Conditions for global minimum through abstract convexity

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This thesis is submitted in total fulfilment of the requirements for the degree of Doctor of Philosophy.

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Submitted in February 2008

Abstract

Subdifferential calculus and separation theorems play a crucial role for applications of classical convex analysis to global optimization. More precisely, they allow the formulation of conditions (necessary or sufficient) for the global minimum of some convex optimization problems. The theory of abstract convexity generalizes ideas of convex analysis by using the notion of global supports and the global definition of subdifferential. In order to apply this theory to optimization, we need to extend subdifferential calculus and separation properties into the area of abstract convexity. This is the main objective of the present thesis.

First, we consider two particular cases. We examine global subdifferentials for convex-along-rays (CAR) functions with respect to different sets of elementary functions and give conditions, which guarantee the non-emptiness of these subdifferentials. The results obtained can be applied for the global minimization of some CAR functions over subsets of $\mathbb{R}^n$ by using numerical methods. We also investigate the weak separability of two star-shaped sets and derive conditions for the global minimum of the so-called star-shaped distance. This is a "best approximation -like" problem.

Then we take a general approach to subdifferential calculus and separation properties in the theory of abstract convexity. We show that the equivalence between local and global definitions of abstract subdifferential can provide certain calculus rules for such subdifferentials. We also investigate the notion of N-connectedness of a topological space with respect to a convexity on this space and investigate separation properties via such type of connectedness.

At the end of the thesis, we generalize the notion of a duality between two complete lattices, to arbitrary partially ordered sets. We introduce and examine conjugations and abstract subdifferentials corresponding to such type of dualities. Conditions for the global minimum in terms of these subdifferentials are given.

Statement of authorship

Except where explicit reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis by which I have qualified for or been awarded another degree or diploma. No other person's work has been relied upon or used without due acknowledgment in the main text and bibliography of the thesis.

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Acknowledgements

I would like to thank my first principal supervisor Professor Alex Rubinov for his guidance and assistance over all period of my study. We began to work with him in 2000. My first papers were written under his unofficial supervision and were prepared for publication with his strong support. During my study at the University of Ballarat he always found time to provide help in my research and suggest new ideas. It was a great pleasure to know him and work with him. Unfortunately, we had time to prepare only two joint papers. They form two chapters of the present thesis.

I thank my associate supervisor Professor Sid Morris for his direction, especially during the confirmation procedure. His remarks were very helpful for me.

I am also very grateful to my provisional principal supervisor Dr. Alex Kruger and associate supervisor Dr. David Yost for their valuable comments regarding my papers and the thesis.

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Chapter 1

Introduction

1.1 Overview

The first monograph on abstract convexity ([29]) by Kutateladze and Rubinov was published in 1976 in Russian. The theory of abstract convexity arises naturally from the convex analysis and is mainly driven by applications to optimization, namely global optimization.

It is a well known fact that every local minimum of a convex function over a convex set coincides with the global one. So we can say that in the convex case there are no differences between local and global optimization.

The structure of the subdifferential in convex analysis has two aspects. On the one hand, every subgradient (that is an element of the subdifferential) allows us to construct a local approximation of a convex function near a given point. Such approximations can give us information about local minimizers. On the other hand, subdifferential as a global notion provides a necessary and sufficient condition for the global minimum. Existence of subdifferential calculus can help to find global minimum of some complicated convex functions.

There are two ways for generalizations of ideas of convex analysis. One of them uses the notion of local approximation and forms nonsmooth analysis, which solves problems of local optimization. The second way leads to the so-called abstract convexity and exploits the notion of global supports and the global definition of subdifferential.

We know that each lower semicontinuous convex function f is the upper envelope of a certain set of affine functions. So the set of all affine minorants of f (the so-called *support set*) contains complete information about the initial function f. Hence conditions for a minimum can be easily reformulated in terms of the support set. Such reformulation can be very convenient. From this point of view it is not important what kind of functions we consider. We can get the same constructions for an arbitrary set H of functions (not necessarily affine). Then we will work with upper envelopes of subsets of H instead of lower semicontinuous convex functions.

On the whole, abstract convex analysis deals with the so-called *closure structures* and generalizes the *outer definition* of convexity for closed sets, which is based on the separation property.

The main results on abstract convexity and its applications can be found in the books by Pallaschke and Rolewicz [37], Singer [57] and Rubinov [41]. Some applications to global optimization (namely, investigation of various dual problems and a survey of numerical methods) are mainly concentrated in [41].

In this thesis we focus on global subdifferentials and separation properties in the framework of abstract convexity. Chapters 2, 3, 4, 6 and the first half of Chapter 5 are based on the corresponding papers [45,46,53–55].

1.2 Abstract convex functions and sets

In this section we consider some notions, which have a central place in the theory of abstract convexity. All definitions presented here can be found in [41].

We begin with the definition of abstract convex functions defined on a set X, which have values in the extended real line $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. A more general case of functions with values in an upper complete semilattice is considered in Chapter 6.

Definition 1.1 Let H be a set of functions defined on X. A function $f : X \to \bar{\mathbb{R}}$ is called abstract convex with respect to H (shortly H-convex) if there exists a set $U \subset H$ such that f is the upper envelope of this set:

$$f(x) = \sup\{h(x) : \ h \in U\} \quad \text{for all } x \in X.$$

Functions $h \in H$ from the above definition are usually called *elementary*. The set H of all elementary functions is called a *supremal generator* of the set of all H-convex functions.

Definition 1.2 Let Y be a set of functions $f : X \to \bar{\mathbb{R}}$. A subset $H \subset Y$ is called a supremal generator of Y if each function $f \in Y$ is H-convex.

There are two main problems that arise in view of these definitions. The first one: how to describe H-convex functions if the set H of elementary functions is given. The second problem: how to find a sufficiently small and simple (in a certain sense) supremal generator H of the given set Y. Unfortunately, there is no good technique for solving these problems in the general case. Nevertheless, there are some results in this direction. The first problem is examined in Section 5.6 in a special case. An attempt to solve the second problem was undertaken in [51].

Recall that a set Y of functions f defined on X is called an upper complete semilattice if a pointwise supremum $\sup_{f \in F} f$ belongs to Y for every subset $F \subset Y$. If we talk about functions $f \in Y$ as H-convex functions then the assumption that Y is an upper complete semilattice seems very natural. Recall also that a subset F of Y is called a chain if for any $f, g \in F$ we have either $f(x) \leq g(x)\ \forall x \in X$ or $f(x) \geq g(x)\ \forall x \in X$. Denote by $H(Y)$ the set of all functions $h \in Y$, for which there exists a point $x^0 = x^0(h) \in X$ such that for any $f \in Y$ the conditions

$$f(x^0) = h(x^0), \quad f(x) \leq h(x) \text{ for all } x \in X$$

imply the equality

$$f(x) = h(x) \text{ for all } x \in X.$$

The following assertions are valid.

Proposition 1.1 *([51], Theorem 6.1) Let Y be an upper complete semilattice of functions $f : X \to \bar{\mathbb{R}}$. If a pointwise infimum $\inf\{f(x) : f \in F\}$ belongs to Y for every chain $F \subset Y$ then $H(Y)$ is a supremal generator of Y.*

Proposition 1.2 *([51], Theorem 6.2) Let Y be an upper complete semilattice of functions $f : X \to \bar{\mathbb{R}}$. If H is a supremal generator of Y then for any function $h \in H(Y)$ there exists a sequence $\{h_n\} \subset H$ such that*

$$h(x) = \lim_{n \to \infty} h_n(x) \textit{ for all } x \in X.$$

So under conditions of Proposition 1.1 the set $H(Y)$ is a supremal generator of Y and it is the smallest in the sense of Proposition 1.2. Unfortunately, as a rule, it is a difficult task to describe $H(Y)$ for a given upper complete semilattice Y.

A classical example of H-convex functions is the usual convex case: every convex lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty} = (-\infty, +\infty]$ is H-convex with respect to the set of all affine functions (i.e., linear functions plus constants). If H contains only linear functions then f is H-convex if and only if it is lower semicontinuous and sublinear.

Many investigations were devoted to positively homogeneous functions. Various kinds of generalized derivatives used in nonsmooth analysis are positively homogeneous of degree one, and representations of such functions as upper envelopes of some sets of sufficiently simple functions can be very useful.

For instance, it was shown in [10] that each continuous positively homogeneous of degree one function f defined on the Euclidean space can be represented as the supremum of a subset of the set $H = \{h : \ h(x) = -a\|x\| + [u, x]; \ a \geq 0, \ u \in \mathbb{R}^n\}$ (here $\|\cdot\|$ is the Euclidean norm and $[u, x]$ is the inner product of vectors u and x).

Supremal generators for the set of all symmetric positively homogeneous of degree two functions (these functions arise in nonsmooth analysis as approximations of the second order) were considered in [16] and [17] for finite-dimensional and Banach spaces respectively.

Another area, where abstract convexity is applicable, is the so-called monotonic analysis. Abstract convexity is a convenient tool for investigation of various classes of monotonic functions. Monotonicity arises in many areas of mathematics and its applications. In particular, production, utility and cost functions, which describe the behaviour of economic agents, are monotonic with respect to the coordinate-wise order relation (see, for example, [19]). There are some works on this theme, where monotonic functions are studied in a very general setting (see [14], [15], [42]). Monotonicity was understood there with respect to a certain order relation induced by a solid closed convex cone $K \subset X$:

$$x \leq y \iff y - x \in K \qquad (x, y \in X).$$

Here we consider only the results related to increasing functions defined on $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : \ x_i > 0 \, \forall i\}$. A function $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ is called increasing if

$$(x, y \in \mathbb{R}^n_{++}, \ \ x \leq y) \implies f(x) \leq f(y),$$

where the inequality $x \leq y$ means that $x_i \leq y_i$ for all $i = 1, \dots, n$. Four classes of increasing functions defined on $\mathbb{R}^n_{++}$ were studied in the framework of abstract convexity: increasing positively homogeneous (IPH), increasing radiant (InR), increasing co-radiant (ICR) and increasing convex-along-rays (ICAR).

Recall that a function $f : \mathbb{R}^n_{++} \to \bar{\mathbb{R}}_+ = [0, +\infty]$ is called radiant if $f(\lambda x) \leq \lambda f(x)$ for all $x \in \mathbb{R}^n_{++}$ and $\lambda \in (0, 1)$. A function $f : \mathbb{R}^n_{++} \to \bar{\mathbb{R}}_+$ is said to be co-radiant if

$f(\lambda x) \geq \lambda f(x)\ \forall\, x \in \mathbb{R}^n_{++}, \forall\, \lambda \in (0,1)$. A function $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ is convex-along-rays provided for each $x \in \mathbb{R}^n_{++}$ the function $f_x(t) = f(tx)$ is convex on $(0, +\infty)$. Note that the sets of all IPH, InR, ICR and ICAR functions are upper complete semilattices.

The following *min-type* functions play a key role in the study of these classes of increasing functions:

$$l(x) = \langle l, x\rangle = \min_{l=1,..,n} l_i x_i, \qquad (l, x \in \mathbb{R}^n_{++}). \tag{1.1}$$

Let us describe supremal generators of the sets (for IPH, ICR and ICAR they can be found in [41]; for the set InR see [51]).

A function $f : \mathbb{R}^n_{++} \to \bar{\mathbb{R}}_+$ is IPH if and only if it is L-convex with respect to the set $L = \{l :\ l(x) = \langle l, x\rangle,\ l \in \mathbb{R}^n_{++}\} \cup \{0\}$.

A function $f : \mathbb{R}^n_{++} \to \bar{\mathbb{R}}_+$ is InR if and only if it is H-convex, where $H = \{h :\ h(x) = c\varphi_l(x),\ l \in \mathbb{R}^n_{++},\ c \geq 0\}$ and the functions φ_l are defined on $\mathbb{R}^n_{++}$ by

$$\varphi_l(x) = \varphi(l, x) = \begin{cases} 0, & \text{if } \langle l, x\rangle < 1, \\ \langle l, x\rangle, & \text{if } \langle l, x\rangle \geq 1. \end{cases} \tag{1.2}$$

A function $f : \mathbb{R}^n_{++} \to \bar{\mathbb{R}}_+$ is ICR if and only if it is H-convex, where $H = \{h :\ h(x) = \min(\langle l, x\rangle, c),\ l \in \mathbb{R}^n_{++},\ c \geq 0\}$.

At last, a function $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ is ICAR if and only if it is H-convex with respect to the set $H = \{h :\ h(x) = l(x) - c,\ l \in L,\ c \in \mathbb{R}\}$.

One of interesting immediate applications of supremal generators is the following *Principle of Preservation of Inequalities* ([28]).

Proposition 1.3 *Let Y be a set of functions defined on a set X and equipped with the natural order relation. Let H be a supremal generator of Y. Furthermore, let ψ be an increasing functional defined on Y and let $x \in X$. Then*

$$(h(x) \leq \psi(h) \textit{ for all } h \in H) \iff (f(x) \leq \psi(f) \textit{ for all } f \in Y).$$

So if we have a set of elementary functions H and a certain inequality holds for all $h \in H$ then the same inequality holds for each H-convex function f. A classical example of increasing functional ψ defined on a set of functions is the integral. Hermite-Hadamard inequality states that for any convex function $f : [a, b] \to \mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2}[f(a) + f(b)]. \tag{1.3}$$

The inequality in the left-hand side of (1.3) is obvious for affine functions f. Hence, by the Principle of Preservation of Inequalities, it holds for convex functions as well. There are many generalizations of Hermite-Hadamard inequalities for different classes of nonconvex functions (see, for example, [12], [38] and the references therein). Consider some of them.

Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and such that

$$f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda} \quad \text{for all } x, y \in [a,b],\ \lambda \in (0,1).$$

Then ([13])

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a}\int_a^b f(x)\,dx$$

and the constant 4 in this inequality is sharp.

If $f : [0,1] \to \mathbb{R}$ is a nonnegative quasiconvex function then for any $u \in (0,1)$ the following inequality holds ([41])

$$f(u) \leq \frac{1}{\min(u, 1-u)}\int_0^1 f(x)\,dx.$$

Let f be an ICAR function defined on $\mathbb{R}^2_{++} = \{(x,y) \in \mathbb{R}^2 :\ x \geq 0,\ y \geq 0\}$. Let $D \subset \mathbb{R}^2_{++}$ be a closed domain (i.e., a bounded set such that $\operatorname{cl}\operatorname{int} D = D$). Denote by $A(D)$ the area of D. Let $(\bar{x}, \bar{y}) \in D$ be a point such that

$$\frac{1}{A(D)}\int_D \min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right)\,dxdy = 1.$$

Then ([11])

$$f(\bar{x}, \bar{y}) \leq \frac{1}{A(D)}\int_D f(x,y)\,dxdy,$$

and this inequality is sharp.

If $f : \mathbb{R}^n_{++} \to \mathbb{R}_+$ is an increasing radiant function and $D \subset \mathbb{R}^n_{++}$ is a closed domain (in topology of $\mathbb{R}^n_{++}$) then we have the inequality ([52])

$$f(\bar{x}) \leq \frac{1}{A(D)}\int_D f(x)\,dx$$

for all points $\bar{x} \in D$ satisfying the equality

$$\frac{1}{A(D)}\int_D \varphi\left(\frac{1}{\bar{x}}, x\right)\,dx = 1,$$

where the function φ is defined by (1.2) and $1/\bar{x} = (1/\bar{x}_1, \ldots, 1/\bar{x}_n)$.

Now consider further notions related to abstract convexity.

Definition 1.3 Let $f : X \to \bar{\mathbb{R}}$. The set

$$\text{supp}\,(f, H) = \{h \in H : \ h \leq f\} \tag{1.4}$$

of all H-minorants of f is called the support set of the function f with respect to the set of elementary functions H.

Here $h \leq f$ means that $h(x) \leq f(x)$ for all $x \in X$. The definition of H-convex functions can be reformulated as follows: a function $f : X \to \bar{\mathbb{R}}$ is H-convex if and only if its support set $\text{supp}\,(f, H)$ is not empty and $f(x) = \sup\{h(x) : \ h \in \text{supp}\,(f, H)\} \ \forall x \in X$.

A general approach of abstract convex analysis to global optimization problems is based on the calculation of support sets. Namely, we need to have a description of the following sets:

$$\partial_H^* f(y) = \{h \in H : \ h \in \text{supp}\,(f, H), \ h(y) = f(y)\}. \tag{1.5}$$

Sufficient condition for the global minimum of the function f over X can be formulated in terms of the set $\partial_H^* f(y)$: if there exists a function $h \in \partial_H^* f(y)$ such that $h(y) = \inf_{x \in X} h(x)$ then $f(y) = \inf_{x \in X} f(x)$. If H contains all constants then we have a necessary and sufficient condition: $f(y) = \inf_{x \in X} f(x)$ if and only if the set $\partial_H^* f(y)$ contains the constant $f(y)$.

At the same time, in some numerical methods (see ([41], Chapter 9) and the references therein), it is sufficient to know for each $y \in X$ at least one elementary function $h \in \text{supp}\,(f, H)$ such that $h(y) = f(y)$. For example, in the so-called Φ-*bundle method*, which was studied in detail by Pallaschke and Rolewicz in [37], a global minimizer of f over X is represented as a limit point of a sequence of solutions of auxiliary global minimization problems. In order to construct these auxiliary problems we need to have for each $y \in X$ at least one function $h \leq f$ with $h(y) = f(y)$.

Support sets can be useful also in the study of some constrained problems. A dual characterization of the problem of global minimization of an objective function subject to some inequality constraints can be given in terms of support sets (see, for example, [44]).

Methods of abstract convex analysis are mostly adapted for needs of global optimization. Nevertheless, they are applicable for problems of local optimization as well. In general, we have the following condition for the global minimum over a subset $Z \subset X$: if there exists a function $h \in \partial_H^* f(y)$ such that $h(y) = \inf_{x \in Z} h(x)$ then $f(y) = \inf_{x \in Z} f(x)$. For example, the problem of global minimization of IPH functions over the unit simplex was examined in [1]. In particular, if Z is a neighbourhood of y then we get: if there exists a function $h \in \partial^*_H f(y)$ such that h attains at y its local minimum then y is also a local minimizer of f.

Definition 1.4 Let $f : X \longrightarrow \bar{\mathbb{R}}$. Assume that the support set $\text{supp}\,(f, H)$ is not empty. Then the function $\text{co}_H f$ defined by

$$\text{co}_H f(x) = \sup\{h(x) : \ h \in \text{supp}\,(f, H)\}, \qquad (x \in X) \tag{1.6}$$

is called the H-convex hull of the function f.

It is clear that H-convex hull of a function f coincides with the greatest H-convex function which minorizes f. So a function $f : X \longrightarrow \bar{\mathbb{R}}$ is H-convex if and only if $f = \text{co}_H f$.

Definition 1.5 A set $U \subset H$ is called abstract convex with respect to X (or (H, X)-convex) if there exists a function $f : X \longrightarrow \bar{\mathbb{R}}$ such that $U = \text{supp}\,(f, H)$.

In other words, a set $U \subset H$ is called (H, X)-convex if

$$U = \{h \in H : \ h(x) \le f_U(x) \ \forall x \in X\},$$

where

$$f_U(x) = \sup_{u \in U} u(x) \tag{1.7}$$

is the *support function* of the set U.

If U' is a (H, X)-convex set then for any nonempty subset U of H the following assertions are equivalent

$$(U \subset U') \iff (f_U \le f_{U'}).$$

Every H-convex function f gives us its support set $U = \text{supp}\,(f, H)$ (which is (H, X)-convex) and coincides with the support function f_U of this set: $f = f_U$. Conversely, each (H, X)-convex set U determines its support function f_U (which is H-convex) and coincides with the support set $\text{supp}\,(f_U, H)$ of this function: $U = \text{supp}\,(f_U, H)$. This one-to-one correspondence between abstract convex functions and sets is called the *Minkowski duality*.

Definition 1.6 The intersection of all (H, X)-convex sets containing a set $U \subset H$ is called the abstract convex hull or (H, X)-convex hull of the set U. This hull is denoted by $\text{co}_{H,X} U$ (or shortly $\text{co}_H U$).

It is easy to see that for any set $U \subset H$ its (H, X)-convex hull $\mathrm{co}_H U$ is equal to the support set of its support function:

$$\mathrm{co}_H U = \mathrm{supp}\,(f_U, H). \tag{1.8}$$

Indeed, due to the Minkowski duality

$$\begin{aligned}
\mathrm{co}_H U &= \bigcap \{U' : \ U' \text{ is } (H,X)\text{-convex}, \ U \subset U'\} \\
&= \bigcap \{\mathrm{supp}\,(f,H) : \ f \text{ is } H\text{-convex}, \ U \subset \mathrm{supp}\,(f,H)\} \\
&= \bigcap \{\mathrm{supp}\,(f,H) : \ f \text{ is } H\text{-convex}, \ f_U \le f\} \\
&= \{h \in H : \ h \le f \text{ for all } H\text{-convex functions } f \text{ such that } f_U \le f\} \\
&= \{h \in H : \ h \le f_U\} = \mathrm{supp}\,(f_U, H).
\end{aligned}$$

Thus, abstract convex sets are exactly the support sets of their support functions. Since optimality conditions can be formulated in terms of support sets then it is very important to have a description of abstract convex sets. For example, if $(h_1 + h_2) \in H$ for all $h_1, h_2 \in H$ then for any H-convex functions f_1, f_2 the following holds (see [41])

$$\mathrm{supp}\,(f_1 + f_2, H) = \mathrm{co}_H(\mathrm{supp}\,(f_1, H) + \mathrm{supp}\,(f_2, H)).$$

Hence, if the set $(\mathrm{supp}\,(f_1, H) + \mathrm{supp}\,(f_2, H))$ is (H, X)-convex then we deduce that the support set of the sum coincides with the sum of support sets. Note also that the maximum of two abstract convex functions is always abstract convex, and the support set of the maximum coincides with the abstract convex hull of the union of support sets of given functions:

$$\mathrm{supp}\,(\max(f_1, f_2), H) = \mathrm{co}_H(\mathrm{supp}\,(f_1, H) \cup \mathrm{supp}\,(f_2, H)).$$

Consequently the problem of describing abstract convex hull of the union of abstract convex sets is of exceptional importance.

As a rule, describing abstract convex functions is much easier than describing abstract convex sets. In Chapter 5 we derive some conditions, which guarantee a description of both abstract convex functions and sets. We also give a description of the abstract convex hull of a finite union of abstract convex sets.

Consider two examples of abstract convex sets.

If H consists of all linear functions $h : \mathbb{R}^n \to \mathbb{R}$ then a set $U \subset H \equiv \mathbb{R}^n$ is $(H, \mathbb{R}^n)$-convex if and only if it is closed (in the topological space $\mathbb{R}^n$) and convex.

If H consists of all min-type functions defined by (1.1) then a set $U \subset H \equiv \mathbb{R}^n_{++}$ is $(H, \mathbb{R}^n_{++})$-convex if and only if it is closed and normal. The latter property means that U contains every $u' \in H$ such that $u' \leq u$ for certain $u \in U$.

1.3 Abstract subdifferential and conjugation

In this section we discuss classical versions of conjugation and abstract subdifferential for real-valued abstract convex functions. Some more general types of conjugations and subdifferentials based on the notion of duality are examined in [57] for functions with values in a complete lattice. In Chapter 6 we generalize the notion of duality and investigate corresponding conjugation and subdifferential for functions with values in an upper complete semilattice and a partially ordered set respectively.

Let L be a set of finite functions $l : X \to \mathbb{R}$. Denote by H_L the set of all functions $h(x) = l(x) - c$ defined on X, where $l \in L$ and $c \in \mathbb{R}$. Here we are interested in H_L-convex functions. Note that the set H_L is closed under vertical shifts, that is, $(h + c) \in H_L$ for all $h \in H_L$ and $c \in \mathbb{R}$. This property of the set of elementary functions allows one to investigate abstract convex functions via the notions of a conjugate function and abstract subdifferential.

Definition 1.7 Let $f : X \to \bar{\mathbb{R}}$. The function $f_L^* : L \to \bar{\mathbb{R}}$ defined by

$$f_L^*(l) = \sup_{x \in X}(l(x) - f(x)), \qquad (l \in L) \tag{1.9}$$

is called the Fenchel-Moreau L-conjugate of f.
The Fenchel-Moreau second L-conjugate $f_L^{**} : X \to \bar{\mathbb{R}}$ is defined as follows:

$$f_L^{**}(x) = \sup_{l \in L}(l(x) - f_L^*(l)), \qquad (x \in X). \tag{1.10}$$

We see that for any function $f : X \to \bar{\mathbb{R}}$ its second L-conjugate f_L^{**} is either H_L-convex or identically equal to $-\infty$ (if $f_L^* \equiv +\infty$). Since $\inf\{h(x) : h \in H_L\} \equiv -\infty$ then it is natural to accept the following: if a function $f : X \to \bar{\mathbb{R}}$ has empty support set supp $(f, H_L) = \emptyset$ then its H_L-convex hull $\mathrm{co}_{H_L} f$ is identical $-\infty$.

The Fenchel-Moreau theorem (see, for example, [57]) states that for an arbitrary function $f : X \to \bar{\mathbb{R}}$ its second L-conjugate coincides with the H_L-convex hull: $f_L^{**} = \mathrm{co}_{H_L} f$.

In particular, we get the following characterization of H_L-convexity: a function $f : X \to \mathbb{R}_{+\infty}$ is H_L-convex if and only if $f = f_L^{**}$.

Another important property of the Fenchel-Moreau conjugation is related to the notion of support set. For more convenience we will identify every pair $(l, c) \in L \times \mathbb{R}$ with the function $h(x) = l(x) - c$. Then for any function $f : X \to \bar{\mathbb{R}}$ its support set supp (f, H_L) with respect to H_L coincides with the epigraph epi $f_L^* = \{(l, c) \in L \times \mathbb{R} : \ f_L^*(l) \le c\}$ of the function f_L^*. Indeed,

$$\begin{aligned} \text{supp}\,(f, H_L) &= \{(l, c) \in L \times \mathbb{R} : \ l(x) - c \le f(x)\ \forall x \in X\} \\ &= \left\{(l, c) \in L \times \mathbb{R} : \ \sup_{x \in X}(l(x) - f(x)) \le c\right\} = \text{epi}\, f_L^*. \end{aligned}$$

Thus, if we have a calculus of conjugate functions then a calculus of support sets is known as well, and vice versa.

Definition 1.8 Let $f : X \to \mathbb{R}_{+\infty}$ and $y \in X$ be such that $f(y) < +\infty$. The set

$$\partial_L f(y) = \{l \in L : \ l(x) - l(y) \le f(x) - f(y)\ \forall x \in X\} \tag{1.11}$$

is called the abstract subdifferential (or L-subdifferential) of the function f at the point y. Elements of L-subdifferential are called L-subgradients.

It is easy to see that the subdifferential $\partial_L f(y)$ consists of all functions $l \in L$ such that the supremum in (1.9) is attained at the point y. Thus, the notions of L-conjugate function and L-subdifferential are related.

It follows from (1.11) that there is a one-to-one correspondence between the subdifferential $\partial_L f(y)$ and the set $\partial^*_{H_L} f(y) = \{h \in H_L : \ \ h \in \text{supp}\,(f, H_L), \ \ h(y) = f(y)\}$. Every abstract subgradient $l \in \partial_L f(y)$ forms the elementary function $h(x) = l(x) - l(y) + f(y)$ which belongs to the support set supp (f, H_L) and coincides with f at the point y: $h(y) = f(y)$. On the other hand, if $h(x) = l(x) - c$ ($l \in L$), $h(y) = f(y)$ and $h \le f$ then $h(x) = l(x) - l(y) + f(y)$ and $l \in \partial_L f(y)$. Thus, we have

$$\partial^*_{H_L} f(y) = \{h \in H_L : \ \ h(x) = l(x) - l(y) + f(y), \ \ l \in \partial_L f(y)\}. \tag{1.12}$$

In particular, for a finite function $f : X \to \mathbb{R}$, nonemptiness of all L-subdifferentials $\partial_L f(y)$ implies that f is H_L-convex. This means that the notion of L-subdifferential is a natural tool for examination of H_L-convex functions. It should be mentioned that the reverse statement is not true in general.

In the previous section we considered conditions for the global minimum of a function f in terms of the set $\partial^*_H f(y)$. Formula (1.12) shows that, if the set of elementary functions is closed under vertical shifts, then the notion of abstract subdifferential allows one to simplify the description of the set $\partial^*_{H_L} f(y)$. Indeed, to ensure that a function $h \in H_L$ belongs to $\partial^*_{H_L} f(y)$ we need to check the equality $h(y) = f(y)$ and the inequality $h \leq f$. In contrast to the set $\partial^*_{H_L} f(y)$, the definition of L-subdifferential contains only the inequality $h \leq f$ with the elementary function $h \in H_L$ defined by $h(x) = l(x) - l(y) + f(y)$. Therefore the description of abstract subdifferential is easier than description of the set $\partial^*_{H_L} f(y)$. In view of (1.12), the conditions for global minimum can be easily reformulated in terms of subdifferentials.

First, assume that L contains identical zero. Then the abstract subdifferential provides the following necessary and sufficient condition for the global minimum of a function $f : X \to \mathbb{R}_{+\infty}$ over X:

$$f(y) = \inf_{x \in X} f(x) \quad \text{if and only if} \quad 0 \in \partial_L f(y). \tag{1.13}$$

If L does not contain identical zero then we can use the following sufficient condition: if a function $l \in \partial_L f(y)$ exists such that $l(y) = \inf_{x\in X} l(x)$ then $f(y) = \inf_{x \in X} f(x)$.

It is convenient to consider also the set $\mathcal{D}_L f(y)$ along with the L-subdifferential $\partial_L f(y)$ (see [41] p. 364, where this set was denoted by $\mathcal{D} f(y)$):

$$\mathcal{D}_L f(y) = \{h \in H_L : \ h(x) = l(x) - l(y), \ l \in \partial_L f(y)\}. \tag{1.14}$$

Then the above statement takes the form: if $\mathcal{D}_L f(y)$ contains a nonnegative function then y is a global minimizer of f over X.

In order to apply these optimality conditions, we need to have a calculus of abstract subdifferentials. There are two approaches to this problem. The first one means immediate calculation of subdifferentials for some abstract convex functions. The second approach is based on finding certain calculus rules, which allow to describe subdifferentials of some combinations of abstract convex functions via subdifferentials of given functions. For example, the maximum of abstract convex functions is always abstract convex with respect to the same set of elementary functions. So the question "How is the subdifferential of the maximum of some functions via the subdifferentials of given functions expressed?" is natural.

As a rule it is a difficult task to describe the whole subdifferential, and we have only some abstract subgradients. Then only numerical methods and sufficient conditions for a minimum are applicable. For example, in Chapter 2 we have a situation, when obtaining the description of the whole subdifferential seems unlikely.

It seems there is still no general theory of subdifferential calculus for abstract convex functions. We try to fill this gap in Chapter 4 based on the recent paper [53]. We use there an approximation function, which is a little bit different from that in [53]. In order to get some calculus rules it is assumed that the set H_L has the so-called strong globalization property.

Nevertheless, there have been separate investigations of abstract subdifferentials for different classes of abstract convex functions. They allow to derive conditions for global optimality of some particular problems (see, for example, [21,22,61]).

Moreover, in some cases we can get exact and sufficiently simple formulas for the L-subdifferentials. Consider one nonconvex example. Let L be the collection of all min-type functions defined by (1.1). Let p be a proper (that is finite and non-zero) IPH function defined on $\mathbb{R}^n_{++}$ and $y \in \mathbb{R}^n_{++}$. Then ([41], Theorem 2.4)

$$\partial_L p(y) = \left\{ l : \ l \geq \frac{p(y)}{y}, \ p\left(\frac{1}{l}\right) = 1 \right\},$$

where $p(y)/y = (p(y)/y_1, \ldots, p(y)/y_n)$ and $1/l = (1/l_1, \ldots, 1/l_n)$.

If a function f is ICAR and strictly increasing at $y \in \mathbb{R}^n_{++}$ (this means that $f(x) < f(y)$ for each $x < y$) then (see [43])

$$\partial_L f(y) = \{t/y : \ t \in \partial f_y(1)\},$$

where $f_y(\alpha) = f(\alpha y)$ for $\alpha > 0$ and $\partial f_y(1) = \{t \geq 0 : \ t\alpha - t \leq f_y(\alpha) - f_y(1) \ \forall \alpha > 0\}$.

1.4 Separation properties in axiomatic and abstract convexity

Axiomatic convexity deals with families of sets, which have some properties of usual convex sets. A general theory of convex structures can be found, for example, in [58] and [60]. Here we use the following definition (see [60], p. 3).

A collection $\mathcal{G}$ of subsets of a set X is called a convexity on X if

(1) $\emptyset, X \in \mathcal{G}$

(2) $\bigcap \mathcal{A} \in \mathcal{G}$ for every $\mathcal{A} \subset \mathcal{G}$

(3) $\bigcup \mathcal{A} \in \mathcal{G}$ whenever $\mathcal{A} \subset \mathcal{G}$ is a chain with respect to the inclusion.

Members of $\mathcal{G}$ are called convex sets and the pair $(X, \mathcal{G})$ is called a convexity space. For any subset $A \subset X$ its convex hull $\text{conv}_{\mathcal{G}}A$ is defined by $\text{conv}_{\mathcal{G}}A = \bigcap\{G \in \mathcal{G} : \ A \subset G\}$.

Along with convexity spaces consider also the so-called closure spaces ([60], p. 4). The pair $(X, \mathcal{G})$ is a closure space provided that $\emptyset, X \in \mathcal{G}$ and $\mathcal{G}$ is stable with respect to intersections, that is, $\bigcap \mathcal{A} \in \mathcal{G}$ for every $\mathcal{A} \subset \mathcal{G}$. Members of $\mathcal{G}$ are called closed sets and $\mathcal{G}$ is called a protopology (Moore family) on X. Closure spaces go back to Moore [35]. They have a central place in lattice theory ([4]).

Abstract convex analysis deals with such closure spaces. Indeed, the intersection of any family of epigraphs of abstract convex functions is also the epigraph of an abstract convex function, and the intersection of any family of abstract convex sets is abstract convex as well.

Note that each convexity space $(X, \mathcal{G})$ is also a closure space. As a rule, in abstract convex analysis we are interested in closure spaces, which are not convexity spaces. For example, in classical convex case the convexity space consists of all convex sets. At the same time, in the framework of abstract convexity, it is convenient to investigate only closed convex sets.

Separation properties in axiomatic and abstract convexity are based on separation of complicated sets by sufficiently simple sets. Here we consider a strong version of separability for disjoint sets. Let $A, B, H \subset X$. We say that H separates A from B provided that $A \subset H$ and $B \subset X \backslash H$. The following two cases are the most interesting: A and B are convex sets; A is convex and B is one-point set. It is important that the convex set H, which separates A from B, should be simple enough.

Consider an interesting example of convexity on $I \times \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval.

Example 1.1 Let $\mathcal{F}$ be a family of continuous functions $\varphi : I \rightarrow \mathbb{R}$. Assume that $\mathcal{F}$ is a two-parameter family (see [2]). Last means that for any two points $(x_1, y_1), (x_2, y_2) \in I \times \mathbb{R}$ with $x_1 \neq x_2$ there exists exactly one $\varphi \in \mathcal{F}$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Let $\varphi_{(x_1,y_1)(x_2,y_2)}$ be the function determined by the point (x_1, y_1) and (x_2, y_2).

Beckenbach [2] introduced the notion of generalized convex functions in the following way: a function $f : I \rightarrow \mathbb{R}$ is said to be $\mathcal{F}$-convex if for any $x_1, x_2 \in I$, $x_1 < x_2$

$$f(x) \leq \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x), \qquad x_1 \leq x \leq x_2.$$

Note that this definition does not coincide with the Definition 1.1. Namely, each $\mathcal{F}$-convex in the sense of Definition 1.1 function $f : I \to \mathbb{R}$ is $\mathcal{F}$-convex in the sense of Beckenbach. The reverse is not true.

Using a similar idea, Krzyszkowski [26] introduced the notion of generalized convex sets. First, for each $a = (x_1, y_1), b = (x_2, y_2) \in I \times \mathbb{R}$ define the generalized segment $[a, b] \subset I \times \mathbb{R}$:

$$[a, b] = \{(x, \varphi_{(x_1,y_1)(x_2,y_2)}(x)) : \min\{x_1, x_2\} \le x \le \max\{x_1, x_2\}\}, \qquad \text{if} \quad x_1 \ne x_2$$

and

$$[a, b] = \{(x_1, y) : \min\{y_1, y_2\} \le y \le \max\{y_1, y_2\}\}, \qquad \text{if} \quad x_1 = x_2.$$

Then a set $A \subset I \times \mathbb{R}$ is said to be $\mathcal{F}$-convex (see [26]) if for any $a, b \in A$ we have $[a, b] \subset A$. It is easy to check that the collection of all $\mathcal{F}$-convex sets is a convexity on $I \times \mathbb{R}$.

Such type of generalized convex functions and sets possesses strong separation properties. Here we present only the result for sets, which was proved in [36]. Let $A, B \subset I \times \mathbb{R}$ be disjoint $\mathcal{F}$-convex sets. Then there exists an $\mathcal{F}$-convex set H which separates A from B and such that its complement $(I \times \mathbb{R})\backslash H$ is also $\mathcal{F}$-convex.

Let us consider separation properties in abstract convex analysis. If a function $f : Y \to \bar{\mathbb{R}}$ is L-convex then for any $(y, c) \notin \operatorname{epi} f$ a function $l \in L$ exists such that $\operatorname{epi} f \subset \operatorname{epi} l$ and $(y, c) \notin \operatorname{epi} l$. If a set $U \subset L$ is (L, Y)-convex and a function $l \in L$ does not belong to U then a point $y \in Y$ exists such that $l(y) > \sup_{u \in U} u(y)$. Hence $U \subset H$ and $l \notin H$, where H is an (L, Y)-convex set defined by $H = \{h \in L : \ h(y) \le c\}$ and c is a number such that $l(y) > c > \sup_{u \in U} u(y)$. In order to use these separation properties efficiently we need a description of abstract convex functions and sets. This is the main problem related to separability in abstract convex analysis.

It appears that, in the framework of the notions of abstract convex functions and sets, we usually deal with the separation of a set from a point in its complement. However, in some cases, which refer to abstract convexity, the separation property (in a certain sense) is also valid for pairs of sets. Separability of two sets can be very useful for formulation of optimality conditions of special global optimization problems. Namely, this finds applications in some best approximation problems (see, for example, [31–34, 47, 48, 56]). In

Chapter 3 we discuss the weak separability of two star-shaped sets and derive conditions for the global minimum of the so-called star-shaped distance.

Now consider some general results from axiomatic convexity. We begin with the separation theorem of Kubiś [27] concerning two arbitrary convexities on a set. This result is a common generalization of results of Ellis [18] and Chepoi [6].

Theorem 1.1 *([27], Theorem 3.1) Let $\mathcal{G}$ and $\mathcal{H}$ be two convexities on a set X. The following conditions are equivalent:*

(a) *For every $x, y, z \in X$ and finite sets $S, T \subset X$ such that $x \in \operatorname{conv}_{\mathcal{G}}(\{z\} \cup S)$ and $y \in \operatorname{conv}_{\mathcal{H}}(\{z\} \cup T)$ it holds that $\operatorname{conv}_{\mathcal{G}}(\{y\} \cup S) \cap \operatorname{conv}_{\mathcal{H}}(\{x\} \cup T) \neq \emptyset$.*

(b) *If $A \in \mathcal{G}$ and $B \in \mathcal{H}$ are disjoint then there exist disjoint sets $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $A \subset G$, $B \subset H$ and $G \cup H = X$.*

Equalities $G \cap H = \emptyset$ and $G \cup H = X$ in condition (b) mean that $H = X \backslash G$. So we have that G is convex in convexity $\mathcal{G}$ and its complement is convex in convexity $\mathcal{H}$. This allows to hope that H and G are sufficiently simple.

In the classical convex case (see [24]) two disjoint convex sets in a real vector space can be separated by a halfspace (i.e, a convex set with the convex complement). We can generalize the notion of a halfspace in the following natural way: a subset $H \subset X$ of convexity space $(X, \mathcal{G})$ is called a halfspace provided $H \in \mathcal{G}$ and $(X \backslash H) \in \mathcal{G}$. There are some separation properties of convexity spaces formulated in terms of such halfspaces. We consider the following separation axioms due to Jamison [20]:

S_3 : If $A \subset X$ is convex and $x \in X \backslash A$ then there is a halfspace H of X with $A \subset H$ and $x \notin H$.

S_4 : If $A, B \subset X$ are disjoint convex sets then there is a halfspace H of X with $A \subset H$ and $B \subset X \backslash H$.

The next theorem gives a characterization of properties S_3 and S_4 via screening of polytopes (i.e., convex hulls of finite sets) by convex sets. Recall that two sets $A, B \subset X$ are said to be screened by $C, D \subset X$ if $A \subset C \backslash D$, $B \subset D \backslash C$ and $C \cup D = X$ (cf. [59]). It is easy to see that screening by two convex sets is a weaker property than separation by a halfspace. Indeed, if a halfspace H separates A from B then A and B are screened by the convex sets H and $X \backslash H$.

Theorem 1.2 *([60], Theorem 3.8) (Polytope Screening Characterization)*

1.) A convexity space satisfies S_3 if and only if each polytope and each point in its complement can be screened by convex sets.

2.) A convexity space satisfies S_4 if and only if each pair of disjoint polytopes can be screened by convex sets.

The situation becomes easier if the convexity is of finite arity. Let N be a positive integer and $(X, \mathcal{G})$ be a convexity space. Then $\mathcal{G}$ is called N-ary (or of arity N) (see [60]) provided that a set $A \subset X$ is convex if and only if $\text{conv}_{\mathcal{G}}\{a_1, \ldots, a_N\} \subset A$ for all $a_1, \ldots, a_N \in A$.

Theorem 1.3 *([27], Theorem 4.2) Let $(X, \mathcal{G})$ be a convexity space. If $\mathcal{G}$ is N-ary then the following conditions are equivalent:*

(i) *Every two disjoint convex sets can be separated by a halfspace.*

(ii) *Every two disjoint N-polytopes can be separated by a halfspace.*

(iii) *Every two disjoint N-polytopes can be screened by convex sets.*

(iv) *If $x \in \text{conv}_{\mathcal{G}}\{c, a_1, \ldots, a_{N-1}\}$ and $y \in \text{conv}_{\mathcal{G}}\{c, b_1, \ldots, b_{N-1}\}$ then*

$$\text{conv}_{\mathcal{G}}\{y, a_1, \ldots, a_{N-1}\} \bigcap \text{conv}_{\mathcal{G}}\{x, b_1, \ldots, b_{N-1}\} \neq \emptyset.$$

Note that Theorems 1.1 and 1.2 do not imply any description of convex sets. On the other hand, if such a description is given by definition of considered convexity, then there is no clear description of halfspaces and sets $G \in \mathcal{G}$ with $(X \backslash G) \in \mathcal{H}$. Moreover, condition (a) in Theorem 1.1 and the polytope screening in Theorem 1.2 are complicated for verification, because they involve arbitrary finite subsets of X.

