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# Minimizing the Probability of Ruin in Exchange Rate Markets

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MINIMIZING THE PROBABILITY OF RUIN  
IN EXCHANGE RATE MARKETS

A Professional Master's Project Report  
submitted to the Faculty  
of the  
WORCESTER POLYTECHNIC INSTITUTE  
in partial fulfillment of the requirements for the  
Degree of Master of Science  
in  
Financial Mathematics  
by

Tyler A. Chase

April 2009

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This report represents the work of one or more WPI graduate students  
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## **Abstract**

The goal of this paper is to extend the results of Bayraktar and Young (2006) on minimizing an individual's probability of lifetime ruin; i.e. the probability that the individual goes bankrupt before dying. We consider a scenario in which the individual is allowed to invest in both a domestic bank account and a foreign bank account, with the exchange rate between the two currencies being modeled by geometric Brownian motion. Additionally, we impose the restriction that the individual is not allowed to borrow money, and assume that the individual's wealth is consumed at a constant rate. We derive formulas for the minimum probability of ruin as well as the individual's optimal investment strategy. We also give a few numerical examples to illustrate these results.

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# 1 Introduction

In the current state of the American economy, it is natural for individuals to be concerned about their financial well-being in the present as well as in the future. In particular, some have reason to be concerned about the possibility of bankruptcy during retirement. In a situation where the American dollar is a bit weaker or more unstable, it is reasonable to suppose that some individuals may be interested in investing in a potentially more stable foreign currency.

In this paper we begin to consider an extension of the work of Bayraktar and Young (2006) in determining how an individual should invest her wealth in order to minimize the probability that she ruins before death. We focus on the scenario in which the individual's rate of consumption is constant and borrowing constraints are imposed. However, we look at a financial market model in which the individual has the option of investing some of her wealth in a domestic bank account and some in a foreign bank account; the risk is introduced by the random exchange rate between the two currencies.

The Foreign Exchange market is an interesting model to consider. For some investors, the possibility of trading in a currency market can be more appealing than trading in a stock exchange. The Foreign Exchange offers high market liquidity, and has a high trading volume. Margins of profit are lower than in other, possibly riskier markets, but there is still the potential for significant earnings. So while not every investor would be interested in this market, its applications are relevant.

The most common criterion for optimization problems in financial literature is the maximization of expected utility of consumption, and there has been a substantial amount of work done on that subject. Bayraktar and Young (2006) note that these methods generally depend on a subjective utility function for consumption, whereas minimizing the probability of lifetime ruin may be more appealing and comprehensible to individuals since that criterion is more objective. And indeed, this technique has seen increased application in recent years.

Our work closely follows that of Bayraktar and Young (2006), since our market model is closely related. We consider only the “no-borrowing” case with constant consumption; after that, it should not be difficult to see how the other cases would follow. Before presenting the main results, we review a few of the definitions and theorems from probability and stochastic calculus, that the reader may have a suitable reference. Later we give a few numerical examples to demonstrate our results.

## 2 Background

### 2.1 Probability

To begin with, it would be helpful to establish the setting in which our work takes place. Specifically, we will assume the existence of a continuous-time filtered probability space. We will give a brief definition of most of the relevant fundamental concepts. While this is not strictly necessary, it helps to assure that the reader is able to follow the reasoning presented in the paper.

**Definition 2.1** ( $\sigma$ -algebra). *Let  $\Omega$  be a nonempty set, and let  $\mathcal{F}$  be a collection of subsets of  $\Omega$ .  $\mathcal{F}$  is a  $\sigma$ -algebra (also known as a  $\sigma$ -field) on  $\Omega$  if the following conditions are satisfied:*

- $\Omega \in \mathcal{F}$
- $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
- $A_1, A_2, \dots \in \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

**Definition 2.2** (Probability measure). *Let  $\Omega$  be a nonempty set, and  $\mathcal{F}$  a  $\sigma$ -algebra of subsets of  $\Omega$ . A probability measure is a function  $P : \mathcal{F} \rightarrow [0, 1]$  such that*

- $P(\Omega) = 1$ .
- If  $A_1, A_2, \dots \in \mathcal{F}$  are disjoint, then  $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ .

**Definition 2.3** (Probability Space). *A triple  $(\Omega, \mathcal{F}, \mathbf{P})$ , consisting of a sample space  $\Omega$ , a  $\sigma$ -algebra  $\mathcal{F}$  on  $\Omega$ , and a probability measure  $P$  on  $\mathcal{F}$ , is called a probability space.*

**Definition 2.4** (Random Variable). *Given a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , a random variable is a function  $X : \Omega \rightarrow \mathbb{R}$  with the property that for any Borel set  $B \in \mathcal{B}$ , the inverse image  $X^{-1}(B)$  belongs to  $\mathcal{F}$ .*

**Definition 2.5** (Measurability). *A random variable  $X$  is said to be measurable with respect to a  $\sigma$ -algebra  $\mathcal{G}$  (or,  $X$  is  $\mathcal{G}$ -measurable) if for any Borel set  $B \in \mathcal{B}$  the inverse image  $X^{-1}(B)$  belongs to  $\mathcal{G}$ .*

These are the basic assumptions of most models in probability. To discuss results relying on stochastic calculus, we also would like to review the concepts of filtrations and stopping times.

**Definition 2.6** (Filtration). *Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a probability space, and let  $T$  be a fixed positive number. A continuous filtration is a collection of  $\sigma$ -algebras with the following properties:*

- $\forall t \in [0, T], \exists \mathcal{F}_t \subset \mathcal{F}$ .
- $s \leq t \implies \mathcal{F}_s \subset \mathcal{F}_t$

Moreover,  $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T})$  is called a filtered probability space.

**Definition 2.7** (Stopping Time). *A stopping time  $\tau$  is a random variable satisfying the following property:  $\forall t \in [0, T], \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ .*



**Definition 2.8** (Stopped  $\sigma$ -algebra). Let  $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T})$  be a filtered probability space, and let  $\tau$  be a stopping time. The stopped  $\sigma$ -algebra (stopped at  $\tau$ ) is defined as:

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t\}.$$

## 2.2 Stochastic Calculus

In addition to the material on probability theory, we also provide a reference for fundamental definitions and theorems in stochastic calculus, as these are necessary tools in the proofs of our results.

**Definition 2.9** (Stochastic Process). A continuous stochastic process is a collection of random variables  $\{X_t, t \in [0, T]\}$ . For each  $\omega \in \Omega$ ,  $X_t(\omega)$  is a deterministic function called the sample path, or trajectory.

**Definition 2.10** (Adaptedness). A stochastic process  $X_t$  is said to be adapted to a filtration  $\{\mathcal{F}_t\}$  if for every  $t$ ,  $X_t$  is  $\mathcal{F}_t$ -measurable.

**Definition 2.11** (Càdlàg Process). A stochastic process  $X_t$  is called càdlàg if it has sample paths satisfying the following conditions almost surely:

- $X_t(\omega)$  is right-continuous, i.e.  $\lim_{t \rightarrow a^+} X_t(\omega) = X_a(\omega)$  for all  $a$ .
- $X_t(\omega)$  has left-limits, i.e.  $\lim_{t \rightarrow a^-} X_t(\omega)$  exists for all  $a$ .

The word “càdlàg” is a French acronym, standing for *continue à droite, limitée à gauche*, literally “continuous on the left, limited on the right.”

