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An extended lifetime measure for telecommunications networks: improvements and implementations

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This paper is dedicated to Professor Alexander Rubinov

Abstract

Predicting the lifetime of a network is a stochastic and very hard task. Sensitivity analysis of a network in order to identify the weakest points in the network, provides valuable knowledge to draw an optimum investment strategy for the expansion of the networks for the network carriers. To achieve this goal, a new measure, called topology lifetime, was recently proposed for measuring the performance of a telecommunication network. This measure not only allows to perform a sensitivity analysis of the networks, but also provides the means to compare the different topologies with respect to the ability of the network in supporting growth in network traffic before new capacity/facility is installed. This paper addresses some improvements upon the previously defined measures and presents the implementation results of the various lifetime measure methodologies. Computational analysis on some commonly used topologies show how the new measure can be utilized in assessing network performance.

Keywords: Telecommunications, network performance, lifetime, linear programming.

1 Introduction

One of the challenging issues in the investment strategies associated with the telecommunication networks is the stochastic behavior of the traffic in the network. It is a well-known network design phenomenon that even sophisticated traffic forecast analysis models might have very large error margin. This is due to uncertainty in 1) growth patterns of the metropolitan areas, 2) distribution and growth of population in the suburbs, 3) unpredicted establishments of technology farms, 4) technology evolution or enhancements 5) upcoming deregulation law (nationally, or local by councils) 6) new greedy applications and so many other factors. As the result of such uncertainty, the investment in the network infrastructure is generally a risky business.

In the case of any abnormality in the network status or any change in traffic load, the residual capacity in the network should be used to compensate. Load sharing, traffic rerouting and other traffic engineering capabilities in the network provide the mean to achieve an optimum operation of the network. One way of testing network resilience is to simulate a series of scenarios to measure how the network resources can cope under these scenarios. Since the number of potential scenarios is infinite, then it would be practical to limit the number of scenarios and measure the network performance under such circumstances. One important class of such scenarios are those constructed based on the changes in the traffic pattern. In other words, one assumes that the traffic load might change in a various ways and the network throughput is then analyzed by checking the feasibility of the carriage of the traffic under each event.

The ability of a telecommunication network to support the expected growth in demand is an important characteristic of the network. However, it is not enough to consider only the expected growth: an unexpected growth can often occur due to technological innovations and the increasing popularity of the internet. A quantitative measure for telecommunication topology design, named *topology lifetime* was suggested by [3], taking into account possible unexpected changes in load. It is noteworthy that the approach suggested by [3] is *multiplicative* in the sense that the unexpected growth occurs as multiples of a given traffic matrix, and considers unexpected changes in traffic of origin-destination (OD) pairs, or nodes, or both.

The lifetime measure methodology provides the capability of analyzing the networks' flexibility in the case of traffic pattern change, sudden traffic shift, link or node failure in the network and other possible events. However, the authors of [1; 2] mentioned that the topology lifetime measure described above is based on a strong assumption with possible loss of generality. The first of these is the unexpected growth of traffic on only one OD pair, which may not be valid on real-life applications. Further, the calculation of the lifetime measure needs to be repeated for each time period (e.g., year) of a multi-period time horizon, whereas it is more desirable to have an evaluation of the network topology that spans across all the periods of a time horizon. Based on these reasons, an extended lifetime measure was described in [2] that is based on an *additive* approach and extends over a multiple time horizon. The analysis in [2] was mainly theoretical, where the implementation of the extended measure on real networks was recommended as a topic of further research.

The goal of this paper is to present the application results of the extended lifetime measure described in [2] on a number of different network structures. While doing so, we propose a modification to the procedure that simplifies the calculation of the lifetime measure.

2 Preliminaries

Consider a network defined by the graph $G = \{V, E\}$ with the set V of nodes and the set E of edges. Our approach is suitable for both directed and undirected topologies. For the sake of definiteness we will consider undirected topologies, so we consider G as a non-oriented graph. Let $c_{k,l}$ be the capacity of the link $(k, l) \in E$. For an undirected topology, we have $c_{k,l} = c_{l,k}$. Let r_{ij} is the traffic demand (requirements) between nodes $i, j \in V$, with $i \neq j$. For the sake of simplicity we assume in this paper that there are no Service Level Agreements, so we consider only one type of requirements. We denote by $R = (r_{ij})_{i,j\in V}$ the matrix of all requirements (traffic matrix). Since we consider undirected traffic, we have $r_{ij} = r_{ji}$ so the matrix R is symmetric. Clearly $r_{ii} = 0\ \forall i$. Following [3] we assume that the present traffic demand is given by a finite collection $\mathcal{R}$ of traffic matrices R. Each $R \in \mathcal{R}$ describes the traffic demand between all origin-destination (OD) pairs at a certain time period. We present below the notation related to paths and flows.

