Graphs and Subgraphs with Bounded Degree

PhD Thesis

Jakub Teska

Certificate of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other university or institution.

Jakub Teska

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Publications arising from this thesis

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- R. Kužel, J. Teska: Trestles in $K_{1,r}$-free graphs, *Proceedings of the 16th Australian Workshop on Combinatorial Algorithms* (2005), Ballarat, Australia (2005), 147-152.

- R. Kužel, M. Miller, J. Teska: Divisibility conditions in almost Moore digraphs with selfrepeats, *Elect. Notes of Discrete Math.* **24** (2006), 161-163.

- J. Teska: On 2-walks in chordal planar graphs, *Proceedings of the 16th Australian Workshop on Combinatorial Algorithms* (2005), Ballarat, Australia, (2005), 333-340.

- J. Teska: On 2-walks in chordal planar graphs, submitted to *Discrete Mathematics.*

- R. Kužel, J. Teska: r-Trestles in $K_{1,r}$-free graphs, submitted *Graphs and Combinatorics.*

- R. Kužel, M. Miller, J. Teska: Divisibility conditions in almost Moore digraphs, in preparation.

Abstract

The topology of a network (such as a telecommunications, multiprocessor, or local area network, to name just a few) is usually modeled by a graph in which vertices represent ‘nodes’ (stations or processors) while undirected or directed edges stand for ‘links’ or other types of connections, physical or virtual.

A cycle that contains every vertex of a graph is called a hamiltonian cycle and a graph which contains a hamiltonian cycle is called a hamiltonian graph. The problem of the existence of a hamiltonian cycle is closely related to the well known problem of a travelling salesman. These problems are NP-complete and NP-hard, respectively. While some necessary and sufficient conditions are known, to date, no practical characterization of hamiltonian graphs has been found. There are several ways to generalize the notion of a hamiltonian cycle. In this thesis we make original contributions in two of them, namely, k-walks and r-trestles.

In particular, as our main results, we present several new sufficient conditions for the existence of k-walks and r-trestles in a graph. Additionally, we also give some new results in the degree/diameter problem, which is to determine the largest graphs or digraphs of given maximum degree d and given diameter k. We present new structural properties of almost Moore digraphs with selfrepeats and we prove the nonexistence of infinitely families of these digraphs for some combinations of values d and k.

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Chapter 1

Introduction

1.1 Structure of the thesis

In Chapter 1 we give basic definitions of graph theory and an introduction to the thesis.

In Chapter 2 we give an overview of significant results concerning the existence of a hamiltonian cycle in a graph.

In Chapter 3 we give a short overview of results concerning k-walks. Moreover, we present new results in this area, in particular, we obtain new sufficient conditions for the existence of 2-walks.

In Chapter 4 we give a short overview of results concerning r-trestles and we present a new result in this area. It is a new general sufficient condition for the existence of r-trestles in graphs.

In Chapter 5 we give an overview of the degree/diameter problem for undirected and directed graphs. We provide new structural properties of almost Moore digraphs, from which we derive the non-existence of almost Moore digraphs for certain values of d and k.

1.2 Motivation

The topology of a network (such as a telecommunications, multiprocessor, or local area network, to name just a few) is usually modelled by a graph in which vertices represent 'nodes' (stations or processors), while undirected or directed edges stand for 'links' or other types of connections.

A cycle that contains every vertex of a graph is called a *hamiltonian cycle* and a graph which contains a hamiltonian cycle is called a *hamiltonian graph*. This terminology is used in honor of Sir William Rowan Hamilton. The problem of existence of a hamiltonian cycle is closely related to the well known problem of a travelling salesman, which can be described as follows. A travelling salesman wishes to visit a number of towns and then return to his starting point. Given the travelling times between towns, how should he plan his itinerary so that he visits each town exactly once and travels in all for as short a time as possible? In graphical terms, the aim is to find a minimum-weight hamiltonian cycle in a weighted graph.

Since the problem of recognizing hamiltonian graphs is NP-complete, the problem of a travelling salesman is NP-hard, and no practical solution to either problem has been found. However, there has still been a considerable amount of information discovered about hamiltonian graphs.

While the problem of characterizing hamiltonian graphs is hard, it is natural to generalize the concept of a hamiltonian cycle. There are several ways how to generalize the notion of a hamiltonian cycle. In this thesis we deal with two of them, namely, k-walks and r-trestles.

A *k-walk* is a closed spanning walk which uses every vertex of a graph at most k times. The concept of a k-walk is a generalization of a hamiltonian cycle, while a 1-walk is exactly a hamiltonian cycle.

Since the idea of a k-walk, for $k > 1$ 'sounds' easier than that of a hamiltonian cycle, one might expect that, for some fixed k, the problem of recognizing graphs containing a k-walk will not be NP-complete any more. This, however, is not the case. In fact, problem of recognizing graphs containing a k-walk is still NP-complete for arbitrary fixed k (see [58]), and no practical characterization for graphs with k-walks has been found.

The study of k-walks was initiated by Jackson and Wormald in 1990. In their pioneering paper [58] they found various sufficient conditions for a graph to have a k-walk. In particular, they generalized the result of Oberly and Sumner [78] (see the next section). Apart from this pioneering paper, there are very few other results concerning k-walks. In 1996, Favaron, Flandrin, Li, and Ryjáček [41] proved the existence and some properties of 2-walks in connected, almost claw-free graphs. In 2000, Ellingham and Zha [37] obtained a new sufficient condition for the existence of a 2-walk in a graph.

In this largely unexplored area we obtained some new results concerning k-walks and, in particular, 2-walks. Motivated by results for hamiltonian cycles (see Chapter 3) and, in particular, by the result of Böhme et al. proving the existence of a hamiltonian cycle in more than 1-tough chordal planar graphs, we prove the existence of a 2-walk in chordal planar graphs with toughness greater then $\frac{3}{4}$. We also find the toughness threshold for the existence of 2-walks in K_4-minor free graphs.

An *r-trestle* is a 2-connected graph with maximum degree at most r. We say that a graph G has an r-trestle if G contains a spanning subgraph which is an r-trestle. The concept of an r-trestle is a generalization of a hamiltonian cycle, while a 2-trestle is exactly a hamiltonian cycle.

Spanning subgraphs with bounded degree have been studied deeply, for example, k-spanning trees, spiders, etc. But, surprisingly, almost nobody dealt with 2-connected spanning subgraphs with bounded degree. As far as we know, there are three papers dealing with trestles.

We obtained one new result concerning r-trestles. We prove that every 2-connected, $K_{1,r}$-free graph has an r-trestle. Moreover, we present graphs that show that our result is sharp, so that the result cannot be improved.

The last chapter of this thesis is devoted to Moore graphs.

The design of large interconnection networks has become of growing interest due to recent advances in very large scale integrated technology. Theoretical research includes discovering optimal designs for network topology.

One of the natural questions in designs for network topology is the following: what is then the largest number of nodes in a network with a limited degree and diameter? In graph theoretical terms we get the well known degree/diameter problem:

Given natural numbers Δ and D, find the largest possible number of vertices $n_{\Delta,D}$ in a graph of maximum degree Δ and diameter at most D.

In this thesis we focus on the proofs of nonexistence of digraphs of order close to the Moore bound.

Although we give an overview of both undirected and directed case, our contributions concern only the directed case. Since Moore digraphs exist only in trivial cases, we deal with almost Moore digraphs. We present new properties of almost Moore digraphs with selfrepeats and using these properties we prove the nonexistence of infinitely many almost Moore digraphs for some combinations of values of k and d.

The main contributions of this thesis are presented in Chapters 3, 4 and 5. All original results are indicated by the symbol $\bigstar$.

Finally, in the Conclusion chapter, we summarize our results and list some open problems and conjectures.

1.3 Basic definitions

A *graph* is a pair $G = (V, E)$ of sets satisfying $E \subseteq [V]^2$; thus, the elements of E are 2-element subsets of V. To avoid notational confusion, we shall always assume that $V \cap E = \emptyset$. The elements of V are the *vertices* (or *nodes*, or *points*) of the graph G, the elements of E are its *edges* (or *lines*). The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices form an edge. Just how these dots and lines are drawn is considered irrelevant: all that matters is the information concerning which vertices form edges and which do not.

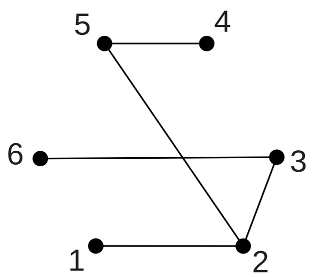

Figure 1.1: The graph on six vertices $V = \{1, ..., 6\}$ and with edge set $E = \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{3, 6\}, \{4, 5\}\}$.

A graph with vertex set V is said to be a graph *on* V. The vertex set of a graph G is referred to as $V(G)$, its edge set as $E(G)$. We shall not always distinguish strictly between a graph and its vertex or edge set. For example, we may speak of a vertex $v \in G$ (rather than $v \in V(G)$), an edge $e \in G$, and so on. A *weighted graph* is a graph in which each edge is given a numerical weight (which is usually taken to be positive).

The number of vertices of a graph G is its *order*, written as $|G|$. Graphs are finite or infinite, according to their order; unless otherwise stated, the graphs we consider are all finite.

A vertex v is *incident* with an edge e if $v \in e$; then also e is an *edge at* v. The two vertices incident with an edge are its *endvertices* or *ends*, and an edge *joins* its ends. An edge $\{x, y\}$ is usually written as xy (or yx). Two vertices x, y of G are *adjacent*, or *neighbours*, if xy is an edge of G. Two edges $e \neq f$ are *adjacent* if they have an end in common. If all the vertices of G are pairwise adjacent then G is *complete*. A complete graph on n vertices is denoted as K_n; in particular, K_3 is called a *triangle*. Pairwise non-adjacent vertices or edges are called *independent*. More formally, a set of vertices or edges is *independent* if no two of its elements are adjacent.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' *isomorphic*, and write $G \simeq G'$, if there exists a bijection $\varphi : V \to V'$ with $xy \in E \Leftrightarrow \varphi(x)\varphi(y) \in E'$, for all $x, y \in V$. Such a map φ is called an *isomorphism*; if $G = G'$ then φ is called an *automorphism*. We do not normally distinguish between isomorphic graphs. Thus, we usually write $G = G'$ rather than $G \simeq G'$.

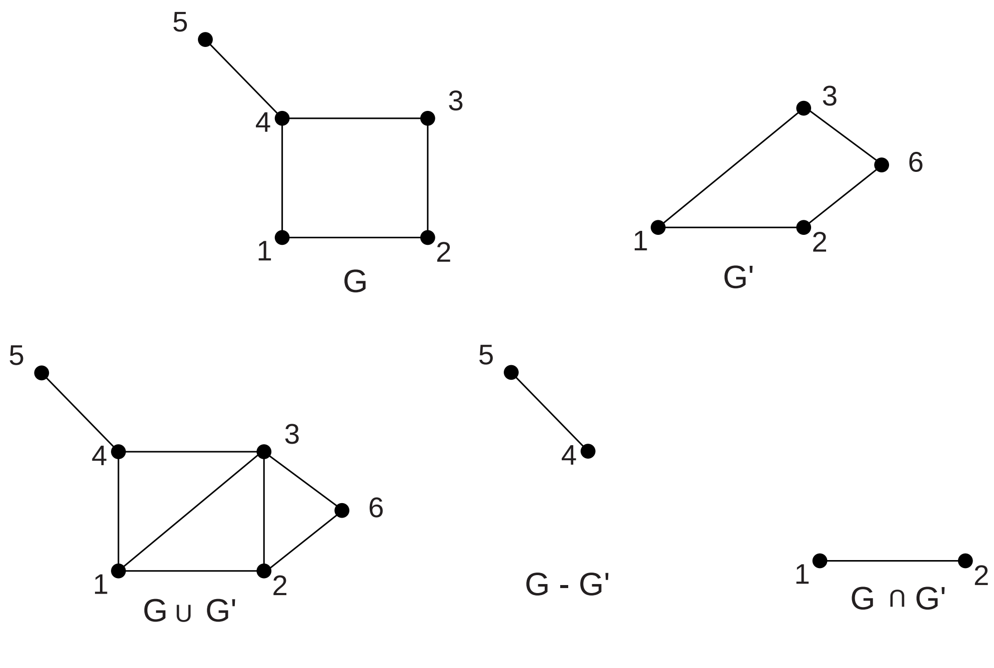

Figure 1.2: Union, difference and intersection of graphs G and G'.

We set $G \cup G' := (V \cup V', E \cup E')$ and $G \cap G' := (V \cap V', E \cap E')$. If $G \cap G' = \emptyset$ then G and G' are *disjoint*. If $V' \subseteq V$ and $E' \subseteq E$ then G' is a *subgraph* of G, written as $G' \subseteq G$. Less formally, we say that G *contains* G'.

