Worcester Polytechnic Institute
Digital WPI

Masters Theses (All Theses, All Years)   Electronic Theses and Dissertations

2008-02-14

Pseudo-Triangulations On Closed Surfaces

John R. Potter
Worcester Polytechnic Institute

Follow this and additional works at: https://digitalcommons.wpi.edu/etd-theses

Pseudo-Triangulations on Closed Surfaces

by

John Potter

A Thesis

submitted to the Faculty of the

WORCESTER POLYTECHNIC INSTITUTE

in partial fulfillment of the requirements for the

Degree of Master of Science in

Applied Mathematics

February 2008

APPROVED:

Brigitte Servatius, Major Advisor

Abstract

Shifting attention from the plane to the sphere and torus, we extend the study of pseudo-triangulations. Planar representations of each surface are used. A number of theorems and concepts are taken from the plane and applied to the sphere and torus not only for pseudo-triangulations but for triangulations as well. We found the number of edges and faces in a triangulation on n vertices in the plane, on the sphere and on the torus.

Contents

List of Figures

1

Chapter 1

Introduction

We define a *graph G* to be composed of a non-empty set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, where the relation we call *incidence* associates each edge with two vertices [2]. The *degree* of a vertex is the number of edges incident to it. A *path* is an alternating sequence of vertices and edges that begins and ends with a vertex and where no vertex or edge appear in the sequence more than once. A special case of a path is a *cycle*, where the beginning and ending vertex are the same. It is important to note that in this paper any graph described in the plane will be assumed to be a *simple* graph, that is there are no loops or multiple edges, whereas graphs on the torus will be allowed to have loops and multiple edges. By *loops* we mean an edge whose two ends are both incident to the same vertex, and by *multiple edges* we mean two or more edges that are incident to the same pair of vertices.[2] Though we allow loops and multiple edges on the sphere and torus, we apply a restriction to their use. We require that any loops or multiple edges be *essential*, by which we mean that their inclusion should split the sphere or torus into hemispheres or a cylinder, respectively, if we cut along the loop or multiple edges. This is an extended definition of essential taken from [11].

We are only concerned with graphs that are *cellular embeddings* on a given surface, that is, the graph must satisfy the Euler characteristic for the surface on which it is embedded. The general form of the Euler characteristic is

$$\chi = v - e + f$$

where v, e and f are the number of vertices, edges and faces of the embedded graph, respectively. For graphs in the plane and on the sphere $\chi = 2$, while on the torus we have $\chi = 0$. The number of vertices and edges are easy to determine

in finite graphs, but it is important to note that the number of faces in the plane includes the unbounded face which extends infinitely in all directions.

The embeddings of graphs studied here, in addition to being cellular, should also be straight line embeddings. A *straight line embedding* is simply an embedding of a graph such that every edge can be drawn as a straight line segment between its two vertices. In [4] Fáry demonstrated that given a simple planar graph, in the plane, there exists an embedding such that no edges cross and every edge is straight line segment in the plane. Notice that Fáry not only shows the existence of a straight line embedding, but also that it can be done so that no edges cross. A graph that can be embedded in this way is called a *planar* graph. We will use this when embedding graphs in the plane, and extend it for use on the sphere and torus, and in doing so it is clear that loops and multiple edges are admissible so long as they are essential.

1.1   Surfaces

In dealing with three different surfaces, the plane, sphere and torus, it becomes necessary to fully understand the similarities and differences between them. We will view the plane to be the Euclidean plane, consisting of vertices and straight lines. Moreover, we will only be concerned with the line segments connecting vertices and disregard the infinite lines that are often used in the Euclidean plane. Embeddings of graphs onto the plane will keep the vertices and edges and will add faces, also known as *cycles* or *2-cells*, whose boundaries are determined by the vertices and edges.

The sphere we will consider, for ease of discussion, is defined as the set of points equidistant from an origin point. This sphere is a smooth and regular surface, there are no waves or bumps on the surface nor any stretching or pinching of any collection of points on the surface, and if there were the surface would still be considered a sphere in the topological sense. Again, graphs will consist of vertices and edges, though on the sphere the embedding of edges are curves, and just as in the plane, there will be the addition of faces.

When describing a torus, one can think of it as a circle of radius r that is rotated 2π radians about a line that is coplanar to the circle and a distance R from the center of the circle, as in Figure1.1. In Cartesian coordinates, the torus can be represented by the explicit equation
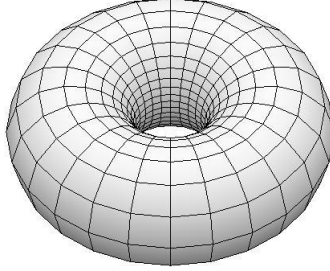
$$\left(R + \sqrt{x^2 + y^2}\right)^2 = r^2 \tag{1.1}$$

Figure 1.1: A Model of the Torus

or by the parametric equations

$$x = (R + r\cos(v))\cos(u) \tag{1.2}$$
$$y = (R + r\cos(v))\sin(u) \tag{1.3}$$
$$z = r\sin(v) \tag{1.4}$$

where u,v are bound by $[0, 2\pi)$. For our purposes, torus will mean a ring torus, that is $R > r$, as opposed to a horn torus where $R = r$ and the central hole is reduced to a point or a spindle torus where $R < r$ and the surface has self-intersections. There are a number of alternative ways to represent the torus, but for simplicity of discussion we will use the torus as we have already described it. When graphs are embedded on the torus, we again add faces to the vertices and edges that came from the graph. Similar to graphs embedded on the sphere, edges are actually curves along the surface.

1.2 Types of Pseudo-Triangulations

A *pseudo-triangle* in the plane is a cycle that is embedded so that exactly three of the interior angles are smaller than π radians, and we find interior angles by measuring the arc between two consectutive edges of the cycle incident to the same vertex so that the arc is contained within the cycle. If the cycle has only three interior angles, then all three must be smaller than π radians. This special case, or degenerate form, of a pseudo-triangle is known simply as a *triangle*. Using pseudo-triangles as a base, the concept can be extended in the following manner.

