COLOCALITY AND TWISTED SUMS OF BANACH SPACES

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ABSTRACT. There is a nontrivial twisted sum of Schreier's space, S, with c_0. That is, there is a Banach space X containing an uncomplemented copy of S, so that X/S is isomorphic to c_0. This and other new examples of nontrivial twisted sums are given by introducing the concept of colocality, which is dual to the existing concept of locality. Given a nontrivial twisted sum, this guarantees the existence of another nontrivial twisted sums with the same quotient but a different subspace. An interesting point is that no restrictions are imposed on the quotient space, only on the two subspaces.

Introduction

A short exact sequence is a diagram $0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{q} Z \longrightarrow 0$ of quasi Banach spaces and bounded linear operators such that the kernel of each arrow coincides with the image of the preceding one. The open mapping theorem implies that X contains a subspace isomorphic to Y, namely $i(Y)$, and the quotient $X/i(Y)$ is isomorphic to Z. We shall also say that X is a **twisted sum** of Y and Z. Following the convention in algebra [15], we will also say that X **is an extension of** Z **by** Y. The twisted sum X is then said to be **trivial** if $i(Y)$ is complemented in X; otherwise, it is **nontrivial**.

Two exact sequences $0 \longrightarrow Y \longrightarrow X_1 \longrightarrow Z \longrightarrow 0$ and $0 \longrightarrow Y \longrightarrow X_2 \longrightarrow Z \longrightarrow 0$ are said to be **equivalent** if there is a bounded linear operator T making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & T\downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

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commutative. The three-lemma and the open mapping theorem imply that T must be an isomorphism [5, p. 525]. An exact sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$ is said to **split** if it is equivalent to the trivial sequence $0 \longrightarrow Y \longrightarrow Y \oplus Z \longrightarrow Z \longrightarrow 0$, that is, X is trivial, in this case, the twisted sum X is isomorphic to $Y \oplus Z$ (the converse is not true). We denote by $Ext\,(Z, Y)$ the space of all equivalence classes of (exact sequences of) locally convex twisted sums of Y and Z. Thus $Ext\,(Z, Y) = 0$ means that all locally convex twisted sums of Y and Z are equivalent to the direct sum $Y \oplus Z$, or every exact sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$ splits. An operator $T : X \longrightarrow Y$ of Banach spaces is an **isomorphism** if it is an invertible bounded linear map, T is an **isometry** if $\|Tx\| = \|x\|$ for every $x \in X$, it is a **λ-isomorphism,** $\lambda > 1$, if T is an isomorphism and $\|T\| < \lambda$, $\|T^{-1}\| < \lambda$ [14, II.6]. **The distance** between two homogeneous maps T_1 and T_2 acting between the same spaces is given by

$$dist\,(T_1, T_2) = \sup\{\|T_1x - T_2x\| : \|x\| \leq 1\}.$$

A related concept is the *Banach-Mazur distance* $d_{BM}\,(B, E)$ between two Banach spaces B and E, defined by $d_{BM}\,(B, E) = \inf\{\|T\|\,\|T^{-1}\|\,;$ $T : X \longrightarrow Y$ is an isomorphism of X onto $Y\}$. We will only consider this for finite dimensional Banach spaces, although it is used more generally.

Note that bounded maps are simply those at finite distance from the zero map. We emphasize that linear maps are not assumed to be bounded. Following the notation in [9], we say that a homogeneous map $F : Z \longrightarrow Y$ acting between two Banach spaces is **z-linear** if it satisfies, for some constant k and all $z_i \varepsilon Z$

$$\left\| F\left(\sum_{i=1}^{n} z_i\right) - \sum_{i=1}^{n} F\left(z_i\right)\right\| \leq k \sum_{i=1}^{n} \|z_i\|.$$

The infimum of the constants k satisfying the above inequality is denoted by $Z\,(F)$ and is called **the z-linearity constant** of F. Given a z-linear map $F : Z \longrightarrow Y$, it is possible to construct a twisted sum of Y and Z, denoted by $Y \oplus_F Z$, by endowing the product space $Y \times Z$ with the quasi-norm $\|(y, z)\| = \|y - F\,(z)\| + \|z\|$. On the other hand, given a short exact sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$, a z-linear map $F : Z \longrightarrow Y$ can be obtained such that X is equivalent to $Y \oplus_F Z$; see Kalton and Peck [19, Theorem 2.4] or Castillo and González [8, 1.6.e]. Two z-linear maps F and G of a Banach space Z into a Banach space Y are said to be **equivalent** if the corresponding exact sequences $0 \longrightarrow Y \longrightarrow Y \oplus_F Z \longrightarrow Z \longrightarrow 0$ and $0 \longrightarrow Y \longrightarrow Y \oplus_G Z \longrightarrow Z \longrightarrow 0$ are equivalent. Therefore, there is a one to one correspondence between the classes of locally convex twisted sums $Y \oplus_F Z$ and the classes of z-linear maps $F : Z \longrightarrow Y$; see Cabello and Castillo [4] or Castillo and González [8, 1.6.e]). A z-linear map $F : Z \longrightarrow Y$ is said to be **trivial** if the exact sequence $0 \longrightarrow Y \longrightarrow Y \oplus_F Z \longrightarrow Z \longrightarrow 0$ splits; by a result of Kalton [18, Proposition 3.3], this is equivalent to F being at finite distance from a linear map $L : Z \longrightarrow Y$.

Throughout this paper $\mathcal{E}$ and $\mathcal{F}$ denote families of finite dimensional Banach spaces and $\mathcal{E}^*$ denotes the family of the duals of the spaces in $\mathcal{E}$.