**Definition 2.12** (Conditional Expectation). *The conditional expectation of a random variable  $X$  with respect to a  $\sigma$ -algebra  $\mathcal{G} \subset \mathcal{F}$ , denoted  $E[X|\mathcal{G}]$ , is itself a random variable with the following properties:*

- $E[X|\mathcal{G}]$  is  $\mathcal{G}$ -measurable.
- $\forall A \in \mathcal{G}, \int_A X dP = \int_A E[X|\mathcal{G}] dP$

**Definition 2.13** (Martingale). *Let  $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T})$  be a filtered probability space. An adapted stochastic process  $M_t$  is called a martingale with respect to  $\mathcal{F}_t$  if it satisfies the following property:*

$$E[M_t|\mathcal{F}_s] = M_s \text{ for all } 0 \leq s \leq t \quad (2.1)$$

**Definition 2.14** (Stopped Process). *If  $X$  is a stochastic process and  $\tau$  is a stopping time, then we can define the stopped process*

$$X_{t \wedge \tau} = \begin{cases} X_t & \text{if } t \leq \tau \\ X_\tau & \text{if } t > \tau \end{cases} \quad (2.2)$$

In that case, the stochastic process is said to be “stopped” at time  $\tau$ . The stopped process is equal to the original process until time  $\tau$ , and becomes constant after that time, equal to the value of  $X_\tau$ .

A similar notion is the “killed” process, which instead of taking on the constant value  $X_\tau$  at its killing time, takes a value  $\Delta$  (outside the range of  $X$ ) called the “coffin state”.

**Definition 2.15** (Brownian Motion). *A standard Brownian motion (or Wiener process) is a continuous stochastic process  $W_t$  with independent increments which are normally distributed: For all  $0 = t_0 < t_1 < t_2 < \dots < t_m$ , the increments*

$$W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \dots, W_{t_m} - W_{t_{m-1}} \quad (2.3)$$

*are independent and normally distributed with mean 0 and variance  $t_i - t_{i-1}$ .*

**Definition 2.16** (Itô Integral). *The Itô integral  $\int_0^t H_s dW_s$  of a càdlàg process  $H$  is defined as:*

$$\int_0^t H_s dW_s = \lim_{n \rightarrow \infty} \sum_{i=1}^n H_{t_{i-1}} (W_{t_i} - W_{t_{i-1}}). \quad (2.4)$$

*where  $0 = t_0 < t_1 < t_2 < \dots < t_n$  is a partition of  $[0, t]$ , growing finer as  $n$  increases.*

**Theorem 2.17** (Properties of the Itô Integral). *The Itô integral  $I_t = \int_0^t H_s dW_s$  has the following properties:*

- $I_t$  has continuous sample paths.
- $I_t$  is  $\mathcal{F}_t$ -adapted.
- $I_t$  is a martingale.
- $I_t$  has quadratic variation  $[I, I]_t = \int_0^t H_s^2 ds$ .

**Definition 2.18** (Itô Process). *An Itô Process is a stochastic process  $X$  of*

the following form:

$$X_t = X_0 + \int_0^t \mu_s dW_s + \int_0^t \nu_s ds \quad (2.5)$$

where  $\mu$  and  $\nu$  are adapted stochastic processes, and  $X_0$  is a nonrandom initial value.

Alternatively, this can be written in the differential form:

$$dX_t = \mu_s dW_s + \nu_s ds \quad (2.6)$$

Also note that the quadratic variation of the Itô process  $X$  is given by (in integral and differential form):

$$[X, X]_t = \int_0^t \mu_s^2 ds \quad (2.7)$$

$$d[X, X]_t = \mu_t^2 dt \quad (2.8)$$

**Theorem 2.19** (Itô's Formula). *Let  $X_t$  be an Itô process and let  $f(t, x)$  be a function for which the partial derivatives  $f_t$ ,  $f_x$ , and  $f_{xx}$  are defined and continuous. Then*

$$\begin{aligned} f(T, X_T) &= f(0, X_0) + \int_0^T f_t(t, X_t) dt + \int_0^T f_x(t, X_t) dX_t \\ &\quad + \frac{1}{2} \int_0^T f_{xx}(t, X_t) d[X, X]_t \end{aligned} \quad (2.9)$$

A Poisson process is a stochastic process which takes nonnegative integer values characterized by a rate parameter  $\lambda$ . It is typically used to model the

number of events which occur in a given time interval.

**Definition 2.20** (Poisson Process). *The Poisson process  $N_t$  with rate parameter  $\lambda$  obeys a Poisson distribution with parameter  $\lambda t$ :*

$$P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad (2.10)$$

The Poisson process is an example of what is called a “pure jump” process. It has stationary, independent increments. Moreover, the times between successive jumps are independent and follow an exponential distribution with parameter  $\lambda$  (i.e., an exponential distribution with mean  $1/\lambda$ ).

**Theorem 2.21** (Compensated Poisson Process). *If  $N_t$  is a Poisson process with rate parameter  $\lambda$ , we define the compensated Poisson process  $M_t = N_t - \lambda t$ .  $M_t$  is a martingale.*

**Definition 2.22** (Jump Process). *Let  $X_t$  be an Itô process, and let  $J_t$  be a “pure jump” process. That is,  $J_t$  is an adapted, càdlàg process with finitely many jumps on the interval  $(0, T]$ , and is constant between jumps. We will call a process of the following form a jump process:*

$$Y_t = X_t + J_t \quad (2.11)$$

When discussing processes with jumps, Itô’s Formula takes a slightly altered form.

**Theorem 2.23** (Itô’s Formula for Jump Processes). *Let  $Y_t$  be a jump process, and let  $f(y)$  be a function which is twice continuously differentiable. Then*

*we have the following:*

$$\begin{aligned} f(Y_t) = f(Y_0) &+ \int_0^t f'(Y_s) dY_s^c + \frac{1}{2} \int_0^t f''(Y_s) d[Y, Y]_s^c \\ &+ \sum_{0 < s \leq t} [f(Y_s) - f(Y_{s-})] \end{aligned} \quad (2.12)$$

Here, the quantity  $Y^c$  denotes the continuous part of the jump process  $Y$ . If  $Y$  is given in the form of (2.11), then  $Y^c = X$ .

### 3 Probability of Lifetime Ruin

We will be considering the problem of minimizing an individual's probability of ruin under the condition that borrowing is forbidden. In section 3.1 we outline the financial market model used in the analysis. In 3.2 we present and prove the main results.

#### 3.1 Model

We will assume a model in which an individual has the option of investing in two assets: A domestic bank account  $B$ , and a foreign bank account  $F$ . Each of these banks will have its own fixed interest rate, and the exchange rate  $X$  between the two currencies will be modeled by a geometric Brownian motion. So our assets have the following dynamics:

$$\begin{aligned}dB &= r_d B dt \\dF &= r_f F dt \\dX &= \mu X dt + \sigma X dW\end{aligned}\tag{3.1}$$

where  $\alpha$  and  $\sigma$  are constants, and  $W$  is a standard Brownian motion. We will prove a lemma which states that the above formulation is equivalent to a situation in which the individual is allowed to invest in the domestic bank account and in a domestic risky asset whose price is given by  $\tilde{F} = FX$ .

**Lemma 3.1.** *The model in (3.1) is equivalent to the following model:*

$$\begin{aligned} dB &= r_d B dt \\ d\tilde{F} &= (\mu + r_f) \tilde{F} dt + \sigma \tilde{F} dW \end{aligned} \quad (3.2)$$

where  $\tilde{F} = FX$ .

