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Stationarity and Regularity of Infinite Collections of Sets

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Abstract This article investigates extremality, stationarity, and regularity properties of infinite collections of sets in Banach spaces. Our approach strongly relies on the machinery developed for finite collections. When dealing with an infinite collection of sets, we examine the behaviour of its finite subcollections. This allows us to establish certain primal-dual relationships between the stationarity/regularity properties some of which can be interpreted as extensions of the Extremal principle. Stationarity criteria developed in the article are applied to proving intersection rules for Fréchet normals to infinite intersections of sets in Asplund spaces.

Keywords subdifferential · normal cone · optimality · extremality · stationarity · regularity · extremal principle · Asplund space

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1 Introduction

Starting with the pioneering work by Dubovitskii and Milyutin [1], it has become natural, when dealing with optimization and other related problems, to reformulate optimality or some other property under investigation as a kind of extremal behaviour of a certain collection of sets. Considering collections of sets is a rather general scheme of investigating extremal problems. For instance, any set of extremality conditions leads to some optimality conditions for the original problem.

The concept of a finite *extremal collection of sets* (see Definition 2.1) was introduced and investigated in [2–4]. This is a very general model embracing many optimality notions. A dual necessary extremality condition in terms of

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Fréchet ε-normal elements was established in [3, 4] (formulated without proof in [2]) for a collection of closed sets in the setting of a Banach space admitting an equivalent norm Fréchet differentiable away from zero. It was extended in [5] to general Asplund spaces and is now known as the *Extremal principle* (see Theorem 2.1). This result can be considered as a generalization of the convex *separation theorem* to collections of nonconvex sets and is recognized as one of the cornerstones of the contemporary variational analysis. It can substitute the latter theorem, when proving optimality conditions and subdifferential calculus formulas. We refer the reader to [6] for other applications and historical comments.

In recent years, finite collections of sets have been a subject of intensive research [7–29]. Similar to the classical analysis, besides extremality, the concepts of *stationarity* and *regularity* have been introduced and investigated. It was established in [17, 18] that the conclusion of the Extremal principle actually characterizes a much weaker than local extremality property of *approximate stationarity* (see Definition 2.2). Several versions of this property (under various names) can be found in [14–21].

Replacing in the Extremal principle local extremality with approximate stationarity produces a stronger statement – the *Extended extremal principle*: approximate stationarity of a finite collection of closed sets in an Asplund space is equivalent to its *separability* (*Fréchet normal approximate stationarity*) (see Theorem 2.2). Some earlier formulations of this result can be found in [14–16].

If a collection of sets is not approximately stationary, it is *uniformly regular* [21] (see also Definition 3.1 (UR)). The latter property is the direct analogue for collections of sets of the *metric regularity* of multifunctions. The corresponding dual property is called *Fréchet normal uniform regularity* [21] (see Definition 3.1 (FNUR)).

This article extends the discussed above extremality, stationarity, and regularity properties of collections of sets to infinite collections in Banach spaces having in mind applications to problems of infinite and semi-infinite programming that are developed in the forthcoming article [30]. The definitions of regularity properties for infinite collections of sets suggested in this article provide a partial answer to a question on the list of open problems compiled at the Metric Regularity Days Workshop, Paris, October 25–26, 2011[1].

Recently, there have appeared a few other attempts to consider regularity properties of infinite collections of sets [31–33]. The authors of these three articles study the so called *linear regularity* (which is in general weaker than uniform regularity considered in the current article) and several related regularity properties for a collection of infinitely many convex or subsmooth sets.

In [34], necessary optimality conditions are established for broad classes of semi-infinite programs where the feasible set is given by a parameterized system of infinitely many linear inequalities. The optimality conditions in this article are formulated in *asymptotic form*, involving the weak* closure of the so-called *second moment cone*. Under the so-called *Farkas-Minkowski type constraint qualification* (FMCQ, in short), ordinary KKT optimality conditions are easily derived.

A FMCQ was previously applied in [35] to a *convex* optimization problem with constraints. If the constraint system enjoys the FMCQ, then every continuous linear consequence of the system is also a consequence of a finite subsystem, and the converse holds if the system is linear [35, Proposition 1].

In [35], a weaker *local Farkas-Minkowski constraint qualification* (LFMCQ, in brief) is introduced. It can be proved that FMCQ implies LFMCQ. This property is also closely related (equivalent, in fact, under quite natural assumptions) to the *basic constraint qualification* (BCQ, in short), introduced in [36, p. 307] relatively to an ordinary convex programming problem with equality/inequality constraints and extended to systems of infinitely many convex constraints in [37] (see also [22]).

FMCQ and LFMCQ are quite strong properties as they entail a kind of finite reducibility, allowing for KKT-type necessary optimality conditions in infinitely constrained optimization. A very deep study of constraint qualifications

[1] M. Théra, personal communication.

related to BCQ is carried out in [38]. An attempt to bring some order into the variety of existing constraint qualifications was undertaken in [39, 40].

Out of the convex scenario, in [24] a general optimization problem with *countable inequality constraints* is approached by applying some tangential extremal principles and related calculus rules for infinite intersections. Asymptotic and non-asymptotic KKT conditions are derived in [24] in the locally Lipschitz case under certain constraint qualifications (CHIP, SQC and SCC).

Our approach in this article strongly relies on the machinery developed for finite collections. When dealing with an infinite collection of sets, we examine the behaviour of its finite subcollections. In all the original definitions, we introduce an additional parameter – a finite subset of the given set of indices (see e.g. Definition 2.3). This allows us to establish the primal-dual relationships between the stationarity/regularity properties of infinite collections of sets (see Theorem 2.3) using the techniques developed for finite collections.

An important feature of the proposed approach is the fact that the proof of the primal-dual relationships does not depend on the method of choice of finite subcollections of sets (as long as primal and dual conditions are considered for the same subcollection). This gives us freedom to define rules governing the choice of such subcollections. When dealing with families (sequences) of subcollections, it can be important to impose growth restrictions on the size (cardinality of the set of indices) of subcollections. This is done in the article by using an abstract *gauge* function Φ (see Definition 2.4). The primal-dual relationships between the stationarity/regularity properties of infinite collections of sets remain valid for corresponding Φ-stationarity/Φ-regularity properties (see Theorem 2.4). Specific Φ-stationarity/Φ-regularity properties depend on the choice of the gauge function.

The plan of the article is as follows. Section 2 contains a more detailed list of important definitions and theorems partially mentioned above and needed in the sequel together with the preliminary discussion of the new developments which are the subject of the current article.

In Section 3, we summarize and partially modernize stationarity and regularity conditions for finite collections of sets from [19–21]. All the properties are defined in terms of certain constants characterizing the mutual arrangement of the sets in space. Among new results, note Proposition 3.1 providing conditions guaranteeing nontriviality of the normal elements corresponding to a certain subcollection of sets, and Theorem 3.1 which refines the core arguments from the proofs of [21, Theorem 4] and [18, Theorem 1] and provides the tools for proving the primal/dual relationships between stationarity and regularity properties of finite and infinite collections of sets.

In Section 4, the definitions and relationships of Section 3 are extended to infinite collections of sets utilizing the idea of replacing an infinite index set by a sequence of its finite subsets with and without growth restrictions on the cardinality of the subsets of indices.

Section 5 is devoted to applications of stationarity criteria from Section 4 to developing several intersection rules for Fréchet normals to infinite intersections of sets in Asplund spaces. Besides the general form of the intersection rule, we formulate also its normal form under the assumption of Fréchet normal regularity of the collection of sets from Section 4.

Other applications of the results of the current article (mostly to optimality conditions) will be presented in the forthcoming article [30].

While preparing this article for publication, we came across the article [23] by Mordukhovich and Phan where the authors also consider infinite collections of sets and establish so called *rated extremal principles*. Rated extremality investigated in this article is a useful property which ensures approximate stationarity of the collection of sets. The main results of [23] follow from the corresponding theorems of the current article as the appropriate in-text remarks point out.

2 Preliminaries

This Section contains a list of important definitions and theorems partially mentioned in the Introduction and needed in the sequel together with the preliminary discussion of the new developments which are the subject of the current article. It illustrates the evolution of the main ideas.

Definition 2.1 [4,6] A collection of sets $\{\Omega_i\}_{i\in I}$, $1<|I|<\infty$, in a normed linear space X, is called *locally extremal* at $\bar{x}\in\bigcap_{i\in I}\Omega_i$ iff there exists a $\rho>0$ such that for any $\varepsilon>0$ there are $a_i\in X$ $(i\in I)$ such that

$$\max_{i\in I}\|a_i\|<\varepsilon \quad\text{and}\quad \bigcap_{i\in I}(\Omega_i-a_i)\bigcap B_\rho(\bar{x})=\emptyset. \tag{1}$$

Condition (1) means that an appropriate arbitrarily small shift of the sets makes them unintersecting in a neighbourhood of $\bar{x}$. This is a very general model embracing many optimality notions.

Theorem 2.1 [2,4–6] *If a collection of closed sets $\{\Omega_i\}_{i\in I}$, $1<|I|<\infty$, in an Asplund space, is locally extremal at $\bar{x}\in\bigcap_{i\in I}\Omega_i$, then for any $\varepsilon>0$ there exist $x_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ and $x_i^*\in N^F_{\Omega_i}(x_i)$ $(i\in I)$ such that*

$$\left\|\sum_{i\in I}x_i^*\right\|<\varepsilon\sum_{i\in I}\|x_i^*\|, \tag{2}$$

where $N^F_{\Omega_i}(x_i)$ is the Fréchet normal cone to Ω_i at x_i.

This result can be considered as a generalization of the convex *separation theorem* to collections of nonconvex sets.

Similar to the classical analysis, besides extremality, the concepts of *stationarity* and *regularity* have been introduced and investigated. The conclusion (2) of the Extremal principle actually characterizes a much weaker than local extremality (1) property which can be interpreted as kind of stationary behaviour of the collection of sets.

Definition 2.2 [21] A collection of sets $\{\Omega_i\}_{i\in I}$, $1<|I|<\infty$, is *approximately stationary* at $\bar{x}\in\bigcap_{i\in I}\Omega_i$ iff for any $\varepsilon>0$ there exist $\rho\in]0,\varepsilon[$; $\omega_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ and $a_i\in X$ $(i\in I)$ such that

$$\max_{i\in I}\|a_i\|<\varepsilon\rho \quad\text{and}\quad \bigcap_{i\in I}(\Omega_i-\omega_i-a_i)\bigcap(\rho\mathbb{B})=\emptyset. \tag{3}$$

Conditions (3) look more complicated than (1): here, instead of the common point $\bar{x}$, each of the sets Ω_i is considered near its own point ω_i and the size of the "shifts" is related to that of the neighbourhood in which the sets become unintersecting, namely $\max_{i\in I}\|a_i\|/\rho\to 0$ as $\varepsilon\downarrow 0$.

Replacing in the Extremal principle local extremality with approximate stationarity produces a stronger statement – the *Extended extremal principle.*

Theorem 2.2 [17,18] *A collection of closed sets $\{\Omega_i\}_{i\in I}$, $1<|I|<\infty$, in an Asplund space, is approximately stationary at $\bar{x}\in\bigcap_{i\in I}\Omega_i$, if and only if for any $\varepsilon>0$ there exist $x_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ and $x_i^*\in N^F_{\Omega_i}(x_i)$ $(i\in I)$ such that (2) holds true.*

In the subsequent sections, we extend the discussed above extremality, stationarity, and regularity properties of collections of sets to infinite collections in Banach spaces. In all the original definitions, we introduce an additional parameter – a finite subset of the given set of indices. For example, the definition of approximate stationarity takes the following form.

Definition 2.3 A collection of sets $\{\Omega_i\}_{i\in I}$, $|I|>1$, is *approximately stationary* at $\bar{x}\in\bigcap_{i\in I}\Omega_i$ iff for any $\varepsilon>0$ there exist $\rho\in]0,\varepsilon[$; $J\subset I$, $|J|<\infty$; $\omega_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ and $a_i\in X$ $(i\in J)$ such that

$$\max_{i\in J}\|a_i\|<\varepsilon\rho \quad\text{and}\quad \bigcap_{i\in J}(\Omega_i-\omega_i-a_i)\bigcap(\rho\mathbb{B})=\emptyset. \tag{4}$$

This allows us to establish the primal-dual relationships between the stationarity/regularity properties of infinite collections of sets using the techniques developed for finite collections. In particular, the Extended extremal principle holds.