1. **Colocality and Basic Examples**

Given a family $\mathcal{E}$ of finite dimensional Banach spaces, a Banach space X is said to **contain $\mathcal{E}$ uniformly complemented** if there exists a constant c such that for every $E \in \mathcal{E}$, there is a c-complemented subspace A of X which is c- isomorphic to E. In [5], Cabello and Castillo introduced the concept of **locality**; a Banach space X is said to be **λ-locally $\mathcal{E}$** (or, if no quantitative estimate is needed, **locally $\mathcal{E}$**) if there exists a constant $\lambda > 1$ such that every finite dimensional subspace A of X is contained in a finite dimensional subspace B of X such that $d_{BM}(B, E) < \lambda$, for some $E \in \mathcal{E}$. In the case when $\mathcal{E}$ is the family (ℓ_p^n), this reduces to the notion of $\mathcal{L}_p$ spaces first defined by Lindenstrauss and Rosenthal [21]. Cabello and Castillo used the locality of a family to determine the existence of nontrivial twisted sums of certain Banach spaces. More precisely, they proved that if Y is a Banach space complemented in its bidual, and if all the locally convex twisted sums $Ext\,(W, Y)$ of Y and some Banach space W containing a family $\mathcal{E}$ uniformly complemented are trivial, then $Ext\,(Z, Y) = 0$ for every Banach space Z which is locally $\mathcal{E}$ [5, Theorem 2]. Then they used this fact to show that there are nontrivial twisted sums of ℓ_1 and ℓ_2, [5, Example 4.1], of ℓ_2 and c_0, and to give a new proof that $Ext(c_0, \ell_1) \neq 0$ [5, Examples 4.2, 4.3].

It is elementary that subspaces of Banach spaces are related to quotients of their duals, and vice versa; recall that if $G = X/D$ is a quotient of a Banach space X, then $G^* = (X/D)^* = D^\perp$ is a subspace of its dual X^*, and if B is w^*-closed in X^*, in particular if B is finite dimensional, then $B = (B_\perp)^\perp = (X/B_\perp)^*$ so that $B^* \equiv (X/B_\perp)^{**} = X/B_\perp$ is a quotient of X, where $D^\perp$ and $B_\perp$ denote the annihilators of D and B (see [23, 4.7, 4.8]). Thus it is natural to introduce a notion analogous to locality which involves quotients rather than subspaces, and then attempt to dualize the known results concerning the construction of new nontrivial twisted sums from old ones. We say that a Banach space X is **λ-colocally $\mathcal{E}$** (or **colocally $\mathcal{E}$**) if there exists a constant $\lambda > 1$ such that every finite dimensional quotient A of X is a quotient of another finite dimensional quotient B of X satisfying $d_{BM}(B, E) < \lambda$ for some $E \in \mathcal{E}$.

It is not hard find examples of Banach spaces which are colocally (ℓ_p^n). To show that our results are significant, some naturally occuring examples involving other finite dimensional families will now be discussed. In particular, Theorem 4 shows that a Banach space X with a shrinking Schauder basis has a natural family of subspaces $\mathcal{E}$ for which it is locally and colocally $\mathcal{E}$.

Recall that a Schauder basis $\{x_i\}_{i=1}^{\infty}$ of a Banach space X is said to be **shrinking** if the sequence $\{x_i^*\}_{i=1}^{\infty}$ of the biorthogonal functionals of $\{x_i\}_{i=1}^{\infty}$ is a basis of X^*, [10, p. 52] or [22, 1.b.1].

Lemma 1. *Let X be a Banach space and let A be an n-dimensional subspace of X with Auerbach basis $\{a_i : 1 \leq i \leq n\}$. If for some $0 < \varepsilon < \frac{1}{2n}$ there is a finite dimensional subspace B of X such that for every $i \in \{1, ..., n\}$, there is an element $b_i \in B$ with $\|a_i - b_i\| < \varepsilon$, then there is an isomorphism $T : X \longrightarrow X$ such that $T(A) \subseteq B$ and $\|I - T\| \leq n\varepsilon$. In particular, for any subspace C of X containing B, there is a subspace D of X containing A such that $d_{BM}(C, D) < \frac{1+n\varepsilon}{1-n\varepsilon}$.*

Proof. Let $\{a_i^* : 1 \leq i \leq n\}$ be the dual basis of $\{a_i : 1 \leq i \leq n\}$ and define $T : X \to X$ by

$$Tx = x + \Sigma_{i=1}^{n} a_i^*(x)(b_i - a_i) \qquad x \in X$$

Then it is easy to see that T is linear and

$$\|Tx - x\| \leq n\varepsilon \|x\|$$
$$\text{whence } (1 - n\varepsilon)\|x\| \leq \|Tx\|.$$

Therefore T is an isomorphism into X with $\|I - T\| \leq n\varepsilon$. Since $Ta_i = b_i$, $T(A) \subseteq B$. Now if B is contained in a subspace C of X, taking $D = T^{-1}(C)$, we see that $A \subseteq D$, and the conclusion follows.

We will call a family $\mathcal{E} = \{E_\alpha\}_{\alpha\in\Lambda}$ of finite dimensional Banach spaces **ordered** if the index set Λ is a directed set and $E_\alpha \subseteq E_\beta$ for every α and β therein with $\alpha \leq \beta$.

