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Pricing American Options on Leveraged Exchange Traded Funds in the Binomial Pricing Model

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Abstract

This paper describes our work pricing options in the binomial model on leveraged exchange traded funds (ETFs) with three different approaches. A leveraged exchange traded fund attempts to achieve a similar daily return as the index it follows but at a specified positive or negative multiple of the return of the index. We price options on these funds using the leveraged multiple, predetermined by the leveraged ETF, of the volatility of the index. The initial approach is a basic time step approach followed by the standard Cox, Ross, and Rubinstein method. The final approach follows a different format which we will call the Trigeorgis pricing model. We demonstrate the difficulties in pricing these options based off the dynamics of the indices the ETFs follow.

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1. Introduction

An exchange traded fund (ETF) is a security that tracks a specified index. The goal of many of these funds is to acquire the same percentage return for the day as the index it follows. Unlike an index, an ETF is traded like a stock in the market. A leveraged ETF tracks an index but has a goal of returning a fixed multiple of the return on the index it follows.

Given historical data on the index an ETF follows, we look at different approaches to the binomial option pricing model to price options on leveraged ETFs. This paper demonstrates that three common approaches to pricing options on leveraged exchange traded funds using the historical data of the index have similar price results. We will determine the volatility of the leveraged ETF by first calculating the volatility of the index and scaling it by the leverage factor. We compare these results with the bid and ask prices of the leveraged ETFs along with the Black Scholes pricing model and the Cox, Ross, and Rubinstien binomial asset pricing model.

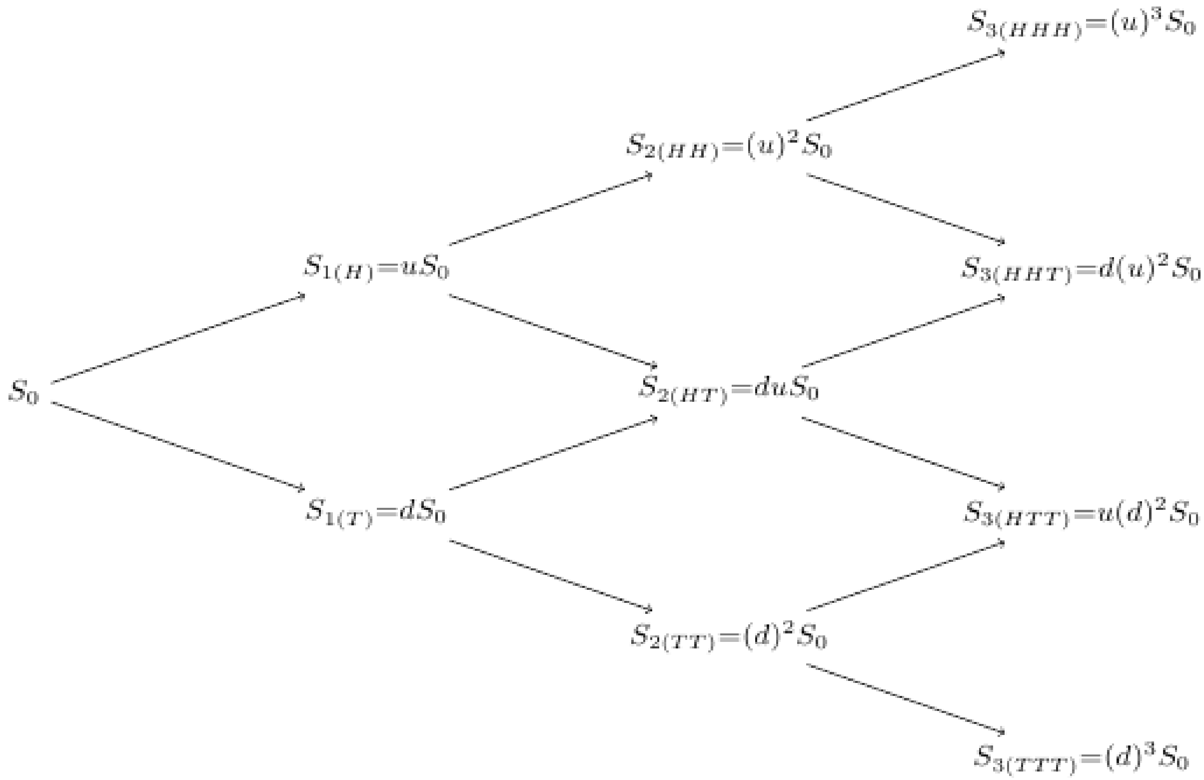
2. Binomial Asset Pricing Model

The binomial asset pricing model is an asset valuation model that allows us to price financial derivatives (options) over discrete time periods. In order to price an option over a specific time period, the price dynamics of the underlying asset over that period must first be established. The binomial pricing model assumes that the price of a stock can change by only two means at each time step. The price of the asset can have an upward movement or it can have a downward movement. These movements can be compared to flipping a coin at each time step.

Assume that each upward movement or "heads" coin flip will increase the stock price by a multiple of u and each downward movement or "tails" coin flip will decrease the stock price by a multiple of d where $0 < d < 1 < u$. The upward and downward movements are derived using σ, the underlying asset's volatility, and T, time until maturity. We assume the initial asset price is S_0.

After one time step the price will change from S_0 to either $d * S_0$ or $u * S_0$. To illustrate the model we will look at an underlying pricing model with a three period maturity date.

A three period binomial underlying stock price tree will be:



Tree 1: Basic Underlying Pricing Tree

3. Returns

In order to begin pricing an option, we must first determine the returns on the underlying asset. Returns measure the change in price of an asset as a fraction of the original price. We will be using daily closing prices in order to determine the returns on the assts. Daily data is the most widely available market data. We will use the adjusted daily close price which includes adjustments for splits and dividends.

a. Net Returns

Assuming that there are no dividends, the *net returns* between times $t-1$ and t are determined as follows:

$$R_t = \frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}} \text{ where } P_i \text{ is the price of the asset at time } t.$$

The revenue between times $t-1$ and t is $P_t - P_{t-1}$ and therefore the revenue can be positive or negative. The initial investment at time $t-1$ is P_{t-1}. The *net revenue* is therefore considered the relative revenue rate. The gross returns over k periods is the product of the$k$ returns. For the period from time $t-k$ to t, the gross return is as follows (Ruppert 75):

$$1 + R_t(k) = \frac{P_t}{P_{t-k}} = \left(\frac{P_t}{P_{t-1}}\right)\left(\frac{P_{t-1}}{P_{t-2}}\right) \dots \left(\frac{P_{t-k+1}}{P_{t-k}}\right) = (1+R_t)(1+R_{t-1}) \dots (1+R_{t-k+1}).$$

b. Log Returns

Continuously compounded returns or *log returns* between time $t-1$ and t are defined as follows:

$$r_t = \ln(1+R_t) = \ln\left(\frac{P_t}{P_{t-1}}\right) \text{ where } \ln(X) \text{ the natural logarithm of X}$$

Log returns have an advantage over net returns because a log return for a specified time period is only the sum of that periods log returns. This is a simpler model than the gross net returns. The gross log returns from time $t-k$ to t the gross return is as follows (Ruppert 77):

$$\begin{aligned} 1+R_t(k) &= ln\{1+R_t(k)\} \\ &= ln\{(1+R_t)(1+R_{t-1}) \dots (1+R_{t-k+1})\} \\ &= ln(1+R_t)ln(1+R_{t-1}) \dots ln(1+R_{t-k+1}) \\ &= ln(1+R_t) + ln(1+R_{t-1}) + \cdots + ln(1+R_{t-k+1}). \end{aligned}$$

4. Volatility

The volatility of an asset is a measure of variation in the price of the daily returns of the asset. Since volatility is a risk measurement, a higher volatility implies a riskier asset. We explore two different approaches in determining volatility.

a. Historical Volatility

The first approach is historical volatility. In order to use historical data to estimate the volatility, we assume that the underlying price has a constant mean and variance. In other words, we assume that the underlying price of the index is a stationary stochastic process. Historical volatility uses historical data for the past m time steps, or days in our case, in order to estimate the volatility. The historical volatility of the daily returns is given by:

$$\sigma = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(R_i - \overline{R}\right)^2}$$

where R_i is the log return at time step i, m is the number of observations and $\overline{R}$ is the average of the daily log returns (Björk 105).

b. Implied Volatility

Another approach to estimate the volatility of the asset is to determine the assets *implied volatility*. *Implied volatility* is calculated by estimating the volatility in the near future using the market expectation of the volatility. To calculate implied volatility, one must first observe the current price of the asset and the market price of an option on this asset. Consider a European call option with current observed market price of C^{obs}. One can determine the implied volatility by setting the current price to the Black Scholes European call option pricing formula. The Black-Scholes pricing formula is a one to one function in σ and can therefore be inverted to estimate the volatility of the underlying asset. The Black-Scholes formula also assumes that the asset returns are normally distributed and the returns have a constant variance (Ruppert 274). In this study, historical data is used to estimate the volatility of the assets.

5. Underlying Price Dynamics

The upward and downward underlying price movements in the binomial model are proportions the price of an asset is going to move up by a factor of $u > 1$ and move down by a factor of $d < 1$. We will define the probability of an upward movement as p and a downward movement as $1 - p$. We explore three different methods that can be used to determine the upward and downward movements of the underlying asset price. Initially we, define σ as the volatility of the asset, n as the number of time steps before expiry, r as the daily interest rate, and Δt as the time step size.

a. Basic Time Step Pricing Model

For the first approach, we assume that the asset price will increase or decrease by a factor of $\frac{\sigma}{\sqrt{n}}$ (Ruppert 273). Therefore it follows that the up and down factors are

$$u = 1 + \frac{\sigma}{\sqrt{n}},$$

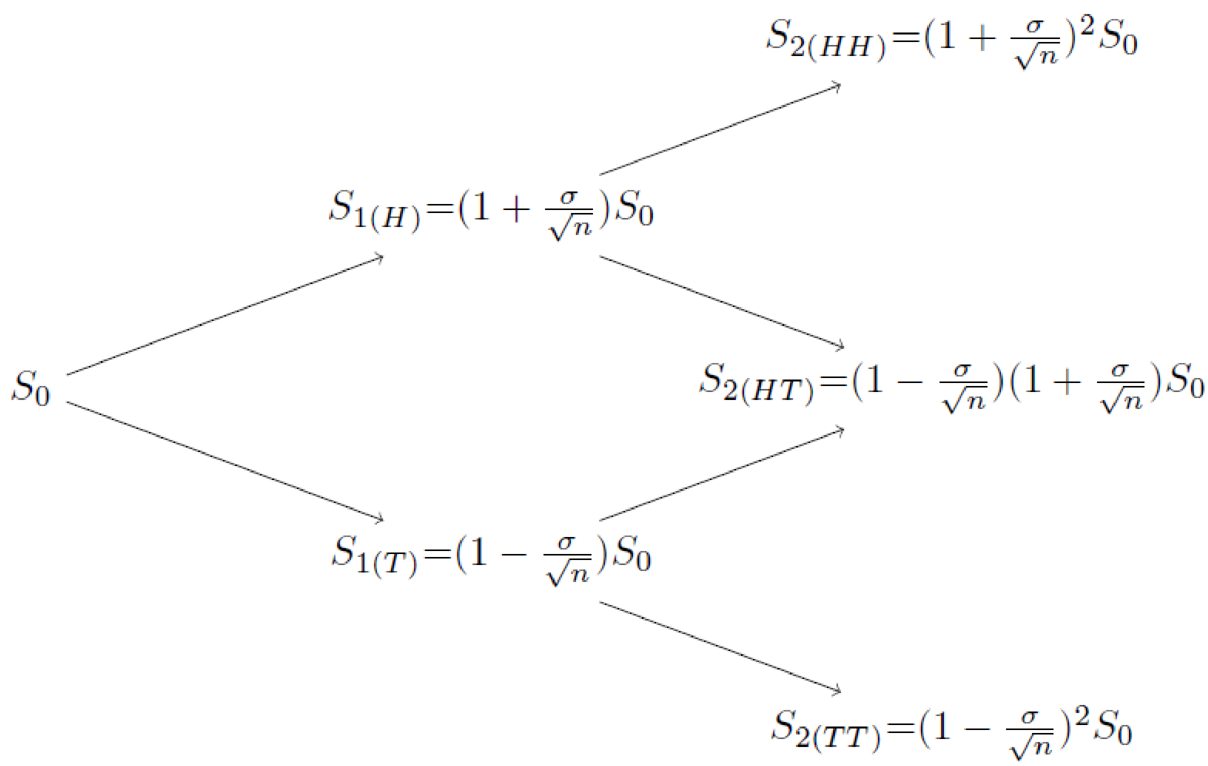
$$d = 1 - \frac{\sigma}{\sqrt{n}},$$

And $p = \frac{1+r-d}{u-d}$.