- $P(G)$ is the set of all paths generated by G,
- $P^*(G) \subset P(G)$ is a given set of working paths,
- $P^*_{k,l}(G) \subset P^*(G)$ is the set of paths containing the link $(k, l) \in E$,
- $P^*(i, j; G) \subset P^*(G)$ is the set of paths having i and j as end-points,
- x_p is the flow transmitted through a working path p,
- $(x_p)_{p\in P^*(G)}$ is the traffic generated by working paths $P^*(G)$.

Next, we consider the notion of feasibility. A traffic matrix $R = (r_{ij})$ is said to be *feasible* if the following system of linear inequalities has a solution:

$$\sum_{p\in P^*(i,j;G)} x_p \geq r_{i,j} \qquad i, j \in V, i \neq j \tag{2.1}$$

$$\sum_{p\in P^*_{k,l}(G)} x_p \leq c_{k,l} \qquad (k, l) \in E \tag{2.2}$$

$$x_p \geq 0 \qquad p \in P^*(G). \tag{2.3}$$

In this set, (2.1) ensures that the traffic requirement constraints are satisfied, (2.2) guarantees that the total flow transmitted through each link $(k, l) \in E$ does not exceed the capacity $c_{k,l}$ of this link, and (2.3) ensure that the traffic flows are nonnegative.

The *growth factor* $\psi_*(R)$ of the traffic matrix R is the the largest number ψ such that the matrix $\psi R = (\psi r_{i,j})$ is feasible. Note that $\psi_*(R)$ depends not only on R but also on the set of working paths $P^*(G)$. $\psi_*(R)$ is a solution of the linear programming problem:

$$\text{maximize } \psi \tag{2.4}$$

subject to

$$\begin{aligned} \sum_{p \in P^*(i,j)} x_p &\geq \psi r_{i,j}, & i,j \in V, i \neq j \\ \sum_{p \in P^*_{k,l}(G)} x_p &\leq c_{k,l}, & (k,l) \in E \\ x_p &\geq 0 & p \in P^*(G). \end{aligned} \tag{2.5}$$

The number $\psi_*(R)$ indicates the largest possible *uniform* growth of the traffic matrix R. The number,

$$\psi^*_{\mathcal{R}} = \min\{\psi_*(R) : R \in \mathcal{R}\},$$

is the largest possible uniform growth of the traffic represented by the collection $\mathcal{R}$ and the set of working paths P^*. The definition of the growth factor is based on a multiplicative approach to traffic extension, since we consider the products of the form $\psi r_{i,j}$.

2.1 The lifetime measure of [3]

An unexpected growth of traffic was discussed by [3] separately for OD pairs and for nodes. For the sake of definiteness we consider only OD pairs here. Let R be a traffic matrix and (i,j) be an arbitrary OD pair. Assume that the traffic between q and s increases by U. Consider a new traffic matrix $R' = (r'_{i,j})$ with $r'_{q,s} = (1+U)r_{q,s}$, $r'_{s,q} = r'_{q,s}$ and $r'_{i,j} = e_{q,s}r_{i,j}$, $(i,j) \neq (q,s)$, $(i,j) \neq (s,q)$. Here $e_{q,s}$ is the coefficient which provides the equality

$$\sum_{i,j \in V,} r'_{i,j} = \sum_{i,j \in V} r_{i,j}.$$

Thus R' describes a *shift in load without growth.* It is assumed that an unexpected growth of traffic can occur only for one OD pair, and this pair is unknown. This means that a family of matrices $R'(i,j)$ corresponding to each OD pair (i,j) is considered.