A *pseudo-triangulation* in the plane is a geometric embedding of an entire graph such that each face, except possibly the unbounded one, is a pseudo-triangle.

The measure of angles in the embedding of a graph are important for assuring that the embedding is in fact a pseudo-triangulation, so we take special notice of embeddings where every vertex has exactly one angle that is at least π radians. This type of pseudo-triangulation in the plane is known as a *pointed pseudo-triangulation*. In a more abstract sense, we can take a graph and give each vertex a *rotation system* that determines the order of the edges counterclockwise around it and then assign angles between consecutive edges in each rotation system to be either big or small, where big and small refer to being at least as big or smaller than π radians, respectively. If the assignment of the rotation systems and angles is such that each face created is pseudo-triangle, then the graph is called a *combinatorial pseudo-triangulation*. We can then ask whether a combinatorial pseudo-triangulation can be realized on a particular surface. Combining the previous two ideas regarding pseudo-triangulations, we arrive at what are called *combinatorial pointed pseudo-triangulation*, which have the properties of both pointed and combinatorial pseudo-triangulations.

1.3   Relevance of Pseudo-Triangulations

Pseudo-triangulations are a relatively new area of study in mathematics that has been developed and studied over the last 25 years. One particular area of interest that is tied in very closely with pseudo-triangulations is rigidity theory. The connection of these two fields stems from choosing embeddings of pseudo-triangulations where edges are given particular lengths that can neither be stretched nor compressed and vertices are treated like joints that the edges can rotate around freely so long as the rotation of an edge does not cause it to cross another edge or vertex. These specifications on a graph, pseudo-triangulation or not, creates what is often referred to in rigidity theory as a bar framework.

Using terminology from [7], we begin by defining the *motion* of a framework as any deformation that preserves edge lengths but changes the distance between points of the system. A *rigid motion* is a motion that also preserves the distance between points. An example of a rigid motion would be a rotation of π radians about a fixed point of the framework. When using graphs to study rigidity, it can be prudent to study the fewest number of edges needed to embed a graph so that it is rigid. Rigid graphs or frameworks that satisfy this property of minimizing the number of edges used are referred to as being *minimally rigid*. For more information regarding rigidity see [6], [7] and [13].

One of more recent and spectacular connections between rigidity theory and

pseudo-triangulations was the proof that pseudo-triangulations are rigid and pointed pseudo-triangulations are minimally rigid and that the converse is also true: any planar (minimally) rigid graph can be drawn as a (pointed) pseudo-triangulation. This result was first shown in [8] and later generalized in [12].

For a further look at the applications of pseudo-triangulations as well as a deeper look at their history see [8] and [14].

1.4 Previous Work

One of the most comprehensive sources of information pertaining to pseudo-triangulations can be found in [14]. Topics discussed include basic properties of pseudo-triangulations, reciprocal diagrams, rigidity of pseudo-triangulations and polytopes of pseudo-triangulations.

The following theorem comes from [14], we will explore it in later sections by examining changes that occur when moving from the plane to the sphere and torus.

Theorem 1.4.1. *Let P be a point set with v elements. Then, every pseudo-triangulation of P has 2v-3+v_n edges, where v_n is the number of non-pointed vertices in it.*

Notice that when $v_n = 0$ the pseudo-triangulation is pointed. Also, when finding a pseudo-triangulation of a point set, it is often done by first forming the *convex hull*, which is simply the smallest convex polygon made by connecting vertices by edges so that every vertex is a corner of the polygon or contained inside the polygon, equivalently this means that no vertices remain in the unbounded face. Using the convex hull is not necessary, but it is frequently used as it reduces the number of choices that need to be made with regard to the edges that need to be added, and it guarantees that the vertices of the convex hull are pointed.

The following is a theorem from [8].

Theorem 1.4.2. *A planar Laman graph with a given topological embedding and a designated outer face can be realized as a pointed pseudo-triangulation with the same face structure.*

A graph is a *Laman graph* if $e = 2v - 3$ and every subset of $k \geq 2$ vertices spans at most $2k - 3$ edges. A Laman graph is also known as a generically minimally rigid graph.

Chapter 2

Convexity of Embedded Graphs

Convexity is an important concept throughout mathematics, but we will confine ourselves to convexity as it relates to graphs and their faces. The aim here is to demonstrate a popular method introduced by Tutte for embedding graphs in the plane so that they are planar and every face is convex. We then contrast this with pseudo-triangulations, which when embedded create faces that are as nonconvex as possible.

2.1   Tutte's Convex Representations

We begin with a look at the work done by Tutte in [15]. Take note that for clarity, we use common graph theory terminology when possible, rather than the labels used by Tutte. Tutte begins with graphs that have a planar embedding and states that under necessary and sufficient conditions such a graph has an embedding which additionally has all convex faces, except the unbounded one, Tutte refers to this kind of embedding as a *convex representation*. Tutte then introduces a number of terms, which are ultimately necessary for understanding his theorems. All necessary terms are listed here and come from [15].

Given a set U with subsets U_1,...,U_k, not necessarily all distinct, Tutte defines the *mod 2 sum* as the set of all $u \in U$ so that the number of subsets satisfying $u \in U_i$ is odd. A *planar mesh* of a graph G is a set $M = S_1, ..., S_k$ of faces of G, not necessarily all distinct, that satisfy the conditions:

i) If an edge of G belongs to a set S_i then that edge belongs to exactly two sets.

ii) Each non-empty cycle of G can be expressed as a mod 3 sum of members

of M.

We say a graph G is *separable* if it has two proper subgraphs H and K, that share no edges and at most one vertex, that between them contain every edge and vertex of G. Given two disjoint faces C and $C\prime$ of G, let $\lambda(C, C\prime)$ be the smallest integer n so that there exist complementary subsets S and $S\prime$ of $E(G)$ that satisfy the conditions:

i) $C \subseteq S$ and $C\prime \subseteq S\prime$.

ii) The number of vertices incident with edges of both S and $S\prime$ is n.