Let $i = 1, \dots, N$ be the time steps up to expiry and S_0 be the initial price of the asset. Also let j be the level of the variable S at time i corresponding to number of upward movements plus one.

Therefore, the price of the asset at the node (i, j) is

$$S_{i,j} = S_0(u^j)(d^{i-j}) = S_0\left(1 + \frac{\sigma}{\sqrt{n}}\right)^j\left(1 - \frac{\sigma}{\sqrt{n}}\right)^{i-j}.$$



Tree 2: Basic Time Step Pricing Model Underlying Pricing Tree

b. Cox, Ross, and Rubinstien Pricing Model

The Cox, Ross, and Rubinstien model is considered a standard approach for selecting the upward and downward movements. These movements are determined as follows:

$$u = e^{(\sigma\sqrt{\Delta t})},$$

$$d = e^{(-\sigma\sqrt{\Delta t})},$$

$$p = \frac{e^{r\Delta t} - d}{u - d} \text{ (Fusai 78).}$$

Then let $i = 1, \dots, N$ be the time steps up to expiry, j be the level of the variable S at time i, and S_0 be the initial price of the asset. Therefore, the price of the asset at the node (i, j) is

$$S_{i,j} = S_0(u^j)(d^{i-j}) = S_0\left(e^{\sigma\sqrt{\Delta t}}\right)^j\left(e^{-\sigma\sqrt{\Delta t}}\right)^{i-j}.$$

$$\begin{tikzcd}
 & & S_{2(HH)} = e^{2\sigma\sqrt{\Delta t}} S_0 \\
 & S_{1(H)} = e^{\sigma\sqrt{\Delta t}} S_0 \arrow[ur] \arrow[dr] & \\
S_0 \arrow[ur] \arrow[dr] & & S_{2(HT)} = e^{\sigma\sqrt{\Delta t}} e^{-\sigma\sqrt{\Delta t}} S_0 \\
 & S_{1(T)} = e^{-\sigma\sqrt{\Delta t}} S_0 \arrow[ur] \arrow[dr] & \\
 & & S_{2(TT)} = e^{-2\sigma\sqrt{\Delta t}} S_0
\end{tikzcd}$$

Tree 3: Cox, Ross, and Rubinstien Pricing Model Underlying Pricing Tree

c. Trigeorgis Pricing Model

Determining the upward and downward movements in the previous two methods may be acceptable for only small time refinements. The error in the first two approaches can get very large if $N < T\left[\frac{\left(r-\frac{\sigma^2}{2}\right)}{\sigma}\right]^2$ where N is the number of time steps and T is the maturity of the option, so this approach is not always stable. The following formations can be used to overcome this problem since this method is consistent and stable at each time step (Trigeorgis 319). In other words, the error approaches zero as the time steps get smaller, and two independent samples of the data have the same distribution. Let x be the logarithmic price on the asset:

$$x = \ln S_0.$$

Then x will increase by $\Delta x_u$to $x + \Delta x_u$with a probability of p_u. Likewise x will decrease by Δx_d to $x - \Delta x_d$ with a probability of $p_d = 1 - p_u$. Assuming that each movement is of equal size, we obtain:

$$\Delta x = \sqrt{\sigma^2 \Delta t + \left(r - \frac{\sigma^2}{2}\right)^2 \Delta t^2},$$

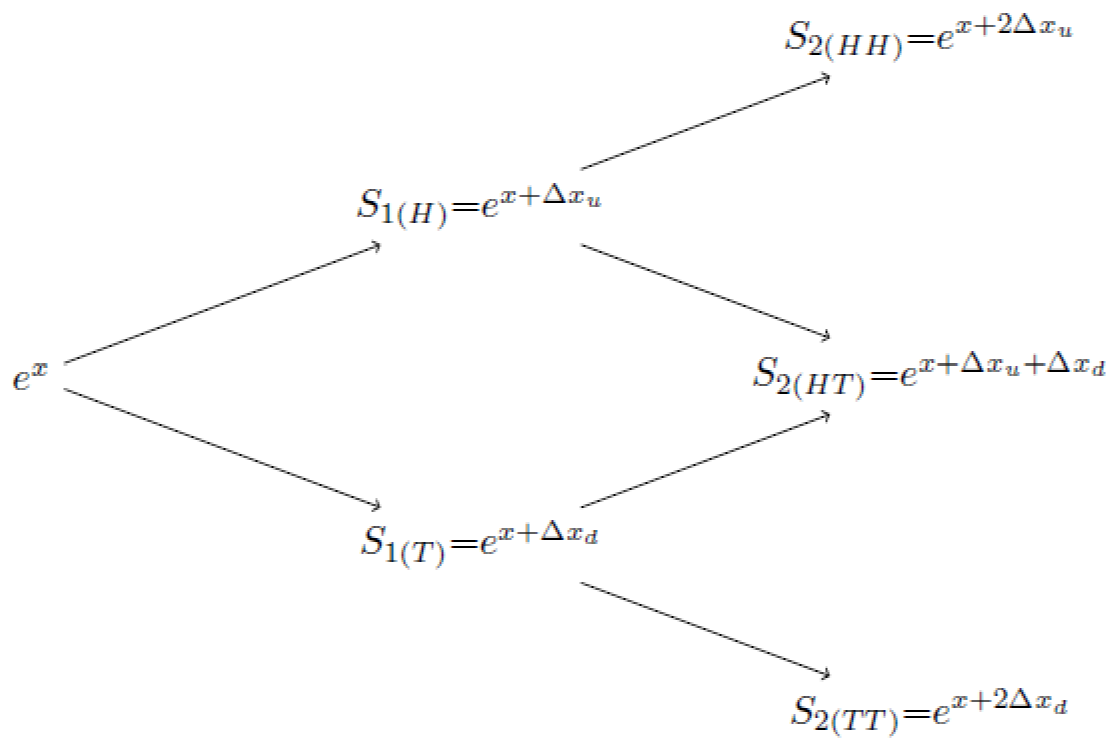
$$p_u = \frac{1}{2} + \frac{\left(r - \frac{\sigma^2}{2}\right)\Delta t}{2\Delta x},$$

$$p_u = \frac{1}{2} - \frac{\left(r - \frac{\sigma^2}{2}\right)\Delta t}{2\Delta x}.$$

Let $i = 1, \dots, N$ be the time steps up to expire and j be the level of the variable S at time i.

Then the price of the asset at the node (i, j) is

$$S_{i,j} = e^{(x_{i,j})} = e^{(x+j\Delta x_u+(j-i)\Delta x_d)} = S_0 e^{(j\Delta x_u+(j-i)\Delta x_d)}$$ (Fusai 2008:78).



Tree 4: Trigeorgis Pricing Model Underlying Pricing Tree

6. European Options

A stock option is a contract between a buyer and seller of an asset in a specific time period. A *European call option* is a financial contract where at expiry the buyer of the option has the choice to purchase the agreed upon asset, the underlying asset, for an agreed upon amount, the strike price. A *European put option* is a financial contract equivalent to a *European call option* with one major difference. The buyer of the option has the right to sell the underlying asset to the writer for the strike price at expiry instead of buying the asset at expiry (Blais). The strike price K is designated during the writing of the contract. Let the price at expiry be S_T. The payoffs of European options at maturity T are as follows:

Chart 1: European Call Option Payoff: $\boldsymbol{max(S_T - K, 0)}$

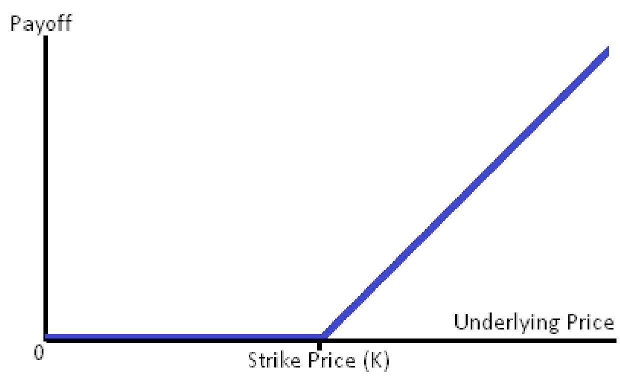
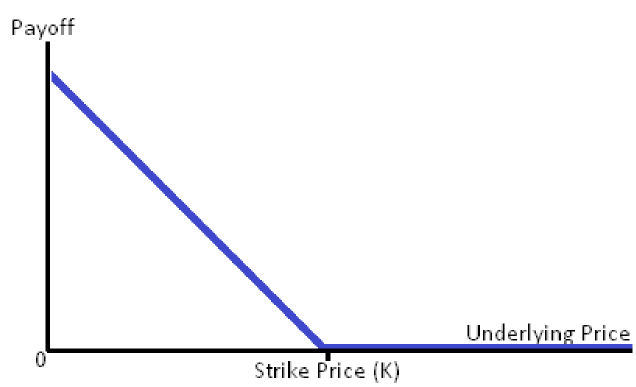


Chart 2: European Put Option Payoff: $\boldsymbol{max(K - S_T, 0)}$



7. American Options

An American option is equivalent to a European option with one major difference; an American option can be exercised any time before the maturity date of the option. At expiry, the payoff of an American option is equivalent to that of the corresponding European option with the same expiry, underlying asset, and strike price. We will discuss the difference in pricing an American option and a European option in the next section.

8. Pricing Options

Now that we have the value of the option at maturity date, we can price back the option. The price of the option is calculated in a backwards manner using the following theorem:

Theorem: (Replication in the Multi-Period Binomial Model)

Consider an N-period binomial asset pricing model, with $0 < d < 1 < u$, and with risk neutral measures p and $q = 1 - p$ determined earlier. Then "let V_n be a random variable that depends on the first N coin tosses $w_1 w_2 \ldots w_n$. Define recursively backward in the time sequence of random variables $V_{N-1}, V_{N-2}, \ldots, V_0$ by

$$V_{n(w_1 w_2 \ldots w_n)} = \frac{1}{1+r}\left[pV_{n+1(w_1 w_2 \ldots w_n H)} + pV_{n+1(w_1 w_2 \ldots w_n T)}\right],$$

so that each V_n depends on the first n coin tosses $w_1 w_2 \ldots w_n$, where n ranges between $N - 1$ and 0" (Shreve 12).

a. European Option Pricing

To price an option we start by looking at the payoff at maturity. Consider a European put option where the payoff is $V_n = \max\left(K - S_{n(w_1 w_2 \ldots w_n)}, 0\right)$, K is the strike price of the option, and $(w_1 w_2 \ldots w_n)$ are the first n coin flips. Then the 2 period put option pricing tree will be:

$$
\begin{array}{ccccc}
 & & & & V_{2(HH)} = max(K-(u)^2S_0, 0) \\
 & & V_{1(H)} = \frac{1}{1-r}[pV_{3(HH)} + qV_{3(HT)}] & & \\
V_0 = \frac{1}{1-r}[pV_{1(H)} + qV_{1(T)}] & & & & V_{2(HT)} = max(K-(ud)S_0, 0) \\
 & & V_{1(T)} = \frac{1}{1-r}[pV_{3(HT)} + qV_{3(TT)}] & & \\
 & & & & V_{2(TT)} = max(K-(d)^2S_0, 0)
\end{array}
$$

Tree 5: European Call Option Pricing Tree

Therefore the initial price of the put option is

$$V_0 = \left(\frac{1}{1+r}\right)^2 [p^2 \max(\mathrm{K} - u^2S_0, 0) + 2pq \max(\mathrm{K} - udS_0, 0) + q^2 \max(\mathrm{K} - d^2S_0, 0)].$$

