On irregular total labellings

Martin Bača[a], Stanislav Jendrol'[b], Mirka Miller[c], Joseph Ryan[c]

[a] Department of Mathematics, Faculty of Engineering, Technical University Košice, Letná 9, SK-040 01 Košice, Slovak Republic
[b] Institute of Mathematics, P.J. Šafárik University, Jesenná 5, 041 54 Košice, Slovak Republic
[c] School of Information Technology and Mathematical Sciences, University of Ballarat, Ballarat, Vic. 3533, Australia

Received 7 October 2002; received in revised form 6 November 2003; accepted 18 November 2005
Available online 28 November 2006

Abstract

Two new graph characteristics, the total vertex irregularity strength and the total edge irregularity strength, are introduced. Estimations on these parameters are obtained. For some families of graphs the precise values of these parameters are proved.

MSC: 05C78

Keywords: Graph labelling; Irregularity strength; Edge assignment; Total assignment; Vertex irregular total labelling; Edge irregular total labelling

1. Introduction

A *labelling* of a graph is a map that carries graph elements to the numbers (usually to the positive or non-negative integers). The most common choices of domain are the set of all vertices (*vertex* labellings), the edge set alone (*edge* labellings), or the set of all vertices and edges (*total* labellings). Other domains are possible. The most complete recent survey of graph labellings is [6].

In many cases it is interesting to consider the sum of all labels associated with a graph element. This will be called the *weight* of the element. As in the study of magic total labellings, see e.g., a recent book of Wallis [13], the weight of a vertex x under a total labelling ∂ of elements of a graph $G = (V, E)$ is

$$wt(x) = \partial(x) + \sum_{xy \in E} \partial(xy), \tag{1}$$

and the weight of the edge xy is

$$wt(xy) = \partial(x) + \partial(xy) + \partial(y). \tag{2}$$

In [2], the following problem was proposed by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba. Assign positive integer labels to the edges of a simple connected graph of order at least 3 in such a way that the graph becomes irregular, i.e., the weights (label sums) at each vertex are distinct. What is the minimum value of the largest label over all such irregular assignments?

This parameter of a graph G is well known as the *irregularity strength* of the graph G, $s(G)$. Finding the irregularity strength of a graph seems to be rather hard even for simple graphs, see [3,4,5,7,9,10]. An excellent survey on the subject is Lehel [11]. For recent results see papers by Amar and Togni [1], Jacobson and Lehel [8] and Nierhoff [12].

Motivated by this problem, a recent excellent book by Wallis [13] and other papers on total labellings, see e.g. [14], we study in this paper irregular total labellings.

For a graph $G = (V, E)$ with vertex set V and edge set E we define a labelling $\partial : V \cup E \to \{1, 2, \ldots, k\}$ to be a total k-labelling. A total k-labelling is defined to be an *edge irregular total k-labelling* of the graph G if for every two different edges e and f of G there is

$$wt(e) \neq wt(f),$$

and to be a *vertex irregular total k-labelling* of G if for every two distinct vertices x and y of G there is

$$wt(x) \neq wt(y).$$

The minimum k for which the graph G has an edge irregular total k-labelling is called the *total edge irregularity strength* of the graph G, $\mathrm{tes}(G)$. Analogously, we define the *total vertex irregularity strength* of G, $\mathrm{tvs}(G)$, as the minimum k for which there exists a vertex irregular total k-labelling of G.

Let $G = (V, E)$ be a (p, q)-graph, that is, the graph with vertex set $V = \{v_1, v_2, \ldots, v_p\}$ and edge set E, $|E| = q$. Let ∂ be a total labelling of G. We associate with G and its total labelling the following $p \times p$ matrix $\mathbf{L} = [l_{ij}]$ defined by

$$l_{ij} = \begin{cases} \partial(v_i v_j) & \text{if } v_i v_j \in E \text{ and } i \neq j, \\ 0 & \text{if } v_i v_j \notin E \text{ and } i \neq j, \\ \partial(v_i) & \text{if } i = j. \end{cases}$$

Let us call this matrix the ∂-*matrix* of the total labelling ∂ of the graph G. This matrix is symmetric with respect to the main diagonal and all its elements are non-negative integers.

The rest of the paper is organised as follows. In Section 2 we investigate properties of edge irregular total labellings. Section 3 is devoted to vertex irregular total labellings.

2. Edge irregular total labellings

Our first result shows that the total edge irregularity strength is defined for all graphs. Namely, we have

Theorem 1. *Let $G = (V, E)$ be a graph with vertex set V and a non-empty edge set E. Then*

$$\left\lceil \frac{|E| + 2}{3} \right\rceil \leqslant \mathrm{tes}(G) \leqslant |E|.$$

Proof. To get the upper bound we label each vertex of G with label 1 and the edges of G consecutively with labels $1, 2, \ldots, |E|$. It is easy to see that $wt(e) \neq wt(f)$ for any two distinct edges e and f of G.

Let φ be an optimal labelling with respect to the $\mathrm{tes}(G)$. Then the heaviest edge e of G has weight $wt(e) \geqslant |E| + 2$. This weight is the sum of three labels. So at least one label is at least $(|E| + 2)/3$. □

The lower bound in Theorem 1 is tight as can be seen from the following theorem.