Lemma 2. *Let X be a Banach space and let $\mathcal{E} = \{E_\alpha\}_\alpha$ be an ordered family of finite dimensional subspaces of X such that $\bigcup E_\alpha$ is dense in X. Then X is $(1+\varepsilon)$-locally $\mathcal{E}$ for every $\varepsilon > 0$.*

Proof. Let $\varepsilon > 0$, let A be an n-dimensional subspace of X and set $\varepsilon_A = \min\left\{\varepsilon, \frac{1}{2n}\right\}$. Let $\{a_1, a_2, ..., a_n\}$ be an Auerbach basis of A, and choose $0 < \delta < \frac{\varepsilon_A}{(2+\varepsilon_A)n}$. Now there exist elements $b_1, b_2, ..., b_n$ in $\cup E_\alpha$ such that $\|b_i - a_i\| < \delta$, since $\cup E_\alpha$ is dense in X. Let $B = span\{b_1, b_2, ..., b_n\}$, then there is some $E \in \mathcal{E}$ that contains B since $\mathcal{E}$ is an ordered family of finite dimensional subspaces of X. Hence by Lemma 1 there is a subspace D of X, $D \supseteq A$ such that $d_{BM}(E, D) < \frac{1+n\delta}{1-n\delta} < 1 + \varepsilon_A < 1 + \varepsilon$, thereby proving the result.

Corollary 3. *Let X be a Banach space and $\{x_i\}_{i=1}^\infty$ be a sequence in X such that $X = [x_i]_{i=1}^\infty$, then X is $(1+\varepsilon)$-locally $\{X_n\}$ for every $\varepsilon > 0$, where $X_n = span\{x_1, x_2, ..., x_n\}$.*

The next theorem is the main result of this section.

Theorem 4. *If $\{x_i\}_{i=1}^\infty$ is a shrinking Schauder basis of a Banach space X and $X_n = span\{x_1, x_2, ..., x_n\}$, then X is locally and colocally the family $\{X_n\}$.*

Proof. By Corollary 3, X is locally $\{X_n\}$. Since $\{x_i^*\}_{i=1}^\infty$ is a basis of X^* then for any $\varepsilon > 0$, X^* is $(1+\varepsilon)$-locally $\{Y_n\}_n$, where $Y_n = span\{x_1^*, x_2^*, ..., x_n^*\}$ by Corollary 3 again. It is clear that $X_n^* = span\{z_1, z_2, ..., z_n\}$, where z_i is the restriction of x_i^* to X_n. So to complete the proof it is enough to show that there is a constant c such that for every n the restriction map $\Psi_n : Y_n \to X_n^*$ given by $\Psi_n(x_i^*) = z_i$, is a c-isomorphism. Let $x = \Sigma_{i=1}^\infty a_i x_i$ be an element in X and let $f = \Sigma_{i=1}^n b_i x_i^* \in Y_n$ then $f(x) = f(b_x)$ where $b_x = \Sigma_{i=1}^n a_i x_i$. If c is the basis constant, that is $\|\Sigma_{i=1}^n a_i x_i\| \le c \|\Sigma_{i=1}^\infty a_i x_i\|$ for any $n \in \mathbb{N}$ then

$$\|f\| = \sup_{\substack{x\in X \\ x\neq 0}} \frac{\|f(x)\|}{\|x\|} \le c \sup_{\substack{x\in X \\ x\neq 0}} \frac{\|f(b_x)\|}{\|b_x\|} = c \sup_{\substack{x\in X_n \\ x\neq 0}} \frac{\|f(x)\|}{\|x\|} = c\|\Psi_n(f)\|$$

so $\|\Psi_n^{-1}\| \le c$. Since $\|\Psi_n\| \le 1$, Ψ_n is a c-isomorphism. Hence X^* is locally $\{X_n^*\}$ and thus the conclusion follows by Theorem 5 below.

Interesting examples of Banach spaces with shrinking bases include Schreier's space S, Tsirelson's space T, the predual B of the James Tree space, and a relatively recent example of Argyros and Deliyanni [1]. These will be discussed further in §3.

2. **Colocality and Twisted Sums**

The last step of the previous proof indicated the need to understand the duality between locality and colocality. We attend to this matter now. Then we will develop our main tool for finding new twisted sums, Theorem 8.

Theorem 5. *Given a Banach space X and a finite dimensional family $\mathcal{E}$, we have:*

(i) X is λ-colocally $\mathcal{E}$ if and only if X^* is λ-locally $\mathcal{E}^*$.

(ii) If X^{**} is λ-locally $\mathcal{E}$ (resp.λ-colocally $\mathcal{E}$) then X is c-locally $\mathcal{E}$ (resp.c-colocally $\mathcal{E}$) for every $c > \lambda$.

Proof. (i) Suppose that X is λ-colocally $\mathcal{E}$, and let B be a finite dimensional subspace of X^*. Then B^*, being a quotient of X, is a quotient of a finite dimensional quotient G of X such that $d_{BM}(G, E) < \lambda$ for some $E \in \mathcal{E}$. Therefore, $B = B^{**}$ is a subspace of $G^* \subseteq X^*$, [23, 4.7, 4.8] and $d_{BM}(G^*, E^*) < \lambda$, and so X^* is λ-locally $\mathcal{E}^*$. Conversely, suppose that X^* is λ-locally $\mathcal{E}^*$ and let G be a finite dimensional quotient of X, then G^* is a subspace of X^*, and so it is contained in a finite dimensional subspace B of X^* such that $d_{BM}(B, E^*) < \lambda$ for some $E \in \mathcal{E}$. Hence $G \equiv G^{**}$ is a quotient of B^* which is a quotient of X, that is X is λ-colocally $\mathcal{E}$.

(ii) Suppose that X^{**} is λ-locally $\mathcal{E}$, then X^* is λ-colocally $\mathcal{E}$. Let A be a finite dimensional subspace of X, then A^* is a quotient of a finite dimensional quotient G of X^* such that $d_{BM}(G, E) < \lambda$ for some $E \in \mathcal{E}$, which implies that $A = A^{**} \hookrightarrow G^* \hookrightarrow X^{**}$. If $c > \lambda$ is given, then by the principle of local reflexivity, [11, p. 178] or [17], there is an isomorphism $T : G^* \to X$ such that $Tx = x$ for all $x \in G^* \cap X$ and $\|T\| \|T^{-1}\| \leq c/\lambda$. Thus $A \hookrightarrow T(G^*) \hookrightarrow X$ and $d_{BM}(T(G^*), E^*) < c$, proving that X *is c-locally* $\mathcal{E}$. Using the same argument, we prove the respective result.