Similarly a two period European Call option pricing tree with payoff V_n = $\max\left(S_{n(\,w_1w_2\dots w_n)} - K, 0\right)$ will be:

$$
\begin{array}{ccccc}
 & & & & V_{2(HH)} = max((u)^2S_0 - K, 0) \\
 & & V_{1(H)} = \frac{1}{1-r}[pV_{3(HH)} + qV_{3(HT)}] & & \\
V_0 = \frac{1}{1-r}[pV_{1(H)} + qV_{1(T)}] & & & & V_{2(HT)} = max((ud)S_0 - K, 0) \\
 & & V_{1(T)} = \frac{1}{1-r}[pV_{3(HT)} + qV_{3(TT)}] & & \\
 & & & & V_{2(TT)} = max((d)^2S_0 - K, 0)
\end{array}
$$

Tree 6: European Put Option Pricing Tree

Therefore, the initial price of the put option is

$$V_0 = \left(\frac{1}{1+r}\right)^2 [p^2 \max(u^2 S_0 - K, 0) + 2pq \max(udS_0 - K, 0) + q^2 \max(d^2 S_0 - K, 0)].$$

In a general n-period model, the price of a European put option over n periods is

$$P = \left(\frac{1}{1+r}\right)^n \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \left[\max\left(\mathrm{K} - u^i d^{n-i} S_0, 0\right)\right],$$

and the price of a European call option is

$$C = \left(\frac{1}{1+r}\right)^n \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \left[\max\left(u^i d^{n-i} S_0 - K, 0\right)\right].$$

b. American Option Pricing

Since an American option is exactly the same as a European option with the addition of the privilege to exercise the option at any time, an American option is at least as valuable as a European option on the same underlying asset with the same strike price and maturity. An American call option should never be exercised early because it is in the best interest of the holder of an American call option to wait until expiry to exercise the option (Ruppert 277). Therefore, “American calls are equal in price to European calls with the same exercise price and expiration date” (Ruppert 278). An American put option, however, should not necessarily be exercised only at maturity. Therefore, an American put option price is generally different from the price of the corresponding European put option.

An American put option can be priced as follows, the option value at each node is the greater of the value of holding the option until the next time period and the early exercise value.

$$V_{n(w_1 w_2 \ldots w_n)} = max\left(\frac{1}{1+r}\left[pV_{n+1(w_1 w_2 \ldots w_n H)} + pV_{n+1(w_1 w_2 \ldots w_n T)}\right], K - S_n\right).$$

Therefore, the two period binomial option pricing tree for an American put is as follows:

$$V_{2(HH)} = max(K - (u)^2 S_0, 0)$$

$$V_{1(H)} = max[\tfrac{1}{1-r}(pV_{3(HH)} + qV_{3(HT)}), K - uS_0]$$

$$V_0 = max[\tfrac{1}{1-r}(pV_{1(H)} + qV_{1(T)}), K - S_0]$$

$$V_{2(HT)} = max(K - (ud)S_0, 0)$$

$$V_{1(T)} = max[\tfrac{1}{1-r}(pV_{3(HT)} + qV_{3(TT)}), K - dS_0]$$

$$V_{2(TT)} = max(K - (d)^2 S_0, 0)$$

Tree 7: American Put Option Pricing Tree

9. Leveraged Exchange Traded Funds

An exchange traded fund (ETF) is a traded investment consisting of different assets such as stocks, bonds, or commodities. Many ETFs follow indices on the market such as the Standard and Poor's 500 or the Dow Transportation Average. A standard ETF attempts to achieve the same daily return of the index specified by that ETF. We will, however, be analyzing leveraged exchange traded funds. A leveraged exchange traded fund is more sensitive to the market than a standard ETF. A leveraged exchange traded fund, like a standard ETF, attempts to achieve a similar daily return but at a specified positive or negative multiple of that return. For example, if an index has a return of 2% and the leveraged ETF is a three times leverage (300%), the positive, or bull, ETF will attempt to have a 6% return and the negative, or bear, ETF will attempt to achieve a -6% return.

We will be analyzing daily leveraged ETFs from two different companies: Direxion and ProShare. The exchange traded options on these leveraged ETFs are American options. Therefore, we use our binomial pricing model for American options. The following are the twelve leveraged ETFs we price American options on.

Table 1: Direxion Leveraged Exchange Traded Funds[1]

Fund	Ticker	Bull/Bear	Index	Index Ticker
Large Cap Bull 3X Shares	BGU	300% Bull	Russell 1000	^RUI
Large Cap Bear3X Shares	BGZ	300% Bear	Russell 1000	^RUI
Technology Bull 3X Shares	TYH	300% Bull	Russell 1000 Technology	^R1RGSTEC
Technology Bear 3X Shares	TYP	300% Bear	Russell 1000 Technology	^R1RGSTEC
Small Cap Bull 3X Shares	TNA	300% Bull	Russell 2000	^RUT
Small Cap Bear 3X Shares	TZA	300% Bear	Russell 2000	^RUT

Table 2: ProShare Leveraged Exchange Traded Funds[2]

Fund	Ticker	Bull/Bear	Index	Index Ticker
UltraPro Dow30	UDOW	300% Bull	Dow Jones Industrial Average	^DJI
UltraPro Short Dow30	SDOW	300% Bear	Dow Jones Industrial Average	^DJI
UltraPro S&P500	UPRO	300% Bull	S&P 500	^GSPC
UltraPro Short S&P500	SPXU	300% Bear	S&P 500	^ GSPC
UltraPro QQQ	TQQQ	300% Bull	NASDAQ-100	^NDX
UltraPro Short QQQ	SQQQ	300% Bear	NASDAQ-100	^NDX

[1] http://www.direxionfunds.com
[2] http://www.proshares.com

10. Leveraged Asset Pricing Model

For pricing the leveraged asset, we use the underlying index's volatility to determine the volatility of the leveraged fund. If the index increases by 2% on one day and the leveraged amount is 300%, the leveraged fund will increase by 6%. If we continue this process for every daily return, we notice that the volatility of the leveraged fund is the leveraged multiple of the volatility of the underlying index:

$$\sigma = L\sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(R_i - \overline{R}\right)^2} \ .$$

Therefore the leveraged volatility for a leveraged fund is $L\sigma$ where L is the leverage factor and σ is the index's volatility.

a. Basic Time Step Pricing Model

To price an option on a leveraged asset, we can use a model similar to that of a single asset or stock. We now define the upward and downward movements for the leveraged ETF's underlying index as $u = 1 + \frac{\sigma}{\sqrt{n}}$ and $d = 1 - \frac{\sigma}{\sqrt{n}}$ where σ is the asset's volatility and n is the number of periods before the maturity date. After the transformation of the binomial asset pricing volatility of σ to $(L\sigma)$,

$$u = 1 + L\frac{\sigma}{\sqrt{n}} \text{ and } d = 1 - L\frac{\sigma}{\sqrt{n}}..$$

Define the leveraged ETF, our option's underlying asset, at time n as S_n^L.

$$\begin{tikzcd}[column sep=small]
 & & S^L_{2(HH)}=(1+L\frac{\sigma}{\sqrt{n}})^2S_0 \\
 & S^L_{1(H)}=(1+L\frac{\sigma}{\sqrt{n}})S_0 \arrow[ur] \arrow[dr] & \\
S_0 \arrow[ur] \arrow[dr] & & S^L_{2(HT)}=(1-L\frac{\sigma}{\sqrt{n}})(1+L\frac{\sigma}{\sqrt{n}})S_0 \\
 & S^L_{1(T)}=(1-L\frac{\sigma}{\sqrt{n}})S_0 \arrow[ur] \arrow[dr] & \\
 & & S^L_{2(TT)}=(1-L\frac{\sigma}{\sqrt{n}})^2S_0
\end{tikzcd}$$

Tree 8: Basic Time Step Pricing Model Underlying Pricing Tree

The option pricing tree for an American call with this approach is as follows.

$$\begin{tikzcd}[column sep=small]
 & & V^L_{2(HH)} = max((1+L\frac{\sigma}{\sqrt{n}})^2S_0-K,0) \\
 & V^L_{1(H)} = \frac{1}{1-r}[pV^L_{2(HH)}+qV^L_{2(HT)}] \arrow[ur, dash] \arrow[dr, dash] & \\
V^L_0 = \frac{1}{1-r}[pV^L_{1(H)}+qV^L_{1(T)}] \arrow[ur, dash] \arrow[dr, dash] & & V^L_{2(HT)} = max((1+L\frac{\sigma}{\sqrt{n}})(1-L\frac{\sigma}{\sqrt{n}})S_0-K,0) \\
 & V^L_{1(T)} = \frac{1}{1-r}[pV^L_{2(HT)}+qV^L_{2(TT)}] \arrow[ur, dash] \arrow[dr, dash] & \\
 & & V^L_{2(TT)} = max((1-L\frac{\sigma}{\sqrt{n}})^2S_0-K,0)
\end{tikzcd}$$

Tree 9: Basic Time Step Pricing Model American Call Option Pricing Tree

Similarly, the option pricing tree for an American put is as follows.

$$V^L_{2(HH)} = max(K - (1 + L\tfrac{\sigma}{\sqrt{n}})^2 S_0, 0)$$

$$V^L_{1(H)} = max[\tfrac{1}{1-r}(pV^L_{2(HH)} + qV^L_{2(HT)}), K - (1 + L\tfrac{\sigma}{\sqrt{n}})S_0]$$

$$V^L_0 = max[\tfrac{1}{1-r}(pV^L_{1(H)} + qV^L_{1(T)}), K - S_0]$$

$$V^L_{2(HT)} = max(K - (1 + L\tfrac{\sigma}{\sqrt{n}})(1 - L\tfrac{\sigma}{\sqrt{n}})S_0, 0)$$

$$V^L_{1(T)} = max[\tfrac{1}{1-r}(pV^L_{2(HT)} + qV^L_{2(TT)}), K - (1 - L\tfrac{\sigma}{\sqrt{n}})S_0]$$

$$V^L_{2(TT)} = max(K - (1 - L\tfrac{\sigma}{\sqrt{n}})^2 S_0, 0)$$

Tree 10: Basic Time Step Pricing Model American Put Option Pricing Tree

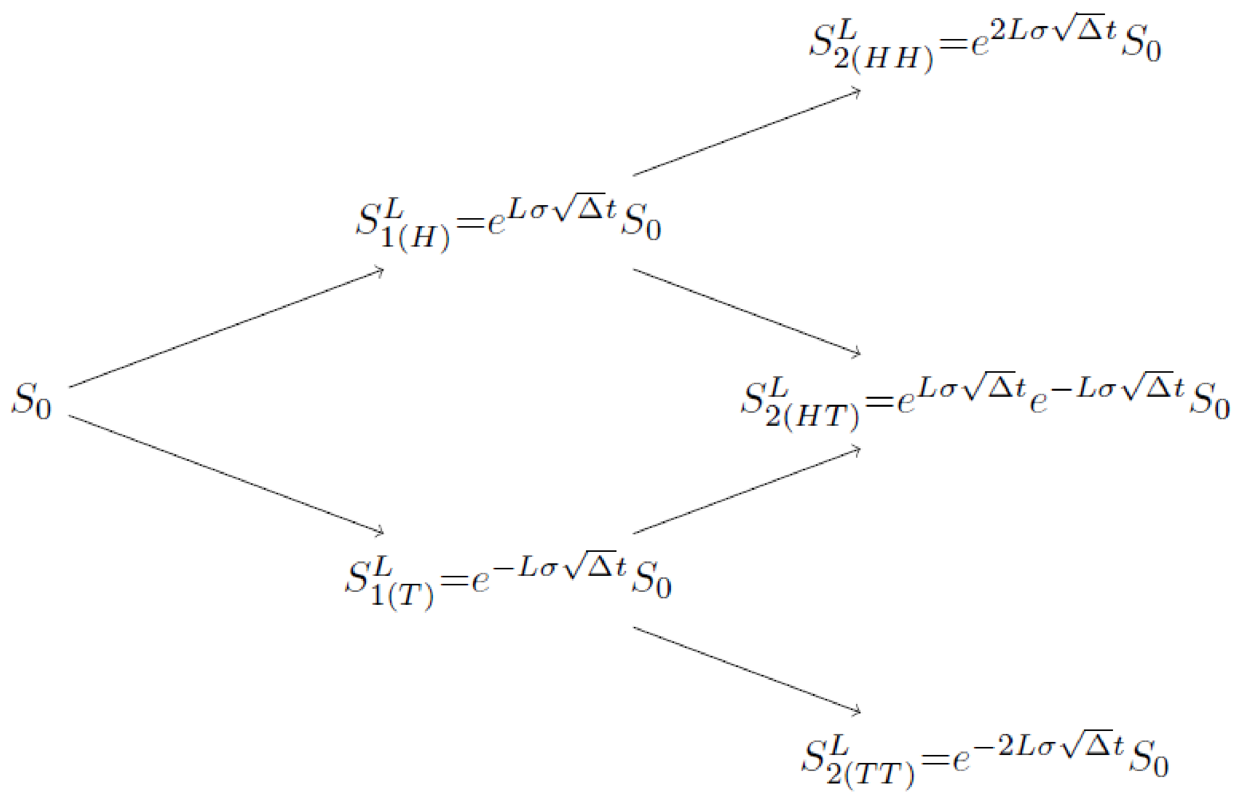
b. Cox, Ross, and Rubinstein Pricing Model

