannot satisfy the stopping criteria, the procedure should be iterated until the goal is reached.

Different optimisation methods (such as gradient-based search algorithms, dynamic programming technique, artificial neural network, hybrid optimisation algorithms, genetic algorithm, particle swarm optimisation and simulated annealing) are applied to solve optimal CPD placement problems [58].

Meanwhile, partitioning techniques are widely used to solve electricity networks optimisation problems. One of these techniques is the Bender's decomposition (BD) algorithm, which decomposes problem into two sub-problems and solves problem using an iterative process [98].

A master sub-problem is first formulated in the BD algorithm as a mixed integer non-linear programming (MINLP) problem in order to determine radial topology of distribution network. In this step, the unserved energy cost is minimised subject to all constraint. The objective function has four items including investment cost, power losses cost, unavailability cost and infeasibility cost.

Then, a slave sub-problem is formulated as a non-linear programming (NLP) problem that is used to check feasibility of the master sub-problem's solution and provide an optimal value for operation variables. A common method to make a problem feasible is to add slack variables to the problem so that these slack variables should be zero in the last iteration. Hence, the objective function of this step is a summation of slack variables.

When the solution obtained by the master problem is feasible and value of the objective function computed in the slave problem (i.e. all slack variables) is zero, the BD algorithm is stopped [98].

Further, a convex geometric programming problem is used in the literature to approximate lengths of lines between nodes. It has been reported that lengths of lines on each voltage level is the sum of length of links (connections between consumers and also between feeding nodes and consumers) [93].

The objective function of this model includes all system costs, including investment cost of substations and lines, cost of losses in substations and lines, non-delivered energy cost and maintenance cost. The objective function is formulated so that system configuration is determined by variables of the cost function. Also, non-negativity of variables, over-loading of transformers and lines, voltage-drop and minimal and maximal ratings for various equipments are often considered as constraints of this problem.

This model is based on uniformly distributed variables and is solved using a random search method. This problem mainly includes a nonlinear multi-objective cost function with nonlinear constraints. Physical feasibility of a system, which means variables should have realistic values, is another important issue that is considered in this problem [93].

2.7 Uncertainty Considerations in Electricity Networks

Many acceptable results have been obtained by minimising the total cost of an electricity power network. However, it's reported that a mono-objective optimisation model cannot yield a compromise result between cost and reliability.

Reliability is one of the most important and complicated issues in electricity networks. Thus, it is accepted that cost minimisation alone cannot be assumed as a comprehensive goal to achieve in optimisation projects and it has been found that a better response will be obtained if uncertainties are considered in the problems [58, 71, 145].

A factor in electricity networks that has a major influence on system, but which is not being observed or cannot be predicted with certainty is generally called an uncertainty. Risk is the hazard to which a utility is exposed because of uncertainty. Uncertainties result in risks. It is widely accepted that simple optimisation is ineffective when there are uncertainties in a system as well as multiple objectives [125].

Maximisation of system loadability has also been considered to propose a new algorithm for electricity power networks reconfiguration [10].

Another basic concept, which must be defined carefully by engineers, is system failure. This definition is used to define a safety level for a system regarding an acceptable level of system performance. For instance, line flow magnitudes must remain below established levels and voltage magnitudes must remain within set limits.

Defining the failure probability of each line in a network as well as the whole system is another way to deal with uncertainties in electricity networks. In this case, failure probability is calculated based on the theory of probability using a predetermined limit of transmission capability for each line or for the whole of the system [151].

A scenario is also a complete set of specified variables (both options and uncertainties), which determines a set of specified options combined with a particular set of outcomes of uncertainties [125].

Uncertainty is often represented by a set of scenarios. Stochastic characteristics are often represented as a set of scenarios. Each scenario is a sequence of possibilities [35].

Uncertainties are significant features of electricity networks. Representation of a network's input data as random variables is one of the accepted approaches to gather sources of uncertainty in a system [151].

Probabilistic load flow is generally defined as a solution of the load flow problem [30]. Random variable values are often assumed as starting data to estimate solutions of the load flow problem. Simulation techniques (such as the Monte Carlo Simulation method) which use deterministic algorithms and analytical techniques (like the method of cumulants) that are based on random variables are widely accepted to solve these problems [12, 108, 150, 152]. Monte Carlo simulation is applied into reliability analysis of an electric power system [154].

The best property justifying the use of above techniques is their computational efficiency [131, 201]. Newly developed electricity networks have increased the need of considering reliability issued in optimisation models [153]. Since various factors influence system reliability, it is very hard to decide how to determine reliability of a system in an optimisation problem [151].

Availability of power plants, loads at nodes, lines out of service, nodes out of service, weather, season, day of the week and hour of the day are different sources of uncertainty in an electricity network [151].

Risk must be considered by attributes such as cost of electricity, capital requirements and environmental effects. Risk is a characteristic of decisions with two dimensions [125]: 1. the likelihood of making a regrettable decision 2. the amount by which the decision is regrettable.

The definition of reliability is one of the most important concepts in power systems. Experts often have different opinions about this definition and its applications, even within one technical field.

However, system reliability in a power network is generally defined as below [36]:

$$R(t) = P([0, t])$$

where P is a probability function and $[0, t]$ is a time period when system does not fail.

It is assumed that failure is indicated by the inability of an item to carry out its particular function. Unreliability or failure is the complement of reliability. An electricity system reliability can be assessed based on the following items [36]: 1. System configuration; 2. System components' reliability; 3. Power delivery to loads of system.

It is reported in the literature that the increasing necessity to deal with uncertainties is one of the significant sources of difficulty and complexity in the optimal planning of electricity power networks. However, the non-linear nature of networks optimisation models, as well as the need to consider a large number of scenarios, have the effects of making these models much more complicated [98].

Uncertainties in electricity power networks can be found in customer demand and failure probability of system. These uncertainties create risk in a system [151].

Reliability of electricity power networks is an important issue that may cause extra difficulties in an optimisation problem [2, 50, 65]. There are generally two approaches to consider uncertainties in electricity networks. Approaches that are based on the probability theory can be summarized as below [125]:

1. If probability distributions are known and problem being studied is consistent with the law of large numbers, uncertainties are often modeled probabilistically. In these approaches, uncertainties are modeled using probability distributions based on the statistical data such as expected values and variances.

2. If probability distributions are not available, then uncertainties can be modeled as unknown-but-bounded variables. This approach does not have a probabilistic structure for uncertainties and contains less information than a probabilistic approach.

Another approach to consider reliability of an electricity network is introduced in the literature as a simplified version of security constraints [31].

As mentioned in this section, various optimisation models and techniques have been introduced for electricity power networks in order to cover reliability issues in the corresponding optimisation models.

However, no probabilistic constraint has yet been introduced for these networks and also there is no reliability-based design optimisation (RBDO) model available for electricity power networks. We introduce an RBDO model for electricity networks in this thesis that considers reliability of the network by evaluating a probabilistic constraint.

2.7.1 Uncertainty Indices

Several indices are suggested by scholars to investigate reliability in electricity networks. A robustness index, which reflects a degree of confidence in or the adequacy of a given plan, is one of the widely accepted indices. In this case, adequacy (the degree of confidence) of each solution is calculated as below [98]:

$$\beta = 1 - \alpha_{max}$$

where α_{max} is the maximum of any possible constraint violations.

In other words, every obtained solution is used to investigate system uncertainties. For this purpose, a robustness index should be calculated for each solution. A given plan is robust with respect to a specific constraint if the constraint holds true for every possible value of the particular variables and constants. In this case, $\alpha_{max} = 0$ and $\beta = 1$.

If any instance of obtained quantities leads to a violation of constraints, β equals the maximum possibility for which the constraint is not violated. The global robustness β is obtained from the minimum value of β_k^j among all n lines [98]:

$$\beta = min(\beta_1^j, ..., \beta_n^j)$$

Reliability of electricity networks is also measured by investigating quality of supply and perceived power. Further, load demand and power injection are two uncertainty sources in electricity networks that have been integrated using a fuzzy power flow and its indices [166].

A new method is introduced in the literature based on the particle swarm optimisation (PSO) algorithm in which reliability of electricity network is considered using a multi-objective model. Reduction/Minimisation in real power losses and improvement/maximisation of electricity network reliability while reducing/minimising non-delivered energy are major aims of this model [11, 128].

Infeasibility rate (IR) is used in the new PSO algorithm as a criterion for choosing a lossless and reliable network as the best network. In other words, IR is a criterion analyzing the number of times that network becomes infeasible. IR is considered as a probability of infeasibility, and the acceptable value for IR is less than 20% [128].

In the mentioned new PSO algorithm, the particle movement has two major components:

a stochastic component and a deterministic component. A particle is attracted toward the position of the current global best (p_i^t) and its own best location (x_i^t), while at the same time it is allowed to move randomly. In this algorithm, a global optimum point (not just a local) is available for all particles [128].

The main aim is to find the global best (g_i^t) among all the current best solutions. It is supposed that x_i^t and v_i^t are the position and velocity vectors, respectively. Thus, new vectors are obtained as following:

$$x_i^{t+1} = x_i^t + v_i^{t+1} \qquad i = 1, 2, ..., n$$

where n is the number of particles and also we have:

$$v_i^{t+1} = w.v_i^t + c_1.r_1(p_i^t - x_i^t) + c_2.r_2(g_i^t - x_i^t)$$

where w is a weight coefficient, c_1 and c_2 are fixed coefficients and r_1 and r_2 are fixed numbers between 0 and 1 [128].

The overall system failure probability is also used in order to determine reliability index of a system. Since it is not yet possible to compute reliability index exactly, only its limits are considered in this thesis. In this case, upper and lower boundaries of reliability index are determined assuming that all contingencies of a set results in either system failure or keeping the system safe, respectively [151].

However, it is accepted that it is very difficult to consider unforeseen contingencies (equipment outages) in a model. Also, the number of simulated contingencies depend on the desired accuracy of obtained results or reliability index limits [123].

System failure probability, failure frequency and expected duration of the failure are the most commonly used indices [123]. Random variables are used to model availability of power generation and load variations at the nodes [151].

2.7.2 Stochastic Optimisation and Multi-Objective Functions

Stochastic optimisation is widely used to design electricity networks [19, 100, 116, 142]. For instance, based on the theory of reliability and Markov models, if failure and repair rates are available, then system reliability assessment can be carried out through the state space method.

In this case, availability of a component is calculated as follows [19]:

$$p = \frac{\mu}{\lambda + \mu} \quad , \quad q = \frac{\lambda}{\lambda + \mu}$$

where p and q are probabilities of safety and failure of the component, respectively, and λ and μ are failure and repair rates, respectively.

A stochastic programming framework is defined in order to use in problems related to renewable energies and corresponding networks. Optimal values are calculated such that fluctuating nature of market prices is taken into account [5].

Moreover, one of the proposed methods to consider uncertainties in optimisation models of electricity networks is to develop a multi-objective optimisation model so that both cost and reliability issues are taken into account in the same time.

In this case, a two-dimensional function is formulated as the objective function. Cost and reliability are two components of this objective function [2, 65]. It's reported in the literature that although more aims can be achieved using multi-objective optimisation models, they are more complicated than the mono-objective optimisation models [3, 50].

2.8 Optimal Power Flow Model

One of the widely used optimisation models for electricity power networks is optimal power flow (OPF). This model was first introduced in the early 1960s. The OPF model is generally known as an extension of conventional economic transmission [32].

The optimal power flow (OPF) problem is intended to compute an optimum point for an electricity power network. A widely accepted cost function of the OPF problem is generation cost function. However, transmission loss is also considered in the literature as a cost function of the OPF problem. Further, various constraints on variables related to power and voltage are often formulated in an OPF problem [130].

The increase need to optimise power networks is due to an increase in the size of power systems and complexity of networks. Load flow constraints are often incorporated into objective function in order to change the constrained optimisation problem into an unconstrained optimisation problem [99].

In an OPF problem, an objective function is minimised based on particular controllable variables such that various physical and operating constraints are satisfied. Cost function is often assumed to be a smooth and quadratic function.

Although the OPF model is a very well-known optimisation model in electricity networks, uncertainties are ignored in this formulation. A chance-constrained (CC) OPF is proposed in the existing literature in order to correct the problem and alleviate dangerous renewable fluctuations, while the current operational procedure has minimum changes [28].

The OPF problem is a large-scale non-linear optimisation problem. An OPF problem can be formulated in polar, rectangular or mixture of polar and rectangular forms. However, the rectangular version has the property that power flow equations do not include trigonometric functions. This property leads directly to the formulation of semi-definite programming (SDP) models [18].

The optimal power flow (OPF) problem looks for decision variable values to yield an optimal operating point for an electric power system [107].

Linear programming, Newton Raphson, quadratic programming, nonlinear programming, Lagrange relaxation, interior point methods, artificial intelligence, artificial neural network, fuzzy logic, genetic algorithms, evolutionary programming and particle swarm optimisation are applied into the OPF problem as different approaches [104, 139].

Several algorithms have been proposed based on the genetic algorithm (GA) to apply in the OPF problem. In these algorithms, which adopt advantages of this evolutionary method, various sets of control variables are used in order to discover their potential usefulness in the OPF solutions [182].

However, since many of the above mentioned approaches are based on the KKT necessary condition, only a local optimal solution is guaranteed as a result of non-convex problem formulations [104].

Total generation cost is a typical objective of the OPF problems. Power flow equations determine relationships between voltages and active and reactive power injections in a power system. Constraints of an OPF problem include engineering limits on active and reactive power generation, bus voltage magnitudes, transmission line and transformer flows [107].

Many researchers have been attracted by OPF problem relevant issues and many algorithms are now available to solve an OPF problem. However, as electricity power networks are getting more complicated issues, the OPF problems turn to be harder to deal with [18].

Moreover, efficient algorithms with guaranteed performance have been developed for the OPF problem [82, 87, 88, 112]. It's also proposed to consider a Lagrangian dual of the OPF problem and solve it in order to recover a desired solution from a dual optimum [102].

Further, a class of quadratic programming problems is proposed in the literature and a connection from this class to non-convex quadratic constraints is studied. It's proved that the OPF problem has a convex semi-definite programming relaxation for DC networks. This relaxation is always equivalent to the main problem [104].

An accurate model of cost function may require a piecewise polynomial form or an optimisation using quadratic, cubic, piecewise linear or piecewise quadratic functions [99].

The only power, which is controlled for cost minimisation, is active power. Active and reactive power flows, voltage magnitudes and phase angles are often used as independent control variables to formulate constraints in an OPF problem.

In optimisation models of electricity power networks, electricity flows are governed by the Kirchhoff's Laws. These laws are the origin of difficulties related to the networks [31].

Commonly used constraints of an OPF problem are as below [99]:

1. Network power balance at each node;

2. Boundaries on all variables;

3. Line-flow limits.

It is reported that an OPF problem is intended to find optimum amounts of active and reactive power flows as well as voltage magnitudes such that operational feasibility constraints are satisfied [103].

Non-linear nature of the OPF problem and its non-convexity leads to convergence difficulties. Comprehensive researches about OPF problem have been done so far and many algorithms have recently been developed with guaranteed performance to solve an OPF problem [79, 171, 178]. Non-linear interior point algorithms are proposed for an equivalent current injection model of the problem [87, 112]. Also, an improved implementation of the automatic differentiation technique for the OPF problem is introduced [88].

Control and system theory, signal processing and communications and also combinatorial optimisation are applied to solve OPF problems [8, 9, 25, 67, 68, 140, 188].

The Jacobian and Hessian matrices (first- and second-order partial derivatives) must be found for each specific problem so that interior point method (IPM) can be applied to solve the OPF problems. Thus, it can be concluded that developing a general and unique software to solve these problems is a hard mission [18].

It has been reported in the literature that non-convexity of the OPF problem originates from non-linear nature of power network parameters, such as active power, reactive power and voltage magnitudes. The OPF problem is NP-hard in the worst case [103].

A semi-definite programming (SDP) formulation and a dual problem for the OPF model are introduced in the literature. These models are developed in order to decrease difficulties of solving an OPF problem.

An active field in numerical optimisation is the semi-definite programming (SDP) on which various algorithms are based. The main priority of the SDP-based interior point method is to avoid deriving and computing partial derivative matrices for each particular problem [1, 18, 127].

Further, solving a dual of an equivalent form of the OPF problem is known as an alternative approach rather than solving the OPF problem itself. The mentioned dual problem is a convex semi-definite problem and can hence be solved efficiently [103].

A necessary and sufficient condition is introduced in the literature in order to guarantee zero duality gap for dual OPF problem. The dual formulation can also be used to convexify practical system problems. Obtained results for the dual problem are monitored by considering a resistive network, which has only resistive and constant active power loads, and a network without any limitation on reactive power flows [103].

Further, it has been shown that there is an unbounded region so that duality gap is zero (if the imaginary part of admittance matrix Y belongs to this region), when the real part of Y is fixed. Zero duality gap of classic OPF problem conveys zero duality gap of general OPF problem which might involve more variables and constraints [103].

Moreover, many algorithms to solve the OPF problem are based on the KKT conditions. In these algorithms, a dual OPF problem is formulated and solved. An important difference between the KKT-based methods to solve dual OPF problems and optimisation algorithms based on the well-known KKT conditions is that the latter methods are built on both primal and dual variables, while the dual OPF problem is only dependent to dual variables [103].

Further, it can be shown that the OPF problem may have many solutions (all satisfying the KKT conditions), but a global optimum of the OPF problem can be obtained by solving the dual OPF when duality gap is zero [103]. A global solution of the OPF problem is sought and a semi-definite programming (SDP) method is proposed for this purpose [89].

It is commonly accepted that a positive semi-definite matrix must be chosen in order to optimise a linear function subject to linear constraints in the semi-definite programming problems. In other words, the LP problem may be generalized by replacing vector of variables by a symmetric matrix and also non-negative constraints by a positive semi-definite constraint [18].

This generalization is convex and has a rich duality theory. Although a semi-definite programming problem can be written in different forms, the primal form and its dual are considered in the literature.

Also, it is reported that all IPM improvements, which are applied to linear programming, are useful for SDP. The OPF problem is reformulated as an SDP model and then an algorithm is introduced based on interior point method (IPM) for SDP [18].

2.8.1 A Basic OPF Formulation

The optimal power flow (OPF) problem has been formulated for various electricity networks. In this subsection, a formulation of this problem is explained based on a three bus example system. Figure (2.6) displays the three bus example system. In this figure, nodes 1 and 2 are considered as generating stations and node 3 indicates demand.

Generators in this model are often considered as marginal cost curves that are smooth functions. This function is designated by function mc and written as below [31]:

$$mc_i(g_i) = lc_i + g_i qc_i \quad g_i \geq 0 \quad i = 1, 2$$

where lc_i and qc_i are linear and quadratic cost components of total cost, respectively. Also, g_i is the generated electricity power at the generator i.

Further, consumers are considered by demand curves as below:

$$p_3 = \alpha_3 - \beta_3 \omega_3$$

where $\omega_3 \geq 0$ is electricity consumption at bus 3.

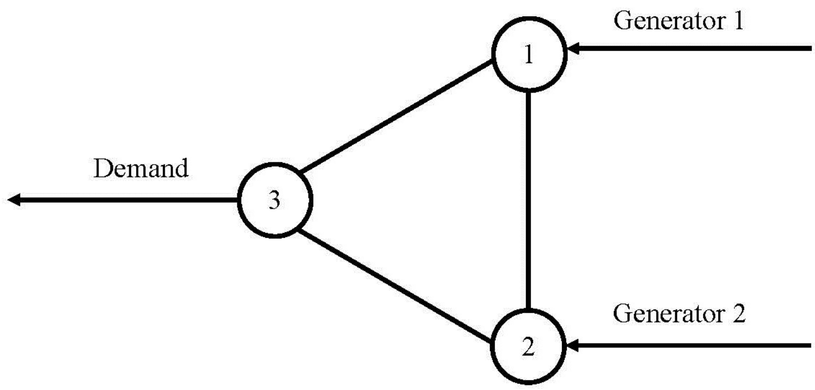

Figure 2.6: Three Bus System

Hence, an optimal power flow (OPF) model based on the given three bus example system is written as following [31]:

$$
\begin{aligned}
& Min\ vc_1(g_1) + vc_2(g_2) - u_3(w_3) \\
& s.t.\ \ g_1 + g_2 = w_3 \quad (Kirchhoff's\ Circuit\ Law) \\
& \qquad \frac{g_1 - g_2}{3} \leq \overline{f}_{12} \quad (Thermal\ Limit) \\
& \qquad g_1, g_2 \geq 0, w_3 \geq Demand
\end{aligned}
\tag{2.30}
$$

where g_1 and g_2 are generated electricity at the nodes 1 and 2, respectively. Also, w_3 is electricity consumption at node 3.

Functions $vc_i(g_i)$ $(i = 1, 2)$ are variable cost functions of the nodes 1 and 2. These functions originate from the explained marginal cost curves using the following:

$$\frac{dvc_i}{dg_i} = mc_i(g_i) = lc_i + g_i qc_i \quad i = 1, 2$$

Also, if a function is considered for willingness to pay and denoted by $u_3(\omega_3)$, then we'll have [31]:

$$\frac{du_3}{\omega_3} = mu_3(\omega_3) = \alpha_3 - \beta_3\omega_3$$

Thermal limit in the above OPF problem originates from the Ohm's Laws. Thermal limit on the line 1-2 is expressed as following [31]:

$$-\overline{f}_{12} \leq f_{12} \leq \overline{f}_{12}$$

where f_{12} is total flow on this line and $-\overline{f}_{12}$ and $\overline{f}_{12}$ are its lower and upper boundaries, respectively.

Thermal limits are intended to limit temperature of lines. They are not a function of line length and usually determine the maximum power flow for lines less than 50 miles in length.

Since the indirect path 1-2-3 in the Figure (4.1) is twice of the direct path 1-3, a unit injection of electricity flow in node 1 that is withdrawn at node 3 requires flows of $\frac{1}{3}$ and $\frac{2}{3}$ on the paths 1-2-3 and 1-3, respectively. Then, the total flow on the line 1-2 can be written as below:

$$f_{12} = \frac{g_1 - g_2}{3}$$

For simplicity, it is assumed that the flow in the line 1-2 (with no contingency) always goes from node 1 to node 2. Then, the thermal limit constraint $-\overline{f}_{12} \leq \frac{g_1 - g_2}{3}$ is always slack and can be dropped.

Moreover, it must be mentioned that although reliability related issues have high importance in electricity power networks, these factors are not considered in the above model. In this case, security constraints are proposed in the literature to cover reliability issues in the OPF problems [31].

A general formulation of the OPF problem will be illustrated in the next subsection.

2.8.2 Mathematical OPF Problem

A classical formulation of the OPF problem is explained in this subsection. The following are assumed in this formulation [107]:

1. $\widetilde{N} = \{1, 2, ..., n\}$ is the set of all buses. In other words, it is assumed that the network includes n buses.