If $G' \subseteq G$ and G' contains all the edges $xy \in E$, with $x, y \in V'$, then G' is an *induced subgraph* of G; we say that V' *induces* G' in G, and write $G' = \langle V' \rangle_G$.

Thus, if $U \subseteq V$ is any set of vertices then $\langle U \rangle_G$ denotes the graph on U whose edges are precisely the edges of G with both ends in U. If H is a subgraph of G, not necessarily induced, we abbreviate $\langle V(H) \rangle_G$ to $\langle H \rangle_G$. Finally, $G' \subseteq G$ is a *spanning* subgraph of G if V' spans all of G, i.e., if $V' = V$.

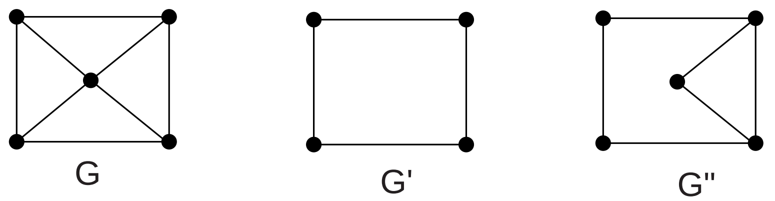

Figure 1.3: Graph G with subgraphs G' and G'' : G' is an induced subgraph of G, but G'' is not.

If U is any set of vertices of G, we write $G - U$ for $\langle V \backslash U \rangle_G$. In other words, $G - U$ is obtained from G by *deleting* all the vertices in U and their incident edges. If $U = \{v\}$ is a single vertex, called *singleton*, we write $G - v$ rather than $G - \{v\}$. Instead of $G - V(G')$, we simply write $G - G'$. By a *join of* two graphs G and H we mean a graph $G \cup H$ plus all the edges between G and H.

The *complement* $\overline{G}$ of G is the graph on the same vertex set $V(G)$ with the edge set $[V(G)]^2 \backslash E(G)$. The *line graph* $L(G)$ of G is the graph on E in which $x, y \in E$ are adjacent as vertices if and only if they are adjacent as edges in G.

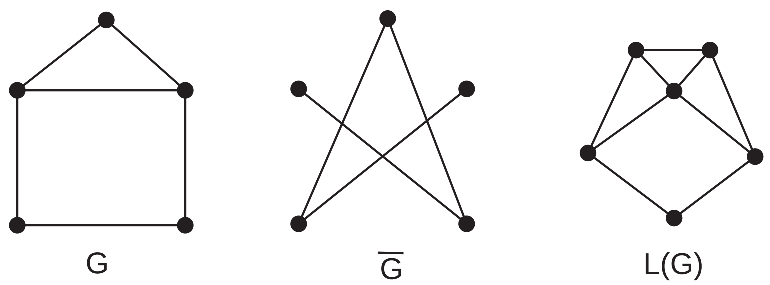

Figure 1.4: Graph G with its complement and line graph.

Let $G = (V, E)$ be a non-empty graph, that is, $V \neq \emptyset$. The set of neighbours of a vertex v in G is denoted by $N_G(v)$ or, briefly, by $N(v)$. More generally, for $U \subseteq V$, the neighbours in $V \backslash U$ of vertices in U are called the *neighbours of* U; their set is denoted by $N(U)$. The *degree* $d_G(v) = d(v)$ of a vertex v is the number of edges at v; by our definition of a graph, this is equal to the number of neighbours of v. A vertex of degree 0 is *isolated*. The number $\delta(G) = min\{d(v)|v \in V\}$ is the *minimum degree* of G; similarly, the number $\Delta(G) = max\{d(v)|v \in V\}$ is its *maximum degree*. If all the vertices of G have the same degree k then G is k-*regular* or, simply, *regular*. A 3-regular graph is called *cubic*.

A *path* is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, ..., x_k\} \quad E = \{x_0x_1, x_1x_2, ..., x_{k-1}x_k\},$$

where the x_i are all distinct. The vertices x_0 and x_k are *linked* by P and are called its *ends*; the vertices $x_1, ..., x_{k-1}$ are the *inner* vertices of P. The number of edges of a path is its *length*, and a path of length k is denoted by P_{k-1}. Note that k is allowed to be zero; thus, $P_0 = K_1$.

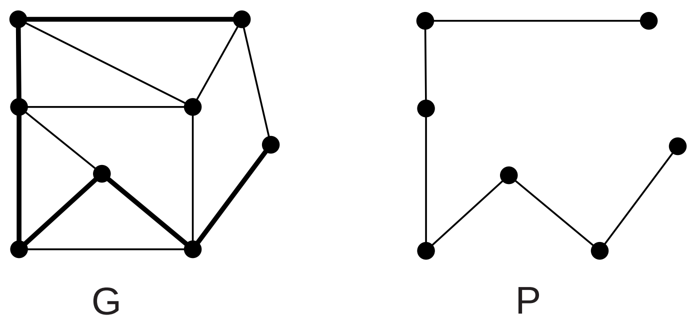

Figure 1.5: Path $P = P_6$ in G.

We often refer to a path by the natural sequence of its vertices, writing, say, $P = x_0x_1...x_k$, and calling P a path *from* x_0 *to* x_k (as well as *between* x_0 *and* x_k). For $0 \leq i \leq j \leq k$ we write,

$$\begin{aligned} Px_i &= x_0...x_i, \\ x_iP &= x_i...x_k, \\ x_iPx_j &= x_i...x_j, \end{aligned}$$

for the appropriate subpaths of P. We use similar intuitive notation for the concatenation of paths; for example, if the union $Px \cup xQy \cup yR$ of three paths is again a path, we may simply denote it by $PxQyR$. Given sets A, B of vertices, we call $P = x_0...x_k$ an *AB path* if $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$. As before, we write aB path rather than $\{a\}B$ path, etc. Two or more paths are *independent* if none of them contains an inner vertex of the other path(s). Two ab paths, for instance, are independent if and only if a and b are their only common vertices.

If $P = x_0...x_{k-1}$ is a path, and $k \geq 3$, then the graph $C = P + x_{k-1}x_0$ is called a *cycle*. As with paths, we often denote a cycle by its (cyclic) sequence of vertices; the above cycle C might be written as $x_0...x_{k-1}x_0$. The *length* of a cycle is its number of edges (or vertices); a cycle of length k is called a *k-cycle* and denoted by C_k.

The minimum length of a cycle (contained) in a graph G is the *girth* $g(G)$ of G; the maximum length of a cycle in G is its *circumference*, denoted $c(G)$. An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle. Thus, an *induced cycle* in G, that is, a cycle in G forming an induced subgraph, is one that has no chords (Fig. 1.6).

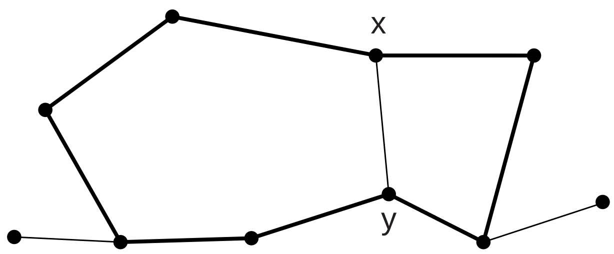

Figure 1.6: Cycle C_8 with chord xy, and induced cycles C_4, C_6.

The *distance* $d_G(x, y)$ in G of two vertices x, y is the length of a shortest xy path in G; if no such path exists then we set $d(x, y) = \infty$. The greatest distance between any two vertices in G is the *diameter* of G, denoted by $diam(G)$. A vertex is *central* in G if its greatest distance from any other vertex is as small as possible. This distance is the *radius* of G, denoted by $rad(G)$. A *walk* (of *length* k) in a graph G is a non-empty alternating sequence $v_0e_0v_1e_1...e_{k-1}v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$ then the walk is *closed*. If the vertices in a walk are all distinct then the walk, clearly, is a path in G. In general, every walk between two vertices contains a path between these vertices.

We say that a graph G is *connected* if any two of its vertices are linked by a path in G. If $U \subseteq V(G)$ and $\langle U \rangle_G$ is connected, we also say that U itself is connected (in G). Let $G = (V, E)$ be a graph. A maximal connected subgraph of G is called a *component* of G. Note that a component, being connected, is always non-empty; the empty graph, therefore, has no components. Integer $\omega(G)$ denotes the number of components in the graph G. If $A, B \subseteq V$ and $X \subseteq V \cup E$ are such that every AB path in G contains a vertex or an edge from X then we say that X *separates* the sets A and B in G. This implies, in particular, that $A \cap B \subseteq X$. More generally, we say that X *separates* G, and we call X a *separating set* in G if X separates two vertices of $G - X$ in G. On the other hand, a vertex which separates two other vertices of the same component is a *cutvertex*, and an edge separating its ends is a *bridge*. Thus, the bridges in a graph are precisely those edges that do not lie on any cycle (Fig. 1.7).

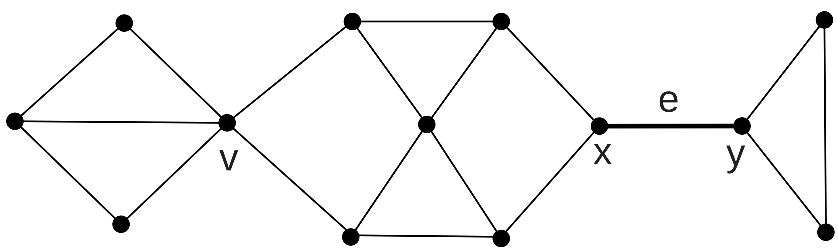

Figure 1.7: Graph with cutvertices x, y, v and bridge $e = xy$.

We say that a graph G is *k-connected* (for $k \in N$) if $|G| > k$ and if $G - X$ is connected, for every set $X \subseteq V$ with $|X| < k$. In other words, no two vertices of G are separated by fewer than k other vertices. Every (non-empty) graph is 0-connected, and the 1-connected graphs are precisely the non-trivial connected graphs. The greatest integer k such that G is k-connected is the *connectivity* $\kappa(G)$ of G. Thus, $\kappa(G) = 0$ if and only if G is disconnected or $G = K_1$. If $|G| > 1$ and $G - F$ is connected, for every set $F \subseteq E$ of fewer than ℓ edges, then G is

ℓ-edge-connected. The greatest integer ℓ such that G is ℓ-edge-connected is the *edge-connectivity* $\lambda(G)$ of G. In particular, we have $\lambda(G) = 0$ if G is disconnected.

An *acyclic* graph, one not containing any cycles, is called a *forest.* A connected forest is called a *tree.* Thus, a forest is a graph whose components are trees. The vertices of degree 1 in a tree are its *leaves.* Every nontrivial tree has at least two leaves, for example, the ends of a longest path. This fact often comes in handy, especially in induction proofs about trees: if we remove a leaf from a tree, what remains is still a tree.

Theorem 1.3.1 *The following assertions are equivalent for a graph T.*

(*i*) *T is a tree.*

(*ii*) *Any two vertices of T are linked by a unique path in T.*

(*iii*) *T is minimally connected, i.e., T is connected but $T - e$ is disconnected, for every edge $e \in T$.*

(*iv*) *T is maximally acyclic, i.e., T contains no cycle but $T + xy$ does, for any two non-adjacent vertices $x, y \in T$.*

A *spanning tree* T of a graph G is a spanning subgraph, i.e., $|V(T)| = |V(G)|$, which is a tree. It is obvious that every connected graph G contains a spanning tree (see Fig. 1.8 for an example).

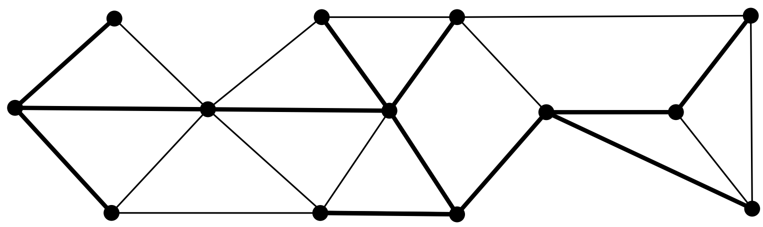

Figure 1.8: Spanning tree (drawn in bold) in a graph.

Subgraph K of a graph G is called a *clique* if K is a complete graph and whenever $K \subseteq K' \subseteq G$ and K' is also complete then $K = K'$. Thus a clique is a maximal complete subgraph (see Fig. 1.9 for an example). The greatest integer $r(G)$ such that $K_r \subseteq G$ is the *clique number* $r(G)$ of G, and the greatest integer $\alpha(G)$ such that $\overline{K_\alpha} \subseteq G$ (induced) is the independence number $\alpha(G)$ of G.

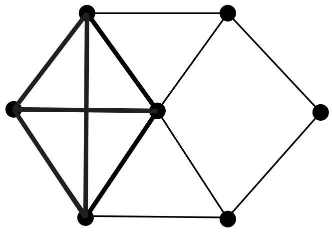

Figure 1.9: Clique K_4 (drawn in bold) in a graph.