*Proof.* The proof follows from the multidimensional version of Itô's lemma. The two-dimensional case is as follows: If  $U$  and  $V$  are Itô processes, and if  $f(t, u, v)$  is a function which is twice continuously differentiable, then

$$\begin{aligned} df(t, U, V) &= f_t(t, U, V) dt + f_u(t, U, V) dU + f_v(t, U, V) dV \\ &\quad + \frac{1}{2} f_{uu}(t, U, V) dU dU + \frac{1}{2} f_{vv}(t, U, V) dV dV \\ &\quad + f_{uv}(t, U, V) dU dV \end{aligned} \quad (3.3)$$

where  $f_u$  denotes the first partial derivative of  $f$  with respect to  $u$ , and so on. Using  $f(t, u, v) = uv$ , this reduces to

$$\begin{aligned} df(t, U, V) &= 0dt + VdU + UdV + 0dUdU + 0dVdV + 1dUdV \\ &= VdU + UdV + dUdV \end{aligned}$$



Substituting  $U = X$  and  $V = F$ , we obtain

$$\begin{aligned}
d\tilde{F} &= FdX + XdF + dXdF \\
&= F(\mu Xdt + \sigma XdW) + X(r_f Fdt) + (\mu Xdt + \sigma XdW)(r_f Fdt) \\
&= \tilde{F}(\mu dt + \sigma dW) + \tilde{F}(r_f dt) + 0
\end{aligned}$$

And this can be rearranged as

$$d\tilde{F} = (\mu + r_f)\tilde{F}dt + \sigma\tilde{F}dW \quad (3.4)$$

So the possibility of investing in the domestic bank  $B$  and the foreign bank  $F$  is equivalent to investing in the domestic bank and a domestic asset  $\tilde{F}$  with dynamics as given in (3.4).  $\square$

We will henceforth assume that the individual may act by investing a portion of her wealth in  $\tilde{F}$ , with the remainder invested in  $B$ . In this formulation,  $\tilde{F}$  can be interpreted as a risky asset while  $B$  is a risk-free asset; we assume that  $\mu + r_f > r_d$  (indeed, we are not interested in the problem otherwise, since no investor should invest in a risky asset with lower expected return than the risk-free asset).

We will also assume that the individual's total wealth is continuously consumed at a *constant* rate  $c$ .

Let  $V_t$  denote the wealth of the individual at time  $t$ , and denote by  $\pi_t$  the amount she invests in the risky asset  $\tilde{F}$ . Then the amount invested in the risk-free asset  $B$  is  $V_t - \pi_t$ . Therefore the wealth process obeys the following

dynamics:

$$\begin{aligned}
dV_t &= \frac{\pi_t}{\tilde{F}_t} d\tilde{F}_t + \frac{(V_t - \pi_t)}{B_t} dB_t - cdt \\
\Rightarrow dV_t &= \frac{\pi_t}{\tilde{F}_t} \left[ (\mu + r_f) \tilde{F}_t dt + \sigma \tilde{F}_t dW_t \right] + \frac{(V_t - \pi_t)}{B_t} r_d B_t dt - cdt
\end{aligned}$$

This can be simplified as:

$$\begin{cases} dV_t = [r_d V_t + (\mu + r_f - r_d) \pi_t - c] dt + \sigma \pi_t dW_t \\ V_0 = v \end{cases} \quad (3.5)$$

We now wish to define what is meant by “lifetime ruin”. We let  $\tau_0$  denote the first time that  $V = 0$ , and let  $\tau_d$  denote the individual’s time of death. Lifetime ruin is defined as the event in which the wealth process reaches zero before the individual dies, i.e. the event  $\{\tau_0 < \tau_d\}$ . Here we will assume that  $\tau_d$  follows an exponential distribution with parameter  $\lambda$ , so that the expected value of  $\tau_d$  is  $1/\lambda$  (later, we will model this using a Poisson process with rate parameter  $\lambda$ , since the time between jumps in the Poisson process follows an exponential distribution).

The minimum probability of ruin will be denoted by  $\psi(v)$  (the argument  $v$  indicates that this probability is conditional on  $V_0 = v$ ). So we are minimizing the probability that  $\tau_0 < \tau_d$ , with respect to the set of admissible trading strategies (denoted by  $\mathcal{A}$ ). For this paper, we also impose the restriction that  $0 \leq \pi_t \leq V_t$  (i.e. no borrowing or short-selling is possible).

Therefore the probability  $\psi(v)$  is given by:

$$\psi(v) = \inf_{\pi} \mathbf{P}[\tau_0 < \tau_d | V_0 = v] \quad (3.6)$$

For each real number  $\alpha$ , we can define a second-order differential operator  $\mathcal{L}^\alpha$  which is associated with the minimization problem. For each open subset  $G$  of  $\mathbb{R}^+$  and for each  $h \in C^2(G)$ , define the function  $\mathcal{L}^\alpha h : G \rightarrow \mathbb{R}$  as follows:

$$\mathcal{L}^\alpha h(v) = [r_d v + (\mu + r_f - r_d)\alpha - c] h'(v) + \frac{1}{2} \sigma^2 \alpha^2 h''(v) - \lambda h(v) \quad (3.7)$$

The operator  $\mathcal{L}^\alpha$  will be used in the following sections to characterize  $\psi$  in a compact manner.

## 3.2 Minimum Probability of Ruin

In this section we will present the verification theorem which states the necessary and sufficient conditions that  $\psi$  must satisfy. First, note that when the individual's wealth is above  $c/r_d$ , the probability of lifetime ruin is equal to 0; the individual can invest all of her wealth in the domestic (risk-free) bank account and consume continuously at rate  $c$  with no possibility of running out of money. To see this, consider the dynamics of the wealth process when all of the individual's wealth is invested in the domestic bank:

$$dV_t = r_d V_t dt - c dt$$

In this case, there is no stochastic integral involved; so it can be expressed as an ordinary differential equation:

$$\frac{dV}{dt} = r_d V - c$$

So for all  $V \geq c/r_d$ , we have that  $\frac{dV}{dt} \geq 0$ , i.e. there is no chance that the wealth process will decrease in this situation, let alone reach zero.

Thus in addition to the stopping times  $\tau_0$  and  $\tau_d$  which we already defined, we also introduce the stopping time  $\tau_{c/r_d} = \inf \{t > 0 : V_t \geq c/r_d\}$ , that is, the first time that the individual's wealth reaches  $c/r_d$  (or more). If we now define the stopping time  $\tau = \tau_d \wedge \tau_{c/r_d}$ , it follows that we can express  $\psi$  as follows:

$$\psi(v) = \inf_{\pi} \mathbf{P} [\tau_0 < \tau \mid V_0 = v] \quad (3.8)$$

We can now present the verification theorem:

**Theorem 3.2.** *Suppose  $h : \mathbb{R}^+ \rightarrow [0, 1]$  is a decreasing function, and  $\alpha_0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which satisfy the following conditions:*

- (i)  $h \in C^2$  on  $[0, c/r_d)$
- (ii)  $\alpha_0 \in \mathcal{A}$
- (iii)  $\mathcal{L}^\alpha h(v) \geq 0$ , for  $0 \leq \alpha \leq v < c/r_d$
- (iv)  $\mathcal{L}^{\alpha_0(v)} h(v) = 0$ , for  $v \in (0, c/r_d)$
- (v)  $h(0) = 1$  and  $h(v) = 0$  for  $v \geq c/r_d$

Then the minimum probability of lifetime ruin  $\psi$  is given by:

$$\psi(v) = h(v), \quad v \geq 0 \quad (3.9)$$

And the optimal investment strategy  $\pi^*$  in the risky asset  $\tilde{F}$  is given by:

$$\pi^*(v) = \alpha_0(v), \quad v \in [0, c/r_d] \quad (3.10)$$

*Proof.* Suppose we have  $h$  which satisfies the properties stated above. Let  $N$  be a Poisson process (independent of  $W$ ) with rate parameter  $\lambda$ . The stopping time  $\tau_d$  will be defined as the time of the first jump of the process  $N$ . Let  $\alpha$  be a function on the interval  $[0, c/r_d]$  with  $0 \leq \alpha(v) \leq v$ , and let  $V^\alpha$  denote the wealth process under the investment strategy  $\alpha$ . We denote  $\alpha_s = \alpha(V_s^\alpha)$ . We will kill the wealth process at time  $\tau_d$  and assign  $W_{\tau_d} = \Delta$ , (the coffin state). Our convention will be that for any function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , we let  $f(\Delta) = 0$ . In particular, note that  $h(c/r_d) = 0$  and  $h(V_{\tau_d}^\alpha) = 0$ . Using the wealth process dynamics as stated in 3.5 and Itô's formula as in 2.23, we have the following:

$$\begin{aligned} h(V_{t \wedge \tau \wedge \tau_0}^\alpha) - h(v) &= \int_0^{t \wedge \tau \wedge \tau_0} h'(V_s^\alpha) dV_s^\alpha + \frac{1}{2} \int_0^{t \wedge \tau \wedge \tau_0} h''(V_s^\alpha) d[V^\alpha, V^\alpha]_s \\ &+ \sum_{0 < s \leq t \wedge \tau \wedge \tau_0} [h(V_s^\alpha) - h(V_{s^-}^\alpha)] \end{aligned} \quad (3.11)$$

$$\begin{aligned}
&= \int_0^{t \wedge \tau \wedge \tau_0} h'(V_s^\alpha) \{ [r_d V_s^\alpha + (\mu + r_f - r_d) \alpha_s - c] ds + \sigma \alpha_s dW_s \} \\
&+ \frac{1}{2} \int_0^{t \wedge \tau \wedge \tau_0} h''(V_s^\alpha) (\sigma^2 \alpha_s^2) ds + \sum_{0 < s \leq t \wedge \tau \wedge \tau_0} [h(V_s^\alpha) - h(V_{s^-}^\alpha)] \quad (3.12)
\end{aligned}$$

Since the process jumps only at time  $\tau_d$ , then the jump at time  $s$  can be expressed as  $h(V_s^\alpha) - h(V_{s^-}^\alpha) = -h(V_{s^-}^\alpha) \Delta N_s$  and so we write

$$\sum_{0 < s \leq t \wedge \tau \wedge \tau_0} [h(V_s^\alpha) - h(V_{s^-}^\alpha)] = - \int_0^{t \wedge \tau \wedge \tau_0} h(V_{s^-}^\alpha) dN_s \quad (3.13)$$

In order to write the expression in a compact manner, we add and subtract the term  $\lambda \int_0^{t \wedge \tau \wedge \tau_0} h(V_{s^-}^\alpha)$  in the right hand side:

$$\begin{aligned}
h(V_{t \wedge \tau \wedge \tau_0}^\alpha) &= h(v) + \int_0^{t \wedge \tau \wedge \tau_0} \{ [r_d V_s^\alpha + (\mu + r_f - r_d) \alpha_s - c] h'(V_s^\alpha) + \frac{1}{2} \sigma^2 \alpha_s^2 h''(V_s^\alpha) \} ds \\
&- \int_0^{t \wedge \tau \wedge \tau_0} \lambda h(V_{s^-}^\alpha) ds + \int_0^{t \wedge \tau \wedge \tau_0} \sigma \alpha_s h'(V_s^\alpha) dW_s \\
&- \int_0^{t \wedge \tau \wedge \tau_0} h(V_{s^-}^\alpha) dN_s + \lambda \int_0^{t \wedge \tau \wedge \tau_0} h(V_{s^-}^\alpha) ds \quad (3.14)
\end{aligned}$$

So the expression can be simplified to:

$$\begin{aligned}
h(V_{t \wedge \tau \wedge \tau_0}^\alpha) &= h(v) + \int_0^{t \wedge \tau \wedge \tau_0} \mathcal{L}^\alpha h(V_s^\alpha) ds + \int_0^{t \wedge \tau \wedge \tau_0} \sigma \alpha_s h'(V_s^\alpha) dW_s \\
&- \int_0^{t \wedge \tau \wedge \tau_0} h(V_{s^-}^\alpha) d(N_s - \lambda s) \quad (3.15)
\end{aligned}$$

Taking the expectation of both sides, the third and fourth terms vanish (it can be shown that the integrands satisfy sufficient conditions). So, following

from assumption (iii) in the theorem statement, we have:

$$\mathbf{E}_v[h(V_{t \wedge \tau \wedge \tau_0}^\alpha)] = h(v) + \mathbf{E}_v \left[ \int_0^{t \wedge \tau \wedge \tau_0} \mathcal{L}^\alpha h(V_s^\alpha) ds \right] \geq h(v) \quad (3.16)$$

Here,  $\mathbf{E}_v$  indicates that the expectation is conditional on  $V_0 = v$ . Therefore, the process  $h(V_{t \wedge \tau \wedge \tau_0}^\alpha)$ ,  $t \geq 0$ , is a submartingale. Since  $h(0) = 1$ ,  $h(V_{\tau_0 \wedge \tau}^\alpha) = 0$ , and  $h(V_{c/r_d}^\alpha)$ , it follows (where  $\mathbf{1}$  denotes the indicator function) that

$$h(V_{\tau_0 \wedge \tau}^\alpha) = \mathbf{1}_{\{\tau_0^\alpha < \tau\}}. \quad (3.17)$$

Now, taking expectations of both sides and applying the optional sampling theorem gives

$$\mathbf{E}_v h(V_{\tau_0 \wedge \tau}^\alpha) = \mathbf{P}_v(\tau_0^\alpha < \tau) \geq h(v), \quad (3.18)$$

since  $h(V_{t \wedge \tau \wedge \tau_0}^\alpha)$  is a submartingale. Therefore

$$\inf_{\alpha} \mathbf{P}_v(\tau_0^\alpha < \tau) = \psi(v) \geq h(v). \quad (3.19)$$

If we consider  $\alpha_0$  as specified in the theorem statement (namely, property (iv), i.e.  $\alpha_0$  is the minimizer of  $\mathcal{L}^\alpha h$ ), then it follows that  $h(V_{t \wedge \tau \wedge \tau_0}^{\alpha_0})$  is a martingale. So we have that

$$\mathbf{E}_v h(V_{\tau_0 \wedge \tau}^{\alpha_0}) = \mathbf{P}_v(\tau_0^{\alpha_0} < \tau) = h(v). \quad (3.20)$$

We have therefore shown that the statements in 3.9 and 3.10 are true for

$v \in [0, c/r_d)$ . Together with the assumption in (v) and the fact that  $\psi(v) = \inf_{\pi \in \mathcal{A}} \mathbf{P}[\tau_0 < \tau | V_0 = v]$ , the proof is complete.  $\square$

The forms of the functions  $h$  and  $\alpha$  are discovered by imposing a few additional properties that are not stated explicitly in the preceding theorem. However, because that theorem asserts that the function  $h$  is unique on the interval  $[0, c/r_d)$ , if we find  $h$  and  $\alpha$  which satisfy the assumptions of Theorem 3.2 and the additional assumptions, then the additional properties are implicit.

We make the following additional hypotheses as well: In the constrained case, we assume that  $\alpha_0$  (the amount invested in the foreign bank  $\tilde{F}$ ) is a continuous function of  $v$ , and that there exists a wealth level  $v_l$  such that  $v - \alpha_0(v) = 0$  for  $v < v_l$  and  $v - \alpha_0(v) > 0$  for  $v > v_l$ . The idea here is that when the individual has more wealth, it is wiser to invest a portion of it in a risk-free asset. The subscript  $l$  denotes that the individual is, upon reaching this level, “lending” some amount of money to the domestic bank.

We will consider the intervals  $[0, v_l]$  and  $(v_l, c/r_d]$  separately. First we look at the interval  $(v_l, c/r_d]$ , and we assume that the borrowing constraint is non-binding.