2. $\widetilde{G} = \{1, 2, ..., g\}$ is the set of generation stations; i.e. it is assumed that the network includes g generators.

3. $\widetilde{L}$ is the set of all existing lines. The number of all existing lines is at most $\frac{n(n-1)}{2}$.

4. P_{D_k} and Q_{D_k} are active and reactive power loads at bus $k \in \widetilde{N}$, respectively. These are real values and given as fixed demands.

5. P_{G_k} and Q_{G_k} are active and reactive powers generated at generator $k \in \widetilde{G}$, respectively. These are real values and considered as optimisation variables.

6. $V_k = V_{d_k} + jV_{q_k}$ is an optimisation variable which shows the voltage magnitude at bus $k \in \widetilde{N}$. V_{d_k} and V_{q_k} are real and imaginary parts of voltage magnitude.

7. S_{lm} is apparent power flow on the line $(l, m) \in \widetilde{L}$.

8. $Y = G + jB$ is the network admittance matrix where G and B are conductance and susceptance, respectively.

Further, operating cost function associated with generator k is often written as below:

$$f_k(P_{G_k}) = c_{k_2} P_{G_k}^2 + c_{k_1} P_{G_k} + c_{k_0}$$

where c_{k_2}, c_{k_1} and c_{k_0} are nonnegative numbers.

The above cost function is a quadratic function that is also formulated for the whole network as following:

$$F(P_G) = \sum_{i=1}^{g} (\alpha_i + \beta_i P_{G_i} + \gamma_i P_{G_i}^2)$$

where P_{G_i} is active power generation at unit i and α_i , β_i and γ_i are cost function parameters of unit i. Thus, cost function and inequality constraints (boundaries) of the OPF problem are formulated as follows [62, 107]:

$$Min \sum_{k \in G} f_k(P_{G_k}) \tag{2.31}$$

$$s.t. \quad P_{G_k}^{min} \leq P_{G_k} \leq P_{G_k}^{max} \qquad k \in \widetilde{G} \tag{2.32}$$

$$Q_{G_k}^{min} \leq Q_{G_k} \leq Q_{G_k}^{max} \qquad k \in \widetilde{G} \tag{2.33}$$

$$(V_k^{min})^2 \leq V_{dk}^2 + V_{qk}^2 \leq (V_k^{max})^2 \qquad k \in \widetilde{N} \tag{2.34}$$

$$|S_{lm}| \leq S_{lm}^{max} \qquad (l,m) \in \widetilde{L} \tag{2.35}$$

Equality constraints of the OPF problem may be written in the rectangular form or in polar form. Rectangular formulation of these constraints is given as below (in both constraints, we have: $k \in \widetilde{N}$):

$$P_{G_k} - P_{D_k} = V_{d_k} \sum_{i=1 \neq k}^{n} (G_{ik} V_{d_i} - B_{ik} V_{q_i}) + V_{q_k} \sum_{i=1 \neq k}^{n} (B_{ik} V_{d_i} - G_{ik} V_{q_i}) \tag{2.36}$$

$$Q_{G_k} - Q_{D_k} = V_{d_k} \sum_{i=1 \neq k}^{n} (-B_{ik} V_{d_i} - G_{ik} V_{q_i}) + V_{q_k} \sum_{i=1 \neq k}^{n} (G_{ik} V_{d_i} - B_{ik} V_{q_i}) \tag{2.37}$$

The equality constraints are formulated in polar form as follows:

$$P_{G_k} - P_{D_k} = |V_k| \sum_{j=1 \neq k}^{n} |V_j||Y_{kj}| cos(\delta_k - \delta_j - \theta_{kj}) \qquad k \in \widetilde{N} \tag{2.38}$$

$$Q_{G_k} - Q_{D_k} = |V_k| \sum_{j=1 \neq k}^{n} |V_j||Y_{kj}| sin(\delta_k - \delta_j - \theta_{kj}) \qquad k \in \widetilde{N} \tag{2.39}$$

where δ_i is phase angle at bus i, θ_{kj} is angle of $(kj)^{th}$ element in the admittance matrix and Y_{kj} is magnitude of admittance of line between buses k and j.

The above problem limits the apparent power flow measured at each end of a given line. In an OPF problem, individual solutions are easily calculated using Newton's method. The main challenge is to find all solutions [107].

Power flow equations (equality constraints in the OPF problem; i.e. Equalities (2.36) and (2.37)) relate active and reactive power injected at each bus to voltage phasor at the same bus. There are different variables associated with each bus that are formulated using an equation. For instance, variables associated with bus $k \in N$ and relevant equations are as follows:

1. Net active power injection, $P_k = P_{G_k} - P_{D_k}$;

2. Net reactive power injection, $Q_k = Q_{G_k} - Q_{D_k}$;

3. Voltage magnitude $V_k = V_{d_k} + jV_{q_k}$ or $|V_k|^2 = V_{d_k}^2 + V_{q_k}^2$;

Three above equations can also be rewritten as below (for sufficiently small $\epsilon > 0$) [107]:

$$P_k - P_{D_k} - \epsilon \leq P_{G_k} \leq P_k - P_{D_k} + \epsilon \quad \forall k \in \{PQ, PV\}$$

$$Q_k - Q_{D_k} - \epsilon \leq Q_{G_k} \leq Q_k - Q_{D_k} + \epsilon \quad \forall k \in PQ$$

$$|V_k|^2 - \epsilon \leq V_{d_k}^2 + V_{q_k}^2 \leq |V_k|^2 + \epsilon \quad \forall k \in \{PV, S\}$$

As can be seen in the above OPF model, probabilistic constraints are ignored when formulating the optimal power flow models. In the following subsection, AC and DC power flow models, as two popular models for electricity networks will be illustrated.

2.8.3 AC and DC Power Flow Formulations

As mentioned before, there are generally two kinds of formulations for electricity networks, which are called alternating current (AC) and direct current (DC) power flows. When a set of linear equations is used to define a DC optimisation model for electricity power networks, this DC model is called linearized DC (LDC) model. A comprehensive LDC model for the planning and control of electricity networks can also be applied to approximate nonlinear AC power flow equations [48].

The LDC model can produce an accurate approximation of the AC power flow equations for active power when normal operating conditions and some adjustments for line losses are considered [164].

In general, an AC power flow equation for bus n is written as below [48]:

$$S_n = \sum_{k \neq m} V_n V_k^* Y_{nk}^* - V_n V_m^* Y_{nm}^*$$

where S_n is AC apparent power of bus n ($S_n = p_n + iq_n$), V_n is AC voltage magnitude of bus n and Y_{nm} is line admittance between buses n and m ($Y_{nm} = g_{nm} + ib_{nm}$). p_n and q_n, as the real and imaginary terms of the AC power, indicate active and reactive powers, respectively.

It must be noted that the above equation is not symmetric. In other words, the following can simply be shown [48]:

$$S_{nm} \neq S_{mn}$$

Considering $Y_{nk}^* = Y_{nk}^b \quad (k \neq m)$ and $Y_{nm}^* = -Y_{nm}^b$, above equation can be rewritten as following:

$$S_n = \sum_n V_n V_k^* Y_{nk}^b$$

Based on the above equation, active and reactive powers can be formulated, respectively, as below:

$$p_n = \sum_m p_{nm} \quad \& \quad q_n = \sum_m q_{nm}$$

where we have:

$$p_{nm} = |V_n||V_m|(g_{nm}^y cos(\theta_n - \theta_m) + b_{nm}^y sin(\theta_n - \theta_m))$$

$$q_{nm} = |V_n||V_m|(g_{nm}^y sin(\theta_n - \theta_m) - b_{nm}^y cos(\theta_n - \theta_m))$$

On the other hand, LDC power flow equations are often written based on the assumptions below [48]:

1. Magnitude of a line susceptance is much larger than magnitude of its conductance; i.e. $|b| \gg |g|$.
2. Phase angles are close to each other so that $sin(\theta_n - \theta_m) \approx \theta_n - \theta_m$.
3. Voltage magnitudes are close to 1 and do not vary significantly; i.e. $|V| \approx 1$.

As $|g| \ll |b|$, it can be shown that $|r| \ll |x|$. In other words, real impedance is much less than imaginary impedance. Based on these assumptions, active power can be reformulated as follows [48]:

$$p_{nm} = -b_{nm}(\theta_n - \theta_m)$$

Further, reactive power flow can be simplified as below:

$$q_{nm} = -b_{nm} + b_{nm} cos(\theta_n - \theta_m) - 0 \Rightarrow q_{nm} = b_{nm}(-1 + cos(\theta_n - \theta_m))$$

Since $\theta_n - \theta_m \approx 0$, we can conclude that $cos(\theta_n - \theta_m) \approx 1$. Thus, $q_{nm} = 0$.

The LDC model is not able to obtain reactive power in an electricity network. Hence, this model cannot be applied in capacitor placement and voltage stability problems [48]. An incorporation of the LDC model in a mixed integer programming (MIP) model is often used in various optimisation applications in electricity networks [29].

A linear programming AC (LPAC) model is introduced in the literature incorporating both reactive power and voltage magnitudes. AC power flow equations include some cosine terms. These terms are approximated in this method using convex functions. Other nonlinear terms are approximated using Taylor series [48].

Optimal transmission switching (OTS) is a natural extension of primal flow problem. Topological changes are also included by the OTS. The total cost of generation is minimised in an OTS, which is often modeled as a quadratic function [49]. However, it is also discussed to apply the AC power flow models into the OTS problems instead of the DC power flow models [160, 162].

An optimal transmission switching (OTS) problem is generally a non-convex mixed integer nonlinear programming problem that is often approximated by using a DC power flow model. A mixed integer linear programming problem is obtained from the OTS problem by the mentioned approximation [21–23, 63, 74–76].

An alternative version of the OTS problem is solved because of computational challenges of solving the AC-OTS. An AC-OTS is often made by using one of the following approaches [49]:

1. Heuristics;

2. Power flow equations approximations;

3. Power flow equations relaxations.

Although heuristic approaches are fast to compute, these do not provide quality guarantees. Approximations are able to computational complexity reduction, but these approaches can also return infeasible solutions in the original space. Relaxation approaches provide dual bounds to solution of original problem. Relaxations can also be used to show that no solution for a network topology is available [23, 160].

Relaxation's solutions are often supposed as a superset of original feasible set. Heuristics and approximations approaches are unable to find dual bounds and prove infeasibility conditions [49].

Benefits of line switching for generation cost reduction are also studied in the literature. It is reported that 29% cost reduction can be obtained by line switching. However, it is not acceptable to use the DC power flow model in order to study line switching [49].

There are three different start models as below [48]:

1. Hot Start Models: An AC solution is available and so model has an additional information, like voltage magnitudes. These models are suitable for applications in which network topology is stable.

2. Cold Start Models: No AC solution is available and it can be very difficult to obtain a solution by simulating of network. These models are used when no operational network is available.

3. Warm Start Models: Model has its target voltages from normal operating conditions, but an actual solution may not exist for these targets.

In characterizing high-level behavior of power systems, phase angles and voltage magnitudes' differences are primary factors in determining active and reactive power flows, respectively [48].

In an active power flow contour, when a fixed voltage magnitude is considered, many lines are crossed inducing significant changes in active power. Also, for a fixed angle difference, varying the voltage has a limited impact on the active power as few lines are crossed.

In a reactive power plot, although varying phase difference results significant changes in reactive power, varying voltage results in even more significant changes in reactive power.

Thus, it can be resulted that:

1. Phase angle differences are the primary factors of active power while voltage differences have only a small effect.

2. Changes in voltage are the primary factors of reactive power, but phase angle differences have also significant influences.

Chapter 3

New Reliability Analysis Methods

During the past decades, many algorithms have been introduced to solve reliability-based design optimisation (RBDO) problems. However, there are still some drawbacks in solution methods for RBDO problems. It seems that more efforts are required to develop new approaches with higher efficiency and stability.

As mentioned in the previous chapter, inner loop of a two-level RBDO approach is concerned with reliability analysis. So far, various algorithms have been proposed to use in this loop in order to evaluate performance functions in an RBDO problem.

Reliability index approach (RIA), as a two-level (double-loop) RBDO approach, includes a first-order reliability analysis problem in its inner loop. This problem is often solved by using the Hasofer and Lind - Rackwitz and Fiessler (HL - RF) method. However, general optimisation algorithms can be used to solve this problem as well.

In inner loop of the RIA, a constrained minimisation problem should be solved to compute a safety reliability index (β_s). In this problem, distance from failure surface ($G_U(u_1, u_2, ..., u_n) = 0$) to origin of the standard normal random space (U-space) is minimised.

Performance measure approach (PMA) is another double-loop RBDO approach that has a first-order inverse reliability analysis problem in its inner loop. In the PMA, reliability analysis problem is a constrained optimisation problem that aims at minimising standard normalised performance function on a circular equality constraint.

A target reliability index (β_t) is considered as radius of this circle. In spite of reliability analysis problem of the RIA, optimum answer of this problem in the PMA has a fixed distance (β_t) from origin of the U-space. In this case, the given performance function is to be minimised.

There are several particular algorithms to solve an inverse reliability analysis problem. It has been reported that advanced mean value (AMV) method is efficient for evaluating convex performance functions, while conjugate mean value (CMV) method has been proposed to evaluate concave performance functions.

Moreover, hybrid mean value (HMV) method has been introduced to adaptively select a suitable method between the AMV and the CMV methods. In this regard, a method can be chosen within the AMV and CMV methods, once the type of performance function is determined. A function type criterion is used in the HMV method to specify type of performance function as either convex or concave.

The MV-based (AMV, CMV and HMV) methods originate from the same concept. All of these methods use the steepest descent direction in order to update the current design point. It has been reported in the literature that the HMV method is the most efficient and stable method to solve reliability analysis problems.

However, there are still many problems that the HMV method does not perform well to solve. In many cases, the HMV method needs a large number of iterations to find an optimum point of a reliability analysis problem. Further, this method sometimes diverges and cannot solve a problem successfully.

In this chapter, two new reliability analysis methods are introduced to evaluate performance functions in reliability analysis loop of the PMA. It must be mentioned that since it is commonly accepted that the PMA solves an RBDO problem better than the RIA, the methods introduced in this chapter are only compared with the existing reliability analysis problems in the PMA.

These new methods (introduced in this chapter), which are named as conjugate gradient analysis (CGA) method and unconstrained polar reliability analysis (UPRA) method, are based on the conjugate gradient direction and the polar co-ordinate system. A number of numerical experiments are performed to compare efficiency and stability of these methods with the existing methods.

3.1 A New Reliability Analysis Method based on Conjugate Gradient Direction

In this section, a new reliability analysis method is introduced to apply in inner loop (reliability analysis loop) of two-level RBDO approaches in order to solve first-order inverse reliability analysis problems.

As a line search method to update a design point, the conjugate gradient direction has been applied to introduce this method. Due to this usage of the conjugate gradient direction, this method is called "Conjugate Gradient Analysis (CGA) Method".

The most stable and efficient existing reliability analysis method, which is hybrid mean value (HMV) method, and its peers, are based on steepest descent direction. It is found that these methods sometimes show instability and inefficient behaviour.

It is found that the HMV method is very sensitive to initial design point so that it diverges in some cases. It will be shown in this chapter that the new CGA method is not sensitive to initial design point. The CGA method converges in all of the numerical experiments mentioned later.

Conjugate gradient direction is used in this method to improve stability and efficiency of the solution process for a reliability analysis problem. In the CGA method, all information (including random variables, performance functions, etc.) should be transformed from the original random space (X-space) to the standard normal random space (U-space).

First order reliability method (FORM) is often used to transform a problem from X-

space to U-space. This transformation is done to reduce non-linearity of problem, because different statistical parameters of random variables in X-space may result in highly non-linear functions.

3.1.1 Conjugate Gradient Analysis Method

Conjugate gradient analysis (CGA) method is a search method that is based on the conjugate gradient direction. This method (like many other search methods) starts with an initial design point that is a simple (and sometimes inexact) estimation of optimum point. In the real world problems, the initial design point is often determined in the basis of different properties of materials used and also on experience and knowledge of previous systems.

It is important to note that an initial design point is the expected value (μ) in the original random space (X-space). Since all information, including initial design point, should be transformed to the standard normal random space (U-space), the original initial design point (in the X-space) is also transformed to the U-space. After this transformation, the origin of the U-space will be considered as a new initial design point; i.e. $x_i^{(0)} = \mu_i \quad (i = 1, 2, ..., n)$ is transformed to $u_i^{(0)} = 0 \quad (i = 1, 2, ..., n)$.

Each design point (in each iteration) must be updated until stopping criteria are satisfied. Hence, we need to find a design change in order to update the current design point.

It is supposed that design change equals to a step size multiplied by a search direction. Like the MV-based methods (discussed in the previous chapter), step size is considered as a fixed number such that satisfies equality constraint of the first-order inverse reliability analysis problem. As this constraint is $\|U\| = \beta_t$, the intended step size must be equal to β_t. In other words, it is supposed that step size (λ) is as big as target reliability index (β_t) in order to satisfy the equality constraint, i.e. $\lambda = \beta_t$.

At the first iteration, conjugate gradient direction is equal to negative of the gradient of the performance function at the initial design point. In other words, the conjugate gradient direction is as same as the steepest descent direction in the first iteration; i.e. $w^{(1)} = -\nabla G_U(0)$.

The main difference between the conjugate gradient direction and the steepest descent direction arises in the subsequent iterations.

In order to update a design point, a unit vector must be calculated by dividing the obtained conjugate gradient direction by its norm. This unit vector is then weighted (multiplied) by target reliability index in order to find a new design point satisfying the equality constraint.

When a new design point is found, value of performance function and its gradient at this point are calculated to check stopping criteria. Stopping criteria of the conjugate gradient method (as a search direction determination method) are applied into this reliability analysis method.

If at least one stopping criterion holds, then the algorithm is stopped. In this case, the new obtained design point is the optimum solution (minimum performance target point or MPTP). Otherwise, the conjugate gradient direction must be updated by using a scale factor (d_i) and gradient of performance function at the new design point. Then, the whole process must be repeated.

3.1.2 Solving a Reliability Analysis Problem by Using the CGA Method

In the reliability analysis method introduced in this section, the conjugate gradient method is used as a line search method to calculate a search direction. In this method, the following vector should be calculated first and then the obtained vector must be divided by its norm in order to obtain a unit vector which yields a search direction. This process is formulated as below:

$$w^{(i+1)} = -\nabla G_U(u^{(i)}) + d_i.w^{(i)} \tag{3.1}$$

where G_U is the standard normalised performance function, ∇G_U is its gradient vector and $u^{(i)}$ is the current design point in the U-space. Also, d_i is a scalar factor that is calculated as follows:

$$d_i = \frac{\|\nabla G(u^{(i)})\|^2}{\|\nabla G(u^{(i-1)})\|^2}$$

Additionally, it must be noted that $d_0 = 0$, $w^{(0)} = 0$ and the initial design point in U-space is the origin, i.e. $u^{(0)} = 0$. Thus, search direction at $(i+1)^{th}$ iteration is obtained as following:

$$SD^{(i+1)} = \frac{w^{(i+1)}}{\|w^{(i+1)}\|} \tag{3.2}$$

Therefore, since design change is a multiplication of step size and search direction, it can be computed as below:

$$D^{(i+1)} = \lambda * SD^{(i+1)} \Rightarrow D^{(i+1)} = \beta_t * \frac{w^{(i+1)}}{\|w^{(i+1)}\|} \tag{3.3}$$

where λ is the step size, which is equal to β_t, and $SD^{(i+1)}$ is the search direction at $(i+1)^{th}$ iteration.

Since the CGA method is carried out in the U-space and the initial design point (in this space) is the origin of the U-space and also the obtained design change in each iteration has to be added to the initial design point, hence the design change is, in fact, a new design point.

Therefore, a new design point is found as below:

$$u^{(i+1)} = u^{(0)} + D^{(i+1)}$$

where $u^{(0)} = 0$. Thus, we will have:

$$u^{(i+1)} = D^{(i+1)}$$

Finally, the following equality is resulted:

$$u^{(i+1)} = \beta_t * \frac{w^{(i+1)}}{\|w^{(i+1)}\|} \tag{3.4}$$

It should be noted that although the CGA method can be implemented in the original random space (X-space), it is easier to carry out this algorithm in the standard normal random space (U-space) due to simplicity of the process in this space.

The main difference between evaluating performance functions (in a reliability analysis problem) in the X- and U-spaces is originated from the equality constraint of the inverse reliability analysis problem. This constraint is written in the U-space as $\|U\| = \beta_t$, while it has some major changes in the X-space that may cause some hardships. For example, if standard deviation of random variables are not the same, then a square root function will be involved in the equality constraint.

It can be seen in this chapter that the CGA method is stable and does not diverge in numerical experiments, while the existing reliability analysis method (the HMV method) exhibits instability. This shortcoming of the HMV method affects the solution process of an RBDO problem, but the CGA method has overcome this difficulty. All steps of this method are briefly illustrated in the following subsection.

3.1.3 Basic Algorithm

The proposed CGA method can be summarised as the following steps. Note that $\cdot$ stands for the scalar product of two vectors and $\| \cdot \|$ stands for the Euclidean norm.

1. Set the iteration counter $i = 0$.

   Select the convergence parameters ε_1, ε_2 and ε_3.

   Suppose:

   $$d_0 = 0, u^{(0)} = 0_{n*1}, w^{(0)} = 0_{n*1}$$

   where n is the number of random variables in the problem.

2. Calculate performance function value at the initial design point as $G_U(u^{(0)})$.

3. Calculate conjugate gradient direction at current design point as below:

$$w^{(i+1)} = -\nabla G_U(u^{(i)}) + d_i \cdot w^{(i)}$$

4. Calculate a unit vector based on the obtained conjugate gradient direction as follows:

$$n(u^{(i)}) = \frac{w^{(i+1)}}{\|w^{(i+1)}\|}$$

5. Calculate a new design point as $u^{(i+1)} = \beta_t . n(u^{(i)})$

6. Calculate gradient vector of performance function at current design point as $\nabla G_U(u^{(i+1)})$.

7. Calculate performance function value at new design point as $G_U(u^{(i+1)})$.

8. Check the following stopping criteria:

$$\|u^{(i)} - u^{(i-1)}\| < \varepsilon_1$$

$$\|\nabla G_U(u^{(i-1)})\| < \varepsilon_2$$

$$|G_U(u^{(i)}) - G_U(u^{(i-1)})| < \varepsilon_3$$

9. If at least one stopping criterion holds, then stop.

Otherwise, calculate a new scalar factor as $d_{i+1} = \frac{\|\nabla G_U(u^{(i+1)})\|^2}{\|\nabla G_U(u^{(i)})\|^2}$, set $i = i + 1$ and then go to step 3.

Convergence of the CGA method is discussed in the next subsection.