Theorem 2.3 *A collection of closed sets $\{\Omega_i\}_{i\in I}$, $|I|>1$, in an Asplund space is approximately stationary at $\bar{x}\in\bigcap_{i\in I}\Omega_i$ if and only if for any $\varepsilon>0$ there exist $J\subset I$, $|J|<\infty$; $x_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ and $x_i^*\in N^F_{\Omega_i}(x_i)$ $(i\in J)$ such that*

$$\left\|\sum_{i\in J}x_i^*\right\|<\varepsilon\sum_{i\in J}\|x_i^*\|. \tag{5}$$

Moreover, for any $\varepsilon>0$, both properties in the above equivalence are satisfied with the same subset J of indices.

When dealing with families (sequences) of subcollections, it can be important to impose growth restrictions on the size (cardinality of the set of indices) of subcollections. This is done in the article by using an abstract *gauge* function $\Phi:\mathbb{R}_+\to\mathbb{R}_+\cup\{+\infty\}$. Appropriate changes in the definitions lead to modified concepts of Φ-stationarity and Φ-regularity.

Definition 2.4 A collection of sets $\{\Omega_i\}_{i\in I}$, $|I|>1$, is *Φ-approximately stationary* at $\bar{x}\in\bigcap_{i\in I}\Omega_i$ iff for any $\varepsilon>0$ there exist $\rho\in]0,\varepsilon[$; $\alpha\in]0,\varepsilon[$; $J\subset I$, $|J|<\Phi(\alpha)$; $\omega_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ and $a_i\in X$ $(i\in J)$ such that

$$\max_{i\in J}\|a_i\|<\alpha\rho \quad\text{and}\quad \bigcap_{i\in J}(\Omega_i-\omega_i-a_i)\bigcap(\rho\mathbb{B})=\emptyset. \tag{6}$$

Note that the parameter α in the above definition determines both the cardinality of the subset J of indices and the upper bound of the size of "shifts" a_i.

The primal-dual relationships between the stationarity/regularity properties of infinite collections of sets remain valid for corresponding Φ-stationarity/Φ-regularity properties. In particular, the Extended Φ-extremal principle can be formulated the following way.

Theorem 2.4 *A collection of closed sets $\{\Omega_i\}_{i\in I}$, $|I|>1$, in an Asplund space is Φ-approximately stationary at $\bar{x}\in\bigcap_{i\in I}\Omega_i$ if and only if for any $\varepsilon>0$ there exist $\alpha\in]0,\varepsilon[$; $J\subset I$, $|J|<\Phi(\alpha)$; $x_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ and $x_i^*\in N^F_{\Omega_i}(x_i)$ $(i\in J)$ such that*

$$\left\|\sum_{i\in J}x_i^*\right\|<\alpha\sum_{i\in J}\|x_i^*\|. \tag{7}$$

Moreover, for any $\varepsilon>0$, both properties in the above equivalence are satisfied with the same number α and subset J of indices.

Specific Φ-stationarity/regularity properties depend on the choice of the gauge function.

Our basic notation is standard, see [6, 41]. Throughout the article, X is a Banach space (although the definitions are valid in a normed linear space). Its topological dual is denoted X^* while $\langle\cdot,\cdot\rangle$ denotes the bilinear form defining the pairing between the two spaces. The closed unit balls in a normed space and its dual are denoted $\mathbb{B}$ and $\mathbb{B}^*$ respectively. $B_\delta(x)$ denotes the closed ball with radius δ and center x.

We say that a set $\Omega\subset X$ is locally closed near $\bar{x}\in\Omega$ iff $\Omega\cap U$ is closed in X for some closed neighbourhood U of $\bar{x}$. Given a set I of indices, its cardinality (the number of elements in I) is denoted $|I|$.

In this article, we consider an abstract subdifferential operator ∂ defined on the class of extended real-valued functions and satisfying the following conditions (axioms):

(A1) For any $f:X\to\mathbb{R}_\infty:=\mathbb{R}\cup\{+\infty\}$ and any $x\in X$, the subdifferential $\partial f(x)$ is a (possibly empty) subset of X^*.

(A2) If f is convex, then ∂f coincides with the subdifferential of f in the sense of convex analysis.

(A3) If $f(u) = g(u)$ for all u near x, then $\partial f(x) = \partial g(x)$.

(A4) If x is a point of local minimum of $f + g$, where $f : X \to \mathbb{R}_\infty$ is lower semicontinuous and $g : X \to \mathbb{R}$ is convex and Lipschitz continuous, then for any $\varepsilon > 0$ there exist $x_1, x_2 \in B_\varepsilon(x)$, $x_1^* \in \partial f(x_1)$, $x_2^* \in \partial g(x_2)$ such that $|f(x_1) - f(x)| < \varepsilon$ and $\|x_1^* + x_2^*\| < \varepsilon$.

The majority of known subdifferentials satisfy conditions (A1)–(A3). The typical examples of subdifferentials satisfying all conditions (A1)–(A4) are Rockafellar-Clarke and Ioffe subdifferentials in Banach spaces and Fréchet subdifferentials in Asplund spaces.

The corresponding to ∂ normal cone mapping N is defined for any $\Omega \subset X$ with the help of its indicator function δ_Ω ($\delta_\Omega(x) = 0$ if $x \in \Omega$ and $\delta_\Omega(x) = \infty$ otherwise): $N_\Omega(x) := \partial \delta_\Omega(x)$ if $x \in \Omega$ and $N_\Omega(x) := \emptyset$ otherwise. Another two natural assumptions about normal cones need to be added to the list of axioms:

(A5) If $x \in \Omega$, then $N_\Omega(x)$ is a cone.

(A6) If $X = X_1 \times X_2$, $x_1 \in \Omega_1 \subset X_1$, $x_2 \in \Omega_2 \subset X_2$, then $N_{\Omega_1 \times \Omega_2}(x_1, x_2) = N_{\Omega_1}(x_1) \times N_{\Omega_2}(x_2)$.

The majority of known normal cones, particularly Fréchet, limiting and Clarke normal cones, satisfy conditions (A5) and (A6).

Throughout this article, we assume that all axioms (A1)–(A6) are satisfied by the subdifferential and normal cone operators ∂ and N.

We will use the denotations ∂^F and N^F for the Fréchet subdifferential and normal cone operators respectively. Recall that

$$\partial^F f(x) = \left\{ x^* \in X^* \middle| \ \liminf_{u \to x} \frac{f(u) - f(x) - \langle x^*, u - x \rangle}{\|u - x\|} \geq 0 \right\}, \tag{8}$$

$$N_\Omega^F(x) = \left\{ x^* \in X^* \middle| \ \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq 0 \right\} \tag{9}$$

if $f(x)$ is finite in the case of the first formula and $x \in \Omega$ in the case of the second one. The denotation $u \xrightarrow{\Omega} x$ in the last formula means that $u \to x$ with $u \in \Omega$. In the convex case, sets (8) and (9) reduce to the subdifferential and normal cone in the sense of convex analysis. In this case, the superscript 'F' will be omitted.

3 Finite Collections of Sets

In this section we summarize stationarity and regularity conditions for finite collections of sets from [19–21].

Given a collection of sets $\mathbf{\Omega} := \{\Omega_i\}_{i \in I} \subset X$, where $1 < |I| < \infty$, and a point $\bar{x} \in \bigcap_{i \in I} \Omega_i$, define nonnegative (possibly infinite) constants:

$$\theta_\rho[\mathbf{\Omega}](\bar{x}) := \sup \left\{ r \geq 0 \middle| \ \bigcap_{i \in I} (\Omega_i - a_i) \bigcap B_\rho(\bar{x}) \neq \emptyset, \ \forall a_i \in r\mathbb{B} \right\}, \quad \rho \in]0, \infty], \tag{10}$$

$$\theta[\mathbf{\Omega}](\bar{x}) := \liminf_{\rho \downarrow 0} \frac{\theta_\rho[\mathbf{\Omega}](\bar{x})}{\rho}, \tag{11}$$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\rho \downarrow 0;\, \omega_i \xrightarrow{\Omega_i} \bar{x},\, i \in I} \frac{\theta_\rho[\{\Omega_i - \omega_i\}_{i \in I}](0)}{\rho}. \tag{12}$$

Evidently $\theta_\rho[\mathbf{\Omega}](\bar{x})$ is nondecreasing as a function of ρ. Moreover, $\lim_{\rho \downarrow 0} \theta_\rho[\mathbf{\Omega}](\bar{x}) = 0$, unless $\bar{x} \in \operatorname{int} \cap_{i \in I} \Omega_i$ [19, Proposition 3].

If $\rho = \infty$, then $B_\rho(\bar{x}) = X$ and

$$\theta_\infty[\mathbf{\Omega}](\bar{x}) := \sup\Big\{r \geq 0 \,\Big|\, \bigcap_{i\in I}(\Omega_i - a_i) \neq \emptyset,\ \forall a_i \in r\mathbb{B}\Big\}.$$

Constants (10)–(12) characterize the mutual arrangement of sets Ω_i $(i \in I)$ in space and are convenient for defining their extremality, stationarity and regularity properties. We demonstrate below that these constants simplify establishing dual characterizations of these properties and provide estimates for the rates/moduli of the regularity properties. The terminology and abbreviations for the properties in the definition below are taken from [21].

Definition 3.1 The collection of sets $\mathbf{\Omega}$ is

(E) *extremal* at $\bar{x}$ iff $\theta_\infty[\mathbf{\Omega}](\bar{x}) = 0$, i.e.,
for any $\varepsilon > 0$ there exist $a_i \in X$ $(i \in I)$ such that $\max_{i\in I}\|a_i\| < \varepsilon$ and

$$\bigcap_{i\in I}(\Omega_i - a_i) = \emptyset;$$

(LE) *locally extremal* at $\bar{x}$ iff $\theta_\rho[\mathbf{\Omega}](\bar{x}) = 0$ for some $\rho > 0$, i.e.,
there exists a $\rho > 0$ such that for any $\varepsilon > 0$ there are $a_i \in X$ $(i \in I)$ such that $\max_{i\in I}\|a_i\| < \varepsilon$ and

$$\bigcap_{i\in I}(\Omega_i - a_i)\bigcap B_\rho(\bar{x}) = \emptyset; \tag{13}$$

(S) *stationary* at $\bar{x}$ iff $\theta[\mathbf{\Omega}](\bar{x}) = 0$, i.e.,
for any $\varepsilon > 0$ there exists a $\rho \in]0, \varepsilon[$ and $a_i \in X$ $(i \in I)$ such that $\max_{i\in I}\|a_i\| < \varepsilon\rho$ and (13) holds true;

(AS) *approximately stationary* at $\bar{x}$ iff $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = 0$, i.e.,
for any $\varepsilon > 0$ there exist $\rho \in]0, \varepsilon[$; $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $a_i \in X$ $(i \in I)$ such that $\max_{i\in I}\|a_i\| < \varepsilon\rho$ and

$$\bigcap_{i\in I}(\Omega_i - \omega_i - a_i)\bigcap(\rho\mathbb{B}) = \emptyset; \tag{14}$$

(R) *regular* at $\bar{x}$ iff $\theta[\mathbf{\Omega}](\bar{x}) > 0$, i.e.,
there exists an $\alpha > 0$ and an $\varepsilon > 0$ such that

$$\bigcap_{i\in I}(\Omega_i - a_i)\bigcap B_\rho(\bar{x}) \neq \emptyset$$

for any $\rho \in]0, \varepsilon[$ and any $a_i \in X$ $(i \in I)$ satisfying $\max_{i\in I}\|a_i\| \leq \alpha\rho$;

(UR) *uniformly regular* at $\bar{x}$ iff $\hat{\theta}[\mathbf{\Omega}](\bar{x}) > 0$, i.e.,
there exists an $\alpha > 0$ and an $\varepsilon > 0$ such that

$$\bigcap_{i\in I}(\Omega_i - \omega_i - a_i)\bigcap(\rho\mathbb{B}) \neq \emptyset$$

for any $\rho \in]0, \varepsilon[$; $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $a_i \in X$ $(i \in I)$ satisfying $\max_{i\in I}\|a_i\| \leq \alpha\rho$.

Extremality properties (E) and (LE) were introduced in [2] and [3,4] respectively as a general model for investigating various settings of optimization problems (see historical comments in [6]). Several modifications of the (AS) property (under different names) can be found in [14–20]. Properties (S), (R), and (UR) were introduced in [19,20]. The definitions of (AS) and (UR) given above follow [20], while the terms 'approximate stationarity' and 'uniform regularity' (and the corresponding abbreviations) were suggested in [21].

The relationships between the extremality, stationarity and regularity properties are straightforward and easily follow from comparing the corresponding constants:

$$\text{(E)} \Rightarrow \text{(LE)} \Rightarrow \text{(S)} \Rightarrow \text{(AS)}, \tag{15}$$

$$\text{(UR)} \Rightarrow \text{(R)}. \tag{16}$$

The regularity properties (R) and (UR) are negations of the corresponding stationarity properties (S) and (AS) respectively. When positive, constants (11) and (12) provide quantitative characterizations of the regularity properties. They coincide with the supremum of all α in the definitions of properties (R) and (UR) respectively.