A graph G is considered *3-linked* with respect to a face C if there is no face $C\prime$ of G so that C and $C\prime$ are disjoint and $\lambda(C, C\prime) \leq 2$. A *vertex of attachment* of a subgraph H of G is a vertex of H that is incident, along with another vertex of G, to an edge of G not found in H. If J is a fixed subgraph of G, then a subgraph H of G is *J-bounded* if all of its vertices of attachment belong to J. Consider the set $\mathbf{J}$ of all subgraphs H of G so that H is J-bounded and not a subgraph of J. A *bridge* of J in G is any member of $\mathbf{J}$ that has has no other member of $\mathbf{J}$ as a subgraph. A *branch* of a graph G is a bridge in G of the edgeless subgraph J satisfying $V(J) = N(G)$, where $N(G)$ is the subset of $V(G)$ containing vertices whose degree are not 2. Given a graph G and one of its faces C, we say that $G \cdot C$ is the subgraph of G where all vertices of C are removed along with any edges incident to them. A subgraph H of G *spans* G if $V(H) = V(G)$.

Having established the necessary definitions, Tutte's first theorem can be summarized as follows:

Theorem 2.1.1. *Let G be a simple graph whose convex representation R has faces C_1,...,C_k with C_1 being the convex hull. Then $\{C_1,...,C_k\}$ is a connected planar mesh of G, G is non-separable and 3-linked with respect to C_1, and no branch of G spanning $G \cdot C_1$ has the same ends as some edge of C_1.*

The second theorem by Tutte then states:

Theorem 2.1.2. *Let M be a planar mesh C_1,...,C_k of a simple graph G. Let G be non-separable and 3-linked with respect to C_1, and let no branch of G spanning $G \cdot C_1$ have the same two ends as any edge of C_1.*

Let P_1 be any $v(G \cdot C_1)$-sided convex polygon in a plane Π, and let h be a 1-1 mapping of $V(G \cdot C_1)$ onto the set of vertices of P_1 which preserves the cyclic order.

Then we can find a 1-1 mapping f of V(G) onto a set of points of Π such that the following conditions are satisfied:

i) $f(x) = h(x)$ *if* $x \in V(G \cdot C_1)$,

ii) f determines a non-singular convex representation R of G, with respect to which G has the convex hull C_1 and the faces $C_1,...,C_k$.

Essentially, this theorem shows that given a planar graph and a choice of a convex hull, one can find an embedding of the graph in the plane so that the convex hull is a convex polygon and every interior face is a convex polygon.

In [16], Tutte takes the existence of convex representations and extends his work to an algorithm that generates a convex representation for a given planar graph. Tutte refers to the convex representations found by his algorithm as *barycentric mappings*. The algorithm Tutte developed begins by selecting a convex hull and creating a regular polygon in the plane. The vertices of the convex hull are set in place and unable to change positions. Masses are then assigned to vertices and the edges are treated as springs. The result is a spring and mass system whose equilibrium creates a convex representation of the given graph.
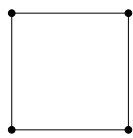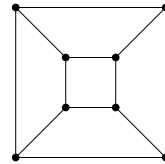
(a) Square   (b) Cube

Figure 2.1: Convex Representations of Some Elementary Graphs

Figures 2.1 and 2.2 show a few examples of planar graphs embedded so that every bounded face is a convex polygon. They do not necessarily reflect the results obtained using Tutte's algorithm, but they still represent the essence of the work done by Tutte to guarantee the convexity of the faces of an embedded planar graph.

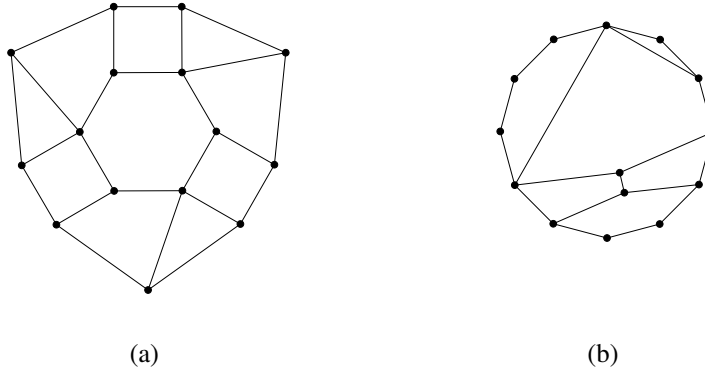(a)   (b)

Figure 2.2: More Complex Convex Representations of Some Graphs

2.2   Convexity of Pseudo-Triangulations

In contrast with Tutte's convex representation of planar graphs, pseudo-triangulations have the property of being as nonconvex as possible. This is due to the fact that each bounded face of a pseudo-triangulation has exactly three small angles, and no polygon can geometrically have fewer than three small angles if every edge is a straight line. Despite this drastic difference in embeddings, it is possible for the same graph to be embedded in the plane using both Tutte's convex representation as well as pseudo-triangulation without changing the number of vertices or edges or their incidences with each other. This is an example of graphs which are isomorphic. Taking the definition from [2], two graphs G and H are *isomorphic* if there are bijections $\theta : V(G) \rightarrow V(H)$ and $\phi : E(G) \rightarrow E(H)$ such that vertex v and edge e are incident in G if and only if vertex $\theta(v)$ and edge $\phi(e)$ are incident in H. The pair of mappings (θ, ϕ) is an *isomorphism* from G to H. The embedded graphs in Figures 2.3 and 2.4 are isomorphic to those in Figures 2.1 and 2.2, the only difference is the embedding as pseudo-triangulations rather than convex representations.
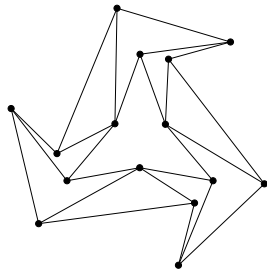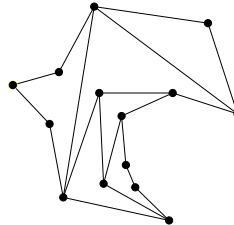
10
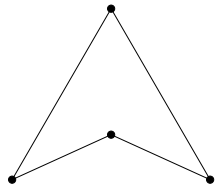
(a) Square
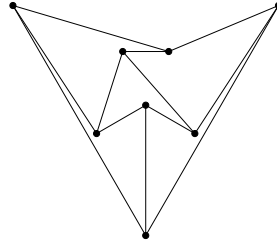
(b) Cube