3.1.4 Convergence of the CGA Method

Three conditions are considered in the introduced CGA method as the stopping criteria of this algorithm. The process is stopped when at least one of these criteria is satisfied. These criteria are formulated as below:

1. $\|u^{(i)} - u^{(i-1)}\| < \varepsilon_1$

2. $\|\nabla G(u^{(i-1)})\| < \varepsilon_2$

3. $|G(u^{(i)}) - G(u^{(i-1)})| < \varepsilon_3$

where ε_1, ε_2 and ε_3 are some pre-defined acceptable tolerances.

Further, convergence of the CGA method is presented via the following theorem. We will denote by $B = \{u : \ \|u\| = \beta_t\}$ the sphere with radius β_t.

Theorem. Suppose that the standard normalized performance function $G_U(u)$ is continuously differentiable and the following two assumptions hold:

(A1) There exists a unit vector p and a positive number ξ such that

$$\nabla G_U(u) \cdot p \geq \xi > 0, \quad \forall u \in B$$

(A2) There exists $\delta > 0$ such that

$$\|\nabla G_U(u)\| \geq \delta > 0, \quad \forall u \in B$$

Then, $\lim_{i\to\infty} \|u^{(i+1)} - u^{(i)}\| = 0$.

Proof. Since $G_U(u)$ is continuously differentiable, there is a number $M < +\infty$ such that

$$\|\nabla G_U(u)\| \leq M, \quad \forall u \in B \tag{3.5}$$

Then, from assumption A2 it follows that

$$\frac{\|\nabla G_U(u_1)\|}{\|\nabla G_U(u_2)\|} \geq \delta_1 > 0, \quad \forall\ u_1, u_2 \in B \tag{3.6}$$

First, we show that

$$\|w^{(i)}\| \to \infty, \quad \text{as } i \to \infty \tag{3.7}$$

From Step 3 of the algorithm, we have $w^{(1)} = -\nabla G_U(0)$ and

$$w^{(i+1)} = -\nabla G_U(u^{(i)}) - \sum_{j=1}^{i}[\nabla G_U(u^{(i-j)}). \prod_{k=1}^{j} d_{i-j+k}]$$

By Assumption 1 for all i the inequality $\nabla G_U(u^{(i)}) \cdot p \geq \xi$ holds. Then

$$w^{(i+1)} \cdot p \leq -\xi - \xi \sum_{j=1}^{i} \prod_{k=1}^{j} d_{i-j+k}$$

On the other hand, from the definition of d_i in Step 9 of the algorithm, we obtain:

$$\prod_{k=1}^{j} d_{i-j+k} = \frac{\|\nabla G_U(u^{(i-j+1)})\|^2}{\|\nabla G_U(u^{(i-j)})\|^2} \cdot \frac{\|\nabla G_U(u^{(i-j+2)})\|^2}{\|\nabla G_U(u^{(i-j+1)})\|^2} \cdots$$

$$\frac{\|\nabla G_U(u^{(i)})\|^2}{\|\nabla G_U(u^{(i-1)})\|^2} = \frac{\|\nabla G_U(u^{(i-j+1)})\|^2}{\|\nabla G_U(u^{(i-1)})\|^2}$$

Thus, (3.6) yields:

$$\prod_{k=1}^{j} d_{i-j+k} \geq \delta_1^2 > 0, \ \forall j = 1, 2, \cdots, i$$

Therefore,

$$w^{(i+1)} \cdot p \leq -\xi(1 + \sum_{j=1}^{i} \delta_1^2) = -\xi(1 + i\,\delta_1^2)$$

or $w^{(i+1)} \cdot p \to -\infty$ as $i \to \infty$.

This means that (3.7) is true.

From $w^{(i+1)} = -\nabla G_U(u^{(i)}) + d_i.w^{(i)}$, it follows that:

$$\|w^{(i+1)}\| \leq \|\nabla G_U(u^{(i)})\| + d_i.\|w^{(i)}\|$$

Dividing this inequality by $\|w^{(i+1)}\|$ we obtain

$$1 \leq \frac{\|\nabla G_U(u^{(i)})\|}{\|w^{(i+1)}\|} + d_i \frac{\|w^{(i)}\|}{\|w^{(i+1)}\|}$$

Clearly, $\lim_{i\to\infty} \frac{\|\nabla G_U(u^{(i)})\|}{\|w^{(i+1)}\|} = 0$; thanks to (3.5) and (3.7). Thus, we have:

$$\liminf_{i\to\infty} d_i \frac{\|w^{(i)}\|}{\|w^{(i+1)}\|} \geq 1 \tag{3.8}$$

Using the formula in Step 3 of the algorithm, we have

$$w^{(i+1)} \cdot w^{(i)} = -\nabla G_U(u^{(i)}).w^{(i)} + d_i\, w^{(i)} \cdot w^{(i)}$$

Then

$$\frac{w^{(i)} \cdot w^{(i+1)}}{\|w^{(i)}\| \cdot \|w^{(i+1)}\|} = -\frac{\nabla G_U(u^{(i)})}{\|w^{(i+1)}\|} \cdot \frac{w^{(i)}}{\|w^{(i)}\|} + d_i\, \frac{\|w^{(i)}\|}{\|w^{(i+1)}\|}$$

and, taking into account (3.8), we obtain

$$\liminf_{i\to\infty} \frac{w^{(i)} \cdot w^{(i+1)}}{\|w^{(i)}\| \cdot \|w^{(i+1)}\|} \geq 1$$

On the other hand $w^{(i)} \cdot w^{(i+1)} \leq \|w^{(i)}\| \cdot \|w^{(i+1)}\|$ for all i and therefore:

$$\lim_{i \to \infty} \frac{w^{(i)} \cdot w^{(i+1)}}{\|w^{(i)}\| \cdot \|w^{(i+1)}\|} = 1 \tag{3.9}$$

Now, from the definition of $u^{(i)}$ in Step 5 of the algorithm, we have:

$$\|u^{(i+1)} - u^{(i)}\| = \beta_t \, \| \frac{w^{(i+1)}}{\|w^{(i+1)}\|} - \frac{w^{(i)}}{\|w^{(i)}\|} \| =$$

$$= \beta_t \sqrt{2 - 2\frac{w^{(i)} \cdot w^{(i+1)}}{\|w^{(i)}\| \cdot \|w^{(i+1)}\|}}$$

Thus, by using (3.9) it can be concluded that:

$$\lim_{i \to \infty} \|u^{(i+1)} - u^{(i)}\| = 0$$

The theorem is proved. □

In the next section, another reliability analysis method will be introduced based on the polar co-ordinate system.

3.2 Solving Reliability Analysis Problems in Polar Space

In this section, a new method is introduced to solve reliability analysis problems. As mentioned earlier, a reliability analysis problem based on the performance measure approach (PMA) is often transformed from original random space (X-space) to standard normalised random space (U-space). In other words, the original performance function ($G_X(x_1, x_2, ..., x_n)$) in inner loop of an RBDO problem must be standard normalised in order to obtain $G_U(u_1, u_2, ..., u_n)$.

In this new method, a new area will be used to evaluate performance functions in a reliability analysis problem of PMA. The main task is to convert performance functions into the polar co-ordinate system. It means that standard normalised performance function must be rewritten with respect to polar co-ordinate system. For this purpose, all random variables are converted to a product of trigonometric functions.

However, it must be noted that a performance function $G_X(x_1, x_2, ..., x_n)$ in inner loop of PMA should first be transformed to the U-space. Then, the obtained standard normalised performance function $G_U(u_1, u_2, ..., u_n)$ is converted to the polar co-ordinate system.

Direct transformation from the X-space to the polar space is difficult and sometimes impossible. In this case, if statistical parameters of random variables were not the same, specially when standard deviations of random variables are different, a square root function will be involved in the problem resulting further difficulties and complexity to direct conversion from the X-space to the polar space.

When this transformation is done in the U-space, the equality constraint (i.e. $\|(u_1, u_2, ..., u_n)\| = \beta_t$) can easily be converted to $\rho = \beta_t$ where ρ is the radius of a circle or a sphere in two or three dimensional spaces, respectively. Also, ρ, which is a fixed number, is used in the conversion from the U-space to the polar co-ordinate system.

The basic idea of this conversion is to reduce the number of variables in the corresponding reliability analysis problem. The equality constraint of reliability analysis problem shows a circular / spherical constraint in two / three dimensional spaces, respectively. Radius of this circular / spherical constraint is a fixed number. Thus, it can be concluded that the number of variables is reduced from n to $n - 1$.

Moreover, it should be noted that reliability analysis problem, which is a constrained minimisation problem in the U-space, will be converted into an unconstrained minimisation problem by converting problem into the polar co-ordinate system.

It happens because there is only one constraint in the original problem that is a circle in two dimensional space, a sphere in three dimensional space, etc. This constraint can generally be seen as a fixed radius. Thus, this constraint will vanish in the polar co-ordinate system and the radius (ρ) is used as a co-efficient in conversions from the U-space to the polar co-ordinate system.

It is anticipated that evaluating functions in the polar co-ordinate system, which is to solve an unconstrained optimisation problem, has less difficulties than evaluating them in the standard normalised random space, which is related to solve a constrained optimisation problem.

Since in this method the constrained reliability analysis problem is converted to an unconstrained reliability analysis problem in the polar co-ordinate system, this new method will be called "Unconstrained Polar Reliability Analysis (UPRA) Method".

All conversions and relevant relationships will be explained in the next subsection.

3.2.1 Unconstrained Polar Reliability Analysis Method

As discussed before, the basic idea of new method introduced in this section is originated from changing (reducing) a constrained optimisation problem to an unconstrained optimisation problem. In this regard, performance function in the U-space ($G_U(u_1, u_2, ..., u_n)$) should be re-written in the polar co-ordinate system.

For this purpose, random variables must first be converted to a combination of trigonometric functions. Hence, some relationships are required to use in conversions. Relevant constraints in two- and three-dimensional spaces are a circle and a sphere, respectively.

Required relationships to convert functions from the U-space to the polar co-ordinate system are as below:

Two dimensional space:

$U_1 = r.cos(\theta)$

$U_2 = r.sin(\theta)$

where r is a constant (radius of the circle) and θ is the polar angle so that:

$$U_1^2 + U_2^2 = r^2$$

Three dimensional space:

$U_1 = \rho.sin(\phi).cos(\theta)$

$U_2 = \rho.sin(\phi).sin(\theta)$

$U_3 = \rho.cos(\phi)$

where ρ is a constant (radius of the sphere), θ is the polar angle and ϕ is the azimuthal angle so that:

$$U_1^2 + U_2^2 + U_3^2 = \rho^2$$

Furthermore, the required relationships for converting problems from four-dimensional space to the polar co-ordinate system can be written as follow:

$U_1 = \rho.sin(\phi).sin(\theta).cos(\beta)$

$U_2 = \rho.sin(\phi).sin(\theta).sin(\beta)$

$U_3 = \rho.sin(\phi).cos(\theta)$

$U_4 = \rho.cos(\phi)$

where ρ is a constant, ϕ, θ and β are various angles so that:

$$U_1^2 + U_2^2 + U_3^2 + U_4^2 = \rho^2$$

These transformations can be expanded to n-dimensional spaces. Therefore, n conversions can generally be formulated for an n-dimensional problem as below:

n **- dimensional space:**

$U_1 = \rho.sin(\theta_1).sin(\theta_2).sin(\theta_3)...sin(\theta_{n-2}).cos(\theta_{n-1})$

$U_2 = \rho.sin(\theta_1).sin(\theta_2).sin(\theta_3)...sin(\theta_{n-2}).sin(\theta_{n-1})$

$U_3 = \rho.sin(\theta_1).sin(\theta_2).sin(\theta_3)...cos(\theta_{n-2})$

$U_4 = \rho.sin(\theta_1).sin(\theta_2)...cos(\theta_{n-3})$

and so on, until:

$U_{n-1} = \rho.sin(\theta_1).cos(\theta_2)$

$U_n = \rho.cos(\theta_1)$

where ρ is a constant. Also, we have:

$$U_1^2 + U_2^2 + ... + U_n^2 = \rho^2$$

3.2.2 Fundamental Concepts of the UPRA Method

The unconstrained polar reliability analysis (UPRA) method, introduced in this section, is proposed on the basis of various facts. It can be seen that the number of variables is decreased from n to $n-1$ by converting the standard normalised reliability analysis problem to the polar co-ordinate space, because ρ is a constant. Thus, one can expect that difficulties of solving the reliability analysis problem is reduced as the number of variables is decreased.

It is worthwhile to mention that reliability analysis problems, which are constrained optimisation problems, are changed to unconstrained optimisation problems by converting to the polar co-ordinate system. Therefore, it can be predicted that evaluating functions in this space can have less difficulties than evaluating them in the standard normalised random space.

Furthermore, it must be noted that when a reliability analysis problem is changed to an unconstrained optimisation problem, it can be solved by using any general optimisation algorithm of unconstrained problems. In this thesis, the obtained function in the polar co-ordinate system will be minimised by using the steepest descent method.

Moreover, it can simply be shown that if a reliability analysis problem has n random variables (after transforming problem to the polar space), only the first k components of the gradient vector are non-zero in iteration k. It happens because of the nature of transformations from the U-space to the polar space that involve trigonometric functions. Thus, in iteration k, just the first k components of design point are changed.

Therefore, one can conclude that in a n-dimensional problem, which is changed to a $(n-1)$-dimensional problem in the polar space, the first iteration in which all components of the gradient vector are non-zero and thus all components of design point are updated is the

$(n-1)^{th}$ iteration.

Further, to integrate the UPRA method with an RBDO problem, it must be mentioned that an RBDO problem, as a two-level problem, deals with reliability analysis and design optimisation problems in its inner and outer loops, respectively.

In general, we start to solve an RBDO problem by using an initial design point. To update this point, it is necessary to solve a reliability analysis problem. The UPRA method is able to solve reliability analysis problems efficiently and in a stable manner. After solving this problem, the obtained information must be used to calculate a new design point by updating the current design point in design optimisation loop.

In this chapter, a number of numerical experiments will be presented and solved in order to compare the performance of the HMV method (as the most stable and efficient existing reliability analysis method) with that of the CGA and UPRA methods as two new analysis analysis methods introduced in this research project.

3.3 Numerical Reliability Analysis Problems Solved by the Existing Methods

A reliability analysis problem is mainly intended to evaluate a probabilistic constraint and a corresponding performance function. For this purpose, performance function is often transformed from original random space (X-space) to standard normalised random space (U-space). Also, new constraint in the U-space is constructed by using a target reliability index (β_t).

The following constrained optimisation problem is used to evaluate a performance function. This problem is called a first-order inverse reliability analysis problem.

$$\begin{aligned} &Min \quad G_U(u_1, u_2, ..., u_n) \\ &s.t. \quad \|(u_1, u_2, ..., u_n)\| = \beta_t \end{aligned} \tag{3.10}$$

where $G_U(u_1, u_2, ..., u_n)$ and β_t are the normalised performance function and target reliability index, respectively.

A number of problems will be solved in the following subsections to get a better idea of performance of the existing reliability analysis methods.

Three problems are stated here in order to show differences between the existing reliability analysis methods (various MV-based methods).

The first two problems are chosen so that performances of the advanced mean value (AMV) and conjugate mean value (CMV) methods vary when solving these problems. The AMV method is more efficient than the CMV method for the first problem, while the CMV method performs better than the AMV method in the second problem.

The third problem includes an uncommon performance function with a changeable behaviour. The hybrid mean value (HMV) method that is used to solve this problem changes the selected method (the AMV or CMV methods) several times.

Also, it must be mentioned that the terms *convex* and *concave* functions, which are used in this section, are based on a particular assumption in the MV-based methods. In this case, the following criterion is supposed to determine nature of a performance function:

$$\zeta^{(k+1)} = (n^{(k+1)} - n^{(k)}).(n^{(k)} - n^{(k-1)}) \tag{3.11}$$

where $\zeta^{(k+1)}$ is the criterion for the performance function type at the $(k + \mathit{1})^{th}$ iteration and $n^{(k)}$ is the steepest descent direction of performance function at k^{th} design point $(u_{HMV}^{(k)})$.

If $\zeta^{(k+1)}$ is positive, then performance function is convex at $u_{HMV}^{(k+1)}$ and the AMV method must be selected. Otherwise, performance function is concave at $u_{HMV}^{(k+1)}$ and the CMV method should be used to solve the problem.

3.3.1 A Convex Performance Function

Consider the following convex performance function:

$$G(x_1, x_2) = -e^{x_1 - 7} - x_2 + 10$$

where both random variables follow the Gaussian distribution and their statistical parameters are $x_i \sim N(6, 0.8), \quad i = 1, 2$. Also, target reliability index (β_t) and convergence parameter (ε) are supposed as 3 and 10^{-6}, respectively.

The corresponding reliability analysis problem with the given data can be solved by using either the AMV or the CMV methods. The given performance function has been evaluated twice by using these methods, separately, and the obtained numerical results are shown in the Table (3.1).

	AMV Method			CMV Method		
Iteration	x_1	x_2	$G(x_1, x_2)$	x_1	x_2	$G(x_1, x_2)$
1	6.0000	6.0000	3.6321	6.0000	6.0000	3.6321
2	6.8286	8.2524	0.9051	6.8286	8.2524	0.9051
3	7.5463	7.8354	0.4376	7.5463	7.8354	0.4376
4	8.0769	7.2027	-0.1383	7.5453	7.8363	0.4386
5	8.2718	6.7739	-0.3412	7.9253	7.4329	0.0444
6	8.3109	6.6478	-0.3574	8.1333	7.0996	-0.2054
7	8.3173	6.6247	-0.3579	8.2061	6.6450	-0.2855
8	8.3183	6.6210	-0.3579	8.2731	6.7699	-0.3420
9	8.3184	6.6204	-0.3579	8.2986	6.6904	-0.3544
10	Converged			8.3087	6.6558	-0.3570
11				8.3145	6.6350	-0.3578
12				8.3167	6.6268	-0.3579
13				8.3176	6.6233	-0.3579
14				8.3181	6.6216	-0.3579
				Converged		

Table 3.1: Minimising A Convex Performance Function

It can be seen in the table that both AMV and CMV methods are stable (i.e. convergent) in this problem, but their efficiencies are not the same. The AMV method converges in 9 iterations, while the CMV method needs 5 more iterations.

Therefore, it can be concluded that although both applied methods are stable and they reach the same performance function values, the AMV method is more efficient than the CMV method for evaluating the given performance function in this problem.

3.3.2 A Concave Performance Function

Suppose that the following performance function is given.

$$G(x_1, x_2) = \frac{e^{0.8x_1 - 1.2} + e^{0.7x_2 - 0.6} - 5}{10}$$

Also, the statistical parameters are $x_1 \sim N(4, 0.8)$ and $x_2 \sim N(5, 0.8)$. Further, target reliability index (β_t) and convergence parameter (ε) are the same as for the previous problem.

	AMV Method			CMV Method		
Iteration	x_1	x_2	$G(x_1, x_2)$	x_1	x_2	$G(x_1, x_2)$
1	4.0000	5.0000	0.2612	4.0000	5.0000	2.0563
2	2.9887	2.8235	-0.1661	2.9887	2.8235	0.2251
3	2.3476	3.2594	-0.2964	2.3476	3.2594	0.2344
4	3.0726	2.7864	-0.1434	2.7870	2.9291	0.2065
...	...	...	...	...	...	...
8	3.2496	2.7203	0.2739	2.6816	2.9946	0.2038
...	...	...	...	...	...	...
11				2.6740	2.9996	0.2038
12				2.6769	2.9976	0.2038
...	...	...	...	Converged		
33	1.9809	3.7027	0.3798			
34	3.4641	2.6606	0.3347			
...	...	...	...			
999	1.9809	3.7027	0.3798			
1000	3.4641	2.6606	0.3347			
...	...	...	...			
9999	1.9809	3.7027	0.3798			
10000	3.4641	2.6606	0.3347			
	Diverged					

Table 3.2: Minimising A Concave Performance Function

This concave performance function has been evaluated by using the AMV and the CMV methods separately and the obtained answers are shown in the Table (3.2).

As mentioned before, the AMV method does not work well for evaluating concave performance functions. It converges very slowly and also sometimes diverges. As shown in the Table (3.2), the AMV method diverges in this problem because of its cyclic behaviour that starts from the *33*rd iteration. This method does not converge even after 10000 iterations.

On the other hand, the CMV method, which has been used in this problem, converges with a reasonable rate (in just 12 iterations).

Thus, it can be concluded from the numerical results shown in the Table (3.2) that the AMV method is unstable for evaluating the given concave performance functions in this problem, while the CMV method is stable and also efficient enough to evaluate the corresponding reliability analysis problem.

3.3.3 The HMV Method

In this subsection, a reliability analysis problem is solved by using the hybrid mean value (HMV) method to show its details. Suppose that a performance function is given as below:

$$G(x_1, x_2) = e^{-0.7x_1+x_2-1.2} + 2x_1^2 - 0.8x_1x_2^2 - 4x_2 + 5$$

where the random variables are normally distributed with the following statistical parameters:

$$x_1 \sim N(3, 0.4) \qquad \& \qquad x_2 \sim N(2, 0.6)$$

Also, target reliability index equals 3.

If this reliability analysis problem is solved by using the HMV method, an unusual behavior of the performance function will be found. Numerical results obtained by the HMV method are shown in the Table (3.3).

As can be seen in this table, behaviour of this performance function changes during the

HMV Method				
Iteration	Method	x_1	x_2	$G(x_1, x_2)$
1	AMV	3.0000	5.0000	-51.5261
2	AMV	3.5580	6.5936	-101.5749
3	AMV	3.9832	6.0320	-95.6200
4	CMV	3.7202	6.4398	-102.5505
5	CMV	3.8128	6.3243	-101.5697
...	...	...	...	...
16	CMV	3.7825	6.3646	-102.0336
17	AMV	3.7831	6.3639	-102.0258
18	AMV	3.7825	6.3646	-102.0336
19	CMV	3.7829	6.3641	-102.0284
...	...	...	...	...
31	CMV	3.7828	6.3643	-102.0297
32	AMV	3.7828	6.3643	-102.0297
Converged				

Table 3.3: The HMV Method

solution procedure. After the first three iterations, which the AMV and the CMV methods are the same, the HMV method is altered to the CMV method until *17*th iteration. In this iteration, the AMV method is used to find the next performance target point.

The HMV method comes back to the CMV method again in iteration 19 (after two iterations). This process (using the CMV method inside the HMV method) is followed until the second last iteration (iteration 31). In the *32*nd iteration (which is in fact the last iteration) the AMV method is used again inside the HMV method and then the iterative process is stopped.

This problem shows that it cannot be assumed that if the HMV method starts with either the AMV or the CMV methods (after the first three iterations), this method will be used in the whole process.

In other words, it can be concluded from this problem that when the HMV method is applied to evaluate any performance function, the function type criterion should be checked in all iterations.

Further, it will be shown in the next sections that the above mentioned behaviour of the HMV method (changing to select the AMV or CMV methods) may result in divergence in for reliability-based design optimisation (RBDO) problem.