The growth factor $\psi_*(R'(i,j))$ is then calculated for each matrix $R'(i,j)$ and the number,

$$\Psi_*(R,U) = \min_{(i,j) \in V} \psi_*(R'(i,j)),$$

is considered as a parameter that characterizes an unexpected traffic growth U corresponding to the matrix R. If a collection $\mathcal{R}$ of traffic matrices R is given, then we need to apply the procedure described to each matrix R. Then we get a new collection of traffic matrices, which consists of all matrices $R'(i,j)$ for all $R \in \mathcal{R}$ and all $i,j \in V$, $i \neq j$. The number $\min\{\Psi_*(R,U) : R \in \mathcal{R}\}$ characterizes an unexpected traffic growth U corresponding to collection $\mathcal{R}$.

The approach described above is useful and can be used for comparison of different topologies. However, this approach is based on some strict assumptions that are stated as follows.

1) The construction of matrices R' is based on the assumption that a uniform redistribution of the amount $2Ur_{i,j}$ is carried out between all OD pairs (i',j') with $(i',j') \neq (i,j)$ and $(i',j') \neq (j,i)$. This uniformity does not always hold in real world networks.

2) The assumption that an unexpected growth can occur only for one OD pair is also not valid in many cases.

The following situation should be also taken into account. Assume that unexpected growth occurs for a pair (i,j) at the end of the first year. This leads to a change of the collection $\mathcal{R}$. So we have a different collection $\mathcal{R}'$ in the second year and we need to recalculate the lifetime measure. However, the proposed lifetime measure is used for the evaluation of the topology design hence it should not be recalculated each year. The extended lifetime measure proposed in [2] is designed to overcome these problems and the outline of the approach is described below.

2.2 The extended lifetime measure of [2]

The lifetime measure introduced in [2] is based on an additive approach to the definition of the lifetime measure rather than multiplicative. Assume that the network under consideration will be in use for m years. First we consider an expected growth. Using forecasts of growth of population and migrations flows we can find coefficients μ_{ij}^q for each OD pair (i,j) and each year $q = 1,\dots m$ such that a traffic matrix $R = (r_{ij})$ will be replaced with the matrix

$$\widetilde{R}_q = (\mu_{ij}^q r_{ij}).$$

Note that it is possible that $\mu_{ij}^q < 1$. (For instance this can occur if the population in both nodes i,j will decrease.) Thus instead of the family $\mathcal{R}$ we will have new families $\widetilde{\mathcal{R}}_q$ that consists of all matrices $\widetilde{R}_q$ with $R \in \mathcal{R}$ $\quad (q = 1,\dots,m)$.

Now we turn to an unexpected traffic growth. Let U be a positive number that indicates an unexpected traffic growth at the year q $(q = 1,\dots,m)$. Consider the system of linear inequalities:

$$\begin{aligned} \sum_{p \in P^*(i,j;G)} x_p &\geq U + \mu_{ij}^q r_{i,j} && i,j \in V, i \neq j \\ \sum_{p \in P^*_{k,l}(G)} x_p &\leq c_{k,l} && (k,l) \in E \\ x_p &\geq 0 && p \in P^*(G). \end{aligned} \tag{2.6}$$

Composing (2.6), we suppose that an unexpected traffic growth U can happen in many arcs and nodes simultaneously. This situation is more realistic than that suggested in [3], where only a single node (or a single arc) becomes more active. Note also that constraint (2.6) is slightly different from that presented in [1], since the expected growth of traffic is reflected on the traffic matrix itself and not on the unexpected traffic.

Indeed, an increase in activity of one node can lead to an increase in activity at many different nodes. One of the main reasons for the unexpected growth is the internet. A server farm which provides a popular service can appear suddenly, and then the load to the corresponding part of the network increases. However, the same reasons that lead to the appearance of this farm may also lead to the appearance of different farms in different parts of the network in different years. So it is important to take into account many nodes simultaneously. Note that U is an upper bound for an unexpected increase of load in year q for each OD pair (i, j). Note that the suggested approach is pessimistic, because we take into account an unexpected increase in all nodes simultaneously.

In [1], the calculation of the lifetime measure is done by solving the following linear programming problem,

$$\text{maximize} \sum_{p \in P^*(G)} x_p \tag{2.7}$$

subject to

$$\begin{aligned} \sum_{p \in P^*(i,j;G)} x_p &\geq U + \mu_{ij}^q r_{i,j} && i, j \in V, i \neq j \\ \sum_{p \in P^*_{k,l}(G)} x_p &\leq c_{k,l} && (k,l) \in E \\ x_p &\geq 0 && p \in P^*(G), \end{aligned}$$

where (2.7) is an arbitrary objective function that is deemed to be the most appropriate one for the problem under consideration which allows one to check whether the system defined by (2.6),(2.2) and (2.3) is feasible. To find the total amount of unexpected traffic growth over the time horizon, one can start with $U = 0$ and incrementally increase its value until the system defined by (2.6),(2.2) and (2.3) is infeasible.