Theorem 1.3 gives an easier characterization of the separation property S_4, however this theorem (as well as Theorem 1.2) does not imply exact description of the collection of all halfspaces. Therefore we have no sufficient information about the sets, which can separate two convex sets.

Thus, there are two main problems concerning separability in axiomatic convexity theory, namely the description of convex sets and the description of collections of sufficiently simple convex sets, which can separate arbitrary convex sets.

In abstract convex analysis we usually deal with the situation, when the collection of elementary sets is given. Then we need to get a description of sets, which can be represented as the intersection of a subfamily of this collection.

It seems there is no solution of these problems in the general case. Hence we need some restrictions. In Chapter 5, as a sort of such restriction, we choose a special type of connectedness of a topological space with respect to a convexity on this space. Although separation properties in axiomatic and abstract convexity have no distinctions in kind, we can say that Theorems 5.4 and 5.5 relate to abstract convexity while all results of Section 5.2 are in the framework of axiomatic convexity. As an application, we give a description of abstract convex functions and sets.

Chapter 2

Subdifferentials of convex-along-rays functions

In this chapter we study lower semicontinuous convex-along-rays (briefly, CAR) functions defined on an Euclidian space $\mathbb{R}^n$ and mapping into semi-extended real line $\mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$. These functions were introduced and examined in [49], see also [41]. Here we examine the existence of abstract subgradients of CAR functions with respect to different sets of elementary functions.

2.1 CAR functions and abstract convexity

First we recall some definitions and results from [41], which are required in the current chapter.

A function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is called convex-along-rays if its restriction f_x on the ray $R_x = \{\lambda x : \lambda \geq 0\}$ is a convex function for each $x \in \mathbb{R}^n$. In other words, f is CAR if the function $f_x : [0, +\infty) \to \mathbb{R}_{+\infty}$ defined by $f_x(\lambda) = f(\lambda x)$ is convex for each $x \in \mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is called convex-along-lines (CAL) if its restriction f_x on the line $L_x = \{\lambda x : \lambda \in \mathbb{R}\}$ is a convex function for each $x \in \mathbb{R}^n$. This special case of CAR functions was investigated in [9].

Positively homogeneous of degree one (briefly, PH) functions are CAR and they are of special interest in this chapter. Recall that $l : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is PH if $l(\lambda x) = \lambda l(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$. If p is PH and $p(0) < +\infty$ then $p(0) = 0$.

CAR functions can be studied by the methods of abstract convexity. We examine here abstract subdifferentials of CAR functions.

Let L be a family of PH functions $l : \mathbb{R}^n \to \mathbb{R}$ and $H_L = \{h_{l,c} : l \in L,\ c \in \mathbb{R}\}$, where $h_{l,c}(x) = l(x) - c$. Consider an H_L-convex function f. It is easy to check and well-known that the subdifferential $\partial_L f(y)$ is not empty if and only if there exists $h \in H_L$ such that $h(x) \le f(x)$ for all $x \in \mathbb{R}^n$ and $h(y) = f(y)$.

If p is PH then (see [41]), p is H_L-convex if and only if p is L-convex and

$$\partial_L p(y) = \{l \in L : l \le p, l(y) = p(y)\}.$$

In this chapter we consider the sets $L := \mathcal{L}_s$ of functions ℓ defined on $\mathbb{R}^n$ by

$$\ell(x) = \min_{i=1,\dots,s} [l_i, x], \qquad x \in \mathbb{R}^n, \tag{2.1}$$

where s is a positive integer and $[l, x]$ stands for the inner product of vectors l and x. As a rule we assume that either $s = n+1$ or $s = n$. The function ℓ defined by (2.1) is PH. It is known (see [41]), that a lower semicontinuous (briefly, lsc) CAR function f with $0 \in \operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ is $H_{\mathcal{L}_{n+1}}$-convex. Some lsc CAR functions are $H_{\mathcal{L}_n}$-convex.

We examine conditions that guarantee the non-emptiness of $\mathcal{L}_{n+1}$-subdifferentials for CAR functions f with $0 \in \operatorname{dom} f$ and $\mathcal{L}_n$-subdifferentials for CAR functions that $H_{\mathcal{L}_n}$-convex.

We start with the existence of $\mathcal{L}_s$-subgradients. This question was investigated in [41], pp. 220-223 for $s = n+1$. Unfortunately some of the results presented in [41] are not correct. We present a revised version of these results in Section 2.3.

2.2 $\mathcal{L}_s$-convexity and $\mathcal{L}_s$-subdifferentiability of positively homogeneous functions

Positively homogeneous of degree one functions form the simplest class of CAR functions. So we shall start with lsc PH functions.

Theorem 2.1 *(see Theorems 5.14 and 5.15 in [41]) A function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is $\mathcal{L}_{n+1}$-convex if and only if this function is PH and lsc. Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a lsc and PH function and $x_0 \in \operatorname{dom} f$, $x_0 \neq 0$. Then the subdifferential $\partial_{\mathcal{L}_{n+1}} f(x_0)$ is not empty if and* *only if f is calm of degree one at x_0, that is*

$$\liminf_{x \to x_0} \frac{f(x) - f(x_0)}{\|x - x_0\|} > -\infty.$$

We now consider nonnegative PH functions p with $0 \in \operatorname{dom} p$. These functions are $\mathcal{L}_n$-convex. In order to study them we need the following well-known definitions (see, for example, [41], Chapter 5).

Let $U \subset \mathbb{R}^n$ be a set. The set

$$\operatorname{kern} U = \{u \in U : u + \lambda(x - u) \in U \quad \text{for all} \quad x \in U \quad \text{and} \quad \lambda \in [0, 1]\} \tag{2.2}$$

is called the kernel of U. A set U is called star-shaped if $\operatorname{kern} U \neq \emptyset$. A star-shaped set U is called radiant if $0 \in \operatorname{kern} U$. If U is a star-shaped set and $u \in \operatorname{kern} U$ then $U - u$ is radiant.

Let $U \subset \mathbb{R}^n$ be a radiant set. The function

$$\mu_U(x) = \inf\{\lambda > 0 : x \in \lambda U\} \tag{2.3}$$

is called the Minkowski gauge of a radiant set U. (It is assumed here that the $\inf \emptyset = +\infty$, $\sup \emptyset = 0$.) The Minkowski gauge of a closed radiant set U is a lsc nonnegative positively homogeneous function. It can be proved that each lsc nonnegative positively homogeneous function $p : \mathbb{R}^n \to [0, +\infty]$ coincides with the Minkowski gauge μ_U of the set $U = \{x \in \mathbb{R}^n : p(x) \leq 1\}$. It follows from the presented geometric interpretation of nonnegative PH functions that each proper nonnegative lsc PH function is $\mathcal{L}_n$-convex (see Theorem 5.13 in [41]).

We can also use this geometric interpretation for examination of $\mathcal{L}_n$-subdifferentials of nonnegative PH functions. We need the following definition (see [41], Definition 5.21).

Definition 2.1 Let $U \subset \mathbb{R}^n$ be a closed set and $x \in U, x \neq 0$. A collection of linearly independent vectors $\ell = (l_1, \dots, l_m)$ is called a support collection to U at x if $[l_i, x] = 1 \, (i = 1, \dots, m)$ and

$$\min_{i=1,\dots,m} [l_i, u] < 1 \quad \text{for all} \quad u \in U, u \neq x. \tag{2.4}$$

Equalities $[l_1, x] = \dots = [l_m, x]$ are used here only for normalization. It is important that $[l_i, x] > 0$ for all i.

Proposition 2.1 *Let p be a nonnegative PH function and $U = \{x : p(x) \le 1\}$. Let $p(x_0) = 1$ and let there exist a support collection $\ell = (l_1, \dots, l_n)$ to U at x_0. Then $\ell \in \partial_{\mathcal{L}_n} p(x_0)$.*

Proof: The equality $\ell(x_0) = 1 = p(x_0)$ holds. So we need to show that $\ell(x) \le p(x)$ for all x. Clearly we can consider only $x \in \operatorname{dom} p$. Let $+\infty > p(x) > 0$. Then $u = x/p(x) \in U$ and $p(u) = 1$, hence $\ell(u) = \min_{i=1,\dots,n}[l_i, u] \le 1 = p(u)$. It follows from this that $p(x) \ge \ell(x)$. If $p(x) = 0$ then $\lambda x \in U$ for all $\lambda > 0$, hence $\ell(\lambda x) < 1$ for all $\lambda > 0$. This means that $\ell(x) \le 0 = p(x)$. □

A function p is called locally Lipschitz at x_0 if the restriction of p to a neighborhood of p is Lipschitz.

Theorem 2.2 *Let p be a nonnegative PH function. Assume that p is locally Lipschitz at a point x_0 with $p(x_0) = 1$ and let $U = \{x : p(x) \le 1\}$. Then there exists a support collection to the set U at the point x_0.*

The proof follows from Theorem 5.7 and Corollary 5.6 in [41].

Theorem 2.3 *Let p be a nonnegative PH function. Assume that p is locally Lipschitz at a point x_0 such that $p(x_0) > 0$. Then $\partial_{\mathcal{L}_n} p(x_0) \neq \emptyset$. If $p(x_0) = 0$ then $0 \in \partial_{\mathcal{L}_n} p(x_0)$. Consequently $\partial_{\mathcal{L}_n} p(x_0) \neq \emptyset$.*

Proof: Let $p(x) > 0$. It is easy to check that $\partial_{\mathcal{L}_n} p(x) = \partial_{\mathcal{L}_n} p(\lambda x)$ for $\lambda > 0$, so we can assume without loss of generality that $p(x_0) = 1$. Then the result follows from Proposition 2.1 and Theorem 2.2. If $p(x_0) = 0$ then the inclusion $0 \in \partial_{\mathcal{L}_n} p(x_0)$ is trivial. □

2.3 Lower affine approximations and $\mathcal{L}_s$-subdifferentials of convex-along-rays functions

Our goal is to extend the results that are known for positively homogeneous functions, to the case of lsc CAR functions with $0 \in \operatorname{dom} f$. For this purpose we need the notion of lower affine approximation of a lsc CAR function corresponding to a number $a \le f(0)$. This notion was introduced in [49], (see also [41]) for arbitrary lsc CAR functions, in particular for functions with $f(0) = +\infty$, and it was assumed there that $a < f(0)$. However, we can accept $a = f(0)$ if $f(0) < +\infty$. Many results from [41, 49] related to the case $a < f(0)$

are also valid if $a = f(0)$. First we need to consider the quantity

$$b^a(x) = \inf_{\alpha>0} \frac{f_x(\alpha) - a}{\alpha} \tag{2.5}$$

Note that b^a is a PH function. Let $a = f(0)$. Since $f(0) = f_x(0)$ and f_x is a convex function then

$$b^a(x) = \inf_{\alpha>0} \frac{f_x(\alpha) - f_x(0)}{\alpha} = \lim_{\alpha\to+0} \frac{f_x(\alpha) - f_x(0)}{\alpha} = f_x'(0) = f'(0, x).$$

Definition 2.2 Let f be a lsc CAR function and $0 \in \operatorname{dom} f$. Let $a \leq f(0)$. Then the function $g^a(x) = a + b^a(x)$ is called a lower affine approximation of f, corresponding to a.

If $a = f(0)$ then $g^a(x) = f(0) + f'(0, x)$. Since b^a is PH it follows that g^a is affine at each ray starting from zero. It is easy to see that $g^a(x) \leq f(x)$ for all $a \leq f(0)$ (see [41], p. 213 where the case $a < f(0)$ was considered). Each lsc CAR function is the upper envelope of its lower affine approximations (see [41], Lemma 5.5). It can be shown that for a lsc CAR function f with $f(0) < +\infty$ the functions b^a are lsc for all $a < f(0)$ (this is the contents of the proof of Theorem 5.16 in [41]). It follows from this result and Theorem 2.1 that the following result holds.

Theorem 2.4 *Each lsc CAR function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(0) < +\infty$ is $H_{\mathcal{L}_{n+1}}$-convex.*

Assume now that f is a lsc CAR function such that $f(0) = \min_{x\in\mathbb{R}^n} f(x)$. Then for each $a \leq f(0)$ we have

$$b^a(x) = \inf_{\alpha>0} \frac{f(\alpha x) - a}{\alpha} \geq \inf_{\alpha>0} \frac{f(0) - a}{\alpha} = 0. \tag{2.6}$$

Since b^a is a nonnegative lsc PH function we can apply ([41], Theorem 5.13) which shows that b^a is $\mathcal{L}_n$-convex. This leads to the following statement.

Theorem 2.5 *Each lsc CAR function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(0) = \min_{x\in\mathbb{R}^n} f(x)$ is $H_{\mathcal{L}_n}$-convex.*

We study relations between $\partial_{\mathcal{L}_s} f(x_0)$ and $\partial_{\mathcal{L}_s} b^a(x_0)$ where b^a is defined by (2.5). Here s is an arbitrary integer, however the results are of interest only when the s-subdifferentials of b^a are nonempty.

Proposition 2.2 *Let f be a lsc CAR function and $x_0 \in \operatorname{int}\operatorname{dom} f$. Let $v \in \partial f_{x_0}(1)$ and $a = f(x_0) - v$. Then $a \leq f(0)$ and $\partial_{\mathcal{L}_s} b^a(x_0) \subset \partial_{\mathcal{L}_s} f(x^0)$.*

Proof: This proposition is of interest only if the set $\partial_{\mathcal{L}_s} b^a(x_0)$ is nonempty. We assume that this is the case. First we show that the number a is well-defined, that is $\partial f_{x_0}(1) \neq \emptyset$. Indeed this follows from $x_0 \in \operatorname{int} \operatorname{dom} f$. Let $v \in \partial f_{x_0}(1)$ and $a = f(x_0) - v = f_{x_0}(1) - v$. For all $\alpha \geq 0$ we have $f_{x_0}(\alpha) - f_{x_0}(1) \geq v\alpha - v1$. In particular if $\alpha = 0$ we get $f_{x_0}(0) - f_{x_0}(1) \geq -v$ that can be rewritten as $a \leq f(0)$. Since $v \in \partial f_{x_0}(1)$ we have $f_{x_0}(\alpha) - v\alpha \geq f_{x_0}(1) - v = a$ for all $\alpha > 0$, therefore $\dfrac{f_{x_0}(\alpha) - a}{\alpha} \geq v = f_{x_0}(1) - a$. It follows from this that

$$b^a(x_0) = f_{x_0}(1) - a = v. \tag{2.7}$$

Since $a \leq f(0)$ we conclude that $g^a(x) = a + b^a(x)$ is a lower affine approximation of f. Let $\ell \in \partial_{\mathcal{L}_s} b^a(x_0)$. Since g^a is a minorant of f and $b^a(x_0) = f(x_0) - a$ we have

$$f(x) - f(x_0) \geq g^a(x) - f(x_0) = b^a(x) - (f(x_0) - a) = b^a(x) - b^a(x_0) \geq \ell(x) - \ell(x_0).$$

Thus the result follows. □

Proposition 2.3 *Let f be a lsc CAR function and $x_0 \in \operatorname{dom} f$. Assume that the $\partial_{\mathcal{L}_s} f(x_0)$ is not empty. Then for each $l \in \partial_{\mathcal{L}_s} f(x_0)$ there exists $a \leq f(0)$ such that $l \in \partial_{\mathcal{L}_s} b^a(x_0)$.*

Proof: Let $\ell \in \partial_{\mathcal{L}_s} f(x_0)$ and $a = f(x_0) - \ell(x_0)$. Then for each $x \in \mathbb{R}^n$ and $\alpha \geq 0$ we have

$$\alpha\ell(x) = \ell(\alpha x) \leq \ell(x_0) + f_x(\alpha) - f(x_0) = f_x(\alpha) - a. \tag{2.8}$$

Setting $\alpha = 0$ we get from (2.8) that $a \leq f(0)$. It follows from (2.8) that

$$b^a(x) = \inf_{\alpha > 0} \frac{f_x(\alpha) - a}{\alpha} \geq \ell(x)$$

and $b^a(x_0) = \ell(x_0)$. This means that $\ell \in \partial_{\mathcal{L}_s} b^a(x_0)$. □

It follows from Propositions 2.2 and 2.3 that calculation of the $\mathcal{L}_s$-subdifferential for CAR functions can be reduced to the calculation of the $\mathcal{L}_s$-subdifferential for PH functions. We also can study the non-emptiness of $\mathcal{L}_s$-subdifferential for CAR functions using results known for PH functions.

We need the following definition (see [41,49], Definition 5.23):

Definition 2.3 A lsc CAR function f is called totally lsc if for all x with $\|x\| = 1$ we have

$$\lim_{\alpha \to +\infty} \frac{f(\alpha x)}{\alpha} \leq \liminf_{x' \to x,\, \alpha \to +\infty} \frac{f(\alpha x')}{\alpha}. \tag{2.9}$$

It is clear that always

$$\lim_{\alpha\to+\infty}\frac{f(\alpha x)}{\alpha}\geq \liminf_{x'\to x,\,\alpha\to+\infty}\frac{f(\alpha x')}{\alpha}$$

so the inequality in (2.9) can be replaced by the equality. The following assertion has been proved in [41] (Lemma 5.8) for $a < f(0)$. It is easy to check that the proof holds for $a = f(0)$ as well.

Lemma 2.1 *Let f be a totally lsc CAR function. Then the lower affine approximation g^a is lsc for each $a \leq f(0)$.*

We also need the following definition:

Definition 2.4 (see [41], p. 220) A function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is called locally Lipschitz on the ray $R_{x_0} = \{\alpha x_0 : \alpha > 0\}$ if there exists a number $\delta > 0$ such that for each $r > 0$ there exists a number L_r, satisfying

$$|f(\alpha x) - f(\alpha x_0)| \leq L_r\alpha\|x - x_0\|, \tag{2.10}$$

whenever $\|x - x_0\| < \delta$ and $0 < \alpha < r$.

Lemma 2.2 *Let f be a totally lsc CAR function with $0 \in \operatorname{dom} f$. Let $x_0 \in dom f$, $x_0 \neq 0$. Suppose that f is locally Lipschitz on the ray R_{x_0} and*

$$\lim_{x\to x_0,\,\alpha\to+\infty}\frac{f(\alpha x)}{\alpha} = +\infty \tag{2.11}$$

Let $a = f(x_0) - v$ where $v \in \partial f_{x_0}(1)$. Then the function b^a is locally Lipschitz at x_0.

Proof: It was shown in the proof of Proposition 2.2 (see (2.7)) that $b^a(x_0) = f(x_0) - a$. Let $\gamma > b^a(x_0)$. In view of (2.10) we conclude that f is continuous at x_0 hence $\gamma > f(x) - a$ for x close to x_0. Therefore $b^a(x) \leq f(x) - a < \gamma$ for such x. It follows from (2.11) that there exist numbers $\delta > 0$ and $r > 0$ such that

$$\frac{f(\alpha x) - a}{\alpha} > \gamma, \qquad \text{if } \|x - x_0\| < \delta,\ \alpha > r.$$

Since $b^a(x) < \gamma$ for x close to x_0 we conclude that

$$\frac{f(\alpha x) - a}{\alpha} > \gamma > b^a(x), \qquad \text{if } \|x - x_0\| < \delta,\ \alpha > r.$$

In view of the definition of b^a we have for x with $\|x - x_0\| < \delta$:

$$b^a(x) = \inf_{0<\alpha<r}\frac{f(\alpha x) - a}{\alpha}. \tag{2.12}$$

Assume that δ so small that (2.10) is valid. Then there exists a number L_r such that

$$\left|\frac{f(\alpha x)-a}{\alpha}-\frac{f(\alpha x_0)-a}{\alpha}\right|=\left|\frac{f(\alpha x)-f(\alpha x_0)}{\alpha}\right|\leq L_r\|x-x_0\|$$

whenever $\|x-x_0\|<\delta$ and $0<\alpha<r$. Due to (2.12) for each $\eta>0$ there is a number $\alpha'<r$ such that $\frac{f(\alpha' x)-a}{\alpha'}<b^a(x)+\eta$. Then

$$b^a(x_0)\leq\frac{f(\alpha' x_0)-a}{\alpha'}\leq b^a(x)+\eta+L_r\|x-x_0\|.$$

Since $\eta>0$ is arbitrary we get $b^a(x_0)\leq b^a(x)+L_r\|x-x_0\|$. The similar argument shows that $b^a(x)\leq b^a(x_0)+L_r\|x-x_0\|$. Hence

$$|b^a(x)-b^a(x_0)|\leq L_r\|x-x_0\|,\qquad \text{if}\quad \|x-x_0\|<\delta. \tag{2.13}$$

□

We now use Proposition 2.2 and Lemma 2.2 for the examination of the non-emptiness of $\mathcal{L}_s$-subdifferential for totally lsc CAR functions.

Proposition 2.4 *Let conditions of Lemma 2.2 hold. Then $\partial_{\mathcal{L}_{n+1}}f(x_0)\neq\emptyset$. If $f(0)=\min_{x\in\mathbb{R}^n}f(x)$ then $\partial_{\mathcal{L}_n}f(x_0)\neq\emptyset$.*

Proof: Let $a=f(x_0)-v$ where $v\in\partial f_{x_0}(1)$. It follows from Lemma 2.1 that lower affine approximation g^a is lsc. Since $b^a(x)=g^a(x)-a$ we conclude that b^a is also lsc. Theorem 2.1 and Lemma 2.2 imply that $\partial_{\mathcal{L}_{n+1}}b^a(x_0)\neq\emptyset$. Applying Proposition 2.2 we conclude that $\partial_{\mathcal{L}_{n+1}}f(x_0)\neq\emptyset$.

Assume now that $f(0)\leq f(x)$ for all $x\in\mathbb{R}^n$. Then (see (2.6)) b^a is a nonnegative PH function. It follows from Lemma 2.2 that b^a is locally Lipschitz at x_0. In view of Theorem 2.3 we conclude that $\partial_{\mathcal{L}_n}b^a(x_0)\neq\emptyset$. Proposition 2.2 demonstrates that $\partial_{\mathcal{L}_n}f(x_0)\neq\emptyset$. □

We now show that condition (2.11) can be weakened.

Theorem 2.6 *Let f be a totally lsc CAR function and $x_0\in$ dom f. Suppose that f is locally Lipschitz on the ray R_{x_0}, where $x_0\neq 0$. If there exist a neighborhood U of x_0 and numbers $\lambda>1$, $\varepsilon>0$ such that*

$$f(x)-f'(\lambda x,x)+\varepsilon\leq f(x_0)-f'(x_0,x_0)\qquad \forall x\in U, \tag{2.14}$$

then the subdifferential $\partial_{\mathcal{L}_{n+1}}f(x_0)$ is nonempty.

If, in addition, $f(0)=\min_{x\in\mathbb{R}^n}f(x)$ then $\partial_{\mathcal{L}_n}f(x_0)$ is nonempty.

Proof: We construct an auxiliary functions for which conditions of Proposition 2.4 hold and then we use this proposition. For the sake of definiteness we consider only $\mathcal{L}_{n+1}$-subdifferentials. The proof for $\mathcal{L}_n$-subdifferentials is similar.

Let U be an open neighborhood of the point x_0 such that (2.14) holds. Let k be a large enough number. Consider the function $f_* : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ defined by:

$$f_*(x) = \begin{cases} f(x) + \|x\|^2 - k & x \in \operatorname{Cone} U, \|x\|^2 > k \\ f(x) & \text{otherwise.} \end{cases}$$

It is easy to check that f_* is a totally lsc function therefore conditions of Proposition 2.4 hold for f_*. In view of this Proposition, $\partial_{\mathcal{L}_{n+1}} f_*(x_0)$ is not empty. Let $\ell_* \in \partial_{\mathcal{L}_{n+1}} f_*(x_0)$. Then

$$\ell_*(x) - \ell_*(x_0) \le f_*(x) - f_*(x_0) \qquad \forall x \in \mathbb{R}^n,$$

Let

$$h_\lambda(x) = f(x) - f(\lambda x) + (\lambda - 1) f'(\lambda x, x).$$

Since f is a CAR function then

$$f'(\lambda x, x) = \lim_{t \to 0+} \frac{f((\lambda + t)x) - f(\lambda x)}{t} = \inf_{t > 0} \frac{f((\lambda + t)x) - f(\lambda x)}{t} \le f((\lambda+1)x) - f(\lambda x),$$

hence

$$h_\lambda(x) \le (f(x) - \lambda f(\lambda x) + (\lambda - 1) f((\lambda + 1)x)). \tag{2.15}$$

Since f is locally Lipschitz on the ray R_{x_0} and $x_0 \ne 0$ then there exists a neighborhood U of x_0 such that $\sup_{x \in U}(f(x) - \lambda f(\lambda x) + (\lambda - 1) f((\lambda + 1)x)) < +\infty$. It follows from this and (2.15) that

$$\sup_{x \in U} h_\lambda(x) < +\infty.$$

Let

$$K = \max\left\{\lambda, \ \frac{1}{\varepsilon} \sup_{x \in U} h_\lambda(x) + 1\right\} \sup_{x \in U} \|x\|, \tag{2.16}$$

where λ and ε as in (2.14). Assume that $k > K^2$ and consider a function $\ell \in \partial_{\mathcal{L}_{n+1}} f_*(x_0)$. We show that $\ell \in \partial_{\mathcal{L}_{n+1}} f(x_0)$, that is $\ell(x) - \ell(x_0) \le f(x) - f(x_0)$ for all $x \in \mathbb{R}^n$. We need to prove this inequality only for $x \in \operatorname{Cone} U$ such that $\|x\|^2 > k$. Let $x = \alpha y$, where $y \in U$, $\alpha > 0$. Then

$$\alpha = \frac{\|x\|}{\|y\|} > \frac{K}{\|y\|} \ge \frac{K}{\sup_{z \in U} \|z\|},$$

so due to (2.16) the following inequalities hold:

$$\alpha > \lambda, \qquad \varepsilon(\alpha - 1) > h_\lambda(z) \qquad \forall z \in U. \tag{2.17}$$

Since $y \in U$ then $\|y\|^2 < \lambda^2 \left(\sup_{z\in U} \|z\|\right)^2 \le K^2 < k$ and $f_*(y) - f_*(x_0) = f(y) - f(x_0)$, therefore

$$\begin{aligned} \ell(x) - \ell(x_0) + f(x_0) &= \ell(\alpha y) - \ell(x_0) + f(x_0) \\ &= \alpha(\ell(y) - \ell(x_0)) + (\alpha - 1)\ell(x_0) + f(x_0) \\ &\le \alpha(f(y) - f(x_0)) + (\alpha - 1)\ell(x_0) + f(x_0). \end{aligned}$$

Let $t > 0$ be a number such that $(1 + t)x_0 \in U$, then

$$t\ell(x_0) = \ell((1 + t)x_0)) - \ell(x_0) \le f((1 + t)x_0) - f(x_0).$$

Hence

$$\ell(x_0) \le \lim_{t\to 0+} \frac{f(x_0 + tx_0) - f(x_0)}{t} = f'(x_0, x_0).$$

We have

$$\begin{aligned} \ell(x) - \ell(x_0) + f(x_0) &\le \alpha(f(y) - f(x_0)) + (\alpha - 1)f'(x_0, x_0) + f(x_0) \\ &= \alpha f(y) - (\alpha - 1)(f(x_0) - f'(x_0, x_0)). \end{aligned}$$

Inequality (2.14) implies

$$\begin{aligned} \ell(x) - \ell(x_0) + f(x_0) &\le \alpha f(y) - (\alpha - 1)(f(y) - f'(\lambda y, y) + \varepsilon) \\ &= f(y) + (\alpha - 1)f'(\lambda y, y) - \varepsilon(\alpha - 1). \end{aligned}$$

Using (2.17) we get

$$\begin{aligned} \ell(x) - \ell(x_0) + f(x_0) &< f(y) + (\alpha - 1)f'(\lambda y, y) - (f(y) - f(\lambda y) + (\lambda - 1)f'(\lambda y, y)) \\ &= f(\lambda y) + (\alpha - \lambda)f'(\lambda y, y). \end{aligned}$$

We need to check that $f(\lambda y) + (\alpha - \lambda)f'(\lambda y, y) \le f(\alpha y) = f(x)$. Since $\alpha > \lambda$ and f is CAR function then

$$f'(\lambda y, y) = \inf_{t>0} \frac{f((\lambda + t)y) - f(\lambda y)}{t} \le \frac{f(\alpha y) - f(\lambda y)}{\alpha - \lambda} \implies$$

$$f(\lambda y) + (\alpha - \lambda)f'(\lambda y, y) \le f(\alpha y).$$

Thus, we have a function ℓ which belongs to $\partial_{\mathcal{L}_{n+1}} f(x_0)$. □

Corollary 2.1 *Let f be a totally lsc CAR function and let $x_0 \in dom\, f$. Suppose that f is locally Lipschitz on the ray R_{x_0}, $x_0 \neq 0$. If there exist a neighborhood U of x_0 and numbers $\lambda > 1$, $\varepsilon > 0$ such that*

$$f'(\lambda x, x) \geq f'(x_0, x_0) + \varepsilon \qquad \forall x \in U, \tag{2.18}$$

then the subdifferential $\partial_{\mathcal{L}_{n+1}} f(x_0)$ is not empty.

Proof: Since f is locally Lipschitz on the ray R_{x_0} then there exists a neighborhood $U(x_0)$ of x_0 such that $f(x) - f(x_0) \leq \varepsilon/2$, hence

$$f(x) - f'(\lambda x, x) + \frac{\varepsilon}{2} \leq f(x_0) - f'(x_0, x_0) \qquad \forall x \in U(x_0) \cap U,$$

where U is from (2.18). Thus we conclude that condition (2.14) holds. □

We now show that the class of functions for which (2.14) holds is broad enough. Let $x_0 \neq 0$ and let F_{x_0} be the class of totaly lsc CAR functions with the properties:

1) f is locally Lipschitz on the ray R_{x_0} and

2) for each $\lambda > 1$ there exists a neighborhood of x_0 such that the function $x \mapsto f'(\lambda x, x)$ is lsc in this neighborhood.

Let $f \in F_{x_0}$. Then the function f_{x_0} is convex, hence its right derivative $f'_{x_0}(\alpha)$ is an increasing function. There are two possibilities:

1) $f'_{x_0}(\lambda)$ is constant for $\lambda \geq 1$, hence f_{x_0} is affine on $[1, +\infty)$ (this means that $f(\lambda x_0) = f'_{x_0}(1)\lambda + f(x_0) - f'_{x_0}(1)$ for $\lambda \geq 1$).

2) there exists $\lambda > 1$ such that $f'_{x_0}(\lambda) > f'_{x_0}(1)$, in other words

$$f'(\lambda x_0, x_0) > f'(x_0, x_0). \tag{2.19}$$

Assume that (2.19) holds. Since $f \in F_{x_0}$ then there exists a neighborhood U of x_0 and $\varepsilon > 0$ such that (2.18) holds.

2.4 Geometric interpretations

We now present geometric interpretations of the $\mathcal{L}_n$-subdifferential for nonnegative PH functions and $\mathcal{L}_{n+1}$-subdifferential for nonpositive PH functions. Using these interpretations we can easily calculate $\mathcal{L}_s$-subgradients (members of $\mathcal{L}_s$-subdifferentials) in some cases. We also show that $\mathcal{L}_s$-subdifferentials $\partial_{\mathcal{L}_s} f(x)$ are very large sets whenever they are nonempty.

Some special classes of $\mathcal{L}_n$-subdifferentials for nonnegative PH functions have been described in Section 2.2 in terms of support collections. First we present a geometric interpretation of a support collection ℓ.

For each collection $\ell = \{l_1, \dots, l_n\}$ of n linearly independent vectors consider the cone $T^\ell = \{y : [l_i, y] \geq 0, i = 1, \dots, n\}$. Then ℓ is the support collection if and only if $U \cap (x + T^\ell) = \{x\}$ and $[l_i, x] = 1$ for all $i = 1, \dots, n$.

It can be proved (see, for example, Proposition 5.32 in [41]) that for each convex cone Q with $\operatorname{int} Q \neq \emptyset$ and for each $x \in \operatorname{int} Q$ there exists a collection ℓ' of n linearly independent vectors $\ell' = (l'_1, \dots, l'_n)$ such that $\operatorname{int} Q \supset T^{\ell'}$ and $[l'_i, x] = 1, i = 1, \dots, n$. Putting $Q = T^\ell$ we obtain the following result from here: for each support collection ℓ there exists a support collection ℓ' such that $T^{\ell'} \subset \operatorname{int} T^\ell$. Consider the set $L(x, U)$ of all support collections to a set U at the point x with the order relation $\geq$. We say that $\ell \geq \ell'$ if $T^\ell \supset T^{\ell'}$.

Let $p : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a nonnegative lsc PH function and $U = \{x \in \mathbb{R}^n : p(x) \leq 1\}$. Then $p = \mu_U$. Let $p(x_0) = 1$ and there exists a support collection $\ell = (l_1, \dots, l_n)$ to U at x_0. It follows from Proposition 2.1 that $\ell \in \partial_{\mathcal{L}_n} p(x_0)$. If $\ell' < \ell$ then ℓ' is also a support collection, hence $\ell' \in \partial_{\mathcal{L}_n} p(x_0)$. Thus the subdifferential $\partial_{\mathcal{L}_n} p(x_0)$ contains a very broad set of all collections $\ell' \leq \ell$. Let $p(x_0) := \lambda > 0$ and $x'_0 = x_0/\lambda$. Then $p(x'_0) = 1$ and $\partial_{\mathcal{L}_n} p(x_0) = \lambda \partial_{\mathcal{L}_n} p(x'_0)$, so we can use the described construction for the examination of $\partial_{\mathcal{L}_n} p(x_0)$. This construction can be used for the examination only of $\partial_{\mathcal{L}_n}$-subgradients $\ell = (l_1, \dots, l_n)$ with the additional properties $[l_1, x] = [l_2, x] = \dots = [l_n, x] > 0$.

The results obtained can be extended for the description of $\mathcal{L}_n$-subgradients for some not necessarily nonnegative PH functions, namely functions p for which there exists a vector l such that $p(x) \geq [l, x]$. Indeed in such a case the function $p'(x) = p(x) - [l, x]$ is nonnegative. It is clear that $\ell = (l_1, \dots, l_n)$ is a $\partial_{\mathcal{L}_n}$-subgradient of p if and only if $(l_1 - l, \dots, l_n - l)$ is a $\partial_{\mathcal{L}_n}$-subgradient of p'.

Now we give a geometric interpretation of $\mathcal{L}_{n+1}$-subdifferentials for nonpositive PH functions by using their support sets with respect to $\mathcal{L}_{n+1}$.

Let q be a nonpositive PH function and $\ell = (l_1, \dots, l_{n+1})$. We have: $\ell \in \operatorname{supp}(q, \mathcal{L}_{n+1})$ if and only if $\min_{i=1,\dots,n+1}[l_i, x] \leq q(x)$ for all $x \in \mathbb{R}^n$. Then $\ell \in \partial_{\mathcal{L}_{n+1}} q(x_0)$ if and only if $\ell \in \operatorname{supp}(q, \mathcal{L}_{n+1})$ and $\min_{i=1,\dots,n+1}[l_i, x_0] = q(x_0)$. Let $p(x) = -q(-x)$. Then p is a nonnegative PH function. Clearly, $\ell \in \operatorname{supp}(q, \mathcal{L}_{n+1})$ if and only if

$$\max_{i=1,\dots,n+1}[l_i, x] \geq p(x), \quad x \in \mathbb{R}^n \tag{2.20}$$

and $\ell \in \partial_{\mathcal{L}_{n+1}} q(x_0)$ if and only if (2.20) holds and $\max_{i=1,\dots,n+1}[l_i, x_0] = p(x_0)$.

We establish the following result:

Proposition 2.5 *Let q be a nonpositive lsc PH function, $p(x) = -q(-x)$ and*

$$U = \{x : p(x) \leq 1\} = \{x : q(-x) \geq -1\}.$$

Then $\operatorname{int} U \neq \emptyset$ *and* $\ell = (l_1, \dots, l_{n+1}) \in \operatorname{supp}(q, \mathcal{L}_{n+1})$ *if and only if*

(i) $0 \in \operatorname{co}\{l_1, \dots, l_{n+1}\}$;

(ii) the set $S_\ell = \{x \in \mathbb{R}^n : [l_i, x] \leq 1, \ i = 1, \dots, n+1\}$ *is contained in* U.

Let $p(x_0) = 1$, *that is* $q(-x_0) = -1$ *and* $\ell \in \operatorname{supp}(q, \mathcal{L}_{n+1})$. *Then* $\ell \in \partial_{\mathcal{L}_{n+1}} q(-x_0)$ *if and only if*

(iii) $x_0 \in S_\ell$.

Proof: Let $\ell = (l_1, \dots, l_{n+1})$ be a collection of vectors. Note that $0 \in \operatorname{int} S_\ell$ where S_ℓ is defined in (ii). We show that (i) is equivalent to

$$\max_{i=1,\dots,n+1}[l_i, x] \geq 0, \qquad x \in \mathbb{R}^n \tag{2.21}$$

Indeed let $p_\ell(x) = \max_{i=1,\dots,n+1}[l_i, x]$. Then p_ℓ is a sublinear function and $\partial p_\ell(0) = \operatorname{co}\{l_1, \dots, l_{n+1}\}$. It is well known that a sublinear function is nonnegative if and only if zero belongs to its subdifferential at zero. This leads to the equivalency of (i) and (2.21).

The set S_ℓ is a radiant and $S_\ell = \cap_i S_i$ where $S_i = \{x : [l_i, x] \leq 1\}$. It is easy to check that $\mu_{S_i}(x) = \max(0, [l_i, x])$. Since the Minkowski gauge of the intersection is equal to the maximum of Minkowski gauges, we get using (2.21):

$$\mu_{S_\ell}(x) = \max_i \max([l_i, x], 0) = \max\left(\max_i [l_i, x], 0\right) = \max_i [l_i, x]. \tag{2.22}$$

The inclusion $S_\ell \subset U$ is equivalent to

$$\max_i [l_i, x] \geq p(x), \qquad x \in \mathbb{R}^n \tag{2.23}$$

Since the support set $\operatorname{supp}(q, \mathcal{L}_{n+1})$ of a lsc PH function q is not empty and

$$\min_{i=1,\dots,n+1}[l_i, x] \leq q(x) \quad \text{for all} \quad x \iff \max_{i=1,\dots,n+1}[l_i, x] \geq p(x) \quad \text{for all} \quad x,$$

we conclude that there exists ℓ such that $S_\ell \subset U$. This implies $\operatorname{int} U \neq \emptyset$.

Assume that $\ell \in \operatorname{supp}(q, \mathcal{L}_{n+1})$. Then (2.23) is valid. Since p is nonnegative we get from (2.23) that (2.21), which is equivalent to (i), is valid. It was also mentioned that (2.23) implies (ii). Assume now that both (i) and (ii) holds. In view of (i) we get (2.22) that together with (2.23) implies $\ell \in \operatorname{supp}(q, \mathcal{L}_{n+1})$.

Let both (i) and (ii) hold and x_0 be an element such that $q(-x_0) = -1$ (in other words, $p(x_0) = 1$). Then $\max_i[l_i, x_0] \geq p(x_0) = 1$. The inclusion $x_0 \in S_\ell$ is equivalent to $\mu_{S_\ell}(x_0) = \max_i[l_i, x_0] \leq 1$, so $\max_i[l_i, x_0] = 1$. This is equivalent to $\min_i[l_i, -x_0] = -1 = q(-x_0)$. Thus $\ell \in \partial_{\mathcal{L}_{n+1}} q(-x_0)$ if and only if $x_0 \in S_\ell$. □

Remark 2.1 1) Let $q(-x_0) < 0$ that is $p(x_0) > 0$. Using element $x_0/p(x_0)$ we can present a geometric interpretation of $\partial_{\mathcal{L}_{n+1}} q(-x_0)$ in this case.

2) Let q be a PH function such that $q(x) \leq [l, x]$ for a vector l and all $x \in \mathbb{R}^n$. Then the function $q'(x) = q(x) - [l, x]$ is nonpositive and $\operatorname{supp}(q', \mathcal{L}_{n+1})$ consists of all collections $\ell' = (l_1 - l, \ldots, l_{n+1} - l)$ where $\ell = (l_1, \ldots, l_n) \in \operatorname{supp}(q, \mathcal{L}_{n+1})$, so we can give a geometric interpretation of $\mathcal{L}_{n+1}$-support sets and $\mathcal{L}_{n+1}$-subdifferentials in this case.

3) Let $q : \mathbb{R}^n \to \mathbb{R}$ be a superlinear function. Then there exists l such that $[l, x] \geq q(x)$ for all x, so the results obtained can be used for a geometric interpretation of $\mathcal{L}_{n+1}$-subdifferentials of q.

Chapter 3

Star-shaped separability with applications

We study the weak separability of star-shaped sets by finite collections of linear functions. One of the main goals of this chapter is to indicate some areas of research, where the star-shaped separability can be used. In particular, we examine a "best approximation - like" problem for star-shaped sets: we introduce a star-shaped distance and consider the minimization of this distance over a star-shaped set. This is a non-convex optimization problem.

3.1 Support collections and weak separability

Separability of two convex sets is one of the fundamental facts of convex analysis that can be considered as a geometrical form of Hanh-Banach theorem. Some attempts to extend the notion of separability to star-shaped sets were undertaken in [56] and [41]. (Recall that a set is star-shaped if it can be represented as the union of a family of convex sets $(U_t)_{t\in T}$, such that $\bigcap_{t\in T} U_t \neq \emptyset$.) The support collection of linear functions at a regular boundary point x of a star-shaped set $U \subset \mathbb{R}^n$ was defined there and the existence of this collection was proved. A separability of two star-shaped sets by means of m linearly independent linear functions (the so-called weak separability) was also defined and studied. Here we introduce also the notion of a conical support collection and discuss some properties of conical collections and weak separability. We also examine some applications of these notions (Sections 3.2 - 3.4).

One of the most challenging questions that arise in modern optimization is the development of a theory of global minimization for some broad classes of non-convex optimization problems. The theory of local optimization is based on calculus and its sophisticated generalizations. Different tools should be used in global optimization. Since separability by a liner function has found applications in convex programming, it is natural to apply separability by a collection of linear functions in star-shaped optimization.