3.4 A Mathematical RBDO Problem

The performance measure approach (PMA) is employed in this section to solve a double-loop RBDO problem. The hybrid mean value (HMV) method, as the most stable and efficient existing reliability analysis method, is used in reliability analysis loop of this problem. Also, design change in the outer loop (design optimisation loop) is calculated by using sequential quadratic programming (SQP) algorithm.

Suppose that the following RBDO problem is given:

$$\begin{aligned} &Min \quad Cost(x_1, x_2) = x_1 + x_2 \qquad (3.12) \\ &s.t. \quad P[G_i(x_1, x_2) \leq 0] \leq \Phi(-\beta_{t_i}) \\ &\qquad 0 \leq x_1 \leq 10 \quad , \quad 0 \leq x_2 \leq 10 \end{aligned}$$

$$(3.13)$$

where target reliability index (β_{t_i}) for all probabilistic constraints is 2 ($i = 1, 2, 3$) and initial design point is $x^{(0)} = [5, 5]^T$ with a standard deviation of $\sigma = 0.6$ for both random variables.

Moreover, three performance functions are given for this problem as below:

$$G_1(x_1, x_2) = \frac{x_1^2 x_2}{20} - 1$$

$$G_2(x_1, x_2) = \frac{(x_1 + x_2 - 5)^2}{30} + \frac{(x_1 - x_2 - 12)^2}{120} - 1$$

$$G_3(x_1, x_2) = \frac{80}{x_1^2 + 8x_2 + 5} - 1$$

It is assumed in this problem that random variables are normally distributed. The maximum number of iterations in the reliability analysis problem is also set as 20.

As discussed earlier, it must be mentioned that variables of an RBDO problem are indeed random variables. Their standard deviations are fixed the throughout solution process, while their expected values are changed in each iteration. In other words, expected values are the design variables in an RBDO problem. Thus, $[x_1, x_2] = [\mu(x_1), \mu(x_2)]$ is design variable.

However, it is commonly accepted in the existing literature to refer to variables of an RBDO problem as random variables. Obviously, the cost function value for the initial design point is 10. Obtained results in the first two iterations are shown in the following subsections.

3.4.1 First Iteration

Reliability analysis

In this loop, each performance function should be evaluated in order to find the minimum performance target point (MPTP). The HMV method is chosen to solve reliability analysis problems. Table (3.4) shows the obtained answers of the reliability analysis problems.

Reliability Analysis with the HMV Method				
Performance Function	Method	x_1	x_2	$G(x_1, x_2)$
1	AMV	3.8979	4.5252	2.4378
2	CMV	4.8346	3.8114	0.4472
3	AMV	5.9982	5.6660	-0.0731

Table 3.4: Obtained Results in Reliability Analysis Loop of the First Iteration

As displayed in the table, the HMV method is changed to the AMV method for evaluating the first and third performance functions because they are convex. But since the second performance function is concave, it has been evaluated by using the CMV method (inside the HMV method).

When all MPTPs are obtained, the information obtained by the HMV method must be used to compute a design change in the design optimisation loop in order to update the current design point.

Design Optimisation

Various optimisation algorithms can be applied into the outer loop of an RBDO problem to find a design change. Sequential quadratic programming (SQP) algorithm is widely used for this purpose.

The obtained results in the reliability analysis loop are often used to formulate a quadratic programming (QP) sub-problem in design optimisation loop in an RBDO problem in order to calculate a design change.

Quadratic function of the QP sub-problem is generally written as $0.5D^THD+c^TD$ where $D = [d_1, d_2]^T$ shows a vector of new variables in the outer loop, c is gradient of the cost function at the current design point (i.e. $c = \nabla f(x^{(0)})$) and H is an approximate Hessian matrix that is initially considered as an $n * n$ identity matrix (n is the number of design variables). However, it is also possible to ignore the Hessian matrix in this function. In this case, the quadratic function is written as $0.5D^TD + c^TD$.

Values of performance functions at the obtained MPTPs and their gradients at these points are also required to evaluate the constraints in the QP sub-problem.

Thus, the following quadratic programming subproblem is obtained, which should be solved to calculate the required design change.

$$Min \quad 0.5d_1^2 + 0.5d_2^2 + d_1 + d_2$$

$$\begin{aligned} s.t. \quad & 3.8979d_1 + 4.5252d_2 \leq 2.4378 \\ & 4.8346d_1 + 3.8114d_2 \leq 0.4472 \\ & 5.9982d_1 + 5.6660d_2 \leq -0.0731 \end{aligned}$$

Hence, the design change would be $[d_1, d_2]^T = [-1.0000, -1.0000]^T$ and so the next design point can be computed as below:

$$x^{(1)} = x^{(0)} + [d_1, d_2]^T \Longrightarrow x^{(1)} = [5,5]^T + [-1.0000, -1.0000]^T$$

Therefore, the next design point and also the cost function's value at this point would be as follows:

$$x^{(1)} = [4.0000, 4.0000]^T \Longrightarrow Cost(x^{(1)}) = 8.0000$$

3.4.2 Second Iteration

Reliability analysis

Now, we have to come back to the inner loop (reliability analysis loop) in order to evaluate the performance functions by using the new design point. The new design point ($x^{(1)}$) must be considered as new expected value (μ) in this iteration. These new expected values create new transformations for performance functions.

All performance functions should be transformed into the U-space by using the new statistical parameters. In other words, $FORM$ transformation (i.e. $T : X \longrightarrow U$) is affected by the new design point ($x = \sigma u + \mu$).

Performance functions must be evaluated again by using new design point to find new MPTPs. The HMV method is used to solve the corresponding reliability analysis problems. Obtained numerical results are displayed in the Table (3.5).

Reliability Analysis with the HMV Method				
Performance Function	Method	x_1	x_2	$G(x_1, x_2)$
1	AMV	2.8887	3.5471	0.4800
2	CMV	4.1766	2.8131	0.0747
3	AMV	4.9315	4.7565	0.1874

Table 3.5: Obtained Results in Reliability Analysis Loop of the Second Iteration

As can be seen in the table, the first and third performance functions are evaluated by using the AMV method. Also, the CMV method is selected inside the HMV method to evaluate the second performance function. The results obtained in this loop are used in the next design optimisation loop in order to find a new design point.

Design Optimisation

The process of solving the design optimisation problem in this iteration is as same as the first iteration's process, except for the approximation of the Hessian matrix. This matrix should be updated by using the quasi-Newton method in this iteration (as well as in subsequent iterations). Information obtained in the first iteration and also the reliability analysis loop of the second iteration are used to update the approximation Hessian matrix.

Therefore, a new QP sub-problem is written by using the obtained information from the reliability analysis loop as below:

$$Min \quad 0.4536d_1^2 + 0.6253d_2^2 + 0.1330d_1d_2 + d_1 + d_2$$

$$\begin{aligned} s.t. \quad & 2.9596d_1 + 3.4021d_2 \leq 0.4900 \\ & 4.1766d_1 + 2.8131d_2 \leq 0.0747 \\ & 4.9315d_1 + 4.7565d_2 \leq 0.1874 \end{aligned}$$

This optimisation problem should be solved to find new design change. Hence, the new design change is $[d_1, d_2]^T = [-0.4207, -0.6932]^T$ and then it can be concluded that the next design point is computed as follows:

$$x^{(2)} = x^{(1)} + [d_1, d_2]^T \Longrightarrow x^{(2)} = [4.0000, 4.0000]^T + [-0.4207, -0.6932]^T$$

Thus, we will have:

$$x^{(2)} = [3.5793, 3.3068]^T \Longrightarrow Cost(x^{(2)}) = 6.8861$$

3.4.3 Convergence

This process should be repeated until convergence. It must be noted that the obtained optimum design may or may not satisfy all the constraints. Feasibility conditions can be checked according to the details in Subsection 2.5.1 (Figures (2.1) and (2.2) and relevant illustrations).

In this problem, an optimum design point satisfying the constraints is found after 5 iterations. The optimum design point and the value of the cost function at this point are:

$$x^{(5)} = x^* = [3.1092, 3.1604]^T \Longrightarrow Cost(x^*) = 6.2696$$

In the next section, performances of all the proposed reliability analysis methods will be compared by solving different reliability analysis problems.

3.5 Performances of the New Reliability Analysis Methods

Two reliability analysis methods are introduced in this chapter; the conjugate gradient analysis (CGA) method and the unconstrained polar reliability analysis (UPRA) method. In order to have a comparison between performances of the existing and new reliability analysis methods, a number of numerical experiments are solved in this section.

Each problem is solved by using three different methods; hybrid mean value (HMV) method as the most stable and efficient existing reliability analysis method and also the CGA and UPRA methods, as two new reliability analysis methods introduced in this chapter.

The reliability analysis methods are compared based on their required number of iterations for convergence and in the obtained performance function values. Also, the elapsed CPU times of all methods are used to make a more comprehensive comparison. In the next section, a conclusion will be made based on the information obtained in this section.

At first, the HMV method, which is changed to the AMV or the CMV methods for evaluating convex and concave performance functions, respectively, is used to solve each problem. It is accepted that the AMV method is efficient enough for evaluating convex performance functions, while the CMV method works better than the AMV method to evaluate concave performance functions.

After that, the first reliability analysis method introduced in this chapter, which is the CGA method, is applied to solve the numerical problems to test the efficiency of this method. The last method that is used to solve all the problems is the UPRA method. In this method, the given performance function must be converted to the polar co-ordinate system by using trigonometric functions.

It must be noted that the HMV and CGA methods are used to solve a constrained minimisation problem and the given performance functions should be minimised subject to an equality constraint. But in the UPRA method, converted performance function to the polar space is minimised as an unconstrained optimisation problem.

Converting performance functions into the polar co-ordinate system should be done in standard normalised random space (U-space); not in the original random space (X-space).

There are three subsections here. The first subsection includes five two- and three dimensional problems. Initial design points are considered as fixed numbers in this subsection.

The second subsection includes the same problems, as the first subsection, but the initial design points are not fixed. 50 initial design points are randomly generated by the MATLAB software for each problem.

Each reliability analysis problem is indeed solved 50 times by using 50 different initial design points in this subsection. An average of the required iterations for 50 different cases of each problem is calculated to compare performances of different reliability analysis methods.

Further, a particular function is introduced and used in the third subsection. The number of design variables in this function varies from 2 to 10. All reliability analysis methods (the HMV, CGA and UPRA methods) are applied to evaluate all various versions of this function.

In all the following problems, accepted tolerances of the stopping criteria in all methods are considered equal to 10^{-5}. Also, all CPU times are shown in milliseconds in this section and all algorithms have been implemented in MATLAB.

3.5.1 Fixed Initial Design Point

Problem 1

Consider the following performance function.

$$G(x_1, x_2) = \frac{(x_1 + x_2 - 5)^2}{30} + \frac{(x_1 - x_2 - 12)^2}{120} - 1$$

where $x_1 \sim N(1, 0.6)$ and $x_2 \sim N(2.3, 0.8)$. Also, target reliability index (β_t) equals 2.

The given performance function in this experiment is minimised by using three different reliability analysis methods. Table (3.6) displays detailed numerical results obtained by these methods.

Table 3.6: Minimising Performance Function ($G(X)$) using Various Methods - Problem 1

	HMV Method			CGA Method			UPRA Method		
i	x_1	x_2	$G(x_1, x_2)$	x_1	x_2	$G(x_1, x_2)$	x_1	x_2	$G(x_1, x_2)$
1	1.0000	2.3000	0.5704	1.0000	2.3000	0.5704	1.0000	2.3000	0.5704
2	2.1418	1.8079	0.1709	2.1418	1.8079	0.1709	2.1418	1.8079	0.1709
3	2.0859	1.6191	0.1644	2.1135	1.7035	0.1661	2.0770	1.5944	0.1643
4	2.1180	1.7187	0.1666	2.1110	1.6954	0.1658	Converged		
5	2.1042	1.6737	0.1652	2.1102	1.6927	0.1657			
6	2.1125	1.7003	0.1660	2.1098	1.6915	0.1657			
7	2.1075	1.6840	0.1655	2.1096	1.6909	0.1657			
8	2.1096	1.6909	0.1657	2.1095	1.6904	0.1657			
9	2.1086	1.6875	0.1656	2.1094	1.6902	0.1656			
10	2.1094	1.6900	0.1656	Converged					
11	2.1090	1.6888	0.1656						
12	2.1091	1.6892	0.1656						
13	2.1090	1.6889	0.1656						
14	2.1091	1.6891	0.1656						
	Converged								
Time	23.213 milliseconds			12.937 milliseconds			28.088 milliseconds		

It can be seen in the Table (3.6) that all methods are convergent in this problem. Various methods have calculated various minimum performance target points (MPTP). However, obtained values for performance function at the optimum points are the same for the HMV and CGA methods, but the UPRA method has found a better (smaller) performance function value.

Further, the number of required iterations of the UPRA method to solve this problem is less than the other methods. In other words, based on the displayed data in the Table (3.6), it can be concluded that although the UPRA method needs longer time for converging, it is more efficient than the other methods. The UPRA method converges after just 3 iterations, while the CGA and HMV methods need 9 and 14 iterations for convergence, respectively.

Moreover, the required CPU time of the CGA method is considerably less than the other methods. The CGA method is able to find MPTP in this problem in less than 13 milliseconds, whereas the HMV and UPRA methods require more than 23 and 28 milliseconds for convergence, respectively.

Problem 2

A performance function is given as below:

$$G(x_1, x_2) = 3e^{-x_1+x_2} + 2x_1^2 - 4x_2$$

where x_1 and x_2 are normally distributed. Their statistical parameters are $(4, 0.7)$ and $(2, 0.5)$, respectively. Also, target reliability index equals 3.

All reliability analysis methods (HMV, CGA and UPRA) are applied to minimise the given performance function in this problem. Detailed results obtained by using these methods are displayed in the Table (3.7).

In this problem, again all methods are stable (convergent) and three different MPTPs are found by various methods. It can be seen in the table that although the HMV and CGA methods have calculated the same value for the performance function, the performance function value obtained by the UPRA method is even less.

Table 3.7: Minimising Performance Function ($G(X)$) using Various Methods - Problem 2

	HMV Method			CGA Method			UPRA Method		
i	x_1	x_2	$G(x_1, x_2)$	x_1	x_2	$G(x_1, x_2)$	x_1	x_2	$G(x_1, x_2)$
1	4.0000	2.0000	24.4060	4.0000	2.0000	24.4060	4.0000	2.0000	24.4060
2	1.9536	2.3369	2.6870	1.9536	2.3369	2.6870	1.9536	2.3369	2.6870
3	1.9143	1.8250	2.7731	1.9034	1.9147	2.6211	1.9044	2.0972	2.5027
4	1.9152	2.1803	2.5256	1.9020	2.0648	2.5062	1.9044	2.0972	2.5027
5	1.9029	2.0783	2.5039	1.9044	2.0973	2.5027	Converged		
...	...	...	...	...	...	...			
11	1.9059	2.1122	2.5034	1.9061	2.1139	2.5036			
12	1.9060	2.1133	2.5035	1.9061	2.1142	2.5036			
...	...	...	...	...	...	...			
17	1.9062	2.1150	2.5038	1.9062	2.1148	2.5037			
18	1.9062	2.1155	2.5038	Converged					
...	...	...	...						
22	1.9062	2.1153	2.5038						
23	1.9062	2.1152	2.5037						
24	Converged								
Time	22.378 milliseconds			12.702 milliseconds			25.583 milliseconds		

Thus, as the problem is a minimisation problem, the function value obtained by the UPRA method can be considered as the best function value obtained in this problem.

Moreover, three different convergence rates (required iterations for convergence) are found in this problem. The HMV method converges at the *24*th iteration, while the CGA method needs 7 iterations less than the HMV method.

However, the UPRA method needs the minimum number of iterations and has converged at the *4*th iteration. Hence, it can be concluded that the UPRA method is more efficient than the HMV and CGA methods in this problem.

Meanwhile, the UPRA method needs the longest time for convergence, even longer than the HMV method, while the CGA method requires half of the time required for the UPRA method for convergence.

Problem 3

Consider the following performance function.

$$G(x_1, x_2) = 0.3x_1^2x_2 - x_2 + 0.8x_1 + 1$$

where $x_1 \sim N(0, 0.55)$ and $x_2 \sim N(6, 0.55)$. Also, it is supposed that $\beta_t = 2$.

Table 3.8: Minimising Performance Function ($G(X)$) using Various Methods - Problem 3

	HMV Method			CGA Method			UPRA Method		
i	x_1	x_2	$G(x_1, x_2)$	x_1	x_2	$G(x_1, x_2)$	x_1	x_2	$G(x_1, x_2)$
1	0.0000	6.0000	-5.0000	0.0000	6.0000	-5.0000	0.0000	6.0000	-5.000
2	-0.6872	6.8590	-5.4371	-0.6872	6.8590	-5.4371	-0.6872	6.8590	-5.4371
3	1.0130	6.4288	-2.6393	-0.0969	7.0957	-6.1533	-0.1549	7.0890	-6.1619
4	-0.5126	6.9733	-5.8337	-0.2628	7.0681	-6.1319	-0.1549	7.0890	-6.1619
5	0.6232	6.9065	-4.6033	-0.0471	7.0990	-6.1319	Converged		
...	...	...	...	...	...	...			
9	0.8311	6.7206	-3.6629	-0.1549	7.0890	-6.1619			
10	-0.8933	6.6419	-4.7666	Converged					
...	...	...	...						
34	-0.9539	6.5477	-4.5234						
35	0.9112	6.6162	-3.2394						
...	...	...	...						
999	0.9112	6.6162	-3.2394						
1000	-0.9539	6.5477	-4.5234						
...	...	...	...						
9999	0.9112	6.6162	-3.2394						
10000	-0.9539	6.5477	-4.5234						
	Diverged								
Time	2740.260 milliseconds			13.079 milliseconds			27.391 milliseconds		

Table (3.8) shows details of numerical results obtained by using different methods. It can be seen in the table that the HMV method is unstable in this problem.

When the HMV method is applied to evaluate this performance function, an oscillating behavior starts from *34*th iteration. Thus, this method cannot find the optimum point and diverges. In this case, this method is stopped after 10,000 iterations.

The other methods are stable and could find an MPTP. The same optimum points and the same performance function values are found by the CGA and UPRA methods.

Further, the elapsed time of the UPRA method is more than the required time for the CGA method. The CGA method has found the MPTP at 13 milliseconds, while the UPRA method needs more than double of this time for convergence.

However, the number of required iterations of the UPRA method is less than the CGA method. The UPRA and CGA methods have found the optimum solution at the 4^{th} and 9^{th} iterations, respectively.

Problem 4

The following performance function is given:

$$G(x_1, x_2, x_3) = 2x_1^3 + 3x_2e^{x_3}$$

where initial design point is [1,4,-2] and standard deviations are 0.3, 0.4 and 0.8, respectively. Also, target reliability index is 3.

Three reliability analysis methods (HMV, CGA and UPRA) are applied to solve the corresponding reliability analysis problem. Table (3.9) displays results obtained by using all methods.

In this problem, which three random variables are involved in the given performance function, all methods are stable and could find MPTPs. However, different methods have found different MPTPs and various performance function values.

The UPRA method has found the best (smallest) performance function value and hence can be considered as the best method to solve this problem, because this problem is intended to minimise the given performance function.

Also, the UPRA method has the minimum number of required iterations for convergence.

Table 3.9: Minimising Performance Function ($G(X)$) using Various Methods - Problem 4

	HMV Method				CGA Method				UPRA Method			
i	x_1	x_2	x_3	$G(x_1, x_2, x_3)$	x_1	x_2	x_3	$G(x_1, x_2, x_3)$	x_1	x_2	x_3	$G(x_1, x_2, x_3)$
1	1.0000	4.0000	-2.0000	3.6240	1.0000	4.0000	-2.0000	3.6240	1.0000	4.0000	-2.0000	3.6240
2	0.1331	3.9218	-2.6257	0.8564	0.1331	3.9218	-2.6257	0.8564	0.1331	3.9218	-2.6257	0.8564
3	0.8919	3.7056	-4.3088	1.5687	0.7823	3.7125	-4.2566	1.1154	0.3632	3.8872	-3.6810	0.3897
4	0.2355	3.8439	-3.2274	0.4835	0.6121	3.7325	-4.0985	0.6445	0.3632	3.8919	-3.6822	0.3897
5	0.3907	3.7798	-3.7104	0.3967	0.4266	3.7711	-3.7924	0.4103	Converged			
...	...	...	...	...	...	...	...	...				
10	0.2039	3.8577	-3.0829	0.5473	0.2884	3.8147	-3.4217	0.4217				
11	0.3582	3.7877	-3.6280	0.5473	Converged							
...	...	...	...	...								
125	0.2889	3.8134	-3.4231	0.4213								
126	0.2889	3.8134	-3.4232	0.4213								
	Converged											
Time	46.442 milliseconds				14.027 milliseconds				32.141 milliseconds			

This method has converged after 4 iterations, while the CGA method and the HMV method have found the MPTPs at the *10*th and *126*th iterations, respectively.

It can also be seen in the Table (3.9) that although the CGA method needs the shortest elapsed time, its performance function value is the worst among all methods.

Problem 5

A performance function is given as below:

$$G(x_1, x_2, x_3) = \frac{e^{-x_1} + 3x_2^3 - 4x_3^2}{2x_2x_3^2 + 5x_1x_2x_3}$$

where $x_1 \sim N(3, 0.5)$, $x_2 \sim N(-2.3, 0.3)$ and $x_3 \sim N(1, 0.8)$. Also, target reliability index is 3.

Table (3.10) is used to show details of answers obtained by various reliability analysis methods. As can be seen in this table, the HMV method diverges in this problem. This method cannot find the optimum point (MPTP) even after 10,000 iterations. Thus, this algorithm is stopped after 10,000 iterations.

However, the other methods are stable and have found different MPTPs. The UPRA method has found the MPTP at the *110*th iteration and needs more than 32 milliseconds for convergence. Also, as shown in the Table (3.10), 36 iterations are enough for the CGA method in order to find an optimum point. Further, this method requires just 18 milliseconds for convergence.

Based on the obtained results, it can be seen that although the number of iterations and also the elapsed time of the CGA method are less than the UPRA method, there is a significant difference between performance function values obtained by the CGA and UPRA methods. The performance function value obtained by the UPRA method is much better than the value found by the CGA method.