Similarly, if we define the upward and downward movements as $u = e^{(\sigma\sqrt{\Delta t})}$ and $d = e^{(-\sigma\sqrt{\Delta t})}$ for the non-leveraged asset, the leveraged asset's upward and downward movements will become

$$u = e^{(L\sigma\sqrt{\Delta t})},$$

$$d = e^{(-L\sigma\sqrt{\Delta t})}.$$

The underlying tree for this model is as follows.



Tree 11: Cox, Ross, and Rubinstein Pricing Model Underlying Pricing Tree

Therefore the two period call and put options would be priced as shown below.

$$V^L_{2(HH)} = max(e^{2L\sigma\sqrt{\Delta t}}S_0 - K, 0)$$

$$V^L_{1(H)} = \frac{1}{1-r}[pV^L_{2(HH)} + qV^L_{2(HT)}]$$

$$V^L_0 = \frac{1}{1-r}[pV^L_{1(H)} + qV^L_{1(T)}]$$

$$V^L_{2(HT)} = max(e^{L\sigma\sqrt{\Delta t}}e^{-L\sigma\sqrt{\Delta t}}S_0 - K, 0)$$

$$V^L_{1(T)} = \frac{1}{1-r}[pV^L_{2(HT)} + qV^L_{2(TT)}]$$

$$V^L_{2(TT)} = max(e^{-2L\sigma\sqrt{\Delta t}}S_0 - K, 0)$$

Tree 12: Cox, Ross, and Rubinstein Pricing Model American Call Option Pricing Tree

$$V^L_{2(HH)} = max(K - e^{2L\sigma\sqrt{\Delta t}}S_0, 0)$$

$$V^L_{1(H)} = max[\frac{1}{1-r}(pV^L_{2(HH)} + qV^L_{2(HT)}), K - e^{L\sigma\sqrt{\Delta t}}S_0]$$

$$V^L_0 = max[\frac{1}{1-r}(pV^L_{1(H)} + qV^L_{1(T)}), K - S_0]$$

$$V^L_{2(HT)} = max(K - e^{L\sigma\sqrt{\Delta t}}e^{-L\sigma\sqrt{\Delta t}}S_0, 0)$$

$$V^L_{1(T)} = max[\frac{1}{1-r}(pV^L_{2(HT)} + qV^L_{2(TT)}), K - e^{-L\sigma\sqrt{\Delta t}}S_0]$$

$$V^L_{2(TT)} = max(K - e^{-2L\sigma\sqrt{\Delta t}}S_0, 0)$$

Tree 13: Cox, Ross, and Rubinstein Pricing Model American Put Option Pricing Tree

c. Trigeorgis Pricing Model

For the last approach, the upward and downward movements are determined by an increase or decrease in x by a factor of Δx. For the leveraged funds, define Δx and the probability of each movement as

$$\Delta x = \sqrt{(L\sigma)^2 \Delta t + \left[r - \frac{(L\sigma)^2}{2}\right]^2 \Delta t^2},$$

$$p_u = \frac{1}{2} + \frac{\left[r - \frac{(L\sigma)^2}{2}\right] \Delta t}{2\Delta x},$$

$$p_u = \frac{1}{2} - \frac{\left[r - \frac{(L\sigma)^2}{2}\right] \Delta t}{2\Delta x}.$$

Let $i = 1, \ldots, N$ be the time steps up to expire and j be the level of the variable S at time i. Then the price of the asset at the node (i, j) is

$$S_{i,j} = e^{(x_{i,j})} = e^{(x + j\Delta x_u + (j-i)\Delta x_d)} = S_0 e^{(j\Delta x_u + (j-i)\Delta x_d)}.$$

11. Results

a. Basic Time Step Pricing Model

The initial approach of pricing American options on leveraged exchange traded funds where the upward and downward movements are defined as $u = 1 + L\frac{\sigma}{\sqrt{n}}$ and $d = 1 - L\frac{\sigma}{\sqrt{n}}$ resulted in prices significantly lower than the market bid and ask prices. The bid price of an option is the highest price a buyer is willing to purchase the option in the market. The ask price of an option is the lowest price a seller is willing to accept for the option. The calculated prices right at the money[3] tend to be quite lower than the bid ask prices while the prices of the options deep in the money[4] and deep out of the money[5] tend to be within the bid ask spread.

We can also compare our estimated option prices with prices calculated using the *binprice* and *blsprice* functions in MATLAB. The *binprice* function prices American call and put options in the binomial model while the *blsprice* function prices European call options using the Black-Scholes pricing model. The built in binomial pricing function in MATLAB resulted in the same prices calculated using the simplified approach of pricing American options on leveraged ETFs, which verifies that our method is giving reasonable results. In the plots below, the *binprice* prices overlap the calculated option prices. The Black- Scholes prices tended to be higher than the bid and ask prices or within the bid and ask prices. The Black- Scholes pricing

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[3] An option is at the money if the current market price of the underlying is equal to the strike price.
[4] An option is in the money for a call if the market price of the underlying is higher than the strike price. An option is in the money for a put if the strike price is above the market price.
[5] An option is out of the money for a call if the market price of the underlying is lower than the strike price. An option is out of the money for a put if the strike price is below the market price.

model requires $\sigma > 0$, so we only price bull leveraged ETFs with this model. Below are four plots demonstrating the relationship between the bid, ask, Black Scholes, *binprice*, and the calculated option prices. More charts demonstrating these relationships can be seen in Appendix A.

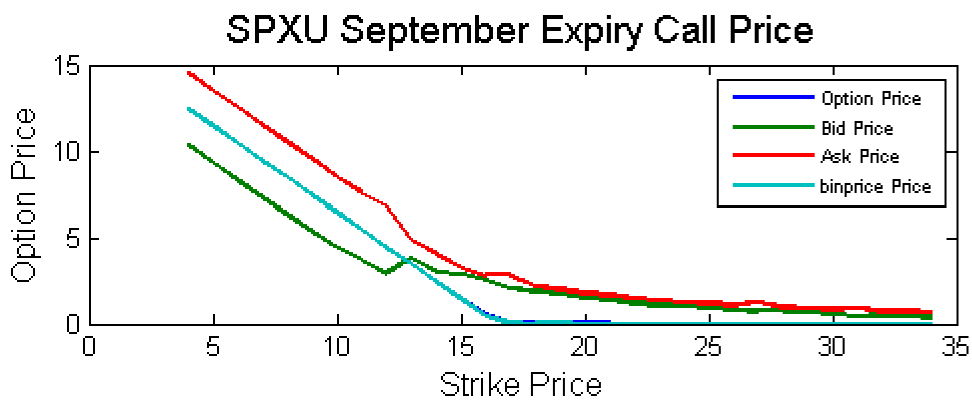

Chart 3: UltraPro Short S&P500 300% bear call option with a September expiry. The life of the option is 176 days.

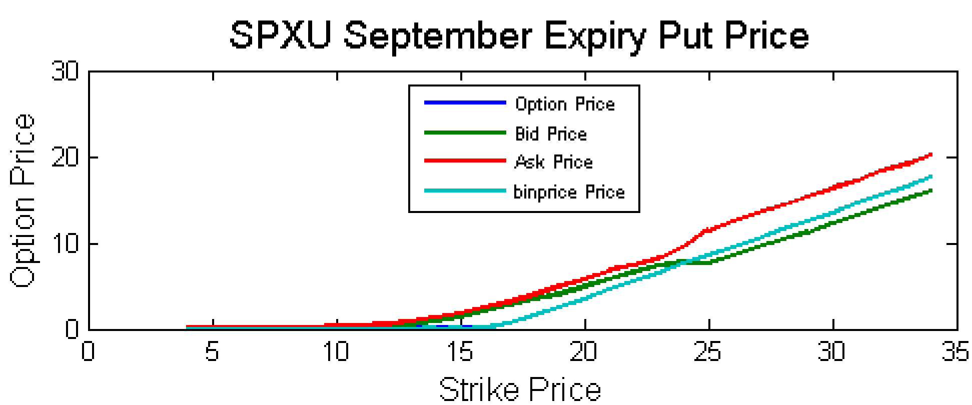

Chart 4: UltraPro Short S&P500 300% bear put option with a September expiry. The life of the option is 176 days.

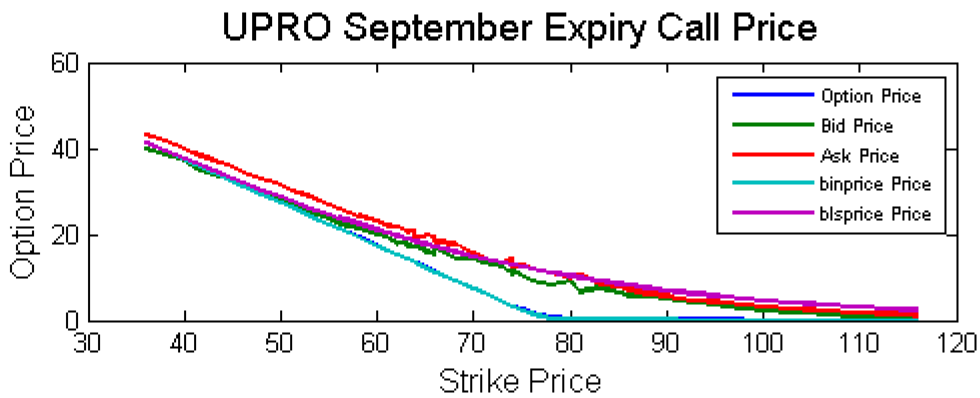

Chart 5: UltraPro S&P500 300% bull call option with a September expiry. The life of the option is 176 days.

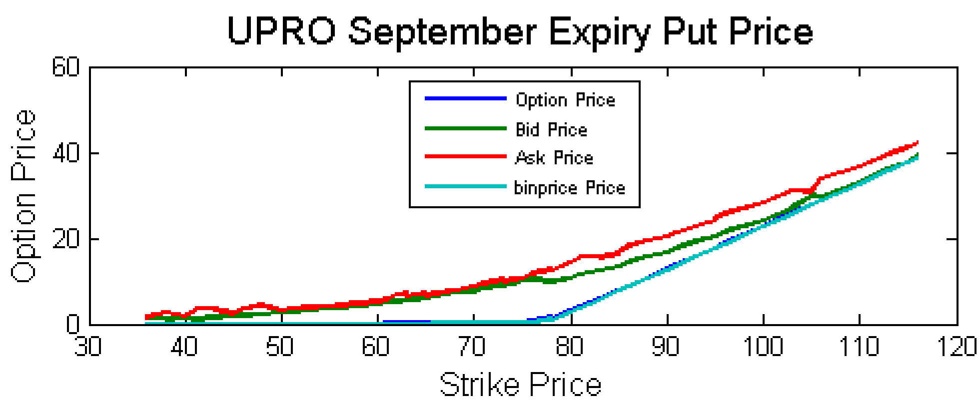

Chart 6: UltraPro S&P500 300% bull put option with a September expiry. The life of the option is 176 days.

b. Cox, Ross, and Rubinstein Pricing Model

Pricing American options with upward and downward movements defined as $u = e^{(L\sigma\sqrt{\Delta t})}$ and $d = e^{(-L\sigma\sqrt{\Delta t})}$ resulted in equivalent prices as those calculated with the first approach and the *binprice* function in MATLAB. Therefore, these prices are also significantly lower than the bid and ask prices when the option is right at the money. When the options are significantly out of the money and significantly in the money, the calculated option prices are within or close to those of the bid and ask prices. It was also possible to compare the calculated option prices to that of the Black -Scholes option pricing model for the bull leveraged ETFs. The estimated prices tended to be slightly higher or right between the bid and ask prices. More charts demonstrating these relationships can be seem in Appendix B.