A set M of independent edges in a graph $G = (V, E)$ is called a *matching*. A matching M in a graph G is called a *perfect matching* if every vertex in G is incident with an edge in M. A spanning subgraph of a graph G is called a *factor* of G and k-regular spanning subgraph is called a k-factor (see Fig. 1.10 for an example of a 2-factor). Thus, a factor H of G is a 1-factor of G if and only if $E(H)$ is a perfect matching.

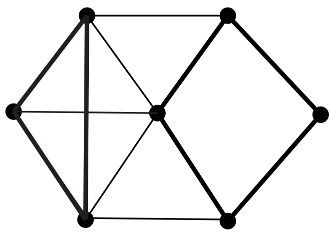

Figure 1.10: 2-factor (drawn in bold) in a graph.

A graph $G = (V, E)$ is called *bipartite* if V admits a partition into two classes such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. A bipartite graph in which every two vertices from different partition classes are adjacent is a *complete bipartite graph*. The complete bipartite graph, where one partition contains m vertices and the other partition

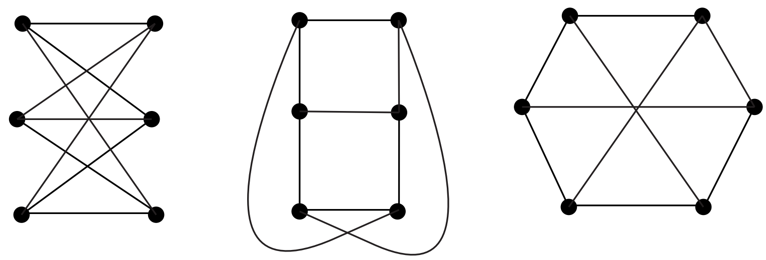

Figure 1.11: Three drawings of the bipartite graph $K_{3,3}$.

contains n vertices, is denoted by $K_{m,n}$. Graphs of the form $K_{1,n}$ are called *stars* and the graph $K_{1,3}$ is called *claw*.

Clearly, a bipartite graph cannot contain an odd cycle, that is, a cycle of odd length. In fact, bipartite graphs are characterized by this property:

Theorem 1.3.2 *A graph is bipartite if and only if it contains no odd cycle.*

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In the remainder of this chapter we state some definitions which are not basic but we will need them in later chapters. A *minor* of a graph G can be obtained from the graph G by first deleting some vertices and edges, and then contracting some further edges (i.e., replacing an induced path by an edge). Conversely, any graph obtained from another by repeated deletions and contractions (in any order) is its minor (see Fig. 1.12). Note that every subgraph of a graph is also its minor, in particular, every graph is its own minor.

If we can draw a graph G in such a way that no two edges meet in a point other than a common end then G is *embeddable in the plane* and G is a *plane* graph. Such a drawing is called a *planar embedding* of G or an *embedding of G into the plane.* Abstract graphs that can be drawn in this way are *planar.* Planar graphs are characterized by the Kuratowski's theorem.

Theorem 1.3.3 [67] *Graph G is planar if and only if G contains neither K_5 nor $K_{3,3}$ as a minor.*

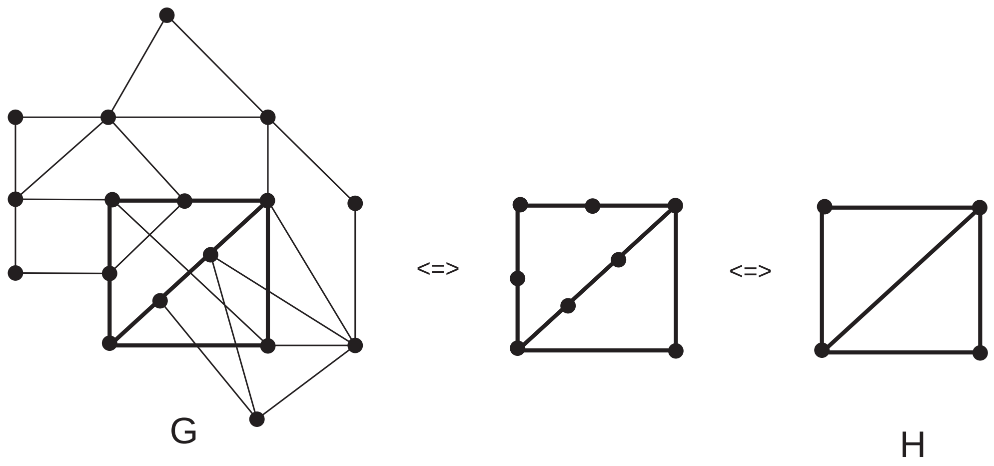

Figure 1.12: Graph G and its minor H.

A graph G is *chordal* if each of its cycles of length at least 4 has a chord, i.e., if G contains no induced cycles other than triangles (see Fig. 1.13).

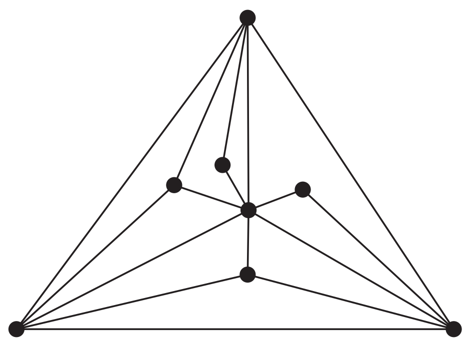

Figure 1.13: Example of a chordal planar graph.

A graph G is called an *interval graph* if there exists a set $\{I_v | v \in V(G)\}$ of real intervals such that $I_u \cap I_v \neq \emptyset$ if and only if $uv \in E(G)$. A class of interval graphs is a subclass of chordal graphs because every interval graph is also a chordal graph.

Another subclass of chordal graphs is the class of split graphs. A graph G is called a *split graph* if $V(G)$ can be partitioned into an independent set and a clique.

The class of k*-trees* can be defined recursively as follows: a complete graph K_{k+1} of order $k+1$ is a k-tree, and if G is a k-tree then a graph obtained from G and K_{k+1} by identifying k vertices contained in a complete subgraph of G and K_{k+1}

is also a k-tree. Hence, 2-trees are obtained from triangles by identifying pairs of edges.

The *tree decomposition* of a graph G is a tree T satisfying the following properties:

(i) $V(T) \subseteq \mathcal{P}(V(G))$ i.e., the nodes of T are subsets of $V(G)$ ($\mathcal{P}(V(G))$ denotes all possible subsets of the set $V(G)$).

(ii) For each edge $uv \in E(G)$, there is a node X_i of T, such that both $u, v \in X_i$.

(iii) For each vertex $u \in V(G)$, the subgraph of T induced by sets containing u is connected (i.e., it is a subtree).

The *width of a decomposition* T is $\max_{X_i \in V(G)} |X_i| - 1$. The *treewidth* of a graph G is the minimum width among all its decompositions.

Before we finish this chapter we state some basic definitions for digraphs. A *digraph* (directed graph) is a pair $G = (V, E)$ of sets satisfying $E \subseteq V \times V$; thus, the elements of E are ordered pairs of V. We say that directed edge xy goes from x to y. Note that a digraph may have several edges between the same two vertices x, y. If they have the same direction (say from x to y), they are *parallel*. If $x = y$ the edge e is called a *loop*. The *out-degree* of a vertex v, denoted by $d^-_G(v)$, is the number of edges going from v. Similarly, the *in-degree* of a vertex v, denoted by $d^+(v)$, is the number of edges going to v. The *out-neighborhood* of a vertex v is a set of vertices $N^-_G(v) = \{u; vu \in E(G)\}$. The *in-neighborhood* of a vertex v is a set of vertices $N^+_G(v) = \{u; uv \in E(G)\}$. We say that a digraph is *diregular* if $|N^-_G(v)| = |N^+_G(v)|$ for every vertex in $V(G)$. Other notions like diameter, path, cycle etc. for digraphs are defined similarly to the notions for undirected graphs.

Chapter 2

Eulerian and hamiltonian cycles

2.1 Eulerian cycles

Probably the oldest and best known of all problems in graph theory centers on the bridges over the river Pregel in the city of Königsberg (presently called Kaliningrad) in Russia. The legend says that the inhabitants of Königsberg amused themselves by trying to determine a route across each of the bridges between the two islands (A and B in Fig. 2.1), both river banks (C and D of Fig. 2.1) and back to their starting point using each bridge exactly once.

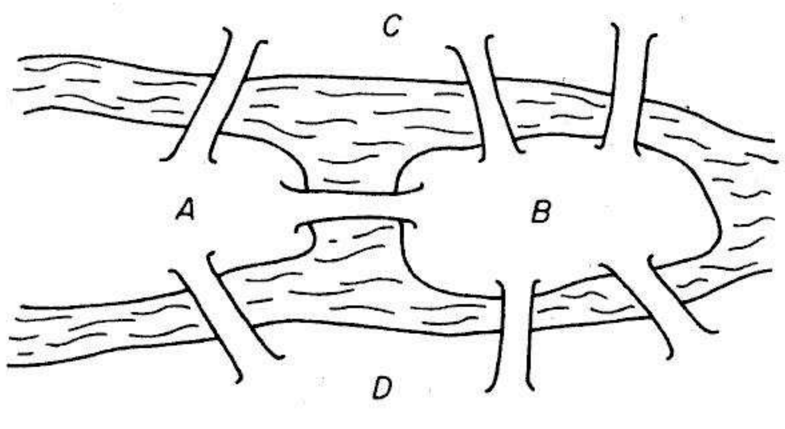

Figure 2.1: Bridges of river Pregel (from [18]).

After many attempts, they all came to believe that such a route was not possible. In 1736, Leonhard Euler [40] published what is believed to be the first paper on graph theory, in which he investigated the Königsberg bridge problem in mathematical terms.

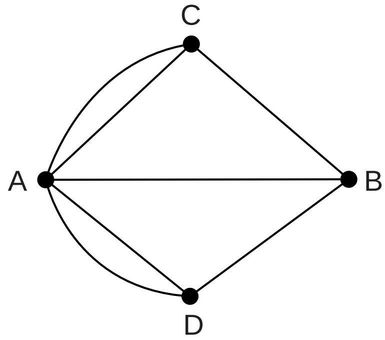

Figure 2.2: Multigraph of the bridges.

The problem seeks a circuit (closed walk) that contains each edge exactly once. Such a circuit is called an *eulerian circuit*. A trail containing every edge of the graph once is called an *eulerian trail*. Several characterizations have been developed for eulerian graphs. The result below is a blend of the works of Euler [40], Hierholzer [55] and Veblen [91].

Theorem 2.1.1 *The following statements are equivalent for a connected graph G:*

(i) *The graph G contains an eulerian circuit.*
(ii) *Each vertex of G has even degree.*
(iii) *The edge set of G can be partitioned into cycles.*

Clearly, since the multigraph representing the bridges of river Pregel cannot be partitioned into cycles, or, equivalently, since the multigraph contains vertices of odd degree (see Fig. 2.2), an eulerian circuit cannot exist.

2.2 Hamiltonian cycles

In the previous section we tried to determine a tour of a graph that would use each edge once and only once. It seems natural to vary this question and try to visit, rather then each edge, each vertex once and only once. A cycle (path) that contains every vertex of a graph is called a *hamiltonian cycle* (*path*). A graph containing a hamiltonian cycle is called a *hamiltonian graph.* This terminology is used in honor of Sir William Rowan Hamilton, who, in 1857, came with a new calculus and exemplified its use by an amusing game. This game, called the *icosian game*, consisted of a wooden dodecahedron (see Fig. 2.3) with pegs inserted at each of the twenty vertices. These pegs supposedly represented the twenty most important cities of the time. The object of the game was to mark a route (following the edges of the dodecahedron) passing through each of the cities exactly once and finally returning to the initial city.

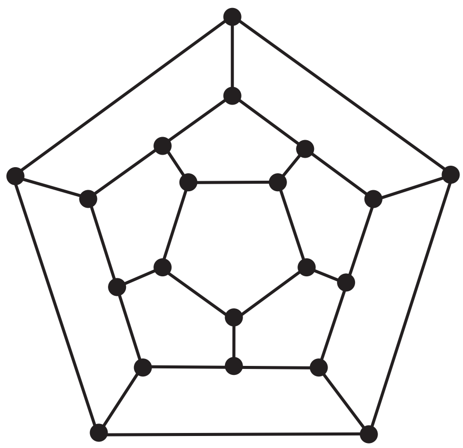

Figure 2.3: Graph of dodechedron.

Since the idea of a hamiltonian cycle 'sounds' analogous to that of an eulerian circuit, one might expect that we will be able to establish some sort of a corresponding theory. This, however, is definitely not the case. In fact, the problem of recognizing hamiltonian graphs is NP-complete (see [45]), and no practical characterization for hamiltonian graphs has been found. However, there has still been a considerable amount of information discovered about hamiltonian graphs. Perhaps the theorem that stimulated most of the subsequent work is that of Ore [79]. It stems from the idea that if a sufficient number of edges are present in the graph then a hamiltonian cycle will exist. To ensure a sufficient number of edges, we try to keep the degree sum of nonadjacent pairs of vertices at a fairly high level. We can see the effect that controlling degree sums provides when we consider a vertex x of 'low' degree. Since x has many nonadjacencies in the graph, the degrees of all these vertices are then forced to be 'high', to ensure that the degree sum remains sufficiently large. Thus, the graph has many vertices of high degree, and so, one hopes the graph contains enough structure to ensure that it is hamiltonian.