Figure 2.3: Pseudo-Triangulations of Some Elementary Graphs

(a)

(b)

Figure 2.4: More Complex Pseudo-Triangulations of Some Graphs

Chapter 3

Planar Representations of Closed Surfaces

One on the primary motivations for studying pseudo-triangulations on the sphere and torus is that they have convenient planar representations. When working with the embeddings of graphs, it can be difficult to visualize in more than two dimensions, and even more difficult to create accurate drawings that reflect all the necessary informatoin. It becomes useful then to find simple ways to represent complicated surfaces. It is important though, that we find representations which are *conformal mappings*, that is they must preserve angles, that is angles do not change when mapped from one surface to another. In the case of the torus, we will loosen the idea of a conformal mapping to mean that angles are preserved to the extent of being big or small. The plane is already a planar representation of itself, so here we will determine appropriate planar representations for the sphere and torus.

3.1 The Sphere

Drawing graphs on the sphere can be quite cumbersome. The main cause of difficulty comes from finding edges of the graph, since they clearly can not be straight lines. To that end, we require edges of graphs on the sphere to be *geodesics*, or shortest paths. This requirement implies that edges between vertices on the sphere are arcs of great circles, where a *great circle* is a circle on the sphere whose center is the center of the sphere and an *arc* is a portion of the circumference of a circle.

Another problem we come across when dealing with graphs on the sphere

and torus, is how to define the convex hull. The difficulty in finding the convex hull of a graph on non-planar surfaces is that there is no unbounded face to work with. Clearly this is an issue with the sphere and torus since they are closed, non-planar surfaces. The solution we use here is to not have a defined convex hull, and therefor no unbounded or outside face, when working with one of these surfaces. In fact, it is shown in [3] that when dealing with the torus, it is often the case that the whole torus is the convex hull, and the probability of this increases with the number of vertices in the graph.

Finding a planar representation of the sphere itself is a relatively simple matter. The typical way of representing the sphere on the plane is via the mapping known as stereographic projection. We take the construction of the planar representation from [9] and summarize it here. Beginning with a graph on a sphere, we rest the sphere on a horizontal plane, which creates a single tangent point that we will refer to as the south pole of the sphere. Then, using the point at the north pole, which is furthest from the plane, we can construct our planar representation. To do so, we find the straight line that connects the north pole with a given vertex of our graph and extend the line until it intersects with the plane. This point of intersection is the vertex in the plane representing the vertex from the sphere. Notice, if there is a vertex at the north pole, then it is not mapped onto the plane and is said to be a vertex at infinity in the plane. Edges on the sphere, which are obviously not straight lines, are mapped to straight lines in the plane incident to the vertices mapped from the original edge's incident vertices. If there is a vertex at the north pole incident to one or more edges, then the stereographic projection of one such edge will consist of an infinite edge that begins at a mapped incident vertex in the plane and extends to the infinity point. Similarly, if an edge passes through the north pole, then its stereographic projection will consist of two infinite edge segments drawn from the mapped incident vertices and both extending to the infinity point. After establishing how to perform a stereographic projection, Hilbert and Cohn-Vossen show that stereographic projections are conformal mappings.

In this paper, graphs drawn as planar representations of the sphere will be drawn within a circle, which represents the infinity point. Edges passing through the infinity point will begin at an incident vertex, extend to the circle boundary, and return π radians around the circle, parallel to the first piece of the edge drawn and ending at the other incident vertex. If a vertex is at the infinity point, it will be drawn on the circle only at the points where its incident edges intersect the circle.

3.2 The Torus

We now show that the torus can be represented in the plane using a boundary composed of a parallelogram, or a hexagon, where graph edges can cross these boundaries, and vertices may exist on these boundaries as well. Moreover, we demonstrate that there is a conformal, or angle preserving, mapping from the torus to the plane, in the sense that big and small angles on the torus are mapped to big and small angles in the plane, respectively.

Unlike the sphere, where we have a simple way to describe edges on its surface, the torus is far more complicated, as shown by the study of geodesics on the torus by Irons in [10]. In fact, Irons shows that there are five different types of geodesics that can occur when finding shortest paths on the torus. Because of this difficulty, we will not establish a method for embedding edges on the torus itself and will only concern ourselves with the embedding of edges in the planar representation of the torus.

Given a torus, we use two circles, one vertical and one horizontal, as a guide for where the torus is sliced open, see Figure 3.1.

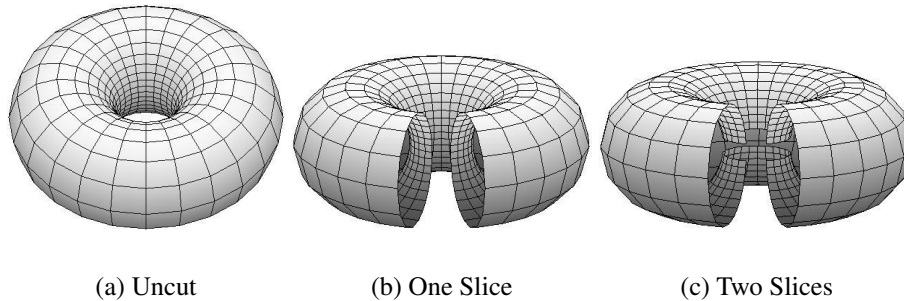(a) Uncut   (b) One Slice   (c) Two Slices

Figure 3.1: Slices of the Torus

Notice that the slices leave the surface with four corners, each connected by a boundary line, and that opposite pairs of boundary lines have equal length. It is natural then to bend our surface into a quadrilateral in the plane, whose opposite boundary lines must be of equal length based on the slicing method used. In addition, the angles at the corners of this quadrilateral must all be $\frac{\pi}{2}$ radians since our slices are made perpendicular to one another. Our quadrilateral must therefore be a rectangle in the plane.