All implications in (15) and (16) can be strict. Some examples can be found in [21]. The chain of implications (15) shows, in particular, that the approximate stationarity property (AS) is the weakest of all extremality and stationarity properties in Definition 3.1. It is in a sense also the most important one: it lies at the heart of the *Extremal principle*. Its direct counterpart – the uniform regularity property (UR) – can be interpreted as a realization (for a collection of sets) of the fundamental in variational analysis property of *metric regularity* (see the comparison of these properties in [19–21]).

The mutual arrangement of sets in space can also be characterized with the help of dual space elements. The next constant plays a crucial role in such characterizations:

$$\hat{\eta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\substack{x_i \overset{\Omega_i}{\to} \bar{x},\, x_i^* \in N_{\Omega_i}(x_i)\ (i\in I) \\ \sum_{i\in I}\|x_i^*\|=1}} \left\| \sum_{i\in I} x_i^* \right\|. \tag{17}$$

It obviously depends on the type of normal cone used in the definition. In the case of the Fréchet normal cone, we will write $\hat{\eta}^F[\mathbf{\Omega}](\bar{x})$.

Definition 3.2 The collection of sets $\mathbf{\Omega}$ is

(NAS) *normally approximately stationary* at $\bar{x}$ iff $\hat{\eta}[\mathbf{\Omega}](\bar{x}) = 0$, i.e.,
for any $\varepsilon > 0$ there exist $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in I)$ such that

$$\left\| \sum_{i\in I} x_i^* \right\| < \varepsilon \sum_{i\in I} \|x_i^*\|; \tag{18}$$

(FNAS) *Fréchet normally approximately stationary* at $\bar{x}$ iff $\hat{\eta}^F[\mathbf{\Omega}](\bar{x}) = 0$, i.e.,
for any $\varepsilon > 0$ there exist $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in I)$ such that (18) holds true;

(NUR) *normally uniformly regular* at $\bar{x}$ iff $\hat{\eta}[\mathbf{\Omega}](\bar{x}) > 0$, i.e.,
there exists an $\alpha > 0$ and an $\varepsilon > 0$ such that

$$\left\| \sum_{i\in I} x_i^* \right\| \ge \alpha \sum_{i\in I} \|x_i^*\| \tag{19}$$

for any $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in I)$;

(FNUR) *Fréchet normally uniformly regular* at $\bar{x}$ iff $\hat{\eta}^F[\mathbf{\Omega}](\bar{x}) > 0$, i.e.,
there exists an $\alpha > 0$ and an $\varepsilon > 0$ such that (19) holds true for any $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in I)$.

The normal approximate stationarity property (NAS) can be interpreted as a kind of separation property for a collection of sets. Its first version (in terms of Fréchet ε-normal elements) was considered in [2–4] as a dual necessary condition of extremality. Later on, the property called here Fréchet normal approximate stationarity has been used in numerous publications. The current formulations of the (FNAS) and (FNUR) properties follow [20, 21]. Constant (17) coincides with the supremum of all α in the definition of property (NUR).

When dealing with normally approximately stationary collections of sets, it can be important to have conditions guaranteeing nontriviality of elements x_i^* in the definition of property (NAS) corresponding to a certain subcollection of sets. Not surprisingly, such conditions are provided by normal uniform regularity of the complement of this subcollection.

Proposition 3.1 *Let a collection of sets $\mathbf{\Omega} = \{\Omega_i\}_{i\in I}$ be normally approximately stationary at $\bar{x}$. Suppose $I = I_1 \cup I_2$, $I_1 \neq \emptyset$, $I_2 \neq \emptyset$ and $I_1 \cap I_2 = \emptyset$. If the collection of sets $\{\Omega_i\}_{i\in I_2}$ is normally uniformly regular at $\bar{x}$, then for any $\varepsilon > 0$ and $\gamma \in]0, 1[$ there exist $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in I)$ such that (18) holds true and*

$$\sum_{i\in I_1} \|x_i^*\| > \gamma c \sum_{i\in I} \|x_i^*\|,$$

where $c := (1 + (\hat{\eta}[\{\Omega_i\}_{i\in I_2}])^{-1})^{-1}$.

Proof Let the collection of sets $\{\Omega_i\}_{i\in I_2}$ be normally uniformly regular at $\bar{x}$ and numbers $\varepsilon > 0$ and $\gamma \in]0, 1[$ be given. Take any $\gamma' \in]\gamma, 1[$. By definition (NUR) and taking into account that $\hat{\eta}$ is the supremum of all α in the definition of property (NUR), there exists an $\alpha > 0$ and a $\delta > 0$ such that $\alpha/(\alpha + 1) = (1 + \alpha^{-1})^{-1} > \gamma' c$ and

$$\left\| \sum_{i\in I_2} x_i^* \right\| \geq \alpha \sum_{i\in I_2} \|x_i^*\|$$

for any $x_i \in \Omega_i \cap B_\delta(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in I_2)$. Chose a $\xi \in]0, \min\{\varepsilon, \delta\}[$ such that $(\alpha - \xi)/(\alpha + 1) > \gamma c$. By definition (NAS), there exist $x_i \in \Omega_i \cap B_\xi(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in I)$ such that

$$\left\| \sum_{i\in I} x_i^* \right\| < \xi \sum_{i\in I} \|x_i^*\|.$$

Then $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, (18) holds true and

$$\begin{aligned}
\sum_{i\in I_1} \|x_i^*\| &\geq \left\| \sum_{i\in I_1} x_i^* \right\| \geq \left\| \sum_{i\in I_2} x_i^* \right\| - \left\| \sum_{i\in I} x_i^* \right\| \\
&> \alpha \sum_{i\in I_2} \|x_i^*\| - \xi \sum_{i\in I} \|x_i^*\| \\
&= \alpha \sum_{i\in I} \|x_i^*\| - \alpha \sum_{i\in I_1} \|x_i^*\| - \xi \sum_{i\in I} \|x_i^*\| .
\end{aligned}$$

Hence,

$$(1 + \alpha) \sum_{i\in I_1} \|x_i^*\| > (\alpha - \xi) \sum_{i\in I} \|x_i^*\| ,$$

yielding

$$\sum_{i\in I_1} \|x_i^*\| > \frac{\alpha - \xi}{1 + \alpha} \sum_{i\in I} \|x_i^*\| > \gamma c \sum_{i\in I} \|x_i^*\| .$$

□

The main tools for comparing primal and dual space stationarity and regularity properties of finite and infinite collections of sets are provided by the next theorem. It refines the core arguments from the proofs of [18, Theorem 1] and [21, Theorem 4].

Theorem 3.1 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, $1 < |I| < \infty$.*

(i) *Suppose $\omega_i \in \Omega_i$, $x_i^* \in N^F_{\Omega_i}(\omega_i)$ $(i \in I)$,*

$$\sum_{i\in I} \|x_i^*\| = 1 \quad \textit{and} \quad \left\| \sum_{i\in I} x_i^* \right\| < \alpha \tag{20}$$

for some $\alpha > 0$. Then for any $\varepsilon > 0$, there exists a $\rho \in]0, \varepsilon[$ and points $a_i \in X$ $(i \in I)$ such that $\max_{i\in I} \|a_i\| < \alpha\rho$ and (14) holds true.

(ii) *Suppose that numbers $\alpha > 0$; $\varepsilon > 0$; $\varepsilon_1 \geq 0$, $\varepsilon_2 > 0$, $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$; and $\rho \in]0, \varepsilon_2/(\alpha+1)[$ and points $\omega_i \in \Omega_i \cap B_{\varepsilon_1}(\bar{x})$ and $a_i \in X$ $(i \in I)$ are given such that the sets $\Omega_i \cap B_\varepsilon(\bar{x})$ $(i \in I)$ are closed near $\bar{x}$, condition (14) is satisfied and $\max_{i\in I} \|a_i\| < \alpha\rho$. Then there exist points $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in I)$ such that conditions (20) are satisfied.*

Proof (i) Chose positive numbers ε_1 and ε_2 such that $\left\|\sum_{i\in I} x_i^*\right\| < \alpha - |I|(\varepsilon_1 + \varepsilon_2)$.

By definition (9) of the Fréchet normal cone, for sufficiently small $\rho < \varepsilon$, the inequalities

$$\langle x_i^*, \omega - \omega_i\rangle \leq \frac{\varepsilon_1}{\alpha+1} \|\omega - \omega_i\| \leq \varepsilon_1\rho$$

hold true for all $\omega \in \Omega_i \cap B_{(\alpha+1)\rho}(\omega_i)$ and all $i \in I$.

For any $i \in I$, chose a point $a_i \in X$, such that

$$\|a_i\| < \alpha\rho \quad \text{and} \quad \langle x_i^*, a_i\rangle > \alpha\rho \left\|x_i^*\right\| - \varepsilon_2\rho. \tag{21}$$

To complete the proof, it is sufficient to show that condition (14) holds true. If it does not, then there exists an $x \in \bigcap_{i\in I}(\Omega_i - \omega_i - a_i) \cap \rho\mathbb{B}$. For any $i \in I$, we have $x = \omega_i' - \omega_i - a_i$ for some $\omega_i' \in \Omega_i$, and $\left\|\omega_i' - \omega_i\right\| = \|x + a_i\| < (\alpha+1)\rho$. Thus, applying (21), we obtain:

$$\langle x_i^*, x\rangle = \langle x_i^*, \omega_i' - \omega_i\rangle - \langle x_i^*, a_i\rangle < -\alpha\rho \left\|x_i^*\right\| + (\varepsilon_1 + \varepsilon_2)\rho,$$

and consequently

$$\sum_{i\in I} \langle x_i^*, x\rangle < -\alpha\rho + |I|(\varepsilon_1 + \varepsilon_2)\rho.$$

On the other hand,

$$\left\langle \sum_{i\in I} x_i^*, x \right\rangle > -(\alpha - |I|(\varepsilon_1 + \varepsilon_2))\rho.$$

A contradiction.

(ii). Put $\gamma := (\alpha+1)^{-1}$ and chose numbers α_1, α_2, satisfying

$$\rho^{-1} \max_{i\in I} \|a_i\| < \alpha_1 < \alpha_2 < \alpha.$$

Note that $\alpha_2 < \gamma^{-1} - 1$, $\gamma\alpha_2 < 1 - \gamma$, and $\rho < \gamma\varepsilon_2$.

Without loss of generality, let $I = \{1, 2, \ldots, n\}$. Consider the Banach space X^{n+1} with the norm $\|\cdot\|_\gamma$ defined by

$$\|(u, v_1, \ldots, v_n)\|_\gamma := \max\{\|u\|, \gamma \max_{1\leq i\leq n} \|v_i\|\}$$

and a function $f_1 : X^{n+1} \to \mathbb{R}_+$:

$$f_1(u, v_1, \ldots, v_n) := \max_{1\leq i\leq n} \|v_i - \omega_i - a_i - u\|.$$

By (14), $f_1(u, v_1, \ldots, v_n) > 0$ for all $u \in \rho\mathbb{B}$ and $v_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $i = 1, 2, \ldots, n$. At the same time,

$$f_1(0, \omega_1, \ldots, \omega_n) = \max_{1\leq i\leq n} \|a_i\| < \alpha_1\rho.$$

Next step is application of the Ekeland variational principle to the restriction of f_1 to the complete metric space $\rho\mathbb{B} \times \Omega_1 \cap B_\varepsilon(\bar{x}) \times \ldots \Omega_n \cap B_\varepsilon(\bar{x})$ (with the induced metric). Take $\rho' := \rho\alpha_1/\alpha_2$. It follows that there exist points $u' \in \rho'\mathbb{B}$ and $\omega_i' \in \Omega_i \cap B_{\rho'/\gamma}(\omega_i)$ such that

$$f_1(u, v_1, \ldots, v_n) - f_1(u', \omega_1', \ldots, \omega_n') + \alpha_2\|(u - u', v_1 - \omega_1', \ldots, v_n - \omega_n')\|_\gamma \geq 0$$

for all $u \in \rho\mathbb{B}$ and $v_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $i = 1, 2, \ldots, n$. Since $\rho' < \rho$, u' is an internal point of $\rho\mathbb{B}$. Since $\varepsilon_1 + \rho'/\gamma < \varepsilon$, ω_i' is an internal point of $B_\varepsilon(\bar{x})$. Hence $(u', \omega_1', \ldots, \omega_n')$ is a local minimum (on X^{n+1}) for the sum $f_1 + f_2 + f_3$, where

$$f_2(u, v_1, \ldots, v_n) := \alpha_2 \|(u - u', v_1 - \omega_1', \ldots, v_n - \omega_n')\|_\gamma,$$

$$f_3(u, v_1, \ldots, v_n) := \begin{cases} 0 & \text{if } v_i \in \Omega_i,\ i = 1, 2, \ldots, n, \\ \infty & \text{otherwise.} \end{cases}$$