Theorem 2.2.1 [79] *If G is a graph of order $n \geq 3$ such that for all pairs of distinct nonadjacent vertices x and y, $d(x) + d(y) \geq n$, then G is hamiltonian.*

We now state an immediate corollary of Ore's theorem that actually preceded it. This result was originally obtained by Dirac [34]. The original proof of this theorem is illustrative and typical.

Corollary 2.2.1 [34] *If G is a graph of order $n \geq 3$ such that $\delta(G) \geq \frac{n}{2}$ then G is hamiltonian.*

Proof. Let $G = (V, E)$ be a graph with $|V(G)| = n \geq 3$ and $\delta(G) \geq n/2$. Then G is connected: otherwise, the degree of any vertex in the smallest component C of G would be less than $|C| \leq n/2$. Let $P = x_0...x_k$ be a longest path in G. By the maximality of P, all the neighbours of x_0 and all the neighbours of x_k lie on P. Hence at least $n/2$ of the vertices $x_0, ..., x_{k-1}$ are adjacent to x_k, and at least $n/2$ of these same $k < n$ vertices x_i are such that $x_0x_{i+1} \in E(G)$. By the pigeonhole principle, there is a vertex x_i that has both properties, so we have $x_0x_{i+1} \in E(G)$ and $x_ix_k \in E(G)$, for some $i < k$ (see Fig 2.4).

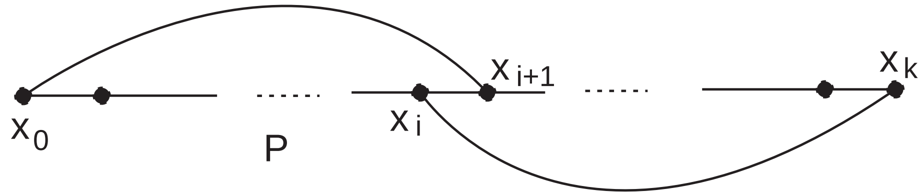

Figure 2.4: Finding a hamiltonian cycle.

We claim that the cycle $C = x_0x_{i+1}Px_kx_iPx_0$ is a hamiltonian cycle of G. Indeed, since G is connected, C would otherwise have a neighbour in $G - C$, which could be combined with a spanning path of C into a path longer than P. ■

Following Ore's theorem, many further generalizations were introduced. This line of investigation culminated in the work of Bondy and Chvátal [17]. The next result stems from their observation that Ore's proof does not need or use the full power of the statement that each nonadjacent pair satisfies the degree sum condition.

Theorem 2.2.2 [17] *Let x and y be distinct nonadjacent vertices of a graph G of order n such that $d(x) + d(y) \geq n$. Then $G + xy$ is hamiltonian if and only if G is hamiltonian.*

This result inspired the following definition. The *closure* of a graph G, denoted $CL(G)$, is that graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains. The first thing we must verify is that this closure is well defined; that is, since no order of operations is specified, we must verify that we always obtain the same graph, no matter what order we use to insert the edges.

Theorem 2.2.3 [17] *If G_1 and G_2 are two graphs obtained by recursively joining pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains, then $G_1 = G_2$; that is, $CL(G)$ is well defined.*

We illustrate the construction of the closure of a graph in Fig. 2.5.

Using Theorem 2.2.2, we see that the next result is immediate.

Theorem 2.2.4 [17] *A graph G is hamiltonian if and only if $CL(G)$ is hamiltonian.*

If $CL(G) = K_n$ then it is immediate that G is hamiltonian.

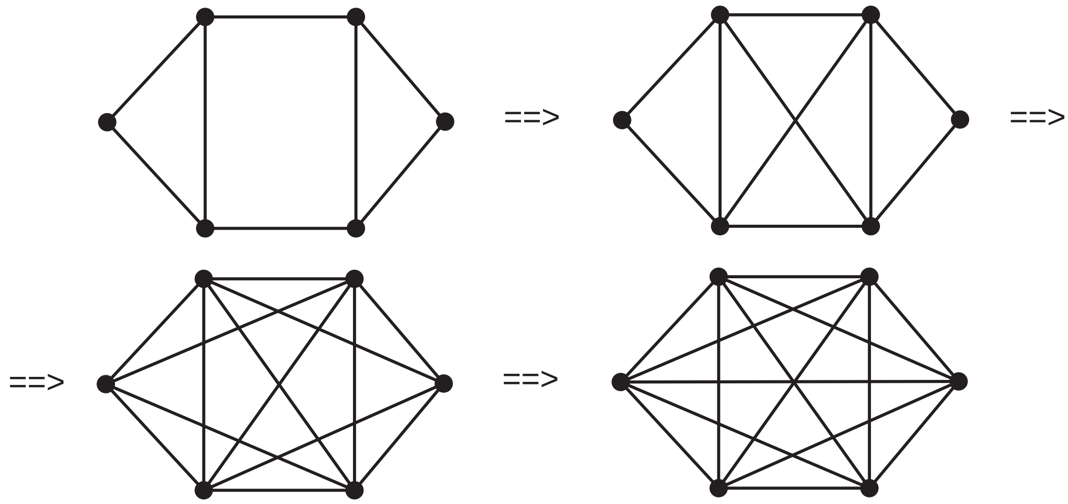

Figure 2.5: Construction of the closure of a graph.

Let us introduce several properties closely related to that of being hamiltonian. Some of these are stronger properties, in the sense that the graphs having these properties are also hamiltonian, while others are weaker. It is not surprising that truly applicable characterizations of these properties are not known. In some cases, very little at all is known about the classes of graphs we will describe.

To begin with, we say a graph is *traceable* if it contains a hamiltonian path. Clearly, every hamiltonian graph is also traceable, and the graphs P_n show that the converse of this statement does not hold. For a graph G, let the *k-closure*, denoted $CL_k(G)$, be the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least k. Bondy and Chvátal [17] proved the following.

Theorem 2.2.5 [17] *A graph G is traceable if and only if $CL_{n-1}(G)$ is traceable.*

There are two other classes of graphs that essentially lie between the hamiltonian and traceable classes. A graph G is *homogeneously traceable* if there is a hamiltonian path beginning at every vertex of G, while G is *hypohamiltonian* if G is not hamiltonian but $G - v$ is hamiltonian, for every vertex v of G. It is easy to see that every hypohamiltonian graph is also homogeneously traceable. The graph of Fig. 2.6 is homogeneously traceable but not hypohamiltonian.

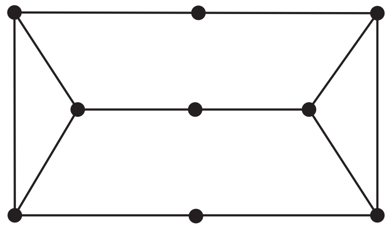

Figure 2.6: Homogeneously traceable nonhamiltonian graph.

Skupien introduced homogeneously traceable graphs in [87], and the existence of homogeneously traceable nonhamiltonian graphs for all orders $n \geq 9$ was shown in [24]. Lindgren [70] and Sousselier [54] independently showed that there are infinitely many hypohamiltonian graphs. However, both the homogeneously traceable and hypohamiltonian classes have remained elusive in the sense that very few results are known about them, especially in view of the vast number of results known about hamiltonian graphs.

More success has been had with properties stronger than that of being hamiltonian. We say a graph G is *hamiltonian connected* if every two vertices of G are joined by a hamiltonian path. Clearly, every hamiltonian connected graph of order at least 3 is hamiltonian; the graphs C_n $(n \geq 4)$ show that the converse is not true. Using the generalization of the idea of the closure, we obtain the following analogue of Theorem 2.2.4, also from Bondy and Chvátal [17].

Theorem 2.2.6 [17] *Let G be a graph of order n. If $CL_{n+1}(G)$ is complete then G is hamiltonian connected.*

The next two corollaries are analogues of the theorems of Ore and Dirac and are immediate from the previous result, although they can also be proved directly.

Corollary 2.2.2 *If G is a graph of order n such that for every pair of distinct nonadjacent vertices x and y in G, $d(x) + d(y) \geq n + 1$, then G is hamiltonian connected.*

Corollary 2.2.3 *If G is a graph of order n such that $\delta(G) \geq \frac{n+1}{2}$ then G is hamiltonian connected.*

Yet another hamiltonian-like property is the following: A connected graph $G = (V, E)$ is said to be *panconnected* if for each pair of distinct vertices x and y, there exists an $x - y$ path of length ℓ, for each ℓ satisfying $d(x, y) \leq \ell \leq |V| - 1$. If a graph G is panconnected, it is clearly hamiltonian connected and thus hamiltonian. It is also easy to see that there are hamiltonian graphs that are not panconnected (for example, cycles). Williamson [92] provided a sufficient condition for a graph to be panconnected in terms of the minimum degree.

Theorem 2.2.7 [92] *If G is a graph of order $n \geq 4$, such that $\delta(x) \geq \frac{n+2}{2}$, then G is panconnected.*

Our final properties are somewhat related. We say that a graph G of order n is *pancyclic* if it contains a cycle of every length ℓ, $3 \leq \ell \leq n$. We say that G is *vertex pancyclic* if each vertex of G lies on a cycle of each length ℓ, $3 \leq \ell \leq n$. Clearly, every pancyclic graph is hamiltonian and every vertex pancyclic graph is pancyclic, hence hamiltonian as well. A sufficient condition for a graph to be pancyclic was provided by Bondy [16].

Theorem 2.2.8 [16] *Let G be a hamiltonian graph with n vertices and m edges. If $m \geq \frac{n^2}{4}$ then either G is pancyclic or n is even and G is isomorphic to $K_{n/2,n/2}$.*

2.3 Local connectivity

It is easy to see that every connected graph satisfying a global degree condition must have a limited diameter, i.e., every global lower bound on the degrees implies at the same time an upper bound on the diameter of G. Note that, in particular, Ore's condition $d(x)+d(y) \geq n$ implies that G must have diameter at most 2. Local conditions, however, are applicable to graphs with arbitrarily large diameter. We state some interesting global consequences of local connectivity conditions. Oberly and Sumner [78] proved the following result. Since the proof of this theorem is interesting and short and several other proofs of related theorems follow the idea of this proof, we include here the complete proof.

Theorem 2.3.1 [78] *If G is a connected, locally connected, claw-free graph on $n \geq 3$ vertices then G is hamiltonian.*

Proof. Suppose to the contrary, that G is a connected, locally connected graph on at least three vertices which is not hamiltonian. Clearly, G contains a cycle. Let C be a largest cycle in G. Then C does not span G and since G is connected, there exists a vertex v not on C which is adjacent to a vertex u lying on C. Let u_1 and u_2 be the vertices neighbouring u on the cycle C. Since G is locally connected, there is a path P in $N(u)$ from v to one of u_1, u_2 but not including the other. Without loss of generality, we suppose that P is a path from v to u_1 and that $u_2 \notin P$.

Now, if $P \cap C = \{u_1\}$ then, by attaching P to C at u_1 and v, we could obtain a cycle larger than C. Hence we may assume that $P \cap C$ contains vertices other than u_1. Also, we cannot have v adjacent to either u_1 or u_2 without producing a cycle larger than C. Thus, since $\{u, u_1, u_2, v\}$ cannot induce a claw ($K_{1,3}$) in G, it must be that u_1u_2 is an edge of G.

For the purposes of this proof we define a *singular* vertex to be a vertex $w \in P \cap C - \{u_1\}$ such that neither of the vertices neighbouring w in C belongs to $N(u)$. We consider two cases.

<u>**Case 1.**</u> Every vertex in $P \cap C - \{u_1\}$ is singular. Then, for any $w \in P \cap C - \{u_1\}$, w is adjacent tu u but neither of the vertices w_1 and w_2 neighbouring w on C belongs

to $N(u)$. Thus, since $\{w, w_1, w_2, u\}$ cannot induce a claw in G, it must be that w_1w_2 is an edge in G. Now traverse C starting at u_2 and moving away from u and, for each $w \in P \cap C - \{u_1\}$, bypass w by taking the edge w_1w_2. Continue until the vertex u_1 is reached. Then follow P from u_1 to v, then to u, and finish at u_2. Then we have passed through each vertex of $C \cup P$ exactly once and have thus constructed a cycle larger than C.

Case 2. $P \cap C - \{u_1\}$ contains nonsingular vertices. Then follow P from v toward u_1 until the first nonsingular vertex w is reached. Let w_1 and w_2 be the vertices neighbouring w along C. Then at least one of w_1 and w_2 is adjacent to u. With no loss of generality, suppose that w_1 is adjacent to u. Now form a new cycle C', containing the same vertices as C, as follows. Delete the edges ww_1, uu_1 and uu_2 and add the edges wu, w_1u and u_1u_2 (see Fig. 2.7).