From a certain point of view, classical best approximation problems are the simplest convex nonlinear problems. Similarly, star-shaped best approximation problems are the simplest star-shaped optimization problems, so we start with best approximation. We show that characterization of best approximation can be done in terms of weak separability of star-shaped sets. A challenging problem is to describe separation collections of linear functions at least in simple cases. This is the topic of the further research.

So, let $U \subset \mathbb{R}^n$ be a set and $x \in U$. Recall the Bouligand tangent cone $\Gamma(x, U)$ consists of all vectors z such that for each $\alpha_0 > 0$ and $\varepsilon > 0$ there exist v and $\alpha > 0$ such that $\|v - z\| < \varepsilon$, $\alpha < \alpha_0$ and $x + \alpha v \in U$.

Let $\mathcal{U}_n$ be a totality of all radiant sets $U \subset \mathbb{R}^n$ that are nontrivial in the sense that $U \neq \{0\}$. (Definitions of radiant and star-shaped sets can be found in Section 2.2.) For $U \in \mathcal{U}_n$ consider the set

$$\Delta(U) = \{x \in U : \mu_U(x) = 1 \quad \text{and} \quad x \notin \Gamma(x, U)\}, \tag{3.1}$$

where the Minkowski gauge μ_U is defined by (2.3). It is easy to see that $\mu_U(x) = 1$ if and only if $x \in \operatorname{bd} U$ and $\lambda x \notin U$ for all $\lambda > 1$. (Here and below $\operatorname{bd} U$ stands for the boundary of a set U.) Thus, the inclusion $\Delta(u) \subset \operatorname{bd} U$ holds. A point $x \in U$ is called a regular boundary point of U if $x \in \Delta(U)$.

Remark 3.1 It is known ([41]) that $0 \in \operatorname{int}\ker U$ if and only if μ_U is Lipschitz (for the definition of kernel see (2.2)). In such a case (see [41], Propositions 5.15 and 5.17) $\operatorname{bd} U = \{x \in \mathbb{R}^n : \mu_U(x) = 1\}$ and (see [41], Corollary 5.6) $x \notin \Gamma(x, U)$ for all $x \in \operatorname{bd} U$. Hence $\Delta(U) = \operatorname{bd}(U)$.

The notion of a support collection was defined in Chapter 2 (see Definition 2.1). The following result holds.

Theorem 3.1 *(see [41], Theorem 5.7). Let $U \in \mathcal{U}_n$ be a closed set and let $x \in \Delta(U)$. Then there exists a support collection $\ell = (l_1, \ldots, l_n)$ to the set U at the point x.*

Sometimes it is convenient to consider a weaker object than a support collection. First we recall the following definition.

Definition 3.1 Vectors $l_1, \dots, l_m \in \mathbb{R}^n$ are said to be conically independent if conditions

$$\alpha_1 l_1 + \dots + \alpha_m l_m = 0, \qquad \alpha_i \geq 0 \quad \text{for all} \quad i = 1, \dots, m$$

imply that $\alpha_1 = \dots = \alpha_m = 0$.

Conical independence of the collection $\ell = (l_1, \dots, l_m)$ means that $-l_i$ does not belong to the cone spanned by $(l_k)_{k \neq i}$ for all i.

Definition 3.2 Let $U \subset \mathbb{R}^n$ be a closed set and $x \in U, x \neq 0$. A collection of vectors $\ell = (l_1, \dots, l_m)$ is called a conical support collection to U at x if $[l_i, x] = 1\,(i = 1, \dots, m)$ and

$$\min_{i=1,\dots,m} [l_i, u] < 1 \quad \text{for all} \quad u \in U, u \neq x. \tag{3.2}$$

A conical support collection $\ell = (l_1, \dots, l_m)$ at x consists of conically independent vectors. Indeed, let $\sum_{i=1}^m \alpha_i l_i = 0$ where $\alpha_i \geq 0$ for all $i = 1, \dots, m$. Then $\sum_{i=1}^m \alpha_i [l_i, x] = \sum_{i=1} \alpha_i = 0$, hence $\alpha_i = 0, \; i = 1, \dots, m$.

It is clear that each support collection is a conical support collection. It follows from Definition 2.1, that a support collection cannot contain more than n vectors, on the other hand a conical collection can contain an arbitrary finite number of vectors.

Let $\ell = (l_1, \dots, l_m)$ be a conical support collection and $T^\ell = \{y : [l_i, y] \geq 0, \; i = 1, \dots, m\}$ be a cone generated by this collection. Then $\operatorname{int} T^\ell$ is nonempty and contains the cone $\{y : [l_i, y] > 0 \; i = 1, \dots, m\}$. It is known (see, for example, Proposition 5.32 and Remark 5.12 in [41]) that for each convex cone Q with $\operatorname{int} Q \neq \emptyset$ there exists a collection $\ell' = (l'_1, \dots, l'_n)$ of n linearly independent vectors such that $[l'_i, x] = 1$ for all $i = 1, \dots, n$ and $T^{\ell'} \subset \operatorname{int} Q$. It follows from this that existence of a conical support collection to U at x implies existence of a support collection to U at x. However, the number of vectors in these collections can be different.

We now discuss some properties of conical support collections.

Proposition 3.1 *Let U be a closed radiant set, and $x_0 \in \operatorname{bd} U$. Let $\ell = (l_1, \dots, l_m)$ be a conical support collection at x_0 and*

$$U_i = \{x \in U : [l_i, x] \leq 1\} = U \cap H_i, \quad \textit{where} \quad H_i = \{x \in \mathbb{R}^n : [l_i, x] \leq 1\}. \tag{3.3}$$

Then $\bigcup_{i=1,\dots,m} U_i = U$ *and*

1) for each $i = 1,\dots,m$ *the set* U_i *is a nonempty radiant set and* $\operatorname{kern} U_i \supset \operatorname{kern} U \cap H_i$*;*
The Minkowski gauge μ_{U_i} *of* U_i *has the form*

$$\mu_{U_i}(x) = \max(\mu_U(x), [l_i, x]); \tag{3.4}$$

2) For each $x \in U$ *there exists* i *such that* $R_x \cap U \subset U_i$ *(here* $R_x = \{\lambda x : \lambda \geq 0\}$*);*
3) Let

$$V = \bigcap_{i=1,\dots,m} U_i = U \cap \left(\bigcap_{i=1,\dots,m} H_i \right). \tag{3.5}$$

Then $V \in \mathcal{U}_n$ *and*

$$\mu_V(x) = \max_i \max(\mu_U(x), [l_i, x]) \tag{3.6}$$

Proof: 1) Let $y \in \operatorname{kern} U$ and $[l_i, y] \leq 1$. Let $u \in U_i$, that is, $u \in U$ and $[l_i, u] \leq 1$. For each $\lambda \in [0, 1]$ we have $\lambda y + (1 - \lambda)u \in U$ and $[l_i, \lambda y + (1 - \lambda)u] \leq 1$. This means that $y \in \operatorname{kern} U_i$. We showed that $\operatorname{kern} U_i \supset \operatorname{kern} U \cap H_i$. It follows from this that $0 \in \operatorname{kern} U_i$, hence U_i is a nonempty radiant set. It is well-known (see, for example [41]) that the Minkowski gauge of the intersection of a finite number of sets is equal to the maximum of the Minkowski gauges of these sets. On the other hand, $\mu_{H_i}(x) = \max(0, [l_i, x])$, where μ_{H_i} is the Minkowski gauge of the half-space H_i. This implies (3.4) and also (3.6).
2) Let $x \in U$ and $\lambda_x = \sup\{\lambda \geq 0 : \lambda x \in U\}$. If $\lambda_x < +\infty$ then $\lambda_x x \in U$. Let i be the index such that $\lambda_x x \in U_i$. Then $R_x \cap [0, \lambda_x]x = R_x \cap U \subset U_i$. Assume now that $\lambda_x = +\infty$. Then $R_x \cap U = R_x$. For each $i = 1, \dots, m$ consider the set $\Lambda_i = \{\lambda \geq 0 : \lambda x \in U_i\}$. There exists at least one i such that Λ_i is unbounded. It easy to check that $R_x = \{\lambda x : \lambda \in \Lambda_i\}$, hence $R_x \subset U$.
3) V is radiant as the intersection of radiant sets. Since $[l_i, x_0] = 1$ for all i, it follows that $x_0 \in V$, therefore $V \neq \{0\}$. Hence $V \in \mathcal{U}_n$. □

We need the following definition (see [56] and also ([41], Definition 5.17).

Definition 3.3 Let U and V be subsets of $\mathbb{R}^n$ and $\ell = (l_i)_{i=1,\dots,m}$ be a collection of linearly independent vectors. The sets U, V are said to be weakly separated by vectors $(l_i)_{i=1,\dots,m}$ if for each pair $u \in U, v \in V$ there exists $i \in I$ such that $[l_i, u] \leq [l_i, v]$. We say that U, V are conically weakly separated if there exists a collection ℓ of conically independent vectors with the indicated property.

Proposition 3.2 *Let U and V be conically weakly separated by vectors $(l_i)_{i=1}^m$ and $\operatorname{int} U \neq \emptyset$. Then $\operatorname{int} U \cap V = \emptyset$.*

Proof: First we show that $U - V \neq \mathbb{R}^n$. Consider the superlinear function $q(x) = \min_{i=1,\dots,m}[l_i, x]$. Weak separability of the sets U and V means that $q(u - v) \leq 0$ for all $u \in U, v \in V$. Let $\bar{\partial} q(0) = \{l \in \mathbb{R}^n : [l, x] \geq q(x) \text{ for all } x \in \mathbb{R}^n\}$ be the superdifferential of q at zero. Then $\bar{\partial} q(0)$ coincides with convex hull $S = \{l = \sum_{i\in 1}^m \alpha_i l_i : \alpha_i \geq 0 \ (i = 1, \dots, m), \sum_{i=1}^m \alpha_i = 1\}$ of vectors $(l_i)_{i=1}^m$. Since these vectors are conically independent we conclude that $0 \notin S$. Then there exists $x \in \mathbb{R}^n$ such that $0 < \inf_{l\in S}[l, x] = q(x)$. This means that $x \notin U - V$.

Assume that there exists $z \in (\operatorname{int} U) \cap V$. Let $B_\varepsilon(z) \subset U$ be a neighborhood of z. Then $B_\varepsilon(z) - z \subset U - V$ is a neighborhood of zero. Since q is positive homogeneous and $q(x) \leq 0$ for $x \in U - V$ it follows that $q(x) \leq 0$ for all $x \in \mathbb{R}^n$, which is a contradiction. □

Remark 3.2 Let $l_1, \dots, l_m$ be a collection of vectors in $\mathbb{R}^n$ such that $0 \in \operatorname{co}(l_1, \dots, l_m)$. Then $q(x) = \min_{i=1,\dots,m}[l_i, x] \leq 0$ for all $x \in \mathbb{R}^n$. This implies the following assertion: let $U, V \subset \mathbb{R}^n$ be two arbitrary sets. Then for each $u \in U, v \in V$ there exists $i \in \{1, \dots, m\}$ such that $[l_i, u] \leq [l_i, v]$. A collection $\ell = (l_i)_{i=1}^m$ does not depend on sets U, V. Note that there exist $n+1$ vectors $(l_i)_{i=1}^{n+1}$ such that $0 \in \operatorname{co}(l_1, \dots, l_{n+1})$.

For weak separability we consider collections $(l_i)_{1=1}^m$ of no more than n vectors and these vectors are linearly independent. It can be shown (see Proposition 3.3 below) that this collection can be chosen as a support collection to a certain set Z at a certain point $\bar{z}$. This means that this collection enjoys an additional property: $[l_i, \bar{z}] > 0$, $i = 1, \dots, m$ and also that strict inequalities can be used instead of nonstrict ones.

A conical collection $(l_i)_{i=1}^m$ can contain more then n vectors. However, for such a collection we have $0 \notin \operatorname{co}(l_1, \dots, l_m)$.

Under some additional assumptions it can be proved that if $(\operatorname{int} U) \cap V = \emptyset$ then U and V can be weakly separated (see [41, 56]).

Theorem 3.2 *(see Theorem 5.8 in [41]). Let U and V be star-shaped sets such that $\operatorname{int} \operatorname{kern} U \neq \emptyset$ and $(\operatorname{int} U) \cap V = \emptyset$. Then U and V are weakly separated.*

Assume that $U \cap V \neq \emptyset$. The proof of this theorem is based on the following construction, which is a modification of the construction from [56]. Let $u \in \operatorname{int} \operatorname{kern} U$, $v \in \operatorname{kern} V$.

Consider the point $z = v - u$ and the set $Z = (U - u) - (V - v) = U - V + z$. Then Z is a radiant set and $0 \in \operatorname{int} \operatorname{kern} Z$. It can be shown that either $z \notin \operatorname{cl} Z$ or $z \in \operatorname{bd} \operatorname{cl} Z$. Since $U \cap V \neq \emptyset$ it follows that $0 \in U - V$, hence $z \in Z$. This implies that z is a boundary point of $\operatorname{cl} Z$. Since $z \in \operatorname{int} \operatorname{kern} Z$ it follows that $z \in \Delta(Z)$. Then there exists a support collection $\ell = (l_1, \dots, l_n)$ to $\operatorname{cl} Z$ at the point z. It is easy to check that ℓ weakly separates U and V. It follows from the aforesaid that the following statement holds.

Proposition 3.3 *Let U and V be star-shaped sets such that* $\operatorname{int} \operatorname{kern} U \neq \emptyset$*, the set $U \cap V$ is nonempty and the set* $(\operatorname{int} U) \cap V$ *is empty. Let $\bar{z} \in \operatorname{kern} V - \operatorname{int} \operatorname{kern} U$ and $Z = U - V + \bar{z}$. Then there exists a support collection $\ell = (l_1, \dots, l_m)$, $(m \leq n)$ to Z at $\bar{z}$ and this collection weakly separates the sets U and V. In other words, the following holds:*
1) $[l_1, \bar{z}] = \dots = [l_m, \bar{z}] = 1$;
2) for each $u \in U, v \in V$ with $u \neq v$ there exists i such that $[l_i, u] < [l_i, v]$.

We only comment the assertion 2). If $u \in U$, $v \in V$ and $u \neq v$, then $u - v + \bar{z} \neq \bar{z}$, hence there exists i such that $[l_i, u - v + \bar{z}] < 1 = [l_i, \bar{z}]$.

3.2 Star-shaped distance and its minimization

The following well-known corollary of Hahn-Banach theorem is a classical result of the approximation theory. Let U be a convex subset of a normed space X and $x \notin U$ and let $\bar{u} \in U$ be the best approximation of x by elements of U, that is, $r := \min\{\|u - x\| : u \in U\} = \|\bar{u} - x\|$. Then there exists a linear function l such that $l(u) \leq l(\bar{u}) \leq l(v)$ for all $u \in U$ and $v \in B(x, r) = \{y : \|x - y\| \leq r\}$. We can present this result in the following form. An element $\bar{u} \in U$ is the best approximation of x if and only if there exists a linear function l such that

$$0 = (-l, l)(\bar{u}, \bar{u}) = \min\{(-l, l)(u, v) : (u, v) \in U \times B(x, r)\}. \qquad (3.7)$$

Here, by definition, $(-l, l)(u, v) = -l(u) + l(v)$.

If U is strictly convex then in addition to (3.7) the following holds:

$$((u, v) \in U \times B(x, r),\ (u, v) \neq (\bar{u}, \bar{u})) \implies (-l, l)(u, v) > 0. \qquad (3.8)$$

We now give a version of (3.7)- (3.8) for star-shaped sets in $\mathbb{R}^n$. We assume that $\mathbb{R}^n$ is equipped with the topology of pointwise convergence. Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$.

Let $U \subset \mathbb{R}^n$ be a star-shaped set and $x \notin U$, let $r = \min\{\|u - x\| : u \in U\}$. Then the intersection $U \cap \{v : \|x - v\| < r\}$ is empty, so the sets U and $\{v : \|x - v\| \le r\}$ can be weakly separated. We do not need to have exactly a norm in order to prove this result, so we consider a more general situation. First consider a function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ = [0, +\infty)$ such that

$$\rho(x, y + \alpha(x - y)) \le \rho(x, y), \qquad x, y \in \mathbb{R}^n, \alpha \in [0, 1]. \tag{3.9}$$

It is easy to check that the function ρ enjoys this property if and only if the "balls" $B(x, r) = \{y : \rho(x, y) \le r\}$ are star-shaped with $x \in \operatorname{kern} B(x, r)$ for all $r > 0$. We need to have star-shaped balls $B(x, r)$ such that

(1) $x \in \operatorname{int} \operatorname{kern} B(x, r)$ for all $r > 0$.

(2) the inequality $\rho(x, y) < r$ holds for interior points of $B(x, r)$.

The following definition takes into account these requirements:

Definition 3.4 A function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is called a star-shaped distance if

(i) $\rho(x, x) = 0$ for all $x \in \mathbb{R}^n$ and $\rho(x, y) > 0$ for all $x, y \in \mathbb{R}^n$ $x \ne y$.

(ii) for each $x \in \mathbb{R}^n$ and $r > 0$ there exists a neighborhood V of x such that

$$\rho(x, y + \alpha(x' - y)) \le \alpha r + (1 - \alpha)\rho(x, y), \qquad y \in \mathbb{R}^n, x' \in V, \alpha \in [0, 1]. \tag{3.10}$$

(iii) for each $x \in \mathbb{R}^n$ the function $\rho_x(y)$ defined by

$$\rho_x(y) = \rho(x, y), \qquad y \in \mathbb{R}^n \tag{3.11}$$

has no local maxima.

(iv) for each $x \in \mathbb{R}^n$ the function ρ_x defined by (3.11) is continuous.

Note that star-shaped distance is not required to be a distance function in usual sense.

Let $x, y \in \mathbb{R}^n$. Then (3.10) with $r = \rho(x, y)$ and $x' = x$ implies (3.9), so sets $B(x, r)$ are star-shaped for all $x \in \mathbb{R}^n$ and $r > 0$. On the other hand (3.10) implies $x \in \operatorname{int} \operatorname{kern} B(x, r)$. Indeed, let $r > 0$, V is a neighborhood of x that is considered in (3.10) and let $\rho(x, y) \le r$. Then

$$\rho(x, \alpha x' + (1 - \alpha)y) \le \alpha r + (1 - \alpha)\rho(x, y) \le \alpha r + (1 - \alpha) r = r,$$

so $V \subset \operatorname{kern} B(x, r)$. This means that $x \in \operatorname{int} \operatorname{kern} B(x, r)$.

Proposition 3.4 *Let ρ be a star-shaped distance. Then for each $x \in \mathbb{R}^n$ and $r > 0$ we have* $\operatorname{int} B(x, r) = \{v \in \mathbb{R}^n : \rho(x, v) < r\}$.

Proof: Since ρ_x is continuous, the set $\{v : \rho(x,v) < r\}$ is open. This implies that $\{v : \rho(x,v) < r\} \subset \operatorname{int} B(x,r)$. Let $\rho(x,v) = r$ and $v \in \operatorname{int} B(x,r)$. Then there exists a neighborhood V of zero such that $v+V \subset \operatorname{int} B(x,r)$. For all $v' \in V$ we have $\rho(x, v'+v) \leq r = \rho(x,v)$. This means that v is a local maximum of ρ_x which contradicts (iii). □

We now give an example of a star-shaped distance.

Proposition 3.5 *Let $(f_t)_{t\in T}$ be a uniformly continuous family of convex functions $f_t : \mathbb{R}^n \to \mathbb{R}_+$ such that $f_t(0) = 0$ and $\inf_{t\in T} f_t(x) > 0$ for $x \neq 0$. Then the function $\rho(x,y) = \inf_{t\in T} f_t(x-y)$ is a star-shaped distance in $\mathbb{R}^n$.*

Proof: We need to check that (i) – (iv) hold.

(i) It follows from properties of the family $(f_t)_{t\in T}$ that $\rho(x,x) = 0$ and $\rho(x,y) > 0$ for $x \neq y$.

(ii) Let us check (3.10). Let $x \in \mathbb{R}^n$ and $r > 0$. Since $(f_t)_{t\in T}$ is uniformly continuous at zero it follows that there exists a neighborhood V_0 of zero such that $f_t(v) \leq r$ for all $v \in V_0$ and $t \in T$. Let $V = x - V_0$ be a neighborhood of x and $x' \in V$. Then

$$\begin{aligned}
\rho(x, y + \alpha(x'-y)) &= \inf_{t\in T} f_t(x - y - \alpha(x'-y)) \\
&= \inf_{t\in T} f_t((1-\alpha)(x-y) + \alpha(x-x')) \\
&\leq \inf_{t\in T}((1-\alpha) f_t(x-y) + \alpha f_t(x-x')) \\
&\leq (\inf_{t\in T}(1-\alpha) f_t(x-y)) + \alpha r = (1-\alpha)\rho(x,y) + \alpha r.
\end{aligned}$$

Thus (3.10) is valid.

(iii) We need to check that for each $x, y \in \mathbb{R}^n$ and small $\varepsilon' > 0$ there exists a direction u such that

$$\rho(x, y + \varepsilon' u) - \rho(x,y) > 0. \tag{3.12}$$

Let $z = x - y$. Then $\rho(x,y) = \inf_{t\in T} f_t(z)$. If $z = 0$ then (3.12) trivially holds, so we assume that $z \neq 0$. Consider a number $\varepsilon = \dfrac{\varepsilon'}{1+\varepsilon'}$ then $1 + \varepsilon' = 1/(1-\varepsilon)$. Let $u = \dfrac{z}{1-\varepsilon}$. Then for each $t \in T$ we have

$$\begin{aligned}
f_t(z) &= f_t((1-\varepsilon)u) = f_t((1-\varepsilon)u + \varepsilon 0) \\
&\leq (1-\varepsilon) f_t(u) \leq f_t(u) - \varepsilon \inf_{\tau\in T} f_\tau(u) = f_t\left(\frac{z}{1-\varepsilon}\right) - \varepsilon\chi,
\end{aligned}$$

where $\chi = \inf_{\tau\in T} f_\tau(u) > 0$. This implies the following:

$$\rho(x,y) = \inf_{t\in T} f_t(z) \leq \inf_{t\in T} f_t((1+\varepsilon')z) - \varepsilon\chi < \rho(x, y - \varepsilon' z).$$

Thus (3.12) holds.

(iv) Since the family $(f_t)_{t\in T}$ is uniformly continuous it follows that $\rho_x(y) = \inf_{t\in T} f_t(x-y)$ is continuous. □

Example 3.1 Consider a family $(\|\cdot\|_t)_{t\in T}$ of norms such that there exist numbers $0 < c < C < +\infty$ such that

$$c\|x\|_* \leq \|x\|_t \leq C\|x\|_*, \tag{3.13}$$

for all $t \in T$, where $\|\cdot\|_*$ is a fixed norm. The right-hand side inequality in (3.13) shows that the family $(\|\cdot\|_t)_{t\in T}$ is uniformly continuous, the left-hand side inequality shows that $\inf_{t\in T}\|x\|_t > 0$ for all $x \neq 0$. Hence the function $d(x,y) = \inf_{t\in T}\|x-y\|_t$ is a star-shaped distance.

Theorem 3.3 *Let ρ be a star-shaped distance on $\mathbb{R}^n$ and $U \subset \mathbb{R}^n$ be a radiant set. Let $x \notin U$, $\bar{u} \in U$ and $r = \rho(x,\bar{u})$. Then*

1) If $r = \min_{u\in U}\rho(x,u)$ then there exist m linearly independent vectors $l_1,\dots,l_m$ such that:

(i) $[l_1,x] = \dots = [l_m,x] = 1$.

(ii) for each $u \in U$ and $v \in B(x,r)$ with $u \neq v$ there exists an index i such that $[l_i,u] < [l_i,v]$.

2) If there exist m conically independent vectors l_i such that the condition (ii') below holds then $r := \rho(x,\bar{u}) = \min_{u\in U}\rho(x,u)$. Here

(ii') $U \times B(x,r) = \bigcup_{i=1}^m (U \times B(x,r))_i$ where

$$(U \times B(x,r))_i = \{(u,v) \in U \times B(x,r) : [l_i,u] \leq [l_i,v]\}.$$

(Condition (ii') means that for every pair (u,v) with $u \in U$ and $v \in B(x,r)$ there exists an i such that $[l_i,u] \leq [l_i,v]$.)

Proof: 1) Let $r := \rho(x,\bar{u}) = \min_{u\in U}\rho(x,u)$. It follows from the properties of the star-shaped distance that the set $B(x,r)$ is star-shaped and $x \in \operatorname{int}\operatorname{kern} B(x,r)$. The intersection $U \cap B(x,r)$ contains $\bar{u}$, hence nonempty. The intersection $U \cap \operatorname{int} B(x,r) = \emptyset$. Indeed, in view of Proposition 3.4 we have $\operatorname{int} B(x,r) = \{v : \rho(x,v) < r\}$. On the other hand $U \subset \{u : \rho(x,u) \geq r\}$.

Consider the set $Z = U - B(x,r)$. Since $0 \in \operatorname{kern} U$ and $x \in \operatorname{int}\operatorname{kern} B(x,r)$ it follows that $\bar{z} := -x \in \operatorname{kern} U - \operatorname{int}\operatorname{kern} B(x,r)$. Then (see Proposition 3.3) there exists

m linearly independent vectors $l'_1, \ldots, l'_m$ such that $[l'_1, -x] = \ldots = [l'_m, -x] = 1$ and for each $u \in U$, $v \in B(x,r)$ with $u \neq v$ there exists i such that $[l'_i, v] < [l'_i, u]$. Thus (i) and (ii) hold for $l_i = -l'_i$.

2) Let (ii') hold. Let $q(x) = \min_{i=1,\ldots,m}[l_i, x]$. Then (ii') is equivalent to

$$q(u-v) \leq 0 \quad \text{for all} \quad u \in U, \ v \in B(x,r). \tag{3.14}$$

We show that

$$U \cap \operatorname{int} B(x,r) = \emptyset. \tag{3.15}$$

Indeed, assume that there exists $u \in U$ and a neighborhood V of zero such that $u - V \subset B(x,r)$. In view of (3.14) we get $q(v) \leq 0$ for all $v \in V$. It follows from positive homogeneity of q that $q(x) \leq 0$ for all $x \in \mathbb{R}^n$, hence $0 \in \bar{\partial} q = \operatorname{co}\{l_1, \ldots, l_m\}$. This contradicts conical independence of vectors $l_1, \ldots, l_m$.

Combining (3.15) and Proposition 3.4 we get $U \subset \{u \in \mathbb{R}^n : \rho(x,u) \geq r\}$. Since $\bar{u} \in U$ and $r = \rho(x, \bar{u})$ it follows that $r = \min_{u \in U} \rho(x, \bar{u})$. □

Theorem 3.3 can be considered as a version of (3.7)- (3.8) for $X = \mathbb{R}^n$. If U is a convex set and $\rho(x,y) = \|x-y\|$ and we replace strict inequalities in (ii) with nonstrict ones, then (3.7) follows from Theorem 3.3 with $m = 1$. We cannot take $m = 1$ for convex sets if we use strict inequalities. However Theorem 3.3 holds with $m = 1$ for a strictly convex set U.

3.3 Star-shapedness and distance to a closed set

In this section we demonstrate that star-shapedness can be used in the study of arbitrary (not necessarily star-shaped) sets.

First we consider an arbitrary closed subset U of $\mathbb{R}^n$ with $0 \in U$. Let $\|\cdot\|$ be an arbitrary norm in $\mathbb{R}^n$ and

$$d_U(x) = \inf\{\|x-u\| : u \in U\}, \qquad x \in \mathbb{R}^n$$

be the distance function generated by this norm. Let β_U be the function defined on X by

$$\beta_U(x) = \|x\| - d_U(x). \tag{3.16}$$

Note that $\beta_U(x) = \|x\|$ for $x \in U$; if $x \notin U$ then $\beta(x) < \|x\|$. The sets $\{x \in \mathbb{R}^n : \beta(x) \leq c\} = \{x \in \mathbb{R}^n : d_U(x) \geq \|x\| - c\}$, $c > 0$ can be useful for examination of the distance function. We study these sets from the point of view of star-shapedness.

We need some preliminaries.

A function $f : X \to \mathbb{R}_+$ is called increasing-along-rays (IAR) if for each $x \neq 0$ the function of one variable $f_x(t) = f(tx)$ is increasing (that is $f_x(t_1) \geq f_x(t_2)$ for $t_1 \geq t_2$ on $[0, +\infty)$). (The definition of the IAR function in a more general situation was introduced in [8].) Note that $f(0) = \min_{x \in X} f(x)$ for an IAR function f. It has been proved in [62], (see also [8]) that a function f is IAR if and only if its level sets $S_r(f) := \{x \in X : f(x) \leq r\}$ are radiant for all $r \geq f(0)$.

Let

$$(d_U)^{\uparrow}_H(x, x) = \limsup_{\alpha \to +0, v \to x} \frac{d_U(x + \alpha v) - d_U(x)}{\alpha}$$

be the Hadamard directional derivative of d_U at a point x in the direction x. It is easy to check that $(d_U)^{\uparrow}_H(x, x) \leq \|x\|$. Indeed, since the distance d_U is Lipschitz continuous with the Lipschitz constant $L = 1$ it follows that $(d_U)^{\uparrow}_H(x, x) \leq \limsup_{v \to x} \|v\| = \|x\|$.

Theorem 3.4 *Let $x_0 \in \mathbb{R}^n \setminus \{0\}$ be a point such that $\|x_0\| > (d_U)^{\uparrow}_H(x_0, x_0)$ and let $V = \{x \in \mathbb{R}^n : \|x\| - d_U(x) \leq \|x_0\| - d_U(x_0)\}$. Then there exist m linearly independent vectors $l_1, \ldots, l_m$ such that*

1) $[l_1, x_0] = \ldots = [l_m, x_0] = 1$;

2) for each $x \in V$ there exists an i such that $[l_i, x] \leq 1$.

The sets $V_i = \{x \in V : [l_i, x] \leq 1\}$ are star-shaped for all i.

Proof: First we show that the function β_U defined by (3.16) is increasing-along-rays. Let $x \in X$ and $\lambda > \mu \geq 0$. Then

$$d_U(\lambda x) = \inf_{u \in U} \|\lambda x - u\| = \inf_{u \in U} \|(\lambda - \mu)x + \mu x - u\| \leq (\lambda - \mu)\|x\| + d_U(\mu x),$$

hence

$$\beta_U(\lambda x) = \lambda\|x\| - d_U(\lambda x) \geq \mu\|x\| - d_U(\mu x) = \beta_U(\mu x),$$

so β_U is IAR. It follows from this that level sets $S_r(\beta_U) = \{x : \beta_U(x) \leq r\}$ of β_U are radiant for all $r \geq \beta(0) = 0$.

Let $x_0 \in \mathbb{R}^n \setminus \{0\}$ be a given point and let $r = \beta(x_0) = \|x_0\| - d_U(x_0) \geq 0$. Then

$$V := S_r(\beta_U) = \{x \in \mathbb{R}^n : \|x\| - d_U(x) \leq \|x_0\| - d_U(x_0)\} \tag{3.17}$$

is a radiant set. Since $x_0 \neq 0$ it follows that $V \in \mathcal{U}_n$. We need to show that x_0 is a regularly boundary point, that is, $x_0 \in \Delta(V)$, where $\Delta(V) = \{x \in V : \mu_V(x) = 1,\, x \notin \Gamma(x, V)\}$

is the set defined for $V = S_r(\beta_U)$ by (3.1). Let us calculate $\mu_V(x_0)$. The inclusion $x_0 \in V$ implies $\mu_V(x_0) \leq 1$. Let us check that $\mu_V(x_0) \geq 1$ and hence $\mu_V(x_0) = 1$.

Since V is a radiant set and $0 \neq x_0 \in V$ then it is sufficiently to check that $\lambda x_0 \notin V$ for $\lambda > 1$. If $\lambda x_0 \in V$ for some $\lambda > 1$ then $(x_0 + \alpha x_0) \in V$ for all $\alpha \in (0, \lambda - 1)$, that is

$$\|x_0 + \alpha x_0\| - d_U(x_0 + \alpha x_0) \leq \|x_0\| - d_U(x_0) \quad \text{for all} \quad \alpha \in (0, \lambda - 1).$$

This fact implies $\|x_0\| \leq (d_U)^{\uparrow}_H(x_0, x_0)$, which is impossible. Hence $\mu_V(x_0) = 1$.

Assume that $x_0 \in \Gamma(x_0, V)$. Then there exist sequences $v_k \to x_0$ and $\alpha_k \to 0$ such that $x_0 + \alpha_k v_k \in V$, that is

$$\|x_0 + \alpha_k v_k\| - \|x_0\| \leq d_U(x_0 + \alpha_k v_k) - d_U(x_0). \tag{3.18}$$

Let $p(x) = \|x\|$. Since $p'(x, x) = \|x\|$ it follows from (3.18) that $\|x_0\| \leq (d_U)^{\uparrow}_H(x_0, x_0)$, which is impossible. Hence $x_0 \notin \Gamma(x_0, V)$.

Applying Theorem 3.1 and Proposition 3.1 we conclude that the desired result holds. □

We now consider bounded subsets of $\mathbb{R}^n$.

Proposition 3.6 *Let U be a bounded subset of $\mathbb{R}^n$. Then the set* hyp d_U *is star-shaped. (Here* hyp $d_U = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : \lambda \leq d_U(x)\}$.*)*

Proof: Let $c > 0$ be a number such that $\|u\| \leq c$ for all $u \in U$. Show that $(0, -c) \in$ kern hyp d_U. Let $t \leq d_U(x)$ and $\alpha \in (0, 1)$. Then

$$\begin{aligned}
-\alpha c + (1 - \alpha)t &\leq -\alpha c + (1 - \alpha)d_U(x) \\
&= \inf_{u \in U}(\|(1 - \alpha)x - (1 - \alpha)u\| - \alpha c) \\
&= \inf_{u \in U}(\|(1 - \alpha)x - u + \alpha u\| - \alpha c) \\
&\leq \inf_{u \in U}(\|(1 - \alpha)x - u\| + \alpha\|u\| - \alpha c) \\
&\leq \inf_{u \in U}\|(1 - \alpha)x - u\| = d_U((1 - \alpha)x).
\end{aligned}$$

Hence

$$\alpha(0, -c) + (1 - \alpha)(x, t) = ((1 - \alpha)x, -\alpha c + (1 - \alpha)t) \in \text{hyp}\, d_U,$$

that means $(0, -c) \in$ kern hyp d_U. □

Corollary 3.1 *Let U be a bounded subset of $\mathbb{R}^n$ and $0 \in U$. Then the sets* epi $\|\cdot\|$, hyp d_U *are weakly separated. (Here* epi $\|\cdot\| = \{(x, \lambda) : \lambda \geq \|x\|\}$.*)*

Proof: Since $0 \in U$ then $d_U(x) \leq \|x\|$, that is $\operatorname{int}\operatorname{epi}\|\cdot\| \cap \operatorname{hyp} d_U = \emptyset$. It follows from convexity of norm that $\operatorname{int}\operatorname{kern}\operatorname{epi}\|\cdot\| = \operatorname{int}\operatorname{epi}\|\cdot\| \neq \emptyset$. Thus we can apply Theorem 3.2.
□

3.4 Degree of strict non-convexity

Consider a radiant set U, which is non strictly convex. It is interesting to classify its boundary points in terms of their strict non-convexity. Conical support collections can be used for such a classification.

Definition 3.5 A positive integer m is called the degree of strict non-convexity of a set $U \in \mathcal{U}_n$ near a point $x \in \Delta(U)$ if there exists a conical support collection ℓ that consists of m conically independent vectors and there is no support collection of $m-1$ conically independent vectors. We denote the degree of strict non-convexity by $\operatorname{nsc}(x, U)$.

A point $x \in \Delta(U)$ will be called a point of strict convexity of U if $\operatorname{nsc}(x, U) = 1$. We now present some simple illustrative examples.

Example 3.2 1) Let $U \subset \mathbb{R}^2$ be a polyhedron with $0 \in \operatorname{int} U$. Then $\operatorname{nsc}(x, U) = 1$ for each vertex x of U and $\operatorname{nsc}(x, U) = 2$ for a point $x \in U$, which is not a vertex.

2) Let $U = U_1 \cup U_2$, where U_1 and U_2 are circles:

$$U_1 = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 1)^2 + x_2^2 \leq 1\}, \quad U_2 = \{(x_1, x_2) : (x_1 + 1)^2 + x_2^2 \leq 1\}.$$

Then U is a radiant set. Consider all boundary points of U. Let $x = (0, 0)$. Then $\mu_U(x) = 0$, so $x \notin \Delta(U)$ and the degree of strict non-convexity is not defined at this point. Let $x = (x_1, x_2)$ be a boundary point of U with either $x_1 < -1$ or $x_1 > 1$. Then $\operatorname{nsc}(x, U) = 1$, so such points are points of strict convexity. Let $x = (x_1, x_2)$ be a boundary point with either $x_1 \in [-1, 0)$ or $x_1 \in (0, 1]$. Then $\operatorname{nsc}(x, U) = 2$.

3) Let $U = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1 x_2| \leq 1\}$. Then U is radiant and $\operatorname{nsc}(x, U) = 2$ for each boundary point x of U.

The second example shows that degree of strict non-convexity of U at x is a global notion: it is possible that $\operatorname{nsc}(x, U) > 1$ and the intersection of a set U with a small neighborhood of x is strictly convex (this means that U is locally strictly convex at x).

We now present a simple assertion about the degree of strict non-convexity.

Proposition 3.7 *Let $U_1, \dots, U_k$ be strictly convex subsets of* $\mathbb{R}^n$ *such that* $0 \in \operatorname{int} U_i$ *for all* i. *Let* $U = \bigcup_i U_i$ *and* $x \in \operatorname{bd} U$. *Then* $\operatorname{nsc}(x, U) \leq k$.

Proof: It easy to check that kern $U \supset \cap_i \operatorname{int} U_i$, therefore $0 \in \operatorname{int} kerU$. In view of Remark 3.1 we conclude that each boundary point of U belongs to $\Delta(U)$, therefore $x \in \Delta(U)$. Since $x \in \operatorname{bd} U$ then $x \notin \operatorname{int} U_i$ for all i. It is well known from convex analysis that there exist vectors $l_1, \dots, l_k \in \mathbb{R}^n \backslash \{0\}$ such that $[l_i, u] < [l_i, x]$ for all i and $u \in U_i$ $(u \neq x)$. Since $l_i \neq 0$ and $0 \in \operatorname{int} U_i$ it follows that $[l_i, x] > 0$ for all i. Denote $l'_i = l_i/[l_i, x]$. Then $[l'_i, x] = 1$ and $[l'_i, u] < 1$ for all i and $u \in U_i$ $(u \neq x)$. This implies $\min_i [l'_i, u] < 1$ for all $u \in U$ $(u \neq x)$. □

Chapter 4

Subdifferential calculus for abstract convex functions

Our main goal in this chapter is to show that the subdifferential calculus is not a privilege of convex analysis only. We indicate some conditions, which guarantee the existence of certain calculus rules in abstract convex case. We are concentrating mainly on the maximum of a finite collection of functions. Subdifferential calculus is important for applications of abstract convex analysis, so it is interesting to find conditions that provide the exact formula for the subdifferential of the maximum. We show that such a formula can be given in terms of abstract convex hull with respect to a certain subset of elementary functions (see Corollary 4.1).

4.1 Subdifferential of the maximum of two abstract convex functions

Let L be a set of functions $l : X \to \mathbb{R}$ defined on a set X. Let H_L be the set of all functions $h(x) = l(x) - c$, where $l \in L$ and $c \in \mathbb{R}$. Consider a function $f : X \to \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ and assume that $y \in \operatorname{dom} f = \{x \in X : \ f(x) < +\infty\}$. In this chapter we work with the set $\mathcal{D}_L f(y)$ defined by (1.14). It is clear that a function $h \in H_L$ belongs to $\mathcal{D}_L f(y)$ if and only if

$$h(y) = 0 \quad \text{and} \quad h(x) \leq f(x) - f(y) \quad \forall x \in X.$$

Remark 4.1 For the sake of convenience we assume that for $f(y) = +\infty$ the sets $\partial_L f(y)$

and $\mathcal{D}_L f(y)$ are defined as empty sets. So if we write "the set $\mathcal{D}_L f(y)$ is nonempty" then we mean, in particular, that $y \in \operatorname{dom} f$.

We use $H_{L,y}$ to denote the collection of all $h \in H_L$ such that $h(y) = 0$, that is $H_{L,y} = \{l - l(y) : \ l \in L\}$. The symbol f_y also denotes the function $f_y(x) = f(x) - f(y)$ (here $y \in \operatorname{dom} f$).

If $T \subset H_{L,y}$ then the $H_{L,y}$-convex hull of T is defined as follows

$$\mathrm{co}_{H_{L,y}} T = \left\{ h \in H_{L,y} : \ h(x) \le \sup_{t \in T} t(x) \ \ \forall\, x \in X \right\}. \tag{4.1}$$

It is more convenient to formulate all statements in terms of the set $\mathcal{D}_L f(y)$. First we present a general inclusion, which does not require additional assumptions.

Proposition 4.1 *Let f_1, f_2 be H_L-convex functions and $f_1(y) = f_2(y)$. Then*

$$\mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)) \subset \mathcal{D}_L(\max\{f_1, f_2\})(y). \tag{4.2}$$

Proof: If $(l - l(y)) \in \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$ then

$$\begin{aligned} l(x) - l(y) &\le \sup_{h \in \mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)} h(x) \\ &= \max\left\{ \sup_{t \in \partial_L f_1(y)} (t(x) - t(y)), \ \sup_{t \in \partial_L f_2(y)} (t(x) - t(y)) \right\} \\ &\le \max\{f_1(x) - f_1(y), f_2(x) - f_2(y)\} \\ &= \max\{f_1(x), f_2(x)\} - \max\{f_1(y), f_2(y)\}. \end{aligned}$$

So $(l - l(y)) \in \mathcal{D}_L(\max\{f_1, f_2\})(y)$. □

For some special types of H_L-convex functions f_1, f_2 we can get equality instead of the inclusion in (4.2).