Functions f_1 and f_2 are convex and Lipschitz continuous. We can apply the fuzzy sum rule (A4). Note that $\max_{1 \le i \le n} \|\omega_i' - \omega_i - a_i - u'\| > 0$. It is easy to check that the subdifferentials of f_1, f_2, and f_3 possess the following properties:

1) If $(u_1^*, v_{11}^*, \ldots, v_{1n}^*) \in \partial f_1(u, v_1, \ldots, v_n)$ then

$$u_1^* = -\sum_{i=1}^n v_{1i}^*, \quad \sum_{i=1}^n \|v_{1i}^*\| = 1 \tag{22}$$

for any $(u, v_1, \ldots, v_n)$ near $(u', \omega_1', \ldots, \omega_n')$. Indeed, f_1 is a composition of the linear mapping $h : X^{n+1} \to X^n$ given by $h(u, v_1, \ldots, v_n) := (v_1 - \omega_1 - a_1 - u, \ldots, v_n - \omega_1 - a_n - u$ and the convex function $g : X^n \to \mathbb{R}$ given by $g(v_1, \ldots, v_n) := \max_{1 \le i \le n} \|v_i\|$. Note that g is a norm on X^n. The corresponding dual norm has the form $(v_1^*, \ldots, v_n^*) \mapsto \sum_{i=1}^n \|v_i^*\|$. Note also that $g(\omega_1' - \omega_1 - a_1 - u', \ldots, \omega_n' - \omega_n - a_n - u') \neq 0$ and, thanks to continuity, $g(\omega_1 - \omega_1 - a_1 - u, \ldots, \omega_n - \omega_n - a_n - u) \neq 0$ for all $(u, v_1, \ldots, v_n)$ near $(u', \omega_1', \ldots, \omega_n')$. The claimed assertion follows from the convex chain rule and the representation of the subdifferential of a norm at a nonzero point [42, Corollary 2.4.16].

2) If $(u_2^*, v_{21}^*, \ldots, v_{2n}^*) \in \partial f_2(u, v_1, \ldots, v_n)$ then

$$\|u_2^*\| + \gamma^{-1} \sum_{i=1}^n \|v_{2i}^*\| \le \alpha_2 \tag{23}$$

for any $(u, v_1, \ldots, v_n) \in X^{n+1}$.

3) $\partial f_3(u, v_1, \ldots, v_n) = \{0_{X^*}\} \times \prod_{i=1}^n N_{\Omega_i}(v_i)$ for any $u \in X$ and $v_i \in \Omega_i$, $i = 1, 2, \ldots, n$ (by (A6)).

Chose a $\xi \in (0, \gamma)$ such that $(\alpha_2 + 2)\xi/(\gamma - \xi) < \alpha - \alpha_2$ and note that $\rho/\gamma < \varepsilon_2$. Applying the fuzzy sum rule, we find three points $(u_1, v_{11}, \ldots, v_{1n})$, $(u_2, v_{21}, \ldots, v_{2n})$, $(u_3, x_1, \ldots, x_n) \in X^{n+1}$ close to $(u', \omega_1', \ldots, \omega_n')$ (We will assume that $\max_{1 \le i \le n} \|x_i - \omega_i'\| < \varepsilon_2 - \rho/\gamma$.) and elements of the three subdifferentials $(u_1^*, v_{11}^*, \ldots, v_{1n}^*) \in \partial f_1(u_1, v_{11}, \ldots, v_{1n})$, $(u_2^*, v_{21}^*, \ldots, v_{2n}^*) \in \partial f_2(u_2, v_{21}, \ldots, v_{2n})$ and $(0_{X^*}, v_{31}^*, \ldots, v_{3n}^*) \in \partial f_3(u_3, x_1, \ldots, x_n)$ such that

$$x_i \in \Omega_i,\ i = 1, 2, \ldots n,$$

$$\|(u_1^* + u_2^*, v_{11}^* + v_{21}^* + v_{31}^*, \ldots, v_{1n}^* + v_{2n}^* + v_{3n}^*)\| < \xi.$$

It follows that $v_{3i}^* \in N_{\Omega_i}(x_i)$, $i = 1, 2, \ldots, n$; (22) and (23) hold true and

$$\|u_1^* + u_2^*\| < \xi, \quad \sum_{i=1}^n \|v_{1i}^* + v_{2i}^* + v_{3i}^*\| < \xi. \tag{24}$$

Then $\|x_i - \bar{x}\| \le \|x_i - \omega_i'\| + \|\omega_i' - \omega_i\| + \|\omega_i - \bar{x}\| < \varepsilon$. Denote $\beta := \sum_{i=1}^n \|v_{2i}^*\|$. By (23), $0 \le \beta \le \gamma\alpha_2 < 1 - \gamma$. By the second inequality in (24) and the second equality in (22), we have

$$\sum_{i=1}^n \|v_{3i}^*\| \ge 1 - \beta - \xi > \gamma - \xi > 0.$$

The second inequality in (24) implies also $\|\sum_{i=1}^n (v_{1i}^* + v_{2i}^* + v_{3i}^*)\| < \xi$, and consequently

$$\left\|\sum_{i=1}^n v_{3i}^*\right\| < \left\|\sum_{i=1}^n v_{1i}^*\right\| + \beta + \xi.$$

Applying successively the first equality in (22), the first inequality in (24), and inequality (23) and recalling the definition of γ, we obtain

$$\left\|\sum_{i=1}^{n} v_{3i}^*\right\| \le \|u_2^*\| + \beta + 2\xi \le \alpha_2 + (1-\gamma^{-1})\beta + 2\xi < \alpha_2(1-\beta) + 2\xi.$$

Put $x_i^* = v_{3i}^*/\sum_{i=1}^n \|v_{3i}^*\|$, $i = 1,2,\ldots,n$. Then obviously $x_i^* \in N_{\Omega_i}(x_i)$, $i = 1,2,\ldots,n$, $\sum_{i=1}^n \|x_i^*\| = 1$, and

$$\left\|\sum_{i=1}^{n} x_i^*\right\| < \frac{\alpha_2(1-\beta)+2\xi}{1-\beta-\xi} = \alpha_2 + \frac{(\alpha_2+2)\xi}{1-\beta-\xi} < \alpha_2 + \frac{(\alpha_2+2)\xi}{\gamma-\xi} < \alpha.$$ □

Next theorem is the limiting form of Theorem 3.1. It establishes the relationship between constants (12) and (17), and consequently between the pairs of primal space properties (AS) and (UR), on one hand, and dual space ones (NAS) and (NUR) (or their Fréchet versions), on the other hand.

Theorem 3.2 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, where $1 < |I| < \infty$.*

(i) $\hat{\theta}[\mathbf{\Omega}](\bar{x}) \le \hat{\eta}^F[\mathbf{\Omega}](\bar{x})$.

(ii) *If the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$, then $\hat{\theta}[\mathbf{\Omega}](\bar{x}) \ge \hat{\eta}[\mathbf{\Omega}](\bar{x})$.*

Part (i) of Theorem 3.2 was proved in [18], while part (ii) was established in [21] in the Asplund space setting and with Fréchet normal cones. A slightly weaker estimate can be found in [18, 20].

Proof (i) Let $\alpha > \hat{\eta}^F[\mathbf{\Omega}](\bar{x})$. By definition (17), for any $\varepsilon > 0$ there exist $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}^F(\omega_i)$ $(i \in I)$, such that conditions (20) hold true. It follows from Theorem 3.1 (i) and definitions (12) and (10) that $\hat{\theta}[\mathbf{\Omega}](\bar{x}) < \alpha$.

(ii). Let $\alpha > \hat{\theta}[\mathbf{\Omega}](\bar{x})$ and $\varepsilon > 0$. By definitions (12) and (10), there exists a positive number $\rho < (\alpha+1)^{-1}\varepsilon/2$, and points $\omega_i \in \Omega_i \cap B_{\varepsilon/2}(\bar{x})$ and $a_i \in X$ $(i \in I)$, such that $\max_{i\in I}\|a_i\| < \alpha\rho$ and (14) holds true. It follows from Theorem 3.1 (ii) and definition (17) that $\hat{\eta}[\mathbf{\Omega}](\bar{x}) < \alpha$. □

There are several important corollaries of Theorem 3.2.

Corollary 3.2.1 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, where $1 < |I| < \infty$.*

(i) *If the collection of sets $\mathbf{\Omega}$ is Fréchet normally approximately stationary at $\bar{x}$, then it is approximately stationary at $\bar{x}$.*

(ii) *If the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$ and the collection of sets $\mathbf{\Omega}$ is*
  (a) *extremal at $\bar{x}$, or*
  (b) *locally extremal at $\bar{x}$, or*
  (c) *stationary at $\bar{x}$, or*
  (d) *approximately stationary at $\bar{x}$,*

  then the collection of sets $\mathbf{\Omega}$ is normally approximately stationary at $\bar{x}$.

Obviously, only assumption (d) is critical in part (ii) of Corollary 3.2.1. Assumptions (a)–(c) are sufficient thanks to the chain of implications (15).

Corollary 3.2.1 (ii) in the Asplund space setting and with Fréchet normal cones under assumption (b) was established in [5] as a generalization of the original theorem in [2] formulated in the setting of a Banach space admitting a Fréchet differentiable renorm and with Fréchet ε-normals under assumption (a) (and in [4] under assumption (b)). This result is now known as the *Extremal principle* and is generally recognized as one of the corner-stones of the contemporary variational analysis (see [6]). Using Corollary 3.2.1 (ii), one can formulate a stronger statement – the *Extended extremal principle* [17, 18] (cf. Theorem 2.2). Some earlier formulations can be found in [14–16].

Corollary 3.2.2 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, where $1 < |I| < \infty$. Suppose the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$ and X is Asplund. The collection of sets $\boldsymbol{\Omega}$ is approximately stationary at $\bar{x}$ if and only if it is Fréchet normally approximately stationary at $\bar{x}$.*

Note that the "if" part of Corollary 3.2.2 is valid in general Banach spaces. On the other hand, the "only if" part cannot be extended beyond Asplund spaces and provides an equivalent extremal characterization of Asplund spaces (see [5, 6]).

One can easily formulate the analogues of Corollaries 3.2.1 and 3.2.2 for regularity properties.

Corollary 3.2.3 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, where $1 < |I| < \infty$.*

(i) *If the collection of sets $\boldsymbol{\Omega}$ is uniformly regular at $\bar{x}$, then it is Fréchet normally uniformly regular at $\bar{x}$.*
(ii) *If the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$ and the collection of sets $\boldsymbol{\Omega}$ is normally uniformly regular at $\bar{x}$, then it is uniformly regular at $\bar{x}$.*

Corollary 3.2.4 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, where $1 < |I| < \infty$. Suppose the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$ and X is Asplund. The collection of sets $\boldsymbol{\Omega}$ is uniformly regular at $\bar{x}$ if and only if it is Fréchet normally uniformly regular at $\bar{x}$.*

Remark 3.3 *If $\dim X < \infty$, then the normal approximate stationarity and uniform regularity conditions can be reformulated equivalently in 'exact' form. It is sufficient to observe that, in finite dimensions, constant* (17) *coincides with the following one:*

$$\bar{\eta}[\boldsymbol{\Omega}](\bar{x}) = \min_{\substack{x_i^* \in \bar{N}_{\Omega_i}(\bar{x}),\, i\in I \\ \sum_{i\in I}\|x_i^*\|=1}} \left\| \sum_{i\in I} x_i^* \right\|, \tag{25}$$

where $\overline{N}_{\Omega}(\bar{x})$ is the limiting normal cone *to Ω at $\bar{x}$:*

$$\overline{N}_{\Omega}(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} N_{\Omega}(x). \tag{26}$$

If $\dim X = \infty$, then the limiting normal cone is still defined by (26)*, where* Lim sup *is understood as the sequential upper/outer limit. However, constants* (17) *and* (25) *are not equal in general. It is still possible to formulate 'exact' versions of the normal approximate stationarity and uniform regularity conditions (not equivalent to the original (NAS) and (NUR)!) in terms of limiting normal cones under the* sequential normal compactness *requirement imposed on all but one sets Ω_i, $i \in I$ (see [6] for the definition and discussion of the sequential normal compactness condition.)*

Remark 3.4 *It is easy to see from the definitions that the approximate stationarity and uniform regularity properties are determined by the ratio of the numbers $r := \max_{i\in I} \|a_i\|$ and ρ in formula* (14)*. For instance, approximate stationarity means the existence of sequences $\rho_k \downarrow 0$, $\omega_{ik} \xrightarrow{\Omega_i} \bar{x}$, and $a_{ik} \to 0$ as $k \to \infty$ such that* (14) *holds and the corresponding sequence r_k satisfies $r_k/\rho_k \downarrow 0$. This is obviously true for the stronger properties (LE) and (E). Some other sufficient conditions, specifying the rate of convergence of r_k/ρ_k to 0 can be of interest in applications. For example, one can consider "rated extremal systems" [23] satisfying $r_k < \gamma\rho_k^{\alpha}$ with some $\gamma > 0$ and $\alpha > 1$ (or equivalently $\rho_k > \gamma r_k^{\alpha}$ with some $\gamma > 0$ and $\alpha \in]0, 1[$). Theorems 3.5, 3.7, 3.8, and 3.9 in [23] follow from Theorem 3.2 (ii) above.*

4 Infinite Collections of Sets

In this section, we still consider a collection of sets $\boldsymbol{\Omega} = \{\Omega_i\}_{i\in I} \subset X$, but now the index set I is not assumed finite. The goal is to extend Theorem 3.2 to this more general setting.

