2006-04-27

Escher's Problem and Numerical Sequences

Matthew Stephen Palmacci
Worcester Polytechnic Institute

Escher's Problem and Numerical Sequences

by

Matthew Palmacci

A Thesis

Submitted to the Faculty

of the

WORCESTER POLYTECHNIC INSTITUTE

In partial fulfillment of the requirements for the

Degree of Master of Science

in

Applied Mathematics

by

May 2006

APPROVED:

Professor Brigitte Servatius, Major Thesis Advisor

Professor Bogdan M. Vernescu, Head of Department

Abstract

Counting problems lead naturally to integer sequences. For example if one asks for the number of subsets of an n-set, the answer is 2^n, or the integer sequence $1, 2, 4, 8, \ldots$.

Conversely, given an integer sequence, or part of it, one may ask if there is an associated counting problem. There might be several different counting problems that produce the same integer sequence.

To illustrate the nature of mathematical research involving integer sequences, we will consider Escher's counting problem and some generalizations, as well as counting problems associated with the Catalan numbers, and the Collatz conjecture. We will also discuss the purpose of the On-Line-Encyclopedia of Integer Sequences.

Acknowledgements

I would like to thank my parents and my family, who have always shown great support for me.

Thanks to Dr. Brigitte Servatius for her support and her guidelines in organizing this thesis.

Last, but not least, I'd like to thank the wonderful professors at WPI who have helped me better myself as well as my skills in mathematics.

Contents

Chapter 1

Introduction

Most integer sequences have certain patterns within their structure. It is interesting to see how two separate counting problems, in turn, count the same sequence. It would then be implied that the two counting problems have very much in common, ignoring their respective disciplines. Studying the common groundwork of two problems will bring a better understanding. This is why the On-line Encyclopedia of Integer Sequences is a helpful tool to one working with counting problems. The ability to see different variations of the same problem can be very beneficial to a researcher. This is also what makes the Catalan numbers so interesting. The Catalan numbers have been found in numerous different counting problems, some of which, are still being developed today.

A problem posed by the graphic artist, M.C. Escher, can be generalized to a few different integer sequences. None of these, so far, have been found in respect to other counting problems. Escher's original problem was found to be in direct relation to concepts in crystallography. The generalizations to his original problem have very interesting properties themselves.

Chapter 2

Neil Sloane's On-line Encyclopedia of Integer Sequences

Neil Sloane is a mathematician who has been maintaining a web page known as the On-line Encyclopedia of Integer Sequences (or OEIS).

(http://www.research.att.com/ njas/sequences/)

This web page has a list of thousands of integer sequences and some interpretations for each of these sequences. Each sequence has its own ID number. Also, if applicable, the sequences have a reference section for personal research, as well as links to other pages. Many sequences have a name and some formulas to derive the numbers. Some sequences also have an algorithm in different computer languages to define the numbers as well as cross-references to similar number sequences. The computer languages included are, but not limited to: MATLAB, Maple, MathCad, and MATHEMATICA. A reference will be displayed in this paper for Sloane's ID number in the context of the sequence.

OEIS refers not only to numerous integer sequences for reference, but also to integer arrays. Pascal's triangle (A7318) is read by rows and becomes

$$1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1, \ldots$$

Square matrices are read by their anti-diagonals, so that the sequence a_n is read

$$\begin{array}{lllll} a_0 & a_2 & a_5 & a_9 & \ldots \\ a_1 & a_4 & a_8 & \ldots & \\ a_3 & a_7 & \ldots & & \\ a_6 & \ldots & & & \end{array}$$

2.1 Purpose

The purpose of the OEIS is to aid in research. An integer sequence could show up in many places, and when it does, the OEIS may give more information than was previously known. If the sequence is not previously known it can then be submitted to the OEIS. Most of the sequences, which have a proof that the sequence counts the objects described, will be accepted by Neil Sloane. If the sequence is previously known, then it can be obtained along with alternate interpretations, computer code, a reference to publications, or a generating function for the sequence.

People with common interests frequently converse with each other using the medium of the internet; the OEIS is another propagation of this. The OEIS can introduce two researchers of different disciplines to work together on a common problem. A sequence can be queried to be better understood, one can read about a sequence previously unknown. Also, the references help find publications on the material. There maybe have been extensive research done on a sequence, but from a different development than was previously known.

Chapter 3

Escher's Problem

3.1 Escher's Life

Maurits Cornelius Escher was born in 1898 in the Netherlands. His father, George Arnold Escher (1843-1939), was a civil engineer who wanted Maurits to train as an architect. Maurits, or Mauk as he was called by his parents, had failed the final examinations in a few high school classes and thus never graduated. Although he passed his mathematics final exam, he was not considered to be highly skilled in the ways of mathematics. Later in his life, even though he was respected by mathematicians for his art, he still would exaggerate that he had difficulty with the subject [BKLW81]:

> *At high school in Arnhem, I was an extremely poor pupil in arithmetic and algebra, and I still have great difficulty with the abstractions of figures and letters.*

Eventually, he was able to enroll in the School for Architecture and Decorative Arts in Haarlem after attempting an alternate school to retake the failed classes. While trying to accomplish this he developed a skin infection that forced him to fall behind in attendance for the lectures. At the school Maurits wanted to try to follow his father's wishes and study architecture.