Proposition 4.2 *Let f_1, f_2 be functions defined on X such that the functions f_{1y}, f_{2y} are $H_{L,y}$-convex and $f_1(y) = f_2(y)$. Then*

$$\mathcal{D}_L(\max\{f_1, f_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)). \tag{4.3}$$

Proof: If is clear that $\mathcal{D}_L f(y) = \mathrm{supp}\,(f_y, H_{L,y})$ for any function f. Since the functions f_{1y} and f_{2y} are $H_{L,y}$-convex then

$$\mathrm{supp}\,(\max\{f_{1y}, f_{2y}\}, H_{L,y}) = \mathrm{co}_{H_{L,y}}(\mathrm{supp}\,(f_{1y}, H_{L,y}) \cup \mathrm{supp}\,(f_{2y}, H_{L,y})).$$

Note also that $(\max\{f_1, f_2\})_y = \max\{f_{1y}, f_{2y}\}$ because $f_1(y) = f_2(y)$. Hence

$$\begin{aligned}\mathcal{D}_L(\max\{f_1, f_2\})(y) &= \text{supp}\,((\max\{f_1, f_2\})_y, H_{L,y}) = \text{supp}\,(\max\{f_{1y}, f_{2y}\}, H_{L,y}) \\ &= \text{co}_{H_{L,y}}(\text{supp}\,(f_{1y}, H_{L,y}) \cup \text{supp}\,(f_{2y}, H_{L,y})) \\ &= \text{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)).\end{aligned}$$

□

The following example demonstrates that the equality (4.3) does not necessarily hold for arbitrary H_L-convex functions f_1, f_2 with $f_1(y) = f_2(y)$.

Example 4.1 Let $X = \mathbb{R}$ and L consists of all linear functions and the function $l(x) = x^2$. Consider the functions f_1, f_2:

$$f_1(x) = \begin{cases} x^2, & x \leq 0, \\ 0, & x \geq 0. \end{cases} \qquad f_2(x) = \begin{cases} 0, & x \leq 0, \\ x^2, & x \geq 0. \end{cases}$$

Note that f_1 and f_2 are H_L-convex and $f_1(0) = f_2(0)$. At the same time, both f_{1y} and f_{2y} are not $H_{L,y}$-convex for $y = 0$. It is clear that $\mathcal{D}_L f_1(0) = \mathcal{D}_L f_2(0) = \{0\}$, hence $\text{co}_{H_{L,0}}(\mathcal{D}_L f_1(0) \cup \mathcal{D}_L f_2(0)) = \{0\}$. But the function $f(x) = \max\{f_1(x), f_2(x)\}$ coincides with elementary function $l(x) = x^2$, therefore $l \in \mathcal{D}_L f(0)$. This means that $\mathcal{D}_L(\max\{f_1, f_2\})(0) \neq \text{co}_{H_{L,0}}(\mathcal{D}_L f_1(0) \cup \mathcal{D}_L f_2(0))$.

Further, consider a multifunction $A : X \times 2^{H_L} \times 2^{H_L} \to 2^{H_L}$, where 2^{H_L} is the set of all nonempty subsets of H_L.

Proposition 4.3 *Let $y \in X$. Assume that the inclusion*

$$A(y, \mathcal{D}_L g_1(y), \mathcal{D}_L g_2(y)) \subset \mathcal{D}_L(\max\{g_1, g_2\})(y) \tag{4.4}$$

holds for all H_L-convex functions g_1, g_2 such that the sets $\mathcal{D}_L g_1(y), \mathcal{D}_L g_2(y)$ are nonempty and $g_1(y) = g_2(y)$. Let f_1, f_2 be H_L-convex functions such that the sets $\mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)$ are nonempty and $f_1(y) = f_2(y)$. If

$$A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) = \mathcal{D}_L(\max\{f_1, f_2\})(y)$$

then

$$\mathcal{D}_L(\max\{f_1, f_2\})(y) = \text{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)). \tag{4.5}$$

Proof: Let f_1, f_2 be H_L-convex functions such that the sets $\mathcal{D}_L f_1(y)$ and $\mathcal{D}_L f_2(y)$ are nonempty, $f_1(y) = f_2(y)$ and $A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) = \mathcal{D}_L(\max\{f_1, f_2\})(y)$. Consider the functions

$$\begin{aligned} g_i(x) &= \sup\{h(x) + f_i(y) : \ h \in \mathcal{D}_L f_i(y)\} \\ &= \sup\{h(x) : \ h(y) = f_i(y), \ h \in \text{supp}\,(f_i, H_L)\}. \end{aligned}$$

It is clear that $g_1(y) = f_1(y) = f_2(y) = g_2(y)$ and g_{1y}, g_{2y} are $H_{L,y}$-convex. Proposition 4.2 implies the equality $\mathcal{D}_L(\max\{g_1, g_2\})(y) = \text{co}_{H_{L,y}}(\mathcal{D}_L g_1(y) \cup \mathcal{D}_L g_2(y))$. Since

$$\begin{aligned} \mathcal{D}_L g_i(y) &= \{h \in H_{L,y} : \ h \le g_i - g_i(y)\} \\ &= \left\{ h \in H_{L,y} : \ h(x) \le \sup_{h' \in \mathcal{D}_L f_i(y)} h'(x) \ \ \forall x \in X \right\} = \mathcal{D}_L f_i(y) \end{aligned}$$

then $A(y, \mathcal{D}_L g_1(y), \mathcal{D}_L g_2(y)) = A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y))$. Hence

$$\begin{aligned} \mathcal{D}_L(\max\{f_1, f_2\})(y) &= A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) = A(y, \mathcal{D}_L g_1(y), \mathcal{D}_L g_2(y)) \\ &\subset \mathcal{D}_L(\max\{g_1, g_2\})(y) = \text{co}_{H_{L,y}}(\mathcal{D}_L g_1(y) \cup \mathcal{D}_L g_2(y)) \quad (4.6) \\ &= \text{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)). \end{aligned}$$

Combining the above inclusion with Proposition 4.1 yields the equality

$$\mathcal{D}_L(\max\{f_1, f_2\})(y) = \text{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)).$$

□

Proposition 4.4 *Let $y \in X$. Assume that*

$$\text{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)) \subset A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) \subset \mathcal{D}_L(\max\{f_1, f_2\})(y)$$

for all H_L-convex functions f_1, f_2 such that the sets $\mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)$ are nonempty and $f_1(y) = f_2(y)$. Then for all such functions f_1, f_2

$$A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) = \text{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)).$$

Proof: Using the same functions g_i as in the proof of Proposition 4.3 we conclude that $A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) \subset \text{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$ (see (4.6)). However, by our assumptions, $\text{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)) \subset A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y))$. So we obtain the desired equality. □

Example 4.1 and Proposition 4.3 show that, in general, the set $\mathcal{D}_L(\max\{f_1, f_2\})(y)$ cannot be described in terms of the sets $\mathcal{D}_L f_1(y)$ and $\mathcal{D}_L f_2(y)$.

At the same time the equality $\mathcal{D}_L(\max\{f_1, f_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$ is valid for broad classes of H_L-convex functions. However the mapping $\mathrm{co}_{H_{L,y}}$ can be very complicated.

Proposition 4.5 *Let $\mathcal{L}$ be a set of functions defined on a set X. Let L consist of all functions $l(x) = \max\{l_1(x), l_2(x) + c\}$, where $l_1, l_2 \in \mathcal{L}$ and $c \in \mathbb{R}$. Then*

$$\mathcal{D}_L(\max\{f_1, f_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$$

for all H_L-convex functions f_1, f_2 and all points $y \in X$ such that the sets $\mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)$ are nonempty and $f_1(y) = f_2(y)$.

Proof: It is clear that $H_{\mathcal{L}} \subset H_L$ and a function is H_L-convex if and only if it is $H_{\mathcal{L}}$-convex. Let f_1 and f_2 be H_L-convex functions (then they are also $H_{\mathcal{L}}$-convex). Let $y \in X$ be a point such that the sets $\mathcal{D}_L f_1(y)$ and $\mathcal{D}_L f_2(y)$ are nonempty and $f_1(y) = f_2(y)$. First prove that

$$\sup\{h_i(x) : \; h_i \in \mathcal{D}_L f_i(y)\} = f_i(x) - f_i(y) \;\; \forall x \in X \; \forall i = 1, 2. \tag{4.7}$$

For this purpose we only need to check that $\sup\{h_i(x) : \; h_i \in \mathcal{D}_L f_i(y)\} \geq f_i(x) - f_i(y)$. For each $i = 1, 2$ choose an arbitrary function $h'_i \in \mathcal{D}_L f_i(y)$. Since $h'_i \in H_L$ then $h'_i(x) = \max\{l_i^1(x), l_i^2(x) + c_i\} + c'_i$, where $l_i^1, l_i^2 \in \mathcal{L}$ and $c_i, c'_i \in \mathbb{R}$. For the sake of definiteness assume that $h'_i(y) = l_i^1(y) + c'_i$. Then $l_i^1(y) + c'_i = 0$ and $l_i^1(x) + c'_i \leq f_i(x) - f_i(y)$ for all $x \in X$. For every $t_i \in \mathrm{supp}\,(f_i, H_{\mathcal{L}})$ consider the function h_{t_i} defined by

$$h_{t_i}(x) = \max\{l_i^1(x) + c'_i, t_i(x) - f_i(y)\}.$$

We see that $h_{t_i} \in H_L$, $h_{t_i}(y) = 0$ and $h_{t_i}(x) \leq f_i(x) - f_i(y) \;\; \forall x \in X$, that is $h_{t_i} \in \mathcal{D}_L f_i(y)$. Since f_i is $H_{\mathcal{L}}$-convex then

$$\sup\{t_i(x) - f_i(y) : \; t_i \in \mathrm{supp}\,(f_i, H_{\mathcal{L}})\} = f_i(x) - f_i(y) \;\; \forall x \in X.$$

Hence

$$\begin{aligned} \sup\{h_i(x) : \; h_i \in \mathcal{D}_L f_i(y)\} &\geq \sup\{h_{t_i}(x) : \; t_i \in \mathrm{supp}\,(f_i, H_{\mathcal{L}})\} \\ &\geq \sup\{t_i(x) - f_i(y) : \; t_i \in \mathrm{supp}\,(f_i, H_{\mathcal{L}})\} \\ &= f_i(x) - f_i(y) \;\; \forall x \in X. \end{aligned}$$

So the equalities (4.7) hold true. This means that

$$\begin{aligned} \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)) &= \{h \in H_{L,y} : \ h(x) \le \max_i (f_i(x) - f_i(y)) \ \forall x \in X\} \\ &= \mathcal{D}_L(\max\{f_1, f_2\})(y). \end{aligned}$$

□

Under the assumptions of Proposition 4.5, in order to describe the sets $\mathcal{D}_L f_1(y)$ and $\mathcal{D}_L f_2(y)$ we need to know all support functions of f_1 and f_2 with respect to $H_{\mathcal{L}}$. In other words, we need to know the values of the functions f_1 and f_2 at each point $x \in X$. These sets can be very complicated, and therefore the set $\mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$ is also complicated.

In the next section we consider one special case, when the subdifferential calculus is possible. Namely, we assume that the subdifferential has local nature. This means that for the description of a set $\mathcal{D}_L f(y)$ we need to know the behaviour of the function f only in a neighbourhood of the point y. This allows us to give a sufficiently simple description of $\mathcal{D}_L f(y)$.

4.2 Subdifferential calculus in the case when H_L has the strong globalization property

In the paper [39] Rolewicz introduced the notion of strong globalization property. He says that a set Φ of functions defined on a topological space X has the strong globalization property if for every Φ-convex function f and for every point $y \in X$ each local Φ-subgradient of f at y can be extended to a global one. Here, into the definition of the strong globalization property, we put a more rigid condition. Namely, we require that each local subgradient is also a global one. We show that in such a case subdifferential calculus can be expressed in terms of special functions that in a sense approximate the given functions.

Let H be a set of functions defined on a topological space X. We say that H has the *strong globalization property* if for any H-convex function f, for any point $y \in X$ and for any $h \in H$ the following implication holds

$$(h(y) = f(y),\ \ h(x) \le f(x) \text{ in a neighbourhood of } y) \implies (h(x) \le f(x) \text{ for all } x \in X). \tag{4.8}$$

For instance, it was shown in [39] (see Example 4.7) that the set H of all continuous affine functions defined on a topological linear space X has the strong globalization property.

Remark 4.2 Assume that H has the strong globalization property. Then every subset $H' \subset H$ also has the strong globalization property since any H'-convex function is H-convex.

Now let L be a set of functions defined on X. As above, H_L denotes the set of all vertical shifts of functions $l \in L$. Assume that H_L has the strong globalization property. Take an H_L-convex function f and a point $y \in X$. Let U be a neighbourhood of y. Then the following equality holds

$$\partial_L f(y) = \{l \in L : \ l(x) - l(y) \leq f(x) - f(y) \ \forall x \in U\}. \tag{4.9}$$

Indeed, let $l \in L$ and $l(x) - l(y) \leq f(x) - f(y) \ \forall x \in U$. Then the function $h(x) = l(x) - l(y) + f(y)$ belongs to H_L. Moreover, $h(y) = f(y)$ and $h(x) \leq f(x) \ \forall x \in U$. Hence $h(x) \leq f(x)$ for all $x \in X$. This implies $l \in \partial_L f(y)$.

Similarly, we have the equality for the set $\mathcal{D}_L f(y)$

$$\mathcal{D}_L f(y) = \{h \in H_L : \ h(y) = 0, \ h(x) \leq f(x) - f(y) \ \forall x \in U\}. \tag{4.10}$$

For $y \in X$ let $\mathcal{U}(y)$ denote the set of all neighbourhoods of y. Let f be an H_L-convex function. Then we can introduce the following function defined on X

$$\text{app}_{f,y}(x) = \inf_{U \in \mathcal{U}(y)} \inf\{p(x) : \ p \text{ is } H_L\text{-convex}, \ p(z) = f(z) \ \forall z \in U\}. \tag{4.11}$$

We will show that the function $\text{app}_{f,y}$ can be considered as an approximation of the function f near the point y. In the classical convex case we can estimate this function using ε-subdifferentials (see Proposition 4.7 and Example 4.2).

Note that another approximation function was considered in [53] (see formula (17) in [53]). It seems that the function (4.11) is more appropriate to our purposes.

First, it is clear that

$$\text{app}_{f,y}(y) = f(y), \qquad \text{app}_{f,y}(x) \leq f(x) \qquad \forall x \in X. \tag{4.12}$$

Let us prove some properties of the function $\text{app}_{f,y}$.

Proposition 4.6 *Let $y \in X$ and $f : X \to \mathbb{R}_{+\infty}$ be an H_L-convex function such that $\mathcal{D}_L f(y) \neq \emptyset$. Assume that H_L has the strong globalization property. Then*

$$\mathcal{D}_L f(y) = \mathcal{D}_L(\text{app}_{f,y})(y),$$

and therefore

$$\sup_{h \in \mathcal{D}_L f(y)} (h(x) + f(y)) \le \mathrm{app}_{f,y}(x) \qquad \forall x \in X. \tag{4.13}$$

If g is an H_L-convex function such that $f(x) = g(x)$ in a neighbourhood of y then $\mathrm{app}_{f,y} = \mathrm{app}_{g,y}$.

Proof: If $U \in \mathcal{U}(y)$ and p is an H_L-convex function such that $p(z) = f(z)$ for all $z \in U$, then, by (4.10), $\mathcal{D}_L p(y) = \mathcal{D}_L f(y)$. Hence

$$\begin{aligned} \mathcal{D}_L f(y) &= \bigcap \{\mathcal{D}_L p(y) : \ U \in \mathcal{U}(y), \ p \text{ is } H_L\text{-convex}, \ p(z) = f(z) \, \forall z \in U\} \\ &= \{h \in H_L : \ h(y) = 0, \ h(x) \le \mathrm{app}_{f,y}(x) - \mathrm{app}_{f,y}(y) \ \forall x \in X\} \\ &= \mathcal{D}_L(\mathrm{app}_{f,y})(y). \end{aligned}$$

In particular, we have

$$\sup_{h \in \mathcal{D}_L f(y)} (h(x) + f(y)) = \sup_{h \in \mathcal{D}_L(\mathrm{app}_{f,y})(y)} (h(x) + \mathrm{app}_{f,y}(y)) \le \mathrm{app}_{f,y}(x) \quad \forall x \in X.$$

It follows directly from (4.11) that $\mathrm{app}_{f,y} = \mathrm{app}_{g,y}$ whenever f and g are H_L-convex and coincide in a neighbourhood of y. □

So if H_L has the strong globalization property and f is an H_L-convex function such that $\mathcal{D}_L f(y) \neq \emptyset$ then, in view of (4.12) and Proposition 4.6, we can say that the function $\mathrm{app}_{f,y}$ approximates the function f near the point y in the following sense: the function $\mathrm{app}_{f,y}$ depends only on the local behaviour of f near y, coincides with f at the point y and does not exceed f on the whole space X. The equality $\mathcal{D}_L(\mathrm{app}_{f,y})(y) = \mathcal{D}_L f(y)$ shows that such an approximation is closely related to the notion of subdifferential. Note that the function $t(x) = \sup_{h \in \mathcal{D}_L f(y)}(h(x) + f(y))$ enjoys all these properties as well. However, due to the inequalities $t(x) \le \mathrm{app}_{f,y}(x) \le f(x)$ (see (4.13)), the approximation $\mathrm{app}_{f,y}(x)$ is better than $t(x)$.

Below we will be interested in conditions which guarantee that the approximations $\mathrm{app}_{f,y}(x)$ and $t(x) = \sup_{h \in \mathcal{D}_L f(y)}(h(x) + f(y))$ coincide on X.

Assume that the space $\mathbb{R}^n$ is equipped with the usual coordinate-wise order relation: $a \le b$ if and only if $a_i \le b_i$ for all $i = 1, \dots, n$ ($a, b \in \mathbb{R}^n$). We will consider increasing continuous mappings $M : \mathbb{R}^n \to \mathbb{R}$. For example, the mappings $M(a) = \sum_i a_i$ and $M(a) = \max_i a_i$ ($a = (a_1, \dots, a_n) \in \mathbb{R}^n$) are increasing and continuous on $\mathbb{R}^n$. Moreover, since the maximum of abstract convex functions is always abstract convex then the mapping $M(a) = \max_i a_i$ satisfies the assumptions of Theorem 4.1 irrespective of the set H_L.

If a mapping $M : \mathbb{R}^n \to \mathbb{R}$ is continuous and increasing then for any sets $A_i \subset \mathbb{R}$ $(i = 1, \dots, n)$

$$\begin{aligned} M\left(\sup_{a_1\in A_1} a_1, \dots, \sup_{a_n\in A_n} a_n\right) &= \sup_{a_i\in A_i} M(a_1, \dots, a_n), \\ M\left(\inf_{a_1\in A_1} a_1, \dots, \inf_{a_n\in A_n} a_n\right) &= \inf_{a_i\in A_i} M(a_1, \dots, a_n). \end{aligned} \tag{4.14}$$

(Here we assume that $M(b_1, \dots, b_j, \dots, b_n) = \lim_{b\to b_j} M(b_1, \dots, b, \dots, b_n)$ for $b_j = \pm\infty$).

Theorem 4.1 *Let $M : \mathbb{R}^n \to \mathbb{R}$ be an increasing continuous mapping such that for all $h_1, \dots, h_n \in H_L$ the function $M(h_1(x), \dots, h_n(x))$ is H_L-convex. Let $y \in X$ and $f_1, \dots, f_n$ be H_L-convex functions. If H_L has the strong globalization property then*

$$\mathcal{D}_L M(f_1, \dots, f_n)(y) = \mathcal{D}_L M\left(\mathrm{app}_{f_1,y}, \dots, \mathrm{app}_{f_n,y}\right)(y). \tag{4.15}$$

Proof: If $h \in \mathcal{D}_L M\left(\mathrm{app}_{f_1,y}, \dots, \mathrm{app}_{f_n,y}\right)(y)$ then

$$h(x) \le M\left(\mathrm{app}_{f_1,y}(x), \dots, \mathrm{app}_{f_n,y}(x)\right) - M\left(\mathrm{app}_{f_1,y}(y), \dots, \mathrm{app}_{f_n,y}(y)\right) \qquad \forall x \in X.$$

Since the mapping M is increasing then, due to (4.12),

$$h(x) \le M(f_1(x), \dots, f_n(x)) - M(f_1(y), \dots, f_n(y)) \qquad \forall x \in X,$$

hence $h \in \mathcal{D}_L M(f_1, \dots, f_n)(y)$.

Conversely, let $h \in \mathcal{D}_L M(f_1, \dots, f_n)(y)$. Let $U_1, \dots, U_n \in \mathcal{U}(y)$ and $p_1, \dots, p_n$ be H_L-convex functions such that $p_i(z) = f_i(z)$ for all $z \in U_i$. Then

$$\begin{aligned} h(z) &\le M(f_1(z), \dots, f_n(z)) - M(f_1(y), \dots, f_n(y)) \\ &= M(p_1(z), \dots, p_n(z)) - M(p_1(y), \dots, p_n(y)) \quad \forall z \in \bigcap_i U_i. \end{aligned} \tag{4.16}$$

Since all functions p_i are H_L-convex then, by (4.14),

$$\begin{aligned} M(p_1(x), \dots, p_n(x)) &= M\left(\sup_{h_1\in \mathrm{supp}\,(p_1,H_L)} h_1(x), \dots, \sup_{h_n\in \mathrm{supp}\,(p_n,H_L)} h_n(x)\right) \\ &= \sup_{h_i\in \mathrm{supp}\,(p_i,H_L)} M(h_1(x), \dots, h_n(x)). \end{aligned}$$

By our assumptions, $M(h_1(x), \dots, h_n(x))$ is H_L-convex for any $h_i \in H_L$. Hence the function $p(x) = M(p_1(x), \dots, p_n(x)) - M(p_1(y), \dots, p_n(y))$ is H_L-convex as well. Since H_L has the strong globalization property then it follows from (4.16) that

$$h(x) \le M(p_1(x), \dots, p_n(x)) - M(p_1(y), \dots, p_n(y)) \quad \forall x \in X.$$

Thus, we deduce that for every $x \in X$

$$\begin{aligned} h(x) &\leq \inf_{U_i \in \mathcal{U}(y)} \inf_{p_i \in T_i(U_i)} [M(p_1(x), \ldots, p_n(x)) - M(p_1(y), \ldots, p_n(y))] \\ &= \inf_{U_i \in \mathcal{U}(y)} \inf_{p_i \in T_i(U_i)} M(p_1(x), \ldots, p_n(x)) - M(\mathrm{app}_{f_1,y}(y), \ldots, \mathrm{app}_{f_n,y}(y)), \end{aligned}$$

where $T_i(U_i)$ is the collection of all H_L-convex functions p_i such that $p_i(z) = f_i(z)$ for all $z \in U_i$. At last, it follows from (4.14) that

$$h(x) \leq M(\mathrm{app}_{f_1,y}(x), \ldots, \mathrm{app}_{f_n,y}(x)) - M(\mathrm{app}_{f_1,y}(y), \ldots, \mathrm{app}_{f_n,y}(y)) \quad \forall x \in X.$$

Therefore $h \in \mathcal{D}_L M\left(\mathrm{app}_{f_1,y}, \ldots, \mathrm{app}_{f_n,y}\right)(y)$. □

Corollary 4.1 *Assume that H_L has the strong globalization property. Let $y \in X$ and $f_1, \ldots, f_n$ be H_L-convex functions such that*

$$\mathrm{app}_{f_i,y}(x) = \sup_{h \in \mathcal{D}_L f_i(y)} (h(x) + f_i(y)) \qquad \text{for all } x \in X, \quad i = 1, \ldots, n.$$

If $f_1(y) = \cdots = f_n(y)$ then

$$\mathcal{D}_L(\max\{f_1, \ldots, f_n\})(y) = \mathrm{co}_{H_{L,y}} \bigcup_{i=1}^{n} \mathcal{D}_L f_i(y). \tag{4.17}$$

If all functions f_i are continuous at y then

$$\mathcal{D}_L(\max\{f_1, \ldots, f_n\})(y) = \mathrm{co}_{H_{L,y}} \bigcup_{i \in I} \mathcal{D}_L f_i(y), \tag{4.18}$$

where $I = \{i : \ f_i(y) = \max\{f_1(y), \ldots, f_n(y)\}\}$.

Proof: Let $M(a_1, \ldots, a_n) = \max\{a_1, \ldots, a_n\}$. Then M satisfies the conditions of Theorem 4.1. Hence, by (4.15),

$$\mathcal{D}_L(\max\{f_1, \ldots, f_n\})(y) = \mathcal{D}_L(\max\{\mathrm{app}_{f_1,y}, \ldots, \mathrm{app}_{f_n,y}\})(y). \tag{4.19}$$

Let $f_1(y) = \cdots = f_n(y)$. Since $\mathrm{app}_{f_i,y}(y) = f_i(y)$ (see (4.12)) then we have

$$\begin{aligned} &\max\{\mathrm{app}_{f_1,y}(x), \ldots, \mathrm{app}_{f_n,y}(x)\} - \max\{\mathrm{app}_{f_1,y}(y), \ldots, \mathrm{app}_{f_n,y}(y)\} = \\ &= \max_i \sup_{h \in \mathcal{D}_L f_i(y)} (h(x) + f_i(y)) - \max\{f_1(y), \ldots, f_n(y)\} \\ &= \max_i \sup_{h \in \mathcal{D}_L f_i(y)} h(x). \end{aligned}$$

So a function $h' \in H_{L,y}$ belongs to $\mathcal{D}_L(\max\{\mathrm{app}_{f_1,y}, \ldots, \mathrm{app}_{f_n,y}\})(y)$ if and only if

$$h'(x) \leq \max_i \sup_{h \in \mathcal{D}_L f_i(y)} h(x) \quad \text{for all } x \in X.$$

In other words (see (4.1))

$$\mathcal{D}_L(\max\{\mathrm{app}_{f_1,y},\dots,\mathrm{app}_{f_n,y}\})(y) = \mathrm{co}_{H_{L,y}} \bigcup_{i=1}^{n} \mathcal{D}_L f_i(y).$$

This and (4.19) give us the required equality (4.17).

If all functions f_i are continuous at the point y then there exists a neighbourhood U of y such that $\max\{f_1(x),\dots,f_n(x)\} = \max_{i\in I} f_i(x)$ for all $x \in U$. Since H_L has the strong globalization property then

$$\mathcal{D}_L(\max\{f_1,\dots,f_n\})(y) = \mathcal{D}_L\left(\max_{i\in I} f_i\right)(y).$$

At the same time, $f_i(y) = f_j(y)$ for any $i,j \in I$. Then it follows from the first part of the proof that

$$\mathcal{D}_L\left(\max_{i\in I} f_i\right)(y) = \mathrm{co}_{H_{L,y}} \bigcup_{i\in I} \mathcal{D}_L f_i(y).$$

Thus the equality (4.18) holds true. □

Corollary 4.2 *Let $y \in X$ and $f_1,\dots,f_n$ be H_L-convex functions such that*

$$\mathrm{app}_{f_i,y}(x) = \sup_{h\in\mathcal{D}_L f_i(y)} (h(x) + f_i(y)) \qquad \textit{for all } x \in X,\ \ i = 1,\dots,n.$$

Let $M : \mathbb{R}^n \to \mathbb{R}$ be an increasing continuous mapping such that $M(h_1,\dots,h_n) \in H_L$ for all $h_i \in H_L$. If H_L has the strong globalization property then

$$\begin{aligned}\mathcal{D}_L M(f_1,\dots,f_n)(y) &= \mathrm{co}_{H_{L,y}}[M(\mathcal{D}_L f_1(y) + f_1(y),..,\mathcal{D}_L f_n(y) + f_n(y)) - \\ &- M(f_1(y),..,f_n(y))],\end{aligned}$$

where $[M(\mathcal{D}_L f_1(y) + f_1(y),\dots,\mathcal{D}_L f_n(y) + f_n(y)) - M(f_1(y),\dots,f_n(y))]$ is the set of all functions of the form

$$h(x) = M(h_1(x) + f_1(y),\dots,h_n(x) + f_n(y)) - M(f_1(y),\dots,f_n(y))$$

with $h_i \in \mathcal{D}_L f_i(y)$ for all $i = 1,\dots,n$.

Proof: It is sufficient to note that, by our conditions, every function

$$h(x) = M(h_1(x) + f_1(y),\dots,h_n(x) + f_n(y)) - M(f_1(y),\dots,f_n(y)) \quad \text{with } h_i \in \mathcal{D}_L f_i(y)$$

belongs to $H_{L,y}$.

Due to (4.15) a function $h' \in H_{L,y}$ belongs to $\mathcal{D}_L M(f_1, \dots, f_n)(y)$ if and only if

$$\begin{aligned} h'(x) &\leq M(\mathrm{app}_{f_1,y}(x), \dots, \mathrm{app}_{f_n,y}(x)) - M(\mathrm{app}_{f_1,y}(y), \dots, \mathrm{app}_{f_n,y}(y)) \\ &= M\left(\sup_{h \in \mathcal{D}_L f_1(y)} (h(x) + f_1(y)), \dots, \sup_{h \in \mathcal{D}_L f_n(y)} (h(x) + f_n(y))\right) - \\ &- M(f_1(y), \dots, f_n(y)) \\ &= \sup_{h_i \in \mathcal{D}_L f_i(y)} [M(h_1(x) + f_1(y), \dots, h_n(x) + f_n(y)) - \\ &- M(f_1(y), \dots, f_n(y)] \quad \forall x \in X. \end{aligned}$$

The proof is completed. □

For example, if $M(a_1, \dots, a_n) = a_1 + \dots + a_n$ then, under the assumptions of Corollary 4.2, the sum $(f_1 + \dots + f_n)$ of H_L-convex functions f_i is H_L-convex as well and

$$\mathcal{D}_L(f_1 + \dots + f_n)(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) + \dots + \mathcal{D}_L f_n(y)).$$

Remark 4.3 An interesting approach to deriving subdifferential sum formula was taken in [23]. Among other results, it was shown that additivity of the mapping $\mathrm{supp}\,(\cdot, H_L)$ implies additivity of the subdifferential. Namely, assume that $(l_1 + l_2) \in L$ for all $l_1, l_2 \in L$. Then for any H_L-convex functions f_1, f_2 we have

$$\mathrm{supp}\,(f_1 + f_2, H_L) = \mathrm{co}_{H_L}(\mathrm{supp}\,(f_1, H_L) + \mathrm{supp}\,(f_2, H_L)).$$

If, moreover, H_L-convex functions f_1, f_2 are such that $(\mathrm{supp}\,(f_1, H_L) + \mathrm{supp}\,(f_2, H_L))$ is (H_L, X)-convex, i.e.

$$\mathrm{supp}\,(f_1 + f_2, H_L) = \mathrm{supp}\,(f_1, H_L) + \mathrm{supp}\,(f_2, H_L), \tag{4.20}$$

then ([23], Corollary 3.2)

$$\partial_L(f_1 + f_2)(x) = \partial_L f_1(x) + \partial_L f_2(x) \quad \forall x \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2. \tag{4.21}$$

It is clear that (4.21) holds if and only if $\mathcal{D}_L(f_1 + f_2)(y) = \mathcal{D}_L f_1(y) + \mathcal{D}_L f_2(y)$ for all $x \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$.

Note that verification of the equality (4.20) is not easy, because we need to have a description of (H_L, X)-convex sets.

The main problem now is to find conditions which guarantee the equality $\mathrm{app}_{f,y}(x) = \sup_{h \in \mathcal{D}_L f(y)}(h(x) + f(y))$. Since $\mathrm{app}_{f,y}(x) \geq \sup_{h \in \mathcal{D}_L f(y)}(h(x) + f(y))$ then we are interested in the inverse inequality. In the following proposition we estimate the function $\mathrm{app}_{f,y}$

using ε-subdifferentials. Let $\varepsilon \geq 0$. Recall that the set

$$\partial_{L,\varepsilon} f(y) = \{l \in L : \; l(x) - l(y) \leq f(x) - f(y) + \varepsilon \;\; \forall x \in X\}$$

is called the ε-subdifferential of the function f at y with respect to L.

Proposition 4.7 *Let $y \in X$. Assume that for any H_L-convex function g the following implication holds:*

$$\limsup_{x \to y} g(x) < +\infty \implies g \text{ is continuous at } y. \tag{4.22}$$

Let a function f be H_L-convex and continuous at y. Then

$$\mathrm{app}_{f,y}(x) \leq \lim_{\varepsilon \to +0} \sup_{l \in \partial_{L,\varepsilon} f(y)} (l(x) - l(y) + f(y)) \qquad \text{for all } x \in X. \tag{4.23}$$

Proof: First prove that for each $\varepsilon > 0$ a neighbourhood U_ε of the point y and a number $\delta = \delta(\varepsilon) > 0$ exist such that

$$l(y) - l(z) + f(z) \geq f(y) - \varepsilon + \delta(\varepsilon) \quad \text{for all } z \in U_\varepsilon, \; l \in \partial_{L,\delta(\varepsilon)} f(z). \tag{4.24}$$

Assume it is not true. Then a number $\varepsilon > 0$ exists such that for any neighbourhood U of the point y and for any $\delta > 0$ we can find $z \in U$ and $l \in \partial_{L,\delta} f(z)$, for which the inequality $l(y) - l(z) + f(z) < f(y) - \varepsilon + \delta$ holds.

Then consider the function

$$g(x) = \sup_{\delta > 0} \sup \{l(x) - l(z) + f(z) - \delta : \\ z \in X, \; l \in \partial_{L,\delta} f(z), \; l(y) - l(z) + f(z) < f(y) - \varepsilon + \delta\}.$$

This function is H_L-convex, $g(x) \leq f(x)$ for all $x \in X$ and $g(y) \leq \sup_{\delta > 0}(f(y) - \varepsilon + \delta - \delta) = f(y) - \varepsilon$. Moreover, due to our assumption, for any neighbourhood U of the point y a point $z \in U$ exists such that $g(z) \geq \sup_{\delta > 0}(f(z) - \delta) = f(z)$, hence $\limsup_{z \to y} g(z) \geq \liminf_{z \to y} f(z)$. Since f is continuous at the point y and $g(y) \leq f(y) - \varepsilon$ then $\limsup_{z \to y} g(z) \geq f(y) > f(y) - \varepsilon \geq g(y)$. Hence g is discontinuous at y and, by (4.22), we conclude that $\limsup_{z \to y} g(z) = +\infty$. On the other hand, since $g \leq f$ and f is continuous at y then $\limsup_{z \to y} g(z) \leq \limsup_{z \to y} f(z) = f(y) < +\infty$, which contradicts the equality $\limsup_{z \to y} g(z) = +\infty$.

So for each $\varepsilon > 0$ a neighbourhood U_ε of y and a number $\delta(\varepsilon) > 0$ exist such that (4.24) holds. Then for any $z \in U_\varepsilon$ and $l \in \partial_{L,\delta(\varepsilon)} f(z)$ we have

$$\begin{aligned} l(x) - l(y) &= (l(x) - l(z) + f(z)) - (l(y) - l(z) + f(z)) \\ &\leq (f(x) + \delta(\varepsilon)) - (f(y) - \varepsilon + \delta(\varepsilon)) = f(x) - f(y) + \varepsilon \quad \forall x \in X. \end{aligned}$$

This implies that $l \in \partial_{L,\varepsilon} f(y)$ for all $l \in \partial_{L,\delta(\varepsilon)} f(z)$ with $z \in U_\varepsilon$. Since $\partial_{L,\delta} f(z) \subset \partial_{L,\delta(\varepsilon)} f(z)$ for any $0 < \delta < \delta(\varepsilon)$ then $l \in \partial_{L,\varepsilon} f(y)$ for all $l \in \partial_{L,\delta} f(z)$ with $z \in U_\varepsilon$ and $0 < \delta < \delta(\varepsilon)$. Therefore for all $\varepsilon > 0$ and $0 < \delta < \delta(\varepsilon)$

$$\sup_{l \in \partial_{L,\delta} f(z),\, z \in U_\varepsilon} (l(x) - l(y) + f(y)) \leq \sup_{l \in \partial_{L,\varepsilon} f(y)} (l(x) - l(y) + f(y)) \quad \text{for all } x \in X. \tag{4.25}$$

At the same time, since $(-l(z) + f(z)) \leq (-l(y) + f(y)) + \delta$ whenever $l \in \partial_{L,\delta} f(z)$ then

$$\sup_{l \in \partial_{L,\delta} f(z),\, z \in U_\varepsilon} (l(x) - l(z) + f(z)) \leq \sup_{l \in \partial_{L,\delta} f(z),\, z \in U_\varepsilon} (l(x) - l(y) + f(y)) + \delta \quad \text{for all } x \in X. \tag{4.26}$$

It follows from the inequalities (4.25) and (4.26) that for all $\varepsilon > 0$ and $0 < \delta < \delta(\varepsilon)$

$$\sup_{l \in \partial_{L,\delta} f(z),\, z \in U_\varepsilon} (l(x) - l(z) + f(z)) \leq \sup_{l \in \partial_{L,\varepsilon} f(y)} (l(x) - l(y) + f(y)) + \delta \quad \text{for all } x \in X. \tag{4.27}$$

Since the function f is continuous at y then it is finite in a neighbourhood U' of y. Since f is H_L-convex then $\partial_{L,\delta} f(z) \neq \emptyset$ for every $z \in U'$ and $\delta > 0$. Hence for any $\varepsilon > 0$ the function

$$p(x) = \sup_{z \in U_\varepsilon \cap U'} \sup_{\delta(\varepsilon) > \delta > 0} \sup_{l \in \partial_{L,\delta} f(z)} (l(x) - l(z) + f(z) - \delta)$$

is H_L-convex and coincides with $f(x)$ for all $x \in U_\varepsilon \cap U'$. Thus, we deduce that

$$\begin{aligned}
\operatorname{app}_{f,y}(x) &= \inf_{U \in \mathcal{U}(y)} \inf\{p(x) : p \text{ is } H_L\text{-convex}, \ p(z) = f(z) \ \forall z \in U\} \\
&\leq \inf_{\varepsilon > 0} \inf\{p(x) : p \text{ is } H_L\text{-convex}, \ p(z) = f(z) \ \forall z \in U_\varepsilon \cap U'\} \\
&\leq \inf_{\varepsilon > 0} \sup_{z \in U_\varepsilon \cap U'} \sup_{\delta(\varepsilon) > \delta > 0} \sup_{l \in \partial_{L,\delta} f(z)} (l(x) - l(z) + f(z) - \delta),
\end{aligned}$$

and, due to (4.27),

$$\operatorname{app}_{f,y}(x) \leq \inf_{\varepsilon > 0} \sup_{l \in \partial_{L,\varepsilon} f(y)} (l(x) - l(y) + f(y)) = \lim_{\varepsilon \to +0} \sup_{l \in \partial_{L,\varepsilon} f(y)} (l(x) - l(y) + f(y)).$$

□

Remark 4.4 Implication (4.22) means that every H_L-convex function g is continuous at y whenever a neighbourhood U of y and a number $c \in \mathbb{R}$ exist such that $g(u) \leq c$ for all $u \in U$. Note that this implication can be false even in the case when all elements of H_L are continuous. For example, let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by: $g(x) = 0$ if $x \leq 0$ and $g(x) = 1$ if $x > 0$. Then g can be represented as the supremum of a family of continuous functions. We see that g is uniformly bounded on $\mathbb{R}$. However g is discontinuous at zero.

Example 4.2 Let L be the set of all linear continuous functions defined on a normed space X. Then every H_L-convex function is convex in the usual sense. It is well known (see, for example, Proposition 2.2.6 in [7]) that a convex function g defined on X is Lipschitz continuous at $y \in X$ provided that g is bounded above in a neighbourhood of y. Thus we conclude that the condition (4.22) is valid in the classical convex case.

The other approach to examining the equality $\mathrm{app}_{f,y}(x) = \sup_{h \in \mathcal{D}_L f(y)}(h(x) + f(y))$ is based on a special property of the mapping $\mathcal{D}_L f(\cdot)$ at the point y. Let X and T be topological spaces. We say that a mapping $D : X \to 2^T$ enjoys property $(*)$ at $y \in X$ if, for any open set $G \subset T$ such that $D(y) \subset G$, a neighbourhood U of y exists such that $D(u) \cap G \neq \emptyset$ for all $u \in U$. Kuratowski [30] gives the following definition of upper and lower semicontinuity of multifunction $D : X \to 2^T$ (see also Borwein and Zhu [5], Definition 5.1.15): D is upper (lower) semicontinuous at y provided that for any open set G in T with $D(y) \subset G$, $(D(y) \cap G \neq \emptyset)$,

$$\{x \in X : \; D(x) \subset G\} \qquad (\{x \in X : \; D(x) \cap G \neq \emptyset\})$$

is an open set in X. Thus, if $D(u)$ is nonempty for all u from a neighbourhood of y then any semicontinuity (upper or lower) of D at y implies the property $(*)$.

Proposition 4.8 *Let H_L be equipped with the topology of pointwise convergence. Let f be an H_L-convex function and $y \in X$. If f is upper semicontinuous at y and $\mathcal{D}_L f(\cdot)$ enjoys property $(*)$ at y then* $\mathrm{app}_{f,y}(x) = \sup_{h \in \mathcal{D}_L f(y)}(h(x) + f(y))$ *for all $x \in X$.*

Proof: Take $x \in X$ and $\varepsilon > 0$. Let $G_\varepsilon = \{h \in H_L : \; \exists\, g \in \mathcal{D}_L f(y) \;\; h(x) < g(x) + \varepsilon\}$. Then G_ε is an open set and $\mathcal{D}_L f(y) \subset G_\varepsilon$. Since the mapping $\mathcal{D}_L f(\cdot)$ possesses property $(*)$ at the point y and f is upper semicontinuous at y then there is a neighbourhood U_ε of y such that $\mathcal{D}_L f(u) \cap G_\varepsilon \neq \emptyset$ and $f(u) < f(y) + \varepsilon$ for all $u \in U_\varepsilon$. Consider the function

$$p_\varepsilon(z) = \sup_{u \in U_\varepsilon} \; \sup_{h \in \mathcal{D}_L f(u) \cap G_\varepsilon} (h(z) + f(u)) \quad \forall\, z \in X.$$

It is clear that p_ε is H_L-convex and $p_\varepsilon(z) = f(z)$ for all $z \in U_\varepsilon$. Hence

$$\begin{aligned}
\mathrm{app}_{f,y}(x) &= \inf_{U \in \mathcal{U}(y)} \inf\{p(x) : \; p \text{ is } H_L\text{-convex}, \; p(z) = f(z) \; \forall\, z \in U\} \\
&\leq \inf_{\varepsilon > 0} p_\varepsilon(x) = \inf_{\varepsilon > 0} \sup_{u \in U_\varepsilon} \; \sup_{h \in \mathcal{D}_L f(u) \cap G_\varepsilon} (h(x) + f(u)) \\
&= \inf_{\varepsilon > 0} \sup_{u \in U_\varepsilon} \; \sup_{h \in \mathcal{D}_L f(u) \cap G_\varepsilon} [(h(x) + f(y)) + (f(u) - f(y))] \\
&\leq \inf_{\varepsilon > 0} \; \sup_{g \in \mathcal{D}_L f(y)} (g(x) + f(y) + 2\varepsilon) = \sup_{g \in \mathcal{D}_L f(y)} (g(x) + f(y)).
\end{aligned}$$

The reverse inequality $\sup_{h\in\mathcal{D}_L f(y)}(h(x)+f(y)) \le \mathrm{app}_{f,y}(x)$ follows from Proposition 4.6. □

4.3 Examples

For beginning, we consider the case, when abstract convex functions are generalized convex in the sense of Beckenbach [2].