Note that the proofs of statements like Theorem 3.2 (ii) (see [18, 20, 21]) strongly rely on the assumption that I is finite. The idea exploited in this section is to extend definitions (10)–(12) and (17), allowing for the infinite index set I to be replaced by a sequence of its finite subsets.

4.1 Finite Subsystems

It is assumed that $|I| > 1$ and $\bar{x} \in \bigcap_{i\in I} \Omega_i$. To simplify the definitions, we are going to use the following notation:

$$\mathcal{J} := \{J \subset I |\ 1 < |J| < \infty\}.$$

Next three constants can be considered as extensions of (10), (12) and (17) respectively.

$$\theta_\rho[\mathbf{\Omega}](\bar{x}) := \inf_{J\in\mathcal{J}} \theta_\rho[\{\Omega_i\}_{i\in J}](\bar{x}), \quad \rho \in]0,\infty], \tag{27}$$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) := \sup_{\varepsilon>0} \inf_{\substack{\rho\in]0,\varepsilon[,\ J\in\mathcal{J}\\ \omega_i\in B_\varepsilon(\bar{x})\cap\Omega_i\ (i\in J)}} \frac{\theta_\rho[\{\Omega_i-\omega_i\}_{i\in J}](0)}{\rho}, \tag{28}$$

$$\hat{\eta}[\mathbf{\Omega}](\bar{x}) := \sup_{\varepsilon>0} \inf_{\substack{J\in\mathcal{J}\\ x_i\in\Omega_i\cap B_\varepsilon(\bar{x}),\, x_i^*\in N_{\Omega_i}(x_i)\ (i\in J)\\ \sum_{i\in J}\|x_i^*\|=1}} \left\|\sum_{i\in J} x_i^*\right\|. \tag{29}$$

Indeed, if I is a finite set, then constants (27), (28) and (29) reduce to (10), (12) and (17) respectively. Constant $\theta[\mathbf{\Omega}](\bar{x})$ can still be defined by (11). Note that N_Ω in (29) is an abstract normal cone mapping discussed in Section 2. In the case of the Fréchet normal cone, we will write $\hat{\eta}^F[\mathbf{\Omega}](\bar{x})$.

Next definition extends Definitions 3.1 and 3.2. We keep the same abbreviations for the corresponding properties.

Definition 4.1 The collection of sets $\mathbf{\Omega}$ is

(E) *extremal* at $\bar{x}$ iff $\theta_\infty[\mathbf{\Omega}](\bar{x}) = 0$, i.e.,

for any $\varepsilon > 0$ there exist $J \in \mathcal{J}$ and $a_i \in X$ $(i \in J)$ such that $\max_{i\in J}\|a_i\| < \varepsilon$ and

$$\bigcap_{i\in J}(\Omega_i - a_i) = \emptyset;$$

(LE) *locally extremal* at $\bar{x}$ iff $\theta_\rho[\mathbf{\Omega}](\bar{x}) = 0$ for some $\rho > 0$, i.e.,

there exists a $\rho > 0$ such that for any $\varepsilon > 0$ there are $J \in \mathcal{J}$ and $a_i \in X$ $(i \in J)$ such that $\max_{i\in J}\|a_i\| < \varepsilon$ and

$$\bigcap_{i\in J}(\Omega_i - a_i)\bigcap B_\rho(\bar{x}) = \emptyset; \tag{30}$$

(S) *stationary* at $\bar{x}$ iff $\theta[\mathbf{\Omega}](\bar{x}) = 0$, i.e.,

for any $\varepsilon > 0$ there exist $\rho \in]0,\varepsilon[$; $J \in \mathcal{J}$; and $a_i \in X$ $(i \in J)$ such that $\max_{i\in J}\|a_i\| < \varepsilon\rho$ and (30) holds true;

(R) *regular* at $\bar{x}$ iff $\theta[\mathbf{\Omega}](\bar{x}) > 0$, i.e.,

there exists an $\alpha > 0$ and an $\varepsilon > 0$ such that

$$\bigcap_{i\in J}(\Omega_i - a_i)\bigcap B_\rho(\bar{x}) \neq \emptyset \tag{31}$$

for any $\rho \in]0,\varepsilon[$; $J \in \mathcal{J}$; and $a_i \in X$ $(i \in J)$ satisfying $\max_{i\in J}\|a_i\| \le \alpha\rho$;

(AS) *approximately stationary* at $\bar{x}$ iff $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = 0$, i.e.,

for any $\varepsilon > 0$ there exist $\rho \in]0,\varepsilon[$; $J \in \mathcal{J}$; $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $a_i \in X$ $(i \in J)$ such that $\max_{i\in J}\|a_i\| < \varepsilon\rho$ and

$$\bigcap_{i\in J}(\Omega_i - \omega_i - a_i)\bigcap(\rho\mathbb{B}) = \emptyset; \tag{32}$$

(UR) *uniformly regular* at $\bar{x}$ iff $\hat{\theta}[\mathbf{\Omega}](\bar{x}) > 0$, i.e.,

there exists an $\alpha > 0$ and an $\varepsilon > 0$ such that

$$\bigcap_{i\in J}(\Omega_i - \omega_i - a_i)\bigcap(\rho\mathbb{B}) \neq \emptyset \tag{33}$$

for any $\rho \in]0,\varepsilon[$; $J \in \mathcal{J}$; $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, and $a_i \in X$ $(i \in J)$ satisfying $\max_{i\in J}\|a_i\| \le \alpha\rho$;

(NAS) *normally approximately stationary* at $\bar{x}$ iff $\hat{\eta}[\mathbf{\Omega}](\bar{x}) = 0$, i.e.,

for any $\varepsilon > 0$ there exist $J \in \mathcal{J}$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in J)$ such that

$$\left\| \sum_{i \in J} x_i^* \right\| < \varepsilon \sum_{i \in J} \|x_i^*\|; \tag{34}$$

(FNAS) *Fréchet normally approximately stationary* at $\bar{x}$ iff $\hat{\eta}^F[\mathbf{\Omega}](\bar{x}) = 0$, i.e.,

for any $\varepsilon > 0$ there exist $J \in \mathcal{J}$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$ such that (34) holds true;

(NUR) *normally uniformly regular* at $\bar{x}$ iff $\hat{\eta}[\mathbf{\Omega}](\bar{x}) > 0$, i.e.,

there exists an $\alpha > 0$ and an $\varepsilon > 0$ such that

$$\left\| \sum_{i \in J} x_i^* \right\| \geq \alpha \sum_{i \in J} \|x_i^*\| \tag{35}$$

for any $J \in \mathcal{J}$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in J)$;

(FNUR) *Fréchet normally uniformly regular* at $\bar{x}$ iff $\hat{\eta}^F[\mathbf{\Omega}](\bar{x}) > 0$, i.e.,

there exists an $\alpha > 0$ and an $\varepsilon > 0$ such that (35) holds true for any $J \in \mathcal{J}$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$.

All the implications in (15) and (16) remain true for these modified extremality, stationarity and regularity properties.

Remark 4.1 *In the normal approximate stationarity definitions (NAS) and (FNAS) above, the small parameter ε is present in the right hand side of* (34). *Sometimes conditions of this type are formulated in a different way (see e.g. [6,23]), with* (34) *replaced by a stronger pair of conditions: $\left\|\sum_{i\in J} x_i^*\right\| = 0$ and $\sum_{i\in J} \|x_i^*\| = 1$ at the expense of relaxing the requirement on x_i^*: $x_i^* \in N_{\Omega_i}(x_i) + \varepsilon\mathbb{B}^*$. It is easy to check that in the case of a finite collection of sets these two settings are equivalent. However, when $|I| = \infty$ the second setting can lead to accumulation of errors and triviality of the Extremal principle as discussed in [23].*

Example 4.2 *Consider the collection $\mathbf{\Omega}$ of sets $\Omega_i = \{(u,v) \in \mathbb{R}^2 |\ u^2 - v \geq -1/i\}$, $\Omega_i' = \{(u,v) \in \mathbb{R}^2 |\ u^2 + v \geq -1/i\}$, $i = 1, 2, \ldots$. Then $\bar{x} := (0,0) \in \operatorname{int}(\Omega_i \cap \Omega_i')$ for all $i = 1, 2, \ldots$, and $(0,0) \in \operatorname{bd} \cap_{i=1}^\infty (\Omega_i \cap \Omega_i')$. We are going to show that collection $\mathbf{\Omega}$ is stationary but not locally extremal at $\bar{x}$.*

Let $\rho > 0$ be given. Chose an $\varepsilon \in]0, \rho[$ such that $(\rho - \varepsilon)^2 - \varepsilon > 0$. Then for any numbers α and β satisfying $|\alpha| < \varepsilon$ and $|\beta| < \varepsilon$, it holds $(\rho + \alpha)^2 - \beta > 0$ and $(\rho + \alpha)^2 + \beta > 0$. Hence, $(\rho, 0) \in (\Omega_i - a_i)$ and $(\rho, 0) \in (\Omega_i' - b_i)$ for any $i \in \mathbb{N}$ and any $a_i, b_i \in \mathbb{R}^2$ satisfying $\|a_i\| < \varepsilon$ and $\|b_i\| < \varepsilon$. This means that collection $\mathbf{\Omega}$ is not locally extremal at $\bar{x}$.

Let $\varepsilon > 0$ be given. Chose a $\rho \in]0, \varepsilon[$ and an index $i > [\rho(\varepsilon - \rho)]^{-1}$. Then $\rho^2 + 1/i < \rho^2 + \rho(\varepsilon - \rho) = \varepsilon\rho$, and one can chose an $\alpha \in]\rho^2 + 1/i, \varepsilon\rho[$. Taking into account the definitions of Ω_i and Ω_i', we have

$$[\Omega_i - (0,\alpha)] \cap [\Omega_i' + (0,\alpha)] \cap (\rho\mathbb{B}) = \{(u,v) \in \rho\mathbb{B} |\ |v| \leq u^2 + 1/i - \alpha\} \subset \{(u,v) \in \mathbb{R}^2 |\ |v| \leq \rho^2 + 1/i - \alpha\} = \emptyset.$$

Hence, collection $\mathbf{\Omega}$ is stationary at $\bar{x}$.

It is easy to check that condition (FNAS) in Definition 4.1 is satisfied too.

The next theorem is an extension of Theorem 3.2. It establishes the relationship between constants (28) and (29).

Theorem 4.3 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, $|I| > 1$.*

(i) $\hat{\theta}[\mathbf{\Omega}](\bar{x}) \leq \hat{\eta}^F[\mathbf{\Omega}](\bar{x})$.

Moreover, if the collection of sets $\mathbf{\Omega}$ is Fréchet normally approximately stationary at $\bar{x}$, then it is approximately stationary at $\bar{x}$ and, for any $\varepsilon > 0$, condition (AS) is satisfied with the same set of indices J the existence of which is guaranteed by condition (FNAS).

(ii) *If the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$, then $\hat{\theta}[\mathbf{\Omega}](\bar{x}) \geq \hat{\eta}[\mathbf{\Omega}](\bar{x})$.*
Moreover, if the collection of sets $\mathbf{\Omega}$ is approximately stationary at $\bar{x}$, then it is normally approximately stationary at $\bar{x}$ and, for any $\varepsilon > 0$, condition (NAS) is satisfied with the same set of indices J the existence of which is guaranteed by condition (AS).