Mauk had showed some of his work to Samuel Jessurun de Mesquita, who was a teacher of graphic arts at the school in Haarlem, and who advised Maurtis to continue his graphic work. With his parents' permission, Mauk then devoted all of his time to "the graphic and decorative arts, in particular woodcuts". While studying the graphic arts, he concentrated largely on landscapes from unusual perspectives and plants.

Maurits traveled for much of the rest of his life, and he went to Italy and Spain from years 1922 to 1935. While in Spain in 1922, Escher made his first visit to the Alhambra Palace in Grenada. Majolica tilings decorate many surfaces in the Palace, this palace truly inspired him to work with the regular divisions of the plane.

In 1923, while living in Italy he met his future wife, Jetta Umiker. Jetta and Mauk got married in 1924 and they had their first child, George, in 1926. Their second child, Arthur, was born in 1928. Together they lived in Italy for just over a decade, taking frequent holidays around Europe. They were forced to leave Italy by the Fascist political regime and then moved to Switzerland. Jetta and Maurits disliked the Swiss surroundings and decided to take a Mediteranean expedition. They traveled the coasts of France, Italy, and Spain. The couple made volumes of sketches for future works. Maurits' second visit to the Alhambra Palace was during this expedition and was his richest source of inspiration. Jetta and Maurits spent days on end creating sketches for the main source of his future work; Escher was commissioned by the De Roos Foundation in 1957 to write an essay on his hobby of regular divisions of the plane. This essay was later published as a book titled *Regelmatige vlakverdeling*; in it, he had this to say about his studies of the regular divisions of the plane [BKLW81]:

> *It remains an extremely absorbing activity, a real mania to which I have become addicted, and from which I sometimes find it hard to tear myself away.*

Unknowingly, he studied areas of crystallography years before any other mathematician working in this area. He showed his work to his brother, Beer, who at the time was a Professor of geology, mineralogy, palaeontology and crystallography at Leiden University. Beer noticed the connections between Maurits' tilings and crystallography. So Beer sent Mauk a list of literature to feed his interest. Maurits then went on to work on numerous different tilings and he developed a systematic and highly mathematical approach using his own symbolism. This research resulted in 1965 with Caroline H. MacGillavry's book *Symmetry Aspects of M.C. Escher's Periodic Drawings*.

He then traveled to Switzerland and Belgium from 1935 to 1941. The Escher family moved to Brussels in 1937. Their third son, Jan, was born in 1938. Escher's father passed away in 1939. The Escher family was forced to leave Brussels because of the invading German forces in Europe. His mother passed away shortly after in 1940. During World War II, Maurits' work drastically slowed since he became very emotional about the war and, in particular, the loss of his parents. In 1944, German forces captured Escher's old teacher, Samuel Jessurun de Mesquita, who was Jewish.

The Escher family then moved to Baarn, Holland in 1941 trying to escape from the clutches of German invading forces. Escher continued to live in Baarn for the rest of his life, while taking frequent trips abroad. Due to his previous artistic practices, he was easily able to return to his work and finally complete what he had put on hold. In 1941, his research led to his notebook entitled, *Regular Division of the Plane with Asymmetric Congruent Polygons*, which would be published later in 1958. In this notebook he had covered all the possible combinations of shape, color and symmetrical properties. From 1945 on, he was asked to give lectures all over the globe. In 1951, two articles were published on Escher, one in *Time* magazine and one in *Life* magazine. He then progressed to study the concept of infinity.

From 1954 to 1961, Mauk made a yearly sea voyage to and/or from Italy. On one of his trips overseas, Escher became friends with H.S.M. Coxeter, professor at the University of Toronto. Coxeter had commented on Escher's new prints and often suggested literature for Escher. Coxeter helped him learn how to understand a circular division of the plane, where the center and the outside border would tend to infinity. Escher once wrote in a letter to his son George, expounding the insight he derived from Coxeter's *A Symposium on Symmetry* as [BKLW81]:

> *His hocus-pocus text is no use to me at all, but the picture can probably help me to produce a division of the plane which promises to become an entirely new variation of my series of divisions of the plane.*

Escher was able to infer the rules pertaining to hyperbolic tessellations from the illustration. In 1960, Escher made plans with Coxeter to organize lectures in Toronto where Escher would speak on the subject.

Escher learned much on the subject of "impossible" objects from Roger Penrose. The knowledge he gained helped him use it in his works of art. *Knots*, and *Ascending and Descending* are a few examples of the spacial concepts developed. Later on in his life he also became interested in doing his Metamorphosis series again. Though Escher started falling ill in 1962, some say he did his best works in his later years.