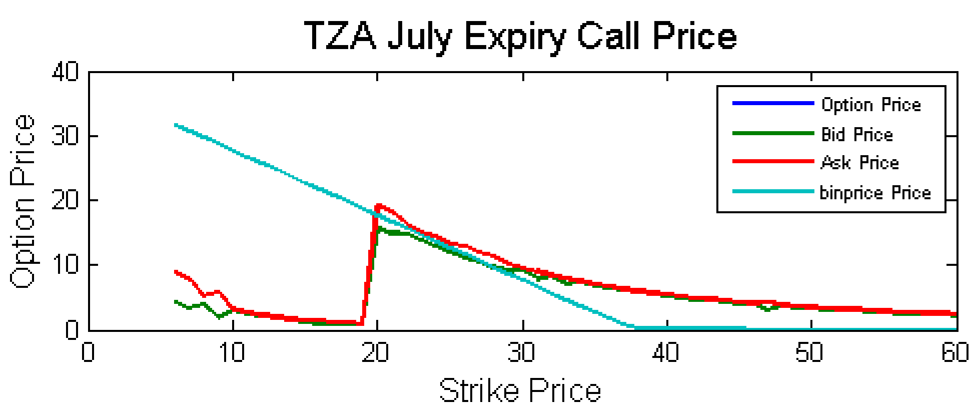

Chart 7: Small Cap Bear 3X Shares call option with a July expiry. The life of the option is 113 days.

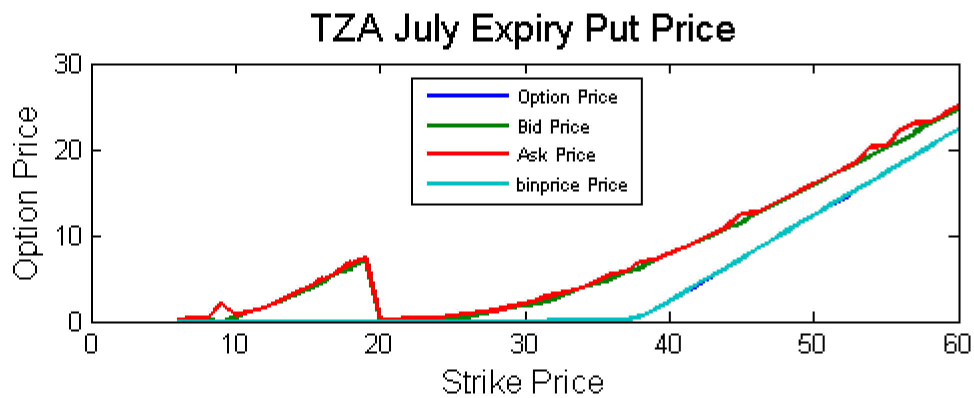

Chart 8: Small Cap Bear 3X Shares put option with a July expiry. The life of the option is 113 days.

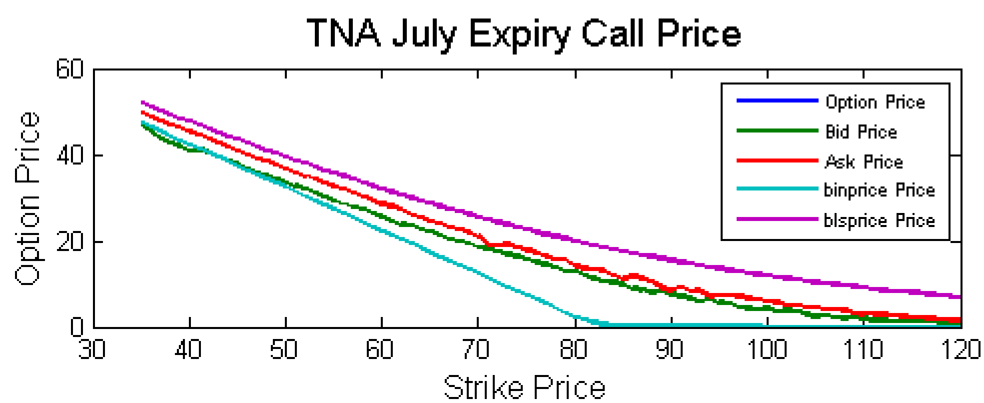

Chart 9: Small Cap Bull 3X Shares call option with a July expiry. The life of the option is 113 days.

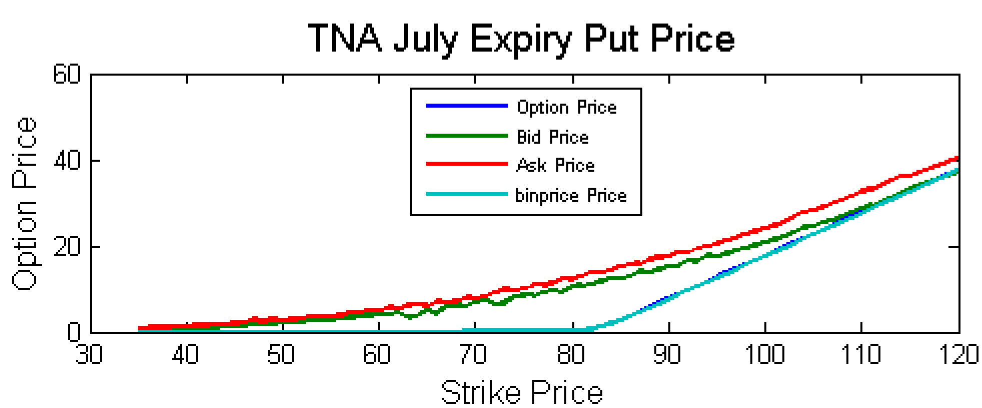

Chart 10: Small Cap Bull 3X Shares put option with a July expiry. The life of the option is 113 days.

c. Trigeorgis Pricing Model

Similarly to the other two approaches, the pricing model with upward and downward movements determined by an increase or decrease in x by a factor of Δx results in prices below the bid and ask prices for options right at the money and above or within the bid ask prices for options far in the money or far out of the money. The call option prices are equivalent to the option prices calculated with the *binprice* function in MATLAB and below the Black Scholes model prices. The estimated put option prices differ from those in the first two approaches. The put prices tended to be higher than the bid, ask, and *binprice* prices when the option is out of the money. The priced options still tend to be lower than the bid, ask, and Black-Scholes prices right at the money. More charts that demonstrate these relationships can be seen in Appendix C.

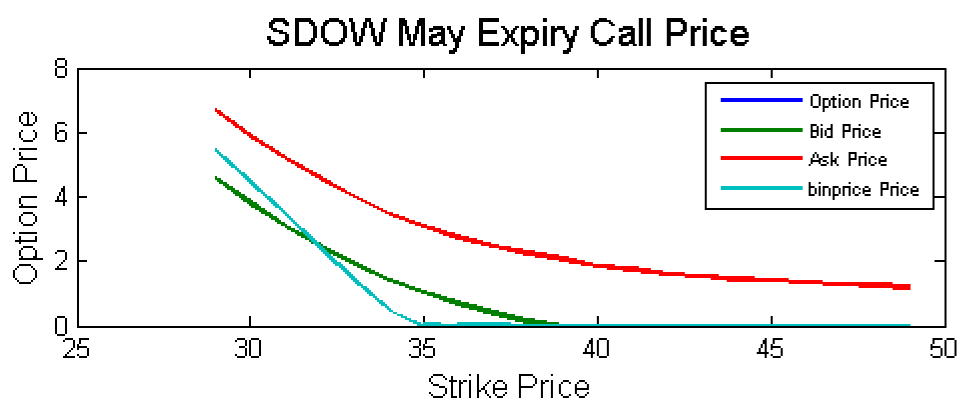

Chart 11: UltraPro Short Dow30 Bear call option with a May expiry. The life of the option is 57 days.

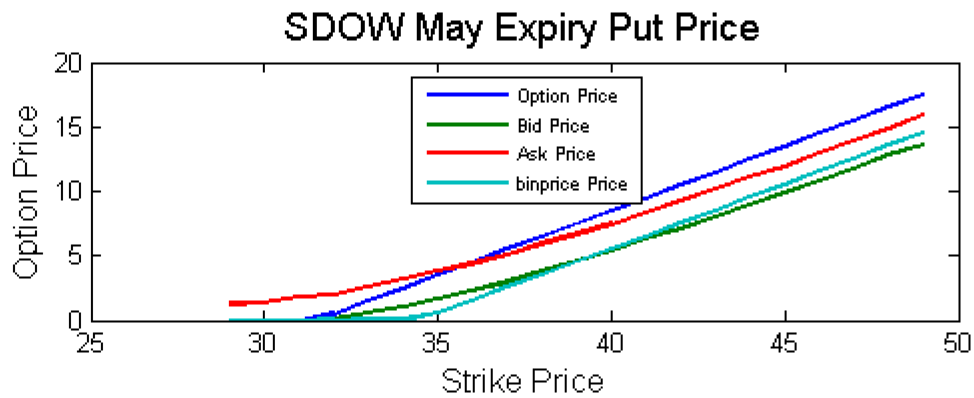

Chart 12: UltraPro Short Dow30 Bear put option with a May expiry. The life of the option is 57 days.

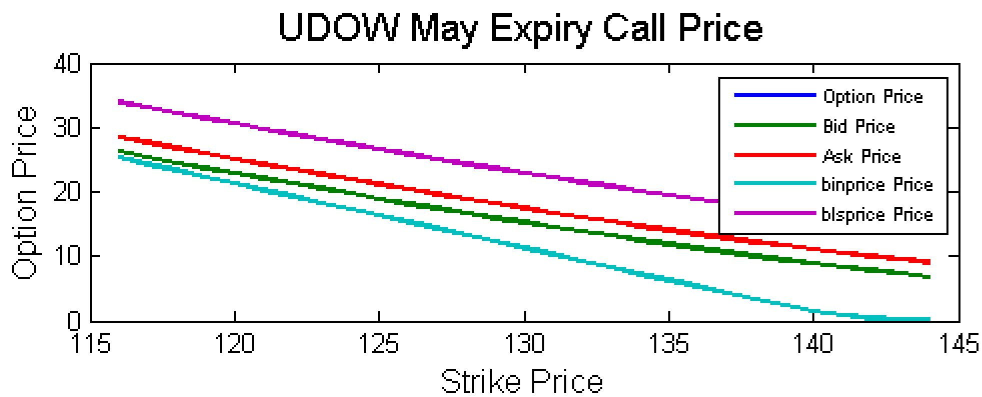

Chart 13: UltraPro Dow30 Bull call option with a May expiry. The life of the option is 57 days.