Referring again the work in [9], we find two important constructions. The first is a transformation of a rectangle into a torus, which is done in the exact opposite manner described above. Opposite edges are matched up parallel to each other. Two of the opposite edges are glued together to form a cylinder, which is then bent so that the end circles of the cylinder can be glued together. A second construction, or deconstruction, can be found in a discussion of conformal mappings from one torus to another. The plane is used as an intermediate surface of the mapping. The process requires a universal covering surface of the torus, which is done by first wrapping the Euclidean plane around a cylinder of infinite length. Notice that the cylinder is wrapped an infinite number of times in this process. The cylinder is then bent into the shape of a torus and the infinite cylinder slides into itself infinitely many times along the way. In this way, the Euclidean plane is mapped topologically onto a surface that covers the torus infinitely many times without and folds, hence the term *universal covering surface*. Now, if we apply slices as in our original construction, we create an infinite number of rectangles which can be reassembled into the Euclidean plane, we refer to this assembly as a *tiling* of the plane. Most importantly in this process, is that Hilbert and Cohn-Vossen show that the universal covering surface of the torus can be conformally mapped to the Euclidean plane.

An interesting point is that the torus can also be represented using a tiling of hexagons where a single hexagon can be used as a planar representation of the torus. A major result in [1] is the following theorem which finds a correlation between the two tilings.

Theorem 3.2.1. *A rectangular symmetric tiling T is also hexagonally symmetric if and only if the basic rectangle of T is hexagonally symmetric.*

One method for drawing graphs on the torus can be found in [11]. The motivation for graph drawings here comes from computer generated embeddings of graphs on a planar representation of the torus. The procedure for embedding a graph is a modified version of Read's algorithm, which essentially assigns coordinates to vertices of a planar graph based on a predefined rotation system. Further work along the same lines can be found in [5] where small graphs with special properties are embedded on the torus and studied.

In this paper, we will draw the planar representations of graphs on the torus using a rectangle whose boundary associates opposite sides, and therefore opposite corners as well. Vertices that lie on one of the slices of the torus will be drawn on both sides of opposite boundary edges of the rectangle, and a vertex that lies

on the intersection of the two slices of the torus will be drawn at all four corners of the boundary rectangle. Edges between vertices will be drawn as straght lines. If an edge intersects one side of the rectangle boundary, it also intersects the boundary horizontally or vertically across from the first point of intersection, depending on whether the first intersection point is on the vertical or horizontal edge boundaries, respectively. If an edge intersects a corner, it also intersects the corner diagonal from it in the rectangle. In both cases, the two segments of the edge are parallel.

Chapter 4

Triangulations

Here, we will discuss triangulations in the plane and work towards describing their properties on the sphere and torus. As discussed in section 1.2, a triangle is a special case of a pseudo-triangle where we have only three angles, which must all be smaller than π radians based on the definition of pseudo-triangles. This implies that a triangle is bounded by exactly three vertices and three edges. *Triangulations*, therefore, are a special case of pseudo-triangulations where every face is a triangle, which is why they are deemed to be degenerate pseudo-triangulations. Clearly, the smallest example of a triangulation in the plane is the complete graph K_3, where a *complete graph* is a graph where every vertex shares an edge with each of the other vertices, as seen in Figure 4.1. It is important to note that tri-
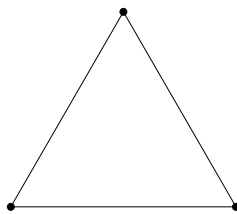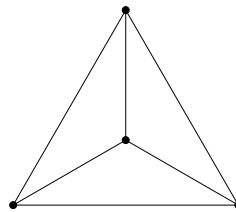
Figure 4.1: Smallest Triangulation in the Plane

angulations in the plane depend on the position of the vertices when the graph is embedded. Take for example, a triangulation on four vertices. If the convex

hull for the graph includes all four vertices, then we get a triangulation comprised of a quadrilateral and one of its diagonals. However, if only three vertices are used in generating the convex hull, we get a triangulation that is isomorphic to the complete graph K_4.
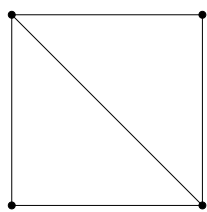
(a) Four Vertex Convex Hull   (b) Three Vertex Convex Hull

Figure 4.2: Two Different Triangulations on Four Vertices

Equally important, is that once we have vertices in place and a convex hull made, it is still possible to create different triangulations. Figure 4.3 shows two graphs, each with a convex hull consisting of six vertices and each having two interior vertices. Despite the fact that different edges were chosen in each of the triangulations, the same number of edges were chosen and the same number of faces resulted.

If we wish to count the number of edges needed for a triangulation on a set of vertices, we can approach in one of two ways. The first approach is through the use of geometry. Given a set of vertices, we first construct the convex hull. It is well known that for a convex polygon with n vertices, and n edges, the sum of the interior angles is $(n\text{-}2)\pi$ radians. Then, for each interior vertex, we add an additional 2π radians. If w is the number of interior vertices, then the sum of the angles within the convex hull of the graph is $2\pi w + (n\text{-}2)\pi$ radians. Since each triangle's sum of interior angles is π radians, the number of triangles in the convex hull of the triangulation is $n+2w\text{-}2$, and the total number of faces in the graph is $n+2w\text{-}1$. With three edges to every triangle, the number of edges is $3(n+2w\text{-}2)$, but the unbounded face is also incident to n edges so the number of edges is
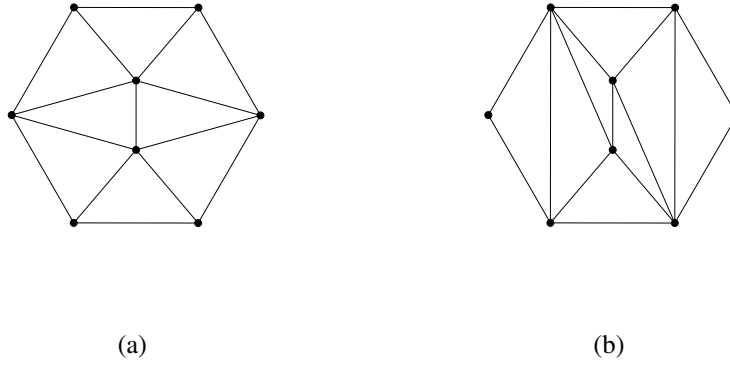
18

(a)   (b)

Figure 4.3: Different Triangulations of a Point Set

$4n$+$6w$-6. However, this counts each edge twice since each edge is incident to two faces, the total number of edges is $2n$+$3w$-3 for a triangulation on n+w vertices.