3 An improved extended lifetime measure calculation

The calculation of the extended lifetime measure described in the previous section implies that a series of linear programs need to be solved by changing the parameter U (either increase or decrease) to find the limit at which the traffic is infeasible. This is doable but cumbersome. The modification we propose here is very simple but effective to perform the same calculation through the solution of a *single* linear programming formulation. Let $R \in \mathcal{R}$. Consider the following linear programming problem (denoted by $\mathcal{LP}$):

$$\text{maximize } U \tag{3.1}$$

subject to

$$\begin{aligned} \sum_{p \in P^*(i,j;G)} x_p &\geq U + \mu_{ij}^q r_{i,j} && i, j \in V, i \neq j \\ \sum_{p \in P^*_{k,l}(G)} x_p &\leq c_{k,l} && (k,l) \in E \\ x_p &\geq 0 && p \in P^*(G), \end{aligned}$$

The solution $\widetilde{U}_q(R)$ of this problem indicates the largest possible unexpected growth in the year q that can be supported by the existing network. We can consider the sequence,

$$\widetilde{U}_1(R), \ldots \widetilde{U}_q(R), \ldots, \widetilde{U}_m(R),$$

as a certain *lifetime measure* of the network under consideration for the traffic matrix R and the given set of working paths P^*. Let a collection $\mathcal{R}$ of traffic matrices R be given. Let us calculate the sequence,

$$\widetilde{U}_1(R), \ldots \widetilde{U}_q(R), \ldots, \widetilde{U}_m(R),$$

for each $R \in \mathcal{R}$. Let $\widetilde{U}_q^{\mathcal{R}} = \min\{\widetilde{U}_q(R) : R \in \mathcal{R}\}$ for $q = 1, \ldots, m$. We consider the sequence,

$$\widetilde{U}_1^{\mathcal{R}}, \ldots \widetilde{U}_q^{\mathcal{R}}, \ldots, \widetilde{U}_m^{\mathcal{R}},$$

as a lifetime measure for the given topology, the given collection $\mathcal{R}$ and the given set $P^*(G)$ of working paths. Thus we suggest the use of a simple one-step procedure for the definition of a lifetime measure, instead of the consecutive two-step procedure suggested in [3], and much less demanding in terms of computational effort as compared to that proposed in [1].

3.1 Extensions of the set of working paths

Let $\widetilde{U}_q(R)$ be a solution of the problem $\mathcal{LP}$. Then for at least one link (k, l) inequality (2.2) holds as an equality. Such links indicate the bottlenecks that do not permit an unexpected load greater than $\widetilde{U}_q(R)$. If the load distribution that occurs due to unexpected load growth exceeds the network possibilities, then new facilities should be installed. In order to support the traffic for the time that is needed for installation of these facilities, the set $P^*(G)$ of working paths needs to be extended. We can incorporate new paths that do not contain bottlenecks.

It is easy to find examples where even a few new paths allow us to significantly increase the capacity to handle an unexpected load. The notion of the order of the path can be useful for indication of paths that should be added to the existing traffic.

4 An illustrative example

We illustrate the application of the proposed approach on a three-node network shown in Figure 1. This is the same directed network used for testing in [3]. The network has a symmetrical traffic matrix, thus an OD pair $[i, j]$ corresponds to both $[i, j]$ and $[j, i]$. Therefore this example has three OD pairs as opposed to six. We assume that for every link we have 30 units of capacity in each direction. The traffic is represented by a single symmetric 3×3 matrix, T. That is, the set $\mathcal{R}$, has only one element T. Let:

$$T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

The goal of our computational experiments is to plot the lifetime function of this network, given a certain (forecasted) expected growth. For illustrative purposes, we consider a uniform growth for