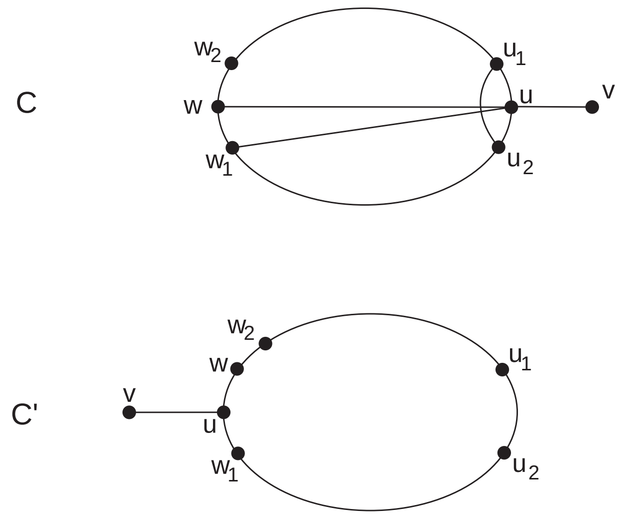

Figure 2.7: Forming of the cycle C' from C.

Note that if w is a neighbour of u_1 or u_2, then not all of these edges may be distinct (e.g., if $w_1 = u_1$, then $uu_1 = uw_1$). But now vertices neighbouring u in C' are w and w_1 and the subpath P' of P from w to v does not include w_1 (as otherwise w_1, being a nonsingular vertex, would have been chosen earlier instead od w). Moreover, from the choice of w, P' cannot contain any nonsingular vertices with respect to C' and w. Hence relative to P' and C', we are back to Case 1.

Hence in any event, C cannot have been a largest cycle and, with this contradiction, Theorem 2.3.1 is proved. ■

This result inspired Ryjáček to introduce a different type of closure. Recall that $c(G)$ is the length of a longest cycle in G. Graph $cl(G)$ of a claw-free graph G is the graph obtained from G by recursively replacing the neighbourhood of a locally connected vertex with a complete graph until no such vertex remains. Ryjáček in [83] proved the following theorem.

Theorem 2.3.2 [83] *Let G be a claw-free graph. Then the following statements hold.*

(i) *The closure $cl(G)$ is well defined.*

(ii) *There is a triangle-free graph H such that $cl(G) = L(H)$.*

(iii) $c(G) = c(cl(G))$.

Using Theorem 2.3.2, the next result is immediate.

Theorem 2.3.3 [83] *A graph G is hamiltonian if and only if $cl(G)$ is hamiltonian.*

Since $cl(G)$ of a connected and locally connected claw-free graph is the complete graph K_n, it is immediate that $cl(G)$ is hamiltonian and hence that G is hamiltonian. Therefore, Theorem 2.3.1 from Oberly and Sumner is a corollary of Theorem 2.3.2. The following theorem summarizes known results from [20], [57] and [84] concerning the stability of other properties with respect to the closure operation. Here we denote by $p(G)$ the length of a longest path in G.

Theorem 2.3.4 *Let G be a claw-free graph. Then the following statements hold.*

(i) $p(cl(G)) = p(G)$.

(ii) *G is traceable if and only if $cl(G)$ is traceable.*

(iii) *G can be covered by k cycles if and only if $cl(G)$ can be covered by k cycles.*

(iv) *G has a 2-factor with at most k components if and only if $cl(G)$ has a 2-factor with at most k components.*

(v) *G can be covered by k paths if and only if $cl(G)$ can be covered by k paths.*

(vi) *G has a path-factor with k components if and only if $cl(G)$ has a path-factor with k components.*

Several authors observed that the assumptions of Theorem 2.3.1 imply stronger cycle properties. Clark [30] (and independently later also Shi Rong Hua [86] and Zhang [94]) proved the following.

Theorem 2.3.5 [30] *Every connected, locally connected claw-free graph on at least three vertices is vertex pancyclic.*

Recall that a graph G is chordal if each cycle C_k in G, of length $k \geq 4$, has a chord. Balakrishnan and Paulraja in [1] showed that every 2-connected chordal graph is locally connected and, from this and the result by Oberly and Sumner, they proved the following.

Theorem 2.3.6 [1] *Every 2-connected claw-free chordal graph is hamiltonian.*

If we assume higher local connectivity, we can obtain stronger cycle properties. First result in this direction is by Chartrand, Gould and Polimeni [25] who proved that if G is a connected, locally 3-connected claw-free graph then G is hamiltonian-connected. This result was improved by Clark [30] who proved that any connected, locally 3-connected claw-free graph is panconnected. Finally, Kanetkar and Rao in [64] proved the following result.

Theorem 2.3.7 [64] *If G is a connected, locally 2-connected claw-free graph then G is panconnected.*

Broersma and Veldman in [22] conjectured that in 3-connected claw-free graphs, the assumption can be further relaxed.

Conjecture 2.3.1 [22] *Let G be a connected, locally connected, claw-free graph of order at least 4. Then G is panconnected if and only if G is 3-connected.*

2.4 Forbidden subgraphs

In this section we consider another approach that has been studied extensively. Goodman and Hedetniemi [48] noticed that forbidding particular induced subgraphs offered a new type of hamiltonian result. If $F = \{H_1, ..., H_k\}$ is a family of graphs, we say that G is *F-free* if G does not contain any of the graphs in the set F as an induced subgraph. If F is a single graph H we simply say that G is *H-free*. We define graph Z_1 as $Z_1 = K_{1,3} + e$.

Theorem 2.4.1 [48] *If G is a 2-connected $\{K_{1,3}, Z_1\}$-free graph then G is hamiltonian.*

Theorem 2.4.1 inspired several others. One of the earliest and strongest of these is stated below. The graph N (known as the *net*) is shown in Fig 2.8.

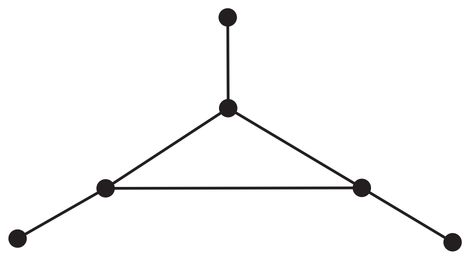

Figure 2.8: The graph N.

Theorem 2.4.2 [35]

(i) *If G is connected and $\{K_{1,3}, N\}$-free then G is traceable.*

(ii) *If G is 2-connected and $\{K_{1,3}, N\}$-free then G is hamiltonian.*

Note that if H is an induced subgraph of some graph S and G is H-free, then G is also S-free, for if G contained an induced S, it clearly would contain an induced H as well. Thus we get the following corollary to Theorem 2.4.2. The graph B is shown in Fig. 2.9.

Corollary 2.4.1 *If G is a 2-connected graph that is $\{R, S\}$-free, where $R = K_{1,3}$ and $S \in \{N, C_3, Z_1, B, P_4, P_3, K_2\ K_1\}$, then G is hamiltonian.*

In the list of graphs given in the previous corollary, we note that if P_3 is forbidden, the graph must clearly be complete, thus it has any hamiltonian property we may wish. Further, if the induced subgraphs of P_3, K_2 or K_1, are forbidden then we either have that G must be K_1 or G is empty. In either case, we arrive at trivial cases that satisfy all hamiltonian properties. For this reason, in what remains we will ignore P_3 and its induced subgraphs as simply being trivial and therefore "uninteresting".

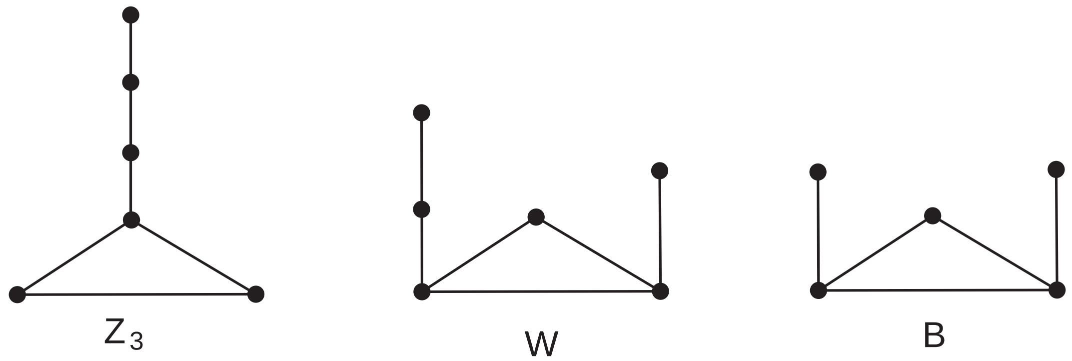

Figure 2.9: Common forbidden subgraphs.

After Theorem 2.4.2 was announced, the search for other families of forbidden subgraphs began in earnest. Over the course of the next decade, a variety of such results were discovered. We summarize the most important of these from [14], [23] and [43] in the next theorem. See Fig. 2.9 for the graphs W and Z_3. More generally, by Z_i we mean a triangle with a path starting from one of its vertices and containing exactly i edges.

Theorem 2.4.3 *If G is a 2-connected graph and G is*

(*i*) $\{K_{1,3}, P_6\}$*-free, or*

(*ii*) $\{K_{1,3}, W\}$*-free, or*

(*iii*) $\{K_{1,3}, Z_3\}$*-free and of order* $n \geq 10$,

then G is hamiltonian.

In 1991, Bedrossian [14] characterized all pairs of graphs which, when forbidden, imply that a 2-connected graph is hamiltonian.

Theorem 2.4.4 [14] *Let R and S be connected graphs ($R, S \neq P_3$) and G a 2-connected graph of order $n \geq 10$. Then G is (R, S)-free implies G is hamiltonian if and only if $R = K_{1,3}$ and $S \in \{C_3,\ P_4,\ P_5,\ P_6,\ Z_1,\ Z_2,\ B,\ N\ W\}$.*

Similar results are known for traceable graphs as well as for several other hamiltonian type properties (see [14], [42] or [43]).

2.5 Toughness condition

We begin this section with the 1973 paper in which Chvátal [29] introduced the definition of toughness. The *toughness* of a non-complete graph is

$$t(G) = min(\frac{|S|}{\omega(G-S)}),$$

where the minimum is taken over all nonempty vertex sets S for which $\omega(G-S) \geq 2$, where $\omega(G-S)$ denotes the number of components in the graph $G-S$. For a complete graph K_n let $t(K_n) = \infty$. A graph G is t-tough if $|S| \geq t\ \omega(G-S)$, for every subset S of the vertex set $V(G)$ with $\omega(G-S) > 1$. From the definition, it is clear that being 1-tough is a necessary condition for a graph to be hamiltonian. In [29] Chvátal conjectured that there exists a finite constant t_0 such that every t_0-tough graph is hamiltonian. For many years, however, the focus was on determining whether all 2-tough graphs are hamiltonian. One reason for this is that if all 2-tough graphs are hamiltonian, a number of important consequences would follow. The results in [38] (for $k = 2$), listed below, seemed to indicate that 2 might be the required threshold for toughness that would imply hamiltonicity.

Theorem 2.5.1 [38] *Let G be a k-tough graph on n vertices, with $n \geq k+1$ and kn even. Then G has a k-factor.*

Theorem 2.5.2 [38] *Let $k \geq 1$. For every $\epsilon > 0$, there exists a $(k-\epsilon)$-tough graph G on n vertices, with $n \geq k+1$ and kn even, which has no k-factor.*

However, it turns out that not all 2-tough graphs are hamiltonian, as indicated by the result below.

Theorem 2.5.3 [11] *For every $\epsilon > 0$, there exists a $(\frac{9}{4}-\epsilon)$-tough nontraceable graph.*

We will describe the construction of the graphs that were used in [11] to prove Theorem 2.5.3. In [11], a construction of a nontraceable graph from non-hamiltonian-connected building blocks was used to show that Chvátal's conjecture on the hamiltonicity of 2-tough graphs is equivalent to several other statements, some seemingly weaker, some seemingly stronger. This construction was inspired by examples of graphs of high toughness without 2-factors occurring in [13]. In [11], the same construction was used to prove Theorem 2.5.3, thereby refuting the 2-tough conjecture. We now give a brief outline of the construction producing these counterexamples.

For a given graph H and $x, y \in V(H)$, we define the graph $G(H, x, y, \ell, m)$ as follows. Take m disjoint copies $H_1, ..., H_m$ of H, with x_i, y_i the vertices in H_i corresponding to the vertices x and y in H ($i = 1, ..., m$). Let F_m be the graph obtained from $H_1 \cup ... \cup H_m$ by adding all possible edges between pairs of vertices in $\{x_1, ..., x_m, y_1, ..., y_m\}$. Let $T = K_\ell$ and let $G(H, x, y, \ell, m)$ be the join $T \vee F_m$ of T and F_m.