Theorem 2. *Let P_n and C_n be a path and a cycle, respectively, with $n \geqslant 1$ edges. Then*

$$\mathrm{tes}(P_n) = \mathrm{tes}(C_n) = \left\lceil \frac{n + 2}{3} \right\rceil.$$

Proof. By Theorem 1 we have $\mathrm{tes}(G) \geqslant \lceil (n + 2)/3 \rceil, n \geqslant 1$ where $G \in \{P_n, C_n\}$. First we prove the bound for paths. We proceed by induction on n. The path P_1 is labelled with label 1 on all three elements. Let P_n be a path $v_1 e_1 v_2 e_2 v_3 \cdots v_n e_n v_{n+1}$; $n \geqslant 1$. Suppose we have labelled the path P_n for $n = 3(k-1)+1$, $k \geqslant 1$, in such a way that the

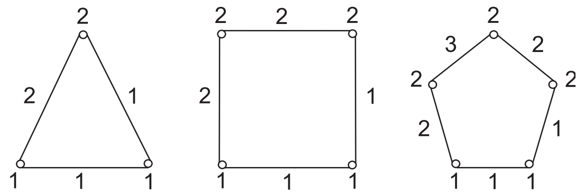

Fig. 1.

edge $e_n = v_n v_{n+1}$ of P_n is labelled with $\partial(v_n) = \partial(v_{n+1}) = \partial(e_n) = k$. For the inductive step the edges $e_1, \ldots, e_n$ and the vertices $v_1, \ldots, v_{n+1}$ are labelled as in P_n and we put $\partial(e_{n+1}) = \partial(e_{n+2}) = \partial(e_{n+3}) = k+1$, $\partial(v_{n+2}) = k$, $\partial(v_{n+3}) = \partial(v_{n+4}) = k+1$. Because $P_{n+1} \subseteq P_{n+2} \subseteq P_{n+3}$, we are done.

Let C_n be a cycle $v_1, e_1, v_2, e_2, v_3, \ldots, v_n, e_n, v_1$. In Fig. 1 we have optimal irregular labellings of C_3, C_4 and C_5 with labels from the set $\{1, 2, 3\}$.

Suppose we have an irregular labelling of C_n for $n = 3(k-1) + 2$, $k \geqslant 2$ with the edge e_{n-1} labelled as follows, $\partial(e_{n-1}) = k+1$, $\partial(v_{n-1}) = \partial(v_n) = k$. To obtain an optimal labelling of C_{n+1} (C_{n+2} and C_{n+3}, respectively) we split the edge e_{n-1} into two (three, or four, respectively) edges by adding one new vertex x_1 (two new vertices x_1 and x_2, or three new vertices x_1, x_2 and x_3, respectively) and label the "new" vertices and edges of the cycle C_{n+1} (C_{n+2}, C_{n+3}, respectively) in the following way, $\partial(v_{n-1}x_1) = k+1$, $\partial(x_1) = k+1$, $\partial(x_1 v_n) = k$ ($\partial(x_1) = \partial(x_2) = k+1$, $\partial(v_{n-1}x_1) = \partial(x_1 x_2) = k+1$, $\partial(x_2 v_n) = k$, and $\partial(x_1) = \partial(x_2) = \partial(x_3) = k+1$, $\partial(v_{n-1}x_1) = \partial(x_2 x_3) = k+1$, $\partial(x_1 x_2) = k+2$, $\partial(x_3 v_n) = k$, respectively). For C_{n+3} we reorder the vertices and edges so that the edge $x_1 x_2$ will play the role of e_{n-1} of the above construction. □

The upper bound in Theorem 1 is not sharp. If we introduce into the play the maximum degree $\Delta = \Delta(G)$ of the graph G, we obtain the following:

Theorem 3. *Let $G = (V, E)$ be a graph with maximum degree $\Delta = \Delta(G)$. Then*

(i)

$$\operatorname{tes}(G) \geqslant \left\lceil \frac{\Delta + 1}{2} \right\rceil \quad \textit{and}$$

(ii)

$$\operatorname{tes}(G) \leqslant |E| - \Delta \quad \textit{if } \Delta \leqslant \frac{|E| - 1}{2}.$$

Proof. (i) Suppose ∂ is an optimal total labelling of G. Let $e_1, e_2, \ldots, e_\Delta$ be the edges incident with a vertex x of maximum degree Δ in G. Let y_i be the other end of the edge e_i, i.e., $e_i = xy_i$. Since $w(e_i) = \partial(x) + \partial(e_i) + \partial(y_i)$ for all i, $1 \leqslant i \leqslant \Delta$, and as $w(e_1), w(e_2), \ldots, w(e_\Delta)$ are all distinct, $\partial(e_i) + \partial(y_i)$ are all distinct for $1 \leqslant i \leqslant \Delta$. So the largest among these values must be at least $\Delta + 1$. Thus, either $\partial(e_i)$ or $\partial(y_i)$ must be at least $(\Delta+1)/2$ for some i, $1 \leqslant i \leqslant \Delta$.

(ii) Let x and $e_1, e_2, \ldots, e_\Delta$ be defined as above. Let φ be the following labelling of the elements from $V \cup E$. $\varphi(v) = 1$ for every $v \in V$, $v \neq x$ and $\varphi(x) = \Delta + 1$. The edges from $E - \{e_1, e_2, \ldots, e_\Delta\}$ are labelled consecutively with labels $1, 2, \ldots, |E| - \Delta$ and $\varphi(e_i) = |E| - 2\Delta + i$, $i = 1, 2, \ldots, \Delta$. It is easy to see that φ is an edge irregular total labelling having the required property. □

The lower bound in Theorem 3 is tight as can be seen from the next theorem.