By the end of 1968 his wife, Jetta, could not bear to live in Baarn anymore and moved to Switzerland without Maurits. Their son, Jan, took care of Jetta in Switzerland while Maurits lived with a housekeeper. His last work, *Snakes*, heads toward infinity at both the center and the border of the picture, with snakes weaving in and out of the tessellation. In 1971, the book *De Werelden van M.C. Escher*(*The*

World of M.C. Escher) was published. Escher died in 1972 at the age of seventy-three.

3.2 Escher's Problem

While Escher was working on the regular division of the plane, he also experimented with repeating patterns to tessellate the plane. For this he would take a one-square design, called a *motif*. Each rotation (or reflection) of a motif is called its *aspect*. Using different aspects of a single motif he would build a 2×2 square array called a *translation block*. The translation block is then used to tessellate the plane.

In order to incorporate the reflected image of the one motif, it would be necessary to carve two separate stamps, the original and the reflected image. The stamps would then be inked and used to produce the translation block. In his sketchbooks Escher would experiment with different translation blocks and see how the resulting plane would differ. Then the question was posed: "How many different patterns can be made with a single motif?" He did restrict the rules for the four motifs of the translation block for two separate cases [Sch97]:

1. The four aspects of the translation block are each either a translation or a rotation of the original motif. (Only one stamp needed.)

2. Two aspects of the translation block are direct images of the original motif and two are opposite (reflected) images. One of the following also applies:

   (A) The two direct images have the same aspect and the two opposite images have the same aspect.

   (B) The two direct images and the two opposite images all have different aspects.

Each element in a translation block can be represented as a number. The list of four numbers will denote (in a clockwise order starting at the upper left hand corner of any unique translation block) the aspects in each position of the translation block. Each number will denote an aspect of a motif. Let the original motif to be the number 1, then consecutive 90° clockwise rotations will be labeled as 2, 3, and 4, respectively. Similarly, for the reflected stamps, Escher used an underline of each of the numbers. So, $\underline{1}$ is the mirror image of 1, etc. The four numbers that represent a translation block is known as that block's *signature*.

3.2.1 Solutions to Escher's Problem and a Generalization

The Original Escher Problem

Case 1 The four aspects of the translation block are each either a translation or a rotation of the original motif.

Schattschneider noted in [Sch97] that Burnside's counting technique can be implemented to solve this problem. First, let C_4 be the group generated by a rotation of 90°, (i.e. $C_4 = \{r,\ r^2,\ r^3,\ r^4 = e\}$). Then, the Klein four-group, K_4, coincides with the action of translation on the set of translation blocks. Let the numbering of positions on the block be

$$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}$$

The possible translations are vertical, horizontal, and diagonal so $K_4 = \{k_0,\ k_1,\ k_2,\ k_3\}$ where $k_0 = e$, $k_1 = (12)(34)$, $k_2 = (14)(23)$, and $k_3 = (13)(24)$. C_4 does not commute with K_4, yet $r^i K_4 r^{-i} \in K_4$, so C_4 normalizes K_4. Also note that $C_4 \cap K_4 = e$. So, let $H = K_4 C_4$ and it follows that H acts on each signature to produce all of the equivalent signatures. Also note that $|H| = 16$.

Burnside's Lemma 3.1 *[vLW01] Let H be a permutation group acting on a set X. For $h \in H$ let $\psi(h)$ denote the number of points of X fixed by h. Then the number of orbits of H is equal to $\frac{1}{|H|}\Sigma_{h\in H}\psi(h)$.*

In the Escher problem the number of orbits is the number of different tessellated patterns. The rotation and translation groups are acting on the set of all translation blocks. If the use of rotations and translations of a certain tessellation arrives at a different signature, then the original tessellating signature and the new signature are equivalent. So, this leaves a less difficult question of what $\psi(h)$ is for each combination of rotation and translation.

To calculate, start by looking at all 16 different elements of H. Here are a few examples, displaying the calculations for $\phi(h)$.

Example 1 $h = k_1 r$

To calculate $\phi(k_1 r)$, look at the action of $k_1 r$ on a translation block. Let (P, Q, R, S) be the translation block where, $P,\ Q,\ R,\ S$ are the arbitrary (clockwise) aspects. Then, $k_1 r(P,\ Q,\ R,\ S) = k_1(S',\ P',\ Q',\ R') = (P',\ S',\ R',\ Q')$, where $P' = r(P)$. Note that in the evaluation of $k_1 r$, the position of P maps directly back to the same position, P, except the aspect is rotated once. Since there is no aspect that, when rotated, remains the same, it is impossible for any translation block to map to its original signature over $k_1 r$. So, $\psi(k_1 r) = 0$.

Example 2 $h = k_2 r^2$

It is easy to see that

$$k_2 r^2(P,\ Q,\ R,\ S) = k_2(R'',\ S'',\ P'',\ Q'') = (Q'',\ P'',\ S'',\ R'').$$

Notice, P and Q switching positions, as well as R and S switch positions with a

180° rotation on each aspect. This implies that $P = Q''$, and $Q = P''$, which can both be true, as well as $R = S''$, and $S = R''$. So, choose any of the four aspects of the original motif for the position P, which leaves the position Q fixed. Also, choose any of the four aspects for the position S, which leaves R fixed. This implies that $\psi(k_2 r^2) = 4 \cdot 4 = 16$.