Proof (i) Let $\hat{\eta}^F[\mathbf{\Omega}](\bar{x}) < \alpha$. By definition (29), for any $\varepsilon > 0$ there exist $J \in \mathcal{J}$; $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$ and $x_i^* \in N^F_{\Omega_i}(\omega_i)$ $(i \in J)$ such that

$$\sum_{i \in J} \|x_i^*\| = 1 \quad \text{and} \quad \left\| \sum_{i \in J} x_i^* \right\| < \alpha. \tag{36}$$

It follows from Theorem 3.1 (i) and definitions (28) and (10) that $\hat{\theta}[\mathbf{\Omega}](\bar{x}) < \alpha$.

(ii). Let $\alpha > \hat{\theta}[\mathbf{\Omega}](\bar{x})$ and $\varepsilon > 0$. By definitions (28) and (10), there exists a positive number $\rho < (\alpha+1)^{-1}\varepsilon/2$, a subset $J \in \mathcal{J}$, and points $\omega_i \in \Omega_i \cap B_{\varepsilon/2}(\bar{x})$ and $a_i \in X$ $(i \in J)$ such that $\max_{i \in J} \|a_i\| < \alpha\rho$ and (32) holds true. It follows from Theorem 3.1 (ii) and definition (29) that $\hat{\eta}[\mathbf{\Omega}](\bar{x}) < \alpha$. □

Theorem 3.2 follows from Theorem 4.3 due to the observation made after the definitions of constants (27)–(29). All the corollaries formulated in Section 3 remain valid with the assumption $|I| < \infty$ omitted. In particular, Theorem 4.3 implies Theorem 2.3.

4.2 **Finite Subsystems with Growth Condition**

In the definitions of stationarity and regularity properties considered above, it is allowed that $|J| \to \infty$. For example, in the definition of property (S), it is required that for any $\varepsilon > 0$ there exists a finite subset J of indices such that the corresponding finite collection of sets satisfies certain properties. When $\varepsilon \to 0$, the cardinality $|J|$ can grow very quickly in order to have (30) fulfilled. It can be important to impose restrictions on the rate of growth of $|J|$. For that purpose, we are going to use a *gauge* function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$. Given $\alpha > 0$, denote:

$$\mathcal{J}_\alpha := \{J \subset I |\ 1 < |J| < \Phi(\alpha)\}.$$

Obviously $\mathcal{J}_\alpha \subset \mathcal{J}$ and $\mathcal{J}_\alpha = \mathcal{J}$ if $\Phi(\alpha) = \infty$.

The following definition introduces modified versions of the stationarity and regularity properties.

Definition 4.2 The collection of sets $\mathbf{\Omega}$ is

(S_Φ) *Φ-stationary* at $\bar{x}$ iff for any $\varepsilon > 0$ there exist $\rho \in]0, \varepsilon[$; $\alpha \in]0, \varepsilon[$; $J \in \mathcal{J}_\alpha$; and $a_i \in X$ $(i \in J)$ such that $\max_{i \in J} \|a_i\| < \alpha\rho$ and (30) holds true;

(R_Φ) *Φ-regular* at $\bar{x}$ iff there exists an $\alpha_0 > 0$ and an $\varepsilon > 0$ such that (31) holds true for any $\alpha \in]0, \alpha_0[$; $\rho \in]0, \varepsilon[$; $J \in \mathcal{J}_\alpha$; and $a_i \in X$ $(i \in J)$ satisfying $\max_{i \in J} \|a_i\| \leq \alpha\rho$;

(AS_Φ) *approximately Φ-stationary* at $\bar{x}$ iff for any $\varepsilon > 0$ there exist $\rho \in]0, \varepsilon[$; $\alpha \in]0, \varepsilon[$; $J \in \mathcal{J}_\alpha$; $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $a_i \in X$ $(i \in J)$ such that $\max_{i \in J} \|a_i\| < \alpha\rho$ and (32) holds true;

(UR_Φ) *uniformly Φ-regular* at $\bar{x}$ iff there exists an $\alpha_0 > 0$ and an $\varepsilon > 0$ such that (33) holds true for any $\alpha \in]0, \alpha_0[$; $\rho \in]0, \varepsilon[$; $J \in \mathcal{J}_\alpha$; $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, and $a_i \in X$ $(i \in J)$ satisfying $\max_{i \in J} \|a_i\| \leq \alpha\rho$;

(NAS_Φ) *normally approximately Φ-stationary* at $\bar{x}$ iff for any $\varepsilon > 0$ there exist $\alpha \in]0, \varepsilon[$; $J \in \mathcal{J}_\alpha$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in J)$ such that

$$\left\| \sum_{i \in J} x_i^* \right\| < \alpha \sum_{i \in J} \|x_i^*\|; \tag{37}$$

(FNAS$_\Phi$) *Fréchet normally approximately Φ-stationary* at $\bar{x}$ iff for any $\varepsilon > 0$ there exist $\alpha \in]0, \varepsilon[$; $J \in \mathcal{J}_\alpha$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$ such that (37) holds true;

(NUR$_\Phi$) *normally uniformly Φ-regular* at $\bar{x}$ iff there exists an $\alpha_0 > 0$ and an $\varepsilon > 0$ such that (35) holds true for any $\alpha \in]0, \alpha_0[$; $J \in \mathcal{J}_\alpha$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in J)$;

(FNUR$_\Phi$) *Fréchet normally uniformly Φ-regular* at $\bar{x}$ iff there exists an $\alpha_0 > 0$ and an $\varepsilon > 0$ such that (35) holds true for any $\alpha \in]0, \alpha_0[$; $J \in \mathcal{J}_\alpha$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$.

The supremum of all $\alpha_0 > 0$ in properties (R$_\Phi$), (UR$_\Phi$), (NUR$_\Phi$), and (FNUR$_\Phi$) (with the convention $\inf \emptyset = 0$) will be denoted $\theta_\Phi[\mathbf{\Omega}](\bar{x})$, $\hat{\theta}_\Phi[\mathbf{\Omega}](\bar{x})$, $\hat{\eta}_\Phi[\mathbf{\Omega}](\bar{x})$, and $\hat{\eta}^F_\Phi[\mathbf{\Omega}](\bar{x})$, respectively.

It would be good to have limiting representations of constants $\theta_\Phi[\mathbf{\Omega}](\bar{x})$, $\hat{\theta}_\Phi[\mathbf{\Omega}](\bar{x})$, and $\hat{\eta}_\Phi[\mathbf{\Omega}](\bar{x})$ similar to formulas (11), (28), and (29). Unfortunately this is not possible in general because in all conditions (R$_\Phi$), (UR$_\Phi$), (NUR$_\Phi$), and (FNUR$_\Phi$), the number α is present twice: in the inequality defining the property in question and in the growth condition $|J| < \Phi(\alpha)$.

The stationarity (regularity) properties in Definition 4.2 are obviously stronger (weaker) than the corresponding properties in Definition 4.1.

If I is a finite set, then one can take a constant function $\Phi(\alpha) = |I| + 1$ for all $\alpha > 0$ (in fact, one can take any function Φ satisfying $\Phi(\alpha) > |I|$). The stationarity and regularity properties in Definition 4.2 will coincide with the corresponding properties in Definitions 3.1 and 3.2. Considering constant functions $\Phi(\alpha) \equiv m$ satisfying $m \le |I|$ can also lead to meaningful conditions. Basically, such functions specify explicitly the number of sets participating in the corresponding stationarity and regularity conditions. For different numbers, the conditions can be significantly different.

Example 4.4 *Consider the collection $\mathbf{\Omega}$ of three halfplanes in $\mathbb{R}^2$:*

$$\Omega_1 := \{(x, y) |\ y \ge 0\}, \quad \Omega_2 := \{(x, y) |\ x \ge 0\}, \quad \text{and} \quad \Omega_3 := \{(x, y) |\ x + y \le 0\}.$$

Obviously $(0,0) \in \Omega_1 \cap \Omega_2 \cap \Omega_3$, and it is easy to establish the representations for the Fréchet normal cones (which coincide in this setting with the normal cones in the sense of convex analysis) to these sets at $(0,0)$: $N^F_{\Omega_1}(0,0) = \{(0, v) |\ v \le 0\}$, $N^F_{\Omega_2}(0,0) = \{(u, 0) |\ u \le 0\}$, and $N^F_{\Omega_3}(0,0) = \{(u, u) |\ u \ge 0\}$.

The collection $\mathbf{\Omega}$ is Fréchet normally approximately stationary at $(0,0)$ (In fact, the sets are convex and this can be interpreted as a separation property.) Indeed, take a positive number c. Then $x_1^* := (0, -c) \in N^F_{\Omega_1}(0,0)$, $x_2^* := (-c, 0) \in N^F_{\Omega_2}(0,0)$, $x_3^* := (c, c) \in N^F_{\Omega_3}(0,0)$, $x_1^* + x_2^* + x_3^* = 0$ while $\|x_1^*\| + \|x_2^*\| + \|x_3^*\| > 0$.

Take a constant gauge function $\Phi(\alpha) \equiv m$.

If $m > 3$, then obviously the collection $\mathbf{\Omega}$ is Fréchet normally approximately Φ-stationary at $(0,0)$.

If $m = 3$, then the collection $\mathbf{\Omega}$ is Fréchet normally uniformly Φ-regular at $(0,0)$. To show this, one needs to consider all pairs of sets from $\mathbf{\Omega}$. For simplicity, we will assume that the primal space $\mathbb{R}^2$ is equipped with the maximum norm: $\|x, y\| = \max\{|x|, |y|\}$. Then the dual norm is of the sum type: $\|u, v\| = |u| + |v|$.

Consider arbitrary $x_1^* := (0, v) \in N^F_{\Omega_1}(0,0)$ and $x_2^* := (u, 0) \in N^F_{\Omega_2}(0,0)$ such that $\|x_1^*\| + \|x_2^*\| = 1$, that is, $|u| + |v| = 1$. Then $\|x_1^* + x_2^*\| = \|u, v\| = |u| + |v| = 1$.

Consider arbitrary $x_1^* := (0, v) \in N^F_{\Omega_1}(0,0)$ and $x_2^* := (u, u) \in N^F_{\Omega_3}(0,0)$ such that $\|x_1^*\| + \|x_2^*\| = 1$, that is, $2|u| + |v| = 2u - v = 1$. Then $\|x_1^* + x_2^*\| = \|u, u + v\| = u + |u + v| = u + |3u - 1| \ge 1/3$.

Similarly, consider arbitrary $x_1^* := (v, 0) \in N^F_{\Omega_2}(0,0)$ and $x_2^* := (u, u) \in N^F_{\Omega_3}(0,0)$ such that $\|x_1^*\| + \|x_2^*\| = 1$, that is, $2|u| + |v| = 2u - v = 1$. Then $\|x_1^* + x_2^*\| = \|u + v, u\| \ge 1/3$.

Thus in all three cases, it holds $\|x_1^* + x_2^*\| \ge 1/3$ as long as $\|x_1^*\| + \|x_2^*\| = 1$. Since in the convex case the normal cone mapping is upper semicontinuous, there is no need to consider points in the neighbourhood of $(0,0)$, since (35) holds true for any $0 < \alpha < 1/3$.

When dealing with infinite systems, it seems reasonable to consider gauge functions Φ such that $\Phi(\alpha) \to \infty$ as $\alpha \downarrow 0$. For instance, this assumption is necessary for the implication (LE) $\Rightarrow$ (S$_\Phi$) to be true. However, the definitions of stationarity and regularity properties as well as their characterizations in the statements below are valid without this requirement.

The implication (S$_\Phi$) $\Rightarrow$ (AS$_\Phi$) is always true (with the same ρ, α, J, and a_i). The next theorem establishes the relationship between the approximate Φ-stationarity properties. It complements Theorem 4.3.

Theorem 4.5 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, $|I| > 1$.*

(i) *If the collection of sets $\mathbf{\Omega}$ is Fréchet normally approximately Φ-stationary at $\bar{x}$, then it is approximately Φ-stationary at $\bar{x}$.*

Moreover, for any $\varepsilon > 0$, condition (AS$_\Phi$) is satisfied with the same number α and set of indices J the existence of which is guaranteed by condition (FNAS$_\Phi$).

(ii) *Suppose the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$. If the collection of sets $\mathbf{\Omega}$ is approximately Φ-stationary at $\bar{x}$, then it is normally approximately Φ-stationary at $\bar{x}$.*

Moreover, for any $\varepsilon > 0$, condition (NAS$_\Phi$) is satisfied with the same number α and set of indices J the existence of which is guaranteed by condition (AS$_\Phi$).