It would be of great interest to know whether the converse of Theorem 5(ii) is true. Thanks to Lindenstrauss and Rosenthal [21, Theorem III (a)] we may conclude that this is true for $\mathcal{L}_p$ spaces. They proved that a Banach space X is locally $\{\ell_p^n\}_{n=1}^{\infty}$ if and only if X^* is locally $\{\ell_q^n\}_{n=1}^{\infty}$, where q is the conjugate of p . Thus we have the following:

Corollary 6. *A Banach space X is locally $\{\ell_p^n\}_{n=1}^{\infty}$ if and only if X is colocally $\{\ell_p^n\}_{n=1}^{\infty}$ where $1 \leq p \leq \infty$.*

Our next theorem is the analogue of a result of Cabello and Castillo [5, Theorem 2]; it indicates how new nontrivial twisted sums can be derived from known examples, without imposing any assumptions on the quotient space. But first we need the following:

Lemma 7. *Let Y, W be Banach spaces and let $\mathcal{E}$ be a family of finite dimensional Banach spaces such that W contains $\mathcal{E}$ uniformly complemented. If $Ext\,(Y,W)=0$, then there is a constant $c\geq 1$ such that for every $E\in\mathcal{E}$ and every z-linear map $F:Y\to E$ there is a linear map $L:Y\to E$ with $dist\,(F,L)\leq cZ\,(F)$.*

Proof. Let $E\in\mathcal{E}$ and let $F:Y\to E$ be a z-linear map. Since W contains $\mathcal{E}$ uniformly complemented there is a k -complemented subspace B of W and a k-isomorphism T of E onto B. A fundamental result of Kalton [18, Proposition 3.3(iii)] gives us a linear map $\tilde{L}:Y\to W$ and a constant t such that $dist\left(TF,\tilde{L}\right)\leq tZ\,(TF)$; note that the composition map $Y\overset{F}{\longrightarrow}E\overset{T}{\longrightarrow}B\hookrightarrow W$ is obviously z-linear and $Ext\,(Y,W)=0$. If p_B is a k-projection of W onto B, then for every $y\in Y$ we have

$$\begin{aligned}
\left\|F\,(y)-T^{-1}p_{_B}\tilde{L}\,(y)\right\| &= \left\|T^{-1}TF\,(y)-T^{-1}p_{_B}\tilde{L}\,(y)\right\| \\
&\leq \left\|T^{-1}\right\|\left\|TF\,(y)-p_{_B}\tilde{L}\,(y)\right\| \\
&= k\left\|p_{_B}TF\,(y)-p_{_B}\tilde{L}\,(y)\right\| \\
&\leq k\left\|p_{_B}\right\|\left\|TF\,(y)-\tilde{L}\,(y)\right\| \\
&\leq k^2tZ\,(TF) \\
&\leq k^2\left\|T\right\|tZ\,(F) \\
&\leq k^3tZ\,(F)
\end{aligned}$$

which implies that $dist\left(F,T^{-1}p_{_B}\tilde{L}\right)\leq k^3tZ\,(F)$. The proof is completed by putting $L=T^{-1}p_{_B}\tilde{L}$ and $c=k^3t$.

Theorem 8. *Let $\mathcal{E}$ be a family of finite dimensional Banach spaces and let W be a Banach space containing $\mathcal{E}$ uniformly complemented. If Y is a Banach space such that $Ext\,(Y,W)=0$, then $Ext\,(Y,Z)=0$ for every Banach space Z which is complemented in its bidual and colocally $\mathcal{E}$.*

Proof. Let Z be a Banach space complemented in its bidual and λ-colocally $\mathcal{E}$ for some $\lambda>1$, and let $F:Y\to Z$ be a z-linear map. We shall show that F is trivial by constructing a linear map

$L : Y \to Z^{**}$ such that $dist\,(F, L) < \infty$. So, let $\mathcal{C}$ be the net of all finite codimensional subspaces A of Z ordered by reverse inclusion, then for every $A \in \mathcal{C}$, there is $B_A \in \mathcal{C}$ and $E_A \in \mathcal{E}$ such that Z/A is a quotient of Z/B_A and $d_{BM}\,(Z/B_A, E_A) < \lambda$. Let $T_A : Z/B_A \to E_A$ be a λ-isomorphism and let $q_A : Z \to Z/B_A$ be the canonical quotient map. Hence the composition map $Y \xrightarrow{F} Z \xrightarrow{q_A} Z/B_A \xrightarrow{T_A} E_A$ is obviously a z-linear map. Since $Ext\,(Y, W) = 0$, there is a constant $c \geq 1$ (which does not depend on E_A) and a linear map $L_A : Y \to E_A$ such that $dist\,(T_A q_A F, L_A) \leq cZ\,(F)$ by Lemma 7. Therefore

$$\left\| q_A F\,(y) - T_A^{-1} L_A\,(y) \right\| \leq c\lambda Z\,(F)\,\|y\|\,, \qquad y \in Y.$$