The other approach is strictly combinatorial. We have n vertices in the convex hull and n edges connecting those vertices to create the convex hull. Additionally, there are n-3 edges that can be drawn between nonadjacent vertices of the convex hull, which creates a triangulation of the convex hull. We then add the w interior vertices one at a time. If an interior vertex is added in the middle of a face already made, then it needs 3 additional edges to keep the triangulation. If an interior vertex falls on an interior edge, then that edge is removed, leaving a quadrilateral, and the new vertex receives an edge connecting it to each of the four vertices of the quadrilateral, which overall adds 3 edges to the triangulation. Taking the sum of all the edges we have n+(n-3)+$3w$, or simply $2n$+$3w$-3.

In a similar fashion we can show that the number of faces in a triangulation is n+$2w$-1. An easier way to find the number of faces, since we know the number of vertices and edges, is to use the Euler characteristic. We know that v-e+f=2 in the plane, and we know that v=n+w and e=$2n$+$3w$-3. Substituting we get n+w-($2n$+$3w$-3)+f=2 and rearranging we find that f=n+$2w$-1. All these calculations depend on the number of vertices, in the convex hull, being at least 3.

4.1 On the Sphere

Moving from the plane to the sphere, we continue our exploration of triangulations. Recall from section 3.1 that we lose our concept of convex hull when embedding graphs on the sphere. This forces us to make every face of the triangulation a triangle without exception. On the other hand, we no longer have to consider different triangulations that can occur due to differing convex hulls. It is still possible though for a set of vertices to have more than one triangulation. We begin with the smallest possible triangulation on the sphere, which consists of three vertices and three edges, and is in fact K_3 from the plane embedded on the sphere. Notice that the embedding effectively slices the sphere into two equal hemispheres. In both triangles, every angle is exactly π radians.
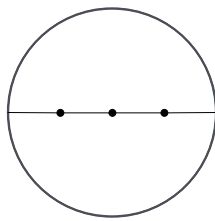

Figure 4.4: Smallest Triangulation on the Sphere

We can again find the number of edges and faces in a triangulation of n vertices on the sphere. Geometrically, each vertex contributes 2π radians to the triangles, and every triangle has between π and 3π radians. So for n vertices we have between $2n$ and $\frac{2}{3}n$ triangles. Since each triangle is comprised of three edges, we get between $6n$ and $2n$ edges, but again this double counts all edges so the number of edges is between $3n$ and n. If the number of edges is n, then Euler's characteristic gives us the number of faces as f=2-n+n=2, but this is only true for n=3, for more vertices we actually have more faces. If the number of edges is $3n$, then Euler's characteristic gives us the number of faces as f=2-n+3n=2n+2. This is not true for n=3 though as the number of faces would be 8. In fact, using some small examples, it is clear that f=2n+2 is always 6 faces too many, so we correct it by saying that f=2n-4. Therefore, using Euler's characteristic again, we find the number of edges to be e=-(2-n-(2n-4))=3n-6.

Combinatorially, we begin with the smallest triangulation and add a vertex. If the vertex lies on an existing edge, then that edge is removed creating a quadri-

lateral and an edge is added between the new vertex and all four corners of the quadrilateral, effectively adding three edges to the count. If the new vertex lies inside an existing triangle face, then three edges are added connecting the vertex to the three corners of the triangle. We can continue this process inductively until we reach our desired number of vertices, n. Since we began with 3 edges and added 3 more for every vertex beyond the first three, the total number of edges in a triangulation of n vertices on the sphere is 3+3(n-3), or simply 3n-6. Thus, for a triangulation of $n \geq 3$ vertices on the sphere, the graph has 3n-6 edges and 2n-4 faces.

4.2 On the Torus

We finally turn our attention to the torus and consider triangulations that occur on it. On the torus, as with the sphere, we have no convex hull to work with, nor any interior vertices. Just as in the case of the sphere and plane, it is possible for more than one triangulation to be made from a particular set of vertices. Figure 4.5 shows the smallest triangulation that can occur on the torus. Notice that allowing for essential loops, the smallest triangulation on the torus has only a single vertex, unlike the smallest examples on the sphere and in the plane where three vertices were needed for the smallest examples.
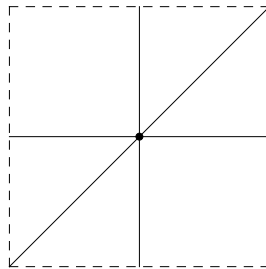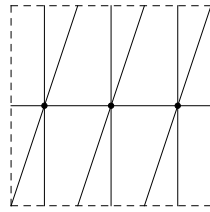

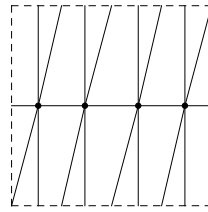Figure 4.5: Smallest Triangulation on the Torus

In Figure 4.6 we see examples of triangulations on the torus with 3, 4 and 5 vertices, again allowing for loops and also for multiple edges. Referring back to [11], there is a corollary which states:

Corollary 4.2.1. *Let G_n be a triangulation of the torus on n¿4 vertices, with no loops, satisfying 1.1. Then we can transform G_n into one of T_3, T_4, or a 6-regular triangulation by successively deleting vertices of degree 3, 4, or 5, and adding diagonals.*