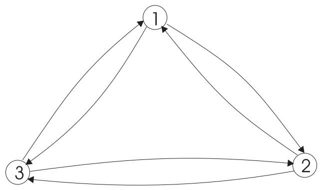

Figure 1: The three node topology

all OD pairs, i.e.i $\mu_{ij}^{q} = \mu$, although the proposed approach is able to handle different (forecasted) growth values for each OD pair. The rate of growth μ is increased from 5 to 11 (implying a growth rate of 5 to 11 times the current flow of traffic between every OD pair), and the corresponding values of $\widetilde{U}$ are shown in Figure 2. Note that in our approach, only a single linear programming formulation has to be solved for a given μ to find the maximum value of the unexpected growth that the network can handle. The lifetime function given in Figure 2 shows that the network, without

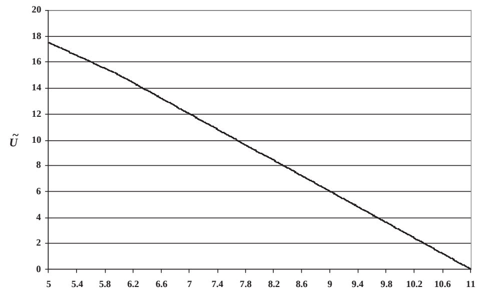

Figure 2: $\widetilde{U}$ values for varying network traffic growth

addition of any extra capacity, is able to support an expected traffic growth of up to 11 times of the current flow, with no feasible flow at any value $\mu > 11$. The upper bound on the unexpected traffic growth $\widetilde{U}$ decreases linearly with the increasing rate of expected growth. At $\mu = 11$, we have $\widetilde{U} = 0$, meaning that the network cannot handle any unexpected traffic growth at this particular point.

5 Implementation Results

The network topologies we have considered in this work are the same of those considered in [3] and are given in Appendix I for the sake of completeness. We will reintroduce them here for the sake of completeness: dual ring, chordal ring, Manhattan Street and hierarchical network, all with 8 nodes. The first three networks are regular topologies. The hierarchical network has six nodes in the lower layer, and two in the upper layer. All of the networks have 8 nodes and bidirectional links. For each link $c_{ij} = 1000$, so that the capacity in each direction is 500. All of the nodes have switching capabilities, except in the hierarchical network where only the nodes in the upper layer can forward traffic between nodes.

Our first implementation results pertain to the four network topologies using the traffic matrices proposed in [3] (shown in Appendix II for the sake of completeness). Each network was evaluated against these two matrices by imposing hop limits from 1 to 7. For each hop limit, model $\mathcal{LP}$ was solved to obtain the corresponding lifetime measure. The results of this experiment are presented in Table 1. In this table, the second column shows the hop limits imposed on each topology and the third column presents the resulting number of working paths. The two subsequent columns show the values of $U_1(R_1)$ and $U_1(R_2)$ obtained using traffic matrices R_1 and R_2, respectively, and the last column presents the lifetime measure $\widetilde{U}_1^{\mathcal{R}}$ as calculated by $\min\{U_1(R_1), U_1(R_2)\}$.

Table 1: Results of the implementation of the new lifetime measure on various network topologies

Topology	Hop limit	No. of working paths	$U_1(R_1)$	$U_1(R_2)$	$\widetilde{U}_1^{\mathcal{R}}$
Dual Ring	1	16	0	0	0
	2	32	0	0	0
	3	48	0	0	0
	4	64	52.9375	51.375	51.375
	5	80	52.9375	51.375	51.375
	6	96	52.9375	51.375	51.375
	7	112	52.9375	51.375	51.375
Chordal Ring	1	16	0	0	0
	2	48	0	0	0
	3	112	62.25	60.607143	60.607143
	4	192	62.25	60.607143	60.607143
	5	304	62.25	60.607143	60.607143
	6	368	62.25	60.607143	60.607143
	7	400	62.25	60.607143	60.607143
Manhattan Street	1	16	0	0	0
	2	40	0	0	0
	3	80	53.4375	51.375	51.375
	4	120	53.4375	51.375	51.375
	5	168	53.4375	51.375	51.375
	6	200	53.4375	51.375	51.375
	7	232	53.4375	51.375	51.375
Hierarchical	1	38	0	0	0
	2	194	190.666667	188.466667	188.466667
	3	686	190.666667	188.466667	188.466667
	4	1862	190.666667	188.466667	188.466667
	5	3878	190.666667	188.466667	188.466667
	6	6182	190.666667	188.466667	188.466667
	7	7478	190.666667	188.466667	188.466667