There are 4 elements of H that have $\psi(h) = 4$, 6 elements with $\psi(h) = 16$, and 1 element, the identity, that fixes all the signatures, $\psi(e) = 256$. The other 5 elements of H do not fix any signatures. Hence,

$$\frac{4 \cdot 4 + 6 \cdot 16 + 256}{|H|} = \frac{368}{16} = 23$$

This is the same number that Escher derived sketching each of the patterns and removing the signatures that he knew were equivalent to ones he had already drawn. This counting technique is quicker to use, and in many ways, leads to easier calculations.

Case 2 Two aspects of the translation block are direct images of the original motif and two are reflected images.

Again, the use of Burnside's lemma is a very efficient method. The only difference in the calculations are that the set of aspects are larger and the groups that act on them include reflection. So, where C_4 is the set of rotations in Case 1, consider the group generated by reflections and a rotation. The symmetries of the square, D_4, are the group actions for rotations and reflections in this case. For the sake of notation, let $G = K_4 D_4$. Note that $|G| = 32$.

Case 2A The two direct images have the same aspect and the two opposite images have the same aspect.

There are $6 * 4 * 4 = 96$ different signatures to consider (there are 6 ways to position the two direct and the two reflected images, and 4 possible aspects for each.) It is already known that

$$\sum_{g \in G} \psi(g) = 320$$

This implies that the number of equivalence classes for this case is 10. Escher also got this exact result in his notebook.

Case 2B The two direct images and the two opposite images all have different aspects.

There are $6 * (4 * 3)(4 * 3) = 864$ different signatures.

$$\sum_{g \in G} \psi(g) = 1248$$

Hence, the solution to Escher's Case 2B is 39. Escher discovered 37 unique tessellations. He made a few mistakes to arrive at this solution, one of which was that he didn't notice two of his solutions were, in fact, the same tessellation just rotated and reflected.

Burnside's lemma is a very useful counting technique and can be used in the generalizations of this problem as outlined in the following sections.

The One-Dimensional Case

Here, instead of tessellating a 2-D plane with a 2×2 block, an infinite 1-D strip will be created by repeating a $1 \times n$ strip. Repeating a $1 \times n$ block an infinite amount of times on one strip is similar to creating a closed ring with n sides. The group of symmetries that act on this ring are not as straight forward as in the original cases. There are still three group actions that will generate the group of symmetries: rotation, translation, and mirror. The notation was denoted in [PSS06].

Translation of the strip corresponds to a rotation of the ring with n sides. The rotation is about the vertical axis through the center of the ring. Each is a rotation is of $\frac{360}{n}^{\circ}$. This group, denoted by T, is equal to $\left\langle \frac{360}{n}^{\circ} \right\rangle$, and has order n. So, the action of T on an arbitrary signature $Q_1Q_2 \ldots Q_n$, where Q_i is any aspect of the given asymmetric motif for all i, looks like

$$T(Q_1Q_2 \ldots Q_n) = Q_2Q_3 \ldots Q_nQ_1.$$

Look at the action of translation on the ring to calculate the number of fixed signatures by that action. So, a translation that moves the strip i positions corresponds to a rotation of $\frac{i}{n}360^{\circ}$. The orbits of $\frac{i}{n}360^{\circ}$ have $\frac{n}{gcd(i,\ n)}$ elements in each and there are $gcd(i,\ n)$ orbits. So, for any i we can freely pick $gcd(i,\ n)$ aspects, the rest of the aspects are fixed by the action of consecutive translations. If we denote $\varphi(k)$ to be the *Euler phi function*, then the number of fixed signatures due to translations for any n is

$$\sum_{k|n} \varphi(k) 4^{n/k}.$$

Where $\varphi(k)$ refers to how many different translations have an orbit of size k.

Rotation can only be 180° about a horizontal axis that travels through the center of a face of the ring, or the center of the intersection of two faces. This is equivalent to rotating the infinite strip 180° about a single position (the only two choices for this one position are either to rotate around one aspect or between two adjacent aspects.) So, this action is not order-preserving, is denoted by R, and the rotation about the position between Q_1 and Q_n looks like

$$R(Q_1Q_2\dots Q_n) = R(Q_n)\dots R(Q_2)R(Q_1)$$

Where $R(Q_i)$ is just a 180° rotation of the aspect of Q_i.

Let's now calculate how many signatures a rotation will fix. A rotation about the axis that passes through the center of one of the faces will fix no signatures. This is so since any aspect of the motif is not equal to its 180° rotation. So, need only look at rotations about the axis that passes through the center of the intersection of two faces. Except, this axis does not exist when n is odd. So, with n even, and with 4 initial aspects to choose from there are

$$\binom{n}{2} 4^{n/2}$$

fixed signatures. This notes that half of the faces are freely chosen, leaving the other half denoted by the first halves 180° rotation. Then for each position of the $\frac{n}{2}$ there are 4 initial aspects to choose from. This number is the same for a combination of a translation with a rotation about the horizontal axis.