**Proposition 3.3.** *Assume that  $0 \leq \alpha_0(v) < v$  on the interval  $(v_l, c/r_d]$ . The function  $h$  has the following form (with  $\beta \geq 1$ ):*

$$h(v) = \beta \left(1 - \frac{r_d}{c} v\right)^d, \quad (3.21)$$



where

$$d = \frac{1}{2r_d} \left[ (r_d + \lambda + m) + \sqrt{(r_d + \lambda + m)^2 - 4r_d\lambda} \right] > 1, \quad (3.22)$$

and

$$m = \frac{1}{2} \left( \frac{\mu + r_f - r_d}{\sigma} \right)^2. \quad (3.23)$$

The corresponding  $\alpha_0$  on  $(v_l, c/r_d]$  is given by:

$$\alpha_0(v) = \frac{\mu + r_f - r_d}{\sigma^2} \frac{1}{d-1} \left( \frac{c}{r_d} - v \right). \quad (3.24)$$

*Proof.* Items (iii), (iv), (v) of Theorem 3.2 require that we solve

$$\lambda h(v) = (r_d v - c)h'(v) + \min_{\alpha} \left[ (\mu + r_f - r_d)\alpha h'(v) + \frac{1}{2}\sigma^2\alpha^2 h''(v) \right] \quad (3.25)$$

with the boundary condition  $h(c/r_d) = 0$ . We can show that we also have the boundary condition  $h'(c/r_d) = 0$ : Consider the solution  $\phi$  of (3.25) with  $\lambda = 0$  (corresponding to the event that the individual never dies). So  $h \leq \phi$  on some interval  $(c/r_d - \delta, c/r_d]$ , since the probability of ruin before death is necessarily less than the probability of ruin before infinity (i.e.  $\mathbf{P}(\tau_0 < \tau_d) \leq \mathbf{P}(\tau_0 < \infty)$ ). So it is enough to show that  $\phi'(c/r_d) = 0$ , which would imply  $h'(c/r_d) = 0$ . Note that  $\phi$  is a solution to the following (with  $\phi(c/r_d) = 0$ ):

$$0 = (r_d v - c)\phi'(v) + \min_{\alpha} \left[ (\mu + r_f - r_d)\alpha\phi'(v) + \frac{1}{2}\sigma^2\alpha^2\phi''(v) \right] \quad (3.26)$$

Pestien and Suddherth (1985) showed that the optimal investment strategy  $\alpha^*$  maximizes (in our case) the quantity:

$$f(\alpha) = \frac{(\mu + r_f - r_d)\alpha - (c - r_d v)}{\alpha^2}. \quad (3.27)$$

By ordinary calculus, it is easily checked that the value of  $\alpha$  which maximizes that expression is  $\alpha^* = 2(c - r_d v)/(\mu + r_f - r_d)$ . However, we also have (again from ordinary calculus, this time applied to the minimization problem in (3.26)) that

$$\alpha^*(v) = -\frac{\mu + r_f - r_d}{\sigma^2} \frac{\phi'(v)}{\phi''(v)} \quad (3.28)$$

Therefore, for  $v \in (c/r_d - \delta, c/r_d]$ , we have (for some  $k < 0$ )

$$\phi'(v) = k(c - r_d v)^{m/r_d} \quad (3.29)$$

So we showed that  $\phi'(c/r_d) = 0$ , which implies that  $h'(c/r_d) = 0$ .

To be consistent with the hypothesis that the borrowing constraint is non-binding on  $(v_l, c/r_d]$ , it must be true that  $h$  is convex on  $(v_l, c/r_d]$ . To see this, note that if  $h$  is not convex in some neighborhood of a point  $v^* \in (v_l, c/r_d]$  (i.e.  $h''(v) < 0$  in that neighborhood), then  $\alpha_0$  is as large as possible on that neighborhood, which contradicts the hypothesis that the borrowing constraint is non-binding. So we have that  $h$  is convex on  $(v_l, c/r_d]$ , and can

therefore consider its Legendre transform  $\tilde{h}$ :

$$\tilde{h}(u) = \min_v [h(v) + vu] \quad (3.30)$$

$h$  can be recovered from  $\tilde{h}$  by

$$h(v) = \max_u [\tilde{h}(u) - uv] \quad (3.31)$$

From ordinary calculus, the value of  $v$  which minimizes the quantity in (3.30) is  $v = (h')^{-1}(-u) = \tilde{h}'(u)$ . Therefore the value of  $u$  which maximizes the expression in (3.31) is  $u = -h'(v)$ . We can then make substitutions into (3.25). Using  $v = \tilde{h}'(u)$ , it follows that

$$h(v) = \tilde{h}(u) - u\tilde{h}'(u), \quad h'(v) = -u, \quad \text{and} \quad h''(v) = -\frac{1}{\tilde{h}''(u)}. \quad (3.32)$$

Additionally, as in (3.28), we use

$$\alpha = -\frac{\mu + r_f - r_d}{\sigma^2} \frac{h'(v)}{h''(v)} \quad (3.33)$$

Making these substitutions in (3.25) gives

$$\begin{aligned} \lambda h(v) &= (r_d v - c)h'(v) - \frac{1}{2} \frac{(\mu + r_f - r_d)^2 (h'(v))^2}{\sigma^2 h''(v)} \\ \Rightarrow \lambda[\tilde{h}(u) - u\tilde{h}'(u)] &= (r_d \tilde{h}'(u) - c)(-u) + m u^2 \tilde{h}''(u) \end{aligned} \quad (3.34)$$

Here,  $m$  is as given in (3.23). Simplifying further gives the following differ-

ential equation:

$$\lambda \tilde{h}(u) + (r_d - \lambda)u \tilde{h}'(u) - mu^2 \tilde{h}''(u) = cu \quad (3.35)$$

The general solution of this

$$\tilde{h}(u) = D_1 u^{B_1} + D_2 u^{B_2} + \frac{c}{r}u \quad (3.36)$$

where  $D_1$  and  $D_2$  are constants, and  $B_1$  and  $B_2$  are the roots of

$$-\lambda - (r_d - \lambda + m)B + mB^2 = 0, \quad (3.37)$$

so

$$B_1 = \frac{1}{2m} \left[ (r_d - \lambda + m) + \sqrt{(r_d - \lambda + m)^2 + 4\lambda m} \right] > 1 \quad (3.38)$$

$$B_2 = \frac{1}{2m} \left[ (r_d - \lambda + m) - \sqrt{(r_d - \lambda + m)^2 + 4\lambda m} \right] < 0 \quad (3.39)$$

Let  $u_c = -h'(c/r_d) = 0$ , so  $\tilde{h}'(0) = c/r_d$ . From the definition of  $\tilde{h}$  and because  $h(c/r_d) = 0$ , we have at  $u = u_c = 0$ ,

$$\tilde{h}(0) = 0. \quad (3.40)$$

From this it follows that  $D_2 = 0$ . We can then use (3.36) and (3.31) to recover  $h$ :

$$h(v) = \max_u \left[ D_1 u^{B_1} + \frac{c}{r}u - vu \right]. \quad (3.41)$$

The maximizing value of  $u$  here (by ordinary calculus) is:

$$u = \left( \frac{v - c/r_d}{D_1 B_1} \right)^{1/(B_1-1)}. \quad (3.42)$$

Substituting this back into (3.41) gives:

$$\begin{aligned} h(v) &= D_1 \left( \frac{v - c/r_d}{D_1 B_1} \right)^{\frac{B_1}{B_1-1}} - (v - c/r_d) \left( \frac{v - c/r_d}{D_1 B_1} \right)^{\frac{1}{B_1-1}} \\ h(v) &= \left[ \frac{D_1}{(D_1 B_1)^{\frac{B_1}{B_1-1}}} - \frac{1}{(D_1 B_1)^{\frac{1}{B_1-1}}} \right] \left( -\frac{c}{r_d} \right)^{\frac{B_1}{B_1-1}} \left( 1 - \frac{r_d}{c} v \right)^{\frac{B_1}{B_1-1}}. \end{aligned} \quad (3.43)$$

And we simplify this by denoting the leading constant quantity by  $\beta$  and noting that  $\frac{B_1}{B_1-1} = d$ , so we obtain

$$h(v) = \beta \left( 1 - \frac{r_d}{c} v \right)^d. \quad (3.44)$$

Using this expression for  $h$ , the optimal investment strategy  $\alpha_0$  is found by minimizing (using ordinary calculus) the expression:

$$\left[ (\mu + r_f - r_d) \alpha h'(v) + \frac{1}{2} \sigma^2 \alpha^2 h''(v) \right]. \quad (3.45)$$