Let $\mathcal{U}$ be an ultrafilter which refines the corresponding order filter on $\mathcal{C}$, and note that $\left\{A^{\perp} : A \in \mathcal{C}\right\}$ is the net of all finite dimensional subspaces of Z^*, and $A^{\perp} \subseteq B^{\perp}$ when $B \subseteq A$, $A, B \in \mathcal{C}$. By [11, 8.8] or [23, 4.7, 4.8], there is a canonical isometric embedding $J : Z^* \to \left(\Pi_{A\in\mathcal{C}} A^{\perp}\right)_{\mathcal{U}} \subseteq \left(\Pi_{A\in\mathcal{C}} B_A^{\perp}\right)_{\mathcal{U}} \equiv \left(\Pi_{A\in\mathcal{C}}\,(Z/B_A)^*\right)_{\mathcal{U}}$ given by $J\,(f) = (f_A)_{\mathcal{U}}\,,\ f \in Z^*$, where $f_A = f$ if $f \in B^{\perp}$ and $f_A = 0$ otherwise. Therefore, by setting

$$(f_A)_{\mathcal{U}}\,((z + B_A)_{\mathcal{U}}) = \lim_{\mathcal{U}}\,(f_A\,(z + B_A))\,,$$

$\left(\Pi_{A\in\mathcal{C}}\,(Z/B_A)^*\right)_{\mathcal{U}}$ embeds isometrically into $\left(\Pi_{A\in\mathcal{C}}\,(Z/B_A)\right)^*_{\mathcal{U}}$ [11, 8.3], where the norm satisfies $\|(f_A)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|f_A\|$ and so we have

$$J : Z^* \to \left(\Pi_{A\in\mathcal{C}} B_A^{\perp}\right)_{\mathcal{U}} \equiv \left(\Pi_{A\in\mathcal{C}}\,(Z/B_A)^*\right)_{\mathcal{U}} \hookrightarrow \left(\Pi_{A\in\mathcal{C}}\,(Z/B_A)\right)^*_{\mathcal{U}}\,.$$

If $Q : \left(\Pi_{A\in\mathcal{C}}\,(Z/B_A)\right)_{\mathcal{U}} \to Z^{**}$ is the restriction of the adjoint operator $J^* : \left(\Pi_{A\in\mathcal{C}}\,(Z/B_A)\right)^{**}_{\mathcal{U}} \to Z^{**}$ then

$$(Q\,((z + B_A)_{\mathcal{U}}))\,(f) = (J(f))\,((z + B_A)_{\mathcal{U}}) = \lim_{\mathcal{U}} f_A\,(z + B_A) = \lim_{\mathcal{U}}\,(f_A\,(z))\,,$$

for every $f \in Z^*$ and $(z + B)_{\mathcal{U}} \in \left(\Pi_{A\in\mathcal{C}_Z}\,(Z/B_A)\right)_{\mathcal{U}}$. Thus, if $f \in Z^*$ and $y \in Y$

$$(Q((q_A F\,(y))_{\mathcal{U}})\,(f) = \lim_{\mathcal{U}}\,(f_A\,(F\,(y))) = f\,(F\,(y)) = F\,(y)\,(f)\,.$$

Identifying Z with $\hat{Z} \subseteq Z^{**}$ we have $Q\,((q_A F\,(y))_{\mathcal{U}}) = F\,(y)$ for every $y \in Y$. Now define $L : Y \to Z^{**}$ by $L\,(y) = Q(\left(T_A^{-1} L_A\,(y)\right)_{\mathcal{U}})$, and let

$\pi : Z^{**} \to Z$ be a bounded projection of Z^{**} onto Z, then $\pi \circ L$ is a linear map from Y into Z with

$$\begin{aligned}\|F(y) - L(y)\| &= \left\|Q\left((q_A F(y))_{\mathcal{U}}\right) - Q\left(T_A^{-1} L_A(y)\right)_{\mathcal{U}}\right\| \\ &\leq \|Q\| \lim_{\mathcal{U}} \left\|q_A F(y) - T_A^{-1} L_A(y)\right\| \\ &\leq c\lambda Z(F)\, \|Q\|\, \|y\|.\end{aligned}$$

Hence, $dist\,(F, \pi L) < \infty$, where. According to Kalton and Peck [19, Theorem 2.5], F is trivial and the theorem is proved.

The following new result is immediate.

Corollary 9. *A Banach space Z satisfies $Ext\,(\ell_1, Z) = 0$ if and only if it satisfies $Ext\,(L_1(\mu), Z) = 0$ for every measure μ.*

Well known examples, the first due to Lindenstrauss [20], show that this does not imply $Ext\,(L, Z) = 0$ for every $\mathcal{L}_1$ space L.

The following result is well known, but our techniques give a particularly simple proof.

Corollary 10. *If a Banach space Z is locally $\{\ell_\infty^n\}_n$ and complemented in its bidual, then $Ext\,(Y, Z) = 0$ for any Banach space Y.*

Proof. Since $Ext\,(Y, \ell_\infty) = 0$ for any Banach space Y, [10, Chapter VII, Theorem 3], then $Ext\,(Y, Z) = 0$ for any Banach space Z complemented in its bidual and colocally $\{\ell_\infty^n\}_n$,by Theorem 8, that is locally $\{\ell_\infty^n\}_n$ by Corollary 6.

It is important to note that in the above theorem, the condition that Z is complemented in its bidual is essential. Indeed $Ext\,(c_0, c_0) = 0$ since c_0 is complemented in every separable Banach space that contains it, [10, Chapter VII, Theorem 4]. On the other hand [13] we know that $Ext\,(c_0, C\,[0,1]) \neq 0$, where $C\,[0,1]$ is locally $\{\ell_\infty^n\}_n$, and hence is colocally $\{\ell_\infty^n\}_n$ as we observed in Corollary 6.