The mirror (or reflection) symmetry, M, is either a reflection about the horizontal plane that runs through the middle of each face of the ring, M_1, or a reflection of a vertical plane that divides the ring in half, M_2. The reflection about the horizontal plane is an order-preserving action. Taking a horizontal mirror of a ring (or strip) will induce a reflection on each face of the ring about the horizontal plane. To summarize

$$M_1(Q_1Q_2\dots Q_n) = M_1(Q_1)M_1(Q_2)\dots M_1(Q_n)$$

Where $M_1(Q_i)$ is a horizontal mirror image of the aspect.

The horizontal mirror, alone, fixes no signatures for any n. Although a rotary reflection, which is a combination of translations and a horizontal reflection does fix some signatures. If the orbit of the translation is of odd size, then the rotary reflection will fix no signatures. If it is of odd size, then the translation will lead to an initial aspect will be mapped to itself, the mirror will then make the initial aspect map to its own mirror image, which is impossible. So, the number of fixed signatures for rotary reflections is

$$\sum_{k|n,\ 2|k} \varphi(k) 4^{n/k}.$$

The reflection of a vertical plane is order reversing. This corresponds to a reflection about the vertical line that divides the 1 dimensional strip into two equally sized sections. Notice, that when n is odd, there are no fixed signatures for this action. When n is odd, the middle position is reflected onto itself and we have assumed that the motif is asymmetric. When n is even, the vertical reflection could pass through the center of a motif, or could pass through the boundaries of the motifs. With the vertical (reflection) plane passing through the center of a motif the action fixes no signatures, for the same reason as when n is odd. When the vertical plane passes through the boundary of motifs there are again,

$$\left(\frac{n}{2}\right) 4^{n/2}$$

fixed signatures. This is the same number as fixed signatures by 180° rotations about the horizontal axis. The reason for this is similar to the explanation for rotations. This also represents the number of fixed signatures for any combination of translations, rotation, and vertical reflection, so long as at least a rotation or a vertical reflection are present.

Now, for 4 aspects without including mirror symmetries we have

$$\frac{n}{2}4^{n/2} + \sum_{k|n} \varphi(k) 4^{n/k} \frac{1}{2n}$$

fixed signatures total. Since $2n$ is the order of the group in this case. If we include mirror symmetry we then have

$$\left(2\left(\frac{n}{2}4^{n/2}\right) + \left(\sum_{k|n,\ 2|k} \varphi(k)4^{n/k} \right) + \sum_{k|n} \varphi(k)4^{n/k} \right) \frac{1}{4n}$$

total fixed signatures.

Other Generalizations

The two-dimensional case is one where an $n \times n$ block is used to tessellate the plane. Intuitively, this problem is easy to understand, yet computationally, this problem is not an easy one. For each n, $\varphi(n)$ be calculated fairly easily, and the number of unique patterns is therefore also easy. The difficult part is to derive a formula to describe the number of unique tessellations for some general n. When looking at the answers for small n, it is not clear, at this time, what pattern exists.

The three-dimensional case was posed by Doris Schattschneider [Sch97]. This can be described as a $2 \times 2 \times 2$ supercube that is used to tessellate the space. There will be 6 possible aspects for rotation of the cube. There is also a 'reflection' of an aspect, which pertains to central inversion of the cube. We can then pose Escher's question to this set up. Not much has been written about its solutions as of this time.

Chapter 4

Catalan Numbers (A108)

Euler discovered a number sequence when studying the question:

> How many distinct ways are there to triangulate a convex $(n+2)$-gon with non-intersecting diagonals, where $n \geq 1$?

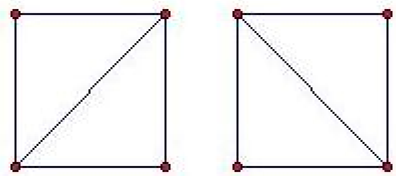
Denote the resulting sequence by E_n. Although Euler deduced the pattern for E_n, he thought his method was cumbersome and left the problem open to others to work on. Eugene Charles Catalan was a Belgian mathematician who lived from 1814 to 1849. Eugene Catalan solved a different combinatorial problem in which these numbers came up again. Catalan noticed the similarities in the two resulting sequences and therefore became motivated to try to discover other interpretations to describe this same integer sequence.

Let a sequence of numbers be defined as :

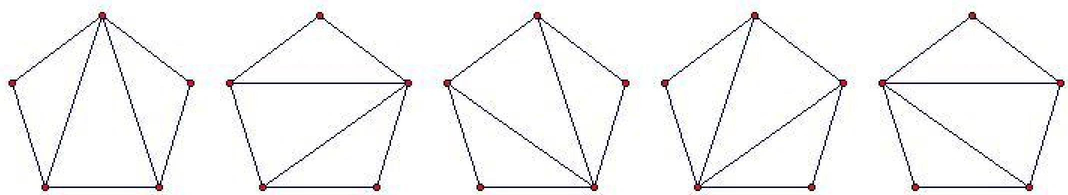
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

The result is the sequence $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, etc. This sequence is known as the Catalan numbers (sometimes called the Segner numbers).