Example 4.3 Let $\mathcal{F}$ be a two-parameter family of continuous functions defined on an interval $I \subset \mathbb{R}$ (see Example 1.1). It is easy to check that each $\mathcal{F}$-convex function $f : I \to \mathbb{R} \cup \{+\infty\}$ possesses the inequality

$$f(x) \le \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x), \qquad x_1 \le x \le x_2 \tag{4.28}$$

for every $x_1, x_2 \in I$ such that $f(x_1)$ and $f(x_2)$ are finite. Let $f : I \to \mathbb{R} \cup \{+\infty\}$ be $\mathcal{F}$-convex and $x_1 \in I$. Let $\varphi \in \mathcal{F}$ be such that $\varphi(x_1) = f(x_1)$ and $\varphi(x) \le f(x)$ in a neighbourhood of x_1. We show that $\varphi(x) \le f(x)$ for all $x \in I$. Assume it is not true. Take an arbitrary $x_2 \in I$ with $\varphi(x_2) > f(x_2)$. For the sake of definiteness, let $x_2 > x_1$. Since $\varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x_1) = f(x_1) = \varphi(x_1)$ and $\varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x_2) = f(x_2) < \varphi(x_2)$ then $\varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x) < \varphi(x)$ for all $x \in (x_1, x_2)$ (see [2]). Hence $\varphi_{(x_1,f(x_1))(x_2,f(x_2))}(x) < f(x)$ for all $x > x_1$ close to x_1, which contradicts (4.28).
Thus, the family $\mathcal{F}$ has the strong globalization property.

Let X and Y be topological spaces and $\omega : X \to Y$ be an open continuous mapping. Let $\mathcal{L}$ be a set of functions defined on $\omega(X) = \{\omega(x) : \; x \in X\}$. Let L be the set of all functions $l(x) = \ell(\omega(x))$ defined on X, where $\ell \in \mathcal{L}$. Then the set of all H_L-convex functions coincides with the set of functions $f(x) = g(\omega(x))$, where g is $H_{\mathcal{L}}$-convex.

Proposition 4.9 *If $H_{\mathcal{L}}$ has the strong globalization property then also H_L has the strong globalization property.*
If g is an $H_{\mathcal{L}}$-convex function, $y = \omega(x)$ and

$$\mathrm{app}_{g,y}(z) = \sup_{h\in\mathcal{D}_{\mathcal{L}} g(y)} (h(z) + g(y)) \qquad \forall z \in \omega(X),$$

then the following equality holds for the function $f = g \circ \omega$

$$\mathrm{app}_{f,x}(z) = \sup_{h\in\mathcal{D}_L f(x)} (h(z) + f(x)) \qquad \forall z \in X.$$

Proof: Assume that $H_{\mathcal{L}}$ has the strong globalization property, and let $h \in H_L$. Let $f(x) = g(\omega(x))$ be an H_L-convex function such that

$$h(y) = f(y), \qquad h(x) \leq f(x) \quad \forall x \in U,$$

where U is a neighbourhood of y. Since $h(x) = \ell(\omega(x)) - c$ then $\ell(\omega(y)) - c = g(\omega(y))$, $\ell(\omega(x)) - c \leq g(\omega(x)) \quad \forall x \in U$. Since ω is an open mapping then $U' = \omega(U)$ is a neighbourhood of the point $\omega(y)$. Because $H_{\mathcal{L}}$ has the strong globalization property, we have $\ell(z) - c \leq g(z)$ for all $z \in \omega(X)$ and $h(x) \leq f(x)$ for all $x \in X$. So we proved that H_L has the strong globalization property.

Let us prove the second part of proposition. Let g be $H_{\mathcal{L}}$-convex, $y = \omega(x)$ and $\mathrm{app}_{g,y}(z) = \sup_{h \in \mathcal{D}_{\mathcal{L}} g(y)}(h(z) + g(y))$ for all $z \in \omega(X)$. Since every H_L-convex function p has the form $p(x) = q(\omega(x))$, where q is $H_{\mathcal{L}}$-convex, then

$$\begin{aligned}\mathrm{app}_{f,x}(z) &= \inf_{U \in \mathcal{U}(x)} \inf\{p(z) : p \text{ is } H_L\text{-convex}, \ p(u) = f(u) \ \forall u \in U\} \\ &= \inf_{U \in \mathcal{U}(x)} \inf\{q(\omega(z)) : q \text{ is } H_{\mathcal{L}}\text{-convex}, \ q(\omega(u)) = g(\omega(u)) \ \forall u \in U\}.\end{aligned}$$

Since the mapping ω is continuous and open then we get

$$\begin{aligned}\mathrm{app}_{f,x}(z) &= \inf_{U' \in \mathcal{U}(\omega(x))} \inf\{q(\omega(z)) : q \text{ is } H_{\mathcal{L}}\text{-convex}, \ q(u') = g(u') \ \forall u' \in U'\} \\ &= \mathrm{app}_{g,y}(\omega(z)) = \sup_{h \in \mathcal{D}_{\mathcal{L}} g(y)} (h(\omega(z)) + g(y)) \\ &= \sup_{h \in \mathcal{D}_{\mathcal{L}} g(\omega(x))} (h(\omega(z)) + g(\omega(x))) = \sup_{h \in \mathcal{D}_L f(x)} (h(z) + f(x)).\end{aligned}$$

□

Note that, under the conditions of Proposition 4.9, we have a simple isomorphism between $H_{\mathcal{L}}$-convex and H_L-convex functions. If $f = g \circ \omega$ then $\inf_{x \in X} f(x) = \inf_{y \in \omega(X)} g(y)$. So if $H_{\mathcal{L}}$ has the strong globalization property but the elementary functions $h \in H_{\mathcal{L}}$ seem difficult then we can use such isomorphism in order to get a more convenient equivalent form of abstract convex functions.

Proposition 4.10 *Let X and V be topological spaces. Let H be a set of functions $h : X \to \mathbb{R}$. Assume that for each two points $x, y \in X$ there exists a continuous mapping $\omega : V \to X$ such that $x, y \in \omega(V)$ and H^ω has the strong globalization property, where H^ω is the set of all functions $h' : V \to \mathbb{R}$ defined by $h'(v) = h(\omega(v))$, $(h \in H)$. Then H has the strong globalization property.*

Proof: Let $f : X \to \mathbb{R}_{+\infty}$ be H-convex function. Let $y \in X$ and $h \in H$ be a function such that $h(y) = f(y)$ and $h(x) \le f(x)$ for all x from a neighbourhood U of the point y. Take a point $x \in X$ and consider a mapping $\omega : V \to X$, which satisfies the conditions of our proposition for the points x, y. Let $\omega(v_1) = y$ and $\omega(v_2) = x$. Consider the functions h', f' defined on V by the formulas: $h'(v) = h(\omega(v))$, $f'(v) = f(\omega(v))$. Then h' belongs to H^ω, and f' is H^ω-convex. Since ω is continuous then a neighbourhood U' of the point v_1 exists such that $\omega(v) \in U$ for all $v \in U'$. Hence $h'(v_1) = h(y) = f(y) = f'(v_1)$ and $h'(v) = h(\omega(v)) \le f(\omega(v)) = f'(v)$ for all $v \in U'$. Since H^ω has the strong globalization property then $h'(v) \le f'(v)$ for all $v \in V$. In particular, $h(x) = h'(v_2) \le f'(v_2) = f(x)$. □

Now consider the simplest case $X = \mathbb{R}$.

Proposition 4.11 *Let L be a set of continuous functions defined on $\mathbb{R}$. Assume that for any functions $h_1, h_2 \in H_L$ and for any points $x_1, x_2 \in X$ the following implication holds*

$$(h_1(x_1) = h_2(x_1),\ \ h_1(x_2) = h_2(x_2),\ \ x_1 \ne x_2) \implies (h_1 = h_2). \tag{4.29}$$

Let $y \in \mathbb{R}$ and f be an H_L-convex function such that the sets $\mathcal{D}_L f(z)$ are nonempty in a neighbourhood U of y. Then for any $h \in H_L$ implication (4.8) holds.

Proof: Let U be a neighbourhood of y such that $\mathcal{D}_L f(z) \ne \emptyset$ for all $z \in U$. Let $h \in H_L$ be an elementary function such that $h(y) = f(y)$ and $h(x) \le f(x)$ for all $x \in U'$, where U' is a neighbourhood of y. We need to check that $h(x) \le f(x)$ for all $x \in \mathbb{R}$.

First show that $h(x) \le f(x)$ for any $x > y$. Let $x > y$. Then a point $z \in U \cap U'$ exists such that $x > z > y$. Since $z \in U$ then $\mathcal{D}_L f(z) \ne \emptyset$. Take an arbitrary function $h_z \in \mathcal{D}_L f(z)$. Then $h_z(y) + f(z) \le f(y) = h(y)$. Moreover, since $z \in U'$ then $h(z) \le f(z) = h_z(z) + f(z)$. Consider the function $h'(t) = h_z(t) + f(z)$. Since H_L is closed under vertical shifts and $h_z \in H_L$ then $h' \in H_L$. So for these z, y and $h, h' \in H_L$ we have

$$z > y,\ \ h'(y) \le h(y),\ \ h(z) \le h'(z). \tag{4.30}$$

Note that, under our assumptions, H_L consists of continuous functions. Then, due to (4.30), a point $t_1 \in [y, z]$ exists such that $h'(t_1) = h(t_1)$.

Now suppose that $h(x) > h'(x)$. This means, in particular, that $h \ne h'$. It follows from (4.29) that $h'(t) \ne h(t)$ for any $t \ne t_1$. Then, by (4.30), either $h'(y) < h(y)$ or $h(z) < h'(z)$. If $h(z) < h'(z)$ then a point $t_2 \in (z, x)$ exists such that $h'(t_2) = h(t_2)$, which contradicts our assumption. Hence $h'(y) < h(y)$ and $y < t_1$. Take a positive number ε such that $\varepsilon < \min\{h(y) - h'(y), h(x) - h'(x)\}$ and consider the function $h_\varepsilon(t) = h'(t) + \varepsilon$. Then $h_\varepsilon \in H_L$. Moreover, the following inequalities hold

$$h_\varepsilon(t_1) > h(t_1), \quad h_\varepsilon(y) < h(y), \quad h_\varepsilon(x) < h(x). \tag{4.31}$$

Since $y < t_1 < x$ and the functions h_ε and h are continuous then, by (4.31), we can find two different points $a \in (y, t_1)$ and $b \in (t_1, x)$ such that $h_\varepsilon(a) = h(a)$ and $h_\varepsilon(b) = h(b)$. Then, by (4.29), $h_\varepsilon = h$, which contradicts (4.31).

So we conclude that $h(x) \leq h'(x)$. Since $h'(x) = h_z(x) + f(z)$ and $h_z \in \mathcal{D}_L f(z)$ then $h'(x) \leq f(x)$. Thus we have proved that $h(x) \leq f(x)$ for any $x > y$.

The same arguments show that $h(x) \leq f(x)$ for all $x < y$. □

Proposition 4.12 *Let L be a set of continuous functions defined on $\mathbb{R}$ such that (4.29) is valid for H_L. Assume also that for any sequence $\{h_i\} \subset H_L$ the following holds: if a function $h \in H_L$ and an interval $(a, b) \subset \mathbb{R}$ exist such that* $\lim_{i \to +\infty} h_i(x) = h(x)$ *for all $x \in (a, b)$ then* $\lim_{i \to +\infty} h_i(x) = h(x)$ *for all $x \in \mathbb{R}$. Then H_L has the strong globalization property.*

Proof: Let f be an H_L-convex function and $y \in \mathbb{R}$. Let $h \in H_L$ be an elementary function such that $h(y) = f(y)$ and $h(x) \leq f(x)$ in a neighbourhood U of the point y. We need to check that $h(x) \leq f(x)$ for all $x \in \mathbb{R}$. Here we show only that $h(x) \leq f(x)$ for all $x < y$. The proof of the inequality $h(x) \leq f(x)$ for $x > y$ is analogous.

First suppose that a sequence $\{y_i\} \subset \mathbb{R}$ exists such that $y_i < y \ \forall i$, $\lim_{i \to +\infty} y_i = y$ and $h(y_i) < f(y_i)$ for all i. Since f is H_L-convex then for each i a function $h_i \in \operatorname{supp}(f, H_L)$ exists such that $f(y_i) \geq h_i(y_i) > h(y_i)$. We have for each i

$$y_i < y, \quad h_i(y_i) > h(y_i), \quad h_i(y) \leq f(y) = h(y). \tag{4.32}$$

Since the functions h_i and h are continuous then we can find a point $t \in (y_i, y]$ such that $h_i(t) = h(t)$. Assume that $h_i(x) < h(x)$ for certain $x < y_i$. Then a point $t' \in (x, y_i)$ exists such that $h_i(t') = h(t')$, and therefore, by (4.29), $h_i = h$, which contradicts (4.32). Hence $h(x) \leq h_i(x) \leq f(x)$ for all $x < y_i$. Since $y_i \to y$ then $h(x) \leq f(x)$ for all $x < y$.

Now suppose that such a sequence $\{y_i\}$ does not exist. Since $h(x) \leq f(x)$ for all $x \in U$ then $h(x) = f(x)$ for all $x \in [a, y]$, where a is a point from the neighbourhood U and $a < y$. Assume that a point $y_0 < a$ exists such that $h(y_0) > f(y_0)$. We will get some contradictions for such a situation. So take a small enough $\varepsilon > 0$ such that $h(y_0) - f(y_0) > 2\varepsilon$. Let $\{\varepsilon_i\}$ be a decreasing sequence of positive numbers and $\lim_{i\to+\infty} \varepsilon_i = 0$, $\varepsilon_1 = \varepsilon$. Since f is H_L-convex and H_L is closed under shifts then a sequence $\{h_i\} \subset \text{supp}\,(f, H_L)$ exists such that $h_i(a) = f(a) - \varepsilon_i$ for each i. Consider two cases:

1.) Let a point $y' \in (a, y)$ and an index i exist such that $f(y') - h_i(y') > f(a) - h_i(a) = \varepsilon_i$. Choose a positive number δ such that $\min\{f(y') - h_i(y'), 2\varepsilon_i\} > \delta > f(a) - h_i(a) = \varepsilon_i$. Then consider the function $h'(x) = h_i(x) + \delta$. We have

$$h'(y') = h_i(y') + \delta < f(y') = h(y'), \qquad h'(a) = h_i(a) + \delta > f(a) = h(a),$$

$$h'(y_0) = h_i(y_0) + \delta < f(y_0) + 2\varepsilon_i \leq f(y_0) + 2\varepsilon < h(y_0).$$

Since $y_0 < a < y'$, these inequalities contradict (4.29) and the continuity of the elementary functions.

2.) Let $f(y') - h_i(y') \leq f(a) - h_i(a) = \varepsilon_i$ for all i and $y' \in (a, y)$. Since $f(y') - h_i(y') \geq 0$ then

$$\lim_{i\to+\infty} h_i(x) = f(x) = h(x) \qquad \text{for all } x \in (a, y).$$

Due to the assumptions of this proposition $\lim_{i\to+\infty} h_i(x) = h(x)$ for all $x \in X$. Hence $h(y_0) = \lim_{i\to+\infty} h_i(y_0) \leq f(y_0)$ because $h_i \in \text{supp}\,(f, H_L)$. But this contradicts the assumption $h(y_0) > f(y_0)$. □

Example 4.4 Let $a_0 > 0$ and $X = \mathbb{R}$. Let L be the set of all functions $l(x) = -a_0(x-a)^2$, where $a \in \mathbb{R}$. Then the conditions of Proposition 4.12 hold for H_L, and therefore H_L has the strong globalization property. But we do not have tools here for establishing necessary or sufficient conditions for global minimum of H_L-convex functions, since H_L does not contain any constant and each function $h(x) = -a_0(x-a)^2 - c$ has no global minimum over X.

So we should consider only examples where some elementary functions attain their global minimum. In the following example zero belongs to L. Hence we have necessary and sufficient condition for the global minimum.

Example 4.5 Let $l_1(x)$ and $l_2(x)$ be continuous strictly decreasing and strictly increasing functions respectively ($x \in \mathbb{R}$). Assume that L consists of all the functions $al_1(x)$, $al_2(x)$

with $a \geq 0$. It is easy to check that the set H_L satisfies the assumptions of Proposition 4.12. For example, we can take

$$l_1(x) = -e^x, \qquad l_2(x) = -e^{-x}.$$

We see that the set H_L here is closed under horizontal and vertical shifts. Moreover, the set of all H_L-convex functions is bigger than the set of all lower semicontinuous convex functions defined on $\mathbb{R}$. Indeed, let $t(x) = ax - c$ be an affine function. If $a = 0$ then $t \in H_L$. If $a > 0$ then for each $y \in \mathbb{R}$ we have that $(-ae^y e^{-x} + a + t(y)) \leq t(x)$ for any $x \in \mathbb{R}$, the function $h(x) = -ae^y e^{-x} + a + t(y)$ coincides with t at the point y and belongs to H_L. The same can be done for $a < 0$. Hence every affine function is H_L-convex.

Example 4.6 Let $l_1, \ldots, l_m, a_1, \ldots, a_m$ be strictly increasing continuous functions defined on $\mathbb{R}$. Let L denote the set of all functions $l^t(x) = a_1(t)l_1(x) + \cdots + a_m(t)l_m(x)$ with $t \in \mathbb{R}$. Check that (4.29) is valid for H_L. So let

$$h_1(x) = a_1(t_1)l_1(x) + \cdots + a_m(t_1)l_m(x) - c_1, \quad h_2(x) = a_1(t_2)l_1(x) + \cdots + a_m(t_2)l_m(x) - c_2.$$

Let $x \neq y$ and $h_1(x) = h_2(x)$, $h_1(y) = h_2(y)$. Then $(h_1(x) - h_1(y)) - (h_2(x) - h_2(y)) = 0$, that is

$$(a_1(t_1) - a_1(t_2))(l_1(x) - l_1(y)) + \cdots + (a_m(t_1) - a_m(t_2))(l_m(x) - l_m(y)) = 0. \quad (4.33)$$

Since $x \neq y$ and the functions l_i are strictly increasing then all the quantities $(l_i(x) - l_i(y))$ are not equal to zero and have the same sign. Since all a_i are strictly increasing then the equality (4.33) is possible only for $t_1 = t_2$. It follows from the equality $h_1(y) = h_2(y)$ that $c_1 = c_2$, hence $h_1 = h_2$.

Now let the sequences $\{t_k\}$, $\{c_k\}$ and an interval (a, b) be such that

$$\lim_{k \to +\infty} \left(\sum_{i=1}^{m} a_i(t_k)l_i(x) - c_k \right) = \sum_{i=1}^{m} a_i(t_0)l_i(x) - c_0 \qquad \text{for all } x \in (a, b).$$

Let $x, y \in (a, b)$ and $x > y$. Then

$$\lim_{k \to +\infty} \sum_{i=1}^{m} a_i(t_k)(l_i(x) - l_i(y)) = \sum_{i=1}^{m} a_i(t_0)(l_i(x) - l_i(y)),$$

consequently

$$\lim_{k \to +\infty} \sum_{i=1}^{m} (a_i(t_k) - a_i(t_0))(l_i(x) - l_i(y)) = 0.$$

Since all the quantities $(l_i(x) - l_i(y))$ are positive and all the functions a_i are continuous and strictly increasing then $\lim_{k\to+\infty} t_k = t_0$. The equality $\lim_{k\to+\infty} c_k = c_0$ is valid as well. Hence, due to Proposition 4.12, H_L has the strong globalization property.

Now consider the usual convex functions defined on a topological linear space.

Example 4.7 Let L be the set of all linear continuous functions defined on a topological linear space X. Let $\mathcal{L}$ be the set of all linear functions defined on $\mathbb{R}$. It follows from Example 4.6 (with $m = 1$, $a_1(t) = t$, $l_1(x) = x$) that the set $H_{\mathcal{L}}$ of all affine functions defined on $\mathbb{R}$ has the strong globalization property. Take two arbitrary points $x, y \in X$ and consider the function $\omega : \mathbb{R} \to X$ defined by $\omega(v) = vx + (1 - v)y$. Then $\omega(0) = y$ and $\omega(1) = x$. Moreover, ω is continuous and for any $h \in H_L$ the function $h'(v) = h(\omega(v))$ belongs to $H_{\mathcal{L}}$. Indeed, if $h(z) = l(z) + c \;\; \forall z \in X$, where $l \in L$ and $c \in \mathbb{R}$, then $h'(v) = l(vx + (1-v)y) + c = v(l(x) - l(y)) + (l(y) + c)$. Thus, by Proposition 4.10 (see also Remark 4.2), H_L has the strong globalization property.

Chapter 5

Separation properties via connectedness of topological convexity spaces

In this chapter we investigate separation of convex sets by elements of a subbase. In order to get required results, we apply a restriction on the choice of a subbase in terms of a special type of connectedness of topological convexity spaces. Among other results, we give a description of convex sets, which can be represented as the intersection of a subfamily of subbase (see Theorem 5.4). In particular, this allows to describe abstract convex functions and sets. We also obtain a description of the abstract convex hull of a finite collection of abstract convex sets.

5.1 Subbases for convexities and topologies

Recall that a collection $\mathcal{G}$ of subsets of a set X is called a convexity on X if

(1) $\emptyset, X \in \mathcal{G}$

(2) $\bigcap \mathcal{A} \in \mathcal{G}$ for every $\mathcal{A} \subset \mathcal{G}$

(3) $\bigcup \mathcal{A} \in \mathcal{G}$ whenever $\mathcal{A} \subset \mathcal{G}$ is a chain with respect to the inclusion.

Members of $\mathcal{G}$ are called convex sets and the pair $(X, \mathcal{G})$ is called a convexity space.

There are two main ways to introduce a convexity on a set. First, we can say that a set $G \subset X$ is convex if it satisfies certain properties. In this case we should require that the collection $\mathcal{G}$ of all such sets $G \subset X$ satisfies axioms (1)-(3). Another way is based on a notion of a subbase for convexity.

It is clear that the intersection of any family of convexities on a given set X is a convexity as well. This fact allows us to talk about subbases for convexities. A set $\mathcal{H} \subset \mathcal{G}$ is called a subbase for the convexity $\mathcal{G}$ if $\mathcal{G}$ is the intersection of all convexities, which contain $\mathcal{H}$ (we will say also that $\mathcal{G}$ is generated by $\mathcal{H}$). Note that topologies enjoy the same property: intersection of any family of topologies on a given set X is also a topology on X. So we can consider subbases for topologies as well.

Let $\mathcal{H}$ be a subbase for topology $\mathcal{T}$. Then open sets can be described in the following way. First we construct the collection $\mathcal{B}$ of all intersections of finite subfamilies of $\mathcal{H}$. Then $\mathcal{T}$ consists of the empty set, whole X and all unions of subfamilies of $\mathcal{B}$.

If A is a subset of X then its convex hull $\mathrm{conv}_{\mathcal{G}} A$ with respect to the convexity $\mathcal{G}$ is defined as follows:

$$\mathrm{conv}_{\mathcal{G}} A = \bigcap \{G \in \mathcal{G} : \ A \subset G\}.$$

For any points $x, y \in X$ denote by $[x, y]_{\mathcal{G}}$ their convex hull $\mathrm{conv}_{\mathcal{G}}\{x, y\}$. We will also use symbol $[A]^{<\omega}$ for the collection of all finite subsets of A.

Recall two results of axiomatic convexity. The following one is well known as the *finitary property.*

Proposition 5.1 *([60], p. 31, Proposition 2.1) Let $(X, \mathcal{G})$ be a convexity space. Then for every subset $A \subset X$*

$$\mathrm{conv}_{\mathcal{G}} A = \bigcup_{F \in [A]^{<\omega}} \mathrm{conv}_{\mathcal{G}} F. \tag{5.1}$$

Proposition 5.2 *([60], p. 10, Proposition 1.7.3) Let $(X, \mathcal{G})$ be a convexity space. If $\mathcal{H}$ is a subbase for the convexity $\mathcal{G}$ then for every finite subset $F \subset X$*

$$\mathrm{conv}_{\mathcal{G}} F = \bigcap \{H \in \mathcal{H} : \ F \subset H\}. \tag{5.2}$$

In the right-hand side of (5.2) it is assumed that the intersection over the empty set is equal to X. In other words, if $F \not\subset H$ for any $H \in \mathcal{H}$ then we set $\mathrm{conv}_{\mathcal{G}} F = X$.

It follows from the formulas (5.1) and (5.2) that for every $A \subset X$ its convex hull $\mathrm{conv}_{\mathcal{G}} A$ can be described via elements of $\mathcal{H}$ in the following way:

$$\mathrm{conv}_{\mathcal{G}} A = \bigcup_{F \in [A]^{<\omega}} \bigcap \{H \in \mathcal{H} : \ F \subset H\}. \tag{5.3}$$

Due to Proposition 5.1, a set $A \subset X$ is convex (belongs to convexity $\mathcal{G}$) if and only if

$$A = \bigcup_{F \in [A]^{<\omega}} \mathrm{conv}_{\mathcal{G}} F.$$

This means that $A \in \mathcal{G}$ whenever $\mathrm{conv}_{\mathcal{G}} F \subset A$ for all $F \in [A]^{<\omega}$.

The so-called N-ary convexities form one of the most important subclasses of convexities. Let N be a positive integer. Let $[A]^{\leq N}$ denote the collection of all subsets $F \subset A$, which contain no more than N points. A convexity $\mathcal{G}$ is called N-ary (or of arity N) (see [60]) if $A \in \mathcal{G}$ whenever $\mathrm{conv}_{\mathcal{G}} F \subset A$ for all $F \in [A]^{\leq N}$. Thus, if $\mathcal{G}$ is N-ary and the number N is not very large then we have a sufficiently simple description of convex sets.

In this chapter we are concentrated on subbases for N-ary convexities and separation of convex sets by the elements of a subbase.

Let $\mathcal{H}$ be a collection of subsets of a set X. In this chapter we use the following notations:

- $\mathcal{H}' = \{X \backslash H : \ H \in \mathcal{H}\}$ is the collection of all complements of sets $H \in \mathcal{H}$;
- $\mathcal{H}_x = \{H \in \mathcal{H} : \ x \in H\}$ for every $x \in X$;
- $\mathcal{H}^*$ is the collection of all sets $\mathcal{H}_x$ with $x \in X$;
- $\mathcal{H}^{*\prime} = \{\mathcal{H} \backslash \mathcal{H}_x : \ x \in X\}$ is the collection of all complements of sets $\mathcal{H}_x \in \mathcal{H}^*$.

We introduce the following convexities and topologies on X:

- $\mathcal{G}$ is the convexity on X generated by $\mathcal{H}$;
- $\bar{\mathcal{G}}$ is the convexity on X generated by the union $\mathcal{H} \cup \mathcal{H}'$;
- $\mathcal{T}_X$ is the topology on X generated by $\mathcal{H}$;
- $\mathcal{T}'_X$ is the topology on X generated by $\mathcal{H}'$.

Note that $\mathcal{H}^*$ and $\mathcal{H}^{*\prime}$ are collections of subsets of the set $\mathcal{H}$. Hence we can introduce the following:

- $\bar{\mathcal{G}}^*$ is the convexity on $\mathcal{H}$ generated by the union $\mathcal{H}^* \cup \mathcal{H}^{*\prime}$;
- $\mathcal{T}_{\mathcal{H}}$ is the topology on $\mathcal{H}$ generated by $\mathcal{H}^*$;
- $\mathcal{T}'_{\mathcal{H}}$ is the topology on $\mathcal{H}$ generated by $\mathcal{H}^{*\prime}$.

Here we use the same definition of a subbase as above. For example, $\bar{\mathcal{G}}^*$ is the intersection of all convexities, which contain $\mathcal{H}^* \cup \mathcal{H}^{*\prime}$.

We first give a description of convex hulls $\mathrm{conv}_{\bar{\mathcal{G}}}$ and $\mathrm{conv}_{\bar{\mathcal{G}}^*}$ of finite subsets of X and $\mathcal{H}$ respectively.

Proposition 5.3 *Let F be a finite subset of X. Then a point $x \in X$ belongs to* $\text{conv}_{\bar{\mathcal{G}}} F$ *if and only if for every set $H \in \mathcal{H}$ the following implications hold*

$$\begin{aligned} F \subset H \quad &\Longrightarrow \quad x \in H, \\ x \in H \quad &\Longrightarrow \quad F \cap H \neq \emptyset. \end{aligned}$$

Proof: Since $\mathcal{H} \cup \mathcal{H}'$ is a subbase for convexity $\bar{\mathcal{G}}$ and F contains a finite number of points of X then its convex hull $\text{conv}_{\bar{\mathcal{G}}} F$ can be described via elements of $\mathcal{H} \cup \mathcal{H}'$ (see Proposition 5.2):

$$\text{conv}_{\bar{\mathcal{G}}} F = \left(\bigcap \{H \in \mathcal{H} : \ F \subset H\}\right) \bigcap \left(\bigcap \{X \backslash H : \ H \in \mathcal{H}, \ F \subset (X \backslash H)\}\right).$$

So a point $x \in X$ belongs to $\text{conv}_{\bar{\mathcal{G}}} F$ if and only if for any $H \in \mathcal{H}$

$$(x \in H \quad \text{whenever } F \subset H) \quad \text{and} \quad (x \notin H \quad \text{whenever } F \cap H = \emptyset).$$

□

Proposition 5.4 *Let $\mathcal{E}$ be a finite subset of $\mathcal{H}$. Then*

$$\text{conv}_{\bar{\mathcal{G}}^*} \mathcal{E} = \left\{ H \in \mathcal{H} : \ \bigcap_{E \in \mathcal{E}} E \subset H \subset \bigcup_{E \in \mathcal{E}} E \right\}. \tag{5.4}$$

Proof: Since $\mathcal{E}$ is a finite subset of $\mathcal{H}$ and $\mathcal{H}^* \cup \mathcal{H}^{*\prime}$ is a subbase for $\bar{\mathcal{G}}^*$ then

$$\begin{aligned} \text{conv}_{\bar{\mathcal{G}}^*} \mathcal{E} &= \bigcap \{\mathcal{A} : \ \mathcal{A} \in \mathcal{H}^* \cup \mathcal{H}^{*\prime}, \ \mathcal{E} \subset \mathcal{A}\} \\ &= \left(\bigcap \{\mathcal{H}_x : \ x \in X, \ \mathcal{E} \subset \mathcal{H}_x\}\right) \bigcap \left(\bigcap \{\mathcal{H}'_x : \ x \in X, \ \mathcal{E} \subset \mathcal{H}'_x\}\right). \end{aligned} \tag{5.5}$$

We have

$$\mathcal{E} \subset \mathcal{H}_x \iff x \in \bigcap_{E \in \mathcal{E}} E, \qquad \mathcal{E} \subset \mathcal{H}'_x \iff x \notin \bigcup_{E \in \mathcal{E}} E.$$

Hence for every set $H \in \mathcal{H}$

$$\begin{aligned} H \in \bigcap \{\mathcal{H}_x : \ x \in X, \ \mathcal{E} \subset \mathcal{H}_x\} &\iff \bigcap_{E \in \mathcal{E}} E \subset H, \\ H \in \bigcap \{\mathcal{H}'_x : \ x \in X, \ \mathcal{E} \subset \mathcal{H}'_x\} &\iff H \subset \bigcup_{E \in \mathcal{E}} E. \end{aligned} \tag{5.6}$$

Thus, the required formula (5.4) follows from (5.5) and (5.6). □

Let Y be a topological space. We will need the following interpretation of continuity of a mapping $\omega : Y \to \mathcal{H}$ in cases, when $\mathcal{H}$ is equipped with one of the topologies: $\mathcal{T}_{\mathcal{H}}$ or $\mathcal{T}'_{\mathcal{H}}$.

Proposition 5.5 *Let $y_0 \in Y$ and $\omega : Y \rightarrow \mathcal{H}$ be a mapping. If $\mathcal{H}$ is equipped with the topology $\mathcal{T}_\mathcal{H}$ then ω is continuous at y_0 if and only if for each $x \in \omega(y_0)$ a neighbourhood U of y_0 exists such that $x \in \omega(y)$ for all $y \in U$. If $\mathcal{H}$ is equipped with the topology $\mathcal{T}'_\mathcal{H}$ then ω is continuous at y_0 if and only if for each $x \notin \omega(y_0)$ a neighbourhood U of y_0 exists such that $x \notin \omega(y)$ for all $y \in U$.*

Proof: Let $\mathcal{H}$ be equipped with the topology $\mathcal{T}_\mathcal{H}$, and assume that ω is continuous at the point y_0. Take a point $x \in \omega(y_0)$. Then $\omega(y_0) \in \mathcal{H}_x \in \mathcal{H}^* \subset \mathcal{T}_\mathcal{H}$. Hence the set $\mathcal{H}_x$ is a neighbourhood of $\omega(y_0)$. Since ω is continuous at y_0 then we can find a neighbourhood U of y_0 such that $\omega(y) \in \mathcal{H}_x$ for all $y \in U$. In other words, $x \in \omega(y)$ for all $y \in U$.

Conversely, assume that for each $x \in \omega(y_0)$ a neighbourhood U of y_0 exists such that $x \in \omega(y)$ for all $y \in U$. Let $\mathcal{S}$ be a neighbourhood of $\omega(y_0)$. Since the topology $\mathcal{T}_\mathcal{H}$ is generated by $\mathcal{H}^*$ then a finite collection $\{\mathcal{H}_{x_1}, \ldots, \mathcal{H}_{x_k}\}$ of elements of $\mathcal{H}^*$ exists such that $\omega(y_0) \in \bigcap_{i=1}^k \mathcal{H}_{x_i} \subset \mathcal{S}$. This implies $x_i \in \omega(y_0)$ for all $i = 1, \ldots, k$. By our assumption, there exist a neighbourhoods $U_1, \ldots, U_k$ of the point y_0 such that $x_i \in \omega(y)$ for all $y \in U_i$. Then the set $U = \bigcap_{i=1}^k U_i$ is a neighbourhood of y_0 and $\omega(y) \in \bigcap_{i=1}^k \mathcal{H}_{x_i} \subset \mathcal{S}$ for all $y \in U$. So the mapping $\omega : Y \rightarrow \mathcal{H}$ is continuous at y_0.

We omit the second part of the proof since all arguments are the same as in the first one.

□

A similar interpretation of continuity of a mapping $\omega : Y \rightarrow X$ is valid for the topologies $\mathcal{T}_X$ and $\mathcal{T}'_X$.

Proposition 5.6 *Let $y_0 \in Y$ and $\omega : Y \rightarrow X$ be a mapping. If X is equipped with the topology $\mathcal{T}_X$ ($\mathcal{T}'_X$) then ω is continuous at y_0 if and only if for each $H \in \mathcal{H}$ such that $\omega(y_0) \in H$ ($\omega(y_0) \notin H$) a neighbourhood U of y_0 exists such that $\omega(y) \in H$ ($\omega(y) \notin H$) for all $y \in U$.*

Proof: The proof is straightforward. □

Remark 5.1 Let X be equipped with a topology $\mathcal{T}$. Then all sets $H \in \mathcal{H}$ are open (closed) in the topology $\mathcal{T}$ if and only if $\mathcal{T}_X \subset \mathcal{T}$ ($\mathcal{T}'_X \subset \mathcal{T}$). As one can see from Proposition 5.5, it is natural to apply the topology $\mathcal{T}_\mathcal{H}$ in the case, when all sets $H \in \mathcal{H}$ are open. At the same time, the topology $\mathcal{T}'_\mathcal{H}$ on $\mathcal{H}$ can be suitable when all sets $H \in \mathcal{H}$ are closed.

5.2 Subbases for N-ary convexities

First we define N-connectedness of a topological space with respect to a convexity on this space.

Definition 5.1 Let $(X, \mathcal{T})$ be a topological space and $\mathcal{G}$ a convexity on X. We say that $(X, \mathcal{T})$ is N-connected with respect to $\mathcal{G}$ if N subsets $X_1, \ldots, X_N \subset X$ exist such that $X = X_1 \cup \cdots \cup X_N$ and for each $i = 1, \ldots, N$ the following condition holds: for any two points $x, y \in X_i$ a continuous mapping $\omega : [0,1] \to [x,y]_{\mathcal{G}}$ exists such that $\omega(0) = x$ and $\omega(1) = y$. We say that $(X, \mathcal{T})$ is connected with respect to $\mathcal{G}$ in the case, when $N = 1$.

It should be mentioned that the number N above is not minimal possible. In other words, if $(X, \mathcal{T})$ is N-connected with respect to $\mathcal{G}$ then it is also n-connected for any $n > N$.

Remark 5.2 It is easy to see that N-connectedness of a topological space with respect to a convexity on this space remains valid if the topology or the convexity decreases. This means the following. Assume that $(X, \mathcal{T})$ is N-connected with respect to $\mathcal{G}$. Let $\mathcal{T}_1$ and $\mathcal{G}_1$ be a topology and a convexity on X such that $\mathcal{T}_1 \subset \mathcal{T}$ and $\mathcal{G}_1 \subset \mathcal{G}$. Then $(X, \mathcal{T}_1)$ is N-connected with respect to $\mathcal{G}_1$ as well.

The following two theorems give important information about convex hulls of finite subsets of X.

Theorem 5.1 *Assume that one of the spaces $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Let F be a finite subset of X. Then for any points $x, y \in F$ and for each $z \in [x,y]_{\bar{\mathcal{G}}}$ the following holds:*

$$\mathrm{conv}_{\mathcal{G}} F = \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{x\})) \bigcup \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{y\})). \tag{5.7}$$

Proof: Since the set F is finite then

$$\mathrm{conv}_{\mathcal{G}} F = \bigcap \{H \in \mathcal{H} : \ F \subset H\}. \tag{5.8}$$

Let $F_1 = \{z\} \cup (F \backslash \{x\})$ and $F_2 = \{z\} \cup (F \backslash \{y\})$. Since $z \in [x,y]_{\bar{\mathcal{G}}} \subset [x,y]_{\mathcal{G}} \subset \mathrm{conv}_{\mathcal{G}} F$ then

$$\mathrm{conv}_{\mathcal{G}} F \supset \mathrm{conv}_{\mathcal{G}} F_1 \bigcup \mathrm{conv}_{\mathcal{G}} F_2.$$

Now we need to check the inclusion $\mathrm{conv}_{\mathcal{G}} F \subset \mathrm{conv}_{\mathcal{G}} F_1 \bigcup \mathrm{conv}_{\mathcal{G}} F_2$. We have

$$\begin{aligned}\mathrm{conv}_{\mathcal{G}} F_1 \bigcup \mathrm{conv}_{\mathcal{G}} F_2 &= \Big(\bigcap\{H_1 \in \mathcal{H}:\ F_1 \subset H_1\}\Big) \bigcup \Big(\bigcap\{H_2 \in \mathcal{H}:\ F_2 \subset H_2\}\Big)\\ &= \bigcap\{H_1 \cup H_2 :\ H_1, H_2 \in \mathcal{H},\ F_1 \subset H_1,\ F_2 \subset H_2\}.\end{aligned}$$

If either $\{H_1 :\ H_1 \in \mathcal{H},\ F_1 \subset H_1\} = \emptyset$ or $\{H_2 :\ H_2 \in \mathcal{H},\ F_2 \subset H_2\} = \emptyset$ then $\mathrm{conv}_{\mathcal{G}} F_1 \bigcup \mathrm{conv}_{\mathcal{G}} F_2 = X$ and the inclusion $\mathrm{conv}_{\mathcal{G}} F \subset \mathrm{conv}_{\mathcal{G}} F_1 \bigcup \mathrm{conv}_{\mathcal{G}} F_2$ becomes trivial.

So we need to show that $\mathrm{conv}_{\mathcal{G}} F \subset H_1 \cup H_2$ whenever $F_1 \subset H_1$ and $F_2 \subset H_2$ $(H_1, H_2 \in \mathcal{H})$. Consider such sets H_1 and H_2. Since the space $\mathcal{H}$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$ then a continuous mapping $\omega : [0,1] \to [H_1, H_2]_{\bar{\mathcal{G}}^*}$ exists such that $\omega(0) = H_1$ and $\omega(1) = H_2$. If either $H_1 \supset F$ or $H_2 \supset F$ then, due to (5.8), $\mathrm{conv}_{\mathcal{G}} F \subset H_1 \cup H_2$. Assume that $H_1 \not\supset F$ and $H_2 \not\supset F$. Then $x \notin H_1$ and $y \notin H_2$.

Let $\mathcal{H}$ be equipped with the topology $\mathcal{T}_{\mathcal{H}}$. Since $y \in H_1 = \omega(0)$ and the mapping ω is continuous then, by Proposition 5.5, a positive number ε exists such that $y \in \omega(t)$ for all $t < \varepsilon$.

Let $\bar{t} = \sup\{\varepsilon \in (0,1) :\ y \in \omega(t)\ \forall t \in [0,\varepsilon)\}$. Then $y \notin \omega(\bar{t})$. Indeed, if $\bar{t} = 1$ then $y \notin \omega(\bar{t}) = H_2$. If $\bar{t} \in (0,1)$ and $y \in \omega(\bar{t})$ then a positive number δ exists such that $y \in \omega(t)$ for all $t \in (\bar{t} - \delta, \bar{t} + \delta)$, which contradicts the definition of $\bar{t}$. Thus the point y does not belong to $\omega(\bar{t})$.

Since $\omega(\bar{t}) \in [H_1, H_2]_{\bar{\mathcal{G}}^*}$ then it follows from the formula (5.4) that $H_1 \cap H_2 \subset \omega(\bar{t})$. This implies $z \in \omega(\bar{t}) \in \mathcal{H}$. Since $z \in [x,y]_{\mathcal{G}}$ then, due to the Proposition 5.3, $\{x,y\} \cap \omega(\bar{t}) \neq \emptyset$. We proved before that $y \notin \omega(\bar{t})$. Hence $x \in \omega(\bar{t})$. Since $\bar{t} > 0$ then a positive number δ exists such that $x \in \omega(t)$ for all $t \in (\bar{t} - \delta, \bar{t})$. Take an arbitrary $t_0 \in (\bar{t} - \delta, \bar{t})$ and consider the set $H_0 = \omega(t_0) \in \mathcal{H}$. Then $x, y \in H_0$.