Theorem 4. *Let $S_n = K_{1,n}$ be a star on $n+1$ vertices, $n > 1$. Then*

$$\operatorname{tes}(S_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. The inequality $\operatorname{tes}(S_n) \geqslant \lceil (n+1)/2 \rceil$ holds by Theorem 3. What follows is an edge irregular total labelling showing the converse inequality. Let $e_1 = xv_1, e_2 = xv_2, \ldots, e_n = xv_n$ be edges of the star S_n. The following total labelling φ is optimal:

$$\varphi(x)=1, \quad \varphi(v_i)=\left\lfloor \frac{i+1}{2} \right\rfloor, \quad \varphi(e_i)=\left\lceil \frac{i+1}{2} \right\rceil \quad \text{for } i=1,2,\ldots,n. \qquad \square$$

The upper bound in Theorem 3 seems to be far from the best possible.

The idea of the proof of Theorem 3(ii) is used in the proof of:

Theorem 5. *Let $G=(V,E)$ be a graph with $|V|=p$ and $|E|=q$. Let $I=\{u_1,u_2,\ldots,u_t\}$ be an independent set of vertices in G such that $\deg_G(u_i)=d_i$ for any $i=1,2,\ldots,t$. If $\sum_{i=1}^{t} d_i \leqslant (q-1)/2$ then*

$$\operatorname{tes}(G)\leqslant q-\sum_{i=1}^{t} d_i.$$

Proof. Let $A_i=\{e_1,e_2,\ldots,e_{d_i}\}$ be the set of edges incident with the vertex u_i, $i=1,2,\ldots,t$. Set $A_0=E-\bigcup_{i=1}^{t} A_i$. The following labelling ∂ is an edge irregular total labelling to the set $\{1,2,\ldots,r\}$, where $r=q-\sum_{i=1}^{t} d_i$. We put $\partial(v)=1$ for every $v\in V-I$,

$$\partial(u_i)=1+d_1+d_2+\cdots+d_i \quad \text{for every } i=1,2,\ldots,t.$$

The edges from the set A_0 are labelled consecutively with labels $1,2,\ldots,r$. The edges $\{e_1,e_2,\ldots,e_{d_i}\}$ from the set A_i get labels $r-d_i+1, r-d_i+2,\ldots,r$.

It is easy to see that the largest integer used in our labelling ∂ is $r=q-\sum_{i=1}^{t} d_i$.

It is a routine matter to verify that each edge of G has a distinct weight. In fact, our labelling has been chosen in such a way that the resulting edge weights form a consecutive sequence of integers from 3 to $q+2$. $\square$

Let ∂ be an edge irregular total labelling of a graph G, and let H be a subgraph of G. The restriction of ∂ to H is also an edge irregular total labelling of H. This means that $\operatorname{tes}(H)\leqslant\operatorname{tes}(G)$. As a consequence of this inequality we have the following:

Lemma 6. *Let $G=(V,E)$ be a graph with $|V|=p$. Then*

$$\operatorname{tes}(G)\leqslant\operatorname{tes}(K_p).$$

Lemma 6 points out that it is very important to know the exact value of $\operatorname{tes}(K_p)$ for any $p\geqslant 2$. We know that

$$\left\lceil \frac{p^2-p+4}{6} \right\rceil \leqslant \operatorname{tes}(K_p) \leqslant \frac{(p-1)(p-2)}{2}. \tag{3}$$

The left inequality is from Theorem 1, the right side one is a consequence of Theorem 3(ii).

From Fig. 1 we know that $\operatorname{tes}(K_3)=2$. The total edge irregularity strength of K_4 equals 3 as can be seen from (3) and from the following matrix $\mathbf{L}(4)$ expressing a suitable labelling ∂:

$$\mathbf{L}(4)=\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \\ 2 & 1 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}.$$

For K_5 the inequality (3) gives $\operatorname{tes}(K_5)\geqslant 4$. However, we have:

Theorem 7. $\operatorname{tes}(K_5)=5$.

Proof. The upper bound 5 is given by the labelling ∂ expressed by ∂-matrix of K_5 below

$$\mathbf{L}(5)=\begin{bmatrix} 1 & 1 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 2 & 5 & 5 \\ 4 & 1 & 5 & 4 & 5 \\ 4 & 1 & 5 & 5 & 3 \end{bmatrix}.$$

To prove that $\text{tes}(K_5) \geqslant 5$, let us proceed by contradiction. Suppose $\text{tes}(K_5)=4$. Then the edges of K_5 must have weights $3, 4, 5, \ldots, 11, 12$. The existence of an edge $e = ab$ with $wt(e) = 12$ forces $\partial(e) = \partial(a) = \partial(b) = 4$. Also, in K_5, there must be an edge $f = xy$ with $wt(f) = 3$ and $\partial(f) = \partial(x) = \partial(y) = 1$. Because the edges ax, ay, bx, and by have one vertex labelled with label 1 and the second vertex labelled with label 4 their labels must be distinct. As $\text{tes}(K_5) = 4$ we have $\{\partial(ax), \partial(ay), \partial(bx), \partial(by)\}=\{1, 2, 3, 4\}$ and so $\{wt(ax), wt(ay), wt(bx), wt(by)\}=\{6, 7, 8, 9\}$. The remaining four edges must receive the weights 4, 5, 10 and 11. The fifth vertex of K_5 cannot be labelled with a label larger than 2, otherwise the weight 4 cannot be attained and it cannot be labelled with a label less than 3, otherwise the weight 11 cannot be attained. This produces a contradiction. □

We believe that the following holds

Conjecture 1. $\text{tes}(K_p) = \lceil (p^2 - p + 4)/6 \rceil$ for any $p \geqslant 6$.