And the value of  $\alpha$  which minimizes this expression is

$$\alpha_0(v) = \frac{\mu + r_f - r_d}{\sigma^2} \frac{1}{d-1} \left( \frac{c}{r_d} - v \right). \quad (3.46)$$

□

**Corollary 3.4.** *The lending level  $v_l$  takes the following form:*

$$v_l = \frac{x}{1+x} \frac{c}{r_d}, \quad (3.47)$$

where

$$x = \frac{\mu + r_f - r_d}{\sigma^2} \frac{1}{d-1}. \quad (3.48)$$

*Proof.* This follows from the assumption that  $\alpha_0$  is continuous. Substituting our value of  $x$  into (3.46), and setting  $\alpha_0(v_l) = v_l$ , we have

$$x \left( \frac{c}{r_d} - v_l \right) = v_l \quad (3.49)$$

which simplifies to the expression in (3.47).  $\square$

We therefore have an explicit expression for the lending level  $v_l$ . Recall that when wealth lies below this level, all of the wealth should be invested into the foreign bank. The quantity  $v_l$  varies nontrivially with changes in most of the parameters, but there are a few things we can note about its behavior. For instance, we can see that as  $c$  approaches zero,  $v_l$  also approaches zero; that is, if we have a low rate of consumption then we should invest most of our money in the domestic bank. Indeed, this agrees with our intuition. However, the behavior of  $v_l$  with respect to the other parameters is more difficult to analyze.

Next we consider the interval  $[0, v_l]$ , on which  $\alpha_0(v) = v$ .

**Proposition 3.5.** *Under the assumption that  $\alpha_0(v) = v$  on  $[0, v_l]$ , the func-*

tion  $h$  solves the following:

$$\lambda h = [(\mu + r_f)v - c] h' + \frac{1}{2}\sigma^2 v^2 h'' \quad (3.50)$$

with boundary conditions

$$h(0) = 1 \quad \text{and} \quad \frac{h(v_l)}{h'(v_l)} = -\frac{1}{d} \left( \frac{c}{r_d} - v_l \right). \quad (3.51)$$

*Proof.* From part (iv) of Theorem 3.2, and with the substitution  $\alpha_0(v) = v$  on  $[0, v_l]$ , we have

$$\begin{aligned} \mathcal{L}^{\alpha_0(v)} h &= 0 \\ \Rightarrow [r_d v + (\mu + r_f - r_d)v - c] h' + \frac{1}{2}\sigma^2 v^2 h'' - \lambda h &= 0 \end{aligned} \quad (3.52)$$

So we have

$$\lambda h = [(\mu + r_f)v - c] h' + \frac{1}{2}\sigma^2 v^2 h''. \quad (3.53)$$

The boundary condition  $h(0) = 1$  is directly from part (v) of Theorem 3.2. The other boundary condition arises from the fact that  $h \in C^2$  on the interval  $[0, c/r_d)$  (part (i) of Theorem 3.2). At  $v = v_l$ , the boundary between the two regions,  $h$  should satisfy:

$$h(v_l) = \beta \left( 1 - \frac{r_d}{c} v_l \right)^d \quad (3.54)$$

$$\Rightarrow h'(v_l) = -\beta d \frac{r_d}{c} \left( 1 - \frac{r_d}{c} v_l \right)^{d-1} \quad (3.55)$$

Combining these conditions gives

$$\frac{h(v_l)}{h'(v_l)} = -\frac{1}{d} \left( \frac{c}{r_d} - v_l \right) \quad (3.56)$$

□

Using these boundary conditions, it is possible to solve the ordinary differential equation (3.53) numerically. Then the continuity condition  $h(v_l-) = h(v_l+)$  can be used to determine the unknown parameter  $\beta$ . We now need only to show that if  $h$  has the properties stated in Proposition 3.5, then  $\alpha_0 = v$ .

**Proposition 3.6.** *Suppose  $h$  satisfies the equations (3.50) and (3.51) on  $[0, v_l]$ . Then*

$$\arg \min_{0 \leq \alpha \leq v} \left[ (\mu + r_f - r_d)\alpha h'(v) + \frac{1}{2}\sigma^2\alpha^2 h''(v) \right] = v, \quad v \in [0, v_l]. \quad (3.57)$$

*Proof.* Define a function  $f$  by

$$f(\alpha) = (\mu + r_f - r_d)\alpha h'(v) + \frac{1}{2}\sigma^2\alpha^2 h''(v) \quad (3.58)$$

for  $v \in [0, v_l]$ . In order to prove this proposition, it suffices to show that  $f'(v) \leq 0$  for  $v \in [0, v_l]$ . That is,

$$f'(v) = (\mu + r_f - r_d)h'(v) + \sigma^2 v h''(v) \leq 0 \quad (3.59)$$



Solving for  $h''(v)$  in equation (3.50) and substituting into this inequality gives

$$\begin{aligned} (\mu + r_f - r_d)h'(v) + \frac{2}{v} \{ \lambda h(v) - [(\mu + r_f)v - c] h'(v) \} &\leq 0 \\ \Rightarrow [-(\mu + r_f + r_d)v + 2c] h'(v) + 2\lambda h(v) &\leq 0 \end{aligned} \quad (3.60)$$

Rearranging, this can be put in the following form:

$$\frac{h(v)}{h'(v)} \geq \frac{(\mu + r_f + r_d)}{2\lambda} v - \frac{c}{\lambda} \quad (3.61)$$

We define functions  $y(v)$  and  $z(v)$  as follows:

$$y(v) = \frac{h(v)}{h'(v)} \quad (3.62)$$

$$z(v) = \frac{(\mu + r_f + r_d)}{2\lambda} v - \frac{c}{\lambda} \quad (3.63)$$

And we complete the proof by proving the following lemma (which asserts that  $y \geq z$  on  $[0, v_l]$ ). □

**Lemma 3.7.** *With  $y$  and  $z$  as given in (3.62) and (3.63),  $y > z$  on  $(0, v_l)$  and  $y = z$  at  $v = 0$  and  $v = v_l$ .*

*Proof.* The equation in (3.53) can be rearranged as:

$$\lambda \frac{h(v)}{h'(v)} = [(\mu + r_f)v - c] + \frac{1}{2} \sigma^2 v^2 h''(v) \quad (3.64)$$

Note that

$$y'(v) = \frac{h'(v)^2 - h(v)h''(v)}{h'(v)^2}, \quad (3.65)$$

so we can solve for the quantity  $\frac{h''(v)}{h'(v)}$  as well:

$$\frac{h''(v)}{h'(v)} = \frac{1 - y'(v)}{y(v)} \quad (3.66)$$

Substituting these into (3.64) and rearranging yields the following:

$$\sigma^2 v^2 (y'(v) - 1) = -2\lambda y(v)^2 + 2[(\mu + r_f)v - c]y(v) \quad (3.67)$$

The function  $z(v)$  satisfies a similar ODE (this is easily verified by substitution):

$$\sigma^2 v^2 \left( z'(v) - \frac{\mu + r_f + r_d}{2\lambda} \right) = -2\lambda z(v)^2 + 2 \left[ \frac{\mu + r_f + r_d}{2} v - c \right] z(v) \quad (3.68)$$

We have that  $y(0) = z(0) = -c/\lambda$  (to see this, set  $v = 0$  in (3.67)), and that  $y(v_l) = z(v_l) = -(1/d)(c/r_d - v_l)$ . First we show that  $y'(v_l) < z'(v_l)$ . If we substitute  $y(v_l)$  into (3.67), we have after simplification

$$y'(v_l) = 1 + \frac{r_d + m}{\lambda} - \frac{r_d}{\lambda} d. \quad (3.69)$$

Substituting the value of  $d$  in this expression, we can show that  $y'(v_l) < z'(v_l)$

if and only if

$$-(\mu + r_f) + \lambda + m < \sqrt{(r_d + \lambda + m)^2 - 4r\lambda}. \quad (3.70)$$

And since  $\mu + r_f > r_d$ , this is true if

$$-r_d + \lambda + m < \sqrt{(r_d + \lambda + m)^2 - 4r\lambda}. \quad (3.71)$$

This inequality is true, and can be checked by squaring both sides. Therefore, we have that  $y'(v_l) < z'(v_l)$ . This means that  $y > z$  on the interval  $(v_l - \delta, v_l)$ , for some  $\delta > 0$ . The remainder of the proof will be done by contradiction: Suppose that there exists  $\tilde{v} \in (0, v_l)$  such that  $y(\tilde{v}) = z(\tilde{v})$  and  $y > z$  on  $(\tilde{v}, v_l)$ . If we can show that no such  $\tilde{v}$  exists, the proof will be complete.