The condition of satisfying 1.1 simply refers to the fact that loops and multiple edges must be essential and faces can be no smaller than triangles. Triangulations on the torus of 3 and 4 vertices are T_3 and T_4 respectively. The key piece of this corollary that we would like to focus on however, is the 6-regular triangulations. For a graph to be 6-regular means that every vertex is incident to exactly 6 edges. There is even an algorithm of sorts that is introduced shortly after the corollary, which essentially consists of choosing a collection of cycles based on the rotation system of the vertices. An example of a 6-regular triangulation on 4 vertices on the torus can be seen in Figure 4.7

(a) Three Vertices   (b) Four Vertices
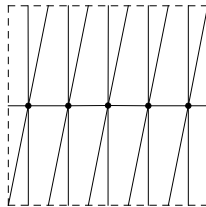
(c) Five Vertices

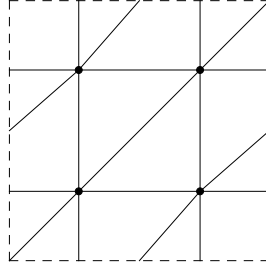Figure 4.6: More Triangulations on the Torus

Figure 4.7: A 6-Regular, 4 Vertex Triangulation on the Torus

Just as with the sphere and torus, we can find the number of edges and faces in a triangulation on n vertices. Using a geometric approach, we begin with n vertices, each contributing 2π radians for a total of $2\pi n$ radians. Unlike the sphere, and like the plane, triangles on the torus have angles that sum to π radians. Thus, the number of triangles that occur is $2n$, which is also the total number of faces. With each triangle requiring three edges, we have $6n$ edges, but this double counts all edges, so the actual number of edges is $3n$.

From a combinatorial standpoint, we begin with the smallest triangulation, which has 1 vertex, 3 edges and 2 faces. Adding a vertex, if it lies in the middle of a face, three edges can be added connecting the new vertex to the corners of the triangle. If the new vertex lies on an edge, remove the edge and add four edges connecting the new vertex with each corner of the created quadrilateral. Thus, every vertex added creates three new edges, so the number of edges for a triangulation on n vertices is $3n$. The Euler characteristic tells us that f=0-n+3n=2n. Therefore, a triangulation on the torus consisting of $n \geq 1$ vertices will have $3n$ edges and $2n$ faces.

Chapter 5

Pseudo-Triangulations

We now examine pseudo-triangulations and their occurrences in the plane, on the sphere and on the torus. Beginning in the plane, recall the definition and types of pseudo-triangulations from section 1.2. In dealing with pseudo-triangulations here we will presume that at least one face is a pseudo-triangle but not also a triangle as we have already studied triangulations. We refer to these pseudo-triangulations as being nondegenerate. In Figure 5.1 we see the smallest pseudo-triangulation in the plane. Not only is it a pseudo-triangulation, it is also pointed and has a combinatorial embedding.
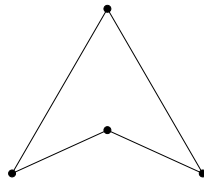
Figure 5.1: Smallest Nondegenerate Pseudo-Triangulation in the Plane

As with triangulations, pseudo-triangulations depend on the position of the embedded vertices, and even with a given embedding of vertices there are still different possible pseudo-triangulations.

Just like triangulations, we can find the number of edges and faces for a given number of vertices n. Since we have already analyzed triangulations, it seems prudent to look at the extreme opposite in terms of edge and face counts, pointed pseudo-triangulations, all other pseudo-triangulations will have edge and face counts somewhere between these two cases. We will also only consider the combinatorial study of the pointed pseudo-triangulations, as the measuring of angles becomes very complicated when allowing, or forcing, large angles in addition to small ones.

Given a pointed pseudo-triangulation on $n+w$ vertices, we first take the convex hull. In doing so, have n vertices which we know are all pointed and we have introduced n edges. We then create a triangulation of the convex hull, adding n-3 edges. Now we add the interior vertices one at a time. If the new vertex lies inside a triangle, then the vertex is connected to two of the three corners of the triangle, which two does not matter. By connecting to only two vertices, the new vertex is guaranteed to be pointed, and because it is inside a triangle there is no risk of breaking a big angle into two small angles. If a new vertex lies inside a pseudo-triangule, then the vertex is connected to one of the small angle corners and to a second small angle corner if possible. For the same reasons as before, the new vertex will be pointed and there is no chance of turning a big angle into two small angles. If the new vertex can only connect to one small angle vertex, then temporarily draw an edge to a small angle vertex. This edge intersects exactly one edge that is a part of the pseudo-triangle the vertex was placed inside of. Following the path that traces the face, choose the vertex incident to the intersected edge that is further away from the small angle already connected to the new vertex and connect it to the new vertex. Lastly, remove the temporary edge. The new vertex is necessarily pointed since it is incident to only two edges. The small angle vertex used is still pointed because its large angle was not touched. The other vertex has its big angle split into a big and a small angle due to the way we chose the point. If the new vertex lies on an edge between two triangles, remove the edge and replace it with two edges from the new vertex to the vertices whose edge was removed and add an edge to one of the two remaining vertices not already connected to the new vertex. All of the vertices of the original triangles remain pointed as their big angles are not impacted and the new vertex has a big angle of π radians. If the new vertex lies on an edge between a triangle and a pseudo-triangle, remove the edge and add three edges from the new vertex to all three vertices of the triangle. Again, no big angles are impacted by the new vertex, and the new vertex has a big angle of π radians. Finally, if the new vertex lies on an edge between two pseudo-triangles, remove the edge and replace it with two edges from the new vertex to

the vertices the old edge connected. A third edge is then added in one of two ways, eiher one of the two remaining small angle vertices is connected to the new vertex, or the method from above is used to find and big angle vertex so that when connected the big angle stays big. Ultimately, each interior vertex adds 2 edges to the count. Therefore, the number of edges in a pointed pseudo-triangulation on $n+w$ vertices is $e=n+(n$-$3)+2w=2n+2w$-$3=2v$-3. The number of faces is then $f=2$-$v+2v$-$3=v$-1. This result for the number of edges can be verified in [12], [8] and [14] when talking about whether or not a graph is a Laman graph.