Table 1 provides interesting results on the comparison of the four topologies, implying that for the two traffic matrices considered here, the hierarchical network topology is the one that is able to support a significantly more amount of unexpected traffic growth than the other topologies. This result is quite to the contrast of that of [3], who had concluded this specific topology to be inferior to the remaining three using their multiplicative lifetime measure. This is due to fact that the extended lifetime measure suggested here, assumes that an unexpected traffic growth U can happen in many arcs and nodes simultaneously as dictated by (2.6). The table also indicates that chordal ring topology is superior to dual ring and Manhattan street, with the latter two showing similar performances in terms of supporting unexpected traffic growth. We believe that these results are strongly correlated with the number of working paths in the graph induced by the hop limits.

It was mentioned above that the new lifetime measure is designed to calculate the lifetime measure over a multi-period time horizon. The second set of results are related to the experiments performed to test the four network topologies over a 30-year time horizon, where, for each year, two random traffic matrices were generated. To incorporate expected traffic growth into the experiments, the generation process was done such that the traffic was increased by a factor of ζ such that the overall growth of traffic from year one to year 30 was three-fold. The results of this experiment are given in Figure 3, where the lifetime measures of the four topologies are plotted against a 30-year time line, showing the maximum amount of traffic growth they can support for each year. The results shown in Figure 3 are in line with those presented in Table 1, showing the

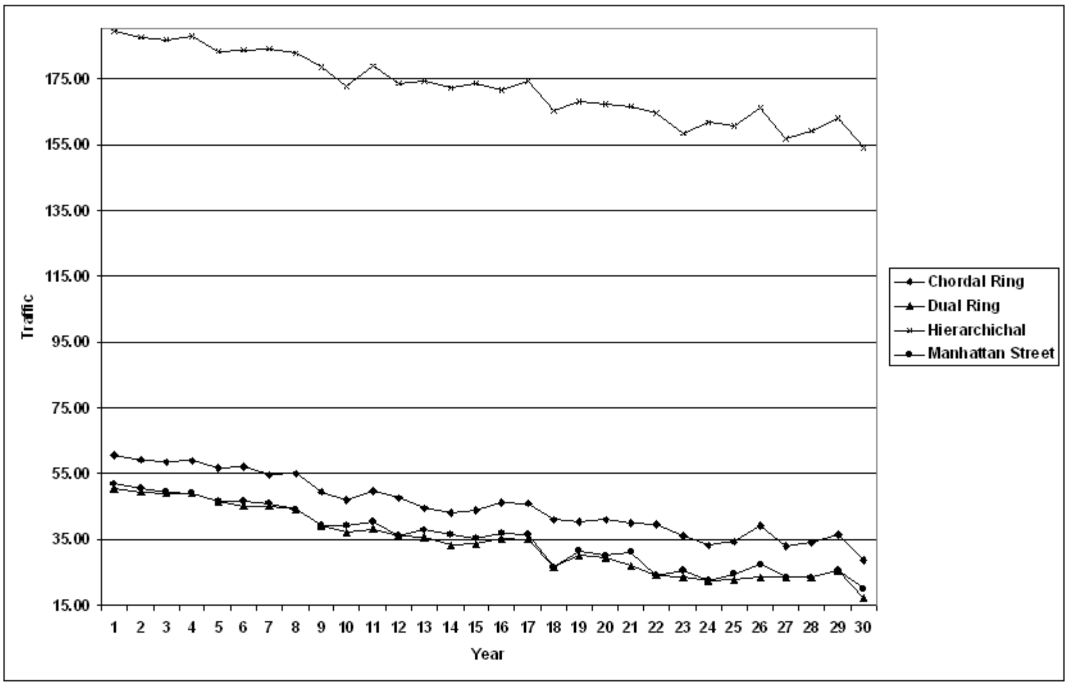

Figure 3: Comparison of the four network topologies over a 30-year time horizon

superiority of the hierarchical network topology over the other three. More importantly, the figure implies that soon after the 30-year horizon, neither of the three topologies (chordal ring, dual ring and Manhattan street) will be able to support unexpected traffic growth provided that the network is not supplemented with additional capacity. The hierarchical network, on the other hand, will sustain more growth without the need for additional capacity.