Since $H_0 \in [H_1, H_2]_{\bar{\mathcal{G}}^*}$ then, due to (5.4), $H_1 \cap H_2 \subset H_0 \subset H_1 \cup H_2$. Since $x, y \in H_0$ and $F \backslash \{x,y\} \subset H_1 \cap H_2 \subset H_0$ then $F \subset H_0$. This implies $\mathrm{conv}_{\mathcal{G}} F \subset H_0 \subset H_1 \cup H_2$.

Now assume that $\mathcal{H}$ is equipped with the topology $\mathcal{T}'_{\mathcal{H}}$. Since $x \notin H_1 = \omega(0)$ then, by Proposition 5.5, a positive number ε exists such that $x \notin \omega(t)$ for all $t < \varepsilon$. Let $\bar{t} = \sup\{\varepsilon \in (0,1) :\ x \notin \omega(t)\ \forall t \in [0,\varepsilon)\}$. Then $x \in \omega(\bar{t})$. Indeed, if $\bar{t} = 1$ then $x \in \omega(\bar{t}) = H_2$. If $\bar{t} \in (0,1)$ and $x \notin \omega(\bar{t})$ then a positive number δ exists such that $x \notin \omega(t)$ for all $t \in (\bar{t} - \delta, \bar{t} + \delta)$, which contradicts the definition of $\bar{t}$. Since $\{x,y\} \cap \omega(t) \neq \emptyset$ for any $t \in [0,1]$ and $x \notin \omega(t)$ for all $t < \bar{t}$ then $y \in \omega(t)$ whenever $t < \bar{t}$. Due to continuity of

ω, the point y belongs to $\omega(\bar{t})$. Hence $F \subset \omega(\bar{t})$, and therefore $\text{conv}_{\mathcal{G}} F \subset \omega(\bar{t}) \subset H_1 \cup H_2$. □

Theorem 5.2 *Assume that one of the spaces $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Let X be equipped with a topology $\mathcal{T}$ such that*

$$\bigcap_{i=1}^{\infty} \text{conv}_{\mathcal{G}}(S \cup \{z_i\}) = \text{conv}_{\mathcal{G}} S \quad \textit{whenever} \quad S \in [X]^{<\omega} \textit{ and } z_i \textit{ converges to a point in } S. \tag{5.9}$$

Let F be a finite subset of X and $x, y \in F$. Let $\omega : [0,1] \to [x,y]_{\bar{\mathcal{G}}}$ be a continuous mapping such that $\omega(0) = x$ and $\omega(1) = y$. Then

$$\text{conv}_{\mathcal{G}} F = \bigcup_{t \in [0,1]} \text{conv}_{\mathcal{G}}(\{\omega(t)\} \cup (F \backslash \{x,y\})). \tag{5.10}$$

Proof: Inclusion

$$\bigcup_{t \in [0,1]} \text{conv}_{\mathcal{G}}(\{\omega(t)\} \cup (F \backslash \{x,y\}) \subset \text{conv}_{\mathcal{G}} F$$

is obvious because $\omega(t) \in [x,y]_{\bar{\mathcal{G}}} \subset [x,y]_{\mathcal{G}} \subset \text{conv}_{\mathcal{G}} F$ for each $t \in [0,1]$.

Let $a \in \text{conv}_{\mathcal{G}} F$. We need to find a number $\bar{t} \in [0,1]$ such that

$$a \in \text{conv}_{\mathcal{G}}(\{\omega(\bar{t})\} \cup (F \backslash \{x,y\})). \tag{5.11}$$

It follows from the Theorem 5.1 that

$$a \in \text{conv}_{\mathcal{G}}(\{\omega(t)\} \cup (F \backslash \{x\})) \bigcup \text{conv}_{\mathcal{G}}(\{\omega(t)\} \cup (F \backslash \{y\})) \quad \text{for each} \quad t \in [0,1]. \tag{5.12}$$

Define a sequence of segments $[c_i, d_i] \subset [0,1]$. Let $c_1 = 0$ and $d_1 = 1$. We set:

$$c_{i+1} = \begin{cases} (c_i + d_i)/2, & \text{if } a \in \text{conv}_{\mathcal{G}}(\{\omega((c_i + d_i)/2)\} \cup (F \backslash \{x\})) \\ c_i, & \text{otherwise} \end{cases} \tag{5.13}$$

$$d_{i+1} = \begin{cases} (c_i + d_i)/2, & \text{if } a \in \text{conv}_{\mathcal{G}}(\{\omega((c_i + d_i)/2)\} \cup (F \backslash \{y\})) \\ d_i, & \text{otherwise} \end{cases} \tag{5.14}$$

Then $[c_{i+1}, d_{i+1}] \subset [c_i, d_i]$ for any integer $i \geq 1$. Moreover, due to (5.12), $(d_{i+1} - c_{i+1}) \leq (d_i - c_i)/2$. Hence there exists a unique point $\bar{t} \in [0,1]$ such that $\{\bar{t}\} = \bigcap_i [c_i, d_i]$. Since the mapping ω is continuous on $[0,1]$ then $\lim_{i \to \infty} \omega(c_i) = \lim_{i \to \infty} \omega(d_i) = \omega(\bar{t})$.

It is clear that for any $i \geq 1$

$$a \in \text{conv}_{\mathcal{G}}(\{\omega(c_i)\} \cup (F \backslash \{x\})) \bigcap \text{conv}_{\mathcal{G}}(\{\omega(d_i)\} \cup (F \backslash \{y\})). \tag{5.15}$$

Indeed, (5.15) is obvious for $i = 1$. Then, by induction, inclusion (5.15) follows from the formulas (5.13) and (5.14). Since $\omega(\bar{t}) = \lim_{i\to\infty} \omega(c_i) = \lim_{i\to\infty} \omega(d_i)$ then, due to (5.9),

$$\begin{aligned}\bigcap_{i\geq 1} \mathrm{conv}_{\mathcal{G}}(\{\omega(c_i)\} \cup (F\backslash\{x\})) &\subset \bigcap_{i\geq 1} \mathrm{conv}_{\mathcal{G}}(\{\omega(c_i)\} \cup \{\omega(\bar{t})\} \cup (F\backslash\{x\}))\\ &= \mathrm{conv}_{\mathcal{G}}(\{\omega(\bar{t})\} \cup (F\backslash\{x\})),\end{aligned}$$

$$\begin{aligned}\bigcap_{i\geq 1} \mathrm{conv}_{\mathcal{G}}(\{\omega(d_i)\} \cup (F\backslash\{y\})) &\subset \bigcap_{i\geq 1} \mathrm{conv}_{\mathcal{G}}(\{\omega(d_i)\} \cup \{\omega(\bar{t})\} \cup (F\backslash\{y\}))\\ &= \mathrm{conv}_{\mathcal{G}}(\{\omega(\bar{t})\} \cup (F\backslash\{y\})).\end{aligned}$$

Hence

$$a \in \mathrm{conv}_{\mathcal{G}}(\{\omega(\bar{t})\} \cup (F\backslash\{x\})) \bigcap \mathrm{conv}_{\mathcal{G}}(\{\omega(\bar{t})\} \cup (F\backslash\{y\})). \tag{5.16}$$

Check the inclusion (5.11). Since $\{\omega(\bar{t})\} \cup (F\backslash\{x, y\})$ is a finite subset of X then it is sufficient to show that $a \in H$ whenever $H \in \mathcal{H}$ and $\{\omega(\bar{t})\} \cup (F\backslash\{x, y\}) \subset H$. So let $\{\omega(\bar{t})\} \cup (F\backslash\{x, y\}) \subset H$. Since $\omega(\bar{t}) \in [x, y]_{\bar{\mathcal{G}}}$ and $\omega(\bar{t}) \in H$ then $\{x, y\} \cap H \neq \emptyset$ (see Proposition 5.3). If $x \in H$ then $\{\omega(\bar{t})\} \cup (F\backslash\{y\}) \subset H$, and therefore $\mathrm{conv}_{\mathcal{G}}(\{\omega(\bar{t})\} \cup (F\backslash\{y\})) \subset H$. If $y \in H$ then $\mathrm{conv}_{\mathcal{G}}(\{\omega(\bar{t})\} \cup (F\backslash\{x\})) \subset H$. In any event the point a belongs to H (see (5.16)). □

Now we can formulate the main result of this section.

Theorem 5.3 *Assume that one of the spaces $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Let $\mathcal{T}$ be a topology on X such that (5.9) holds true. Let $N \geq 2$. Assume that the space $(X, \mathcal{T})$ is N-connected with respect to the convexity $\bar{\mathcal{G}}$. Then the convexity $\mathcal{G}$ is of arity N.*

Proof: Let A be a subset of X such that $\mathrm{conv}_{\mathcal{G}} F \subset A$ whenever $F \in [A]^{\leq N}$. We need to check that A belongs to the convexity $\mathcal{G}$. Due to Proposition 5.1, we have

$$A \in \mathcal{G} \iff \mathrm{conv}_{\mathcal{G}} A \subset A \iff (\mathrm{conv}_{\mathcal{G}} F \subset A \quad \text{for each} \quad F \in [A]^{<\omega}).$$

Let F be a finite subset of A. If $F \in [A]^{\leq N}$ then the inclusion $\mathrm{conv}_{\mathcal{G}} F \subset A$ is valid.

Now assume that F consists of n different points of A and $n > N$. We need to show that for each point $a \in \mathrm{conv}_{\mathcal{G}} F$ a set $F_{n-1} \in [A]^{\leq(n-1)}$ exists such that $a \in \mathrm{conv}_{\mathcal{G}} F_{n-1}$. Then, by induction, we can find a set $F_N \in [A]^{\leq N}$ such that $a \in \mathrm{conv}_{\mathcal{G}} F_N$. Therefore $\mathrm{conv}_{\mathcal{G}} F \subset A$.

So take a point $a \in \text{conv}_{\mathcal{G}}F$. Since the space $(X, \mathcal{T})$ is N-connected with respect to $\bar{\mathcal{G}}$ and F contains more than N points of $A \subset X$ then a points $x, y \in F$ $(x \neq y)$ and a continuous function $\omega : [0,1] \to [x,y]_{\bar{\mathcal{G}}}$ exist such that $\omega(0) = x$ and $\omega(1) = y$. Then, by Theorem 5.2, there is a number $\bar{t} \in [0,1]$ with $a \in \text{conv}_{\mathcal{G}}(\{\omega(\bar{t})\} \cup (F \setminus \{x,y\}))$.

Consider the set $F_{n-1} = \{\omega(\bar{t})\} \cup (F \setminus \{x,y\})$. Since $\omega(\bar{t}) \in [x,y]_{\bar{\mathcal{G}}} \subset [x,y]_{\mathcal{G}}$ and $\{x,y\} \in [A]^{\leq 2} \subset [A]^{\leq N}$ then $\omega(\bar{t}) \in A$. This implies $F_{n-1} \in [A]^{\leq (n-1)}$. □

Below we will show that the estimate of arity number in Theorem 5.3 is sharp (see Example 5.3).

Remark 5.3 Recall that a convexity space $(X, \mathcal{G})$ is called join-hull commutative (see [25]) provided for each finite set $F \subset X$ and for each $x \in X$ we have

$$\text{conv}_{\mathcal{G}}(F \cup \{x\}) = \bigcup_{y \in \text{conv}_{\mathcal{G}}F} [x,y]_{\mathcal{G}}.$$

Assume that one of the spaces $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Assume also that $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$, where $\mathcal{T}$ is a topology on X, which enjoys (5.9). Then the convexity space $(X, \mathcal{G})$ is join-hull commutative.

Indeed, let F be a finite subset of X and $x \in X$. Take an arbitrary $a \in \text{conv}_{\mathcal{G}}(F \cup \{x\})$. Let $y_1, y_2 \in F$. Since $(X, \mathcal{T})$ is one-connected with respect to $\bar{\mathcal{G}}$ then a continuous mapping $\omega : [0,1] \to [y_1, y_2]_{\bar{\mathcal{G}}}$ exists such that $\omega(0) = y_1$ and $\omega(1) = y_2$. Theorem 5.2 implies that

$$\text{conv}_{\mathcal{G}}(F \cup \{x\}) = \bigcup_{t \in [0,1]} \text{conv}_{\mathcal{G}}(\{\omega(t)\} \cup \{x\} \cup (F \setminus \{y_1, y_2\})).$$

Hence there is a point $y_0 \in \text{conv}_{\mathcal{G}}F$ such that $a \in \text{conv}_{\mathcal{G}}(\{y_0\} \cup \{x\} \cup (F \setminus \{y_1, y_2\}))$. Since F is finite then, by induction, we can find a point $y \in \text{conv}_{\mathcal{G}}F$ with $a \in [x,y]_{\mathcal{G}}$.

Since condition (5.9) is not easy for verification, we present a simpler condition, which implies (5.9).

Proposition 5.7 *Let $\mathcal{T}$ be a topology on X such that*

$$\bigcap \{E \in \mathcal{H} : \ S \subset \text{int}\, E\} \subset H \quad \textit{for each} \quad H \in \mathcal{H} \quad \textit{and} \quad S \in [H]^{<\omega}, \tag{5.17}$$

where $\text{int}\, E$ *is the interior of* E *in topology* $\mathcal{T}$*. Then condition (5.9) holds true for* $\mathcal{T}$*. In particular, (5.9) holds for any topology* $\mathcal{T}$ *such that all sets* $H \in \mathcal{H}$ *are open in* $\mathcal{T}$*.*

Proof: Let S be a finite subset of X and $z_i \xrightarrow{\mathcal{T}} z \in S$. We need to check the inclusion

$$\bigcap_{i=1}^{\infty} \mathrm{conv}_{\mathcal{G}}(S \cup \{z_i\}) \subset \mathrm{conv}_{\mathcal{G}} S.$$

It follows from (5.17) that

$$\bigcap\{E \in \mathcal{H}: \ S \subset \mathrm{int}\, E\} \subset \bigcap\{H \in \mathcal{H}: \ S \subset H\} = \mathrm{conv}_{\mathcal{G}} S.$$

Let $E \in \mathcal{H}$ be such that $S \subset \mathrm{int}\, E$. Since $z_i \xrightarrow{\mathcal{T}} z \in S$ then a number k exists such that $S \cup \{z_i\} \subset E$ for all $i > k$, and therefore $\mathrm{conv}_{\mathcal{G}}(S \cup \{z_i\}) \subset E$ for all $i > k$. Thus we conclude that

$$\bigcap_{i=1}^{\infty} \mathrm{conv}_{\mathcal{G}}(S \cup \{z_i\}) \subset \bigcap\{E \in \mathcal{H}: \ S \subset \mathrm{int}\, E\} \subset \mathrm{conv}_{\mathcal{G}} S.$$

If all sets $E \in \mathcal{H}$ are open in topology $\mathcal{T}$ (i.e. $\mathrm{int}\, E = E$) then (5.17) obviously holds. □

Corollary 5.1 *Assume that $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}_X)$ is N-connected with respect to $\bar{\mathcal{G}}$, where $N \geq 2$. Then the convexity $\mathcal{G}$ is of arity N.*

Proof: It follows from Proposition 5.7 that condition (5.9) is valid for the topology $\mathcal{T} = \mathcal{T}_X$ because all sets $H \in \mathcal{H}$ are open in $\mathcal{T}_X$. Then, by Theorem 5.3, $\mathcal{G}$ is of arity N. □

Unfortunately, condition (5.9) does not necessarily hold for the topology $\mathcal{T} = \mathcal{T}'_X$. To show this consider a simple example.

Example 5.1 Let $X = \mathbb{R}$. Let $\mathcal{H}$ be the collection of all segments $[c, +\infty)$ with $c \in \mathbb{R}$. Since $\mathcal{H}'$ is a subbase for the topology $\mathcal{T}'_X$ then $z_i \xrightarrow{\mathcal{T}'_X} z \in \mathbb{R}$ if and only if $z_i \notin H$ for all sufficiently large i whenever $H \in \mathcal{H}$ and $z \notin H$. In other words,

$$z_i \xrightarrow{\mathcal{T}'_X} z \iff (z_i < c \ \text{ for large } i \text{ whenever } z < c) \iff \limsup_{i \to \infty} z_i \leq z.$$

For example, $0 \xrightarrow{\mathcal{T}'_X} 1$. At the same time, $\mathrm{conv}_{\mathcal{G}}\{0\} = [0, +\infty) \not\subset \mathrm{conv}_{\mathcal{G}}\{1\} = [1, +\infty)$. So, condition (5.9) does not hold in this case.

However, it can be convenient to use a topology $\mathcal{T}$ on X, which possesses (5.9) and contains the topology $\mathcal{T}'_X$.

5.3 Some particular cases

In order to check the connectedness of a topological space with respect to a convexity on this space we need to describe convex hulls of each two elements of this space, or at least to indicate some points of these convex hulls. Here we consider two particular cases, where the sets $H \in \mathcal{H}$ are expressed via real-valued functions, and get some formulas for the convex hulls in terms of these functions.

Subbases of level sets $S_0(l) = \{x \in X: \ l(x) \leq 0\}$

Let L be a family of real-valued functions defined on a set X. Consider the collection $\mathcal{H}$ of all sets $S_0(l) = \{x \in X: \ l(x) \leq 0\}$, where $l \in L$.

Let $x_1, x_2 \in X$. Then, by Proposition 5.3, the set $[x_1, x_2]_{\bar{\mathcal{G}}}$ consists of all points $x \in X$ such that for any $l \in L$ the following implications hold

$$\begin{array}{lcl} \max\{l(x_1), l(x_2)\} \leq 0 & \Longrightarrow & l(x) \leq 0, \\ l(x) \leq 0 & \Longrightarrow & \min\{l(x_1), l(x_2)\} \leq 0. \end{array}$$

In particular, $[x_1, x_2]_{\bar{\mathcal{G}}}$ contains all points $x \in X$ such that

$$\min\{l(x_1), l(x_2)\} \leq l(x) \leq \max\{l(x_1), l(x_2)\} \quad \forall l \in L.$$

Let $l_1, l_2 \in L$. Due to Proposition 5.4, we have

$$[S_0(l_1), S_0(l_2)]_{\bar{\mathcal{G}}^*} = \{S_0(l) \in \mathcal{H}: \ S_0(l_1) \cap S_0(l_2) \subset S_0(l) \subset S_0(l_1) \cup S_0(l_2)\}.$$

In other words, the set $[S_0(l_1), S_0(l_2)]_{\bar{\mathcal{G}}^*}$ consists of all $S_0(l)$ such that for any $x \in X$ the following implications hold

$$\begin{array}{lcl} \max\{l_1(x), l_2(x)\} \leq 0 & \Longrightarrow & l(x) \leq 0, \\ l(x) \leq 0 & \Longrightarrow & \min\{l_1(x), l_2(x)\} \leq 0. \end{array}$$

In particular, $[S_0(l_1), S_0(l_2)]_{\bar{\mathcal{G}}^*}$ contains all $S_0(l)$ such that

$$\min\{l_1(x), l_2(x)\} \leq l(x) \leq \max\{l_1(x), l_2(x)\} \quad \forall x \in X. \tag{5.18}$$

Proposition 5.8 *Assume that L is closed under vertical shifts (this means that for each $l \in L$ and $c \in \mathbb{R}$ the function $h(x) = l(x) + c$ belongs to L). Let $x_1, x_2 \in X$. Then*

$$[x_1, x_2]_{\bar{\mathcal{G}}} = \{x \in X: \ \min\{l(x_1), l(x_2)\} \leq l(x) \leq \max\{l(x_1), l(x_2)\} \quad \forall l \in L\}. \tag{5.19}$$

If, moreover, X is equipped with a topology $\mathcal{T}$ such that

$$\{x \in X : \; l(x) < 0\} \subset \operatorname{int} S_0(l) \qquad \forall l \in L \tag{5.20}$$

then condition (5.9) is valid for $\mathcal{T}$.

Proof: First we check the equality (5.19). Let $x \in [x_1, x_2]_{\bar{\mathcal{G}}}$. Take an arbitrary $l \in L$ and consider the number $c = \max\{l(x_1), l(x_2)\}$. Since L is closed under vertical shifts then the function $h(z) = l(z) - c$ belongs to L. It is easy to see that $x_1, x_2 \in S_0(h)$. Since $x \in [x_1, x_2]_{\bar{\mathcal{G}}}$ then $x \in S_0(h)$, therefore $h(x) = l(x) - \max\{l(x_1), l(x_2)\} \leq 0$.

In order to check the inequality $\min\{l(x_1), l(x_2)\} \leq l(x)$ consider the function $h(z) = l(z) - l(x)$. Since $S_0(h) \in \mathcal{H}$ and $x \in S_0(h)$ then, by Proposition 5.3, $\{x_1, x_2\} \cap S_0(h) \neq \emptyset$. Hence either $h(x_1) \leq 0$ or $h(x_2) \leq 0$. This means that $\min\{l(x_1), l(x_2)\} \leq l(x)$.

Let $\mathcal{T}$ be a topology on X, which enjoys (5.20). We show that (5.17) holds for $\mathcal{T}$. Then, by Proposition 5.7, condition (5.9) holds as well. So let $S_0(l) \in \mathcal{H}$, where $l \in L$. Take a positive number ε and consider the function $h_\varepsilon(x) = l(x) - \varepsilon$. Since L is closed under vertical shifts then $h_\varepsilon \in L$ and, by (5.20),

$$\{x \in X : \; l(x) < \varepsilon\} = \{x \in X : \; h_\varepsilon(x) < 0\} \subset \operatorname{int} S_0(h_\varepsilon).$$

Hence $S_0(l) \subset \operatorname{int} S_0(h_\varepsilon)$ for any positive ε. We have

$$\bigcap \{H \in \mathcal{H} : \; S_0(l) \subset \operatorname{int} H\} \subset \bigcap_{\varepsilon > 0} S_0(h_\varepsilon) = \bigcap_{\varepsilon > 0} \{x \in X : \; l(x) \leq \varepsilon\} = S_0(l).$$

Thus we have proved even a stronger fact then that in (5.17). □

Consider the classical convex case.

Proposition 5.9 *Let $(X, \mathcal{T})$ be a topological linear space and L be the set of all continuous affine functions $l : X \to \mathbb{R}$. Let $\mathcal{H}$ be the collection of all level sets $S_0(l) = \{x \in X : l(x) \leq 0\}$, where $l \in L$ (in other words, $\mathcal{H}$ consists of the empty set, whole X and all closed half-spaces of X). Then the convexity $\mathcal{G}$ generated by $\mathcal{H}$ is of arity* 2.

Proof: Let us prove that the space $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$, and $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$.

Let $l_1, l_2 \in L$. Since l_1 and l_2 are continuous and affine then for every $\alpha \in [0, 1]$ the function $l(x) = (1 - \alpha) l_1(x) + \alpha l_2(x)$ is also continuous and affine. Consider the mapping $\omega : [0, 1] \to \mathcal{H}$ defined by

$$\omega(\alpha) = \{x \in X : \; (1 - \alpha) l_1(x) + \alpha l_2(x) \leq 0\}. \tag{5.21}$$

Then $\omega(0) = S_0(l_1)$ and $\omega(1) = S_0(l_2)$. Moreover, since for any $\alpha \in [0,1]$ and $x \in X$

$$\min\{l_1(x), l_2(x)\} \le (1-\alpha)l_1(x) + \alpha l_2(x) \le \max\{l_1(x), l_2(x)\}$$

then, due to (5.18), $\omega(\alpha) \in [S_0(l_1), S_0(l_2)]_{\bar{\mathcal{G}}^*}$ for all $\alpha \in [0,1]$.

Assume that $\mathcal{H}$ is equipped with the topology $\mathcal{T}'_{\mathcal{H}}$. We need to check that ω is continuous on $[0,1]$. Take an arbitrary $\alpha_0 \in [0,1]$ and $x \notin \omega(\alpha_0)$. Then $(1-\alpha_0)l_1(x) + \alpha_0 l_2(x) > 0$ and we can find a sufficiently small number $\varepsilon > 0$ such that $(1-\alpha)l_1(x) + \alpha l_2(x) > 0$ for all $\alpha \in [0,1] \cap (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$. This implies continuity of ω (see Proposition 5.5). Thus the space $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$.

Let $x_1, x_2 \in X$. Consider the mapping $\omega : [0,1] \to X$ defined by

$$\omega(\alpha) = (1-\alpha)x_1 + \alpha x_2. \tag{5.22}$$

Then $\omega(0) = x_1$ and $\omega(1) = x_2$. Since $l((1-\alpha)x_1 + \alpha x_2) = (1-\alpha)l(x_1) + \alpha l(x_2)$ whenever l is affine then

$$\min\{l(x_1), l(x_2)\} \le l(\omega(\alpha)) \le \max\{l(x_1), l(x_2)\} \quad \forall\, l \in L.$$

Hence, by (5.19), $\omega(\alpha) \in [x_1, x_2]_{\bar{\mathcal{G}}}$ for any $\alpha \in [0,1]$. Since $(X, \mathcal{T})$ is a topological linear space then ω is continuous on $[0,1]$. So the space $(X, \mathcal{T})$ is connected with respect to the convexity $\bar{\mathcal{G}}$.

Note that L is closed under vertical shifts, and all functions $l \in L$ are continuous in topology $\mathcal{T}$ (in particular, they enjoy (5.20)). Then, by Proposition 5.8, condition (5.9) is valid for $\mathcal{T}$. At last, it follows from Theorem 5.3 that the convexity $\mathcal{G}$ generated by the collection of all closed half-spaces of X is of arity 2. □

Now consider the case of affine functions defined on an arbitrary linear space.

Example 5.2 Let X be a linear space and L be the set of all affine functions $l : X \to \mathbb{R}$. As in Proposition 5.9, let $\mathcal{H}$ be the collection of all level sets $S_0(l) = \{x \in X : \ l(x) \le 0\}$ $(l \in L)$ and $\mathcal{G}$ the convexity on X generated by $\mathcal{H}$. Then the space $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Indeed, for any $l_1, l_2 \in L$ the function (5.21) enjoys all required properties. For $x_1, x_2 \in X$ we only need to check that the function (5.22) is continuous on $[0,1]$ if X is equipped with the topology $\mathcal{T}'_X$. Let $\alpha_0 \in [0,1]$ and $l \in L$ be such that $l(\omega(\alpha_0)) > 0$. Since l is affine then $(1-\alpha_0)l(x_1) + \alpha_0 l(x_2) = l(\omega(\alpha_0)) > 0$, hence a positive number ε exists such that

$l(\omega(\alpha)) = (1-\alpha)l(x_1) + \alpha l(x_2) > 0$ for all $\alpha \in [0,1] \cap (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$. This implies continuity of ω (see Proposition 5.6).

Since for any affine function l the function $-l$ is also affine then the set

$$\{x \in X :\ l(x) < 0\} = \{x \in X :\ -l(x) > 0\} = X \backslash S_0(-l) \in \mathcal{H}'$$

is open in the topology $\mathcal{T}'_X$ for all $l \in L$. Hence (5.20) holds true for $\mathcal{T} = \mathcal{T}'_X$, and, by Proposition 5.8, condition (5.9) is also valid. Thus, due to Theorem 5.3, the convexity $\mathcal{G}$ on X is of arity 2.

The following example demonstrates that the estimate of arity number in Theorem 5.3 is sharp.

Example 5.3 Let $N \geq 2$. Choose arbitrary vectors $e^1, \ldots, e^N \in \mathbb{R}^{N-1}$ such that every $(N-1)$ of them are linearly independent and zero is a convex combination of all e^i (for example, we can take the usual orthogonal base of $\mathbb{R}^{N-1}$ and vector $(-1, \ldots, -1)$). Let $X = X_1 \cup \cdots \cup X_N$, where $X_i = \{ae^i :\ \ a \geq 0\}$ for any $i = 1, \ldots, N$. Let L be the set of all affine functions defined on $\mathbb{R}^{N-1}$ and $\mathcal{H}$ be the collection of all level sets $S_0(l) = \{x \in X :\ l(x) \leq 0\}$, $l \in L$. Then $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is N-connected with respect to $\bar{\mathcal{G}}$. Indeed, we can use the same functions $\omega : [0,1] \to \mathcal{H}$ and $\omega : [0,1] \to X$ as in the proof of Proposition 5.9 and Example 5.2. For each $i = 1, \ldots, N$ we have: $\omega(\alpha) = (1-\alpha)x_1 + \alpha x_2$ belongs to $[x_1, x_2]_{\bar{\mathcal{G}}}$ for any $x_1, x_2 \in X_i$ and $\alpha \in [0,1]$. Condition (5.20) is also valid for the topology $\mathcal{T} = \mathcal{T}'_X$. Then, by Proposition 5.8 and Theorem 5.3, convexity $\mathcal{G}$ on $X = X_1 \cup \cdots \cup X_N$ generated by $\mathcal{H}$ is of arity N. Now we show that $\mathcal{G}$ is not of arity $N-1$. Consider the set $A = \{e^1, \ldots, e^N\}$. Then, due to our choice of vectors e^i, $\text{conv}_{\mathcal{G}} F = X \cap \text{conv} F = F \subset A$ for any $F \in [A]^{\leq N-1}$ (here $\text{conv} F$ is the classical convex hull of F in $\mathbb{R}^{N-1}$). However, the set A does not belong to $\mathcal{G}$ since $0 \in \text{conv}_{\mathcal{G}} A$.

Subbases of epigraphs $\text{epi}\, l = \{(y, c) \in Y \times \mathbb{R} :\ l(y) \leq c\}$

Let L be a set of real-valued functions defined on a set Y. Let $X = Y \times \mathbb{R}$. Consider the collection $\mathcal{H}$ of all epigraphs $\text{epi}\, l = \{(y, c) \in Y \times \mathbb{R} :\ l(y) \leq c\}$, where $l \in L$.

Remark 5.4 Just note that each epigraph $\text{epi}\, l$ can be represented as a level set $S_0(h)$ of the function $h(y, c) = l(y) - c$ defined on one higher dimension space $Y \times \mathbb{R}$.

Let $(y_1, c_1), (y_2, c_2) \in Y \times \mathbb{R}$. Then the set $[(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$ consists of all points $(y, c) \in Y \times \mathbb{R}$ such that for any $l \in L$ the following implications hold

$$\begin{aligned} \max\{l(y_1) - c_1, l(y_2) - c_2\} \leq 0 \quad &\Longrightarrow \quad l(y) \leq c, \\ l(y) \leq c \quad &\Longrightarrow \quad \min\{l(y_1) - c_1, l(y_2) - c_2\} \leq 0. \end{aligned}$$

In particular, $[(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$ contains all (y, c) such that

$$\min\{l(y_1) - c_1, l(y_2) - c_2\} \leq l(y) - c \leq \max\{l(y_1) - c_1, l(y_2) - c_2\} \quad \forall l \in L. \quad (5.23)$$

At the same time, we have a very easy description of the set $[\operatorname{epi} l_1, \operatorname{epi} l_2]_{\bar{\mathcal{G}}^*}$ for every $l_1, l_2 \in L$:

$$\begin{aligned} [\operatorname{epi} l_1, \operatorname{epi} l_2]_{\bar{\mathcal{G}}^*} &= \{\operatorname{epi} l : \ l \in L, \ (\operatorname{epi} l_1 \cap \operatorname{epi} l_2) \subset \operatorname{epi} l \subset (\operatorname{epi} l_1 \cup \operatorname{epi} l_2)\} \\ &= \{\operatorname{epi} l : \ l \in L, \ \min_i l_i(y) \leq l(y) \leq \max_i l_i(y) \ \forall y \in Y\}. \end{aligned}$$

Proposition 5.10 *Assume that L is closed under vertical shifts. Let $(y_1, c_1), (y_2, c_2) \in Y \times \mathbb{R}$. Then a point $(y, c) \in Y \times \mathbb{R}$ belongs to $[(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$ if and only if (5.23) holds.*

In other words,

$$[(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}} = \{(y, c) : \ f(y) \leq c \leq g(y)\},$$

where the functions f and g are defined by

$$f(x) = \sup_{l \in L}(l(x) - \max\{l(y_1) - c_1, l(y_2) - c_2\}),$$

$$g(x) = \inf_{l \in L}(l(x) - \min\{l(y_1) - c_1, l(y_2) - c_2\}).$$

Let, moreover, $Y \times \mathbb{R}$ be equipped with a topology $\mathcal{T}$ such that

$$\{(y, c) : \ l(y) < c\} \subset \operatorname{int} \operatorname{epi} l \qquad \forall l \in L.$$

Then condition (5.9) is valid for $\mathcal{T}$.

Proof: Let $(y, c) \in [(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$. Take an arbitrary $l \in L$ and consider the following functions defined on Y:

$$h(z) = l(z) - \max\{l(y_1) - c_1, l(y_2) - c_2\}, \qquad h'(z) = l(z) - l(y) + c.$$

Since L is closed under vertical shifts then $h, h' \in L$. We have

$$h(y_1) \leq c_1, \quad h(y_2) \leq c_2, \quad h'(y) = c.$$

Since $(y, c) \in [(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$ then $h(y) \leq c$ and $\min\{h'(y_1)-c_1, h'(y_2)-c_2\} \leq 0$. This means that $l(y)-c \leq \max\{l(y_1)-c_1, l(y_2)-c_2\}$ and $\min\{l(y_1)-c_1, l(y_2)-c_2\} \leq l(y)-c$.

Let $\mathcal{T}$ be a topology on $Y \times \mathbb{R}$ such as in the statement of proposition. Take an arbitrary $l \in L$. Then for any positive ε the function $h_\varepsilon(z) = l(z) - \varepsilon$ belongs to L and

$$\operatorname{epi} l \subset \{(y, c) : \; h_\varepsilon(y) < c\} \subset \operatorname{int} \operatorname{epi} h_\varepsilon.$$

Thus we have

$$\bigcap\{H \in \mathcal{H} : \; \operatorname{epi} l \subset \operatorname{int} H\} \subset \bigcap_{\varepsilon>0} \operatorname{epi} h_\varepsilon = \bigcap_{\varepsilon>0}\{(y, c) \in Y \times \mathbb{R} : \; l(y) \leq c + \varepsilon\} = \operatorname{epi} l.$$

Proposition 5.7 implies that condition (5.9) is valid for $\mathcal{T}$. □

5.4 Separation theorems

In this section we investigate separation of convex sets by elements of a subbase. In general, we have the following weak version of the separation property, which follows directly from (5.3): if $\mathcal{H}$ is a subbase for convexity $\mathcal{G}$ and $G \in \mathcal{G}$ then for every $g \notin G$ and for every finite subset $F \subset G$ a set $H \in \mathcal{H}$ exists such that $F \subset H$ and $g \notin H$.

If the convexity $\mathcal{G}$ is N-ary and the number N is not very large (as a rule, we are interested in the cases, when $N = 2$) then we have a sufficiently simple description of convex sets. At the same time, if $\mathcal{G}$ is generated by $\mathcal{H}$ then our weak separation property can be applied for every convex set G.

For example, if $(\mathcal{H}, \mathcal{T}_\mathcal{H})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}_X)$ is N-connected with respect to $\bar{\mathcal{G}}$ ($N \geq 2$), then for any $G \subset X$ the following conditions are equivalent:

1.) for every $g \notin G$ and $F \in [G]^{<\omega}$ a set $H \in \mathcal{H}$ exists with $F \subset H$ and $g \notin H$,

2.) $\operatorname{conv}_\mathcal{G}\{g_1, \ldots, g_N\} \subset G$ for all $g_1, \ldots, g_N \in G$.

Indeed, by Corollary 5.1, the convexity $\mathcal{G}$ is of arity N. Then conditions 1.) and 2.) are equivalent because 1.) means that $G \in \mathcal{G}$.

In order to have a stronger version of the separation property, some additional assumptions are required. Namely, we need some topological properties of convex sets.

Here we consider the case, when all sets $H \in \mathcal{H}$ are closed. More precisely, we assume that X is equipped with the topology $\mathcal{T}'_X$ and $\mathcal{H}$ is equipped with the topology $\mathcal{T}'_{\mathcal{H}}$. Moreover, one-connectedness of $(X, \mathcal{T}'_X)$ with respect to $\bar{\mathcal{G}}$ (as well as one-connectedness of $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ with respect to $\bar{\mathcal{G}}^*$) will be essential to get next results.

Let us begin with the following lemma.

Lemma 5.1 *Assume that $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Let $x, y \in X$ and $H_1, H_2 \in \mathcal{H}$.*

1.) If $x \in H_1$, $y \in H_2$ and $[x,y]_{\bar{\mathcal{G}}} \subset H_1 \cup H_2$ then $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 \neq \emptyset$.

2.) If $x \notin H_1$, $y \notin H_2$ and $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 = \emptyset$ then $[x,y]_{\bar{\mathcal{G}}} \not\subset H_1 \cup H_2$.

Proof: Since $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$ then a continuous mapping $\omega : [0,1] \to [x,y]_{\bar{\mathcal{G}}}$ exists such that $\omega(0) = x$ and $\omega(1) = y$. We will use the interpretation of continuity of ω presented in Proposition 5.6.

1.) Assume that $x \in H_1$, $y \in H_2$ and $[x,y]_{\bar{\mathcal{G}}} \subset H_1 \cup H_2$. Consider the number

$$\bar{t} = \sup\{\varepsilon \geq 0 : \ \omega(t) \in H_1 \ \forall\, t \leq \varepsilon\}.$$

Then $\omega(\bar{t}) \in H_1$ (otherwise, if $\omega(\bar{t}) \notin H_1$ then $\omega(t) \notin H_1$ for all t from a neighbourhood of $\bar{t}$, which contradicts the definition of $\bar{t}$). We need to check that $\omega(\bar{t}) \in H_2$ (then $\omega(\bar{t}) \in [x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2$). Assume it is not true, that is $\omega(\bar{t}) \notin H_2$. Since $\omega(1) \in H_2$ then $\bar{t} < 1$. By definition of $\bar{t}$, for any $\delta > 0$ a number $t \in (\bar{t}, \bar{t} + \delta)$ exists such that $\omega(t) \notin H_1$. At the same time, since $\omega(\bar{t}) \notin H_2$ then $\omega(t) \notin H_2$ for all t from a neighbourhood of $\bar{t}$. Hence a number $t \in (\bar{t}, 1)$ exists such that $\omega(t) \notin H_1 \cup H_2$, which contradicts the inclusion $[x,y]_{\bar{\mathcal{G}}} \subset H_1 \cup H_2$.

2.) Now assume that $x \notin H_1$, $y \notin H_2$ and $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 = \emptyset$. If either $x \notin H_2$ or $y \notin H_1$ then $[x,y]_{\bar{\mathcal{G}}} \not\subset H_1 \cup H_2$, because either $x \notin H_1 \cup H_2$ or $y \notin H_1 \cup H_2$. So, assume that $x \in H_2$ and $y \in H_1$. Define the following number

$$\bar{t} = \sup\{\varepsilon \geq 0 : \ \omega(t) \in H_2 \ \forall\, t \leq \varepsilon\}.$$

Since ω is continuous and $\omega(0) \in H_2$ then $\omega(\bar{t}) \in H_2$. If $\bar{t} = 1$ then $y = \omega(\bar{t}) \in H_2$, which contradicts the assumption $y \notin H_2$. Hence $\bar{t} < 1$ and, by definition of $\bar{t}$, for each $\delta > 0$ a number $t \in (\bar{t}, \bar{t} + \delta)$ exists such that $\omega(t) \notin H_2$. On the other hand, since $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 = \emptyset$ and $\omega(\bar{t}) \in H_2$ then $\omega(\bar{t}) \notin H_1$, therefore $\omega(t) \notin H_1$ for all t

from a neighbourhood of $\bar{t}$. Thus, for each $\delta > 0$ a number $t \in (\bar{t}, \bar{t} + \delta)$ exists such that $\omega(t) \notin H_1 \cup H_2$. This implies that $[x, y]_{\bar{\mathcal{G}}} \not\subset H_1 \cup H_2$. □

For any $G \subset X$ let $\mathrm{co}_{(X,\mathcal{H})}G$ denote the set defined by:

$$\mathrm{co}_{(X,\mathcal{H})}G = \bigcap\{H \in \mathcal{H} : \ G \subset H\}. \tag{5.24}$$

If $G \not\subset H$ for all $H \in \mathcal{H}$ then we set $\mathrm{co}_{(X,\mathcal{H})}G = X$.

The following separation theorem gives a description of sets $G \subset X$, which can be represented as the intersection of a subfamily of $\mathcal{H}$.

Theorem 5.4 *Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Let $G \subset X$.*

1.) The following conditions are equivalent:

(i) *For every $g \in X \backslash G$ a set $H \in \mathcal{H}$ exists such that $G \subset H$ and $g \notin H$.*

(ii) *G is closed in topology $\mathcal{T}'_X$ and convex in convexity $\mathcal{G}$.*

(iii) *G is closed in topology $\mathcal{T}'_X$ and $[x, y]_{\mathcal{G}} \subset G$ for all $x, y \in G$.*

(iv) *G is closed in topology $\mathcal{T}'_X$ and $[x, y]_{\bar{\mathcal{G}}} \subset G$ for all $x, y \in G$.*

2.) If $[x, y]_{\bar{\mathcal{G}}} \subset G$ for all $x, y \in G$ then

$$\mathrm{co}_{(X,\mathcal{H})}G = \mathrm{cl}_{\mathcal{T}'_X}G, \tag{5.25}$$

where $\mathrm{cl}_{\mathcal{T}'_X}G$ is the closure of G in topology $\mathcal{T}'_X$.

Proof: We first prove (5.25). Since each set $H \in \mathcal{H}$ is closed in topology $\mathcal{T}'_X$ then $\mathrm{cl}_{\mathcal{T}'_X}G \subset \mathrm{co}_{(X,\mathcal{H})}G$. In order to prove the inclusion $\mathrm{co}_{(X,\mathcal{H})}G \subset \mathrm{cl}_{\mathcal{T}'_X}G$ we will check that $g \notin \mathrm{co}_{(X,\mathcal{H})}G$ whenever $g \notin \mathrm{cl}_{\mathcal{T}'_X}G$. Note that $g \notin \mathrm{co}_{(X,\mathcal{H})}G$ if and only if a set $H \in \mathcal{H}$ exists with $g \notin H$ and $G \subset H$. So let $g \notin \mathrm{cl}_{\mathcal{T}'_X}G$.