Table 3.10: Minimising Performance Function ($G(X)$) using Various Methods - Problem 5

	HMV Method				CGA Method				UPRA Method			
i	x_1	x_2	x_3	$G(x_1, x_2, x_3)$	x_1	x_2	x_3	$G(x_1, x_2, x_3)$	x_1	x_2	x_3	$G(x_1, x_2, x_3)$
1	3.0000	-2.3000	1.0000	1.0346	3.0000	-2.3000	1.0000	1.0346	3.0000	-2.300	1.0000	1.0346
2	3.5520	-1.4642	1.1127	0.4404	3.5520	-1.4642	1.1127	0.4404	3.5520	-1.4642	1.1127	0.4404
3	3.4556	-1.5761	-0.2254	-1.9931	3.5030	-1.5139	0.1524	2.5473	3.9129	-1.6923	0.0107	45.4287
...	...	...	...	...	...	...	...	...	...	...	...	...
36	2.9647	-3.1217	0.0227	86.5544	3.3630	-1.4423	0.5622	0.7033	3.0175	-2.6212	-1.2417	-1.4662
37	3.9440	-1.6585	1.7435	0.3849	Converged				2.2086	-2.3947	-1.3851	-1.2309
...	...	...	...	...					...	...	...	...
109	2.6281	-3.0908	0.0209	103.8045					4.1243	-1.8853	2.1408	0.3822
110	3.8340	-1.8326	-0.5451	-1.0843					Converged			
...	...	...	...	...								
999	2.6281	-3.0908	0.0209	103.8045								
1000	3.8430	-1.8326	-0.5451	-1.0843								
...	...	...	...	...								
9999	2.6281	-3.0908	0.0209	103.8045								
10000	3.8430	-1.8326	-0.5451	-1.0843								
	Diverged											
Time	3165.841 milliseconds				18.375 milliseconds				32.141 milliseconds			

The five previous problems will be solved again in the next subsection by using different initial design points. 50 various initial design points are randomly generated by MATLAB for each problem and then an average of all obtained results are used for a comparison.

3.5.2 Randomly Generated Initial Design Points

The problems solved in the previous subsection are solved in this subsection again by using 50 different initial design points. The design points are randomly generated by the MATLAB software.

Answers obtained by the HMV, CGA and UPRA methods are shown it the Table (3.11). The best performance function values are displayed by bold digits in this table.

The table below includes three parts and each part includes three columns. For each part, the first column is devoted to show numerical details obtained by the existing HMV method. Two other columns at each part are considered for details of results obtained by the CGA and UPRA methods, respectively.

Table 3.11: Reliability analysis problems with randomly generated initial design points - The best performance function values are shown by bold digits.

	Convergence			Iterations			Function Value		
Problem	HMV	CGA	UPRA	HMV	CGA	UPRA	HMV	CGA	UPRA
1	50	50	50	10	9	2	1.2745	0.9253	**0.6902**
2	0	50	50	10000	16.4	2.9	—	11.3112	**4.6657**
3	50	50	50	9	51	3.1	-2.0784	-0.8461	**-2.3836**
4	50	50	50	8	55	6.7	-45.3117	-53.2223	**-55.6486**
5	0	50	50	10000	12.7	7.6	—	0.4294	**0.3853**

The first part of the Table (3.11) displays the number of convergent cases for each method. As 50 different randomly generated initial design points are applied to solve the above mentioned reliability analysis problems, a number between 0 and 50 will be displayed for each problem and each method. This number shows the number of cases in which the relevant reliability analysis method is convergent.

The numbers shown in the second and third parts of the table are related to the number of required iterations for convergence and the obtained performance function values. However, it must be mentioned that these numbers are averages of correspondence results for each method.

Based on the data displayed in the Table (3.11), we can conclude that the CGA and UPRA methods are stable in all problems and all cases, while the HMV method is sometimes divergent.

Furthermore, it can be seen by comparing the averages of required iterations that the UPRA method requires the minimum number of iterations generally.

Moreover, the averages of performance function values obtained by using the UPRA method is much less than the performance function values obtained by the other methods. Since these problems are intended to minimise a given performance function, it can be seen that the UPRA method has found the best performance function values.

Thus, it can be concluded that the UPRA method is more stable and efficient than the HMV and CGA methods to minimise these performance functions.

The elapsed CPU times to solve the above reliability analysis problems by using 50 randomly generated initial design points in this subsection are not displayed in the Table (3.11). However, it can be found from the previous subsection that the required computational times of each iteration in the HMV and CGA methods are approximately the same.

Therefore, it can be concluded that the elapsed time of the CGA method is shorter than the HMV method, because the CGA method needs less iterations than the HMV method.

3.5.3 High dimensional problem

A special function is introduced in this subsection. In this function, the number of random variables varies from 2 to 10. Suppose that the following performance function is given.

$$G(x_i) = \frac{\sum_{i=1}^{n} e^{-x_i^i}}{\prod_{i=1}^{n} x_i^{\frac{i}{i+1}}}$$

where $\sigma = 0.3$ and $\beta_t = 3$. n is the number of random variables and also the initial design point is $x^{(0)} = (2, 2, ..., 2)$.

Table 3.12: Minimising performance function ($G(x_i)$) by using the HMV, CGA and UPRA methods - The best performance function values are shown by bold digits.

	Iteration			$G(x_i)$			Time		
Variables	HMV	CGA	UPRA	HMV	CGA	UPRA	HMV	CGA	UPRA
2	15	8	4	0.1700	**0.0212**	**0.0212**	24.866	18.386	49.686
3	13	4	4	0.1310	0.0124	**0.0120**	24.916	19.327	53.452
4	13	3	4	0.0815	0.0068	**0.0066**	25.444	16.903	58.845
5	12	5	4	0.0487	0.0036	**0.0035**	27.563	18.201	85.286
6	10	3	4	0.0278	0.0019	**0.0018**	29.273	17.350	112.323
7	10	3	4	0.0155	**0.0010**	**0.0010**	25.282	17.412	148.787
8	10	3	3	0.0085	**0.0005**	**0.0005**	30.570	16.803	159.096
9	10	3	3	0.0046	**0.0003**	**0.0003**	32.251	17.856	167.015
10	9	10	3	0.0024	**0.0001**	**0.0001**	34.402	17.145	174.170

This performance function is minimised by using all reliability analysis methods (the HMV, CGA and UPRA methods) for different number of variables. Table (3.12) shows the obtained numerical results using various methods for different conditions.

The first column of the table displays the number of variables in the performance function. Then, there are three categories in the table. The first three columns show the number of required iterations of each method for convergence in each case.

The middle category is devoted to display performance function values obtained by various reliability analysis methods for different cases (different number of variables in each problem). The best performance function value is shown by bold digits. Also, the required CPU times for convergence in each problem are displayed in the last three columns.

Based on the numerical results displayed in this table, it can be concluded that all methods are stable in all cases. Further, the performance function values obtained by the UPRA method are generally less than the values resulted from the other methods. Therefore, it can be resulted that the UPRA method has generally found the best performance function values.

On the other hand, the elapsed CPU time of the UPRA method is longer than the other methods. This has happened because of the transformations involved in the MATLAB code. In all cases, many transformations are required in this experiment in order to calculate gradient vector and performance function values in the UPRA method.

3.6 Conclusion

Two new reliability analysis methods are introduced in this chapter to apply to the reliability analysis problem (inner loop) of the performance measure approach (PMA). As mentioned before, PMA and reliability index approach (RIA) include different reliability analysis problems. Since the PMA (solved by the HMV method) results in better stability and efficiency than the RIA (solved by the HL-RF method), new reliability analysis methods (introduced in this chapter) are compared only with the HMV method.

The first developed method is based on the conjugate gradient direction. In this method, the conjugate gradient direction is used to update a design point at each iteration and hence this method is called conjugate gradient analysis (CGA) method.

Further, a reliability analysis problem, which is a constrained optimisation problem, is converted to an unconstrained optimisation problem in the second reliability analysis method introduced in this chapter. Then, the obtained unconstrained optimisation problem is solved in the polar space. Therefore, this method is called unconstrained polar reliability analysis (UPRA) method.

Stabilities and efficiencies of the newly introduced reliability analysis methods are compared with stability and efficiency of the HMV method (as the most stable and efficient existing reliability analysis method) by application to various performance functions.

It is shown in this chapter that the HMV method is not stable in some cases. This method sometimes diverges and is hence unstable. This instability results in failing to solve an RBDO problem, because a reliability analysis problem must be solved in the inner loop of a double-loop RBDO problem. However, based on the solved numerical experiments, the new reliability analysis methods (CGA and UPRA methods) are stable and convergent in all numerical experiments.

It can be concluded that an RBDO problem can completely be solved when one of the new CGA or UPRA methods is applied to solve a reliability analysis problem in its inner loop. This guarantee to solve an RBDO problem comes from stability of the new methods, while the HMV method is not always stable and hence cannot guarantee a solution of RBDO problem.

In all cases, the UPRA method finds the best values for performance function. Thus, it can be concluded that the UPRA method is the best reliability analysis method regarding their robustness.

Moreover, the newly proposed CGA method often requires a shorter CPU time to solve reliability analysis problems and performs better than the other methods regarding the elapsed time.

In summary, based on the numerical experiments solved in this chapter, it can be resulted that the new CGA and UPRA methods are more stable and efficient than the existing HMV method to solve reliability analysis problems.

Chapter 4

Non-Deterministic Optimisation Models for Electricity Networks

Electricity power networks are known as complicated engineering systems that need modern mathematical optimisation models. Solar energy and wind energy, which are generally called renewable and clean energies, are considered as new sources for electricity generating systems requiring new optimisation models and techniques.

Many efforts have been made to introduce and develop powerful optimisation models and algorithms for electricity power networks. An optimisation model of electricity networks is generally intended to minimise total cost of the network. Manufacturing cost, instalment cost, operation cost and maintenance cost are often considered as different parts of total cost.

Other objectives are also considered for minimisation in electricity power networks. Network's area, energy losses and power flow are different examples that have been minimised in the existing optimisation models.

Cost relevant issues are generally categorised into fixed and variable costs. In this case, manufacturing cost and instalment cost are considered as fixed costs, while variable costs include operation and maintenance costs.

Furthermore, non-deterministic concepts play important roles in electricity networks. However, most existing optimisation models for these networks are deterministic. A typical deterministic optimisation model for electricity power networks is often formulated as below:

$$\begin{aligned} Min \quad & Cost(d) \\ s.t. \quad & f_i(d) = 0 \qquad i = 1, 2, ..., m \\ & h_j(d) \leq 0 \qquad j = 1, 2, ..., p \end{aligned} \tag{4.1}$$

where $d = [d_1, d_2, ..., d_n]$ is design variable and $f(d)$ and $h(d)$ are two sets of equality and inequality technical constraints, respectively. As can be seen in the above model, reliability related issues are not considered in this optimisation model.

However, different approaches are considered in formulating non-deterministic optimisation models of electricity networks. A number of stochastic optimisation models are available for these networks in which reliability issues are formulated based on a Markov model. Various indices, such as an availability index and a robustness index, are also developed in order to consider reliability issues in electricity power networks.

Moreover, multi-objective optimisation models are applied into electricity power networks so that reliability relevant concerns are taken into account. In these models, network's reliability is considered as part of an objective function that must be maximised while cost is considered for minimisation.

Nevertheless, there is still a gap in the literature as there is no comprehensive optimisation model for electricity networks in which network's uncertainty is considered as a constraint. Although probabilistic constraints are developed for particular electricity networks, these constraints have not yet been extended to consider a general electricity network.

Our main aim in this chapter is to introduce non-deterministic optimisation models for electricity power networks by considering their reliability. In this case, a major task is to introduce a probabilistic constraint for a network while cost is minimised. The main goal of formulating a probabilistic constraint is to control network's failure probability.

A general idea of how to formulate a probabilistic constraint for electricity networks will be illustrated first in this chapter. Next, a linear-programming (LP) problem will be supposed as a basic optimisation model for electricity networks in order to formulate a reliability-based design optimisation (RBDO) model for these networks.

Then, we will move to optimal power flow (OPF) model as a widely used optimisation model for electricity networks. A new OPF model will be developed so that a probabilistic constraint is added to the problem. In this case, failure probability of an electricity network is kept below a predetermined and acceptable level while cost function is minimised at the same time.

4.1 An Introduction to Probabilistic Constraints in Power Networks

Optimisation models are generally known as deterministic and non-deterministic. The main difference between these two groups is in their constraints. A non-deterministic optimisation model often includes a probabilistic constraint. This constraint incorporates uncertainties in a system. A probabilistic constraint is often defined as below:

$$P[G(x_1, x_2, ..., x_n) \leq 0] \leq \Phi(-\beta_t)$$

where P and Φ are a probability function and a standard normalised cumulative distribution function (CDF), respectively. Further, $G(x_1, x_2, ..., x_n)$ is a system performance function which is formulated in terms of random variables $(x_1, x_2, ..., x_n)$ in order to define a safety/failure condition for a system.

In this formulation, $G(x_1, x_2, ..., x_n) < 0$ denotes failure region. Safety region and failure surface are also determined by $G(x_1, x_2, ..., x_n) > 0$ and $G(x_1, x_2, ..., x_n) = 0$, respectively.

Since failure surface (i.e. $G(x_1, x_2, ..., x_n) = 0$) indicates a border between failure and safety regions, it is also called limit state function.

β_t in the mentioned probabilistic constraint is target reliability index of a system. Hence, $\Phi(-\beta_t)$ is in fact an intended upper level for system safety. This acceptable level is often assumed as a fixed number that system failure probability must be kept below.

Therefore, it can simply be seen that a probabilistic constraint consists of two general functions (P and Φ), one system function ($G(x_1, x_2, ..., x_n)$) and an index (β_t).

As P and Φ are two functions from theory of probability and β_t is only a number, which is often determined based on experience and knowledge from the existing models, one can conclude that the main step to introduce a probabilistic constraint for a system is to define a safety/failure condition for the system by formulating a performance function $G(x_1, x_2, ..., x_n)$.

In this section, we aim at introducing a general probabilistic constraint for electricity power networks. For this purpose, we first need to know the origin of uncertainties in the mentioned networks. Customer demand is often supposed as the main source of uncertainties in electricity power networks. Thus, it can be resulted that demand of customers must be considered in a probabilistic constraint.

Different approaches can be used to define safety and failure conditions for electricity networks. It is assumed in this section that $GE(x_1, x_2, ..., x_n)$ and $D(x_1, x_2, ..., x_n)$ are two available functions for these networks, which are formulated with respect to a random variable $(x_1, x_2, ..., x_n)$. Suppose that these functions indicate generated electricity power that is ready for distribution in the network and electricity power demand that is supposed to be needed by customers, respectively.

Therefore, a performance function for an electricity network can be defined as follows:

$$G(x_1, x_2, ..., x_n) = GE(x_1, x_2, ..., x_n) - D(x_1, x_2, ..., x_n) \tag{4.2}$$

Hence, safety and failure conditions of an electricity network are defined as below:

1. Safety Condition: One of the most important requirements of every electricity power network is to supply a requested amount of electricity power to customers (i.e. customer demand). Hence, a safety condition can be defined based on the supplied power.

   In this case, an electricity network continues to work safely when all demanded electricity power is supplied to customers. In other words, and based on the above mentioned functions, an electricity network is safe if the following condition is satisfied:

   $$GE(x_1, x_2, ..., x_n) \geq D(x_1, x_2, ..., x_n)$$

   Therefore, in terms of the performance function, which is introduced in the Function (4.2), a safety condition can be defined for an electricity network as following:

   $$GE(x_1, x_2, ..., x_n) \geq D(x_1, x_2, ..., x_n) \simeq G(x_1, x_2, ..., x_n) \geq 0$$

2. Failure Condition: If an electricity network is unable to supply all demanded electricity power to customers, it can be said that the network failed in its mission. In other words, the following condition indicates a failure in an electricity network:

   $$GE(x_1, x_2, ..., x_n) \leq D(x_1, x_2, ..., x_n)$$

   Hence, a failure condition of an electricity power network can be defined by using Function (4.2) as follows:

   $$GE(x_1, x_2, ..., x_n) \leq D(x_1, x_2, ..., x_n) \simeq G(x_1, x_2, ..., x_n) \leq 0$$

Based on the above safety and failure conditions, a failure surface (as a border between safety and failure regions) can be introduced for an electricity power network as below:

$$G(x_1, x_2, ..., x_n) = GE(x_1, x_2, ..., x_n) - D(x_1, x_2, ..., x_n) = 0$$

The above equality results $GE(x_1, x_2, ..., x_n) = D(x_1, x_2, ..., x_n)$. Since this equality is obtained if the distributed and demanded electricity powers are the same, this condition can be supposed as a general optimal condition for the network.

Therefore, the above three conditions divide the whole space into three subregions:

1. Safety Region, indicated by $G(x_1, x_2, ..., x_n) > 0$ or $GE(x_1, x_2, ..., x_n) > D(x_1, x_2, ..., x_n)$;
2. Failure Region, indicated by $G(x_1, x_2, ..., x_n) < 0$ or $GE(x_1, x_2, ..., x_n) < D(x_1, x_2, ..., x_n)$;
3. Limit State Function, indicated by $G(x_1, x_2, ..., x_n) = 0$ or $GE(x_1, x_2, ..., x_n) = D(x_1, x_2, ..., x_n)$;

Thus, failure probability of an electricity network can be formulated based on the above defined failure condition as follows:

$$P_f = P[G(x_1, x_2, ..., x_n) \leq 0] = P[GE(x_1, x_2, ..., x_n) \leq D(x_1, x_2, ..., x_n)]$$

If we suppose that a predetermined and acceptable level for failure probability of an electricity network is assigned based on experience and knowledge from the existing models, a probabilistic constraint can be introduced for the network.

Assuming that $\overline{P}_f$ is the requested level for network's failure probability, a probabilistic constraint can be formulated for electricity network as follows:

$$P[G(x_1, x_2, ..., x_n) \leq 0] = P[GE(x_1, x_2, ..., x_n) - D(x_1, x_2, ..., x_n) \leq 0] \leq \overline{P}_f$$

The acceptable level of a system failure probability is formulated based on a target reliability index (β_t). When an acceptable level is determined for a system (in general) or for an electricity network (in particular), the relevant β_t can be calculated based on the theory of probability and by using function Φ. This index will then be used to formulate probabilistic constraint and solve the corresponding reliability analysis problem.

Thus, a probabilistic constraint for electricity power networks is introduced as below:

$$P_f = P[GE(x_1, x_2, ..., x_n) \leq D(x_1, x_2, ..., x_n)] \leq \Phi(-\beta_t) \quad (4.3)$$

The above introduced probabilistic constraint for electricity power networks can be applied into different non-deterministic optimisation models. We will introduce non-deterministic design optimisation models for electricity power networks in the following sections based on this formulated probabilistic constraint.

4.2 A Novel LP-Based Optimisation Model for Electricity Networks Considering Uncertainties

A general idea was illustrated in the previous section describing how to introduce a probabilistic constraint for electricity power networks. In this section, a new reliability-based design optimisation (RBDO) model is developed for these networks based on a linear programming (LP) problem.

Optimisation problems for electricity power networks are often formulated in different models, such as linear, non-linear, etc. For simplicity, consider an optimisation problem for an electricity network, which can be formulated as a simple LP problem.

Constraints of this LP problem are written based on Kirchhoff's and Ohm's laws. Optimal answer of the LP problem is further modified in this section in order to formulate a probabilistic constraint for the network. This constraint is then used to introduce an RBDO model for electricity networks.

RBDO is a non-deterministic optimisation model that is widely used to take into account reliability issues when minimising cost function. Failure probability of a system is kept below an acceptable level while minimising cost in RBDO models.

An RBDO model, which is illustrated in full details in Chapter 2, is generally formulated as follows:

$$\begin{aligned} & Min\ Cost(x_1, x_2, ..., x_n) \\ & s.t.\quad P[G(x_1, x_2, ..., x_n) \le 0] \le \Phi(-\beta_t) \\ & \qquad (x_1, x_2, ..., x_n)^L \le (x_1, x_2, ..., x_n) \le (x_1, x_2, ..., x_n)^U \end{aligned} \tag{4.4}$$

where $(x_1, x_2, ..., x_n)$ is a random variable. In this model, the cost function can be any function of the random variable, $G(x_1, x_2, ..., x_n)$ is system performance function, and $(x_1, x_2, ..., x_n)^L$ and $(x_1, x_2, ..., x_n)^U$ are lower and upper bounds for the random variable $(x_1, x_2, ..., x_n)$, respectively.

Further, P is probability function, Φ is standard normalised cumulative distribution function (CDF), β_t is target reliability index and $\Phi(-\beta_t)$ is an acceptable level for system failure probability. As mentioned earlier, this type of constraint is called probabilistic constraint.

In the RBDO formulation, $P[G(x_1, x_2, ..., x_n) \le 0]$ is failure probability of system that is statistically defined as below:

$$P[G(x_1, x_2, ..., x_n) \le 0] = \int\int ... \int_{G(x_1, x_2, ..., x_n) < 0} f(x_1, x_2, ..., x_n) d(x_1, x_2, ..., x_n) \tag{4.5}$$

where f is joint probability density function (JPDF).

Probabilistic constraint of an RBDO model, which is constructed by using system failure probability, defines a feasible region by restricting probability of violating limit state $(G(x_1, x_2, ..., x_n) = 0)$ to an admissible failure probability $(\overline{P}_f = \Phi(-\beta_t))$. The probabilistic constraint is often evaluated by solving a first-order inverse reliability analysis problem. Inner loop of a double-loop RBDO problem is designated for solving this problem.

Target reliability index (β_t) has an important role in a reliability analysis problem. Based on the theory of probability, target reliability index (β_t) is defined as following:

$$\beta_t = -\Phi^{-1}(\overline{P}_f) \tag{4.6}$$

where $\overline{P}_f$ is an acceptable level of system failure probability.

β_t is often calculated by solving a first-order reliability analysis problem. Then, a first-order inverse reliability analysis problem is solved by using the obtained reliability index in order to evaluate probabilistic constraint. Reliability analysis methods introduced in Chapter 3 (conjugate gradient analysis (CGA) method and unconstrained polar reliability analysis (UPRA) method) are stable and efficient enough to solve a first-order inverse reliability analysis problem.

In the following subsections, an RBDO problem is introduced for electricity networks based on an LP problem.

4.2.1 An LP Problem for Electricity Networks

We aim at briefly explaining a particular optimisation model for electricity power networks in this subsection. This model, which is illustrated with full details in Section 2.8, is written as below:

$$\begin{aligned} &Min\ vc_1(g_1) + vc_2(g_2) - u_3(w_3) \\ &s.t.\quad g_1 + g_2 = w_3 \\ &\qquad f_{12} \leq \overline{f}_{12} \\ &\qquad g_1, g_2 \geq 0, w_3 \geq cd \end{aligned} \tag{4.7}$$

where g_1 and g_2 are the generated electricity powers at the nodes 1 and 2, respectively. Also, w_3 is electricity consumption at the node 3 and cd is customer electricity power demand. (g_1, g_2, w_3) is considered as a design variable (d) in the above formulation.