Proof (i) Let $\varepsilon > 0$. By Definition 4.2 (FNAS$_\Phi$), there exist $\alpha \in]0, \varepsilon[$; $J \in \mathcal{J}_\alpha$; $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, and $x_i^* \in N^F_{\Omega_i}(\omega_i)$ $(i \in J)$ such that conditions (36) hold true. It follows from Theorem 3.1 (i) that all conditions in Definition 4.2 (AS$_\Phi$) are satisfied.

(ii). Let $\varepsilon > 0$. By Definition 4.2 (AS$_\Phi$), there exist positive numbers $\alpha < \varepsilon$ and $\rho < (\varepsilon+1)^{-1}\varepsilon/2$, a subset $J \in \mathcal{J}_\alpha$, and points $\omega_i \in \Omega_i \cap B_{\varepsilon/2}(\bar{x})$ and $a_i \in X$ $(i \in J)$ such that $\max_{i\in J} \|a_i\| < \alpha\rho$ and (32) holds true. Then $\rho < (\alpha+1)^{-1}\varepsilon/2$ and it follows from Theorem 3.1 (ii) that there exist points $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in J)$ such that conditions (36) hold true. Hence all conditions in Definition 4.2 (NAS$_\Phi$) are satisfied. □

Corollary 4.5.1 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, $|I| > 1$. Suppose X is Asplund and the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$. The collection of sets $\mathbf{\Omega}$ is approximately Φ-stationary at $\bar{x}$ if and only if it is Fréchet normally approximately Φ-stationary at $\bar{x}$.*

Moreover, for any $\varepsilon > 0$, conditions (AS$_\Phi$) and (FNAS$_\Phi$) are satisfied with the same number α and set of indices J.

The above corollary implies Theorem 2.4.

Once again, the "if" part of Corollary 4.5.1 is valid in general Banach spaces while the "only if" part cannot be extended beyond Asplund spaces and provides an equivalent extremal characterization of Asplund spaces.

Since the regularity properties in Definition 4.2 are negations of the corresponding stationarity properties, the assertions of Theorem 4.5 and Corollary 4.5.1 can be reformulated in terms of regularity properties.

Corollary 4.5.2 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, $|I| > 1$.*

(i) *If the collection of sets $\mathbf{\Omega}$ is uniformly Φ-regular at $\bar{x}$, then it is Fréchet normally uniformly Φ-regular at $\bar{x}$.*

(ii) *Suppose the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$. If the collection of sets $\mathbf{\Omega}$ is normally uniformly Φ-regular at $\bar{x}$, then it is uniformly Φ-regular at $\bar{x}$.*

Corollary 4.5.3 *Let $\bar{x} \in \bigcap_{i\in I} \Omega_i$, $|I| > 1$. Suppose X is Asplund and the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$. The collection of sets $\mathbf{\Omega}$ is uniformly Φ-regular at $\bar{x}$ if and only if it is Fréchet normally uniformly Φ-regular at $\bar{x}$.*

Remark 4.6 *In the case of an infinite index set I, stationarity and regularity properties in Definition 4.2 depend in general on the choice of the gauge function Φ, which determines the "growth rate" of the cardinality $|J|$ of finite subsets $J \subset I$. Since the same gauge function participates in the assumptions and conclusions of Theorem 4.5 and its corollaries, when applying them for characterizing stationarity (regularity) of a specific collection of sets, it can be important to find the smallest (largest) function such that the property in question still holds true. Then the theorem or a corollary provides the strongest conclusion.*

Possible choices of Φ that could be of interest:

- $\Phi(\alpha) = +\infty$ *for all* $\alpha > 0$. *This means that no restrictions are imposed on the growth of the cardinality $|J|$, and stationarity and regularity properties in Definition 4.2 reduce to the corresponding properties in Definition 4.1.*
- $\alpha\Phi(\alpha) \to 0$ *as* $\alpha \downarrow 0$. *Φ can be an increasing function, but its growth must be much slower than that of α^{-1}: $\Phi(\alpha) = o(\alpha^{-1})$. A growth condition of this type was used in [23] when defining "R-rated extremal systems", "R-perturbed extremal systems", and the "rated extremal principle". Theorems 4.6 and 4.10 in [23] follow from Theorem 4.5 (ii) above, which allows to establish the conclusions of these two theorems under significantly weaker assumptions.*
- $\Phi(\alpha) = \gamma\alpha^k$ *where* $\gamma > 0$ *and* $k > 0$.

One can consider other growth conditions: exponential, logarithmic, etc.

5 Normals to Infinite Intersections

An important group of calculus results in variational analysis consists of rules which allow to represent normals (of a certain type: convex, Fréchet, limiting or other) to the finite ($|I| < \infty$) intersection $\Omega := \bigcap_{i\in I} \Omega_i$ of a collection of sets at a point $\bar{x} \in \Omega$ via normals to particular sets at or around this point. Such *intersection rules* in the convex and nonconvex settings are well known [4, 6, 27, 28, 41, 43].

Using Theorem 4.5 (ii) (or its Corollary 4.5.1) it is possible to develop an intersection rule for Fréchet normals to infinite intersections $\Omega := \bigcap_{i\in I} \Omega_i$ in Asplund spaces. In this section, we assume that I is a nonempty set of indices, possibly infinite. From now on, we drop the assumption that $|J| > 1$ in the definitions of $\mathcal{J}$ and $\mathcal{J}_\alpha$:

$$\mathcal{J} := \{J \subset I|\ 0 < |J| < \infty\},$$
$$\mathcal{J}_\alpha := \{J \subset I|\ 0 < |J| < \Phi(\alpha)\}.$$

Recalling that Φ-stationarity properties introduced in Definition 4.2 in fact reduce consideration of an infinite collection of sets to that of a sequence of its finite subcollections, it is clear that techniques based on Theorem 4.5 can be applicable not to arbitrary Fréchet normals to the intersection, but only to those which are "approximately normal" to the intersections of certain finite subcollections.

In the definition below, a gauge function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is used. Such functions were discussed in the previous section.

Definition 5.1 An element $x^* \in X^*$ is

(i) *Fréchet Φ-normal* to the intersection $\Omega = \bigcap_{i\in I} \Omega_i$ at $\bar{x} \in \Omega$ iff for any $\varepsilon > 0$ there exist $\rho > 0$, $\alpha \in]0, \varepsilon[$, and $J \in \mathcal{J}_\alpha$ such that

$$\langle x^*, x - \bar{x}\rangle < \alpha\|x - \bar{x}\| \quad \forall x \in \bigcap_{i\in J} \Omega_i \bigcap B_\rho(\bar{x}) \setminus \{\bar{x}\}; \tag{38}$$

(ii) *Fréchet finitely normal* to the intersection $\Omega = \bigcap_{i\in I} \Omega_i$ at $\bar{x} \in \Omega$ iff for any $\varepsilon > 0$ there exists a $\rho > 0$ and a subset $J \in \mathcal{J}$ such that

$$\langle x^*, x - \bar{x}\rangle < \varepsilon\|x - \bar{x}\| \quad \forall x \in \bigcap_{i\in J} \Omega_i \bigcap B_\rho(\bar{x}) \setminus \{\bar{x}\}. \tag{39}$$

Note that Definition 5.1 takes into account that Ω is the intersection of a family of sets and is not applicable to arbitrary sets. Part (ii) of Definition 5.1 is a particular case of part (i) corresponding to $\Phi(\alpha) = \infty$ for all $\alpha > 0$. It is immediate from the definition that every Fréchet Φ-normal element to the intersection $\Omega = \bigcap_{i\in I} \Omega_i$ is Fréchet finitely normal to this intersection, while every Fréchet finitely normal element is Fréchet normal to Ω in the sense of definition (9). If the collection is finite and $\Phi(\alpha) > |I|$ for all $\alpha > 0$, then every Fréchet normal element to Ω is automatically Fréchet Φ-normal to the intersection $\Omega = \bigcap_{i\in I} \Omega_i$. If $|I| = \infty$, then there can be Fréchet normals which are not finitely generated.

Example 5.1 *Let $\Omega_i = \{(u, v) \in \mathbb{R}^2 |\ v \geq iu^2\}$, $i = 1, 2, \ldots$. Then $\bar{x} := (0, 0) \in \Omega := \cap_{i=1}^{\infty}\Omega_i = 0 \times \mathbb{R}_+$ and $N^F_\Omega(\bar{x}) = \mathbb{R} \times \mathbb{R}_-$. At the same time, for any finite set J of natural numbers, $\cap_{i\in J}\Omega_i = \Omega_j$, where j is the maximal number in J. If an element $x^* \in (\mathbb{R}^2)^*$ satisfies (39) for some $\varepsilon > 0$, then $x^* \in 0 \times \mathbb{R}_- + \varepsilon\mathbb{B}^*$. It follows that the set of all Fréchet finitely normal elements to the intersection $\cap_{i=1}^{\infty}\Omega_i$ coincides with $0 \times \mathbb{R}_-$ and is strictly smaller than $N^F_\Omega(\bar{x})$.*

Next theorem provides an intersection rule for Fréchet Φ-normal elements to an infinite intersection of sets.

Theorem 5.2 *Let $\bar{x} \in \Omega = \bigcap_{i\in I} \Omega_i$. Suppose X is Asplund and the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$. If $x^* \in X^*$ is Fréchet Φ-normal to the intersection $\bigcap_{i\in I} \Omega_i$ at $\bar{x}$, then for any $\varepsilon > 0$ there exist $\alpha \in]0, \varepsilon[$; $J \in \mathcal{J}_\alpha$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$; and a $\lambda \geq 0$ such that*

$$\sum_{i\in J} \|x_i^*\| + \lambda\|x^*\| + 2\lambda = 1 \quad \text{and} \quad \left\|\lambda x^* - \sum_{i\in J} x_i^*\right\| < \alpha. \tag{40}$$

Unlike the traditional ways of proving an intersection rule based on some form of extremal principle (see, e.g. [6, Lemma 3.1]), where one of the sets in the collection is modified in a special way to make the extremal principle applicable, in the proof below (based largely on the same ideas), all the sets are left unchanged; instead, another set with a simple structure is added to the collection, and Corollary 4.5.1 is applied. This makes the idea of the proof clearer and the proof itself much shorter.

Proof Let an element $x^* \in X^*$ be Fréchet Φ-normal to the intersection $\Omega = \bigcap_{i\in I} \Omega_i$ at $\bar{x} \in \Omega$.

Consider the Banach space $X \times \mathbb{R}$ with the maximum norm: $\|(x, \mu)\| = \max\{\|x\|, |\mu|\}$, $x \in X$, $\mu \in \mathbb{R}$. For each $i \in I$, introduce a set $\tilde{\Omega}_i := \Omega_i \times \mathbb{R}_+$. Without loss of generality assume that $0 \notin I$ and denote $\tilde{I} := I \cup \{0\}$. Consider now the collection of sets $\{\tilde{\Omega}_i\}_{i\in\tilde{I}}$ in $X \times \mathbb{R}$, where

$$\tilde{\Omega}_0 := \{(x, \mu) |\ \mu \leq \langle x^*, x - \bar{x}\rangle\}.$$

Obviously $(\bar{x}, 0) \in \bigcap_{i\in\tilde{I}} \tilde{\Omega}_i$ and the sets $\tilde{\Omega}_i$, $i \in \tilde{I}$, are locally closed near $(\bar{x}, 0)$. We claim that the collection of sets $\{\tilde{\Omega}_i\}_{i\in\tilde{I}}$ is $\tilde{\Phi}$-stationary at $(\bar{x}, 0)$, where $\tilde{\Phi}(\alpha) = \Phi(\alpha) + 1$. Indeed, by Definition 5.1, for any $\varepsilon > 0$ there exist $\rho \in]0, \varepsilon[$; $\alpha \in]0, \varepsilon[$; $J \in \mathcal{J}_\alpha$, such that (38) holds. Take $a_0 = (0, \alpha\rho/2)$ and $a_i = (0, -\alpha\rho/2)$, $i \in J$. Then $\max_{i\in\tilde{J}} \|a_i\| < \alpha\rho$, where $\tilde{J} = J \cup \{0\}$. Next we show that

$$\bigcap_{i\in\tilde{J}} (\tilde{\Omega}_i - a_i) \bigcap B_\rho(\bar{x}, 0) = \emptyset.$$

If this is not true, then there exists an $(x, \mu) \in B_\rho(\bar{x}, 0)$ such that $(x, \mu) + a_i \in \tilde{\Omega}_i$, $i \in \tilde{J}$. Thus $x \in \Omega_i \bigcap B_\rho(\bar{x})$, $i \in J$; $\mu \geq \alpha\rho/2$, and $\mu + \alpha\rho/2 \leq \langle x^*, x - \bar{x}\rangle$. Hence $x \neq \bar{x}$ and

$$\langle x^*, x - \bar{x}\rangle \geq \alpha\rho \geq \alpha\|x - \bar{x}\|,$$

which contradicts (38).