We have verified this conjecture for $6 \leqslant p \leqslant 20$.

The next two theorems were motivated by our desire to explore labellings of graphs with a vertex of maximum degree beyond the constraints of Theorem 3(ii).

First we present the total edge irregularity strength of W_n, the wheel with $n + 1$ vertices.

Theorem 8. $\text{tes}(W_n) = \lceil (2n + 2)/3 \rceil$ *for* $n \geqslant 3$.

Proof. Let W_n be the wheel with $V(W_n) = \{v\} \cup \{v_i : 1 \leqslant i \leqslant n\}$ and $E(W_n) = \{vv_i : 1 \leqslant i \leqslant n\} \cup \{v_i v_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{v_n v_1\}$. That $\lceil (2n + 2)/3 \rceil$ is a lower bound for $\text{tes}(W_n)$ follows from Theorem 1. To show that $\lceil (2n + 2)/3 \rceil$ is an upper bound for $\text{tes}(W_n)$ we describe a total $\lceil (2n + 2)/3 \rceil$-labelling for W_n.

For $n \geqslant 6$ we construct the function φ as follows:

$$\varphi(v) = \left\lceil \frac{2n+2}{3} \right\rceil,$$

$$\varphi(v_i) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant \left\lceil \frac{2n+2}{3} \right\rceil - 2, \\ \left\lceil \frac{2n+2}{3} \right\rceil - 2 & \text{if } i = \left\lceil \frac{2n+2}{3} \right\rceil - 1 \text{ and } i = n, \\ \left\lceil \frac{2n+2}{3} \right\rceil & \text{if } \left\lceil \frac{2n+2}{3} \right\rceil \leqslant i \leqslant n-1, \end{cases}$$

$$\varphi(v_i v_{i+1}) = \begin{cases} i & \text{if } 1 \leqslant i \leqslant \left\lceil \frac{2n+2}{3} \right\rceil - 3, \\ 1 & \text{if } i = \left\lceil \frac{2n+2}{3} \right\rceil - 2, \\ 5 & \text{if } i = \left\lceil \frac{2n+2}{3} \right\rceil - 1, \\ 2n+2 - \left\lceil \frac{2n+2}{3} \right\rceil - i & \text{if } \left\lceil \frac{2n+2}{3} \right\rceil \leqslant i \leqslant n-2, \\ 4 & \text{if } i = n-1, \\ 2 & \text{if } i = n, \end{cases}$$

$$\varphi(vv_i) = \begin{cases} i & \text{if } 1 \leqslant i \leqslant \left\lceil \frac{2n+2}{3} \right\rceil - 2, \\ 2 & \text{if } i = \left\lceil \frac{2n+2}{3} \right\rceil - 1, \\ i - \left\lceil \frac{2n+2}{3} \right\rceil + 4 & \text{if } \left\lceil \frac{2n+2}{3} \right\rceil \leqslant i \leqslant n-1, \\ 3 & \text{if } i = n. \end{cases}$$

Observe that

$$wt(v_iv_{i+1}) = \varphi(v_i) + \varphi(v_{i+1}) + \varphi(v_iv_{i+1})$$
$$= \begin{cases} i+2 & \text{for } 1 \leqslant i \leqslant \left\lceil \frac{2n+2}{3} \right\rceil - 2, \\ \left\lceil \frac{2n+2}{3} \right\rceil + 1 & \text{for } i = n, \\ 2\left\lceil \frac{2n+2}{3} \right\rceil + 2 & \text{for } i = n-1, \\ 2\left\lceil \frac{2n+2}{3} \right\rceil + 3 & \text{for } i = \left\lceil \frac{2n+2}{3} \right\rceil - 1, \\ 2n+2+\left\lceil \frac{2n+2}{3} \right\rceil - i & \text{for } \left\lceil \frac{2n+2}{3} \right\rceil \leqslant i \leqslant n-2 \end{cases}$$

and

$$wt(vv_i) = \varphi(v) + \varphi(v_i) + \varphi(vv_i) = \begin{cases} \left\lceil \frac{2n+2}{3} \right\rceil + 1 + i & \text{for } 1 \leqslant i \leqslant \left\lceil \frac{2n+2}{3} \right\rceil - 1, \\ 2\left\lceil \frac{2n+2}{3} \right\rceil + 1 & \text{for } i = n, \\ i + \left\lceil \frac{2n+2}{3} \right\rceil + 4 & \text{for } \left\lceil \frac{2n+2}{3} \right\rceil \leqslant i \leqslant n-1. \end{cases}$$

So the weights of edges of W_n under the labelling φ constitute the set $\{3, 4, \dots, 2n+2\}$ and the function φ is a map from

$$V(W_n) \cup E(W_n) \quad \text{into} \quad \left\{1, 2, \dots, \left\lceil \frac{2n+2}{3} \right\rceil\right\}.$$

To take care of W_n, $3 \leqslant n < 6$, we give the following special labellings:

For W_3: $\varphi(v) = 3$, $\varphi(v_1) = \varphi(v_2) = 1$, $\varphi(v_3) = 2$ $\varphi(vv_1) = 2$, $\varphi(vv_2) = \varphi(vv_3) = 3$, $\varphi(v_1v_2) = \varphi(v_2v_3) = 1$ and $\varphi(v_1v_3) = 2$.