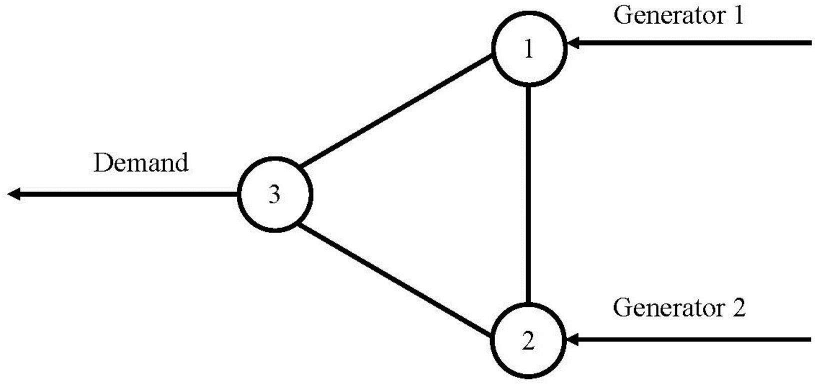

Figure 4.1: Standard Three Nodes Model

Figure 4.1 displays a standard three node model in a basic scheme for which the above optimisation problem can be applied for cost minimisation.

There are two kinds of constraints in the given optimisation problem (Model 4.7). These constraints can be illustrated as below:

1. The first type of constraint is a power balance constraint which is written based on Kirchhoff's laws. This constraint checks whether there is a balance between incoming (generated) electricity power and outgoing (supplied) electricity power in the network.
2. The second constraint, which is an inequality constraint, is formulated based on the Ohm's laws. This constraint is intended to limit thermal property of the line between generating stations (the nodes 1 and 2).

Since the total power flow on the line 1 - 2 is calculated as $f_{12} = \frac{g_1 - g_2}{3}$, the thermal limit on the line 1 - 2 can be rewritten as below:

$$\frac{g_1 - g_2}{3} \leq \overline{f}_{12}$$

As can be seen in the Model (4.7), reliability related issues are ignored in this formulation.

The main concern of this problem is to minimise total cost of a network.

It must be noted that the equivalent of the above optimisation problem (Model (4.7)) for more general networks is often a non-linear programming (NLP) problem. In fact, it is very difficult to formulate behaviour of an electricity network by using a linear programming (LP) problem.

However, for simplicity, we use the mentioned LP model in the next subsection in order to introduce a basic reliability-based design optimisation (RBDO) model for electricity networks. The main aim is to introduce a non-deterministic optimisation model for electricity networks by using a probabilistic constraint.

4.2.2 Definition of New Variables

Consider the optimisation problem introduced in the previous subsection and suppose that $d^* = [d_1^*, d_2^*, d_3^*] = [g_1^*, g_2^*, w_3^*]$ is an optimal solution for this problem. We now define a new design variable ($\hat{d} = [\hat{d}_1, \hat{d}_2, \hat{d}_3]$) based on this optimal solution to formulate a new optimisation problem considering system reliability.

New design variable is defined as below:

$$\hat{d}_i = x_i + d_i^* \quad (i = 1, 2) \quad and \quad \hat{d}_3 = y_3 + d_3^* \tag{4.8}$$

where x_i is considered as a random variable as an adjustment on the optimal solution d_i^*. The idea of this adjustment comes from our current aim to formulate a non-deterministic optimisation model for electricity networks. Hence, random variables are obtained as follows:

$$x_i = \hat{d}_i - d_i^* \quad (i = 1, 2) \quad and \quad y_3 = \hat{d}_3 - d_3^*$$

where d_i^* and $\hat{d}_i$ are the i^{th} components of the optimal solution and new design variable, respectively.

It should be noted that random variables of an electricity network can also be considered as a time-dependent variable. Further, some conditions, such as weather, daylight and other environmental conditions may affect the defined random variable of an optimisation model for electricity power networks. However, we ignore these additional complications at this stage.

As discussed before, our current intention is to introduce an RBDO problem for electricity networks by using a new variable and based on the optimum solution obtained by the LP problem. It is not difficult to predict that additional cost will be experienced in a network when reliability is taken into account. Thus, the main aim would be minimising the additional cost while keeping the system failure probability below an acceptable level.

In other words, failure probability of system should not exceed a predetermined amount ($\Phi(-\beta_t)$), while the additional cost is minimised. A performance function must be defined to calculate system failure probability in the probabilistic constraint.

Probabilistic constraint is an important part of an RBDO problem. A probabilistic constraint, which was generally introduced for electricity power networks in the previous section, is often evaluated by using a target reliability index in a reliability analysis problem.

Considering (g_1^*, g_2^*, w_3^*) as the obtained optimal LP solution and based on Kirchhoff's law (power balance law or equality constraint in the Model (4.7)), we have:

$$g_1^* + g_2^* = w_3^* \tag{4.9}$$

Further, suppose that $\hat{g}_1$, $\hat{g}_2$ and $\hat{w}_3$ are new design variables using which failure probability of the network is kept below an intended level, while experiencing an extra cost. In other words, we suppose that $(\hat{g}_1, \hat{g}_2, \hat{w}_3) = (\hat{d}_1, \hat{d}_2, \hat{d}_3)$. Optimal values of these variables should be calculated such that system can work safely. However, new variables ($\hat{g}_1$, $\hat{g}_2$, $\hat{w}_3$) are not at the optimum LP solution and definitely result in an additional cost.

Thus, new random variables are defined for the nodes 1 and 2 as below:

$$\hat{g}_1 = g_1^* + x_1 \quad , \quad \hat{g}_2 = g_2^* + x_2 \tag{4.10}$$

and for the node 3, we will have:

$$\hat{w_3} = w_3^* + y_3 \tag{4.11}$$

where x_1 , x_2 and y_3 are random variables which indicate differences between initial optimum solutions and new design variables.

Based on the equality constraint in the Model (4.7), the following equation is obtained:

$$\hat{g_1} + \hat{g_2} = \hat{w_3} \tag{4.12}$$

On the other hand, by summation of the Equations (4.10), it can be concluded that:

$$\hat{g_1} + \hat{g_2} = g_1^* + x_1 + g_2^* + x_2 \tag{4.13}$$

Hence, it can simply be shown that:

$$\hat{w_3} = w_3^* + x_1 + x_2$$

Therefore, the following equation is obtained:

$$\hat{w_3} - w_3^* = x_1 + x_2 \Rightarrow y_3 = x_1 + x_2 \tag{4.14}$$

Thus, new random variable y_3 can be eliminated from the RBDO problem by using the above equation. Hence, the new RBDO model includes only two random variables.

Further, regarding the inequality constraint of the Model (4.7) ($g_1 - g_2 \leq 3\overline{f}_{12}$), we can define a feasible set for the RBDO problem as below:

$$g_1 - g_2 \leq 3\overline{f}_{12} \Rightarrow x_1 - x_2 \leq g_2^* - g_1^* + 3\overline{f}_{12} \tag{4.15}$$

This feasibility condition, which is obtained using the thermal limit of the Model (4.7), can also be added to the RBDO cost function as a penalty term as below:

$$max(0, 10^6((x_1 - x_2) - (g_2^* - g_1^* + 3\overline{f}_{12}))) \tag{4.16}$$

The next subsection includes an RBDO problem for electricity power networks that is formulated using the random variables introduced in this subsection. It should be mentioned that the expected values of these random variables are indeed considered in this section to formulate the RBDO model (as is usual for RBDO models).

4.2.3 Formulating the RBDO Problem

A new optimisation problem for electricity networks is discussed in the previous subsections so that network's failure probability is taken into account. A reliability-based design optimisation (RBDO) problem is introduced in this subsection.

The basic idea is to replace the design variables of the optimisation problem illustrated in Subsection 4.2.1 (Model (4.7)) by using random variables introduced in Subsection 4.2.2. In this case, a new cost function is obtained using the original cost function after replacement of variables and corresponding simplifications.

Since the new RBDO model will be formulated based on the optimal solution (d^*) and also considers failure probability of the network by keeping this probability below an acceptable level, we expect additional cost compared to the obtained optimal cost values in the LP problem. Thus, the RBDO problem is intended to minimise this additional cost.

In other words, cost amount obtained by the RBDO problem cannot be less than cost obtained by the LP problem due to additional assumptions and constraints related to considering system reliability.

Moreover, it is anticipated that RBDO cost function would be an increasing function with respect to target reliability index (β_t). Hence, a higher target reliability index, which leads to a lower acceptable level of system failure probability and a higher safety level for the system, results in a bigger extra cost.

Therefore, the only task left now is to formulate a probabilistic constraint in order to obtain an RBDO problem for the standard three node example system (Figure (4.1)).

In the Model (4.7), the condition below must be satisfied in order to find a feasible solution and keep the network in safety conditions:

$$\hat{w_3} > w_3^*$$

This inequality results a safety condition as below:

$$\hat{g_1} + \hat{g_2} > g_1^* + g_2^* \Rightarrow x_1 + x_2 > 0$$

Thus, it can be concluded that system fails if:

$$\hat{g_1} + \hat{g_2} < g_1^* + g_2^* \quad or \quad x_1 + x_2 < 0$$

Therefore, safety and failure conditions can be defined as following:

1. Failure Condition: $x_1 + x_2 < 0$

2. Safety Condition: $x_1 + x_2 > 0$

In this case, $x_1 + x_2 = 0$, which leads to the initial optimal solution (g_1^*, g_2^*, w_3^*), is considered as limit state function or failure surface.

Since $x_1 + x_2 < 0$ results failure in electricity network, the required performance function can be written as a function of both random variables (x_1, x_2). It must be noticed that a necessary condition for the intended performance function is that $x_1 + x_2 < 0$ should result in system failure.

Hence, an RBDO problem for standard three nodes electricity network is written based on the Model (4.7) as following:

$$\begin{aligned} & Min\ vc_1(x_1) + vc_2(x_2) \\ & s.t.\quad P[G(x_1, x_2) < 0] \leq \Phi(-\beta_t) \\ & \qquad (x_1, x_2)^L \leq (x_1, x_2) \leq (x_1, x_2)^U \end{aligned} \tag{4.17}$$

where $G(x_1, x_2)$ is network's performance function, and $(x_1, x_2)^L$ and $(x_1, x_2)^U$ are lower and upper boundaries of the random variables, respectively.

Further, a new cost function is obtained using the original cost function in the Model (4.7) by replacing the initial variables (g_1, g_2, w_3) by new variables (x_1, x_2, y_3). It should be noted that the third variable (y_3) is eliminated based on the Equation (4.14).

Moreover, if the penalty term defined in the Equation (4.16) is added to the cost function, the RBDO problem for the network can be rewritten as below:

$$\begin{aligned} & Min\ vc_1(x_1) + vc_2(x_2) + max(0, 10^6((x_1 - x_2) - (g_2^* - g_1^* + 3\overline{f}_{12}))) \\ & s.t.\quad P[G(x_1, x_2) < 0] \leq \Phi(-\beta_t) \\ & \qquad (x_1, x_2)^L \leq (x_1, x_2) \leq (x_1, x_2)^U \end{aligned} \tag{4.18}$$

In the next section, the above introduced RBDO problem will be investigated by using various performance functions in a numerical experiment.

4.3 Illustrative Experiment Based on an LP Problem

In this section, a numerical experiment is solved in full details to explain all concepts discussed in the previous section. Figure (4.1) is again considered here for the problem.

A linear programming problem is first solved for the three nodes electricity network and then an RBDO problem will be formulated based on the obtained optimal solution.

As discussed earlier, additional cost is anticipated with respect to the obtained optimal solution by the LP problem. The main reason for the additional cost is to take network's failure probability into account.

We aim at minimising this additional cost while keeping the failure probability of system below a predetermined acceptable level. Various target reliability indices (β_ts) are used to investigate their effects on solutions and compare the resulting additional costs.

Suppose that the following LP problem is given for a standard three node electricity power network:

$$Min\ 30g_1 + 40g_2 - 25w_3$$

$$s.t.\quad g_1 + g_2 = w_3 \tag{4.19}$$

$$g_1 - g_2 \leq 90 \tag{4.20}$$

$$w_3 \geq 75 \tag{4.21}$$

$$g_1, g_2 \geq 0 \tag{4.22}$$

It is not difficult to observe that the optimum solution for the above LP problem is as below:

$$g_1^* = 75, g_2^* = 0, w_3^* = 75 \Rightarrow Cost = 375$$

This optimal solution is used in the following subsections in order to introduce new variables, formulate a new cost function and then an RBDO problem.

4.3.1 New Cost Function

The first step to introduce an RBDO problem based on the above mentioned LP problem is to define random variables. As explained in the previous section, three random variables must be defined based on the obtained optimal solution. Two random variables are supposed for the nodes 1 and 2 as follows:

$$\hat{g_1} = 75 + x_1 \quad \& \quad \hat{g_2} = 0 + x_2$$

where x_1 and x_2 indicate differences between optimal solutions ($g_1^* = 75$ and $g_2^* = 0$) and new variables ($\hat{g_1}$ and $\hat{g_2}$). Also, based on the Equation (4.14), the third variable (y_3) has been eliminated.

The initial cost function (used in the LP problem) must be used here to formulate a new cost function for RBDO problem. For this purpose, the original variables (g_1, g_2, w_3) must be replaced by the new random variables (x_1, x_2) and then the cost function can be simplified. Therefore, RBDO cost function is obtained as follows:

$$Cost(x_1, x_2) : 30(75 + x_1) + 40(0 + x_2) - 25(75 + x_1 + x_2)$$

$$\Rightarrow Cost(x_1, x_2) : 5x_1 + 15x_2 + 375$$

Further, based on the thermal limit Equation (4.15) (which is Equation (4.20) in the above solved LP problem), a feasible set of this problem is found as follows:

$$\hat{g_1} - \hat{g_2} \leq 90 \Rightarrow x_1 - x_2 \leq 90 - (75 - 0) \Rightarrow x_1 - x_2 \leq 15$$

This feasibility condition should be added to the new cost function as a penalty term. Thus, the RBDO cost function is rewritten as below:

$$Cost(x_1, x_2) : 5x_1 + 15x_2 + 375 + max\{0, 10^6(x_1 - x_2 - 15)\}$$

The next subsection explains how to formulate a performance function for this network.

4.3.2 Boundaries and Performance Functions

In this subsection, new boundaries are formulated in order to complete the RBDO problem formulation. Also, a number of performance functions are introduced for the three node electricity network so that the network can complete a requested mission safely.

Based on the non-negativity constraint of the LP problem (Inequality (4.22)), the following conditions are obtained:

$$\hat{g}_1 \geq 0 \Rightarrow x_1 \geq -75$$

$$\hat{g}_2 \geq 0 \Rightarrow x_2 \geq 0$$

These conditions are used to find lower and upper boundaries for two new random variables. Boundaries are supposed as below:

$$-10 \leq x_1 \leq 10$$

$$0 \leq x_2 \leq 10$$

Boundaries of the first random variable (x_1) are defined so that both negative and positive numbers are allowed. However, negative numbers cannot be considered for the second random variable (x_2), because it must be non-negative. Otherwise, we will have $g_2 < 0$ leading a contradiction with the non-negativity condition of the LP problem (Inequality (4.22)).

Moreover, a performance function should be defined such that the safety condition is satisfied. We will use several performance functions to evaluate system performance and then solve reliability analysis problems inside the RBDO problem.

The following performance functions will be introduced on the basis of the fact from previous section that if $x_1 + x_2 = 0$, we are at the optimum point of the LP problem and hence on the limit state function. In other words, $G(x_1 + x_2 = 0)$ must be zero. The proposed performance functions are indeed assumed to be increasing functions of $x_1 + x_2$.

Further, it is assumed that rate of additional cost will be high initially (immediately after moving away from $x_1 + x_2 = 0$) and then the rate of this change will decrease.

The first performance function is defined as below:

$$G(x_1, x_2) = \begin{cases} e^{x_1+x_2} - 1 & for \quad x_1 + x_1 \le 0.5578 \\ \sqrt{x_1 + x_2} & for \quad 0.5578 < x_1 + x_2 \end{cases} \tag{4.23}$$

where $x_1 + x_2 < 0$ results system failure and $x_1 + x_2 > 0$ keeps system in safety condition. It can easily be seen that this performance function meets the required safety/failure conditions.

The exponential function is first used to prevent having a complex number around zero in the square root function. Also, the number 0.5578 is used to separate two sections of the performance function because this is the point where both functions intersect.

Another performance function that is used to solve the proposed RBDO problem is formulated as follows:

$$G(x_1, x_2) = arctan(x_1 + x_2) \tag{4.24}$$

It can also be seen that above performance functions have a minimum around zero (i.e. where $x_1 + x_2 \simeq 0$) and then increase or decrease while the variable $(x_1 + x_2)$ is increase or decrease, respectively.

Further, the second performance function is modified to investigate effects of changing gradient vector. Thus, the third performance function is written as following:

$$G(x_1, x_2) = arctan(\frac{x_1 + x_2}{3}) \tag{4.25}$$

Therefore, the required RBDO problem is as below:

$$\begin{aligned} &Min \; 5x_1 + 15x_2 + 375 + max\{0, 10^6(x_1 - x_2 - 15)\} \\ &s.t. \quad P[G(x_1, x_2) \le 0] \le \Phi(-\beta_t) \\ &\qquad -10 \le x_1 \le 10, 0 \le x_2 \le 10 \end{aligned} \tag{4.26}$$

where three performance functions (Functions (4.23), (4.24) and (4.25)) are applied to solve this problem. It must be noticed that the feasibility condition is added to the RBDO cost function as a penalty term.

Also, the statistical parameters are supposed as follows:

$$\mu = [0, 0] \quad and \quad \sigma = [0.65, 0.45]$$

Further, it is widely accepted that a target reliability index (β_t) must be considered for any RBDO problem. This index indicates an acceptable level for system failure probability. When this acceptable level is determined, a target reliability index can be assigned based on the theory of probability.

Four different β_ts are listed in the Table (4.1). These indices are applied to solve the above RBDO problem.

It must be mentioned that it is not common to use a target reliability index greater than 3, because failure probability of system is already zero in this case and so system is absolutely safe.

Table 4.1: Selecting Target Reliability Index

β_t	$\Phi_Z(-\beta_t)$	Probability of Failure	Accepted Safety Level
0.01	$P(Z < -0.01)$	0.4960	0.5040
1	$P(Z < -1)$	0.1587	0.8413
2	$P(Z < -2)$	0.0228	0.9772
3	$P(Z < -3)$	0.0014	0.9986

The above RBDO problem is solved by using various target reliability indices (β_t) and with different performance functions and target reliability indices in the next subsections. Each subsection is dedicated to one performance function. Then, obtained answers will be compared with the initial optimum solution in order to check effects of different β_ts to solve this problem.

Each subsection includes two tables. The first tables show optimum solutions for random variables (x_1, x_2, y_3). Then, new additional and total cost values and also a percentage of increase in the total cost are shown in the second tables.

Moreover, it should be noted that the conjugate gradient analysis (CGA) method is applied to solve all reliability analysis problems inside these RBDO problems. The unconstrained polar reliability analysis (UPRA) method can also be applied to solve these problems, but the CGA method is selected for this purpose because of its higher convergence rate (speed). However, this selection of reliability analysis method does not affect the obtained final solution.

4.3.3 First Performance Function

The RBDO problem introduced in the previous subsection (Problem (4.26)) is solved in this subsection using the first performance function (Function (4.23)). This performance function is as below:

$$G(x_1, x_2) = \begin{cases} e^{x_1+x_2} - 1 & for \quad x_1 + x_2 \le 0.5578 \\ \sqrt{x_1 + x_2} & for \quad 0.5578 < x_1 + x_2 \end{cases}$$

Optimal solutions of the RBDO problem, obtained by using the above performance function, are displayed in the following tables.

Table (4.2) shows optimal solutions of the random variables (x_1, x_2). In all cases, a summation of both random variables (i.e. $y_3 = x_1 + x_2$) is also provided in the last column of the table.

Table 4.2: Optimum Solutions of Random Variables

β_t	Failure Probability	Accepted Level	x_1	x_2	$y_3 = x_1 + x_2$
0.01	0.4960	0.5040	0.0156	0.0000	0.0156
1	0.1587	0.8413	1.5556	0.0000	1.5556
2	0.0228	0.9772	3.1113	0.0000	3.1113
3	0.0014	0.9986	4.6669	0.0000	4.6669

The following table shows the resulting additional and total cost amounts. Additional cost increases while the acceptable level of failure probability of the system is decreased and thus system's safety is improved.

Table 4.3: New Cost Amounts based on Different β_ts

β_t	Failure Probability	Accepted Level	Extra Cost	Total Cost	Percentage
0.01	0.4960	0.5040	0.0779	375.0779	0.02
1	0.1587	0.8413	7.7782	382.7782	2.08
2	0.0228	0.9772	15.5563	390.5563	4.15
3	0.0014	0.9986	23.3345	398.3345	6.22

It can be seen in the first row of the Table (4.3) that there is only a 0.02% increase in the total cost. However, in this case the failure probability of the system is 0.4960 which means that the system is at high risk and its reliability is not improved very much.

4.3.4 Second Performance Function

In this subsection, the following performance function is applied to solve the RBDO problem:

$$G(x_1, x_2) = arctan(x_1 + x_2)$$

The below table displays the obtained optimal solutions of the RBDO problem considering the above performance function.

Table 4.4: Optimum Solutions of Random Variables

β_t	Failure Probability	Accepted Level	x_1	x_2	$y_3 = x_1 + x_2$
0.01	0.4960	0.5040	0.0156	0.0000	0.0156
1	0.1587	0.8413	1.8931	0.0000	1.8931
2	0.0228	0.9772	3.0090	0.0000	3.0090
3	0.0014	0.9986	4.3940	0.0000	5.3940

Also, the corresponding additional and total costs are shown in the Table (4.5). The last column of the next table is the percentage increase in the total cost values.

It can be concluded from these tables that the second performance function (considered in this subsection) results in a better solution than the previous performance function only when $\beta_t = 2$. In this case, the percentage increase in the total cost is 0.14% less than the corresponding percentage increase obtained with the first performance function.

Table 4.5: New Cost Amounts based on Different β_ts

β_t	Failure Probability	Accepted Level	Extra Cost	Total Cost	Percentage
0.01	0.4960	0.5040	0.0779	375.0779	0.02
1	0.1587	0.8413	9.4654	384.4654	2.52
2	0.0228	0.9772	15.0452	390.0452	4.01
3	0.0014	0.9986	26.9701	401.9701	7.19

But there are an additional 0.44% and 0.97% percentage increases in these amounts compared to the results from the previous performance function when $\beta_t = 1$ and $\beta_t = 3$, respectively.

4.3.5 Third Performance Function

The last performance function, which is considered in this subsection, is defined as below:

$$G(x_1, x_2) = arctan(\frac{x_1 + x_2}{3})$$

This performance function is applied to solve the RBDO problem. Table (4.6) displays the obtained optimal solutions for the random variables.