Recall that two families $\mathcal{E}$ and $\mathcal{F}$ of finite dimensional Banach spaces are said to satisfy $Ext\,(\mathcal{F}, \mathcal{E}) = 0$ **uniformly** if there is a constant c such that, for every couple of spaces $A \in \mathcal{E}$ and $B \in \mathcal{F}$ and every z-linear map $F : B \to A$, there is a linear map $L : B \to A$ such that $dist\,(F, L) \leq cZ(F)$ according to Cabello and Castillo [5, p.7]. Now we need another of their result.

Proposition 11. [4, Theorem 3]. *Let $F : Z \longrightarrow Y$ be a z-linear map. Then there is a homogeneous map $H : Y^* \to Z'$ that satisfies*

$$\|H(y^*) - y^* \circ F\| \leq Z(F)\, \|y^*\|$$

for all $y^* \in Y^*$. Given a Hamel basis (f_α) of Y^*, a linear map $L_H : Y^* \longrightarrow Z'$ can be defined by $L_H(f_\alpha) = H(f_\alpha)$. The dual map F^* of F is the map $L_H - H$, it is a z-linear map, $F^*(y^*) \in Z^*$ for every $y^* \in$, Y^* and $Z(F^*) \leq 2Z(F)$. The map F^* is unique up to equivalence.

Lemma 12. *$Ext(\mathcal{E}, \mathcal{F}) = 0$ uniformly if and only if $Ext(\mathcal{F}^*, \mathcal{E}^*) = 0$ uniformly.*

Proof. Suppose that $Ext(\mathcal{E}, \mathcal{F}) = 0$ uniformly, and let $A \in \mathcal{E}$, $B \in \mathcal{F}$, $F : B^* \to A^*$ be a z-linear map. Cabello and Castillo [4, Theorem 3 and Remark 1] proved that the dual z-linear map $F \equiv F^{**} : B^* \equiv (B^*)^{**} \to (A^*)^{**} \equiv A^*$ can be written as $F = F^{**} = L_H - H$ for some homogeneous map $H : B^* \to A'$ that satisfies $\|b^* \circ F^* - H(b^*)\| \leq Z(F^*)\|b^*\|$ for all $b^* \in B^*$ and a linear map $L_H : B^* \to A^*$. Since $Ext(\mathcal{E}, \mathcal{F}) = 0$ uniformly, there is a linear map $T : A \longrightarrow B$ and a constant c depends on $\mathcal{E}$ and $\mathcal{F}$ such that $\|F^*(a) - T(a)\| \leq cZ(F^*)\|a\|$ for all $a \in A$. The dual map $T^* : B^* \longrightarrow A^*$ satisfies $T^*(b^*)(a) = b^*(T(a))$ for all $a \in A$. Therefore,

$$\begin{aligned}
\|F(b^*) - L_H(b^*) + T^*(b^*)\| &= \|-H(b^*) + T^*(b^*) \pm b^* \circ F^*\| \\
&\leq \|b^* \circ F^* - H(b^*)\| + \|T^*(b^*) - b^* \circ F^*\| \\
&\leq Z(F^*)\|b^*\| + \|b^*\|\|T - F^*\| \\
&\leq (c+1)Z(F)\|b^*\|,
\end{aligned}$$

for all $b^* \in B^*$ since $Z(F) \leq Z(F^*)$. Hence $Ext(\mathcal{F}^*, \mathcal{E}^*) = 0$ uniformly. The converse is obvious since $\mathcal{E} = \mathcal{E}^{**}$.

Cabello and Castillo [5, Theorem 3] showed that if $\mathcal{E}$ and $\mathcal{F}$ are families of finite dimensional Banach spaces such that $Ext(\mathcal{E}, \mathcal{F}) = 0$ uniformly, and if Y and Z are Banach spaces such that Y is locally $\mathcal{E}$, Z is both locally $\mathcal{F}$ and complemented in its bidual, then $Ext(Y, Z) = 0$. Similar results involving colocality families are given in the following:

Theorem 13. *If Y and Z are Banach spaces such that Y is colocally $\mathcal{E}$, Z is colocally $\mathcal{F}$ complemented in its bidual, and $Ext(\mathcal{E}, \mathcal{F}) = 0$ uniformly, then $Ext(Y, Z) = 0$.*

Proof. This is immediate from Theorem 5, Lemma 12 and the aforementioned result.

Theorem 14. *Let Y, Z be Banach spaces and let $\mathcal{E}$, $\mathcal{F}$ be two families of finite dimensional Banach spaces such that Y is locally $\mathcal{E}$, Z is colocally $\mathcal{F}$ and is complemented in its bidual. If $Ext\,(\mathcal{E},\mathcal{F}) = 0$ uniformly, then $Ext\,(Y, Z) = 0$.*

Proof. Suppose that Y is λ-locally $\mathcal{E}$, Z is λ'-colocally $\mathcal{F}$ and let $F : Y \to Z$ be a z-linear map. Then there is a cofinal subnet $\mathcal{G}$ of the net of all finite dimensional subspaces of Y such that for every $G \in \mathcal{G}$ there is $E \in \mathcal{E}$ with $d_{BM}\,(G, E) \leq \lambda$. Let $\mathcal{C}$ be the net of all finite codimensional subspaces of Z directed by reverse inclusion, then for each $A \in \mathcal{C}$, let B_A and q_A be as described in the proof of Theorem 8.