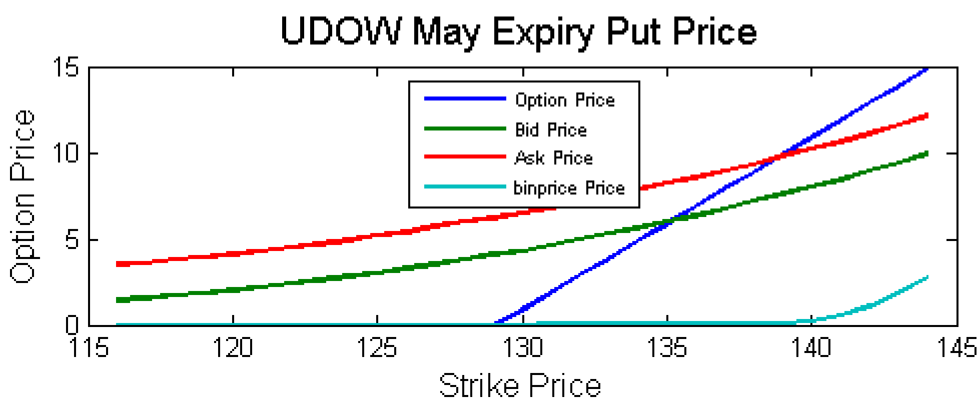

Chart 14: UltraPro Dow30 Bull put option with a May expiry. The life of the option is 57 days.

12. Conclusion and Possible Future Work

In order to be able to price options on leveraged exchange traded funds based on the volatility of the index tracked by the ETF, there must be a relationship between the volatility of the index and the leveraged ETF. Future work could involve option pricing with pricing dynamics estimates using the leveraged ETFs daily data instead of the index's daily data to see if this pricing has similar results.

However, the binomial asset pricing model applied to leveraged exchange traded funds is not the most accurate way to price options in the market. In theory, pricing an option using the binomial pricing model and the Black-Scholes model yield similar results (Shreve 20). With resulting prices that differ between the Black-Scholes model and the binomial model, one can conclude that these options must be traded very differently than other options on the market. Using only daily data to estimate returns and volatility may also result in inaccurate pricing. All three approaches used to calculate prices in the binomial model were precise, but not accurate to market prices. For options at the money all three approaches to the binomial model resulted in call prices below the bid and ask prices of these options in the market. The out of the money and in the money prices tended to be close to or in between the bid and ask prices. On the other hand, the Black-Scholes pricing model resulted in prices higher than the bid and ask prices.

Future work could involve Monte Carlo pricing of American options on leveraged exchange traded funds. This approach generates several thousand pricing paths for the underlying, calculates and averages the payoffs for each path while also discounting back the prices. It is a very common pricing model used in practice today called the Longstaff-Schwartz model.

Looking at different ways to hedge the cost of an option on a leveraged ETF could be involved in future work. One main leverage technique that could be looked at is a bear put spread. A bear put spread involves purchasing a put option on a specific underlying while simultaneously writing a put on the same underlying with a lower strike price. This option strategy is a way to decrease the risk of purchasing a put option on risky funds like leveraged ETFs. Similarly, future work could include pricing bull call spreads on leveraged ETFs which involves purchasing a call option while simultaneously writing a call option on the same underlying with a higher strike price.

13. Appendix A – Basic Time Step Pricing Model

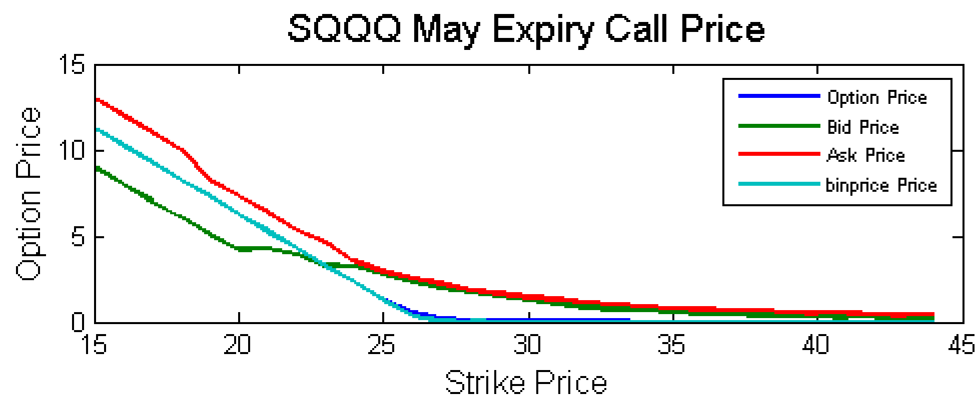

Chart 15: UltraPro Short QQQ 300% bear call option with a May expiry. The life of the option is 57 days.

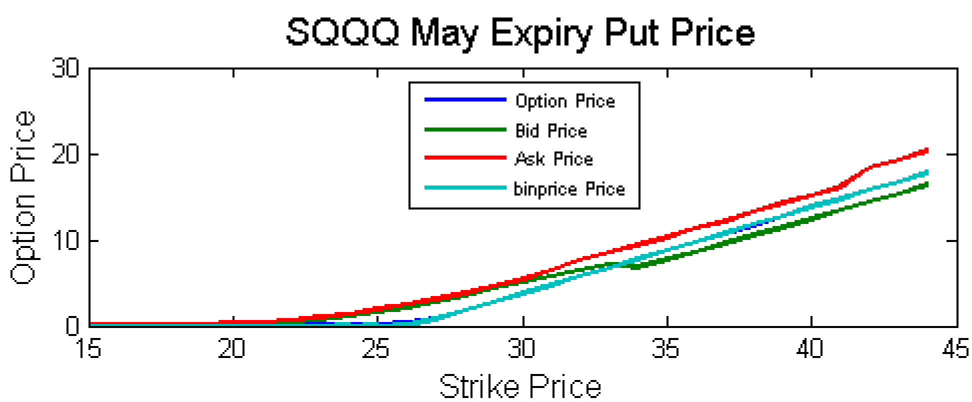

Chart 16: UltraPro Short QQQ 300% bear put option with a May expiry. The life of the option is 57 days.

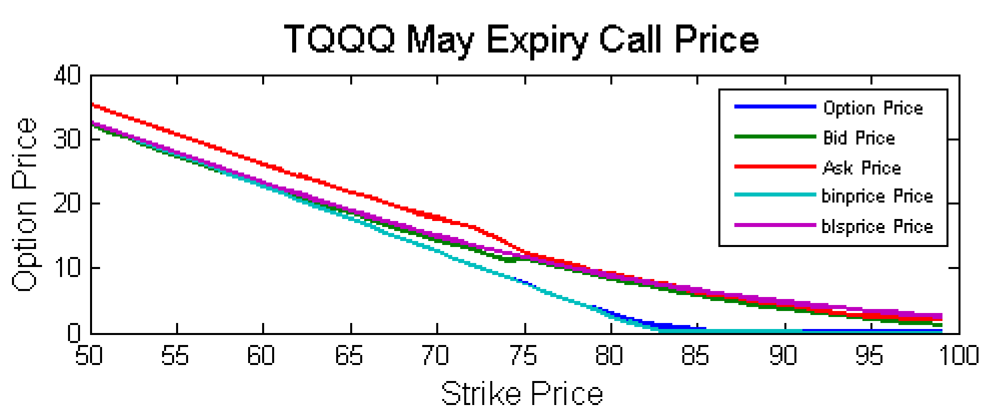

Chart 17: UltraPro QQQ 300% bull call option with a May expiry. The life of the option is 57 days.

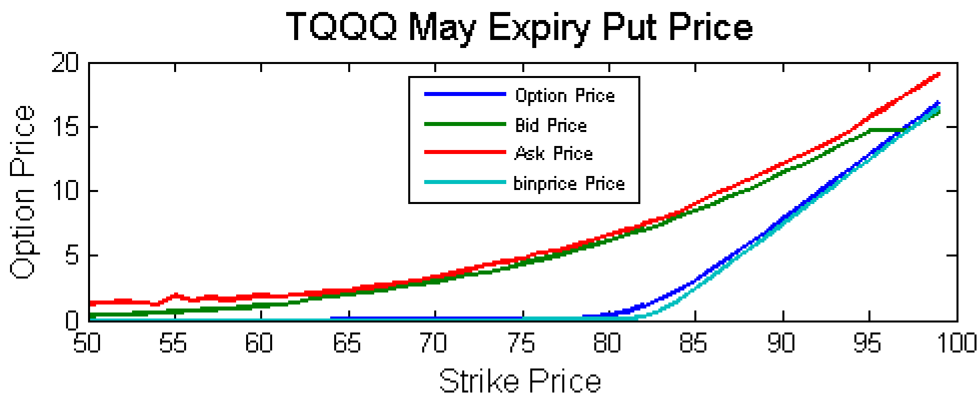

Chart 18: UltraPro QQQ 300% bull put option with a May expiry. The life of the option is 57 days.

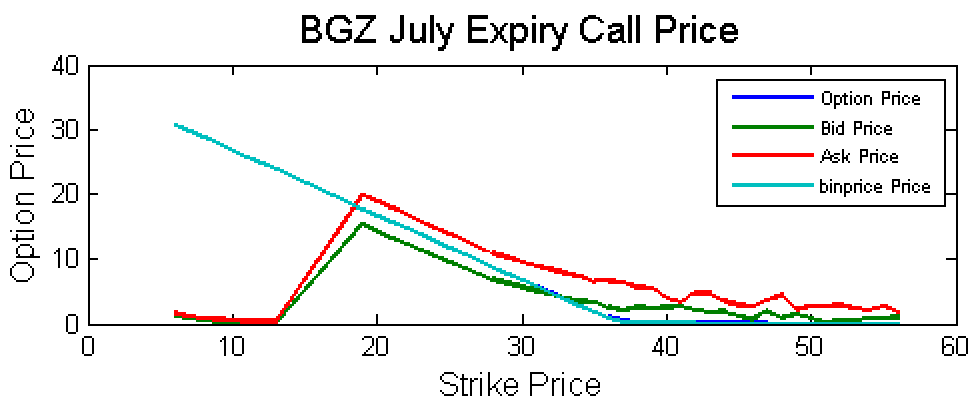

Chart 19: Large Cap Bull 3X Shares call option with a July expiry. The life of the option is 113 days.

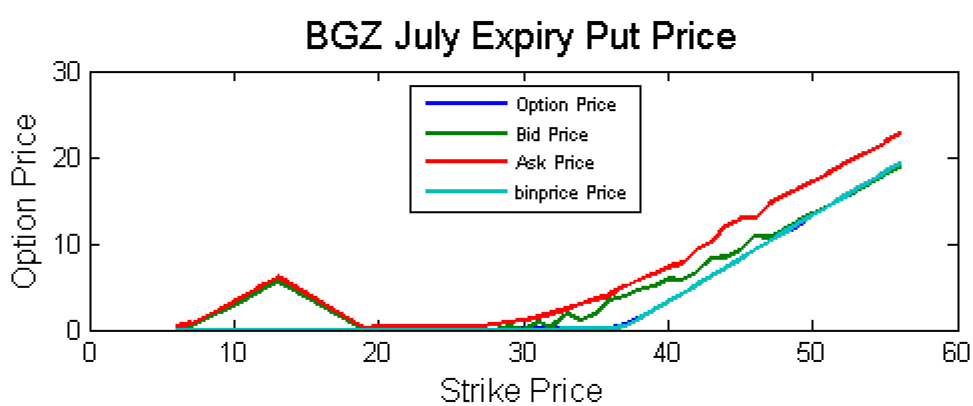

Chart 20: Large Cap Bull 3X Shares put option with a July expiry. The life of the option is 113 days.

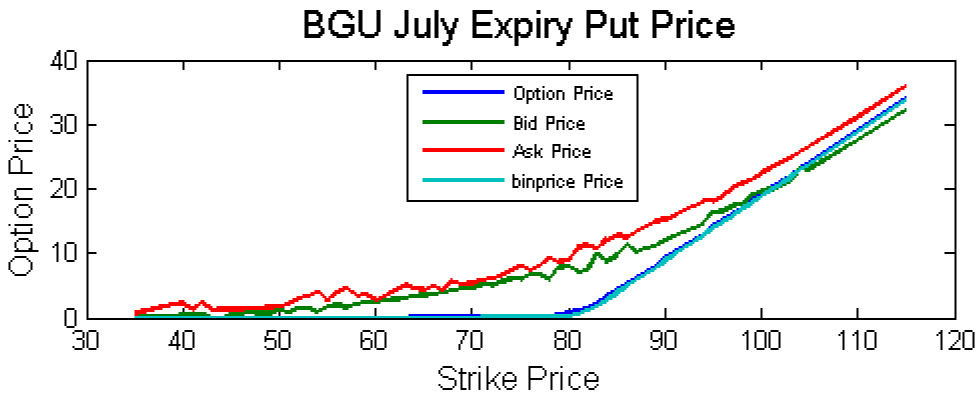

Chart 21: Large Cap Bear 3X Shares call option with a July expiry. The life of the option is 113 days.