Theorem 2.5.4 [11] *Let H be a graph and x, y two vertices of H which are not connected by a hamiltonian path of H. If $m \geq 2\ell + 3$, then $G(H, x, y, \ell, m)$ is nontraceable.*

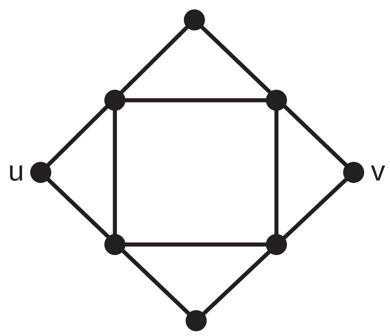

Figure 2.10: The graph L.

Consider the graph L of Fig. 2.10. There is obviously no hamiltonian path in L between u and v. Hence $G(L, u, v, \ell, m)$ is nontraceable for every $m \geq 2\ell + 3$. The toughness of these graphs has been established in [11].

Theorem 2.5.5 [11] *For $\ell \geq 2$ and $m \geq 1$,*

$$t(G(L, u, v, \ell, m)) = \frac{\ell + 4m}{2m + 1}.$$

Combining Theorems 2.5.4 and 2.5.5 for sufficiently large values of m and ℓ, we obtain the next result.

Corollary 2.5.1 [11] *For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$-tough nontraceable graph.*

It is easily seen from the proof in [11] that Theorem 2.5.4 remains valid if '$m \geq 2\ell + 3$' and 'nontraceable' are replaced by '$m \geq 2\ell + 1$' and 'nonhamiltonian', respectively. Thus the graph $G(L, u, v, 2, 5)$ is a nonhamiltonian graph, which by Theorem 2.5.5 has toughness 2. This graph is sketched in Fig. 2.11. It follows that a smallest counterexample to the 2-tough conjecture has at most 42 vertices. Similarly, a smallest nontraceable 2-tough graph has at most $|V(G(L, u, v, 2, 7))| =$ 58 vertices.

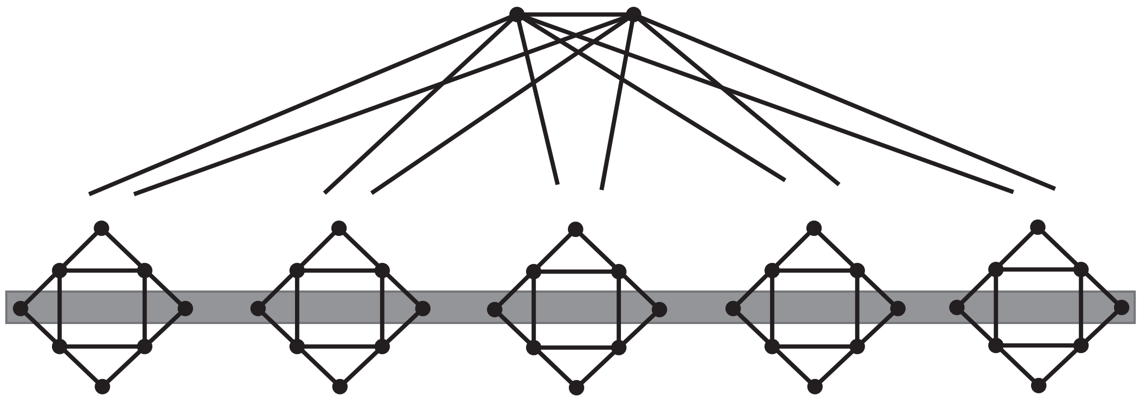

Figure 2.11: The graph $G(L, u, v, 2, 5)$.

In [29] Chvátal also states the following weaker version of the 2-tough conjecture: 'Every 2-tough locally connected graph is hamiltonian'. Since all counterexamples described above are locally connected, this weaker conjecture is also false. It only remains to observe that using the specific graph L as a 'building block' produced a graph with toughness at least 2. Hopefully, other building blocks and/or smarter constructions will lead to counterexamples with a higher toughness.

Chvátal [29] obtained $(\frac{3}{2}-\epsilon)$-tough graphs without a 2-factor for arbitrary $\epsilon > 0$. These examples are all chordal. Recently it was shown in [12] that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Based on this, Kratsch raised the question whether every $\frac{3}{2}$-tough chordal graph is hamiltonian. Using Theorem 2.5.4, it has been shown that this conjecture, too, is false [11]. A key observation in this context is that the graphs $G(H, x, y, l, m)$ are chordal whenever H is chordal, as is easily shown. Consider the graph M of Fig. 2.12.

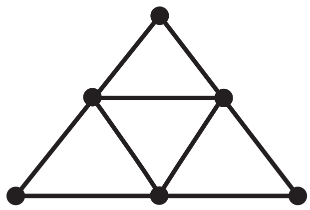

Figure 2.12: The graph M.

The graph M is chordal and has no hamiltonian path with endvertices p and q. Hence, by Theorem 2.5.4, the chordal graph $G(M, p, q, l, m)$ is nontraceable whenever $m \geq 2l + 3$. By arguments similar to those used in the proof of Theorem 2.5.5 (in [11]), the toughness of $G(M, p, q, \ell, m)$ is $\frac{\ell+3m}{2m+1}$ if $\ell \geq 2$. Hence, for $\ell \geq 2$, the graph $G(M, p, q, \ell, 2\ell+3)$ is a chordal nontraceable graph with toughness $\frac{7\ell+9}{4\ell+7}$. This gives the following result.

Theorem 2.5.6 [11] *For every $\epsilon > 0$, there exists a $(\frac{7}{4} - \epsilon)$-tough chordal nontraceable graph.*

Since 1990's, one problem that has received much attention is that of determining the minimum level of toughness to ensure that a chordal graph is hamiltonian. We have seen an infinite class of chordal graphs with toughness close to $\frac{7}{4}$ having no hamiltonian path. Hence, 1-tough chordal graph need not be hamiltonian. However, for other classes of perfect graphs, being 1-tough will ensure hamiltonicity. For example, in [65] it was shown that 1-tough interval graphs are hamiltonian.

However, in [19] it was proven that for chordal planar graphs, 1-toughness does not ensure hamiltonicity.

Theorem 2.5.7 [19] *Let G be a chordal, planar graph with $t(G) > 1$. Then G is hamiltonian.*

To see that being 1-tough will not suffice, we must first define the 'shortness exponent' of a class of graphs. This concept was first introduced in [52] as a way of measuring the size of longest cycles in polyhedral, i.e., 3-connected planar graphs.

Let Σ be a class of graphs. The *shortness exponent* $\sigma(G)$ of the class Σ is given by

$$\sigma(\Sigma) = \liminf H \frac{\log c(H_n)}{\log |V(Hn)|}.$$

The $\liminf$ is taken over all sequences of graphs H_n in Σ such that $|V(H_n)| \to \infty$ as $n \to \infty$. In [19], it is also shown that the shortness exponent of the class of all 1-tough chordal planar graphs is at most $\frac{\log 8}{\log 9}$. Hence there exists a sequence $G_1, G_2, ...$ of 1-tough chordal planar graphs with $\frac{c(G_i)}{|V(G_i)|} \to 0$ as $i \to \infty$. On the other hand, all 1-tough $K_{1,3}$-free chordal graphs are hamiltonian. This follows from the well known result of Matthews and Sumner [71] relating toughness and vertex connectivity in $K_{1,3}$-free graphs, and a result of Balakrishnan and Paulraja [1] showing that 2-connected $K_{1,3}$-free chordal graphs are hamiltonian.

Let us now consider $\frac{3}{2}$-tough chordal graphs. We have already seen that such graphs need not be hamiltonian. However, for a certain subclass of chordal graphs, namely 'split graphs', we have a different result. A graph G is called a *split graph* if $V(G)$ can be partitioned into an independent set and a clique. We have the following.

Theorem 2.5.8 [66] *Every $\frac{3}{2}$-tough split graph is hamiltonian.*

The previous results on tough chordal graphs lead to a very natural question. This question was answered by Chen et al. in the title of their paper 'Tough enough chordal graphs are hamiltonian' [28]. Using an algorithmic proof they were able to prove the result below.

Theorem 2.5.9 [28] *Every 18-tough chordal graph is hamiltonian.*

The authors do not claim that 18 is the best possible. A natural question, in light of the disproof of the 2-tough conjecture for general graphs, is what level of toughness will ensure that a chordal graph is hamiltonian. More specifically, are 2-tough chordal graphs hamiltonian?

2.6 Other results

In this section we consider several other types of results concerning hamiltonian and hamiltonian-like graphs. These results use conditions that are often very different from those we have seen thus far. Let us start with a little bit of history.

The four color problem, whether every map can be colored with four colors so that adjacent countries are shown in different colors, was raised by a certain Francis Guthrie in 1852. This problem is equivalent to the problem whether every planar graph can be colored with four different colors such that adjacent vertices have different colors. In 1879 Kempe published an incorrect proof, which was in 1890 modified by Heawood into a proof of the five color theorem. The first generally accepted proof of the four color theorem was published by Appel and Haken in 1977. The proof builds on ideas that can be traced back as far as Kempe's paper. Very roughly, the proof sets out first to show that every plane triangulation must contain at least one of 1482 certain unavoidable configurations. In the second step, a computer is used to show that each of those configurations is reducible, i.e., that any plane triangulation containing such a configuration can be 4-colored by piecing together 4-colorings of smaller plane triangulations. Taken together, these two steps amount to an inductive proof that all plane triangulations, and hence all planar graphs, can be 4-colored. Appel and Haken's proof has been criticized, not only because of their use of a computer. The authors responded with a 741 page long algorithmic version of their proof which addresses the various criticisms and corrects a number of errors (e.g., by adding more configurations to the unavoidable list). A much shorter proof, based on the same ideas (and, in particular, using a computer in the same way) but more readily verifiable, has been given by Robertson, Sanders, Seymour and Thomas in [82].

It may come as a surprise to find that hamiltonicity for planar graphs is related to the four color problem. The four color theorem is equivalent to the non-existence of a planar 'snark', i.e., to the assertion that every bridgeless planar cubic graph has a 4-flow (for definition, see [32]). It is easily checked that 'bridgeless' can be replaced with '3-connected' in this assertion, and that every hamiltonian graph has a 4-flow. For a proof of the four color theorem, therefore, it would suffice to show that every 3-connected planar cubic graph has a hamiltonian cycle. Unfortunately, this is not the case: the first counterexample was found by Tutte in 1946 (see Fig. 2.13). Ten years later, Tutte proved the following deep theorem as a best possible weakening.

Theorem 2.6.1 [90] *Every 4-connected planar graph has a hamiltonian cycle.*

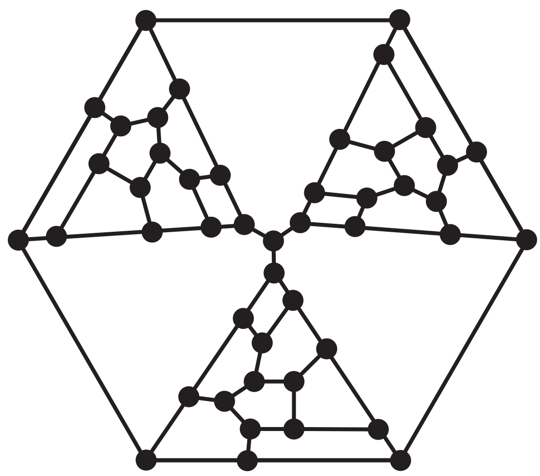

Figure 2.13: Tutte's graph: a non-hamiltonian 3-connected cubic graph.

Now we proceed with an investigation of the powers of a graph. The *nth power* G^n of a connected graph G is the graph with $V(G^n) = V(G)$ and in which uv is an edge of G^n if and only if $1 \leq d_G(u,v) \leq n$. In particular, the graphs G^2 and G^3 are called the *square* and *cube* of G, respectively. Figure 2.14 shows the *subdivision graph* of the graph $K_{1,3}$, that is, the graph $S(K_{1,3})$ obtained by subdividing each edge of $K_{1,3}$. The graph $S(K_{1,3})$ is formed from $K_{1,3}$ when each edge $e = xy$ is removed and a new vertex w is inserted along with the edges wx and wy. Figure 2.14 shows the square of $S(K_{1,3})$. Since higher powers of a graph G tend to contain more edges than G itself, it is reasonable to ask if these powers will eventually become hamiltonian, even if G is not. Nash-Williams and Plummer conjectured that this is the case for the squares of 2-connected graphs. In the now classic paper [44], Fleischner verified that this is indeed true.