For each $G \in \mathcal{G}$, let F_G be the restriction of F to G, then clearly $Z\,(F_G) \leq Z\,(F)$ and the composition map $q_A F_G : G \to Z/B_A$ is a trivial z-linear map. Since $Ext\,(\mathcal{E},\mathcal{F}) = 0$ uniformly, there is a constant c and a linear map $L_G : G \to Z/B_A$ such that for all $y \in G$

$$\begin{aligned} \|q_A F_G\,(y) - L_G\,(y)\| &\leq cZ\,(F_G)\,\|y\| \\ &\leq c\lambda\lambda' Z\,(F)\,\|y\| \end{aligned}$$

$\|q_A F_G\,(y) - L_G\,(y)\| \leq c\lambda\lambda' Z\,(F)\,\|y\|$ for all $y \in G$. Note that

$$\|L_G\,(y)\| \leq \|q_A F_G\,(y_G) - L_G\,(y)\| + \|q_A F_G\,(y)\| \leq c\lambda\lambda' Z\,(F)\,\|y\| + \|F\,(y)\|$$

Let $\mathcal{V}$ be an ultrafilter refining the order filter on $\mathcal{G}$ and define a map $L_A : Y \to Z/B_A$ by

$$L_A\,(y) = w^*\text{-}\lim_{\mathcal{V}} L_G\,(y_G)$$

where $y_G = y$ if $y \in G$, and 0 otherwise. It is easy to see that L_A is a linear map and well defined sincefor all $G \in \mathcal{G}$. Hence

$$\|q_A F\,(y) - L_A\,(y)\| \leq c\lambda\lambda' Z\,(F)\,\|y\|$$

which implies that $q_A F$ is trivial. As in the proof of the Theorem 8, it is possible to find a linear map $L : Y \to Z$ such that $\|F\,(y) - L\,(y)\| \leq k\,\|y\|$ for some constant k, proving that $Ext\,(Y, Z) = 0$.

3. Some Applications

We begin by describing a number of Banach spaces which contain ℓ_1^n uniformly complemented. They are not presented in chronological order. It is worth noting that a Banach space X contains $\mathcal{E}$ uniformly complemented if and only if X^{**} does. We are mainly interested here in the case $\mathcal{E} = \ell_1^n$.

The Schreier Space S is the completion of the space of finite sequences with respect to the following norm:

$$\|x\| = \sup_{A} \ (\Sigma_{j\in A} |x_j|)$$

where the supremum is taken over all "admissible" subsets of $\mathbb{N}$, which are defined as the finite subsets $A = \{n_1, n_2, ..., n_k\}$ of $\mathbb{N}$ such that $n_1 < n_2 < ... < n_k$ and $k \leq n_1$. It is studied in some detail by Castello and González [8, p. 119]. Note that for each $n \in \mathbb{N}$, the subspace $F_n = \{(x_i)_i \in S : x_i = 0 \text{ for all } i < n \text{ and } i \geq 2n\}$ of S is isometrically isomorphic to ℓ_1^n, and clearly norm one complemented.

Remark 15. *The Schreier Space S also contains $\{l_\infty^n\}_n$ uniformly complemented.*

Proof. It is known that S contains c_0, although this fact is not obvious. A simple construction of subspaces isometric to ℓ_∞^n is given in [7, p. 167].

Argyros and Deliyanni constructed two Banach spaces in [1]. Their second example is of more interest to us; we shall denote it by AD. It is asymptotically ℓ_1, which implies that it contains ℓ_1^n uniformly complemented. It is also hereditarily indecomposable.

The famous space T of Tsirelson is studied in great detail in [6]. It is super-reflexive, but does not contain any of the classical sequence spaces. It is also asymptotically ℓ_1, whence it contains ℓ_1^n uniformly complemented.

The Johnson-Lindenstrauss space JL is a well known example in $Ext(\ell_2(\Gamma), c_0)$ with Γ uncountable. Thanks to the lifting property, its dual JL^* contains a complemented copy of ℓ_1.

The James Tree space JT and its predual B are studied in great detail in [12].

Lemma 16. *The predual B of the James tree space JT contains $\{\ell_1^n\}_{n=1}^\infty$ uniformly complemented.*

Proof. As usual, we denote the famous quasi-reflexive space of James by J. Since c_0 is finitely represented in J, [12, 2.b.8], and since JT contains J, [12, 3.a.7], JT contains $\{\ell_\infty^n\}_{n=1}^\infty$ uniformly which implies that it contains $\{\ell_\infty^n\}_{n=1}^\infty$ uniformly complemented. Choose $c > 1$ and let G_n be a subspace of $JT = B^*$ such that G_n is c-isomorphic to ℓ_∞^n, $n \in \mathbb{N}$. Since G_n is w^*-closed in B^*, there is a subspace D_n of B such that $G_n \equiv (B/D_n)^*$, [23, 4.8]. Therefore ℓ_1^n is $c-$isomorphic to B/D_n. Now let ψ_n be a c- isomorphism of B/D_n onto ℓ_1^n and let $q_n : B \to B/D_n$ be the quotient map, then $\psi_n q_n (cB_B(0,t)) \supseteq B_{\ell_1}$, for any prefixed $t > 1$. Let $\{e_i^n : i = 1, 2, ..., n\}$ be the standard basis of ℓ_1^n, then there is $x_i^n \in cB_B(0, t)$ such that $\psi_n q_n(x_i^n) = e_i^n$. It is clear that $x_1^n, x_2^n, ..., x_n^n$ are linearly independent. Let B_n be the subspace of B generated by $\{x_1^n, x_2^n, ..., x_n^n\}$, define $T_n : \ell_1^n \to B$ by $T_n(\Sigma_{i=1}^n \lambda_i e_i^n) = \Sigma_{i=1}^n \lambda_i x_i^n$. Thus

$$\|T_n(\Sigma_{i=1}^n \lambda_i e_i^n)\| = \|\Sigma_{i=1}^n \lambda_i x_i^n\| \leq ct\Sigma_{i=1}^n |\lambda_i| = ct \|\Sigma_{i=1}^n \lambda_i e_i^n\|_1$$

from which we see that $\|T_n\| \leq ct$.