For W_4: $\varphi(v) = 4$, $\varphi(v_1) = \varphi(v_2) = 1$, $\varphi(v_3) = 2$, $\varphi(v_4) = 3$ $\varphi(vv_1) = \varphi(v_1v_2) = \varphi(v_1v_4) = \varphi(v_2v_3) = 1$, $\varphi(vv_2) = 2$, $\varphi(vv_3) = \varphi(vv_4) = \varphi(v_3v_4) = 3$.

For W_5: $\varphi(v) = \varphi(v_4) = 4$, $\varphi(v_1) = \varphi(v_2) = 1$, $\varphi(v_3) = 2$, $\varphi(v_5) = 3$, $\varphi(vv_1) = \varphi(v_1v_2) = \varphi(v_1v_5) = \varphi(v_2v_3) = 1$, $\varphi(vv_2) = \varphi(vv_3) = \varphi(vv_5) = 2$, $\varphi(vv_4) = \varphi(v_3v_4) = \varphi(v_4v_5) = 4$.

It is easy to see that these total labellings have the required properties.

This concludes the proof. □

Next we determine the total edge irregular strength for friendship graphs.

The *friendship graph* F_n may be visualised as n triangles sharing a common vertex (but otherwise independent). Alternatively, F_n may be considered as an even wheel W_{2n} with every alternate rim edge missing. This second conceptualisation justifies us referring to the edges adjacent to the vertex of maximum degree as *spokes* and the remainder of the edges as *rim edges*. Note that $|V(F_n)| = 2n + 1$ and $|E(F_n)| = 3n$.

Theorem 9. $\text{tes}(F_n) = \lceil (3n + 2)/3 \rceil$.

Proof. For the friendship graph F_n our aim is to allocate edge weights from the set $\{3, \dots, 3n + 2\}$ so that each edge receives a distinct weight. Ideally, we would wish to ensure that the largest label is $M = \lceil (3n + 2)/3 \rceil$, the minimum largest label by Theorem 1.

Case 1: $n \equiv 1 \pmod 2$. Label the vertex x of degree $\Delta = 2n$ with $c = (n + 1)/2$. Choose a vertex v_1 and without loss of generality number v_2, v_3 etc clockwise, with $v_1v_2 \in E(F_n)$. Let s_i represent the spoke xv_i and r_i the rim edge

$v_i v_{i+1}$. Note that the subscript of r is always odd. Define a labelling ϕ for the first $(n-1)/2$ triangles as

$$\phi(r_i) = \begin{cases} \phi(v_i)+1, & i \equiv 0 \pmod 3,\\ \phi(v_i), & i \equiv 1 \pmod 3,\\ \phi(v_i)-1, & i \equiv 2 \pmod 3, \end{cases}$$

$$\phi(s_i) = i - \phi(v_i) + 1,$$

where $1 \leqslant i \leqslant n-1$, so we have labelled $(n-1)/2$ complete triangles and allocated weights from $\{3, \ldots, (3n+1)/2\}$ such that each edge has a distinct weight.

For the second part of the labelling we now allocate the largest edge weight, $3n+2$ to a rim edge so that the maximum label is as small as possible. Allocate $\phi(v_{2n}) = n+1 = \phi(v_{2n-1})$ and $\phi(r_{2n-1}) = n$. If $n+1$ is to be the largest label, then the largest spoke weight can be no greater than $(5n+5)/2$. Next we need to allocate weights from the set $\{(5n+7)/2, \ldots, 3n+2\}$ to rim edges. Note that the cardinality of this set is $(n-1)/2$. We may then assign labels to a further $n-1$ spokes so as to allocate weights from the set $\{(3n+9)/2, \ldots, (5n+5)/2\}$. For a separate set of $(n-1)/2$ triangles define the labelling ϕ as

$$\phi(v_i) = \left\lceil \frac{4n+i+2}{6} \right\rceil, \quad n+2 \leqslant i \leqslant 2n,$$

$$\phi(r_i) = \begin{cases} \phi(v_i)+1, & (n+i) \equiv 0 \pmod 3,\\ \phi(v_i), & (n+i) \equiv 1 \pmod 3,\\ \phi(v_i)-1, & (n+i) \equiv 2 \pmod 3, \end{cases}$$

$$\phi(s_i) = i + 2 - \phi(v_i).$$

We have now labelled $n-1$ complete triangles and allocated all weights from the given set except for the weights $(3n+3)/2$, $(3n+5)/2$ and $(3n+7)/2$. To complete the labelling we label the remaining two vertices and rim edge all with the value c and the two remaining spokes with $c+1$ and $c+2$. This completes our labelling.

Case 2: $n \equiv 0 \pmod 2$. Label the vertex of maximum degree with $c = n/2$. The rest of the labelling follows the labelling for Case 1 except that the first part labels only $(n/2)-1$ complete triangles. The second part of the labelling and the remaining triangle are labelled as in Case 1. □

Note 1. The problem of finding the total edge irregularity strength for other classes of graphs such as complete graphs, complete bipartite graphs, complete multipartite graphs, trees and regular graphs remains open.