Solutions to Euler's Triangulation Problem The case where $n = 1$, the triangulation of a triangle can only be done in 1 way. For $n = 2$, there are 2 diagonals that could be added to arrive at the triangulation of a rectangle. Hence, for $n = 2$, the answer to the problem is 2.



The case for $n = 3$ is displayed below.



Graphics made with Geometer's Sketchpad

The solution to the next consecutive cases are 14, 42, 132, 429, etc.

4.1 Alternate Interpretations

4.1.1 Well-formed Sequences of Parentheses

There are numerous problems that result in the Catalan numbers.

> Find the number of ways of writing a well-formed sequence of parentheses using n pairs of parentheses, or $2n$ parentheses.

This can also be seen as a multiplication of elements using $2n$ parentheses, where multiplication is not associative and not commutative. When $n = 1$, there is only one way to pair no parentheses: the null set, λ. When $n = 2$, there is only one way to write 1 pair of well-formed parentheses, (). For the case of $n = 3$, there are exactly two ways to properly use 2 parentheses:

$$(())\qquad ()()$$

The case of $n = 4$ has 5 solutions:

$$()()()\quad ()(())\quad (())()\quad (()())\quad ((()))$$

Proof that this sequence is equivalent to C_n will be described in section 4.2.

4.1.2 Ordered Trees

There are many other ways of interpreting the Catalan numbers. For example, the question could be asked:

> How many ordered trees are there on $n + 1$ vertices?

Note that an ordered tree is defined to be a rooted tree, in which, each tree obtained by removing the root is also an ordered tree with some known order having been assigned. Instead of going through each case of n, describing the bijective function between the number of ordered trees on $n+1$ vertices and the number of well-formed statements on n pairs of parentheses will be sufficient.

If, given an ordered tree, T, then define $\mu(T) = P$ recursively, where P is a well-formed sequence of parentheses. Initially, if T is a tree consisting of a single vertex, then P contains no parentheses. Then, assume that μ is defined on all ordered trees $\widetilde{T}$ for $k + 1$ vertices, where $k < n$. Therefore, $\mu(\widetilde{T})$ is known with $2k$ parentheses. Now, if T is an ordered tree with $n + 1$ vertices and s principle subtrees (the trees obtained by removing the root) then denote the principle subtrees by T_1, T_2,... T_s.

Then let the corresponding well-formed sequence of parentheses be P_1, P_2,... P_s, and $P = (P_1)(P_2)\ldots(P_s)$.

So, for each vertex in T, not including the root, there will be one pair of parentheses associated with it, through μ. This implies that with $n+1$ vertices in T, there is also $2n$ parentheses in P.

Now, to describe μ^{-1}, let the well-formed sequence of parentheses be $P = (P_1)(P_2)\ldots(P_s)$, where each P_i is also a well-formed sequence of parentheses. Then the corresponding principle subtrees are T_1, T_2,... T_s, where each $T_i = \mu^{-1}(P_i)$.

4.1.3 Binary Trees

How many binary trees are there on n vertices?

Let the function μ take a binary tree on n vertices and produce an ordered tree on $n+1$ vertices. Let B be a binary tree. The construction of $T = \mu(B)$ is as follows [SW86]:

(a) The vertices of B are the vertices of T with the root deleted.

(b) The root of B is the first son of the root of T.

(c) Vertex v is a left son of vertex w in B if and only if v is the first son of w in T.

(d) Vertex v is a right son of vertex w in B if and only if v is the brother to the right of w in T.

4.1.4 Full Binary Trees

Define a full binary tree to be a binary tree where every vertex has either 0 or 2 children. Then, the posed question is as follows.

How many full binary trees are there on $2n+1$ vertices?

The bijection will be shown between a binary tree with n vertices and a full binary tree with $2n+1$ vertices. Given a binary tree, B, add a new child to every vertex of B with exactly 1 child. Also, to each vertex of B with 0 children add 2 children to that vertex. This gives the construction of a full binary tree, $\mu(B)$. It can be easily shown, using induction, that the number of terminal vertices (vertices with 0 children) in a full binary tree is one more than the number of internal vertices (vertices with 2 children). Therefore, F has $2n+1$ vertices. The inverse of this function is obtained by "pruning" the terminal vertices of F.

4.2 Equivalence to the Catalan Numbers

From the interpretation of well-formed statements with $2n$ parentheses, create a string from the set $\{1, -1\}$. A left parenthesis is replaced by a 1, and a right parenthesis is replaced by a -1. Now, rewrite C_n as:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \frac{(2n+1)!}{(2n+1)(n+1)!n!} = \frac{1}{2n+1}\binom{2n+1}{n+1}$$

In lieu of this equality, consider Δ_n, the set of $\{1, -1\}$ strings with $2n+1$ digits, and with $n+1$ of those digits being 1. Note, the cardinality of Δ_n is $\binom{2n+1}{n+1}$. More importantly, notice that any string can be cyclically permuted to $2n+1$ distinct strings in Δ_n. Therefore, Δ_n can be partitioned into equivalence classes over cyclic permutation. Each equivalence class will have $2n+1$ elements in it. The number of equivalence classes is exactly C_n, by definition.