6 Conclusions

In this paper, we have presented a simplified way of calculation of a recently proposed lifetime measure and used it to compare a number of various network topologies through computational experimentation. The results provided here are in contrast with those of other earlier studies using a different measure, mainly due to the approach taken in calculating the lifetime measure, either being multiplicative or additive.

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APPENDIX I: Network Topologies (from [3])

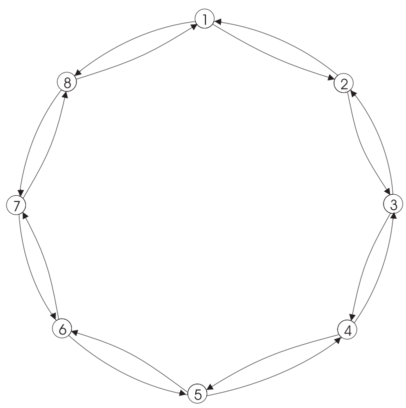

Figure 4: The eight node dual ring topology

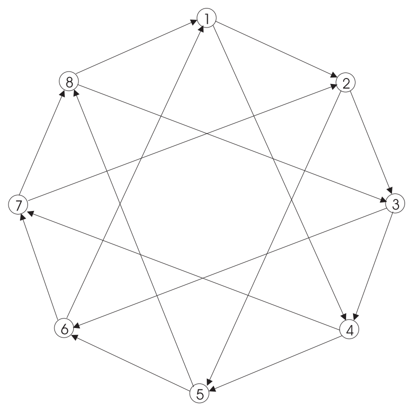

Figure 5: The eight node chordal ring topology

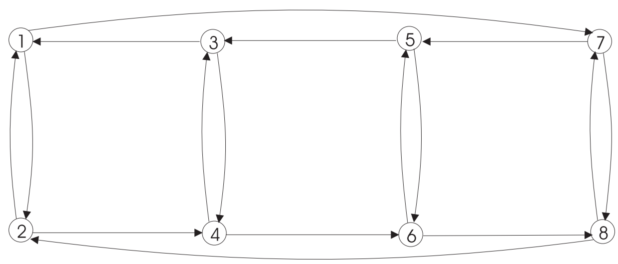

Figure 6: The eight node Manhattan Street Network Topology

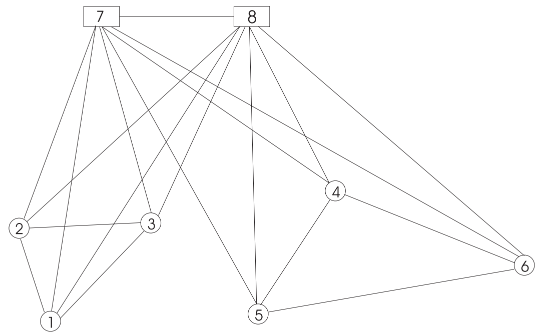

Figure 7: The eight node hierarchical network example

APPENDIX II: Traffic Matrices (from [3])

$$R_1(i,j) = \begin{bmatrix} 0 & 9 & 6 & 2 & 10 & 3 & 7 & 11 \\ 9 & 0 & 11 & 3 & 19 & 6 & 14 & 22 \\ 6 & 11 & 0 & 2 & 13 & 4 & 9 & 15 \\ 2 & 3 & 2 & 0 & 4 & 1 & 3 & 4 \\ 10 & 19 & 13 & 4 & 0 & 7 & 16 & 25 \\ 3 & 6 & 4 & 1 & 7 & 0 & 5 & 8 \\ 7 & 14 & 9 & 3 & 16 & 5 & 0 & 18 \\ 11 & 22 & 15 & 4 & 25 & 8 & 18 & 0 \end{bmatrix}$$

$$R_2(i,j) = \begin{bmatrix} 0 & 9 & 18 & 5 & 12 & 16 & 5 & 21 \\ 9 & 0 & 12 & 3 & 8 & 11 & 3 & 14 \\ 18 & 12 & 0 & 6 & 15 & 21 & 6 & 27 \\ 5 & 3 & 6 & 0 & 4 & 6 & 2 & 7 \\ 12 & 8 & 15 & 4 & 0 & 14 & 4 & 17 \\ 16 & 11 & 21 & 6 & 14 & 0 & 6 & 24 \\ 5 & 3 & 6 & 2 & 4 & 6 & 0 & 7 \\ 21 & 14 & 27 & 7 & 17 & 24 & 7 & 0 \end{bmatrix}$$