3. Vertex irregular total labellings

It is easy to see that irregularity strength $s(G)$ of a graph G is defined only for graphs containing at most one isolated vertex and no connected component of order 2. On the other hand, the total vertex irregularity strength $\mathrm{tvs}(G)$ is defined for every graph G. Our first result in this section is

Theorem 10. *Let G be a graph with no component of order $\leqslant 2$. Then*

$$\mathrm{tvs}(G) \leqslant s(G). \tag{4}$$

Proof. Let φ be an edge labelling providing the irregularity strength $s(G)$, $\varphi : E \to \{1, 2, \ldots, s(G)\}$. If we extend this labelling to the vertex set $V(G)$ of G using $\varphi(v) = 1$ for every $v \in V(G)$, we obtain a vertex irregular total labelling of G. □

Nierhoff [12] recently proved that for all graphs G with no component of order at most 2 and $G \neq K_3$, the irregularity strength $s(G)$ of G is at most $p-1$. Using this result and (4) we obtain

Corollary 11. *Let G be a graph with no component of order $\leqslant 2$, $G \neq K_3$. Then* $\mathrm{tvs}(G) \leqslant p-1$.

Theorem 12. *Let T be a tree with n pendant vertices and no vertex of degree* 2. *Then*

$$\left\lceil \frac{n+1}{2} \right\rceil \leqslant \operatorname{tvs}(T) \leqslant n.$$

Proof. Amar and Togni [1] established that the irregularity strength $s(T)$ of any tree with no vertex of degree 2 is equal to its number of pendant vertices. This provides an upper bound. To prove the lower bound consider the weights of the pendant vertices. The smallest weight among them is at least two and the largest weight has value at least $n+1$. Since the weight of any pendant vertex is the sum of two positive integers, the proof is complete. □

The sharpness of the lower bound in Theorem 12 is given by

Lemma 13. *Let $K_{1,n}$ be a star with n pendant vertices then*

$$\operatorname{tvs}(K_{1,n}) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. It is enough to describe a suitable vertex irregular total labelling. Let x be the central vertex of the star and let $e_1, e_2, \ldots, e_n$ be edges and $v_1, v_2, \ldots, v_n$ the pendant vertices. The following total labelling ∂ fulfills our requirements:

$$\partial(x) = 1, \quad \partial(v_i) = \left\lceil \frac{i+1}{2} \right\rceil \quad \text{for every vertex } v_i, \ i = 1, 2, \ldots, n,$$

and $\partial(e_i) = \lfloor (i+1)/2 \rfloor$ for every edge e_i, $i = 1, 2, \ldots, n$. □

Let K_p denote the complete graph of order p. In [2] it was shown that $s(K_p) = 3$ for every p. Here, we have

Theorem 14. *For $p \geqslant 2$ we have*

$$\operatorname{tvs}(K_p) = 2.$$

Proof. Trivially, $\operatorname{tvs}(K_p) \geqslant 2$ for any $p \geqslant 2$. For the converse, we define a suitable vertex irregular total labelling as follows. Let $V(K_p) = \{v_1, v_2, \ldots, v_p\}$. We define

$$\varphi(v_i) = 1 \quad \text{for } 1 \leqslant i \leqslant \left\lceil \frac{p}{2} \right\rceil$$

and

$$\varphi(v_i) = 2 \quad \text{for } \left\lceil \frac{p}{2} \right\rceil < i \leqslant p$$

and for every i, $1 \leqslant i \leqslant p$, we define

$$\varphi(v_i v_j) = 1 \quad \text{for } 1 \leqslant j \leqslant p - i + 1, \ \ i \neq j$$

and

$$\varphi(v_i v_j) = 2 \quad \text{for } p - i + 2 \leqslant j \leqslant p.$$

To see that this total labelling φ is irregular on the vertices of K_p, consider the corresponding φ-matrix $\mathbf{L}(K_p)$. It is easy to see that the sum of entries of $\mathbf{L}(K_p)$ in the row corresponding to the vertex v_k gives the weight

$$wt(v_k) = p + k - 1 \quad \text{for any } k = 1, \ldots, p. \qquad \square$$

The following simple observation is very useful. We formulate it as

Lemma 15. *Let φ be a vertex irregular total labelling of a graph G and let wt be the corresponding weight function on $V(G)$, the vertex set of G. Let $e = xy$ be an edge of G. Then the weights $wt(x)$ and $wt(y)$ are not changed if φ is transformed into φ^+ or φ^-, respectively, where*

(i) $\varphi^+(v) = \varphi(v)$ *for* $x \neq v \neq y$,
$\varphi^+(x) = \varphi(x) + 1$, $\varphi^+(y) = \varphi(y) + 1$,
$\varphi^+(f) = \varphi(f)$ *for* $f \neq e$ *and*
$\varphi^+(e) = \varphi(e) - 1$.

(ii) $\varphi^-(v) = \varphi(v)$ *for* $x \neq v \neq y$,
$\varphi^-(x) = \varphi(x) - 1$, $\varphi^-(y) = \varphi(y) - 1$ *provided that* $\varphi(x), \varphi(y) \geqslant 2$,
$\varphi^-(f) = \varphi(f)$ *for* $f \neq e$ *and*
$\varphi^-(e) = \varphi(e) + 1$.

Note that in Lemma 15 the case when $\phi^+(e) = 0$ corresponds to the deletion of the edge e.