5.1 On the Sphere

As we have said, since the sphere is a closed surface, there is no unbounded face, so we now require that every face be a pseudo-triangle, without exception. The smallest example of a pseudo-triangulation on the sphere can be seen in Figure 5.2. We see again that the smallest example is both pointed and has a combinatorial embedding.
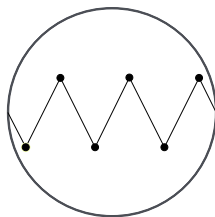

Figure 5.2: Smallest Nondegenerate Pseudo-Triangulation on the Sphere

Again, we can determine the number of edges and faces in a pointed pseudo-triangulation on n vertices on the sphere.

5.2 On the Torus

Turning our attention oncd again to the torus, we must require each face to be a pseudo-triangle. Looking at Figure 5.3, we see that the smallest pseudo-triangulation on the torus is pointed and has a combinatorial embedding as we saw in the smallest examples on the plane and in the torus.
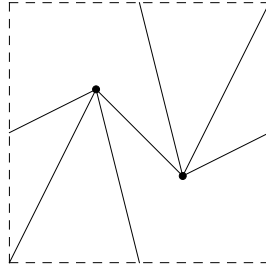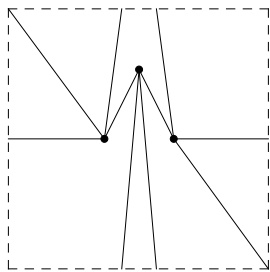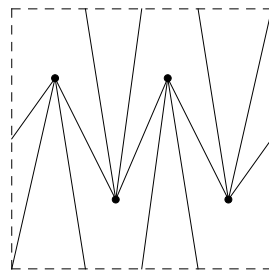
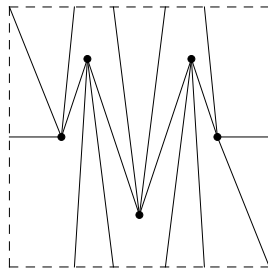Figure 5.3: Smallest Nondegenerate Pseudo-Triangulation on the Torus

Much like triangulations, pointed pseudo-triangulations have regular embeddings on the torus. For pointed pseudo-triangulations, an embedding exists on the torus that is 4-regular, and also has exactly one big angle in every face.

(a) Three Vertices

(b) Four Vertices

(c) Five Vertices

Figure 5.4: More Pseudo-Triangulations on the Torus

Chapter 6

Conclusions

After the developments of pseudo-trianulations in the plane, we extended our study to the sphere and torus where we found both extensions of theorems from the plane and new theorems with interesting properties. We found the number of edges and faces that occur when creating triangulations on n vertices in the plane, on the sphere and on the torus. We also found that both triangulations and pointed pseudo-triangulations have degree-regular embeddings on the torus.

6.1 Future Work

There are still many aspects of pseudo-triangulations that are, as of yet, unexplored or minimally developed. One natural extension is to look at higher dimensional Euclidean or Real Space, beginning with three dimensions. Another natural extension based on the work here is the *n-torus*, or *n-fold torus*, where n is the number of tori connected to one another. A method for the planar representation of the n-torus can be found in [9] where the n-torus is associated with a regular 4n-gon. If we use the 2-torus as an example, we create a regular octagon. If we then label the edges of the octagon clockwise with the numbers 1 through 8, we can find a representation of the n-torus by matching edges 1 and 3, 2 and 4, 5 and 7, and 6 and 8, where matching means that edges which intersect one side of the pair also intersect the other side of the pair. It would also be useful to find a more formal proof of the conformal mapping between the torus and the plane. In particular, there would be extremely useful to find a mapping from the torus to the plane that is equivalent to stereographic projection for mapping the sphere to the plane.

Bibliography

[1] Ergun Akleman, Jianer Chen, and Burak Meric. Intuitive and effective design of periodic symmetric tiles, 2000.

[2] J. A. Bondy. Basic graph theory: paths and circuits. In *Handbook of combinatorics, Vol. 1, 2*, pages 3–110. Elsevier, Amsterdam, 1995.

[3] J. C. Dana, C. I. Grima, and A. Márquez. Convex hull in non-planar surfaces, 1997.

[4] István Fáry. On straight line representation of planar graphs. *Acta Sci. Math. Szeged*, 11:229–233, 1948.

[5] Andrei Gagarin, William Kocay, and Daniel Neilson. Embeddings of small graphs on the torus.

[6] Jack Graver, Brigitte Servatius, and Herman Servatius. *Combinatorial rigidity*, volume 2 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1993.

[7] Jack E. Graver. *Counting on frameworks*, volume 25 of *The Dolciani Mathematical Expositions*. Mathematical Association of America, Washington, DC, 2001. Mathematics to aid the design of rigid structures.

[8] Ruth Haas, David Orden, Günter Rote, Francisco Santos, Brigitte Servatius, Herman Servatius, Diane Souvaine, Ileana Streinu, and Walter Whiteley. Planar minimally rigid graphs and pseudo-triangulations. *Comput. Geom.*, 31(1-2):31–61, 2005.

[9] D. Hilbert and S. Cohn-Vossen. *Geometry and the imagination*. Chelsea Publishing Company, New York, N. Y., 1952. Translated by P. Neményi.

[10] Mark L Irons. The curvature and geodesics of the torus, 2005.

[11] William Kocay, Daniel Neilson, and Ryan Szypowski. Drawing graphs on the torus. *Ars Combin.*, 59:259–277, 2001.

[12] David Orden, Francisco Santos, Brigitte Servatius, and Herman Servatius. Combinatorial pseudo-triangulations. *Discrete Math.*, 307(3-5):554–566, 2007.

[13] Rudi Penne. Relative centers of motion, implicit bars and dead-point positions for planar mechanisms, 2006.

[14] Guenter Rote, Francisco Santos, and Ileana Streinu. Pseudo-triangulations - a survey, 2006.

[15] W. T. Tutte. Convex representations of graphs. *Proc. London Math. Soc. (3)*, 10:304–320, 1960.

[16] W. T. Tutte. How to draw a graph. *Proc. London Math. Soc. (3)*, 13:743–767, 1963.