On the other hand, $T_n^{-1} : B_n \longrightarrow \ell_1^n$ satisfies

$$\begin{aligned}\left\|T_n^{-1}(\Sigma_{i=1}^n \lambda_i x_i^n)\right\| &= \|\Sigma_{i=1}^n \lambda_i e_i^n\| = \|\psi_n q_n(\Sigma_{i=1}^n \lambda_i x_i^n)\| \leq \|\psi_n\| \|q_n\| \|\Sigma_{i=1}^n \lambda_i x_i^n\| \\ &\leq c\|\Sigma_{i=1}^n \lambda_i x_i^n\|\end{aligned}$$

which implies that $\|T_n^{-1}\| \leq c$. Hence ℓ_1^n is ct-isomorphic to B_n. Now consider the following diagram

$$\begin{array}{ccc} B & \xrightarrow{q_n} & B/D_n \\ T_n \uparrow & & \downarrow \psi_n \\ \ell_1^n & \xrightarrow{id_n} & \ell_1^n \end{array}$$

It is easily seen that $\psi_n q_n T_n = id_n$ so $T_n \psi_n q_n : B \to T_n(\ell_1^n)$ is a projection with norm $\leq c^2 t$. Therefore the result holds.

Example 17. *Let Y be one of the classical sequence spaces, c_0 or ℓ_p, where $1 < p < \infty$. Then $Ext(Y, S)$, $Ext(Y, S^{**})$, $Ext(Y, T)$, $Ext(Y, AD)$, $Ext(Y, B)$, $Ext(Y, JT^*)$ and $Ext(Y, JL^*)$ are all non-zero.*

Proof. These all follow from the fact that $Ext(Y, \ell_1) \neq 0$ and the above remarks.

The case $Ext(Y, AD) \neq 0$ is particularly interesting. We do not know whether any such twisted sums is hereditarily indecomposable. Argyros and Felouzis [2] showed that each such Y is isomorphic to a quotient of some hereditarily indecomposable Banach space.

The following will enable us to unearth more nontrivial twisted sums.

Theorem 18. *Let Y and Z be two Banach spaces. Then $Ext(Y, Z^*) = 0$ if and only if $Ext(Z, Y^*) = 0$.*

Proof. Suppose that $Ext(Y, Z^*) = 0$ and let $F : Z \longrightarrow Y^*$ be a z-linear map, then there is a constant c and a linear map $H(g) : Y^{**} \longrightarrow \mathbb{K}$ such that

$$\|H(g)(f) - gF^*(f)\| \leq cZ(F)\|g\| \|f\|$$

for every $g \in Z^{**}$ and $f \in Y^{**}$, where $F^* : Y^{**} \longrightarrow Z^*$ is the dual z-linear map of F (cf. Proposition 11). Let G be the restriction of $F^* : Y^{**} \longrightarrow Z^*$ to Y, then G is trivial. Since

$$\|H(g)(y) - gF^*(y)\| \leq cZ(F)\,\|g\|\,\|y\|$$

for all $y \in Y$, then $G^* |_Z = L_H - H |_Z$ .Since $F^{**} |_Z = F$, then $G^* |_Z$ is equivalent to F. But G^* is trivial, hence so is F, proving that $Ext(Z, Y^*) = 0$. The converse follows by symmetry.

In fact, $Ext(Z, Y)$ admits a natural vector space structure. This was explained in some detail, using pushout and pullback arguments, in a preliminary version of [5], and may also be found in a more general setting in [15]. The relationship between twisted sums and z-linear mappings allows one to see this more clearly. In fact, it can be shown that $Ext(Z, Y^*)$ is isomorphic to $Ext(Y, Z^*)$, which generalizes Theorem 18.

Example 19. *We have nontrivial spaces of extensions* $Ext\,(S, S) \neq 0$, $Ext\,(S^*, S^*) \neq 0$, $Ext\,(S^{**}, S^{**}) \neq 0$, $Ext\,(c_0, JL^*) \neq 0$, $Ext\,(JL, \ell_1) \neq 0$, $Ext\,(JL, \ell_1) \neq 0$, $Ext\,(S, B) \neq 0$, $Ext\,(S, JT^*) \neq 0$, $Ext\,(JL, B) \neq 0$ *and* $Ext\,(JL, JT^*) \neq 0$.

Proof. Since $Ext\,(c_0, \ell_1) \neq 0$, Remark 15 and the result of Cabello and Castillo [5, Theorem 2] tell us that $Ext\,(S, \ell_1) \neq 0$ and $Ext\,(S^{**}, \ell_1) \neq 0$. Therefore by Theorem 8, $Ext\,(S, S) \neq 0$, $Ext\,(S, S^{**}) \neq 0$ and by Theorem 5 $Ext\,(S^*, S^*) \neq 0$. Also $Ext\,(S^{**}, \ell_1) \neq 0$ gives $Ext\,(S^{**}, S) \neq 0, Ext\,(S^{**}, S^{**}) \neq 0$. Since JL^* contains ℓ_1 complemented, we have $Ext\,(c_0, JL^*) \neq 0$ and hence $Ext\,(JL, \ell_1) \neq 0$ by Theorem 5, so $Ext\,(JL, S) \neq 0$, $Ext\,(JL, S^{**}) \neq 0$ by Theorem 8. Applying Lemma 16 and Theorem 8 again yields $Ext\,(c_0, B) \neq 0$ and $Ext\,(c_0, JT^*) \neq 0$. Also $Ext\,(S, \ell_1) \neq 0$ implies $Ext\,(S, B) \neq 0$ and $Ext\,(S, JT^*) \neq 0$. And $Ext\,(JL, \ell_1) \neq 0$ implies $Ext\,(JL, B) \neq 0$ and $Ext\,(JL, JT^*) \neq 0$.

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