Since the topology $\mathcal{T}'_X$ is generated by $\mathcal{H}' = \{X \backslash H : \ H \in \mathcal{H}\}$ then a finite collection $\{H_1, \dots, H_n\} \subset \mathcal{H}$ exists such that $g \in \bigcap_i (X \backslash H_i) \subset X \backslash G$. In other words, $g \notin \bigcup_i H_i$ and $G \subset \bigcup_i H_i$. If $n = 1$ then the set H_1 possesses required properties: $G \subset H_1$ and $g \notin H_1$. Let $n > 1$.

We will prove that a set $H_0 \in \mathrm{conv}_{\bar{\mathcal{G}}^*}\{H_1, H_2\}$ exists such that $G \subset \bigcup_{i \geq 3} H_i \cup H_0$. Then, by induction, there is a set $H \in \mathrm{conv}_{\bar{\mathcal{G}}^*}\{H_1, \dots, H_n\}$ with $G \subset H$. Moreover, $g \notin H$ because $g \notin \bigcup_i H_i$ and $H \subset \bigcup_i H_i$ (see Proposition 5.4), hence $g \notin \mathrm{co}_{(X,\mathcal{H})}G$.

Since $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ then a continuous mapping $\omega : [0,1] \to [H_1, H_2]_{\bar{\mathcal{G}}^*}$ exists such that $\omega(0) = H_1$ and $\omega(1) = H_2$. Consider the number

$$\bar{t} = \sup \left\{ \varepsilon \in [0,1] : \; G \subset \bigcup_{i \geq 2} H_i \cup \omega(t) \;\; \forall t \leq \varepsilon \right\}.$$

Prove the inclusion $G \subset \bigcup_{i\geq 2} H_i \cup \omega(\bar{t})$. If $\bar{t} = 0$ then this inclusion is trivial. If $\bar{t} > 0$ and $G \not\subset \bigcup_{i\geq 2} H_i \cup \omega(\bar{t})$ then a point $y \in G$ exists such that $y \notin \bigcup_{i\geq 2} H_i$ and $y \notin \omega(\bar{t})$. Hence, by Proposition 5.5, $y \notin \omega(t)$ for all $t \in (\bar{t} - \delta, \bar{t})$ with sufficiently small $\delta > 0$. This implies that $G \not\subset \bigcup_{i\geq 2} H_i \cup \omega(t)$ for all $t \in (\bar{t} - \delta, \bar{t})$, which contradicts definition of $\bar{t}$. Thus, we conclude that $G \subset \bigcup_{i\geq 2} H_i \cup \omega(\bar{t})$.

Let $H_0 = \omega(\bar{t})$. If $\bar{t} = 1$ then $H_0 = H_2$, therefore $G \subset \bigcup_{i\geq 2} H_i \cup \omega(\bar{t}) = \bigcup_{i\geq 3} H_i \cup H_0$. Assume that $\bar{t} < 1$.

We need to check that $G \subset \bigcup_{i\geq 3} H_i \cup H_0$. Assume it is not true. Since $G \subset \bigcup_{i\geq 2} H_i \cup \omega(\bar{t})$ then a point $y \in G \cap H_2$ exists such that $y \notin \omega(\bar{t})$ and $y \notin H_i$ for all $i \geq 3$. Since $\bar{t} < 1$ and $y \notin \omega(\bar{t})$ then, due to Proposition 5.5, $y \notin \omega(t)$ for all $t \in (\bar{t}, \bar{t} + \delta)$ with sufficiently small $\delta > 0$. At the same time, by definition of $\bar{t}$, $G \not\subset \bigcup_{i\geq 2} H_i \cup \omega(t)$ for some $t \in (\bar{t}, \bar{t} + \delta)$. Hence a number $t \in (\bar{t}, \bar{t} + \delta)$ and a point $x \in G$ exist such that $x \notin \omega(t)$ and $x \notin H_i$ for all $i \geq 2$. This implies, in particular, that $x \in H_1$, because $G \subset \bigcup_{i\geq 1} H_i$. We have

$$x, y \in G, \quad x, y \notin \omega(t), \quad x, y \notin H_i \;\; \forall i \geq 3, \quad x \in H_1, \quad y \in H_2.$$

It follows from Proposition 5.3 that $[x, y]_{\bar{\mathcal{G}}} \cap \omega(t) = \emptyset$ and $[x, y]_{\bar{\mathcal{G}}} \cap H_i = \emptyset$ for any $i \geq 3$. On the other hand, $[x, y]_{\bar{\mathcal{G}}} \subset G \subset \bigcup_{i\geq 1} H_i$. Therefore $[x, y]_{\bar{\mathcal{G}}} \subset H_1 \cup H_2$. Since $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$ then, by Lemma 5.1, $[x, y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 \neq \emptyset$. Since $\omega(t) \in [H_1, H_2]_{\bar{\mathcal{G}}^*}$ then $H_1 \cap H_2 \subset \omega(t)$. Hence $[x, y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 \cap \omega(t) \neq \emptyset$, which contradicts the equality $[x, y]_{\bar{\mathcal{G}}} \cap \omega(t) = \emptyset$.

Thus, (5.25) is valid. Now prove the equivalence of (i)–(iv). Clearly condition (i) means that $G = \bigcap \{H \in \mathcal{H} : \; G \subset H\} = \mathrm{co}_{(X,\mathcal{H})} G$.

(i) $\Longrightarrow$ (ii) Since all sets $H \in \mathcal{H}$ are closed in topology $\mathcal{T}'_X$ and convex in convexity $\mathcal{G}$ then condition (ii) holds true.

(ii) $\Longrightarrow$ (iii) It is obvious because $[x, y]_{\mathcal{G}} \subset G$ for all $x, y \in G$ whenever $G \in \mathcal{G}$.

(iii) $\Longrightarrow$ (iv) It is sufficient to note that $[x, y]_{\bar{\mathcal{G}}} \subset [x, y]_{\mathcal{G}}$ for all $x, y \in X$.

(iv) $\Longrightarrow$ (i) Since G is closed in topology $\mathcal{T}'_X$ then $\mathrm{cl}_{\mathcal{T}'_X} G = G$. Moreover, by (5.25), $\mathrm{co}_{(X,\mathcal{H})} G = \mathrm{cl}_{\mathcal{T}'_X} G$ because $[x, y]_{\bar{\mathcal{G}}} \subset G$ for all $x, y \in G$. Hence $\mathrm{co}_{(X,\mathcal{H})} G = G$. □

Remark 5.5 If $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$ then for any $G \subset X$

$$\mathrm{co}_{(X,\mathcal{H})} G = \mathrm{cl}_{\mathcal{T}'_X} \mathrm{conv}_{\mathcal{G}} G. \tag{5.26}$$

Indeed, equality $\mathrm{co}_{(X,\mathcal{H})}\mathrm{conv}_{\mathcal{G}} G = \mathrm{cl}_{\mathcal{T}'_X}\mathrm{conv}_{\mathcal{G}} G$ follows from (5.25) because $[x,y]_{\bar{\mathcal{G}}} \subset \mathrm{conv}_{\mathcal{G}} G$ for all $x, y \in \mathrm{conv}_{\mathcal{G}} G$. At the same time, since for every $H \in \mathcal{H}$ inclusions $G \subset H$ and $\mathrm{conv}_{\mathcal{G}} G \subset H$ are equivalent then $\mathrm{co}_{(X,\mathcal{H})}\mathrm{conv}_{\mathcal{G}} G = \mathrm{co}_{(X,\mathcal{H})} G$.

The next theorem states that, under some conditions, two convex sets, one of which is closed in $\mathcal{T}'_X$ and the other one is compact in $\mathcal{T}'_X$, can be separated by a set $H \in \mathcal{H}$.

Theorem 5.5 *Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Assume also that $[H_1, H_2]_{\bar{\mathcal{G}}^*} = [H_1, H]_{\bar{\mathcal{G}}^*} \cup [H, H_2]_{\bar{\mathcal{G}}^*}$ whenever $H_1, H_2 \in \mathcal{H}$ and $H \in [H_1, H_2]_{\bar{\mathcal{G}}^*}$. Let $G, K \subset X$ be such that $G \cap K = \emptyset$. Assume that $[x,y]_{\bar{\mathcal{G}}} \subset G \ \forall x, y \in G$ and $[x,y]_{\bar{\mathcal{G}}} \subset K \ \forall x, y \in K$. If G is closed in topology $\mathcal{T}'_X$ and K is compact in $\mathcal{T}'_X$ then a set $H \in \mathcal{H}$ exists with $G \subset H$ and $K \subset X \backslash H$.*

Proof: Since G and K are disjoint then, by Theorem 5.4, for every $g \in K$ a set $H \in \mathcal{H}$ exists such that $G \subset H$ and $g \notin H$. Hence $K \subset \bigcup\{X \backslash H \in \mathcal{H}' : \ G \subset H\}$. Since K is compact in topology $\mathcal{T}'_X$ and all sets $X \backslash H \in \mathcal{H}'$ are open in $\mathcal{T}'_X$ then there exists a finite collection $\{H_1, \ldots, H_n\} \subset \mathcal{H}$ such that $G \subset \bigcap_i H_i$ and

$$K \subset \bigcup_{i \geq 1} (X \backslash H_i). \tag{5.27}$$

Let $n > 1$.

We need to find a set $H_0 \in \mathrm{conv}_{\bar{\mathcal{G}}^*}\{H_1, H_2\}$, which satisfies inclusion

$$K \subset \bigcup_{i \geq 3} (X \backslash H_i) \cup (X \backslash H_0). \tag{5.28}$$

Due to Proposition 5.4, $H_1 \cap H_2 \subset H_0$ whenever $H_0 \in \mathrm{conv}_{\bar{\mathcal{G}}^*}\{H_1, H_2\}$, therefore $G \subset \bigcap_{i \geq 3} H_i \cap H_0$. Then, by induction, there is a set $H \in \mathcal{H}$ with $G \subset H$ and $K \subset X \backslash H$.

First, note that for any $H \in \mathrm{conv}_{\bar{\mathcal{G}}^*}\{H_1, H_2\}$ we have:

$$\text{either} \quad K \subset \bigcup_{i \geq 3} (X \backslash H_i) \cup (X \backslash H) \cup (X \backslash H_1) \quad \text{or} \quad K \subset \bigcup_{i \geq 3} (X \backslash H_i) \cup (X \backslash H) \cup (X \backslash H_2). \tag{5.29}$$

Assume it is not true. Since $K \subset \bigcup_{i\geq 1}(X\backslash H_i)$ then there exist $x, y \in K$ such that

$$x, y \in H, \quad x, y \in H_i \ \forall i \geq 3, \quad x \in X\backslash H_1, \quad y \in X\backslash H_2.$$

Hence $[x,y]_{\bar{\mathcal{G}}} \subset H$ and $[x,y]_{\bar{\mathcal{G}}} \subset H_i$ for any $i \geq 3$. Since $[x,y]_{\bar{\mathcal{G}}} \subset K \subset \bigcup_{i\geq 1}(X\backslash H_i)$ then $[x,y]_{\bar{\mathcal{G}}} \subset (X\backslash H_1) \cup (X\backslash H_2)$. This means that $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 = \emptyset$ and, due to Lemma 5.1, $[x,y]_{\bar{\mathcal{G}}} \not\subset H_1 \cup H_2$, which contradicts $[x,y]_{\bar{\mathcal{G}}} \subset H$, because $H \subset H_1 \cup H_2$. Thus, (5.29) holds true for every $H \in \text{conv}_{\bar{\mathcal{G}}^*}\{H_1, H_2\}$.

Since $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ then a continuous mapping $\omega : [0,1] \to [H_1, H_2]_{\bar{\mathcal{G}}^*}$ exists such that $\omega(0) = H_1$ and $\omega(1) = H_2$. Consider the following number

$$\bar{t} = \sup\left\{ t \in [0,1] : \ K \subset \bigcup_{i\geq 2}(X\backslash H_i) \cup (X\backslash \omega(t)) \right\}.$$

Prove that

$$K \subset \bigcup_{i\geq 3}(X\backslash H_i) \cup (X\backslash \omega(\bar{t})) \cup (X\backslash H_1). \tag{5.30}$$

If $\bar{t} = 1$ then this inclusion is trivial. Assume it is not true for $\bar{t} < 1$. Then, by (5.29), $K \subset \bigcup_{i\geq 2}(X\backslash H_i) \cup (X\backslash \omega(\bar{t}))$. Since ω is continuous then $x \in X\backslash \omega(t)$ whenever $x \in X\backslash \omega(\bar{t})$ and t is close to $\bar{t}$. Therefore $X\backslash \omega(\bar{t}) \subset \bigcup_{t\in(\bar{t},1)} X\backslash \omega(t)$ and

$$K \subset \bigcup_{i\geq 2,\ t\in(\bar{t},1)} (X\backslash H_i) \cup (X\backslash \omega(t)).$$

Since K is compact then a finite collection $T \subset (\bar{t}, 1)$ exists such that

$$K \subset \bigcup_{i\geq 2,\ t\in T} (X\backslash H_i) \cup (X\backslash \omega(t)). \tag{5.31}$$

Check that for every $t', t'' \in T$

$$\text{either} \quad K \subset \bigcup_{i\geq 2,\ t\in T\backslash t'} (X\backslash H_i) \cup (X\backslash \omega(t)) \quad \text{or} \quad K \subset \bigcup_{i\geq 2,\ t\in T\backslash t''} (X\backslash H_i) \cup (X\backslash \omega(t)).$$

By conditions of theorem, $[H_1, H_2]_{\bar{\mathcal{G}}^*} = [H_1, \omega(t')]_{\bar{\mathcal{G}}^*} \cup [\omega(t'), H_2]_{\bar{\mathcal{G}}^*}$. Hence $\omega(t'')$ belongs to the union $[H_1, \omega(t')]_{\bar{\mathcal{G}}^*} \cup [\omega(t'), H_2]_{\bar{\mathcal{G}}^*}$.

If $\omega(t'') \in [H_1, \omega(t')]_{\bar{\mathcal{G}}^*}$ then $\omega(t'') \subset H_1 \cup \omega(t')$. This means that $(X\backslash H_1) \cap (X\backslash \omega(t')) \subset X\backslash \omega(t'')$, and, in view of inclusions (5.27) and (5.31), we get

$$K \subset \bigcup_{i\geq 2,\ t\in T\backslash t'} (X\backslash H_i) \cup (X\backslash \omega(t)).$$

If $\omega(t'') \in [\omega(t'), H_2]_{\mathcal{G}^*}$ then $\omega(t') \cap H_2 \subset \omega(t'')$, which is equivalent to $X \backslash \omega(t'') \subset (X \backslash \omega(t')) \cup (X \backslash H_2)$. This and (5.31) give the inclusion

$$K \subset \bigcup_{i \geq 2,\ t \in T \backslash t''} (X \backslash H_i) \cup (X \backslash \omega(t)).$$

Since T is finite then, by induction, a number $t \in T$ exists such that

$$K \subset \bigcup_{i \geq 2} (X \backslash H_i) \cup (X \backslash \omega(t)).$$

This contradicts definition of $\bar{t}$ because $t > \bar{t}$. So, we conclude that (5.30) is valid.

By definition of $\bar{t}$, a sequence $\{t_j\} \subset [0, \bar{t}]$ exists such that $t_j \to \bar{t}$ and for any j

$$K \subset \bigcup_{i \geq 2} (X \backslash H_i) \cup (X \backslash \omega(t_j)). \tag{5.32}$$

Since ω is continuous and $t_j \to \bar{t}$ then if follows from (5.30) that

$$K \subset \bigcup_{i \geq 3,\ j \geq 1} (X \backslash H_i) \cup (X \backslash \omega(t_j)) \cup (X \backslash H_1).$$

Since K is compact then a finite collection $T = \{t_{j_1}, \ldots, t_{j_m}\}$ exists such that

$$K \subset \bigcup_{i \geq 3,\ t \in T} (X \backslash H_i) \cup (X \backslash \omega(t)) \cup (X \backslash H_1).$$

By repeating reasoning after formula (5.31) (with H_1 instead of H_2), we deduce that an integer j exists, which enjoys the inclusion

$$K \subset \bigcup_{i \geq 3} (X \backslash H_i) \cup (X \backslash \omega(t_j)) \cup (X \backslash H_1). \tag{5.33}$$

At last, inclusions (5.32) and (5.33) imply that

$$K \subset \bigcup_{i \geq 3} (X \backslash H_i) \cup (X \backslash \omega(t_j)),$$

because $(X \backslash H_1) \cap (X \backslash H_2) \subset (X \backslash \omega(t_j))$. Thus, (5.28) is valid for $H_0 = \omega(t_j)$. □

5.5 Convex hull of a finite union of convex sets

Here we give a description of the convex hull $\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i$, where $\{G_1, \ldots, G_n\}$ is a finite collection of convex sets. Note that the set $\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i$ can be described via convex hulls of unions of two convex sets, because $\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i = G^n$, where $G^1 = G_1$ and $G^i = \mathrm{conv}_{\mathcal{G}}(G^{i-1} \cup G_i)$ for $i = 2, \ldots, n$.

Proposition 5.11 *Assume that one of the spaces $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Assume also that $(X, \mathcal{T})$ is N-connected with respect to $\bar{\mathcal{G}}$, where $\mathcal{T}$ is a topology on X, which enjoys (5.9). Then for any $G_1, \ldots, G_n \in \mathcal{G}$*

$$\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i = \bigcup_{F_i \in [G_i]^{\leq N}} \mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} F_i. \tag{5.34}$$

Proof: If $F_i \in [G_i]^{\leq N}$ for all i then $\bigcup_i F_i \subset \bigcup_i G_i$, hence $\mathrm{conv}_{\mathcal{G}} \bigcup_i F_i \subset \mathrm{conv}_{\mathcal{G}} \bigcup_i G_i$.

Now we need to check the inclusion

$$\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i \subset \bigcup_{F_i \in [G_i]^{\leq N}} \mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} F_i.$$

Let $a \in \mathrm{conv}_{\mathcal{G}} \bigcup_i G_i$. Then, by Proposition 5.1, there exists a finite subset $F \subset \bigcup_i G_i$ with $a \in \mathrm{conv}_{\mathcal{G}} F$.

If $F \cap G_i \in [G_i]^{\leq N}$ for all $i \leq n$ then $a \in \mathrm{conv}_{\mathcal{G}} \bigcup_i F_i$, where $F_i = F \cap G_i \in [G_i]^{\leq N}$.

Let $F \cap G_i \notin [G_i]^{\leq N}$ for certain i. In other words, F contains m different points of G_i and $m > N$. Since $(X, \mathcal{T})$ is N-connected with respect to $\bar{\mathcal{G}}$ then two points $x, y \in F \cap G_i$ and a continuous mapping $\omega : [0,1] \to [x,y]_{\bar{\mathcal{G}}}$ exist such that $\omega(0) = x$ and $\omega(1) = y$. Theorem 5.2 implies that

$$\mathrm{conv}_{\mathcal{G}} F = \bigcup_{t \in [0,1]} \mathrm{conv}_{\mathcal{G}}(\{\omega(t)\} \cup (F \backslash \{x, y\})).$$

Therefore $a \in \mathrm{conv}_{\mathcal{G}}(\{\omega(t)\} \cup (F \backslash \{x,y\}))$ for certain $t \in [0,1]$. Since G_i is convex and $x, y \in G_i$ then $\omega(t) \in G_i$. Hence the set $\{\omega(t)\} \cup (F \backslash \{x,y\})$ contains $(m-1)$ points of G_i.

By induction, there is a set $F_i \in [G_i]^{\leq N}$ such that $a \in \mathrm{conv}_{\mathcal{G}}(F_i \cup (F \backslash G_i))$. By repeating this process for each $i = 1, \ldots, n$, we will find n sets $F_i \in [G_i]^{\leq N}$ with $a \in \mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} F_i$.

□

Now consider a description of the set $\mathrm{co}_{(X,\mathcal{H})} \bigcup_{i=1}^{n} G_i$, where $G_i \in \mathcal{G}$.

Proposition 5.12 *Let $\mathcal{T}$ be a topology on X such that $\mathcal{T}'_X \subset \mathcal{T}$ and (5.9) is valid for $\mathcal{T}$. Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$. Then for any $G_1, \ldots, G_n \in \mathcal{G}$*

$$\mathrm{co}_{(X,\mathcal{H})} \bigcup_{i=1}^{n} G_i = \mathrm{cl}_{\mathcal{T}'_X} \left(\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i \right) = \mathrm{cl}_{\mathcal{T}'_X} \left(\bigcup_{g_i \in G_i} \mathrm{conv}_{\mathcal{G}}\{g_1, \ldots, g_n\} \right). \tag{5.35}$$

Proof: Since $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$ and $\mathcal{T}'_X \subset \mathcal{T}$ then $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$ as well (see Remark 5.2). It follows from (5.26) that

$$\mathrm{co}_{(X,\mathcal{H})} \bigcup_{i=1}^{n} G_i = \mathrm{cl}_{\mathcal{T}'_X} \left(\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i \right),$$

and, by Proposition 5.11 (with $N = 1$),

$$\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i = \bigcup_{g_i \in G_i} \mathrm{conv}_{\mathcal{G}}\{g_1, \dots, g_n\}.$$

□

5.6 Description of abstract convex functions

Let L be a set of functions $l : Y \longrightarrow \mathbb{R}$ defined on a set Y. Let $X = Y \times \mathbb{R}$ and $\mathcal{H}$ be the collection of all epigraphs $\mathrm{epi}\, l = \{(y, c) \in Y \times \mathbb{R} : \ l(y) \leq c\}$ with $l \in L$.

Some formulas for the segments $[(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$ and $[\mathrm{epi}\, l_1, \mathrm{epi}\, l_2]_{\bar{\mathcal{G}}^*}$ in this case were considered in Section 5.3. Below we give a description of L-convex functions by using connectedness of X and $\mathcal{H}$.

We begin with the description of L-convex functions on finite subsets of Y. Let Z be a subset of Y. Recall (see [41]) that a function $f : Y \longrightarrow \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ is called L-convex on Z if a subfamily $T \subset L$ exists such that $f(z) = \sup_{l \in T} l(z)$ for all $z \in Z$.

Proposition 5.13 *Let $N \geq 2$ and $\mathcal{T}$ be a topology on X, which enjoys (5.9). Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T})$ is N-connected with respect to $\bar{\mathcal{G}}$. Then for any function $f : Y \longrightarrow \mathbb{R}_{+\infty}$ the following conditions are equivalent:*

1.) For every $y, y_1, \dots, y_N \in Y$

$$f(y) \leq \sup\{l(y) : \ l \in L, \ l(y_i) \leq f(y_i) \ \forall i = 1, \dots, N\}. \tag{5.36}$$

2.) f is L-convex on every finite subset of Y.

Proof: It follows from Theorem 5.3 that the convexity $\mathcal{G}$ generated by $\mathcal{H}$ is of arity N.

1.) $\Longrightarrow$ 2.) Let a function f enjoys (5.36) for all $y, y_1, \dots, y_N \in Y$. Then its epigraph $\mathrm{epi}\, f$ belongs to the convexity $\mathcal{G}$. Indeed, since $\mathcal{G}$ is N-ary then $\mathrm{epi}\, f$ belongs to $\mathcal{G}$ if and only if $\mathrm{conv}_{\mathcal{G}}\{(y_1, c_1), \dots, (y_N, c_N)\} \subset \mathrm{epi}\, f$ for any $(y_1, c_1), \dots, (y_N, c_N) \in \mathrm{epi}\, f$. So let

$(y_1, c_1), \ldots, (y_N, c_N) \in \text{epi} f$. Then we have

$$\begin{aligned}\text{conv}_{\mathcal{G}}\{(y_1,c_1),\ldots,(y_N,c_N)\} &= \{(y,c): \sup\{l(y): l(y_i) \le c_i \ \forall i \le N\} \le c\} \\ &\subset \{(y,c): \sup\{l(y): l(y_i) \le f(y_i) \ \forall i \le N\} \le c\} \\ &\subset \text{epi} f.\end{aligned}$$

Let Z be a finite subset of Y. If $f(z) = +\infty$ for all $z \in Z$ then also $\sup_{l \in L} l(z) = +\infty$ for all $z \in Z$, and therefore f is L-convex on Z. Indeed, if $f(z) \equiv +\infty$ on Z then it follows from (5.36) that for any $z, y_1, \ldots, y_N \in Z$

$$\sup_{l \in L} l(z) = \sup\{l(z): l \in L, \ l(y_i) \le +\infty \ \forall i = 1, \ldots, N\} \ge f(z) = +\infty.$$

Now assume that the set $F = \{(z, f(z)): z \in Z, \ f(z) < +\infty\}$ is not empty. Since F is a finite subset of $Y \times \mathbb{R}$ then due to (5.2)

$$\text{conv}_{\mathcal{G}} F = \bigcap\{H \in \mathcal{H}: F \subset H\} = \bigcap\{\text{epi}\, l: l \in L, \ l(z) \le f(z) \ \forall z \in Z\}. \quad (5.37)$$

Let T be the collection of all functions $l \in L$ such that $l(z) \le f(z)$ for any $z \in Z$. Since $\text{epi} f \in \mathcal{G}$ and $F \subset \text{epi} f$ then $\text{conv}_{\mathcal{G}} F \subset \text{epi} f$. This means, in view of (5.37), that T is nonempty (otherwise $F \not\subset H$ for any $H \in \mathcal{H}$ and, by Proposition 5.2, we have $\text{conv}_{\mathcal{G}} F = Y \times \mathbb{R} \not\subset \text{epi} f$) and $f(y) \le \sup_{l \in T} l(y)$ for all $y \in Y$. On the other hand, $\sup_{l \in T} l(z) \le f(z)$ for any $z \in Z$ by definition of T. Hence $f(z) = \sup_{l \in T} l(z) \ \forall z \in Z$. In other words, f is L-convex on Z.

2.) $\Longrightarrow$ 1.) Let f be L-convex on every finite subset of Y. Let $y, y_1, \ldots, y_N \in Y$. Since f is L-convex on $\{y, y_1, \ldots, y_N\}$ then

$$\begin{aligned} f(y) &= \sup\{l(y): l \in L, \ l(y) \le f(y), \ l(y_i) \le f(y_i) \ \forall i = 1, \ldots, N\} \\ &\le \sup\{l(y): l \in L, \ l(y_i) \le f(y_i) \ \forall i = 1, \ldots, N\}. \end{aligned}$$

□

Now consider the case, when $(X, \mathcal{T}'_X)$ is connected (one-connected) with respect to $\bar{\mathcal{G}}$. This allows us to give a description of L-convex functions on the whole set Y.

Proposition 5.14 *Let $\mathcal{L}$ be the collection of all functions $\ell(x) = \min_{l \in T} l(x)$ with $T \in [L]^{<\omega}$, where $[L]^{<\omega}$ is the collection of all finite subsets of L. Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Then a function $f: Y \to \mathbb{R}_{+\infty}$ is L-convex if and only if it is $\mathcal{L}$-convex and*

$$f(y) \le \sup\{l(y): l \in L, \ l(y_1) \le f(y_1), \ l(y_2) \le f(y_2)\} \qquad \forall y, y_1, y_2 \in Y. \quad (5.38)$$

Proof: If f is L-convex then inequalities (5.38) obviously hold. Moreover, since $L \subset \mathcal{L}$ then f is $\mathcal{L}$-convex as well.

Conversely, assume that f is $\mathcal{L}$-convex and enjoys (5.38). It is clear that for every $\ell \in \mathcal{L}$ its epigraph $\operatorname{epi} \ell$ is closed in topology $\mathcal{T}'_X$ because it is union of a finite number of epigraphs of functions $l \in L$. Since f is $\mathcal{L}$-convex then the epigraph $\operatorname{epi} f$ is also closed in topology $\mathcal{T}'_X$. Moreover, inequalities (5.38) imply that $[(y_1, c_1), (y_2, c_2)]_{\mathcal{G}} \subset \operatorname{epi} f$ for any $(y_1, c_1), (y_2, c_2) \in \operatorname{epi} f$. Indeed, if $(y_1, c_1), (y_2, c_2) \in \operatorname{epi} f$ and $(y, c) \in [(y_1, c_1), (y_2, c_2)]_{\mathcal{G}}$ then

$$\begin{aligned} f(y) &\leq \sup\{l(y) : \ l \in L, \ l(y_1) \leq f(y_1), \ l(y_2) \leq f(y_2)\} \\ &\leq \sup\{l(y) : \ l \in L, \ l(y_1) \leq c_1, \ l(y_2) \leq c_2\} \\ &\leq \sup\{l(y) : \ l \in L, \ l(y) \leq c\} \leq c. \end{aligned}$$

Due to Theorem 5.4, for each $(y, c) \notin \operatorname{epi} f$ a set $\operatorname{epi} l \in \mathcal{H}$ exists such that $\operatorname{epi} f \subset \operatorname{epi} l$ and $(y, c) \notin \operatorname{epi} l$. This means that f is L-convex. □

Next proposition shows that, in some cases, $\mathcal{L}$-convexity of f can be interchanged with the lower semicontinuity.

Proposition 5.15 *Assume that L is closed under vertical shifts. Let Y be equipped with a topology such that Y is compact and all functions $l \in L$ are continuous. Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Then a function $f : Y \to \mathrm{I\!R}_{+\infty}$ is L-convex if and only if it is lower semicontinuous and possesses (5.38).*

Proof: Since L consists of continuous functions then every L-convex function is lower semicontinuous. Inequalities (5.38) for L-convex functions f are trivial.

Now assume that f is lower semicontinuous and possesses (5.38). Let us prove that f is $\mathcal{L}$-convex, where $\mathcal{L}$ is the collection of all minimums of finite subfamilies of L. Then, by Proposition 5.14, f is L-convex.

Take an arbitrary $y \in Y$. It follows from (5.38) that

$$f(y) \leq \sup\{l(y) : \ l \in L, \ l(y) \leq f(y), \ l(z) \leq f(z)\} \quad \forall z \in Y,$$

hence

$$f(y) = \sup\{l(y) : \ l \in L, \ l(y) \leq f(y), \ l(z) \leq f(z)\} \quad \forall z \in Y. \tag{5.39}$$

Let $\varepsilon > 0$. If $f(y) < +\infty$ then, by (5.39), for each $z \in Y$ a function $l_z \in L$ exists such that $l_z(z) \leq f(z)$ and $f(y) - \varepsilon/2 \leq l_z(y) \leq f(y)$. If $f(y) = +\infty$ then for each $z \in Y$ a function $l_z \in L$ exists such that $l_z(z) \leq f(z)$ and $1/\varepsilon \leq l_z(y) \leq f(y)$. Since L is closed under vertical shifts then every function $h_z(x) = l_z(x) - \varepsilon/2$ belongs to L. We have:

$$h_z(y) \leq f(y) - \varepsilon/2, \qquad h_z(z) \leq f(z) - \varepsilon/2$$

and

$$f(y) - \varepsilon \leq h_z(y) \quad \text{if } f(y) < +\infty, \qquad 1/\varepsilon - \varepsilon/2 \leq h_z(y) \quad \text{if } f(y) = +\infty.$$

Since f is lower semicontinuous, h_z is continuous and $h_z(z) < f(z)$ then for each $z \in Y$ a neighbourhood U_z of z exists such that $h_z(x) < f(x)$ for all $x \in U_z$. Due to compactness of Y, there is a finite collection $\{z_1, \ldots, z_m\} \subset Y$ with $U_{z_1} \cup \ldots \cup U_{z_m} = Y$. Consider the function $\ell(x) = \min_i h_{z_i}(x)$. Then $\ell \in \mathcal{L}$ and $\ell(x) < f(x)$ for all $x \in Y$. Moreover, $f(y) - \varepsilon \leq \ell(y)$ if $f(y) < +\infty$ and $1/\varepsilon - \varepsilon/2 \leq \ell(y)$ if $f(y) = +\infty$.

Thus, we have proved that, for any $y \in Y$ and $\varepsilon > 0$, a function $\ell \in \operatorname{supp}(f, \mathcal{L})$ exists such that $f(y) - \varepsilon \leq \ell(y)$ for $f(y) < +\infty$ and $1/\varepsilon - \varepsilon/2 \leq \ell(y)$ for $f(y) = +\infty$. This means that f is $\mathcal{L}$-convex. □

5.7 Description of abstract convex sets

Let L be a set of functions defined on a set Y. Let $X = L$ and $\mathcal{H}$ be the collection of all subsets $\{l \in L : \ l(y) \leq c\} \subset X$, where $(y, c) \in Y \times \mathbb{R}$.

Then for any $l_1, l_2 \in L$

$$\begin{aligned} [l_1, l_2]_{\mathcal{G}} &= \bigcap \{H \in \mathcal{H} : \ l_1, l_2 \in H\} \\ &= \{l \in L : \ l(y) \leq c \text{ whenever } \max\{l_1(y), l_2(y)\} \leq c\} \\ &= \{l \in L : \ l(y) \leq \max\{l_1(y), l_2(y)\} \ \forall y \in Y\}. \end{aligned}$$

Similarly,

$$[l_1, l_2]_{\bar{\mathcal{G}}} = \{l \in L : \ \min\{l_1(y), l_2(y)\} \leq l(y) \leq \max\{l_1(y), l_2(y)\} \ \forall y \in Y\}.$$

Let $(y_1, c_1), (y_2, c_2) \in Y \times \mathbb{R}$ and $H_i = \{l \in L : \ l(y_i) \leq c_i\} \ \ (i = 1, 2)$. Then, by Proposition 5.4, $[H_1, H_2]_{\mathcal{G}^*} = \{H \in \mathcal{H} : \ H_1 \cap H_2 \subset H \subset H_1 \cup H_2\}$. In other words, a set

$H = \{l \in L: \ l(y) \le c\}$ belongs to $[H_1, H_2]_{\bar{\mathcal{G}}^*}$ if and only if for each $l \in L$ the following implications hold:

$$\begin{array}{rcl} \max\{l(y_1) - c_1, l(y_2) - c_2\} \le 0 & \Longrightarrow & l(y) \le c, \\ l(y) \le c & \Longrightarrow & \min\{l(y_1) - c_1, l(y_2) - c_2\} \le 0. \end{array}$$

Thus, our formulas for $[l_1, l_2]_{\bar{\mathcal{G}}}$ and $[H_1, H_2]_{\bar{\mathcal{G}}^*}$ coincide with the corresponding formulas for $[\mathrm{epi}\, l_1, \mathrm{epi}\, l_2]_{\bar{\mathcal{G}}^*}$ and $[(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$ in the case, when $X = Y \times \mathbb{R}$ and $\mathcal{H} = \{\mathrm{epi}\, l : l \in L\}$ (see the second part of Section 5.3).

Recall that a set $U \subset L$ is called (L, Y)-convex if $U = \mathrm{co}_L U$, where $\mathrm{co}_L U = \{l \in L : l(y) \le \sup_{u \in U} u(y) \ \forall y \in Y\}$. It is easy to see that $\mathrm{co}_L U = \mathrm{co}_{(X,\mathcal{H})} U$, where $\mathrm{co}_{(X,\mathcal{H})} U$ is defined by (5.24). Indeed,

$$\begin{aligned} \mathrm{co}_{(X,\mathcal{H})} U &= \bigcap \{H \in \mathcal{H} : \ U \subset H\} \\ &= \{l \in L : \ l(y) \le c \text{ whenever } u(y) \le c \ \forall u \in U\} \\ &= \left\{ l \in L : \ l(y) \le \sup_{u \in U} u(y) \ \forall y \in Y \right\} = \mathrm{co}_L U. \end{aligned}$$

Proposition 5.16 *Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Then a set $U \subset L$ is (L, Y)-convex if and only if it is closed in the topology $\mathcal{T}'_X$ and*

$$\{l \in L : \ \min\{l_1(y), l_2(y)\} \le l(y) \le \max\{l_1(y), l_2(y)\} \ \forall y \in Y\} \subset U \quad \forall l_1, l_2 \in U. \tag{5.40}$$

Proof: Let $U \subset L = X$. Theorem 5.4 states that $U = \mathrm{co}_{(X,\mathcal{H})} U$ if and only if U is closed in topology $\mathcal{T}'_X$ and $[l_1, l_2]_{\bar{\mathcal{G}}} \subset U$ for all $l_1, l_2 \in U$. Since $\mathrm{co}_L U = \mathrm{co}_{(X,\mathcal{H})} U$ then U is (L, Y)-convex if and only if it is closed in topology $\mathcal{T}'_X$ and possesses (5.40). □

A set $U \subset L$ is closed if and only if it contains each $l \in L$ such that every neighbourhood of l contains an element of U. Since the topology $\mathcal{T}'_X$ is generated by the collection of all sets $\{l \in L : \ l(y) > c\}$ with $(y, c) \in Y \times \mathbb{R}$, then U is closed in topology $\mathcal{T}'_X$ if and only if it contains all $l \in L$ such that for every finite subset $F \subset Y$ and for every $\varepsilon > 0$ a function $u \in U$ exists with $u(y) > l(y) - \varepsilon \ \forall y \in F$.

Let $\mathcal{T}$ be the topology of pointwise convergence on L. Then $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$ if and only if for any $l_1, l_2 \in L$ a mapping $\omega : [0, 1] \times Y \to \mathbb{R}$ exists such that $\omega(\cdot, y)$ is continuous on $[0, 1]$ for each fixed $y \in Y$ and the following holds:

$$\omega(t, \cdot) \in L \ \forall t \in [0, 1], \quad \omega(0, \cdot) = l_1, \quad \omega(1, \cdot) = l_2,$$

$$\min\{l_1(y), l_2(y)\} \le \omega(t,y) \le \max\{l_1(y), l_2(y)\} \;\; \forall t \in [0,1], \;\; y \in Y.$$

It is clear that condition (5.9) is valid for $\mathcal{T}$. Indeed, let U be a finite subset of L and l_i converges to an element $l' \in U$. This means that $l_i(y)$ converges to $l'(y)$ for every $y \in Y$. Then

$$\begin{aligned}\bigcap_{i\ge 1} \mathrm{conv}_{\mathcal{G}}(U \cup \{l_i\}) &= \left\{l \in L: \;\; l(y) \le \max\left\{l_i(y), \max_{u\in U} u(y)\right\} \;\; \forall y \in Y, \; i \ge 1\right\} \\ &= \left\{l \in L: \;\; l(y) \le \max_{u\in U} u(y) \;\; \forall y \in Y\right\} = \mathrm{conv}_{\mathcal{G}} U.\end{aligned}$$

Moreover, since every set $H = \{l \in L: \;\; l(y) \le c\} \in \mathcal{H}$ is closed in topology $\mathcal{T}$ then $\mathcal{T}'_X \subset \mathcal{T}$.

Proposition 5.17 *Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. If L is compact in the topology of poinwise convergence then a set $U \subset L$ is (L,Y)-convex if and only if it is closed in $\mathcal{T}$ and*

$$\{l \in L: \;\; l(y) \le \max\{l_1(y), l_2(y)\} \;\; \forall y \in Y\} \subset U \quad \forall l_1, l_2 \in U. \tag{5.41}$$

Proof: By Theorem 5.4, U is (L,Y)-convex if and only if it is closed in topology $\mathcal{T}'_X$ and $[l_1, l_2]_{\mathcal{G}} \subset U$ for all $l_1, l_2 \in U$.

Inclusions $[l_1, l_2]_{\mathcal{G}} \subset U$ for $l_1, l_2 \in U$ are equivalent to (5.41). If U is closed in topology $\mathcal{T}'_X$ then it is closed in $\mathcal{T}$ as well, because $\mathcal{T}'_X \subset \mathcal{T}$.

Conversely, let $U \subset L$ be closed in the topology of pointwise convergence and enjoy (5.41). Assume that L is compact in $\mathcal{T}$. Then U is also compact in $\mathcal{T}$. We need to check that U is closed in the topology $\mathcal{T}'_X$. Let $l \in L \backslash U$. It follows from (5.41) that for every $u \in U$ a point $y_u \in Y$ exists with $l(y_u) > u(y_u)$. Let $c_u = (u(y_u) + l(y_u))/2$. Then for each $u \in U$ the set $\{l' \in L: \;\; l'(y_u) < c_u\}$ is a neighbourhood of u (i.e. it is open in topology $\mathcal{T}$ and contains u), and $l(y_u) > c_u$. Since U is compact then there is a finite collection $\{(y_1, c_1), \ldots, (y_n, c_n)\} \subset Y \times \mathbb{R}$ such that $\min_i(u(y_i) - c_i) < 0 < \min_i(l(y_i) - c_i)$ for all $u \in U$. Hence $l \notin \bigcup_i H_i$ and $U \subset \bigcup_i H_i$, where $H_i = \{l' \in L: \;\; l'(y_i) \le c_i\} \in \mathcal{H}$. This means that l does not belong to the closure $\mathrm{cl}_{\mathcal{T}'_X} U$, because $\bigcup_i H_i$ is closed in topology $\mathcal{T}'_X$. Thus, U is closed in $\mathcal{T}'_X$. □

Proposition 5.18 *Assume that L is closed under vertical shifts. Let Y be equipped with a topology such that Y is compact and all functions $l \in L$ are continuous on Y. Assume that*

$(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$. Then a set $U \subset L$ is (L, Y)-convex if and only if (5.41) holds and U contains every $l \in L$ such that $(l - \varepsilon) \in U$ for any $\varepsilon > 0$.

Proof: If U is (L, Y)-convex then (5.41) is valid. Moreover, since $l(y) = \sup_{\varepsilon>0}(l(y) - \varepsilon)$ then U contains every $l \in L$ such that $(l - \varepsilon) \in U$ for all $\varepsilon > 0$.

Conversely, assume that $U \subset L$ possesses (5.41) and $l \in U$ whenever $(l-\varepsilon) \in U$ for all $\varepsilon > 0$. Let $l \in L$ be such that $l(y) \leq \sup_{u \in U} u(y)$ for all $y \in Y$. We show that $(l - \varepsilon) \in U$ for any positive ε. Then l belongs to U as well, and therefore U is (L, Y)-convex.

So let $\varepsilon > 0$. Since $l(y) - \varepsilon < \sup_{u \in U} u(y) \quad \forall y \in Y$ then for each $y \in Y$ a function $u_y \in U$ exists with $l(y) - \varepsilon < u_y(y)$. Due to the continuity of functions l and u_y, $l(z) - \varepsilon < u_y(z)$ for all z from a neighbourhood of y. Then, by compactness of Y, a finite collection $\{u_1, \dots, u_n\} \subset U$ exists such that $l(y) - \varepsilon < \max_i u_i(y)$ for all $y \in Y$.