Table 4.6: Optimum Solutions of Random Variables

β_t	Failure Probability	Accepted Level	x_1	x_2	$y_3 = x_1 + x_2$
0.01	0.4960	0.5040	0.0156	0.0000	0.0156
1	0.1587	0.8413	3.0396	0.0000	3.0396
2	0.0228	0.9772	4.4513	0.0000	4.4513
3	0.0014	0.9986	5.5976	0.0000	5.5976

The below table shows optimal solutions of the cost functions and the corresponding percentages. By comparing results shown in these tables, it can be concluded that the third performance function results in the largest additional costs in all cases (except when $\beta_t = 0.01$).

Table 4.7: New Cost Amounts based on Different β_ts

β_t	Failure Probability	Accepted Level	Extra Cost	Total Cost	Percentage
0.01	0.4960	0.5040	0.0779	375.0779	0.02
1	0.1587	0.8413	15.1978	390.1978	4.05
2	0.0228	0.9772	22.2567	397.2567	4.94
3	0.0014	0.9986	28.4882	403.4882	7.60

Moreover, if the obtained results in this subsection and the previous subsection are compared, we will be able to track effects of changes in the gradient vector of performance functions. As can be seen, the worst results are obtained when $x_1 + x_2$ is divided by 3 in the performance function.

Thus, based on the above obtained optimal solutions, we can conclude that different performance functions may provide different solutions and additional costs for the same acceptable level for network's failure probability.

Further, it can be resulted that although a performance function may work well for a particular safety level, any change in the required safety level may require us to alter the performance function.

4.4 An Enhancement for OPF Model Using Random Variables

In this section, it is intended to extend RBDO formulation to a very well-known optimisation model for electricity power networks. This model, which is fully illustrated in Chapter 2, Section 8, is called optimal power flow (OPF) model. OPF is a classical non-linear optimisation model that is often used, as a powerful tool, to optimise power flows in electricity networks.

The main objective of an OPF problem is to find values of decision variables in order to minimise power generation cost. Constraints of this problem are based on the Kirchhoff's and Ohm's laws. Engineering limits on active and reactive power generation, bus voltage magnitudes, transmission lines and transformer flows are widely considered as different technical constraints in an OPF problem.

The non-linear nature of parameters in an electricity power network may result in non-convexity in an OPF problem. This non-linearity can be found in active power, reactive power and voltage magnitudes. The OPF problem limits the apparent power flow measured at each end of a given line.

However, as discussed earlier, safety issues have always been of concern in electricity networks. Many efforts have been made to reinforce the existing OPF models to cover these issues. But there is still a lack of recognition of an electricity network's failure probability in many existing OPF models.

We aim in this section at modifying the existing OPF model so that failure probability of an electricity network is also considered in the model. For this purpose, a probabilistic constraint should be introduced first. It is intended to keep failure probability of electricity power networks below a predetermined level in our model.

In the RBDO model explained in Chapter 2, a random variable must be defined to formulate a probabilistic constraint. In the meantime, it should be considered what properties of an electricity network are taken into account when defining random variables.

When a random variable is determined, a performance function $G(x_1, x_2, ..., x_n)$ should be formulated so that a safety/failure condition can be defined for the system. The formulated performance function is then used to find a safety region, a failure region, and a limit state function.

In general, an OPF problem, which is fully illustrated in Subsection 2.8.2, is written as below:

$$
\begin{aligned}
& Min\ F(P_G) = \sum_{k \in G} f_k(P_{G_k}) \\
& s.t.\ \ P_{G_k} - P_{D_k} = V_k \sum_{j=1 \neq k}^{N} V_j Y_{kj} cos(\delta_k - \delta_j - \theta_{kj}) \quad k = 1, 2, ..., N \\
& \qquad Q_{G_k} - Q_{D_k} = V_k \sum_{j=1 \neq k}^{N} V_j Y_{kj} sin(\delta_k - \delta_j - \theta_{kj}) \quad k = 1, 2, ..., N \\
& \qquad P_{G_k}^{min} \leq P_{G_k} \leq P_{G_k}^{max} \qquad k = 1, 2, ..., g \\
& \qquad Q_{G_k}^{min} \leq Q_{G_k} \leq Q_{G_k}^{max} \qquad k = 1, 2, ..., g \\
& \qquad V_k^{min} \leq V_k \leq V_k^{max} \qquad k = 1, 2, ..., n \\
& \qquad |S_k| \leq S_k^{max} \qquad k = 1, 2, ..., l
\end{aligned}
\tag{4.27}
$$

where G is the set of generator buses, V_k is the voltage magnitude at bus $k \in N$ and S_k is apparent power flow on the existing line k.

The OPF cost function is often expanded as follows:

$$F(P_G) = \sum_{i \in G} (\alpha_i + \beta_i P_{G_i} + \gamma_i P_{G_i}^2)$$

As this function is a function of active power P_G and since the active power is itself a function of voltage magnitudes (V) and phase angle (θ), hence it can be resulted that the OPF cost function is indeed a function of V and θ. In other words, it can be concluded that:

$$Cost = h(V, \theta)$$

Random variables of the intended RBDO problem will then be defined based on the above mentioned variables (i.e. V and θ). Hence, it can be concluded that cost function can be defined in terms of random variables.

As discussed in the previous sections, an important step of formulating an RBDO problem is to define a performance function. A performance function is a function of defined random variables. In an OPF problem, either equality or inequality constraints may be used to define a safety/failure condition and introduce a performance function.

Each performance function is assumed to formulate a probabilistic constraint (and then to formulate a reliability analysis problem). In other words, if the network had g generators, n nodes and l lines, we can introduce $4g+2n+l$ probabilistic constraints by using the existing boundaries and also $2g$ probabilistic constraints by using the equality constraints of an OPF problem.

It will be shown in the coming subsections how new random variables must be defined to formulate a probabilistic constraint.

4.4.1 Random Variable Definition Based on Boundaries

The first step of formulating a probabilistic constraint for an electricity network is to define random variable(s). For this purpose, it is assumed in this subsection that the bounds of an OPF problem are used to define performance functions.

As can be seen in the OPF formulation (Model (4.27)), there are four groups of inequality constraints (including seven sets of bounds) in an OPF problem. If n and g indicate the number of all nodes and generators, respectively, and also l is the number of existing lines, then the maximum number of bounds is $4g + 2n + l$, because there are $2g$ bounds on active power flows in generating stations, $2g$ bounds on reactive power flows in generating stations, $2n$ bounds on voltage magnitudes in all nodes and l bounds on apparent powers on the existing lines in an OPF problem.

Suppose that (V^*, θ^*) denotes the optimal solution of the OPF problem such that:

$$V^* = [V_1^*, V_2^*, ..., V_n^*]^T$$

$$\theta^* = [\theta_1^*, \theta_2^*, ..., \theta_n^*]^T$$

where n is the number of all nodes in electricity network. Random variables are defined as a perturbation of this optimal solution.

Two new variables are considered for this adjustment. These new variables are $(\hat{V}, \hat{\theta})$ and will be used to formulate a performance function for electricity network. Based on the difference between the defined variables $(\hat{V}, \hat{\theta})$ and the obtained optimal solutions (V^*, θ^*), we are now able to define new random variables. These random variables will be used to formulate system performance functions.

Random variables are denoted by x_1 and x_2 and defined as follows:

$$x_1 = \hat{V} - V^* \tag{4.28}$$

$$x_2 = \hat{\theta} - \theta^* \tag{4.29}$$

The original cost function, which was a function of V and θ, can now be reformulated as a function of x_1 and x_2. These random variables can generally be positive or negative depending on conditions of electricity network and OPF problem. In this case, if x_1 and/or x_2 equals zero, then it can be concluded that the relevant variable(s) has (have) reached the optimum amount obtained by the OPF problem.

Different inequalities in the OPF problem may be used to define random variables. Three possible approaches for random variable definitions based on various bounds are illustrated in this subsection as follows:

1. Based on Active and Reactive Power Flows:

Active and reactive power flows are two functions of voltage magnitudes (V) and phase angle (θ). These functions are given below:

$$P = VIcos(\theta)$$

$$Q = VIsin(\theta)$$

When an OPF problem is solved, the optimum solution is obtained based on the above mentioned variables; i.e. V and θ. Hence, it is easy to compute active and reactive power flows (P and Q, respectively) using the obtained values for V and θ.

Each bound of active and reactive power flows can be considered to formulate a performance function. For instance, if we consider upper boundary of active power flow at generator k, then we have:

$$P_{G_k} \leq P_{G_k}^{max}$$

The above inequality can also be rewritten by using the definition of active power flow as follows:

$$V_k I_k cos(\theta_k) \leq P_{G_k}^{max}$$

Therefore, the defined random variables can now be used to formulate a performance function. If the latter inequality is written in temrs of the new variables $(\hat{V}, \hat{\theta})$, then we have:

$$\hat{V}_k I_k cos(\hat{\theta}_k) \leq P_{G_k}^{max}$$

and then by replacing these variables with random variables using Equalities (4.28) and (4.29), it follows that:

$$(V_k^* + x_{k_1}) I_k cos(\theta_k^* + x_{k_2}) \leq P_{G_k}^{max}$$

where the only variables (unknowns) in the above inequality are x_{k_1} and x_{k_2}.

Moreover, based on the first inequality constraint of the OPF problem, it can be concluded that the following condition results in failure of an electricity network:

$$P_{G_k} > P_{G_k}^{max}$$

and then this failure condition can simply be rewritten as follows:

$$(V_k^* + x_{k_1}) I_k cos(\theta_k^* + x_{k_2}) > P_{G_k}^{max}$$

Therefore, a performance function can now be formulated based on the above failure condition as following:

$$G_1(x_{k_1}, x_{k_2}) = P_{G_k}^{max} - (V_k^* + x_{k_1}) I_k cos(\theta_k^* + x_{k_2}) \quad (k = 1, 2, ..., g) \tag{4.30}$$

This performance function is formulated using two newly defined random variables; i.e. x_{k_1} and x_{k_2}.

A similar process can be implemented in order to define a performance function based on the lower bound of active power flow at node k. A failure condition can be determined by using this boundary as follows:

$$(V_k^* + x_{k_1}) I_k cos(\theta_k^* + x_{k_2}) < P_{G_k}^{min}$$

Thus, the corresponding performance function is formulated as below:

$$G_2(x_{k_1}, x_{k_2}) = (V_k^* + x_{k_1}) I_k cos(\theta_k^* + x_{k_2}) - P_{G_k}^{min} \quad (k = 1, 2, ..., g) \tag{4.31}$$

Furthermore, bounds on reactive power flow at node k may be used to define performance functions. Two failure conditions are considered for reactive power flow at this node as below:

$$(V_k^* + x_{k_1}) I_k sin(\theta_k^* + x_{k_2}) > Q_{G_k}^{max}$$

$$(V_k^* + x_{k_1}) I_k sin(\theta_k^* + x_{k_2}) < Q_{G_k}^{min}$$

and thus the following are the corresponding performance functions:

$$G_3(x_{k_1}, x_{k_2}) = Q_{G_k}^{max} - (V_k^* + x_{k_1}) I_k sin(\theta_k^* + x_{k_2}) \quad (k = 1, 2, ..., g) \tag{4.32}$$

$$G_4(x_{k_1}, x_{k_2}) = (V_k^* + x_{k_1}) I_k sin(\theta_k^* + x_{k_2}) - Q_{G_k}^{min} \quad (k = 1, 2, ..., g) \tag{4.33}$$

It is not difficult to see that $G_i(x_{k_1}, x_{k_2}) < 0$ and $G_i(x_{k_1}, x_{k_2}) > 0 \quad (i = 1, 2, 3, 4)$ indicate failure and safety regions, respectively. Also, $G_i(x_{k_1}, x_{k_2}) = 0$ shows failure surface or limit state function for electricity power network.

2. Based on Voltage Magnitudes:

There are $2n$ bounds for voltage magnitudes in an OPF problem. Suppose that V_k^* is the optimal solution for voltage magnitude at node k, obtained by solving the OPF problem. There is an inequality constraint for voltage magnitude at node k as below:

$$V_k^{min} \leq V_k \leq V_k^{max}$$

Suppose that $\hat{V}_k$ is a new variable based on voltage magnitude at node k to consider system's failure probability. Also, suppose that we aim at introducing a performance function for this node using, for example, its upper bound. Hence, a new random variable can be defined at this node as follows:

$$y_k = \hat{V}_k - V_k^* \Rightarrow \hat{V}_k = V_k^* + y_k$$

Thus, we have:

$$\hat{V}_k = V_k^* + y_k \leq V_k^{max}$$

Hence, system fails if $V_k^{max} < V_k^* + y_k$. Therefore, a performance function can be formulated as below:

$$G_5(y_k) = V_k^{max} - (y_k + V_k^*) \Rightarrow G_5(y_k) = (V_k^{max} - V_k^*) - y_k \quad (k = 1, 2, ..., n) \tag{4.34}$$

The above linear function can be supposed as a safety condition to formulate a performance function. Also, this function itself may be considered as a performance function or to define a new bound for the RBDO problem.

By using a similar process, a failure condition and performance function can be defined based on the lower bound of voltage magnitude at node k. This bound is written as below:

$$V_k^{min} \leq \hat{V}_k = V_k^* + y_k$$

Hence, the following inequality results in system failure:

$$V_k^* + y_k < V_k^{min}$$

Therefore, a performance function is formulated as follows:

$$G_6(y_k) = y_k + V_k^* - V_k^{min} \Rightarrow G_6(y_k) = y_k + (V_k^* - V_k^{min}) \quad (k = 1, 2, ..., n) \tag{4.35}$$

It can simply be shown that $G_i(y_k) < 0$ and $G_i(y_k) > 0 \quad (i = 5, 6)$ are held as failure and safety conditions. In both cases, limit state function is determined by $G_i(y_k) = 0$.

3. Based on Apparent Power Flow:

Each existing line has an upper bound for its apparent power flow. Having l lines in a network, we will have l inequalities for the existing lines' apparent power flows. As $S = P + jQ$, we have:

$$|S| = \sqrt{P^2 + Q^2} \Rightarrow |S| = VI$$

Therefore, inequality constraint of an OPF problem can be rewritten as follows:

$$V_k I_k \leq S_k^{max}$$

Again, assuming V_k^* as the obtained optimal solution, a new variable is defined for voltage magnitude as below:

$$\hat{V}_k = V_k^* + z_k$$

Hence, various conditions for the network can be obtained based on apparent power flow on the line k as following:

1. Safety Condition: $V_k^* + z_k < S_k^{max}$
2. Failure Condition: $V_k^* + z_k > S_k^{max}$
3. Failure Surface: $V_k^* + z_k = S_k^{max}$

Thus, a performance function can be defined as below:

$$G_7(z_k) = S_k^{max} - (V_k^* + z_k) \quad (k = 1, 2, ..., l) \tag{4.36}$$

All bounds of an OPF problem are now considered for random variable definitions in order to formulate performance functions and probabilistic constraints.

It can easily be seen that if the new variables defined for voltage magnitudes in various boundaries of P, Q, V and S are the same, all performance functions can be formulated using only two random variables. In this case, x_1 is the first random variable that is defined for voltage magnitude. Also, the second random variable is defined for phase angle and denoted by x_2.

Further, it should be noted that two above random variables (i.e. x_1 and x_2) should be defined individually for each node. In other words, each node needs two random variables and hence the whole network requires $2n$ random variables.

When all required random variables are defined and also all performance functions are formulated, the resulting reliability analysis problem(s) can be solved by using either method introduced in Chapter 3. This step, which is generally called probabilistic constraint evaluation in the solution process of a non-deterministic design optimisation problem, plays a significant role in finding optimal solutions for a system subject to uncertainties.

It must also be noted that before solving a reliability analysis problem, we need to define or determine various standard deviations (σ); each σ for one random variable.

In this case, there is no need to define/determine relevant expected values (μ), because all expected values must be equal to the obtained optimum solutions of the OPF problem so that initial design point in the standard normalised random space (U-space) would be origin of the U-space. These expected values are in fact considered as design variables of non-deterministic design optimisation problem.

In the next subsection, equality constraints of the OPF model will be considered for introducing probabilistic constraints and formulating a new non-deterministic design optimisation problem for electricity networks.

4.4.2 Formulating Performance Functions Using Equality Constraints

It is discussed in the previous subsection how to formulate performance functions for an OPF problem using its bounds. This concept is extended here to equality constraints of an OPF problem.

There are two groups of equality constraints in an OPF problem; the first group is for active power flows in generating stations and the second group is for reactive power flows in generating stations. Hence, if g denotes the number of generators in an electricity network, there could be $2g$ equality constraints in the corresponding OPF problem.

Similar to the previous subsection, suppose that $V^* = [V_1^*, V_2^*, ..., V_n^*]^T$ and $\theta^* = [\theta_1^*, \theta_2^*, ..., \theta_n^*]^T$ are the obtained optimum solution for an OPF problem. These can also be described as a single vector as shown below:

$$d^* = [d_1^*, d_2^*, ..., d_{2n}^*]^T = [V_1^*, V_2^*, ..., V_n^*, \theta_1^*, \theta_2^*, ..., \theta_n^*]^T$$

Thus, a random variable should be defined as a modification of this optimum point in order to formulate new performance functions.

Considering $\hat{d} = [\hat{d}_1, \hat{d}_2, ..., \hat{d}_{2n}]^T$ as a new variable based on a modification on optimum solution d^*, a random variable can be defined as below:

$$x_i = \hat{d}_i - d_i^* \qquad i = 1, 2, ..., 2n$$

Since the existing equality constraints of the OPF problem are initially written with respect to the variables V and θ, they can now be rewritten in terms of the variable $\hat{d} = [\hat{V}, \hat{\theta}]$. In this case, new constraints are obtained (after this change of variables and required simplifications) with respect to newly defined random variable $X = [x_1, x_2, ..., x_n]^T$.

The new equations, which are written using the random variable X, can be considered as new performance functions. It can simply be shown that setting the obtained performance functions to zero (i.e. $G(x_1, x_2, ..., x_n) = 0$) yields deterministic constraints in the OPF problem.

In this case, $G(x_1, x_2, ..., x_n) \neq 0$ indicates system failure and thus new failure regions for the network can be defined based on the above mentioned concepts. These failure regions and related failure probabilities will then be used to formulate a probabilistic constraint for the network in order to apply in a new RBDO problem.

The next sections are dedicated to solving some numerical experiments in order to illustrate the introduced constraints in his section and their effects on the optimal solution of an OPF problem.

4.5 Numerical OPF Examples

The optimal power flow (OPF) problem was selected for further modification in the previous section in order to introduce a probabilistic constraint for this popular optimisation problem of electricity power networks.

An OPF problem consists of various constraints that can generally be divided into two groups as equality and inequality constraints. Equality constraints are based on the power balance laws and inequality constraints are effectively bounds for different parameters.

In this section, details of an OPF problem and its solution are explained in two subsections. Equality constraints of an OPF problem are illustrated in the first subsection. The second subsection includes an OPF problem that is written based on a three bus example network. An RBDO problem will then be formulated and solved in the next section.

4.5.1 Equality Constraints

Figure (4.2) shows a five bus example network. As can be seen in the figure, this network includes only one generating station (Node 1), because this node is the only node in which P and $|V|$ are given. Based on the explanations given in Chapter 1, Section 1.2, Nodes 2, 3 and 4 are load nodes and Node 5 is a slack bus in this network.

Using the given data in the Figure (4.2), various pieces of information are obtained for this network. This information is displayed in the Table (4.8).

Since there is only one generating station in this network (Node 1), there will be just two equality constraints in the relevant OPF problem; one constraint for active power flow at this node and one constraint for reactive power flow.

Equality constraints of the OPF problem for the mentioned five bus system are written in polar form as follows:

$$P_{G_1} - P_{D_1} = |V_1|\{|V_2||Y_{12}|cos(\delta_1 - \delta_2 - \theta_{12}) + |V_3||Y_{13}|cos(\delta_1 - \delta_3 - \theta_{13}) + ...$$
$$|V_4||Y_{14}|cos(\delta_1 - \delta_4 - \theta_{14}) + |V_5||Y_{15}|cos(\delta_1 - \delta_5 - \theta_{15})\}$$

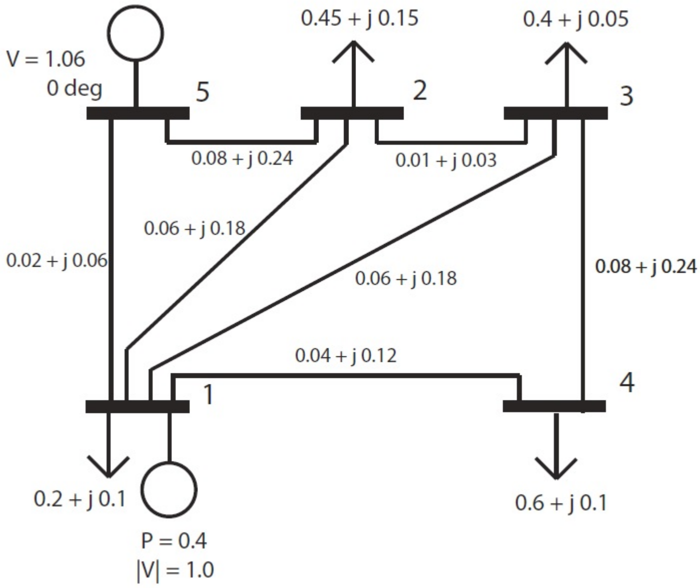

Figure 4.2: Five Bus System

$$Q_{G_1} - Q_{D_1} = |V_1|\{|V_2||Y_{12}|sin(\delta_1 - \delta_2 - \theta_{12}) + |V_3||Y_{13}|sin(\delta_1 - \delta_3 - \theta_{13}) + ...$$
$$|V_4||Y_{14}|sin(\delta_1 - \delta_4 - \theta_{14}) + |V_5||Y_{15}|sin(\delta_1 - \delta_5 - \theta_{15})\}$$

Table 4.8: Conductance (G), Susceptance (B), Admittance (Y) and Admittance Angles (θ)

From Bus	To Bus	G	B	Y	$\|Y\|$	θ (degrees)
1	2	0.06	0.18	$0.06 + i0.18$	0.1897	71.5
1	3	0.06	0.18	$0.06 + i0.18$	0.1897	71.5
1	4	0.04	0.12	$0.04 + i0.12$	0.1265	71.5
1	5	0.02	0.06	$0.02 + i0.06$	0.0632	71.5
2	3	0.01	0.03	$0.01 + i0.03$	0.0316	71.5
2	5	0.08	0.24	$0.08 + i0.24$	0.2530	71.5
3	4	0.08	0.24	$0.08 + i0.24$	0.2530	71.5

Using the provided data, these constraints can be rewritten as below:

$$P_{G_1} - 0.2 = |V_1|\{0.1897|V_2|cos(\delta_1 - \delta_2 - 71.5) + 0.1897|V_3|cos(\delta_1 - \delta_3 - 71.5) + ...$$
$$0.1265|V_4|cos(\delta_1 - \delta_4 - 71.5) + 0.0632|V_5|cos(\delta_1 - \delta_5 - 71.5)\}$$
$$Q_{G_1} - 0.5095 = |V_1|\{0.1897|V_2|sin(\delta_1 - \delta_2 - 71.5) + 0.1897|V_3|sin(\delta_1 - \delta_3 - 71.5) + ...$$
$$0.1265|V_4|sin(\delta_1 - \delta_4 - 71.5) + 0.0632|V_5|sin(\delta_1 - \delta_5 - 71.5)\}$$

An optimum solution that satisfies these two constraints is displayed in the Table (4.9).