Let P be the string with $(n+1)$ 1's. Then, $P = P_1P_2\cdots P_{2n+1}$, where $P_i \in \{1, -1\}$ for each i. Also, $S_0 = 0$, $S_1 = P_1$, $S_2 = P_1 + P_2$, $\ldots$, $S_{2n+1} = P_1 + P_2 +$

$\cdots + P_{2n+1}$ are the string's associated partial sums.

Consider one of the equivalence classes described above, Θ. Take any string, $w \in \Theta$. Define $\gamma(w)$ to be the index of the last minimal partial sum of w. So, if $w \in \Theta$, and $S_i \geq 0 \ \forall i$, $S_0 = 0$, $S_6 = 0$, and $S_8 = 0$, then $\gamma(w) = 8$.

Notice what happens to $\gamma(w^+)$, with w^+ being equal to w cyclically permuted one position to the right. For sake of wording, if $w = -1, 1, 1, -1, 1$, then $w^+ = 1, -1, 1, 1, -1$. It easily seen that

$$\gamma(w^+) = \begin{cases} \gamma(w) + 1 & \text{if } \gamma(w) \neq 2n \\ 0 & \text{if } \gamma(w) = 2n \end{cases}$$

Therefore, there is exactly one string, w^*, such that $\gamma(w^*) = 0$. Note that the first digit in w^* must be a 1. If the initial 1 is removed, what remains is the string that corresponds to the sequence of well-formed parentheses. This is also conversely true, adding a 1 to the beginning of a string, $\hat{w}$, that corresponds to a well-formed sequence of parentheses implies $\gamma(\hat{w}) = 0$. This shows that the number of well-formed statements with $2n$ parentheses is exactly equal to the Catalan numbers, C_n.

Chapter 5

Collatz Conjecture

The Collatz conjecture has been credited to Lothar Collatz, who worked on it in the 1930's while he was a student at the University of Hamburg.

Define a function, $\pi : \mathbb{N}^+ \to \mathbb{N}^+$ (A6370), where

$$\pi(n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 (\text{mod } 2); \\ \frac{3n+1}{2}, & \text{if } n \equiv 1 (\text{mod } 2) \end{cases}$$

This is known as the "Collatz" function.

Collatz Conjecture 5.0.1 *For every n, there exists an iterate of the Collatz function $\pi^{(k)}(n)$ that equals* 1.

The Collatz conjecture is also known as: the Syracuse conjecture, the "$3x + 1$" problem, Kakutani's problem, Hasse algorithm, and Ulam's problem. The problem had spread through word of mouth throughout the mathematical community. There have been many researchers who have studied this conjecture, yet none who have obtained a complete proof. There are a list of prizes that have been offered for the proof or counter example of this conjecture. Paul Erdős once said, "Mathematics is not yet ready for such problems." Still, numerous mathematicians have done hours of work in the quest for a proof. There are a many different ways of attacking this problem. Here are a few of the different interpretations.

The remaining sequence after one iteration of the Collatz number can be read on the OEIS at (A6370). The sequence that is the largest output value for each n from repeated Collatz iterations is denoted at (A6884). (A74473) is the smallest number of iterations of the Collatz function is necessary such that $\pi^k(n) < n$.

5.1 Methods for the Collatz Conjecture

5.1.1 Directed Graph Interpretation

Assume an infinite directed graph where each vertex is a positive integer such that each vertex has at least one directed edge from n to $\pi(n)$. Now, the sequence of iterates $(n,\ \pi(n),\ \pi^{(2)}(n),\ \pi^{(3)}(n), \ldots)$ is called the *trajectory* of n. There are three possible behaviors for trajectories [Lag85]:

1. *Convergent trajectory.* $\exists\ k \geq 1 : \pi^{(k)}(n) = 1$.

2. *Non-trivial cyclic trajectory.* The sequence becomes periodic and $\nexists\ k \geq 1 : \pi^{(k)}(n) = 1$.

3. *Divergent trajectory.* $\lim_{k \to \infty} \pi^{(k)}(n) = \infty$.