Theorem 2.6.2 [44] *Let G be a 2-connected graph. Then G^2 is hamiltonian.*

This work opened the door for others to investigate properties of powers of graphs. In [26], Fleischner's result was strengthened to show that the square of

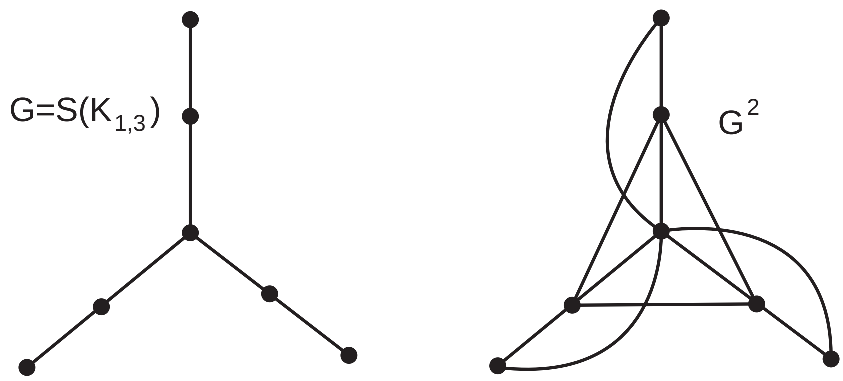

Figure 2.14: The graph $S(K_{1,3})$ and its square.

every 2-connected graph is actually hamiltonian connected. Let us next switch attention to hamiltonian properties of line graphs. In [53], Harary and Nash-Williams characterized graphs whose line graphs are hamiltonian. In order to do this, we define a *dominating circuit* C of a graph G to be a circuit with the property that every edge of G is incident to a vertex of C.

Theorem 2.6.3 [53] *Let G be a graph without isolated vertices. Then $L(G)$ is hamiltonian if and only if G is isomorphic to $K_{1,n}$, $n \geq 3$, or G contains a dominating circuit.*

The major impact of Theorem 2.6.3 has been its use in proving other results. In [49], the idea of dominating circuits is used to help establish a result that combines ideas presented throughout this section. We define $L^{i+1}(G) = L(L^i(G))$.

Theorem 2.6.4 [49] *Let G be a connected graph of order $n \geq 3$ which does not contain a vertex cutset consisting only of vertices of degree 2. Then $L^2(G)$ is hamiltonian.*

From this theorem, the following result, originally discovered by Chartrand and Wall [27], is immediate.

Corollary 2.6.1 *If G is a connected graph such that $\delta(G) \geq 3$ then $L^2(G)$ is hamiltonian.*

Chapter 3

k-walks

3.1 Introduction

In this chapter we generalize the concept of hamiltonicity. There are several ways how to generalize this concept. The first generalization of hamiltonicity is considering 2-factors with a specific number of components. It is obvious that a 2-factor with exactly one component is a hamiltonian cycle. So if a graph has a 2-factor with a small number of components, we can say that it is not so far from being hamiltonian. There are several results about 2-factors, see [13], [10] and [38]. As a motivation for this chapter, we mention the following interesting result from Enomoto, Jackson, Katerinis and Saito [38].

Theorem 3.1.1 [38] *Suppose $|V(G)| > k+1$, $k|V(G)|$ even, and $t(G) \geq k$. Then G has a k-factor.*

Recall that a hamiltonian cycle is a cycle that goes through every vertex exactly once. Another approach how to generalize the concept of hamiltonicity is to allow going through a vertex more than once. A *k-walk* is a closed spanning walk which uses every vertex of a graph at most k times. The concept of a k-walk is a generalization of a hamiltonian cycle, while a 1-walk is exactly a hamiltonian cycle.

Since the idea of a k-walk 'sounds' easier than that of a hamiltonian cycle, for $k > 1$, one might expect that, for some fixed k, the problem of recognizing graphs that have a k-walk will not be NP-complete anymore. This, however, is not the case.

In fact, the problem of recognizing graphs that have a k-walk is still NP-complete for arbitrary fixed k (see [58]), and so far no practical characterization for graphs with k-walks has been found.

The study of k-walks was initiated by Jackson and Wormald in 1990. In their pioneering paper [58], they obtained various sufficient conditions for a graph to have a k-walk. In particular, they generalized the result of Oberly and Sumner [78] (see the next section). Apart from this pioneering paper, there are very few results concerning k-walks. In 1996, Favaron, Flandrin, Li and Ryjáček [41] proved the existence and some properties of 2-walks in connected, almost claw-free graphs. In 2000, Ellingham and Zha [37] obtained a new interesting sufficient condition for the existence of a 2-walk in a graph and, in the same year, Zemin Jin and Xueliang Li in [60] gave examples to show that a conjecture on k-walks of graphs, proposed by Jackson and Wormald [58], is false.

In this unexplored area we obtained some new results concerning k-walks and, in particular, 2-walks. Motivated by results for hamiltonian cycle (see Chapter 3) and, in particular, by result of Böhme et al., proving the existence of a hamiltonian cycle in more than 1-tough chordal planar graphs, we prove the existence of a 2-walk in chordal planar graphs with toughness greater then $\frac{3}{4}$. We also find the toughness threshold for the existence of 2-walks in K_4-minor free graphs. Note that the case of hamiltonian cycles is rather trivial for K_4-minor free graphs: it is easy to show by induction based on the construction of series-parallel graphs (see Section 3.4 for details) that 1-tough K_4-minor free graphs are hamiltonian. The bound is optimal since any hamiltonian graph is at least 1-tough. Additionally, in this chapter we propose several conjectures.

3.2 Literature review

As mentioned earlier, the concept of a k-walk is a generalization of hamiltonicity. Jackson and Wormald in [58] gave several results related to the results presented in Chapter 2. We start with a necessary condition for the existence of a k-walk in a graph G.

Theorem 3.2.1 [58] *If G has a k-walk then $\omega(G - S) \leq k|S|$ for all nonempty proper subsets of $V(G)$.*

Using the concept of toughness introduced by Chvátal [29], this theorem can be restated as follows: If G has a k-walk then G is $\frac{1}{k}$-tough. It is not hard to see that the condition in this theorem is not also sufficient. We show this later. In the same paper, the authors found several sufficient conditions. To state them, we need the following definition. A *spanning k-tree* is a spanning tree with maximum degree at most k.

Theorem 3.2.2 [58]

(i) *If G contains a spanning k-tree then G has a k-walk.*

(ii) *If G has a k-walk then G contains a spanning $(k+1)$-tree.*

Now we can use the following result of Sein Win [93].

Theorem 3.2.3 [93] *If G is connected, $k \geq 2$ and, for any subset S of $V(G)$, $\omega(G - S) \leq (k-2)|S| + 2$, then G has a spanning k-tree.*

Combining Theorems 3.2.2 and 3.2.3, we get the following.

Corollary 3.2.1 [58] *If G is connected, $k \geq 2$ and, for any subset S of $V(G)$, $\omega(G - S) \leq (k-2)|S| + 2$, then G has a k-walk.*

For $k = 2$, we get the following class of graphs: complete graphs K_n without a matching. It is easy to see why. Removing an arbitrary subset of vertices from a complete graph, we always get only one component. Removing arbitrary subset of vertices from a complete graph without an edge, we get either one or two components. But if we take a complete graph without a triangle abc then by removing all the vertices except a, b, c, we get three components, namely, vertices a, b, c. Therefore, for $k = 2$, this result is not very interesting. The authors feel that this corollary can be probably improved. They stated the following conjecture.

Conjecture 3.2.1 [58] *If $k \geq 2$ then every $\frac{1}{k-1}$-tough graph has a k-walk.*

For $k = 2$, they conjectured that every 1-tough graph has a 2-walk. The best result in this direction is due to Ellingham and Zha [37] from 2000.

Theorem 3.2.4 [37] *Every 4-tough graph has a 2-walk.*

Ellingham and Zha also give a lower bound on the toughness for the existence of a k-walk.

Theorem 3.2.5 [37] *For every $\epsilon > 0$ and every $k \geq 1$, there exists a $(\frac{8k+1}{4k(2k-1)} - \epsilon)$-tough graph with no k-walk.*

To prove this theorem, they first modify the graph L from Fig. 2.10 and then rely on the same basic construction that is used in [11] (for illustration, see Fig 2.11).

Another interesting result obtained by Jackson and Wormald in [58] is the following.

Theorem 3.2.6 [58] *Let G be a connected, $K_{1,k+1}$-free graph. Then G has a k-walk.*

This theorem is sharp in the following sense. Since graph $K_{1,k}$ has no $(k-1)$-walk, the theorem is not true when k is replaced by $k-1$. Observe that this theorem is valid even for $k = 1$, i.e., for hamiltonian cycle. Since the only $K_{1,2}$-free graph on n vertices is the complete graph K_n, then it is hamiltonian. We prove a similar result for r-trestles (see Chapter 4 for details). In 1996, Favaron, Flandrin, Li and Ryjáček [41] generalized Theorem 3.2.6 by proving the existence of a 2-walk in every connected, almost claw-free graph (for definition of an almost claw-free graph see [41]). Furthermore, in the same paper [41], they proved some other properties of a 2-walk in an almost claw-free graph.

Next we recall a famous result of Oberly and Summner [78]. They proved that every connected, locally connected claw-free ($K_{1,3}$-free) graph is hamiltonian. Again, Jackson and Wormald in [58] generalized this result for k-walks.

Theorem 3.2.7 [58] *For $k \geq 1$, every connected, locally connected $K_{1,k+2}$-free graph with at least two vertices has a k-walk.*

As mentioned earlier, the result of Oberly and Sumner motivated Ryjáček to introduce a new closure concept. He proved that the property of 'having a hamiltonian cycle' is stable under the closure operation in the class of claw-free graphs. Very recently Kužel generalized this result for k-walks. He proved that the property of 'having a k-walk' is stable under the same closure operation in the class of $K_{1,k+2}$-free graphs.

Jackson and Wormald in [58] also examined graphs with higher global connectivity. They proved the following.

Theorem 3.2.8 [58] *If $j \geq 1$, $k \geq 3$, G is j-connected and $K_{1,j(k-2)+1}$-free then G has a k-walk.*

Note that when $j = 1$, Theorem 3.2.6 is stronger than Theorem 3.2.8. On the other hand, Theorem 3.2.8 improves Theorem 3.2.7, whenever $k \geq 6$ in Theorem 3.2.8, because all locally connected graphs other then K_2 are also 2-connected. Jackson and Wormald believed that Theorem 3.2.8 can be improved as follows.

Conjecture 3.2.2 [58] *If $j \geq 1$, $k \geq 2$, G is j-connected and $K_{1,jk+1}$-free then G has a k-walk.*

In 2000 Zemin Jin and Xueliang Li [60] constructed counterexamples to Conjecture 3.2.2, for $j \geq 3$. Additionally, they also found a minimally 2-connected graph to show that the conjecture is also false for $j = 2$.

As a motivation for the next three sections, we briefly mention some known results concerning toughness and the existence of a hamiltonian cycle. One of the most famous conjectures in this area is the Chvátal's conjecture. Its particular early version asserts that every 2-tough graph G is hamiltonian. This version of the conjecture was disproved by Bauer et al. [11] who constructed $(9/4-\varepsilon)$-tough graphs which are not hamiltonian, but it remains open whether there exists a constant α_0, such that every α_0-tough graph is hamiltonian.

Although Chvátal's conjecture remains open in general, it is known to be true for several special classes of graphs. We mention such results on chordal graphs as an example. Recall that a graph is chordal if it does not contain an induced cycle of length four or more. Every 18-tough chordal graph is hamiltonian [28]. It is conjectured [12] that the bound of 18 can be reduced to 2. There is almost nothing known about the existence of 2-walks in chordal graphs. Therefore, we propose the following weaker conjecture.

★ **Conjecture 3.2.3** *Every 2-tough chordal graph has a 2-walk.*

Better bounds are known for several subclasses of chordal graphs: 1-tough interval graphs [65], $\frac{3}{2}$-tough split graphs [66] (see also [62]) and $(1+\varepsilon)$-tough planar chordal graphs are hamiltonian. The two latter results are known to be the best possible. In the case of planar graphs, the existence of a Tutte cycle implies that every $(\frac{3}{2}+\varepsilon)$-tough planar graph is hamiltonian and Böhme et al. [19] constructed $(\frac{3}{2}-\varepsilon)$-tough planar graphs with no hamiltonian cycle.

3.3 2-walks in chordal planar graphs

In this section we present one of the main results of this thesis. As mentioned earlier, motivated by results for hamiltonian cycle and, in particular, by result of Böhme et al., proving the existence of a hamiltonian cycle in more than 1-tough chordal planar graphs, we prove the following theorem.

★ **Theorem 3.3.1** *Every chordal planar graph G with toughness $t(G) > \frac{3}{4}$ has a 2-walk.*

We use somewhat similar proof technique as the one in [19], but our proof is much more complex and longer. Note that this result is not sharp (see Section 3.5).