Since the topology $\mathcal{T}$ enjoys condition (5.9) then, by Theorem 5.3, the convexity $\mathcal{G}$ is of arity 2. It follows from (5.41) that $[l_1, l_2]_{\mathcal{G}} \subset U$ for any $l_1, l_2 \in U$. Hence U is convex. This implies that $\text{conv}_{\mathcal{G}}\{u_1, \dots, u_n\} \subset U$. In other words,

$$\left\{u \in L : \ u(y) \leq \max_i u_i(y) \ \ \forall y \in Y\right\} \subset U.$$

In particular, U contains the function $h(y) = l(y) - \varepsilon$. □

At last, we give a formula for the (L, Y)-convex hull of a finite union of (L, Y)-convex sets. This is important for the description of the support set and the subdifferential of the maximum of a finite collection of abstract convex functions. Indeed, for every L-convex functions $f_1, \dots, f_n$ we have

$$\text{supp}\left(\max_{i=1,\dots,n} f_i, L\right) = \text{co}_L \bigcup_{i=1}^{n} \text{supp}\,(f_i, L).$$

Subdifferential of the maximum of a finite collection of abstract convex functions have been considered in Chapter 4 (see formulas (4.17) and (4.18)).

Proposition 5.19 *Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$. Then for any (L, Y)-convex sets $U_1, \dots, U_n$*

$$\text{co}_L \bigcup_{i=1}^{n} U_i = \text{cl}_{\mathcal{T}'_X}\left(\bigcup_{u_i \in U_i}\left\{l \in L : \ l(y) \leq \max_{i=1,\dots,n} u_i(y) \ \forall y \in Y\right\}\right). \qquad (5.42)$$

Proof: Since $\mathcal{T}'_X \subset \mathcal{T}$ and condition (5.9) is valid for $\mathcal{T}$ then we can apply Proposition 5.12. Let $U_1, \ldots, U_n \subset L$ be (L, Y)-convex. Then $U_1, \ldots, U_n \in \mathcal{G}$ and, by (5.35),

$$\begin{aligned} \mathrm{co}_L \bigcup_{i=1}^{n} U_i &= \mathrm{co}_{(X,\mathcal{H})} \bigcup_{i=1}^{n} U_i = \mathrm{cl}_{\mathcal{T}'_X} \left(\bigcup_{u_i \in U_i} \mathrm{conv}_{\mathcal{G}} \{u_1, \ldots, u_n\} \right) \\ &= \mathrm{cl}_{\mathcal{T}'_X} \left(\bigcup_{u_i \in U_i} \{ l \in L : \; l(y) \leq \max_{i=1,\ldots,n} u_i(y) \;\; \forall y \in Y \} \right). \end{aligned}$$

□

Chapter 6

On generalized conjugations and subdifferentials

Applications of abstract convex analysis to global optimization problems are based on the description of support sets or, at least, its maximal elements (see [41]). The most convenient tools for this are the notions of a conjugate function and abstract subdifferential. In this chapter we consider a type of conjugate functions, which is an evident generalization of the notion of conjugation examined in [57]. We also consider abstract subdifferentials and give conditions for the global minimum in terms of these subdifferentials.

6.1 Optimality conditions and the role of the abstract subdifferential and conjugation

The notion of a support set of real-valued functions can be easily generalized for functions with values in an arbitrary partially ordered set. Let H be a collection of functions $h : X \to U$, where X is a set and U is a partially ordered set. The set

$$\text{supp}\,(f, H) = \{h \in H : \; h(x) \leq f(x) \;\; \forall x \in X\} \tag{6.1}$$

of all H-minorants of f is called the support set of the function f with respect to H.

If, moreover, U is an upper complete semilattice then we can define also H-convex functions. Namely, a function $f : X \to U$ is called abstract convex with respect to H (or H-convex) if its support set $\text{supp}\,(f, H)$ is not empty and $f(x) = \sup\{h(x) : \; h \in \text{supp}\,(f, H)\} \; \forall x \in X$. If $f : X \to U$ is a function such that $\text{supp}\,(f, H) \neq \emptyset$ then its

H-convex hull $\text{co}_H f$ is defined by

$$\text{co}_H f(x) = \sup\{h(x) : \ h \in \text{supp}\,(f, H)\} \qquad (x \in X). \tag{6.2}$$

Note that the family of all H-convex functions is an upper complete semilattice as well.

Let $f : X \to U$, where U is a partially ordered set. For each point $y \in X$ define the following set

$$\partial^*_H f(y) = \{h \in H : \ h \in \text{supp}\,(f, H), \ \ h(y) = f(y)\}. \tag{6.3}$$

Consider conditions for the global minimum and global minimal element in terms of the set $\partial^*_H f(y)$. Recall that an element $\bar{w} \in W$ of a partially ordered set W is called minimal if there is no $w \in W$ such that $w < \bar{w}$ (in other words, $\bar{w} \not> w$ for all $w \in W$). Let S be a subset of X. The following statements are obvious.

1. If there exists a function $h \in \partial^*_H f(y)$ such that $h(y) \not> h(x)$ for all $x \in S$ then $f(y) \not> f(x)$ for all $x \in S$.

2. If there exists a function $h \in \partial^*_H f(y)$ such that $h(y) \leq h(x)$ for all $x \in S$ then $f(y) \leq f(x)$ for all $x \in S$.

3. Assume that H contains all constants. Then $f(y) \leq f(x)$ for all $x \in X$ if and only if the set $\partial^*_H f(y)$ contains the constant $f(y)$.

Let $U = \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ and H be a set of functions $h : X \to \mathbb{R}$.

First, assume that H is closed under vertical shifts. Then it is convenient to represent H as $H_L = \{h : \ h(x) = l(x) - c, \ \ l \in L, \ \ c \in \mathbb{R}\}$, where L is a subset of H such that $(l - c) \notin L$ whenever $l \in L$ and $c \neq 0$. For any function $f : X \to \mathbb{R}_{+\infty}$ we can define its L-conjugate function f^*_L and L-subdifferential $\partial_L f(y)$ in usual manner:

$$f^*_L(l) = \sup_{x \in X}(l(x) - f(x)), \qquad (l \in L), \tag{6.4}$$

$$\partial_L f(y) = \{l \in L : \ l(x) - l(y) \leq f(x) - f(y) \ \ \forall x \in X\}. \tag{6.5}$$

Then the support set of f with respect to H coincides with the epigraph of the conjugate function f^*_L. Thus, conjugate function accumulates whole information about the support set of the initial function. Furthermore, as it was discussed in Section 1.3, the description of the abstract subdifferential is easier than the description of the set $\partial^*_{H_L} f(y)$. At the same time, we have:

$$\partial^*_H f(y) = \{h : \ h(x) = l(x) - l(y) + f(y), \ l \in \partial_L f(y)\}. \tag{6.6}$$

So if H is closed under vertical shifts then we can simplify the description of the set $\partial^*_H f(y)$. Moreover, conditions for global minimum can be easily reformulated in terms of subdifferentials.

Now assume that H is not closed under vertical shifts and consider its extension $H' = \{h' : \ h'(x) = h(x) + c, \ h \in H, \ c \in \mathbb{R}\}$. Define also the conjugation f^*_L and the subdifferential $\partial_L f(y)$ for this new set H'. Then both f^*_L and $\partial_L f(y)$ contain an excess information about the function f. For example, the set $\partial^*_{H'} f(y)$ is in one-to-one correspondence with $\partial_L f(y)$, while we need only the description of the set $\partial^*_H f(y) \subset \partial^*_{H'} f(y)$. Hence we have an excess complexity of f^*_L and $\partial_L f(y)$. Moreover, if the functions $h \in H$ are not real-valued (for example, if they have values in a nonlinear space) then the notions of conjugate function (6.4) and subdifferential (6.5) are not applicable. This is the reason why we need a generalization of these notions.

Some types of conjugations and subdifferentials for functions $f : X \to U$, where U is a complete lattice, were considered in [57]. Here we examine the case, when the set U is either partially ordered or upper complete semilattice.

6.2 Involutions, subinvolutions and dualities

Definition 6.1 Let T be a partially ordered set. A mapping $I : T \to T$ is called an involution of T if it is decreasing and $I(I(t)) = t$ for all $t \in T$. We will say that I is a subinvolution of T if it is decreasing and $I(I(t)) \leq t$ for all $t \in T$.

The notion of involution can be found, e.g., in [4]. Consider a classical example.

Example 6.1 Let M be a set and T be the collection of all subsets of M (including the empty set and whole M). Assume that T is equipped with the following order relation: $t_1 \leq t_2 \iff t_1 \subset t_2$. Then the mapping $I(t) = M \backslash t = \{m \in M : \ m \notin t\}$ is an involution of T.

Remark 6.1 Equalities $I(I(t)) = t$ in the definition of involution mean that $I : T \to T$ is a bijective mapping and $I^{-1} = I$. Indeed, since $I(t) \in T$ and $I(I(t)) = t$ then $I(T) = T$. If $I(t_1) = I(t_2)$ then $t_1 = I(I(t_1)) = I(I(t_2)) = t_2$. So, every involution is strictly decreasing.

The following characterizations of involutions and subinvolutions can be useful.

Proposition 6.1 *Let T be a partially ordered set.*
1.) A mapping $I : T \to T$ is a subinvolution of T if and only if for each $t_1, t_2 \in T$

$$I(t_1) \le t_2 \iff I(t_2) \le t_1. \tag{6.7}$$

2.) A mapping $I : T \to T$ is an involution of T if and only if for each $t_1, t_2 \in T$

$$(I(t_1) \le t_2 \iff I(t_2) \le t_1) \quad \text{and} \quad (I(t_1) = t_2 \iff I(t_2) = t_1). \tag{6.8}$$

Proof: 1.) Let I be a subinvolution of T. Since I is decreasing and $I(I(t)) \le t$ then inequality $I(t_1) \le t_2$ implies that $I(t_2) \le I(I(t_1)) \le t_1$. Conversely, let (6.7) be valid. Let $t \in T$. If follows from (6.7) and the inequality $I(t) \le I(t)$ that $I(I(t)) \le t$. If $t \le t'$ then $I(I(t)) \le t'$, hence $I(t') \le I(t)$.

2.) If I is an involution of T then I is also a subinvolution and (6.7) is valid. If $I(t_1) = t_2$ then $I(t_2) = I(I(t_1)) = t_1$. So we get (6.8). Conversely, assume that (6.8) holds. Then I is decreasing on T. Equality $I(I(t)) = t$ follows from the right part of (6.8) and the equality $I(t) = I(t)$. □

The next proposition states that every subinvolution is an involution of its image.

Proposition 6.2 *Let $I : T \to T$ be a subinvolution of T. Then its restriction $I : I(T) \to I(T)$ on the image $I(T) = \{I(t) : t \in T\}$ is an involution of $I(T)$. In particular, $I(I(I(t))) = I(t)$ for all $t \in T$.*

Proof: We only need to check that $I(I(t')) \ge t'$ for all $t' \in I(T)$. So let $t' = I(t)$, where $t \in T$. Since I is decreasing on T then the mapping $I \circ I$ is increasing, hence $I(I(I(I(t)))) \le I(I(t)) \le t$. Due to (6.7) we get $t' = I(t) \le I(I(I(t))) = I(I(t'))$. □

Recall two definitions from the book [57]. Let U and V be complete lattices.

Definition 6.2 ([57], Definition 5.1) A mapping $\Delta : U \to V$ is called a duality if for each nonempty subset $S \subset U$

$$\Delta\left(\inf_{u \in S} u\right) = \sup_{u \in S} \Delta(u) \tag{6.9}$$

and $\Delta(U_{\max}) = V_{\min}$, where $U_{\max}$ is the maximal element of U and $V_{\min}$ is the minimal element of V.

Definition 6.3 ([57], Definition 5.2) Let $\Delta : U \to V$ be a mapping. Then the mapping $\Delta' : V \to U$ defined by

$$\Delta'(v) = \inf\{u \in U : \ \Delta(u) \leq v\} \tag{6.10}$$

is called the dual of Δ. If $\{u \in U : \ \Delta(u) \leq v\} = \emptyset$ then we set $\Delta'(v) = U_{\max}$.

If $\Delta : U \to V$ is a duality then its dual mapping $\Delta' : V \to U$ is a duality as well and $\Delta'' = \Delta$, where $\Delta'' = (\Delta')' : U \to V$ (see [57], Theorem 5.3).

Here we show that dualities can be described via subinvolutions. At the same time, involutions correspond to bijective dualities.

For complete lattices U and V consider the following partially ordered set:

$$T = (U, 1) \cup (V, 2), \tag{6.11}$$

where $(U, 1) = \{(u, 1) : \ u \in U\}$ and $(V, 2) = \{(v, 2) : \ v \in V\}$. We assume that $(w, i) \leq (w', j)$ if and only if $i = j$ and $w \leq w'$ (here $(w, i), (w', j) \in T, i, j = 1, 2$).

Proposition 6.3 *Let U and V be two complete lattices and $\Delta : U \to V$. Consider the mapping $I : T \to T$ defined by the formulas*

$$I((u, 1)) = (\Delta(u), 2) \quad \forall u \in U, \qquad I((v, 2)) = (\Delta'(v), 1) \quad \forall v \in V, \tag{6.12}$$

where T is defined by (6.11). Then

1.) I is a subinvolution of T if and only if Δ is a duality.

2.) I is an involution of T if and only if Δ is a bijective duality.

Proof: 1.) If I is subinvolution of T then, by Definition 6.1, Δ is decreasing and $\Delta\Delta'(v) \leq v$ for all $v \in V$. It follows from ([57], Proposition 5.3) that Δ is a duality. Conversely, let Δ be a duality. Then, due to ([57], Corollary 5.3),

$$\Delta(u) \leq v \iff \Delta'(v) \leq u \qquad (u \in U, \ v \in V),$$

and, by Proposition 6.1, I is subinvolution of T.

2.) If I is involution of T then it is subinvolution, and therefore Δ is a duality. In view of Remark 6.1 Δ is a bijective mapping. Conversely, let $\Delta : U \to V$ be a bijective duality. Then I is subinvolution of T. Since Δ is bijective then Proposition 6.2 implies that I is involution of T. □

Now consider the case of arbitrary partially ordered sets U and V.

Proposition 6.4 *Let U and V be two partially ordered sets and $\Delta : U \to V$. Let I_1 and I_2 be subinvolutions of T, where T is defined by (6.11). If*

$$I_1((u,1)) = I_2((u,1)) = (\Delta(u), 2) \quad \forall u \in U$$

then $I_1 = I_2$.

Proof: Let $\Delta_1, \Delta_2 : V \to U$ be the mappings defined by

$$(\Delta_1(v), 1) = I_1((v,2)), \qquad (\Delta_2(v), 1) = I_2((v,2)).$$

If I_1 and I_2 are subinvolutions of T then, by Proposition 6.1, we get

$$\Delta_1(v) \leq u \iff \Delta(u) \leq v \iff \Delta_2(v) \leq u.$$

Hence $\Delta_1 = \Delta_2$. □

Proposition 6.4 allows one to extend the notion of duality to arbitrary partially ordered sets U and V.

Definition 6.4 Let U and V be two partially ordered sets. We will say that a mapping $\Delta : U \to V$ is a duality if there exists a subinvolution I of the set T (see (6.11)) such that $I((u,1)) = (\Delta(u), 2)$ for all $u \in U$. If Δ is a duality then the mapping $\Delta' : V \to U$ defined by $(\Delta'(v), 1) = I((v,2))$ is called the dual of Δ.

In other words, a mapping $\Delta : U \to V$ is called a duality if there exists a mapping $\Delta' : V \to U$ such that

$$\Delta(u) \leq v \iff \Delta'(v) \leq u. \tag{6.13}$$

If Δ is a duality then Δ' is called the dual of Δ.

Proposition 6.5 *Let U and V be two partially ordered sets. If $\Delta : U \to V$ is a duality then its dual mapping $\Delta' : V \to U$ is a duality as well, and $\Delta'' = (\Delta')' = \Delta$. Moreover, both mapping Δ and Δ' are decreasing and*

$$\Delta'\Delta(u) \leq u \quad \forall u \in U, \qquad \Delta\Delta'(v) \leq v \quad \forall v \in V, \tag{6.14}$$

$$\Delta\Delta'\Delta(u) = \Delta(u) \quad \forall u \in U, \qquad \Delta'\Delta\Delta'(v) = \Delta'(v) \quad \forall v \in V. \tag{6.15}$$

In particular, if $\Delta : U \to V$ is a bijective duality then we have equalities

$$\Delta'\Delta(u) = u \quad \forall u \in U, \qquad \Delta\Delta'(v) = v \quad \forall v \in V, \tag{6.16}$$

that is $\Delta' = \Delta^{-1}$.

Proof: It follows from the Definitions 6.4 and 6.1 that the inequalities (6.14) hold and the mappings Δ and Δ' are decreasing. Due to equivalent definition of duality (see (6.13)) the mapping Δ' is duality and $\Delta'' = \Delta$. Equalities (6.15) follow from the Proposition 6.2, where the mapping $I : T \to T$ is defined by (6.12). If $\Delta : U \to V$ □

Remark 6.2 It is easy to see that Proposition 6.3 holds true for any partially ordered sets U and V.

6.3 L-subdifferentials with respect to a mapping $\Phi : X \times L \times U \to V$

Let U and V be two partially ordered sets. Let X and L be sets.

Consider a mapping $\Phi : X \times L \times U \to V$ such that for each fixed $x \in X$ and $l \in L$ the mapping $\Delta_{x,l} : U \to V$ defined by $\Delta_{x,l}(u) = \Phi(x, l, u)$ is a duality. Denote by H_X^Φ the set of all functions $h : L \to V$ defined by $h(l) = \Phi(x, l, u)$, where $x \in X$ and $u \in U$.

Let $\Phi' : X \times L \times V \to U$ be the mapping defined by $\Phi'(x, l, v) = \Delta'_{x,l}(v)$ (here $\Delta'_{x,l}$ is the dual of $\Delta_{x,l}$). By the symbol $H_L^{\Phi'}$ we denote the set of all functions $h : X \to U$ defined by $h(x) = \Phi'(x, l, v)$, where $l \in L$ and $v \in V$.

We introduce L-subdifferentials of functions $f : X \to U$ in the following way (compare it with the definition of the subdifferential considered in ([57], p. 359)).

Definition 6.5 Let $f : X \to U$ and $y \in X$. We say that the set

$$\partial_L f(y) = \{l \in L : \ \Phi(x, l, f(x)) \leq \Phi(y, l, f(y)) \ \forall x \in X\} \tag{6.17}$$

is the L-subdifferential of the function f at the point y with respect to the mapping Φ. Elements $l \in \partial_L f(y)$ will be called L-subgradients of f at y with respect to Φ.

Since $\Phi'(x, l, \cdot) : V \to U$ is the dual of $\Phi(x, l, \cdot) : U \to V$ then, due to (6.13),

$$\Phi(x, l, f(x)) \leq \Phi(y, l, f(y)) \iff \Phi'(x, l, \Phi(y, l, f(y))) \leq f(x).$$

Therefore

$$\partial_L f(y) = \{l \in L : \ \Phi'(x, l, \Phi(y, l, f(y))) \leq f(x) \ \forall x \in X\}. \tag{6.18}$$

Proposition 6.6 *Let $f : X \to U$ and $y \in X$. Let L-subgradient $l \in \partial_L f(y)$ be such that the mapping $\Phi(y, l, \cdot) : U \to V$ is bijective. Then the function $h(x) = \Phi'(x, l, \Phi(y, l, f(y)))$ belongs to the set $\partial^*_{H_L^{\Phi'}} f(y)$ defined by*

$$\partial^*_{H_L^{\Phi'}} f(y) = \{h \in H_L^{\Phi'} : \ h(y) = f(y), \ \ h \in \operatorname{supp}(f, H_L^{\Phi'})\}.$$

Therefore, if $\Phi(y, l, \cdot) : U \to V$ is bijective for all $l \in L$ then

$$\partial^*_{H_L^{\Phi'}} f(y) = \{h : \ h(x) = \Phi'(x, l, \Phi(y, l, f(y))), \ \ l \in \partial_L f(y)\}. \tag{6.19}$$

Proof: Formula (6.18) implies that the function $h(x) = \Phi'(x, l, \Phi(y, l, f(y)))$ belongs to $\operatorname{supp}(f, H_L^{\Phi'})$. If the mapping $\Phi(y, l, \cdot) : U \to V$ is bijective then, due to (6.16), we get $h(y) = f(y)$. Hence $h \in \partial^*_{H_L^{\Phi'}} f(y)$.

Assume that $\Phi(y, l, \cdot)$ is bijective for all $l \in L$. Take a function $h(x) = \Phi'(x, l, v)$ from the set $\partial^*_{H_L^{\Phi'}} f(y)$. Since $\Phi'(y, l, v) = h(y) = f(y)$ then $v = \Phi(y, l, \Phi'(y, l, v))) = \Phi(y, l, f(y))$. Since $h \leq f$ then $h(x) = \Phi'(x, l, \Phi(y, l, f(y))) \leq f(x)$ for all $x \in X$, hence $l \in \partial_L f(y)$. □

Corollary 6.1 *Assume that U is upper complete semilattice. Consider a function $f : X \to U$. Assume that for each $y \in X$ and $l \in L$ the duality $\Phi(y, l, \cdot) : U \to V$ is bijective. If all subdifferentials $\partial_L f(y)$ $(y \in X)$ are nonempty then the function f is $H_L^{\Phi'}$-convex.*

Consider conditions for the global minimum and global minimal element in terms of abstract subdifferentials defined in this section.

Denote by L_0 the following set

$$L_0 = \{l \in L : \ \Phi'(x, l, \Phi(y, l, u)) = u \ \ \forall x, y \in X, \ \forall u \in U\}. \tag{6.20}$$

For example, in the classical convex case we have: $X = Y = \mathbb{R}^n$, $U = V = \mathbb{R}$, $\Phi(x, l, u) = \Phi'(x, l, u) = (x, l) - u$. Therefore $L_0 = \{0\}$.

Remark 6.3 If $l \in L_0$ then $\Phi(x, l, u) \leq \Phi(y, l, u)$ for all $x, y \in X$, $u \in U$, therefore we get equalities $\Phi(x, l, u) = \Phi(y, l, u)$ $(x, y \in X, u \in U)$.

Proposition 6.7 *Let $f : X \to U$, $y \in X$ and $S \subset X$.*
1.) Suppose that the set L_0 is nonempty. Then

$$f(y) \leq f(x) \ \forall x \in X \quad \textit{if and only if} \quad L_0 \subset \partial_L f(y).$$

2.) If there exists a subgradient $l \in \partial_L f(y)$ such that

$$f(y) \leq \Phi'(x, l, \Phi(y, l, f(y))) \quad \textit{for all} \quad x \in S$$

then $f(y) \leq f(x)$ for all $x \in S$.
3.) If there exists a subgradient $l \in \partial_L f(y)$ such that

$$f(y) \not> \Phi'(x, l, \Phi(y, l, f(y))) \quad \textit{for all} \quad x \in S$$

then $f(y) \not> f(x)$ for all $x \in S$.

Proof: The proof follows directly from the formula (6.18). □

6.4 Conjugate functions with respect to Φ and Φ'

Assume that U and V are upper complete semilattices.

Definition 6.6 Let $f : X \to U$. We say that the function $f_L^* : L \to V$ defined by the formula

$$f_L^*(l) = \sup_{x \in X} \Phi(x, l, f(x)) \tag{6.21}$$

is the L-conjugate to f with respect to Φ.
For a function $g : L \to V$ the function $g_X^* : X \to U$ defined by

$$g_X^*(x) = \sup_{l \in L} \Phi'(x, l, g(l)) \tag{6.22}$$

will be called the X-conjugate to g with respect to Φ'.

So we define a conjugation for all functions $f : X \to U$ and $g : L \to V$. Sometimes for more convenience we will use the term "conjugate functions" for both cases (without symbols L or X). Thus the term "second conjugate function" seems natural. The second conjugate functions $f^{**} : X \to U$ and $g^{**} : L \to V$ are defined as follows:

$$f^{**}(x) = (f_L^*)_X^*(x), \qquad g^{**}(l) = (g_X^*)_L^*(l). \tag{6.23}$$

Proposition 6.8 *Let $f : X \to U$ and $g : L \to V$. Let $x \in X$, $l \in L$, $u \in U$ and $v \in V$. Then*

$$\begin{aligned} \Phi'(\cdot, l, v) \in \operatorname{supp}(f, H_L^{\Phi'}) &\iff f_L^*(l) \le v, \\ \Phi(x, \cdot, u) \in \operatorname{supp}(g, H_X^{\Phi}) &\iff g_X^*(x) \le u. \end{aligned} \tag{6.24}$$

Proof: Consider $f : X \to U$. Since $\Phi(x, l, \cdot) : U \to V$ is a duality then (see (6.13))

$$\Phi'(x, l, v) \le f(x) \ \forall x \in X \iff \Phi(x, l, f(x)) \le v \ \forall x \in X \iff f_L^*(l) \le v.$$

Similarly, for a function $g : L \to V$, we have

$$\Phi(x, l, u) \le g(l) \ \forall l \in L \iff \Phi'(x, l, g(l)) \le u \ \forall l \in L \iff g_X^*(x) \le u.$$

□

Since U and V are upper complete semilattices then we can define elements $U_{\max} = \max\{u : \ u \in U\}$ and $V_{\max} = \max\{v : \ v \in V\}$. Since each function $\Delta_{x,l}(u) = \Phi(x, l, u)$ is a duality then

$$\begin{aligned} \Phi(x, l, u) \le V_{\max} \ \forall u \in U &\implies \Phi'(x, l, V_{\max}) \le u \ \forall u \in U, \\ \Phi'(x, l, v) \le U_{\max} \ \forall v \in V &\implies \Phi(x, l, U_{\max}) \le v \ \forall v \in V. \end{aligned}$$

Hence there exist elements $U_{\min} \in U$ and $V_{\min} \in V$ such that $U_{\min} \le u \ \forall u \in U$ and $V_{\min} \le v \ \forall v \in V$. We have equalities:

$$\Phi(x, l, U_{\max}) = V_{\min}, \quad \Phi'(x, l, V_{\max}) = U_{\min}.$$

Therefore for any functions $f : X \to U$ and $g : L \to V$ their support sets $\operatorname{supp}(f, H_L^{\Phi'})$ and $\operatorname{supp}(g, H_X^{\Phi})$ are nonempty, and abstract convex hulls $\operatorname{co}_{H_L^{\Phi'}} f$ and $\operatorname{co}_{H_X^{\Phi}} g$ are well defined.

Proposition 6.9 *Let $f : X \to U$ and $g : L \to V$. Then second conjugate functions f^{**} and g^{**} coincide with $H_L^{\Phi'}$-convex and H_X^{Φ}-convex hulls respectively:*

$$f^{**} = \operatorname{co}_{H_L^{\Phi'}} f, \qquad g^{**} = \operatorname{co}_{H_X^{\Phi}} g. \tag{6.25}$$

Proof: We consider only function f. We have:

$$f^{**}(x) = (f_L^*)_X^*(x) = \sup_{l \in L} \Phi'(x, l, f_L^*(l)) = \sup_{l \in L} \Phi'\left(x, l, \sup_{y \in X} \Phi(y, l, f(y))\right).$$

Since each function $\Phi'(x, l, \cdot)$ is decreasing and $\Phi'(x, l, \Phi(x, l, u)) \leq u$ (see Proposition 6.5) then

$$f^{**}(x) \leq \sup_{l \in L} \Phi'(x, l, \Phi(x, l, f(x))) \leq f(x).$$

Since the function $f^{**} : X \to U$ is $H_L^{\Phi'}$-convex then $f^{**} \leq \mathrm{co}_{H_L^{\Phi'}} f$.

On the other hand, due to Proposition 6.8,

$$\begin{aligned} \mathrm{co}_{H_L^{\Phi'}} f(x) &= \sup\{\Phi'(x, l, v) : \ \Phi'(\cdot, l, v) \in \mathrm{supp}\,(f, H_L^{\Phi'})\} \\ &= \sup\{\Phi'(x, l, v) : \ f_L^*(l) \leq v\} \leq \sup_{l \in L} \Phi'(x, l, f_L^*(l)) = f^{**}(x). \end{aligned}$$

□

So a function $f : X \to U$ is $H_L^{\Phi'}$-convex if and only if $f = f^{**}$.

Now we present a formula for conjugation of elementary functions from the sets $H_L^{\Phi'}$ and H_X^{Φ}. Let $x \in X$, $l \in L$, $u \in U$ and $v \in V$. Consider the following functions defined on X and L respectively:

$$\begin{aligned} \psi_{x,u}(y) &= \sup\{h(y) : \ h \in H_L^{\Phi'}, \ h(x) \leq u\} \qquad (y \in X), \\ \psi_{l,v}(t) &= \sup\{h(t) : \ h \in H_X^{\Phi}, \ h(l) \leq v\} \qquad (t \in L). \end{aligned} \tag{6.26}$$

Proposition 6.10 *Let $x \in X$, $l \in L$, $u \in U$ and $v \in V$. Then*

$$(\Phi(x, \cdot, u))_X^* = \psi_{x,u} \qquad \text{and} \qquad (\Phi'(\cdot, l, v))_L^* = \psi_{l,v}. \tag{6.27}$$

Proof: Let us prove the second equality in (6.27). Let $f(x) = \Phi'(x, l, v)$. We need to check the equality $f^* = \psi_{l,v}$. Since the function f^* is H_X^{Φ}-convex then, due to (6.26),

$$f^* \leq \psi_{l,v} \iff f^*(l) \leq v \iff \Phi(x, l, \Phi'(x, l, v)) \leq v \ \ \forall x \in X.$$

Now we show that $f^* \geq \psi_{l,v}$. Let $\Phi(\bar{x}, \cdot, \bar{u}) \in H_X^{\Phi}$ be a function such that $\Phi(\bar{x}, l, \bar{u}) \leq v$. Then $\Phi'(\bar{x}, l, v) \leq \bar{u}$, and therefore for all $t \in L$

$$\Phi(\bar{x}, t, \bar{u}) \leq \Phi(\bar{x}, t, \Phi'(\bar{x}, l, v)) \leq \sup_{x \in X} \Phi(x, t, \Phi'(x, l, v)) = f^*(t).$$

Hence $\psi_{l,v} \leq f^*$. □

Denote by F the set of all $H_L^{\Phi'}$-convex functions $f : X \to U$. By the symbol G we denote the set of all H_X^{Φ}-convex functions $g : L \to V$. It is clear that F and G are upper complete semilattices.

Consider the mapping $\Delta : U^X \to V^L$ defined by $\Delta(f) = f_L^*$, where U^X is the set of all functions $f : X \to U$ and V^L is the set of all functions $g : L \to V$.

Proposition 6.11 *The mapping $\Delta : U^X \to V^L$ is a duality. Its dual duality $\Delta' : V^L \to U^X$ is defined by $\Delta'(g) = g_X^*$. The duality $\Delta : F \to G$ is bijective with $\Delta^{-1} = \Delta'$.*

Proof: Let $f \in U^X$ and $g \in V^L$. Then

$$\begin{aligned}\Delta(f) \leq g \quad &\Longleftrightarrow \quad f_L^*(l) \leq g(l)\ \forall l \in L \Longleftrightarrow \Phi(x, l, f(x)) \leq g(l)\ \forall x \in X\ \forall l \in L \\ &\Longleftrightarrow \quad \Phi'(x, l, g(l)) \leq f(x)\ \forall x \in X\ \forall l \in L \Longleftrightarrow g_X^* \leq f.\end{aligned}$$

Hence $\Delta : U^X \to V^L$ is a duality and $\Delta'(g) = g_X^*$.

Due to Proposition 6.9, the duality $\Delta : F \to G$ is bijective. Indeed, for any function $g \in G$ we have $g = g^{**} = \Delta(g_X^*)$, where $g_X^* \in F$. This means that $\Delta(F) = G$. If $\Delta(f_1) = \Delta(f_2)$ $(f_1, f_2 \in F)$ then $f_1 = f_1^{**} = \Delta'\Delta(f_1) = \Delta'\Delta(f_2) = f_2^{**} = f_2$. Thus, the mapping $\Delta : F \to G$ is bijective. □

Now assume that U and V are complete lattices. Let $l \in L_0$, where L_0 is defined by (6.20). Then Remark 6.3 and Definition 6.2 imply that for any $f : X \to U$ and $y \in X$

$$f_L^*(l) = \sup_{x \in X} \Phi(x, l, f(x)) = \sup_{x \in X} \Phi(y, l, f(x)) = \Phi\left(y, l, \inf_{x \in X} f(x)\right).$$

So for arbitrary $y \in X$ and $l \in L_0$ the following equality holds (see (6.20))

$$\inf_{x \in X} f(x) = \Phi'\left(y, l, \Phi\left(y, l, \inf_{x \in X} f(x)\right)\right) = \Phi'(y, l, f_L^*(l)). \tag{6.28}$$

6.5 Inverse results

Let U and V be two upper complete semilattices such that there exist elements $U_{\min} \in U$ and $V_{\min} \in V$: $U_{\min} \leq u\ \forall u \in U$, $V_{\min} \leq v\ \forall v \in V$. Let F be an upper complete semilattice of functions $f : X \to U$ and G be an upper complete semilattice of functions $g : L \to V$. Assume that $F_{\min} \in F$ and $G_{\min} \in G$, where $F_{\min}(x) \equiv U_{\min}$ and $G_{\min}(l) \equiv V_{\min}$.

Here we show that each duality $\Delta : F \to G$ can be represented as a conjugation with respect to a certain mapping Φ. A similar results was obtained in ([57], Theorem 7.3) in the case, when U and V are complete lattices.

Let $x \in X$, $l \in L$, $u \in U$ and $v \in V$. Due to our assumptions we can define the following functions

$$\begin{aligned} \psi_{x,u}(y) &= \sup\{h(y) : \ h \in F, \ h(x) \leq u\} \qquad (y \in X), \\ \psi_{l,v}(t) &= \sup\{h(t) : \ h \in G, \ h(l) \leq v\} \qquad (t \in L). \end{aligned} \tag{6.29}$$

Since F and G are upper complete semilattices then $\psi_{x,u} \in F$ and $\psi_{l,v} \in G$.

Proposition 6.12 *Let $\Delta : F \to G$ be a duality. Consider the mapping $\Phi : X \times L \times U \to V$ defined by*

$$\Phi(x, l, u) = \Delta(\psi_{x,u})(l). \tag{6.30}$$

Then for each fixed $x \in X$ and $l \in L$ the mapping $\Phi(x, l, \cdot) : U \to V$ is a duality. The dual duality $\Phi'(x, l, \cdot) : V \to U$ is defined by

$$\Phi'(x, l, v) = \Delta'(\psi_{l,v})(x). \tag{6.31}$$

Moreover, for any functions $f \in F$ and $g \in G$

$$\Delta(f) = f_L^*, \qquad \Delta'(g) = g_X^*, \tag{6.32}$$

where f_L^* is the L-conjugate to f with respect to Φ, and g_X^* is the X-conjugate to g with respect to Φ'.

Proof: We have

$$\Delta(\psi_{x,u})(l) \leq v \iff \Delta(\psi_{x,u}) \leq \psi_{l,v} \iff \Delta'(\psi_{l,v}) \leq \psi_{x,u} \iff \Delta'(\psi_{l,v})(x) \leq u.$$

This means that the mapping $\Phi'(x, l, \cdot) : V \to U$ defined by (6.31) is the dual of $\Phi(x, l, \cdot)$. Hence the mapping $\Phi(x, l, \cdot) : U \to V$ is duality.

Take a function $f \in F$. Then for any $l \in L$ and $v \in V$

$$\begin{aligned} f_L^*(l) \leq v \quad &\iff \quad \Phi(x, l, f(x)) \leq v \ \ \forall x \in X \iff \Phi'(x, l, v) \leq f(x) \ \ \forall x \in X \\ &\iff \quad \Delta'(\psi_{l,v}) \leq f \iff \Delta(f) \leq \psi_{l,v} \iff \Delta(f)(l) \leq v. \end{aligned}$$

This leads to the equality $\Delta(f) = f_L^*$. The same arguments show that $\Delta'(g) = g_X^*$ for all $g \in G$. □

Note that, under conditions of Proposition 6.12, the duality Δ is defined only for $f \in F$. At the same time, the notion of L-conjugation is applicable to every function $f : X \to U$. So we can extend the mapping $\Delta : F \to G$ on the whole space U^X by the formula $\Delta(f) = f_L^*$. And, by Proposition 6.11, this extension is also a duality.

Remark 6.4 If $F = U^X$ and $G = V^L$ then the functions (6.29) have the following simple form:

$$\psi_{x,u}(y) = \begin{cases} u, & \text{if } y = x \\ U_{\max}, & \text{if } y \neq x \end{cases} \qquad \psi_{l,v}(t) = \begin{cases} v, & \text{if } t = l \\ V_{\max}, & \text{if } t \neq l \end{cases}$$

where $U_{\max} = \sup_{u \in U} u$ and $V_{\max} = \sup_{v \in V} v$.

Proposition 6.13 *Let $\Delta : F \to G$ be a bijective duality. Consider the mappings Φ and Φ' defined by (6.30) and (6.31). Then a function $f : X \to U$ is $H_L^{\Phi'}$-convex if and only if $f \in F$; a function $g : L \to V$ is H_X^{Φ}-convex if and only if $g \in G$.*

Proof: We prove our statement for the set F. Since $\psi_{l,v} \in G$ and $\Phi'(x, l, v) = \Delta'(\psi_{l,v})(x)$ then for each fixed $l \in L$, $v \in V$ the function $h(x) = \Phi'(x, l, v)$ belongs to F. Therefore the set $H_L^{\Phi'}$ is a subset of F. Since F is upper complete semilattice then every $H_L^{\Phi'}$-convex function $f : X \to U$ belongs to F. Conversely, let $f \in F$. Since the duality $\Delta : F \to G$ is bijective then (see (6.16) and (6.32)) $f = \Delta'\Delta(f) = f^{**}$. Hence the function f is $H_L^{\Phi'}$-convex. □

Since each duality $\Delta : F \to G$ is a conjugation then support sets of functions $f \in F$ can be described via $\Delta(f)$ (see Proposition 6.8).

Indeed, let $\Delta : F \to G$ be a duality and Δ' be the dual of Δ. Then, by Proposition 6.12, for every function $f \in F$

$$\Delta(f)(l) = f_L^*(l) = \sup_{x \in X} \Phi(x, l, f(x)) \quad \forall l \in L,$$

where Φ is defined by (6.30). Proposition 6.8 implies that for every $l \in L$ and $v \in V$ we have the following equivalence

$$\Phi'(\cdot, l, v) \in \operatorname{supp}(f, H_L^{\Phi'}) \iff \Delta(f)(l) \leq v,$$

where Φ' is defined by (6.31).

Conclusion

Throughout the entire thesis we talked about various issues related to abstract subdifferentials and separation properties. Our main aim was to find possible approaches to some global optimization problems. First we examined abstract subdifferentials and separation properties in two particular cases. Then we considered the problems of subdifferential calculus and separation of sets from a general point of view. Finally, we investigated optimality conditions via generalized subdifferentials for non-real-valued functions.

In Chapter 2 we examined abstract subdifferentials of CAR (convex-along-rays) functions defined on $\mathbb{R}^n$ with respect to special sets of elementary functions. We took various approaches to the calculation of abstract subgradients (i.e, the elements of abstract subdifferential) of CAR functions. In particular, we gave some conditions, which guarantee the existence of abstract subdifferential of CAR functions and therefore allow one to describe certain abstract subgradients. The results obtained can be applied for the global minimization of some CAR functions over subsets of $\mathbb{R}^n$ by using numerical methods.

In Chapter 3 we discussed the weak separability of two star-shaped sets by a collection of linear functions. The main result of this chapter is the characterization of a solution of a "best approximation -like" problem for star-shaped sets. We introduced a notion of a star-shaped distance and gave necessary and sufficient conditions for its global minimum over a radiant set in terms of weak separability of star-shaped sets. Note that the class of such problems is quite broad. However the description of separating collections of linear functions is very complicated. Thus, in order to apply this result in practice, a further research is required. We need to describe separation collections at least for some simple star-shaped sets.

Chapter 4 takes a general approach to the problem of subdifferential calculus for abstract convex functions. As used here the subdifferential calculus means the existence of some calculus rules. Such rules allow the description of subdifferentials of some combinations of abstract convex functions via subdifferentials of given functions. Since conditions for the global minimum of an abstract convex function can be given in terms of the abstract subdifferential then the existence of calculus rules is very important.

It turned out, that the so-called strong globalization property can provide subdifferential calculus for different combinations of abstract convex functions, including the maximum. Namely, we proved that, if the set of elementary functions has the strong globalization property, then subdifferential calculus can be expressed in terms of special functions that in a sense approximate the given functions. If, moreover, these approximation functions possess certain equalities, then we get exact calculus rules.

We considered some examples, which demonstrate that there are a lot of sets of nonconvex functions having the strong globalization property. This means that the results obtained can find applications in a broad range of global optimization problems.

In Chapter 5 we investigated separation properties via a special type of connectedness of a topological space with respect to a convexity on this space. We chose a way based on the separation of convex sets by elements of a fixed subbase. In order to get efficient results, we required some restrictions on the choice of a subbase in terms of the connectedness. This approach leads to a weak separation property for arbitrary convex sets and to a stronger one for closed convex sets.

First we indicated the cases, when the convexity is of finite arity. This gives a description of convex sets. Moreover, in view of Theorem 1.3, we can use this result for further research in order to get the separation property S_4 (see Section 1.4).

Then we provided some conditions, which guarantee a description of closed convex sets and the following property: each closed convex set and each point in its complement can be separated by an element of the subbase. As a particular case, this result implies a description of abstract convex functions and sets.

We also proved that each two disjoint convex sets, one of them being closed and the other one compact, can be separated by an element of the subbase. This can be applied for a characterization of solutions of some "best approximation -like" problems.

An important issue is the description of the abstract convex hull of a finite union of abstract convex sets. Our results about this can be applied to the formula, which was obtained in Chapter 4, for the subdifferential of the maximum of a finite collection of abstract convex functions. Furthermore, this gives a description of the support set of the maximum of abstract convex functions. Thus, we can express conditions for the global minimum of the maximum of a finite collection of abstract convex functions in terms of subdifferentials (or support sets) of given functions.

Note that the main assumptions in Chapters 4 and 5 are also valid for usual convex functions and sets. Thus, we picked out some essential properties of the classical convexity and used them for generalization of subdifferential calculus and separation theorems.

In Chapter 6 we introduced generalized conjugations and subdifferentials for functions with values in an upper complete semilattice and a partially ordered set respectively. They are based on the notion of a duality between two partially ordered sets, which generalizes corresponding notion considered in [57] for pairs of complete lattices. The main feature of such broadly defined subdifferentials is that they provide conditions not only for a global minimum, but also for a global minimal element. We also proved that each duality of two upper complete semilattices of functions is a conjugation. This implies that global optimality conditions can be formulated in terms of dualities. Consequently, an further investigation of these dualities is of importance.

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