This leaves the new interpretation to be:

Alternate Conjecture 5.1.1 *All trajectories are convergent trajectories.*

5.1.2 Stopping Time Interpretation

Clearly, it can be seen that the iterations will never arrive at $\pi^{(k)}(n) = 1$ for some k unless $\exists\ l : \pi^{(l)}(n) < n$. So, define $\sigma(n)$ to be the smallest $l : \pi^{(l)}(n) < n$. Call this the *stopping time* of n. If $\nexists\ k \geq 1 : \pi^{(k)}(n) = 1$ then set $\sigma(n) = \infty$. Denote the *total stopping time*, $\sigma_\infty(n)$, to be the smallest value of $k : \pi^{(k)}(n) = 1$. Again, if $\nexists\ k \geq 1 : \pi^{(k)}(n) = 1$, then set $\sigma_\infty(n) = \infty$. The conjecture is now stated as:

Alternate Conjecture 5.1.1 *Every $n \geq 2$ has a finite stopping time.*

5.1.3 Heuristic Argument

This heuristic argument supports the conjecture. Start with an odd integer, n_0, and iterate the Collatz function until the output is another odd integer, n_1. Notice that half of the time $n_1 = \frac{1}{2}(3n_0 + 1)$, a quarter of the time $n_1 = \frac{1}{4}(3n_0 + 1)$, and an eighth of the time $n_1 = \frac{1}{8}(3n_0 + 1)$, etc. Then the expected growth between two consecutive odd integers is the multiplicative factor

$$\left(\frac{3}{2}\right)^{1/2}\left(\frac{3}{4}\right)^{1/4}\left(\frac{3}{8}\right)^{1/8}\cdots = \frac{3}{4} < 1$$

This implies that the iterates, on average, get closer to 1. Hence, no divergent trajectories should exist.

5.2 Current Progress

There is still much research being done for the proof of this conjecture. This section will mention a few of the recent discoveries without proof (see references). There is much more information on this topic than presented here.

The Collatz conjecture has been verified for all $n < 2.702 \times 10^{16}$ by Oliveira e Silva [OeS99], the computations for which extended the bound to all $n < 1.125 \times 10^{17}$. It also easy to see that there is an infinite number of integers that have a finite total stopping time (i.e. 2^i for all i.) Computing $\pi^{-1}(n)$ implies traversing the directed graph interpretation in the opposite direction. Calculating π^{-1} of the family 2^i, will derive other infinite families with the same property.

It was shown by Crandall [Cra78] that if $\pi^k(n) = n$ for some k, then $k > 17985$. Crandall also showed the correlation between the validity of the Collatz conjecture and the diophantine equation $2^x - 3^y = p$, where p is a prime.

Lower bounds on the total stopping time was shown by Applegate and Lagarias [AL03]. They proved that there are an infinitely many numbers having a finite total stopping time such that $\sigma_\infty(n) > 6.14316 \log(n)$.

Bibliography

[AKN00] Ştefan Andrei, Manfred Kudlek, and Radu Ştefan Niculescu. Some results on the Collatz problem. *Acta Inform.*, 37(2):145–160, 2000.

[AL03] David Applegate and Jeffrey C. Lagarias. Lower bounds for the total stopping time of $3x + 1$ iterates. *Math. Comp.*, 72(242):1035–1049 (electronic), 2003.

[BKLW81] F.H. Bool, J.R. Kist, J.L. Locher, and F. Wierda. *M.C. Escher: His Life and Complete Graphic Work*. Harry N. Abrams, Inc., New York, 1981.

[Cof97] Judita Cofman. Catalan numbers for the classroom? *Elem. Math.*, 52(3):108–117, 1997.

[Cos03] G. E. Cossali. A common generating function for Catalan numbers and other integer sequences. *J. Integer Seq.*, 6(1):Article 03.1.8, 8 pp. (electronic), 2003.

[Cra78] R. E. Crandall. On the "$3x + 1''$ problem. *Math. Comp.*, 32(144):1281–1292, 1978.

[Dav97] Dan Davis. On a tiling scheme from M. C. Escher. *Electron. J. Combin.*, 4(2):Research Paper 23, approx. 11 pp. (electronic), 1997. The Wilf Festschrift (Philadelphia, PA, 1996).

[Lag85] Jeffrey C. Lagarias. The $3x + 1$ problem and its generalizations. *Amer. Math. Monthly*, 92(1):3–23, 1985.

[OeS99] Tomás Oliveira e Silva. Maximum excursion and stopping time record-holders for the $3x + 1$ problem: computational results. *Math. Comp.*, 68(225):371–384, 1999.

[PSS06] Tomaž Pisanski, Doris Schattschneider, and Brigitte Servatius. Applyingn burnside's lemma to a one-dimensional escher problem. *Math. Mag.*, pages 167–180, 2006.

[Sch97] Doris Schattschneider. Escher's combinatorial patterns. *Electron. J. Combin.*, 4(2):Research Paper 17, approx. 31 pp. (electronic), 1997. The Wilf Festschrift (Philadelphia, PA, 1996).

[Slo99] N. J. A. Sloane. My favorite integer sequences. In *Sequences and their applications (Singapore, 1998)*, Springer Ser. Discrete Math. Theor. Comput. Sci., pages 103–130. Springer, London, 1999.

[Slo03] N. J. A. Sloane. The on-line encyclopedia of integer sequences. *Notices Amer. Math. Soc.*, 50(8):912–915, 2003.

[SW86] Dennis Stanton and Dennis White. *Constructive combinatorics*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1986.

[vLW01] J. H. van Lint and R. M. Wilson. *A course in combinatorics*. Cambridge University Press, Cambridge, second edition, 2001.