Since  $y(\tilde{v}) = z(\tilde{v})$  and  $y > z$  on  $(\tilde{v}, v_l)$ , we have  $y'(\tilde{v}) \geq z'(\tilde{v})$ . So by substitution in (3.67) and (3.68), we have

$$\begin{aligned} & 1 - \frac{2\lambda}{\sigma^2 \tilde{v}^2} y(\tilde{v})^2 + \frac{2[(\mu + r_f)\tilde{v} - c]}{\sigma^2 \tilde{v}^2} y(\tilde{v}) \\ & \geq \frac{(\mu + r_f + r_d)}{2\lambda} - \frac{2\lambda}{\sigma^2 \tilde{v}^2} z(\tilde{v})^2 + \frac{2[\frac{1}{2}(\mu + r_f + r_d)\tilde{v} - c]}{\sigma^2 \tilde{v}^2} z(\tilde{v}) \end{aligned} \quad (3.72)$$

Note that the middle terms cancel (as  $y(\tilde{v}) = z(\tilde{v})$ ), so substituting the value of  $z(\tilde{v})$ , what remains can be simplified to

$$1 - \frac{\mu + r_f + r_d}{2\lambda} \geq -\frac{\mu + r_f - r_d}{\sigma^2 \tilde{v}} \left( \frac{\mu + r_f + r_d}{2\lambda} \tilde{v} - \frac{c}{\lambda} \right) \quad (3.73)$$

The right hand side of this inequality is positive. We therefore have two

cases. In the case where  $\frac{\mu+r_f+r_d}{2\lambda} \geq 1$ , we directly obtain our contradiction.

In the case when  $\frac{\mu+r_f+r_d}{2\lambda} < 1$ , the inequality in (3.73) can be written as

$$\tilde{v} \geq \frac{2c(\mu + r_f - r_d)}{\sigma^2[2\lambda - (\mu + r_f) - r_d] + [(\mu + r_f)^2 - r_d^2]} \quad (3.74)$$

If we can show that  $v_l$  is less than that quantity, then  $\tilde{v} \in (0, v_l)$  cannot exist. It turns out that if we substitute in the value of  $v_l$  we can show that  $\tilde{v} \geq \frac{2c(\mu+r_f-r_d)}{\sigma^2[2\lambda-(\mu+r_f)-r_d]+[(\mu+r_f)^2-r_d^2]}$  is equivalent to the inequality in (3.70), which was already shown to be true.

Therefore, we have shown that there cannot exist  $\tilde{v} \in (0, v_l)$  such that  $y(\tilde{v}) = z(\tilde{v})$  and  $y > z$  on  $(\tilde{v}, v_l)$ . So  $y > z$  on the whole interval  $(0, v_l)$ .  $\square$

The results of this section are summarized in the following theorem.

**Theorem 3.8.** *The constrained minimum probability of lifetime ruin*

$\psi \in C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+ \setminus \{c/r_d\})$  is given by

$$\psi(v) = \begin{cases} h(v) & \text{if } v \in [0, v_l] \\ \beta \left(1 - \frac{r_d}{c}v\right)^d, & \text{if } v \in (v_l, c/r_d) \\ 0 & \text{if } v > c/r_d \end{cases} \quad (3.75)$$

where  $h$  solves the differential equation specified in (3.50) and (3.51) and where  $v_l$  is as specified in Corollary 3.4, and where

$$\beta = h(v_l) \left(1 - \frac{r_d}{c}v_l\right)^{-d} \quad (3.76)$$

The optimal investment strategy  $\pi^*(v)$  is given by

$$\pi^*(v) = \begin{cases} v & \text{if } v \in [0, v_l] \\ \frac{\mu+r_f-r_d}{\sigma^2} \frac{1}{d-1} \left( \frac{c}{r_d} - v \right) & \text{if } v \in (v_l, c/r_d] \end{cases} \quad (3.77)$$

## 4 Numerical Examples

Here we provide a few examples with numerical data to illustrate the results of section 3.2. We assume the following parameter values:

- $r_d = 0.02$ ; the domestic bank has an interest rate of 2% over inflation.
- $r_f = 0.035$ ; the foreign bank has an interest rate of 3.5% over inflation.
- $\mu = 0.025$ ; the drift of the exchange rate is 2.5%.
- $\sigma = 0.20$ ; the volatility of the exchange rate is 20%.
- $c = 1$ ; wealth is consumed at a rate of one unit per year.
- $\lambda = 0.04$ ; constant hazard rate of 4% such that the individual's expected future lifetime is 25 years.

For these parameter choices, we have an approximate lending level of  $v_l = 14.64$ . So the individual would invest all of her wealth in the foreign bank when  $v \leq 14.64$ , and some amount less than her total wealth when  $v > 14.64$ . Moreover, the wealth level at which she invests only in the domestic bank is  $c/r_d = 50$ . Figure 4.1 shows the amount invested in the foreign bank for wealth levels  $v \in [0, 50]$ , computed using Theorem 3.8.

We can also express the optimal investment strategy in terms of the fraction of total wealth invested. Figure 4.2 shows the fraction of total wealth that the individual would invest in the foreign bank for  $v \in [0, 50]$ .

The function  $\psi(v)$  given in Theorem 3.8 can be evaluated numerically for  $v \leq 14.64$  using a software ODE solver. After that solution is found,

the other piece of  $\psi(v)$  can be evaluated. For this example, we have the boundary conditions (as in (3.51)) of  $h(0) = 1$  and  $\frac{h(14.64)}{h'(14.64)} = -10.36$ . Solving numerically in Maple (using *dsolve* with *numeric, method=bvp* options) gives a curve with  $h(14.64) \approx 0.361$ . This value can then be used to solve for  $\beta$  in Theorem 3.8, using (3.76). The resulting curve is shown in Figure 4.3.

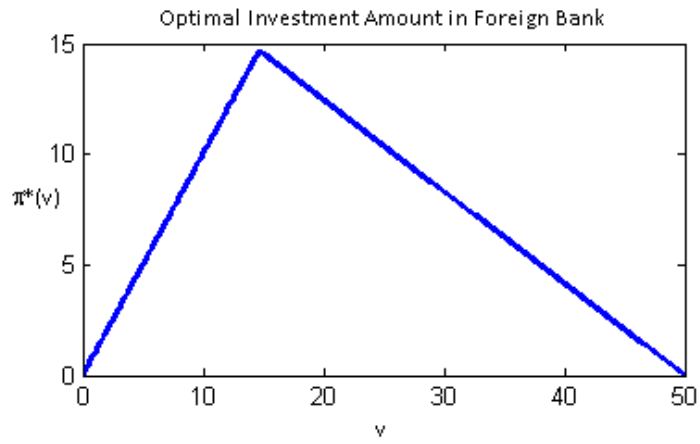


Figure 4.1: Optimal investment strategy. Here  $v_l = 14.64$ .