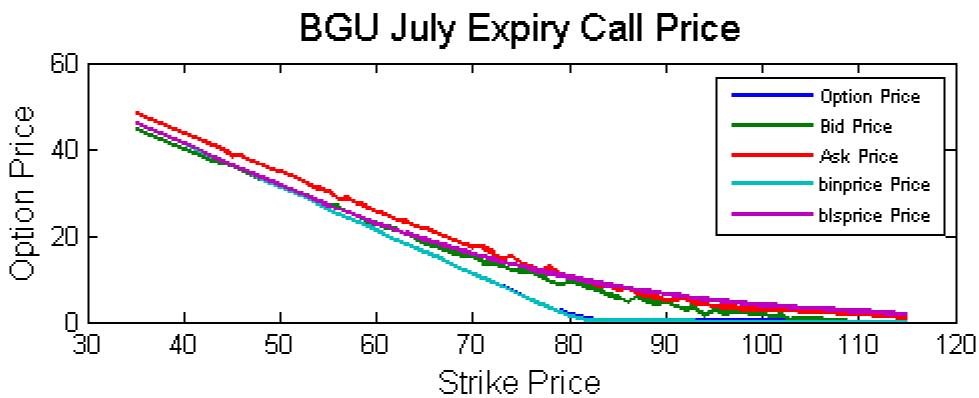

Chart 22: Large Cap Bear 3X Shares put option with a July expiry. The life of the option is 113 days.

14. Appendix B – Cox, Ross, and Rubinstein Pricing Model

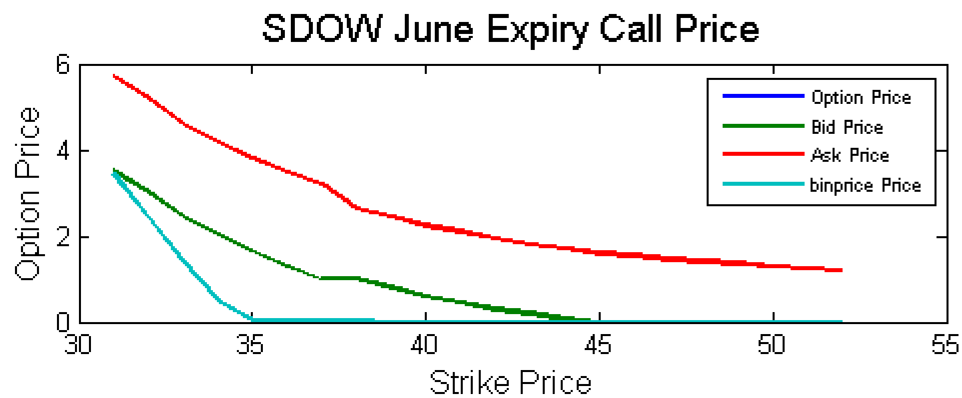

Chart 23: UltraPro Short Dow30 300% Bear call option with a June expiry. The life of the option is 85 days.

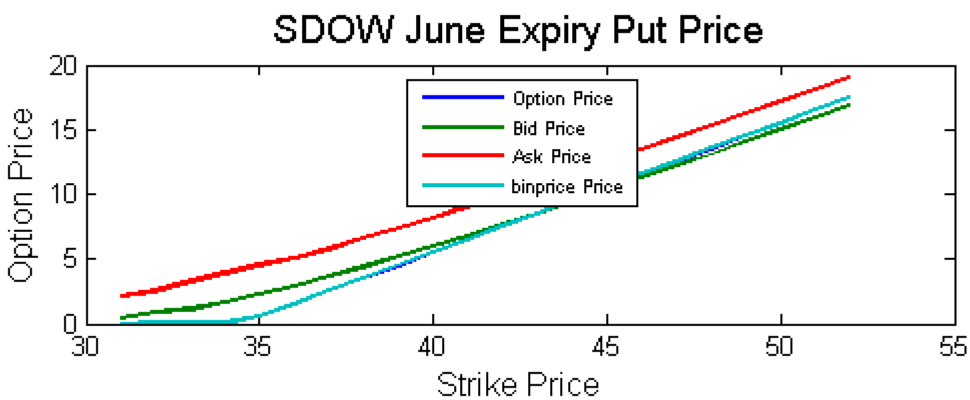

Chart 24: UltraPro Short Dow30 300% Bear put option with a June expiry. The life of the option is 85 days.

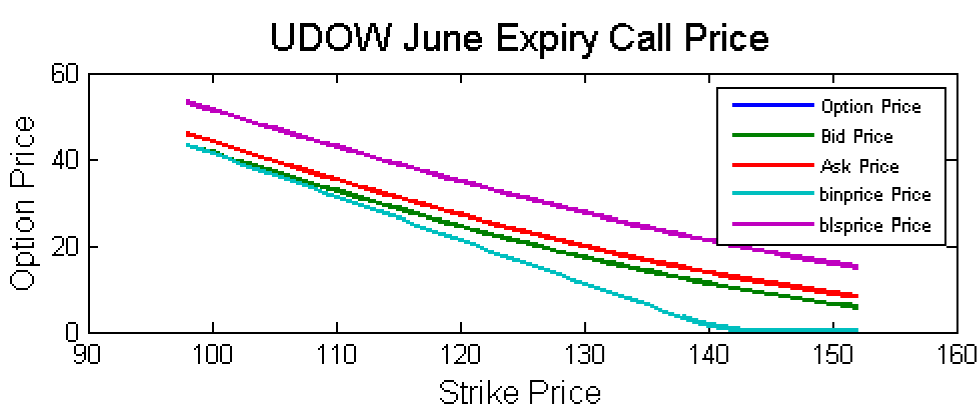

Chart 25: UltraPro Dow30 300% Bull call option with a June expiry. The life of the option is 85 days.

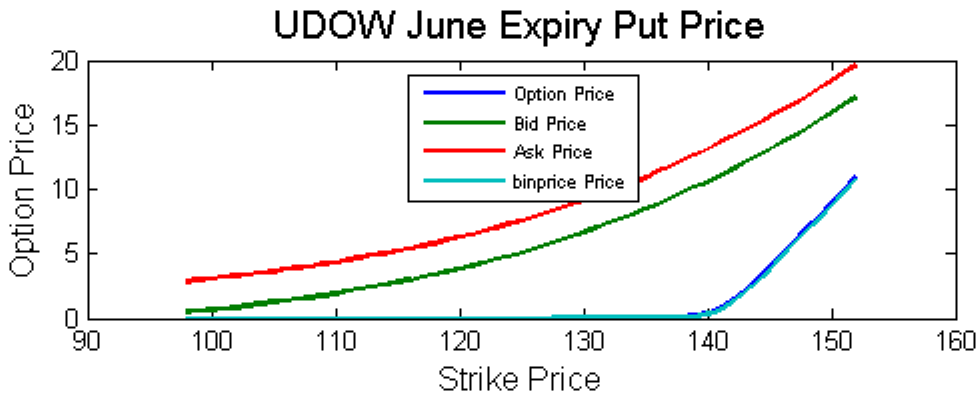

Chart 26: UltraPro Dow30 300% Bull put option with a June expiry. The life of the option is 85 days.

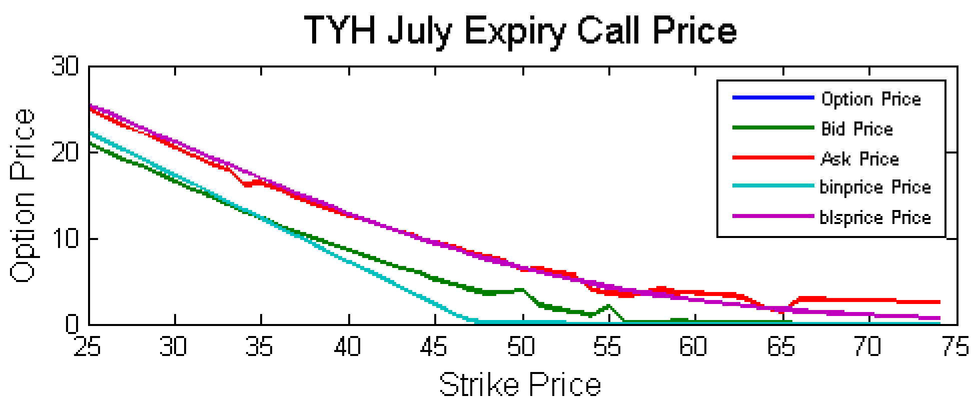

Chart 27: Technology Bear 3X Shares call option with a July expiry. The life of the option is 113 days.

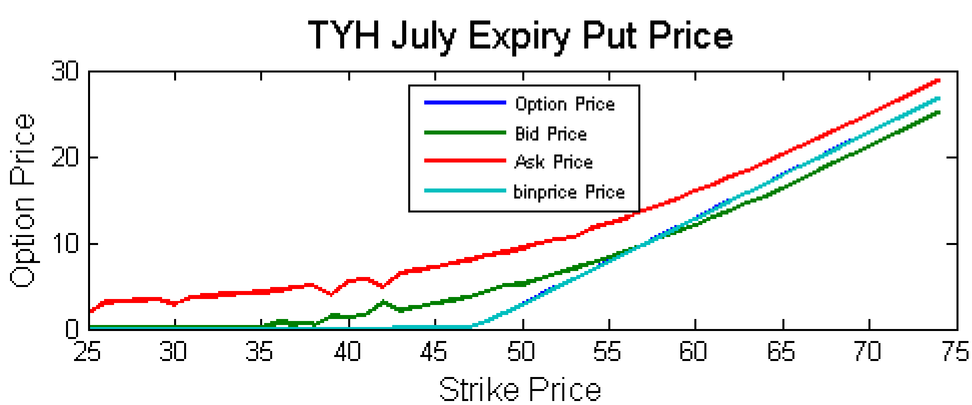

Chart 28: Technology Bear 3X Shares put option with a July expiry. The life of the option is 113 days.

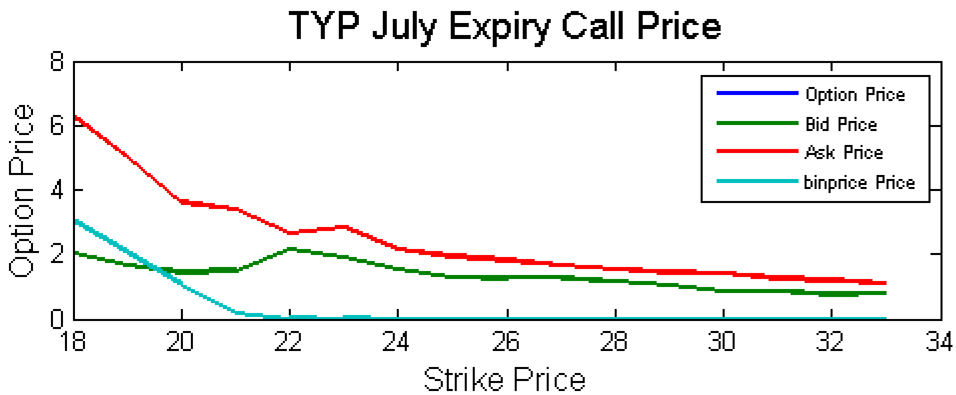

Chart 29: Technology Bull 3X Shares call option with a July expiry. The life of the option is 113 days.