Table 4.9: Solution of the Five Bus Example System

Node	1	2	3	4	5
V	1.0000	0.9805	0.9771	0.9662	1.0600
δ	-2.0675	-4.5358	-4.8535	-5.6925	0.0000

4.5.2 Three Node Network

An OPF problem for a three bus example system is studied and solved in this subsection. Figure (4.3) displays the relevant network.

No active power is generated at Node 3 and hence there is no cost here. In other words, all cost parameters (related to active power flow) at this node are zero. This leads to:

$$P_{G_3} = 0 \Rightarrow c_{0,3} = c_{1,3} = c_{2,3} = 0$$

Cost function parameters for the other nodes (two generating stations at Nodes 1 and 2) are given in the Table (4.10).

Further, Table (4.11) displays active and reactive power loads at all nodes. Conductance, susceptance, admittance and its norms as well as angle of various elements in admittance

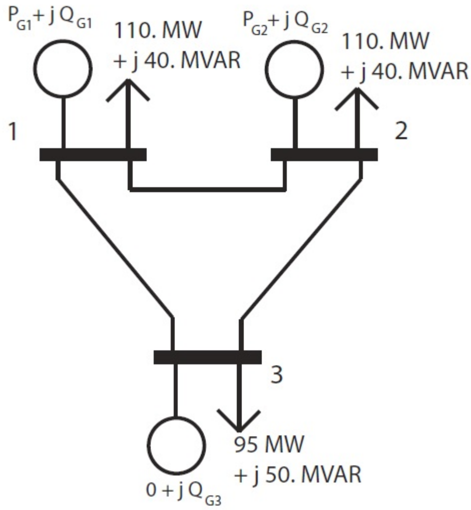

Figure 4.3: Three Bus System

Table 4.10: Cost Function Parameters

Generator	c_0	c_1	c_2
1	0	5	0.11
2	0	1.2	0.085

matrix can also be easily calculated. These are shown in the Table (4.12).

Table 4.11: Active and Reactive Power Loads

Generator	P_D	Q_D
1	110	40
2	110	40
3	95	50

Moreover, there are no bounds for active and reactive power flows. Apparent power flows on the lines 1-2 and 1-3 are also not bounded.

Hence, the OPF problem, based on the shown network and using the provided data, is

Table 4.12: Conductance, Susceptance, Admittance and Admittance Angles

From Bus	To Bus	G	B	Y	$\|Y\|$	θ (degrees)
1	3	0.1673	-1.5954	$0.1673 - i1.5954$	1.6041	-84
3	2	0.0444	-1.3318	$0.0444 - i1.3318$	1.3326	-88
1	2	0.0517	-1.1087	$0.0517 - i1.1087$	1.1099	-87

written in polar form as below:

$$Min\ 5P_{G_1} + 0.11P_{G_1}^2 + 1.2P_{G_2} + 0.085P_{G_2}^2 \tag{4.37}$$

$$\begin{aligned}
s.t.\ \ P_{G_1} - 110 &= V_1\{1.1099V_2cos(\delta_1 - \delta_2 + 87) + 1.6041V_3cos(\delta_1 - \delta_3 + 84)\} \\
P_{G_2} - 110 &= V_2\{1.1099V_1cos(\delta_2 - \delta_1 + 87) + 1.3226V_3cos(\delta_2 - \delta_3 + 88)\} \\
-95 &= V_3\{1.6041V_1cos(\delta_3 - \delta_1 + 84) + 1.3226V_2cos(\delta_3 - \delta_2 + 88)\} \\
Q_{G_1} - 40 &= V_1\{1.1099V_2sin(\delta_1 - \delta_2 + 87) + 1.6041V_3sin(\delta_1 - \delta_3 + 84)\} \\
Q_{G_2} - 40 &= V_2\{1.1099V_1sin(\delta_2 - \delta_1 + 87) + 1.3326V_3sin(\delta_2 - \delta_3 + 88)\} \\
Q_{G_3} - 50 &= V_3\{1.6041V_1sin(\delta_3 - \delta_1 + 84) + 1.3326V_2sin(\delta_3 - \delta_2 + 88)\} \\
& 0.9 \leq V_1, V_2, V_3 \leq 1.1 \quad \& \quad |S_{23}| \leq 60
\end{aligned}$$

This problem can be solved by using various general optimisation methods for constrained non-linear problems such as augmented Lagrangian multipliers. Table (4.13) displays the obtained results.

Table 4.13: Solution to Three Bus System

Node	1	2	3
$\|V\|$	1.069	1.028	1.001
δ (degrees)	0	9.916	-13.561

Thus, active and reactive power flows in different nodes can be obtained. These information are shown in the Table (4.14).

Table 4.14: Active and Reactive Power Flows

Node	1	2	3
P_g	131.09	185.93	0
Q_g	17.02	-3.50	0.06

In the next section, a new RBDO problem will be formulated and solved based on an OPF problem.

4.6 An RBDO Problem Based on an OPF Problem

We now solve a numerical example to illustrate the idea explained in Section (4.4). For this purpose, consider the electricity power network shown in the Figure (4.4).

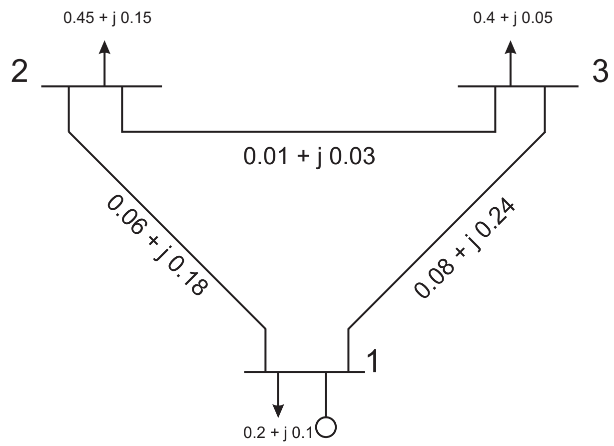

Figure 4.4: Three Bus System

As can be seen in the figure, there are three nodes in this network including only one generating station (Node 1). An OPF problem must first be formulated for this network.

The cost function parameters of the generating station, which will be used to formulate the mentioned OPF problem, are given as below:

$$c_0 = 3 \quad , \quad c_1 = 2 \quad , \quad c_2 = 1.2$$

It should be noticed that all above parameters are for Node 1. Also, based on the displayed network, load flows (active and reactive powers) and admittances as well as corresponding angles are written in the Tables (4.15) and (4.16).

Table 4.15: Active and Reactive Power Loads

Generator	P_D	Q_D
1	0.20	0.10
2	0.45	0.15
3	0.40	0.05

Table 4.16: Admittances and Admittance Angles

From Bus	To Bus	Y	$\|Y\|$	θ (degrees)
1	2	$0.06 + i0.18$	0.1897	71.56
1	3	$0.08 + i0.24$	0.2530	71.56
2	3	$0.01 + i0.03$	0.0316	71.56

Moreover, it must be noted that there are no bounds for any variable in this problem. Thus, the intended OPF problem can be formulated as:

$$Min \ 1.2P_{G_1}^2 + 2P_{G_1} + 3 \tag{4.38}$$

$$s.t. \quad P_1 = 0.1897V_1V_2cos(\delta_1 - \delta_2 - 71.56) + 0.2530V_1V_3cos(\delta_1 - \delta_3 - 71.56)$$

$$Q_1 = 0.1897V_1V_2sin(\delta_1 - \delta_2 - 71.56) + 0.2530V_1V_3sin(\delta_1 - \delta_3 - 71.56)$$

The obtained solution for the above OPF problem is displayed in the Table (4.17). Hence, optimum value for the cost function is calculated as 5.1993.

Table 4.17: Obtained Solution for the Given Network

	Bus 1	Bus 2	Bus 3
$\|V\|$	4.54	0.09	2.38
δ (degrees)	107.83	39.29	282.24

Now, the obtained solution is used to formulate new probabilistic constraints for the given network in order to obtain an RBDO problem. For this purpose, all variables in the above optimisation problem must be redefined in terms of new design variables.

The following new design variables are considered:

$$d = [d_1 = V_1, d_2 = V_2, d_3 = V_3, d_4 = \delta_1, d_5 = \delta_2, d_6 = \delta_3]^T$$

Thus, the above OPF problem can be rewritten using the new design variable as below:

$$\begin{aligned} &Min\ 0.432d_1^2cos^2(d_4) + 1.2d_1cos(d_4) + 3 &(4.39)\\ &s.t.\ \ 0.6d_1cos(d_4) - 0.1897d_1d_2cos(d_4 - d_5 - 71.56) - 0.2530d_1d_3cos(d_4 - d_6 - 71.56) = 0.2 \\ &\quad\ 0.6d_1sin(d_4) - 0.1897d_1d_2sin(d_4 - d_5 - 71.56) - 0.2530d_1d_3sin(d_4 - d_6 - 71.56) = 0.1 \end{aligned}$$

In this notation, the obtained optimum point is denoted by d^*. Hence, we have:

$$d^* = [d_1^* = 4.54, d_2^* = 0.09, d_3^* = 2.38, d_4^* = 107.83, d_5^* = 39.29, d_6^* = 282.24]^T$$

Since our aim is to re-optimise the above system so that its reliability is also taken into account, it is necessary to modify the existing optimum point. Suppose that $\hat{d}$ is a new variable in the reformulated problem.

Then, we need to define a random variable using which performance functions and probabilistic constraints will be formulated. Suppose that $x = [x_1, x_2, ..., x_6]^T$ is the intended random variable. In other words, the random variable x can be considered as a modification on the optimum point d^*.

$$\hat{d}_i = d_i^* + x_i \quad \Rightarrow \quad x_i = \hat{d}_i - d_i^*$$

The following hold for these new variables:

$$x = [\hat{d}_1 - 4.54, \hat{d}_2 - 0.09, \hat{d}_3 - 2.38, \hat{d}_4 - 107.83, \hat{d}_5 - 39.29, \hat{d}_6 - 282.24]^T$$

or

$$\hat{d} = [x_1 + 4.54, x_2 + 0.09, x_3 + 2.38, x_4 + 107.83, x_5 + 39.29, x_6 + 282.24]^T$$

When the random variable x is defined (using the new variable $\hat{d}$), the OPF cost function must be reformulated by replacing the existing variable d by the new variable $\hat{d}$. Hence, this cost function can be rewritten as below:

$$0.432\hat{d}_1^2 cos^2(\hat{d}_4) + 1.2\hat{d}_1 cos(\hat{d}_4) + 3$$

and then:

$$0.432(x_1 + 4.54)^2 cos^2(x_4 + 107.83) + 1.2(x_1 + 4.54)cos(x_4 + 107.83) + 3$$

Moreover, we want to define some probabilistic constraints for this network in order to introduce a new RBDO problem. For this purpose, we must first define performance functions for this network.

In this case, design variable d in the equality constraints of the OPF problem (Problem (4.39)) should be replaced by the new variable $\hat{d}$. By this replacement and after required simplifications, we will have:

$$
\begin{aligned}
&0.6(x_1+4.54)cos(x_4+107.83)-0.1897(x_1+4.54)(x_2+0.09)cos(x_4-x_5-3.02)...\\
&\quad -0.2530(x_1+4.54)(x_3+2.38)cos(x_4-x_6-245.97)=0.2\\
&0.6(x_1+4.54)sin(x_4+107.83)-0.1897(x_1+4.54)(x_2+0.09)sin(x_4-x_5-3.02)...\\
&\quad -0.2530(x_1+4.54)(x_3+2.38)sin(x_4-x_6-245.97)=0.1
\end{aligned}
$$

Now, we use the above equality constraints to introduce new performance functions for this network. Hence, the new performance functions would be defined as following:

$$
\begin{aligned}
G_1(x) &= 0.6(x_1+4.54)cos(x_4+107.83)-0.1897(x_1+4.54)(x_2+0.09)... \qquad (4.40)\\
&\quad cos(x_4-x_5-3.02)-0.2530(x_1+4.54)(x_3+2.38)cos(x_4-x_6-245.97)-0.2\\
G_2(x) &= 0.6(x_1+4.54)sin(x_4+107.83)-0.1897(x_1+4.54)(x_2+0.09)... \qquad (4.41)\\
&\quad sin(x_4-x_5-3.02)-0.2530(x_1+4.54)(x_3+2.38)sin(x_4-x_6-245.97)-0.1
\end{aligned}
$$

Therefore, a new RBDO problem can now be formulated as below:

$$
\begin{aligned}
&Min\ 0.432(x_1+4.54)^2cos^2(x_4+107.83)+1.2(x_1+4.54)cos(x_4+107.83)+3\\
&s.t.\quad P[G_i(x_1,x_2,...x_6)\neq 0]\leq \Phi(-\beta_{t_i}) \qquad i=1,2 \qquad (4.42)\\
&\qquad (x_1,x_2,...x_6)^L \leq (x_1,x_2,...x_6) \leq (x_1,x_2,...x_6)^U
\end{aligned}
$$

where functions (4.40) and (4.42) are two performance functions defined for this problem.

It must be noted that there is no bound in the OPF problem. Hence, the above mentioned performance functions are obtained using the method explained in Subsection (4.4.2).

Suppose that target reliability index for both probabilistic constraints equals 2 (i.e. $\beta_{t_i} =$

2 $(i = 1, 2)$). Also, it is considered that standard deviation (σ) for potential changes in all random variables is 0.4; i.e. $\sigma_j = 0.4 \quad j = 1, 2, ..., 6$.

If the optimal point obtained for the OPF problem is assumed as the initial design point of the RBDO problem, a new optimum point is found as below. This optimum solution is obtained by using the CGA method in the inner loop and the SQP algorithm in the outer loop (same as Section 3.4).

$$x^* = [4.91, 0.24, 2.71, 92.34, 41.52, 217.14]^T$$

Therefore, it can be concluded that with only 1.8416 increase in the total cost, which is 35% of the initial total cost, failure probability of the network is kept below than 2.28% as $\beta_t = 2$.

4.7 Conclusion

Electricity power networks are well known as examples of large scale systems. There are many optimisation models available for these networks. However, failure probability of these networks is not studied comprehensively in the existing literature.

We have studied electricity networks in this thesis and dealt with investigation of their reliability in this chapter. At the beginning of the current chapter, a general idea is explained about probabilistic constraints and how to apply them into electricity networks optimisation problems. It is found that a performance function must be formulated for a network in order to define its safety and failure conditions and considering network's reliability.

Based on the defined performance function and also safety and failure conditions, a probabilistic constraint is introduced for electricity networks so that failure probability of the network is kept below a predetermined and acceptable level.

After that, a reliability-based design optimisation (RBDO) model is introduced in this chapter to apply into electricity networks. Although most of the existing optimisation models of electricity networks are non-linear and it is accepted that this non-linearity comes from the nature of electricity power flow, the introduced RBDO model is based on a linear programming (LP) problem.

It is found that the LP problem must first be solved and then its optimal solution is modified using the introduced RBDO problem. This modification, which is done to consider an electricity networks failure probability, results in extra cost.

The new RBDO model of electricity networks is intended to minimise the extra cost, while network's failure probability is kept below an intended level. This level for failure probability of electricity networks is calculated by using a target reliability index (β_t) and is based on the theory of probability.

Further, an optimal power flow (OPF) model, as an existing and popular optimisation model for electricity power networks, is considered for further modification. Different constraints of the OPF problem are used to introduce probabilistic constraints for electricity networks.

All concepts discussed in this chapter are investigated by using numerical experiments. Various performance functions are applied in these problems. Based on the obtained results, it is concluded that the additional cost is increased when the acceptable level of system failure probability is decreased.

In other words, a bigger target reliability index (β_t), which leads to a lower acceptable level of system failure probability and thus a higher level of system safety, leads to a higher additional cost.

Chapter 5

Conclusion and Future Works

Reliability-based design optimisation (RBDO) is known as a non-deterministic and highly non-linear design optimisation model. This model, which is often solved in two different loops, is studied in this research project. Also, electricity power networks are considered as large-scale systems for which new non-deterministic optimisation models are proposed.

There are three kinds of RBDO approaches, mono-level (single-loop) approaches, two-level (double-loop) approaches and decoupled approaches. Different features of these approaches are summarised as follows:

1. Single-loop approaches are based on some approximations and hence their precision is not at an acceptable level. However, some mono-level formulations are very efficient and simple to implement.

2. Double-loop approaches are simple to implement, but they are usually inefficient for real world structures. In these approaches, reliability analysis and design optimisation problems are solved in two loops.

3. Decoupled approaches are generally efficient and accurate, but they require specific implementations. A sequential process is often used to solve an RBDO problem by using a decoupled approach.

Double-loop RBDO approaches are mainly considered in this thesis for further modification and improvement. The reliability index approach (RIA) and the performance measure approach (PMA) are two double-loop RBDO approaches. As it has been reported in the literature that PMA is generally more stable and efficient than RIA, new reliability analysis methods that are introduced in this research project are compared with reliability analysis methods inside PMA.

Reliability related issues play very significant roles in double-loop RBDO approaches. This importance has led to the introduction of different reliability analysis methods. However, there are still many drawbacks in these approaches.

Moreover, many optimisation models are available for electricity power networks. The main concern in these models is to minimise total cost of the network. In other words, although safety of these networks is very important, there is very little research in this area about safety issues and non-deterministic optimisation models of electricity networks.

In this chapter, a conclusion of this PhD thesis is illustrated in two different sections. The first section includes a brief conclusion about new reliability analysis methods that are introduced in this thesis, and then research aims about electricity networks are addressed in the second section. The last section of this chapter consists of our plans for future research work.

5.1 New Reliability Analysis Methods

Two new reliability analysis methods are introduced in this thesis in order to apply in reliability analysis loop (inner loop) of the performance measure approach (PMA) to evaluate performance functions. These methods are called the conjugate gradient analysis (CGA) method and the unconstrained polar reliability analysis (UPRA) method.

The CGA method is formulated based on the conjugate gradient direction. This vector is employed in the CGA method to find a new search direction and update a design point. In this method, reliability analysis problems are first standard normalised in order to reduce their non-linearity, and then solved by the CGA method.

Another new reliability analysis method is the UPRA method that solves reliability analysis problems in the polar space. In this method, a standard normalised reliability analysis problem is transformed to the polar space. By this transformation, a reliability analysis problem, which is a constrained optimisation problem, is changed to an unconstrained optimisation problem.

Efficiency and stability of these methods are investigated by solving several numerical problems in this thesis. Since it has been reported in the literature that hybrid mean value (HMV) method is the most stable and efficient existing reliability analysis method, stability and efficiency of the CGA and UPRA methods are compared with those of the HMV method.

It is found in this thesis that the newly introduced methods are more stable and efficient than the HMV method. A function type criterion has been used in the HMV method to select advanced mean value (AMV) method or conjugate mean value (CMV) method for solving a problem.

Based on the comparisons between all reliability analysis methods, it is shown that stability and efficiency of the CGA and UPRA methods are much greater if behaviour of the HMV method to choose either AMV or CMA methods changes during iterations and/or amount of the function type criterion is too small.

However, the best property of the CGA and UPRA methods is their stability, not efficiency. They always converge even if their convergence rate was not good, while the HMV method sometimes diverges when evaluating a performance function. In other words, the new methods are always stable and also often more efficient than the existing reliability analysis methods.

Performance function values obtained by the new methods are often smaller than those obtained by the existing HMV method. Also, a smaller number of iterations and a shorter time are required for the new reliability analysis methods for convergence compared with the HMV method.

5.2 Application of RBDO in Electricity Power Networks

One of the most complicated systems ever known in real world problems is an electricity power network. Different optimisation models have been proposed in the literature for these systems. Mono- and multi-objective models as well as a number of stochastic optimisation models and various reliability indices are available for these networks. However, there are still many concerns about reliability related issues in electricity networks and corresponding optimisation models.

In this thesis, a general probabilistic constraint is formulated for these networks. For this purpose, a safety/failure condition is first defined for a network. Based on this, a performance function is introduced for the network. Then, assuming an acceptable level for failure probability of the network, a probabilistic constraint is introduced for electricity power networks.

It must be mentioned that the acceptable level for network's failure probability must be determined by using knowledge and experience from the existing networks. Standard normalised cumulative distribution function (CDF) from the theory of probability, which is called function Φ, and a reliability index are often used to calculate this level.

The introduced probabilistic constraint, which is evaluated in a reliability analysis problem, can be applied in different non-deterministic optimisation models. A reliability-based design optimisation (RBDO) model is proposed in this PhD thesis for electricity networks. This model is based on a linear programming (LP) problem.

Another non-deterministic optimisation model that is developed for these networks is based on the optimal power flow (OPF) model. A probabilistic constraint is added to this model in order to take uncertainties of a network into account and also to formulate a new optimisation model for the network.

The main idea in introducing these non-deterministic design optimisation models for electricity power networks is to modify an obtained optimum solution. In this case, an initial optimisation problem should be solved and then the obtained answer is adjusted to formulate probabilistic constraints and introduce a new model.

Thus, since extra assumptions and constraints are added to the model, an additional cost is experienced. It is found in this thesis that if the acceptable level for network's failure probability is reduced and hence network's reliability is improved, more additional cost is incurred to design the network. In other words, the lower the acceptable level of failure probability, the higher the associated additional cost.

5.3 Future Work

In this section, a number of our main concerns about RBDO models and electricity power networks, which will be considered in our future researches, are addressed.

One of the major drawbacks of the RBDO models is their difficulties in dealing with a large number of variables. The existing reliability analysis methods may not be efficient, or even stable, if a system performance function includes many random variables.

Another shortcoming of the existing methods is originated from probability distribution function of the random variables. In this regard, the only probability distribution that is widely studied in the existing literature is the Gaussian distribution. Hence, a solid research work is required in the future to investigate other probability distributions and their effects on a real world problem.

Further, reliability analysis problem inside RIA (as a double-loop RBDO approach) still needs more work to increase its stability and efficiency. The existing methods for this problem are very sensitive with respect to initial design point and often show singularity behaviour. Hence, our other future work would be dedicated to propose a new method for the inner loop of RIA so that reliability analysis problems can be solved stably and efficiently.

Moreover, electricity power networks require new optimisation models in the future. The main issue in this area is that environmental factors must be considered in the next generation of optimisation models of these networks. Various factors, such as daylight, temperature and season, must be considered in the future when a non-deterministic design optimisation model is introduced for these networks.

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