Theorem 16. *Let G be a (p, q)-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \leqslant \mathrm{tvs}(G) \leqslant p + \Delta - 2\delta + 1.$$

Proof. *Lower bound*: The largest value among the weights of vertices of G is at least $p + \delta$ and this weight is the sum of at most $\Delta + 1$ integers. Hence the largest label contributing to this weight must be at least $\lceil (p + \delta)/(\Delta + 1) \rceil$.

Upper bound: Consider K_p with the optimal total labelling φ from the proof of Theorem 14 providing $\mathrm{tvs}(K_p) = 2$. Choose a subgraph of K_p isomorphic with G and delete from K_p all edges not belonging to G in agreement with Lemma 15. This means that if an edge $e = xy$ is deleted and its label is $\varphi(e) = k$ ($k = 1$ or 2 because $\mathrm{tvs}(K_p) = 2$) then in the new labelling $\varphi^+(x) = \varphi(x) + k$ and $\varphi^+(y) = \varphi(y) + k$. After the deletion of all the edges of $\overline{G}$, the complement of G, the new label $\partial(v)$ of any vertex of degree d is

$$p - d \leqslant \partial(v) \leqslant 2p - 2d.$$

Hence the labels ∂ of vertices of G are at least $p - \Delta$ and at most $2p - 2\delta$. Because of Lemma 15, the weights of vertices of G are the same as in K_p and they are mutually distinct. Now the label $\partial(v)$ of every vertex is decreased by $p - \Delta - 1$. The new vertex irregular total labelling ψ is obtained with the property that

$$1 \leqslant \psi(v) \leqslant 2p - 2\delta - (p - \Delta - 1) = p + \Delta - 2\delta + 1 \quad \text{for any } v \in V(G)$$

and

$$\psi(e) \leqslant 2 \quad \text{for every edge } e \in E(G).$$

The new weight $\overline{wt}(v) = wt(v) - p + \Delta + 1 \geqslant p - p + \Delta + 1 = \Delta + 1$ and, clearly, $\overline{wt}(x) \neq \overline{wt}(y)$ whenever $x \neq y$. □

Corollary 17. *Let G be an r-regular (p, q)-graph. Then*

$$\left\lceil \frac{p+r}{1+r} \right\rceil \leqslant \mathrm{tvs}(G) \leqslant p - r + 1.$$

By using the labelling from Theorem 2, one can easily see that

$$\mathrm{tvs}(C_p) = \mathrm{tes}(C_p) = \left\lceil \frac{p+2}{3} \right\rceil.$$

This observation can be used in the proof of:

Theorem 18. *Let G be a regular hamiltonian (p, q)-graph. Then*

$$\text{tvs}(G) \leqslant \left\lceil \frac{p+2}{3} \right\rceil.$$

Proof. Label totally the hamiltonian cycle of G as in Theorem 2, using labels from the set $\{1, 2, \ldots, \lceil (p+2)/3 \rceil\}$; the remaining edges of G are labelled with label 1. This yields $\text{tvs}(G) \leqslant \lceil (p+2)/3 \rceil$. □

Theorem 19. *Let G be a (p, q)-graph with maximum degree $\Delta = \Delta(G)$ and no component of order $\leqslant 2$. Then*

$$\text{tvs}(G) \leqslant p - 1 - \left\lfloor \frac{p-2}{\Delta+1} \right\rfloor.$$

Proof. If G satisfies the conditions of the Theorem and $G \neq K_3$ then, by Nierhoff [12], G has the irregularity strength $s(G) \leqslant p-1$. This means that there is an edge labelling φ such that for each edge e of $G \varphi(e) \leqslant p-1$. We extend this labelling to a vertex irregular total labelling by setting $\varphi(v) = 1$ for every vertex $v \in V(G)$.

Put $t = \lfloor (p-2)/(\Delta+1) \rfloor$. Lemma 15(i) is now applied t times to each edge $e = xy$ of G with $\varphi(e) > t$. Then the labels of x and y increase by t. Therefore, applying Lemma 15(i) we get new vertex irregular total labelling ∂ with the following properties:

$$\partial(e) = \varphi(e) - t \leqslant p - 1 - t = p - 1 - \left\lfloor \frac{p-2}{\Delta+1} \right\rfloor \quad \text{for every edge } e \in E(G)$$

with

$$\varphi(e) > t, \quad \partial(e) = \varphi(e) \text{ for all other edges,}$$

and for every vertex $v \in V(G)$ we have

$$\partial(v) \leqslant \varphi(v) + t \deg_G(v) \leqslant \varphi(v) + t\Delta = 1 + t\Delta \leqslant p - 1 - t$$

because $t(\Delta+1) \leqslant p-2$. This gives

$$\partial(v) \leqslant p - 1 - \left\lfloor \frac{p-2}{\Delta+1} \right\rfloor$$

and we are done. □

Note that for $G = K_3$ $\text{tvs}(K_3) = 2 = p - 1 - \lfloor (3-2)/(2+1) \rfloor$ by Theorem 14.

It is a natural desire to seek the exact value of any graph characteristic for every member of some family of graphs. In the next part we investigate the total vertex irregular strength of the n-sided prism $D_n, n \geqslant 3$.

The prism $D_n, n \geqslant 3$, is a trivalent graph which can be defined as the cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on n vertices. D_n consists of an outer n-cycle $y_1 y_2 \cdots y_n$, an inner n-cycle $x_1 x_2 \cdots x_n$ and a set of n spokes $x_i y_i$, $i = 1, 2, \ldots, n$.