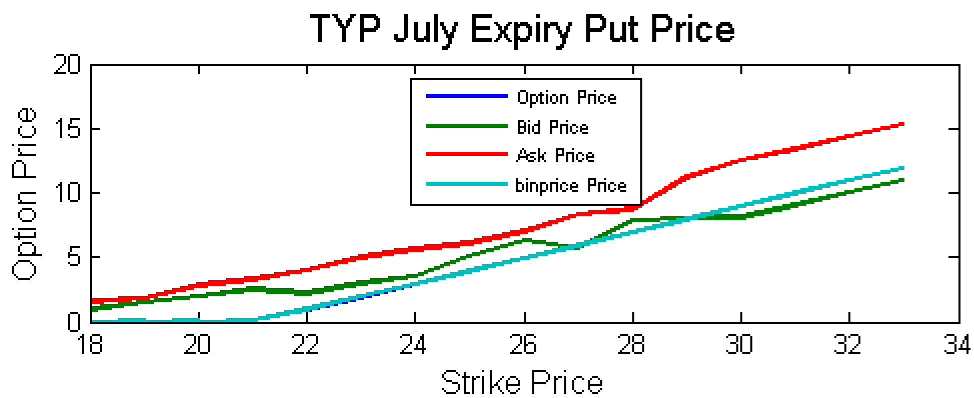

Chart 30: Technology Bear 3X Shares put option with a July expiry. The life of the option is 113 days.

15. Appendix C – Trigeorgis Pricing Model

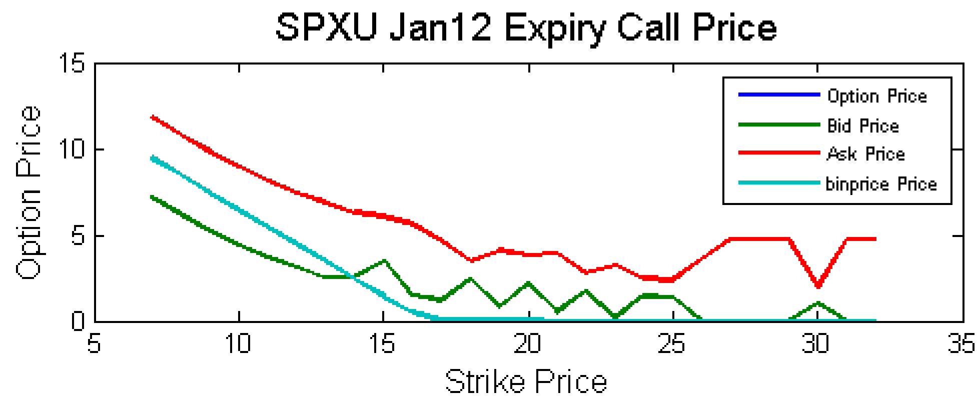

Chart 31: UPRO Short S&P500 300% Bear call option with a January 2012 expiry. The life of the option is 302 days.

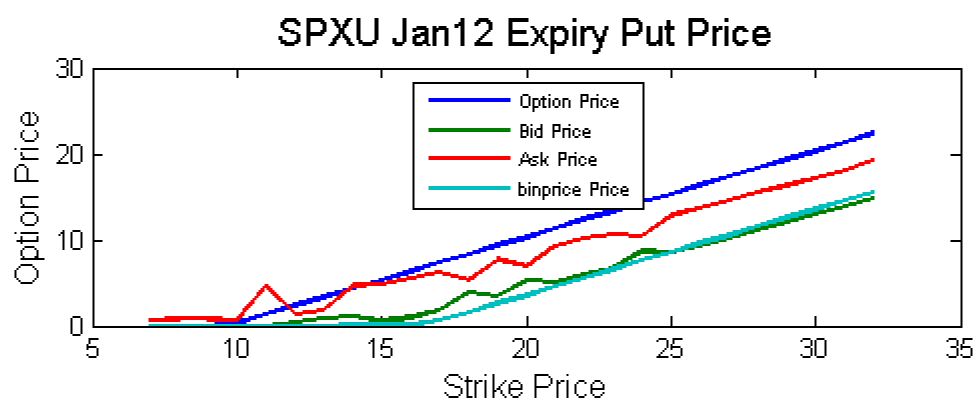

Chart 32: UPRO Short S&P500 300% Bear put option with a January 2012 expiry. The life of the option is 302 days.

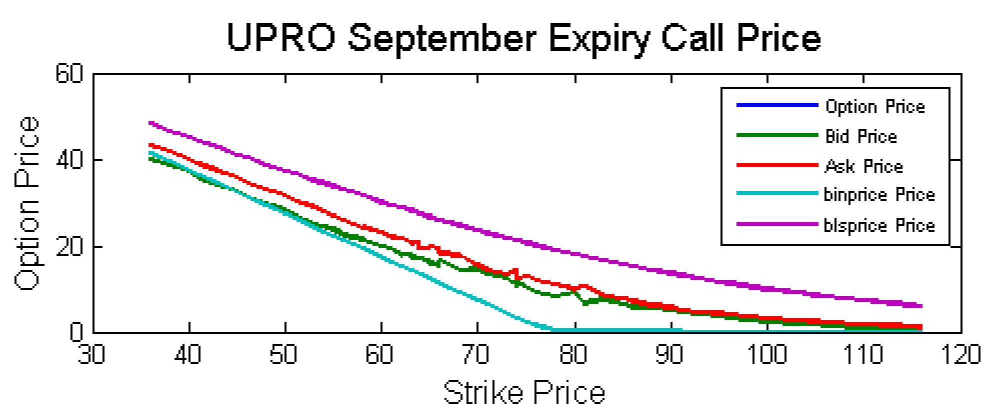

Chart 33: UPRO S&P500 300% Bull call option with a September expiry. The life of the option is 176 days.

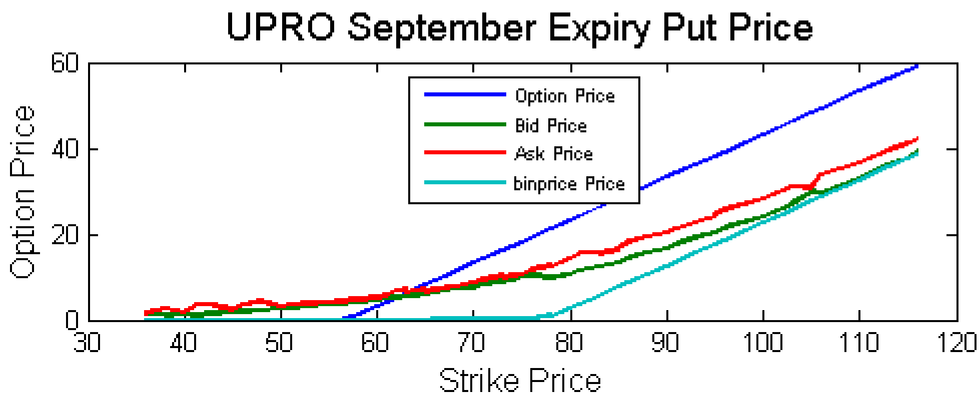

Chart 34: UPRO S&P500 300% Bull put option with a September expiry. The life of the option is 176 days.

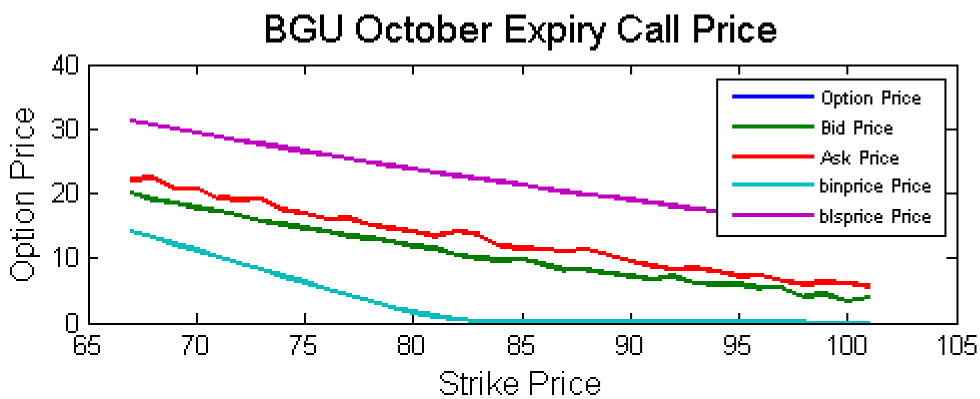

Chart 35: Large Cap Bear 3X Shares call option with an October expiry. The life of the option is 211days.

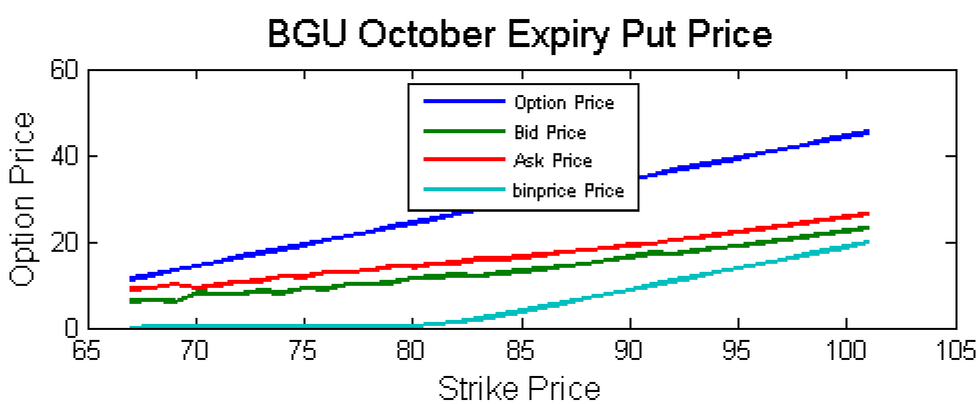

Chart 36: Large Cap Bear 3X Shares put option with an October expiry. The life of the option is 211days.

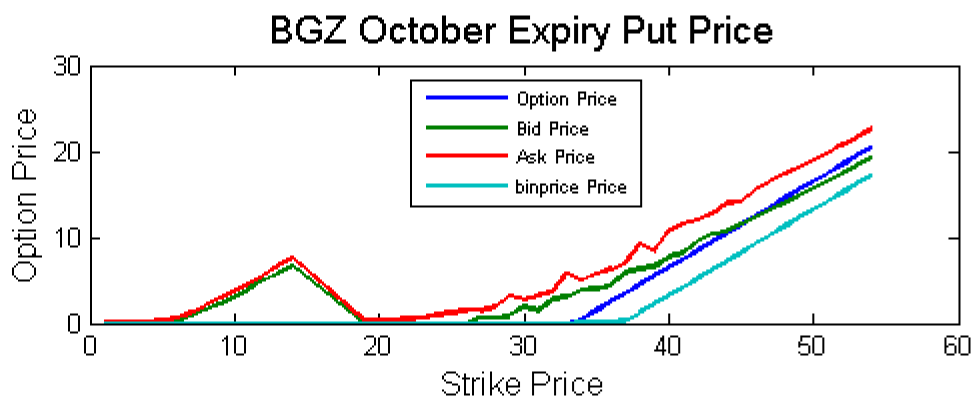

Chart 37: Large Cap Bull 3X Shares call option with an October expiry. The life of the option is 211 days.

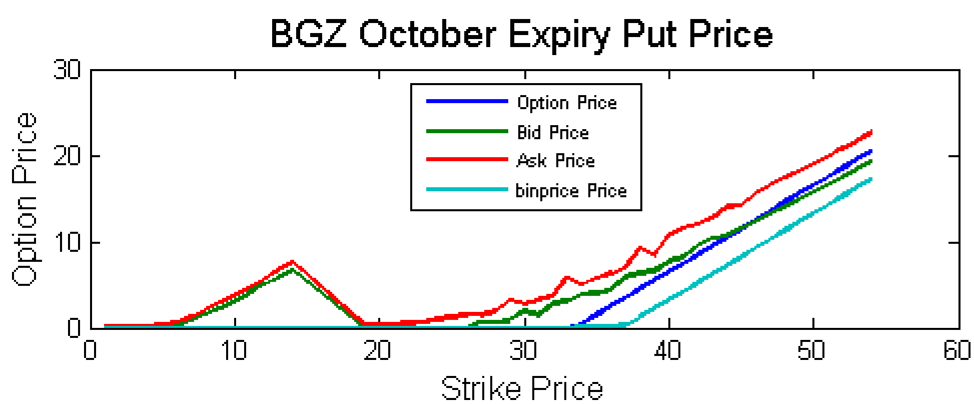

Chart 38: Large Cap Bull 3X Shares put option with an October expiry. The life of the option is 211 days.

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