Since the collection of sets $\{\tilde{\Omega}_i\}_{i\in\tilde{I}}$ is $\tilde{\Phi}$-stationary at $(\bar{x},0)$, it is also approximately $\tilde{\Phi}$-stationary and, by Corollary 4.5.1, Fréchet normally approximately $\tilde{\Phi}$-stationary at this point. For any $\varepsilon\in]0,1[$, there exist $\alpha\in]0,\varepsilon[$; $J\in\mathcal{J}_\alpha$ (the same as in the description of property (AS$_\Phi$)); $(x_i,\mu_i)\in\tilde{\Omega}_i\cap B_\varepsilon(\bar{x},0)$ and $(x_i^*,\lambda_i)\in N^F_{\tilde{\Omega}_i}(x_i,\mu_i)$ $(i\in\tilde{J})$, where $\tilde{J}=J\cup\{0\}$, such that

$$\left\|\sum_{i\in\tilde{J}}(x_i^*,\lambda_i)\right\|<\alpha\sum_{i\in\tilde{J}}\|(x_i^*,\lambda_i)\|.$$

Thus $x_i\in\Omega_i\bigcap B_\varepsilon(\bar{x})$, $\mu_i\ge0$, $x_i^*\in N^F_{\Omega_i}(x_i)$, $\lambda_i\le0$, $\lambda_i\mu_i=0$, $i\in J$; $x_0\in B_\varepsilon(\bar{x})$, $\mu_0\le\langle x^*,x_0-\bar{x}\rangle$, $x_0^*=-\lambda_0x^*$, $\lambda_0\ge0$, and $\lambda_0(\mu_0-\langle x^*,x_0-\bar{x}\rangle)=0$. Hence

$$\left\|\sum_{i\in J}x_i^*-\lambda_0x^*\right\|+\left|\sum_{i\in\tilde{J}}\lambda_i\right|<\alpha\left(\sum_{i\in J}\|x_i^*\|+\lambda_0\|x^*\|+\sum_{i\in\tilde{J}}|\lambda_i|\right).$$

Note that

$$\sum_{i\in\tilde{J}}|\lambda_i|=\left|\sum_{i\in J}\lambda_i\right|+\lambda_0\le\left|\sum_{i\in\tilde{J}}\lambda_i\right|+2\lambda_0,$$

and consequently

$$\left\|\sum_{i\in J}x_i^*-\lambda_0x^*\right\|+(1-\alpha)\left|\sum_{i\in\tilde{J}}\lambda_i\right|<\alpha\left(\sum_{i\in J}\|x_i^*\|+\lambda_0\|x^*\|+2\lambda_0\right).$$

Since $\gamma_0:=\sum_{i\in J}\|x_i^*\|+\lambda_0\|x^*\|+2\lambda_0\ne0$ and $\alpha<1$, the conclusion follows after replacing x_i^* by x_i^*/γ_0 $(i\in I)$ and λ_0 by $\lambda:=\lambda_0/\gamma_0$. □

Remark 5.3 *Given a neighbourhood U of $\bar{x}$, it is sufficient to require in Theorem 5.2 that only those sets Ω_i are closed for which $U\not\subset\Omega_i$.*

The main feature of the first condition in (40) is that the elements x_i^* $(i\in J)$ and number λ cannot be zero simultaneously. This point is expressed clearer in the next corollary with a slightly weaker conclusion.

Corollary 5.3.1 *Let all assumptions of Theorem 5.2 be satisfied. Then for any $\varepsilon>0$ there exist $\alpha\in]0,\varepsilon[$; $J\in\mathcal{J}_\alpha$; $x_i\in\Omega_i\cap B_\varepsilon(\bar{x})$, $x_i^*\in N^F_{\Omega_i}(x_i)$ $(i\in J)$; and a $\lambda\ge0$ such that*

$$\sum_{i\in J}\|x_i^*\|+\lambda=1\quad\textit{and}\quad\left\|\lambda x^*-\sum_{i\in J}x_i^*\right\|<c\alpha,\tag{41}$$

where $c:=\|x^*\|+2$.

Proof It is sufficient to notice that

$$\sum_{i\in J}\|x_i^*\|+\lambda\|x^*\|+2\lambda=\sum_{i\in J}\|x_i^*\|+c\lambda\le c\left(\sum_{i\in J}\|x_i^*\|+\lambda\right).$$

Hence in the conclusion of Theorem 5.2, it holds $\gamma_0:=\sum_{i\in J}\|x_i^*\|+\lambda\ge c^{-1}$. The conclusion of the Corollary follows after replacing in (40) x_i^* by x_i^*/γ_0 $(i\in I)$ and λ by λ/γ_0, respectively. □

The number α and set of indices J in conditions in (40) and (41) are related by the growth condition $|J|<\Phi(\alpha)$. If the growth condition is not important, the intersection rule can be formulated in a more conventional way.

Corollary 5.3.2 *Let $\bar{x} \in \Omega = \bigcap_{i\in I} \Omega_i$. Suppose X is Asplund and the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$. If $x^* \in X^*$ is Fréchet finitely normal to the intersection $\bigcap_{i\in I} \Omega_i$ at $\bar{x}$, then for any $\varepsilon > 0$ there exist $J \in \mathcal{J}$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$; and a $\lambda \geq 0$ such that*

$$\sum_{i\in J} \|x_i^*\| + \lambda = 1 \quad \text{and} \quad \left\| \lambda x^* - \sum_{i\in J} x_i^* \right\| < \varepsilon. \tag{42}$$

The last corollary generalizes the intersection rules for finite collections of sets (see e.g. [6, Lemma 3.1]). It also generalizes and strengthens the recent "fuzzy intersection rule for R-normals" in [23].

Note that, strictly speaking, conditions (40), (41), and (42) do not provide representation formulas for x^* in terms of x_i^*, $i \in J$. It is important to have *normal* versions of these conditions, that is, with $\lambda \neq 0$. To this end, regularity conditions need to be imposed on the collection of sets $\mathbf{\Omega} := \{\Omega_i\}_{i\in I} \subset X$. The next corollary shows that (FNUR) acts as a regularity condition.

Corollary 5.3.3 *Let $\bar{x} \in \Omega = \bigcap_{i\in I} \Omega_i$. Suppose X is Asplund, the sets Ω_i $(i \in I)$ are locally closed near $\bar{x}$ and the collection $\mathbf{\Omega} = \{\Omega_i\}_{i\in I}$ is Fréchet normally uniformly regular at $\bar{x}$. If $x^* \in X^*$ is Fréchet Φ-normal to the intersection $\bigcap_{i\in I} \Omega_i$ at $\bar{x}$, then for any $\varepsilon > 0$ and $\gamma \in]0,1[$, there exist $\alpha \in]0,\varepsilon[$; $J \in \mathcal{J}_\alpha$; and $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$ such that*

$$\left\| x^* - \sum_{i\in J} x_i^* \right\| < c\alpha, \tag{43}$$

where $c := \|x^*\|((\gamma\hat{\eta}^F[\mathbf{\Omega}](\bar{x}))^{-1} + 1) + 2$.

Proof Let $x^* \in X^*$ be Fréchet Φ-normal to the intersection $\bigcap_{i\in I} \Omega_i$ at $\bar{x}$ and let $\varepsilon > 0$ and $\gamma' \in]\gamma, 1[$ be given. Then

$$\|x^*\|(\gamma'\hat{\eta}^F[\mathbf{\Omega}](\bar{x}))^{-1} = (\gamma/\gamma')(c - \|x^*\| - 2) < c - \|x^*\| - 2. \tag{44}$$

Since the collection $\mathbf{\Omega}$ is Fréchet normally uniformly regular at $\bar{x}$, by Definition (FNUR), there exists an $\alpha_1 > \gamma'\hat{\eta}^F[\mathbf{\Omega}](\bar{x})$ and a $\delta > 0$ such that

$$\left\| \sum_{i\in J} x_i^* \right\| \geq \alpha_1 \sum_{i\in J} \|x_i^*\| \tag{45}$$

for any $J \in \mathcal{J}$; $x_i \in \Omega_i \cap B_\delta(\bar{x})$ and $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$.

By Theorem 5.2, there exist $\alpha \in]0,\varepsilon[$ satisfying

$$\frac{(\gamma/\gamma')(c - \|x^*\| - 2) + \|x^*\| + 2}{1 - \alpha\alpha_1^{-1}} < c; \tag{46}$$

$J \in \mathcal{J}_\alpha$; $x_i \in \Omega_i \cap B_{\min\{\varepsilon,\delta\}}(\bar{x})$, $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$; and a $\lambda \geq 0$ such that (40) holds true. The last condition together with (45) implies the following estimates:

$$\lambda\|x^*\| > \left\| \sum_{i\in J} x_i^* \right\| - \alpha \geq \alpha_1 \sum_{i\in J} \|x_i^*\| - \alpha = \alpha_1[1 - \lambda(\|x^*\| + 2)] - \alpha,$$

and consequently, by virtue of (44) and (46),

$$\lambda^{-1} < \frac{\|x^*\| + \alpha_1(\|x^*\| + 2)}{\alpha_1 - \alpha} = \frac{\|x^*\|(\alpha_1^{-1} + 1) + 2}{1 - \alpha\alpha_1^{-1}} < \frac{\|x^*\|((\gamma'\hat{\eta}^F[\mathbf{\Omega}](\bar{x}))^{-1} + 1) + 2}{1 - \alpha\alpha_1^{-1}} < c.$$

The conclusion follows after dividing the inequality in (40) by λ and replacing $\lambda^{-1}x_i^*$ with x_i^*. □

Remark 5.4 *The assumption of Fréchet normal uniform regularity of the collection* $\mathbf{\Omega}$ *in Corollary 5.3.3 can be replaced by a kind of strengthened Fréchet normal uniform* Φ*-regularity: there exists an* $\alpha > 0$ *and an* $\varepsilon > 0$ *such that* (35) *holds true for any* $\alpha' \in]0, \alpha[$*;* $J \in \mathcal{J}_{\alpha'}$*;* $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ *and* $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$*. The last condition is in general weaker than Fréchet normal uniform regularity. If* $\Phi(\alpha) \to \infty$ *as* $\alpha \downarrow 0$*, then the two conditions are equivalent.*

If the growth condition is not important, the intersection rule can be formulated in a conventional way.

Corollary 5.4.1 *Let* $\bar{x} \in \Omega = \bigcap_{i \in I} \Omega_i$*. Suppose* X *is Asplund, the sets* Ω_i $(i \in I)$ *are locally closed near* $\bar{x}$ *and the collection* $\mathbf{\Omega} = \{\Omega_i\}_{i \in I}$ *is Fréchet normally uniformly regular at* $\bar{x}$*. If* $x^* \in X^*$ *is Fréchet finitely normal to the intersection* $\bigcap_{i \in I} \Omega_i$ *at* $\bar{x}$*, then for any* $\varepsilon > 0$ *there exist* $J \in \mathcal{J}$*;* $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ *and* $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$ *such that*

$$\left\| x^* - \sum_{i \in J} x_i^* \right\| < \varepsilon. \tag{47}$$

Thanks to Theorem 4.3 (i), the assumption of Fréchet normal uniform regularity of the collection $\mathbf{\Omega}$ in Corollaries 5.3.3 and 5.4.1 can be replaced by the corresponding primal space uniform regularity condition. For instance, next statement is a consequence of Corollary 5.4.1.

Corollary 5.4.2 *Let* $\bar{x} \in \Omega = \bigcap_{i \in I} \Omega_i$*. Suppose* X *is Asplund and the sets* Ω_i $(i \in I)$ *are locally closed near* $\bar{x}$*. If* $x^* \in X^*$ *is Fréchet finitely normal to the intersection* $\bigcap_{i \in I} \Omega_i$ *at* $\bar{x}$*, then for any* $\varepsilon > 0$ *there exist* $J \in \mathcal{J}$*;* $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ *and* $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in J)$ *such that* (47) *holds true, provided that the collection* $\mathbf{\Omega} = \{\Omega_i\}_{i \in I}$ *is uniformly regular at* $\bar{x}$*.*

6 Concluding Remarks

In this article, we demonstrate how the existing theory of extremality, stationarity and regularity of finite collections of sets can be successfully extended to infinite collections. The full set of definitions together with the primal-dual relations between the corresponding properties are presented in a unified way. Applications of this extended theory to problems of infinite and semi-infinite programming are considered in our forthcoming article.

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