Theorem 20. *For* $n \geqslant 3$ $\text{tvs}(D_n) = \lceil (2n+3)/4 \rceil$.

Proof. We label edges of $D_n, n \geqslant 3$, in the following way:

(i) each edge of the inner n-cycle receives the label 1,
(ii) if n is even then each edge of the outer n-cycle receives the label $\lceil (2n+3)/4 \rceil$,
(iii) if n is odd then the edge $y_1 y_2$ receives the label 1 and the other edges of the outer n-cycle receive the label $\lceil (2n+3)/4 \rceil$,
(iv) spokes $x_i y_i$ receive the label 1, for $1 \leqslant i \leqslant \lceil (2n+3)/4 \rceil$, and the labels $\lceil (2n+3)/4 \rceil$, for $\lceil (2n+3)/4 \rceil + 1 \leqslant i \leqslant n$.

The weights of vertices x_i, $1 \leqslant i \leqslant n$, successively attain values $4, 5, 6, \ldots, n+3$ if all vertices x_i, $1 \leqslant i \leqslant \lceil (2n+3)/4 \rceil$, receive distinct labels from the set $\{1, 2, 3, \ldots, \lceil (2n+3)/4 \rceil\}$ and all vertices x_i, $\lceil (2n+3)/4 \rceil + 1 \leqslant i \leqslant n$, receive distinct labels from the set $\{2, 3, 4, \ldots, n - \lceil (2n+3)/4 \rceil + 1\}$.

If n is even and each vertex y_i, $1 \leqslant i \leqslant n$, receives the same label as vertex x_i then the weights of vertices y_i, $1 \leqslant i \leqslant n$, constitute the set of consecutive integers $\{n+4, n+5, \ldots, 2n+3\}$. If n is odd and

$$\varphi(y_i) = \begin{cases} \left\lceil \frac{2n+3}{4} \right\rceil - 2 + i & \text{for } 1 \leqslant i \leqslant 2, \\ i - 1 & \text{for } 3 \leqslant i \leqslant \left\lceil \frac{2n+3}{4} \right\rceil, \\ i - \left\lceil \frac{2n+3}{4} \right\rceil & \text{for } \left\lceil \frac{2n+3}{4} \right\rceil + 1 \leqslant i \leqslant n \end{cases}$$

then the weights of vertices y_i are $n + 3 + i$, $1 \leqslant i \leqslant n$.

This labelling provides the upper bound on $\mathrm{tvs}(D_n)$. The sharpness of this bound is by Theorem 16. □

Note 2. In the previous parts we determined the total vertex irregularity strength for complete graphs, cycles, stars, paths and prisms. It would be interesting to know the exact value of this parameter for other families of graphs e.g. complete bipartite graphs, regular graphs, n-cubes.

Acknowledgement

The second author acknowledges the support by The University of Newcastle's RMC Visitor Grant VR53806. Support of the Slovak VEGA Grant 1/0424/03 is also acknowledged. The authors wish to thank the referees for their thoughtful suggestions.

References

[1] D. Amar, O. Togni, Irregularity strength of trees, Discrete Math. 190 (1998) 15–38.
[2] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba, Irregular networks, Congr. Numer. 64 (1988) 187–192.
[3] J.H. Dimitz, D.K. Garnick, A. Gyárfás, On the irregularity strength of the $m \times n$ grid, J. Graph Theory 16 (1992) 355–374.
[4] R.J. Faudree, M.S. Jacobson, J. Lehel, R.H. Schlep, Irregular networks, regular graphs and integer matrices with distinct row and column sums, Discrete Math. 76 (1988) 223–240.
[5] R.J. Faudree, J. Lehel, Bounds on the irregularity strength of regular graphs, in: Combinatorics (Eger, (1987)) Colloquia Mathematica Societatis János Bobyai, vol. 52, North-Holland, Amsterdam, 1988, pp. 247–256.
[6] J.A. Gallian, Graph labeling, Electron. J. Combin. 5 (1998) (Dynamic Survey DS6).
[7] A. Gyárfás, The irregularity strength of $K_{m,m}$ is 4 for odd m, Discrete Math. 71 (1998) 273–274.
[8] M.S. Jacobson, J. Lehel, Degree irregularity, available online at ⟨http://athena.louisville.edu/msjaco01/irregbib.html⟩.
[9] S. Jendrol', M. Tkáč, The irregularity strength of tK_p, Discrete Math. 145 (1995) 301–305.
[10] S. Jendrol', M. Tkáč, Z. Tuza, The irregularity strength and cost of the union of cliques, Discrete Math. 150 (1996) 179–186.
[11] J. Lehel, Facts and quests on degree irregular assignment, Graph Theory, Combin. Appl., Wiley, New York, 1991, pp. 765–782.
[12] T. Nierhoff, A tight bound on the irregularity strength of graphs, SIAM J. Discrete Math. 13 (2000) 313–323.
[13] W.D. Wallis, Magic Graphs, Birkhäuser, Boston, 2001.
[14] D.R. Wood, On pseudo vertex-magic and edge-magic total labelings, in: E.T. Baskoro (Ed.), Proceedings of the Twelfth Australasian Workshop on Combinatorial Algorithms, ITB Bandung, July 14–17, 2001, pp. 180–198.