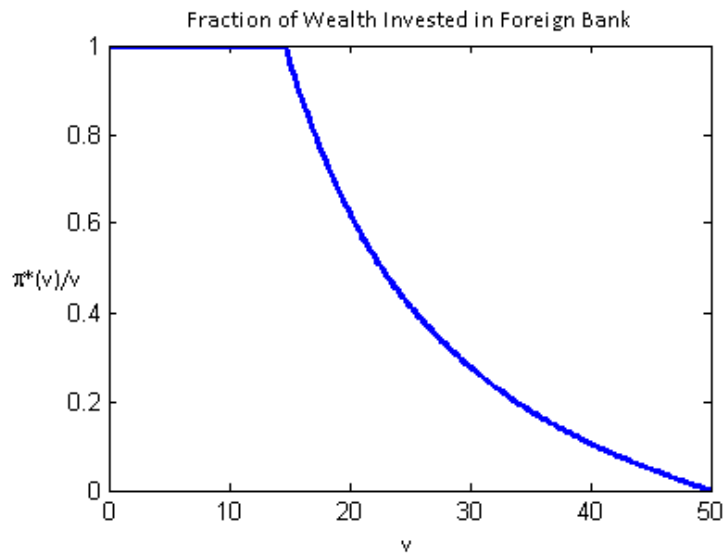


Figure 4.2: Percentage of total wealth invested.

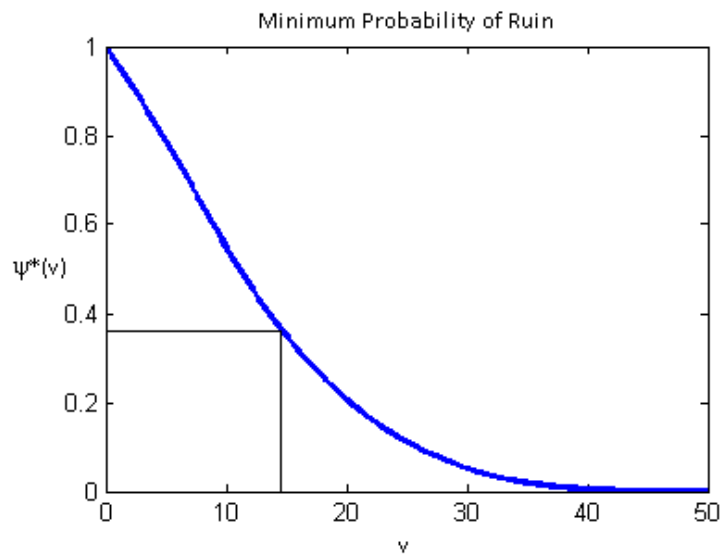


Figure 4.3: Minimum probability of ruin. The level  $v_l = 14.64$  is indicated.



## 5 Summary and Conclusion

In this paper we consider the problem of minimizing the probability of lifetime ruin of an individual investing in a market with foreign and domestic bank accounts. Our model assumes that the investor is not allowed to borrow, and that her consumption remains at a constant level. By extending the work of Bayraktar and Young (2006), we find expressions for the minimum probability of ruin as well as the optimal investment strategy for any given wealth level.

Moreover, part of our goal was to present the arguments leading to our results in a very clear manner. To that end we included a reference of the main concepts from probability theory and stochastic calculus which were applied, and attempted to make the steps of each proof clear and justified.

We find that there exists a “lending level” of wealth at which the investment strategy changes. For wealth below that level, the individual invests all of her wealth in the foreign bank account. For wealth above the lending level, the individual instead is able to reduce her risk of ruin by investing a portion of her wealth into the risk-free domestic bank. Naturally, an individual with a sufficiently high amount of wealth will have zero risk of lifetime ruin as long as her consumption is constant.

We do not address the case in which borrowing is allowed or in which the consumption rate varies with total wealth. Bayraktar and Young (2006) cover these cases, and their results naturally apply to our model as well. The assumptions of this paper are in some ways simplistic, and the results could

be made more realistic by assuming random interest rates or other (possibly random) consumption rates. Additionally, the assumption that only two currencies are tradeable is itself a significant simplification.

Nevertheless, viewing these results in the context of foreign exchange markets can give insight into the behavior of any investor seeking to minimize the risk of bankruptcy.

## 6 Appendix: MATLAB and Maple Code

Maple code used to solve the ODE in (3.50) and (3.51):

```
> sol1 := dsolve(0.04*h(v)-(0.06*v-1)*(diff(h(v),v))
- 0.02*v^2*(diff(diff(h(v),v),v)) = 0,
h(0.01) = 1,
h(14.644660940672622)/(D(h))(14.644660940672622) = -10.355339059327376],
numeric, method = bvp, abserr = 0.001);
sol1(14.6446);
plots[odeplot](sol1, 0.001 .. 15, color = red);
```

MATLAB function for computing the lending level  $v_l$  as in Corollary 3.4:

```
function[vl] = LendingLevel(rd,rf,mu,sigma,c,lambda)

m = 0.5*(mu+rf-rd)^2/sigma^2;
d = ( (rd+lambda+m) + sqrt((rd+lambda+m)^2-4*rd*lambda) )/(2*rd);
x = (mu+rf-rd)/(sigma^2*(d-1));
vl = (x*c)/((1+x)*rd);
```

MATLAB function for computing the function  $\pi^*(v)$  as in Theorem 3.8:

```
function[pi] = OptimalStrategyPlot(rd,rf,mu,sigma,c,lambda)

pi = zeros(1,251);
v = 0:c/(rd*250):c/rd;

m = 0.5*(mu+rf-rd)^2/sigma^2;
d = ( (rd+lambda+m) + sqrt((rd+lambda+m)^2-4*rd*lambda) )/(2*rd);

vl = LendingLevel(rd,rf,mu,sigma,c,lambda);

for i = 1:251
    if v(i) < vl,
        pi(i) = v(i);
    else
        pi(i) = (mu+rf-rd)/(sigma^2*(d-1)) * (c/rd - v(i));
    end
end
```

MATLAB function used to compute the function  $\psi(v)$  as in Theorem 3.8. Here, HVECEX was a global variable consisting of the data imported from Maple regarding the first half of the function (solved by the Maple code shown above).

```
function[h] = h_example_plot()

global HVECEX
h_first = HVECEX;

lambda = 0.04; mu = 0.025; r_f = 0.035; r_d = 0.02; c = 1; sigma = 0.2;
m = 0.5*((mu+r_f-r_d)/sigma)^2;
d = (1/(2*r_d))*( r_d+lambda+m) + sqrt((r_d+lambda+m)^2-4*r_d*lambda) );
vl = LendingLevel(r_d,r_f,mu,sigma,c,lambda);

v_first = (vl/100:vl/100:vl)';
v_rest = (vl+vl/100:vl/100:50)';

beta = h_first(length(h_first))*(1-r_d*vl/c)^(-d);

h_rest = beta*(1-r_d*v_rest/c).^d;

v = [ v_first ; v_rest ];
h = [ h_first ; h_rest ];
```

## References

- [1] Erhan Bayraktar and Virginia R. Young. Minimizing the Probability of Lifetime Ruin under Borrowing Constraints. *Insurance: Mathematics and Economics*, 41(1), 2006.
- [2] Björk, Tomas. *Arbitrage Theory in Continuous Time*. Oxford University Press Inc, 2004.
- [3] V. C. Pestien and W. D. Suddherth. Continuous-time red and black: How to control a diffusion to a goal. *Mathematics of Operations Research*, 10(4):599–611, 1985.
- [4] Philip E. Protter. *Stochastic Integration and Differential Equations*. Springer, 2004.
- [5] Steven E Shreve. *Stochastic Calculus for Finance II*. Springer, 2004.
- [6] V. R. Young. Optimal investment strategy to minimize the probability of lifetime ruin. *North American Actuarial Journal*, 8(4):105–126, 2004.