As an aside, together with Král and Dvořák, we improved Theorem 3.3.1, proving that every chordal planar graph with toughness $t(G) \geq \frac{3}{5}$ has a 2-walk. This has been achieved with the help of a computer. The idea of the proof is not difficult. We classified several types of walks and several configurations in a proper subgraph of a chordal planar graph with toughness $t(G) \geq \frac{3}{5}$. Then we proved that for every possible configuration of the proper subgraph, there exist all types of the walk through this subgraph. It turns out that there are roughly 200 cases. We proved some of them by hand, but the proof of just one case took about half an hour. With the help of a computer, the proof was almost immediate. Since the proof would be too long and the result is not so significant, we will not publish the result (the paper would have more than 100 pages).

Before proceeding with the proof of Theorem 3.3.1, several results about chordal planar graphs are given. Throughout the rest of this section, whenever we consider a planar graph G, we always mean a fixed embedding of G into the plane. The following theorem is due to Dirac [34].

Theorem 3.3.2 [34] *Every chordal graph G has a simplicial vertex v. Furthermore, the graph $G - v$ is chordal.*

We will also need the following result from [19].

Theorem 3.3.3 [19] *Let G be an ℓ-connected chordal graph and let v be a simplicial vertex in G. Then the graph $G - v$ is either ℓ-connected or complete.*

We will prove Theorem 3.3.1 by induction. Before we start the induction, we prove the following two useful lemmas. Lemma 3.3.1 shows that our induction will be well defined and Lemma 3.3.2 shows the way the toughness changes during the induction.

★ **Lemma 3.3.1** *Let G be a 2-connected chordal planar graph. Then there is a sequence of graphs $G_0, ..., G_k$ and a sequence of sets $S_0, ..., S_{k-1}$ such that:*

(i) $G_0 = K_3$,

(ii) $V(G_{i+1}) = V(G_i) \cup S_i$, *where* $S_i \cap V(G_i) = \emptyset$, $\langle V(G_i)\rangle_{G_{i+1}} = G_i$, $N_{G_{i+1}}(S_i) \subset V(G_i)$ *and, for every* $x \in S_i$, $\langle N_{G_{i+1}}(x)\rangle_{G_{i+1}}$ *is complete,* $i = 0, \ldots, k-1$,

(iii) $G_k = G$.

(iv) *The integer k, the graphs G_i ($i = 0, \ldots, k$) and the sets S_i ($i = 0, \ldots, k-1$) can be chosen so that, for every $i = 0, \ldots, k-1$,*

A) there is a vertex $v_i \in V(G_i)$ such that v_i is simplicial in G_i and $S_i \subset N_{G_{i+1}}(v_i)$;

B) if $x \in S_i$ is of degree $d_{G_{i+1}}(x) = 3$ then x lies in the inner face of the triangle $\langle N_{G_{i+1}}(x)\rangle_{G_{i+1}}$.

Proof. First we show that there is a sequence $G_0, ..., G_k$ satisfying the statements (i), (ii), (iii) and (iv-A). By Theorem 3.3.2, $G = G_k$ has at least one simplicial vertex. Let S be the set of all simplicial vertices in G_k. By Theorem 3.3.2, the graph $G - S$ is a chordal graph and there exists a simplicial vertex x in $G - S$. Let S'_{k-1} be the set of all simplicial vertices in G_k adjacent to x. If all the vertices in S'_{k-1} are independent in G_k then we set $S'_{k-1} = S_{k-1}$ and $v_{k-1} = x$. Otherwise there exist in S'_{k-1} two vertices u_1 and u_2 that are adjacent in G_k and we set $S_{k-1} = \{u_1\}$ and $v_{k-1} = u_2$. By Theorem 3.3.2, the graph $G - S_{k-1} = G_{k-1}$ is again chordal and the vertex v_{k-1} is simplicial in G_{k-1}. We can repeat this procedure until we obtain K_3. If we reverse this procedure we can construct an arbitrary chordal graph from K_3 such that the statements (i), (ii), (iii) and (iv-A) hold.

B) Suppose that statement (*iv-B*) holds for every G_j, for $1 \leq j \leq i$. We prove that the statement also holds for G_{i+1}. Assume otherwise. Then there is a vertex $u_2 \in S_i$ of degree 3, which lies in the outer face of the triangle $\langle N_{G_{i+1}}(u_i)\rangle_{G_{i+1}}$. Then $d_{G_i}(v_i) = 2$. Otherwise, we would get a contradiction with the planarity of G. Let v', v'' be the neighbours of v_i in G_i. If u_1 is the only vertex in S_i of degree 3 then we can place u_1 into the inner face of the triangle $\langle N_{G_{i+1}}(u_i)\rangle_{G_{i+1}}$. There cannot be three vertices of degree 3 in S_i, otherwise $K_{3,3}$ is a subgraph of G_{i+1}. Next assume that there are two vertices of degree 3, namely, $u_1, u_2 \in S_{i+1}$. Since u_1 and u_2 are simplicial vertices in G_{i+1}, then $N(u_1)_{G_{i+1}} = N(u_2)_{G_{i+1}} = \{v_i, v', v''\}$. We may assume that u_1 lies in the inner face of the triangle $\langle\{v_i, v', v''\}\rangle_{G_i}$ and u_2 lies in its outer face. Then we separate the construction step from G_i to G_{i+1} into two steps G_i to G_i' and G_i' to G_{i+1}', in such a way that the statement (*iv-B*) will hold. That is, we define $S_i' = \{u_2\}$ and $G_{i+1}' = \langle V(G_i) \cup \{u_2\}\rangle_{G_{i+1}}$. Additionally, we define $S_{i+1}' = S_i \backslash \{u_2\}$. We connect vertices from S_{i+1}' with vertices in G_{i+1}' as in G_{i+1}, but every vertex will be incident with u_1 instead of v_i. Then $G_{i+1} \cong G_{i+2}'$. See Fig. 3.1.

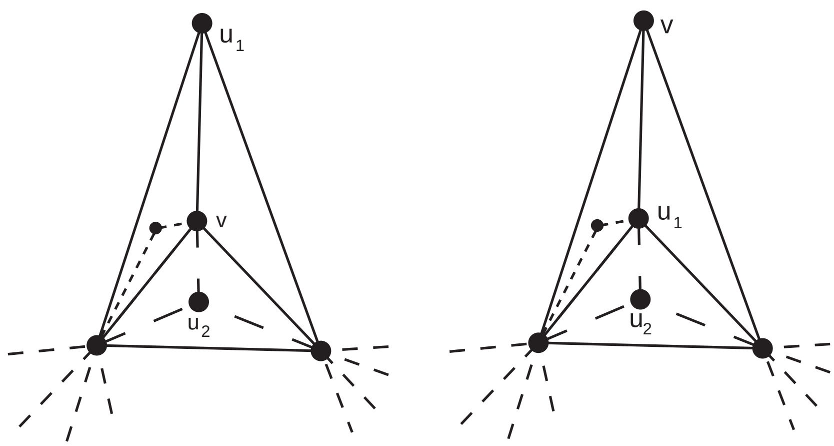

Figure 3.1: Modification of construction.

■

Let G be a 2-connected chordal planar graph and let $G_0, ..., G_k$ and $S_0, ..., S_{k-1}$ be any sequences of graphs and sets, respectively, satisfying the conditions of Lemma 3.3.1. Then we say that the sequence $(G_0, ..., G_k;\ S_0, ..., S_{k-1})$ is a *convenient construction of* G and, for any $x \in S_i$, $i = 0, \ldots, k-1$, a vertex v_i with the properties given in part *iv-A*) of Lemma 3.3.1 will be said to be a *parent* of the vertex x, denoted $v_i = p(x)$. From Lemma 3.3.1, it is obvious that every vertex, except vertices in G_0, has exactly one parent. Furthermore, the vertex v_i is the parent of all vertices in S_i and there are no other vertices in G such that v_i is their parent. If $p(x) \notin V(G_0)$ then by $p^2(x)$ we denote the parent of the parent of a vertex x. More generally, if $p^{j-1} \notin V(G_0)$, for some $j \geq 2$, then we denote $p^j(x) = p(p^{j-1}(x))$.

In the rest of the paper we use the following notation: For an arbitrary non-simplicial vertex u in a graph G_i from a convenient construction, we define an integer $\varphi(u)$, $0 \leq \varphi(u) < k$, as the integer such that u is simplicial in $G_{\varphi(u)}$ and is not simplicial in $G_{\varphi(u)+1}$ (i.e., we added some new simplicial vertices into the neighbourhood of u).

It is clear that for every vertex u from G, except the three vertices in G_0, the construction step $\varphi(p(u))$ is exactly the step in which vertex u was added into the graph $G_{\varphi(p(u))}$. Therefore, $u \in S_{\varphi(p(u))}$, for every $u \in V(G) \backslash V(G_0)$.

★ **Lemma 3.3.2** *Let* $(G_0, ..., G_k; S_0, ..., S_{k-1})$ *be a convenient construction of a* t*-tough chordal planar graph* G*. Then every graph* G_i*,* $i = 0, \ldots, k-1$*, is also* t*-tough.*

Proof. Let G_j a graph from the convenient construction, $0 < j \leq k$. Assume that there exists a set of vertices P such that $\omega(G_j - P) < \omega(G_{j-1} - P)$. Then there are two components C_1, C_2 of $G_{j-1} - P$ such that both C_1 and C_2 are in the same component of $G_j - P$. We get G_j by adding new simplicial vertices to G_{j-1}. Then there must be a simplicial vertex v in G_j which has two neighbours v_1, v_2, such that $v_1 \in C_1$ and $v_2 \in C_2$. This is a contradiction because v_1 and v_2 are not adjacent, which contradicts the fact that v is a simplicial vertex.

Hence for any subset of vertices P, $\omega(G_j - P) \geq \omega(G_{j-1} - P)$. Therefore, if G_j is t-tough, G_{j-1} is also t-tough. ■

The following definitions will be useful in the proof of Theorem 3.3.1. Suppose we have a graph G with a 2-walk T. Then we can define, for every vertex v in G, the *multiplicity* of v in T as: $m_T(v) = 1$ if v is used once in the 2-walk T, and $m_T(v) = 2$ if v is used twice in the 2-walk T. For every vertex v with multiplicity $m_T(v) = 1$, the predecessor of the vertex v in the 2-walk T will be denoted v_T^- and the successor of v in T will be denoted v_T^+. Note that possibly $v_T^+ = v_T^-$. Also, for every vertex v with multiplicity $m_T(v) = 1$, we define $e_T(v) = |\{v_T^+, v_T^-\}|$.

Assume G is a 2-connected chordal planar graph and $(G_0, ..., G_k;\ S_0, ..., S_{k-1})$ its convenient construction. We say that a 2-walk T_i in a graph G_i $(0 \leq i \leq k)$ is a *good* 2-walk if there exists a sequence of 2-walks $T_0, ...T_i$ such that, T_j is a 2-walk in G_j, $0 \leq j \leq i$, with the following properties.

For every simplicial vertex x in G_i, different from vertices in G_0, we have

(*i*) $m_{T_i}(x) = 1$.

(*ii*) If $|S_{\varphi(x)}| < 4$ then $x_{T_i}^+ = p(x)$ or $x_{T_i}^- = p(x)$.

(*iii*) If $d_{G_i}(x) = 3$ and $e_{T_i}(x) = 1$ then $d_{G_{\varphi(p(x))}}(p(x)) = 3$ and
- A) $e_{T_{\varphi(p(x))}}(p(x)) = 1$ and $p^2(x) \notin N_{G_i}(x)$ or
- B) $e_{T_{\varphi(p(x))}}(p(x)) = 2$ and in the set $S_{\varphi(p(x))}$ there are three vertices of degree 3 in the graph $G_{\varphi(p(x))+1}$ (x is one of them) or
- C) $e_{T_{\varphi(p(x))}}(p(x)) = 2$ and $x_{T_i}^+ \neq p(x)$ and $x_{T_i}^- \neq p(x)$.

(*iv*) If $d_{G_i}(x) = 3$, $e_{T_i}(x) = 2$, $m_{T_i}(p(x)) = 2$ and $x_{T_i}^- x_{T_i}^+ \notin E(T_i)$, then either:
- A) $d_{G_{\varphi(p(x))}}(p(x)) = 2$ and $|S_{\varphi(p(x))}| = 2$, or
- B) $d_{G_{\varphi(p(x))}}(p(x)) = 3$ and either
  - $|S_{\varphi(p(x))}| \geq 3$ or
  - $|S_{\varphi(p(x))}| = 2$ and there is a vertex $x' \in S_{\varphi(p(x))}$, $x' \neq x$, such that either
    - $|S_{\varphi(x')}| = 4$ or
    - $N_{G_{\varphi(x')}}(x') \subseteq N_{G_i}(x)$.

(*v*) Subject to the properties (*i*) - (*iv*), the number of simplicial vertices of degree 3 with $e_{T_i} = 2$ is maximal.

Now we proceed with a crucial lemma.

★ **Lemma 3.3.3** *Let G be a chordal planar graph with toughness $t(G) > \frac{3}{4}$. Let $(G_0, ..., G_r; S_0, ..., S_{r-1})$ be a convenient construction of G and suppose that for some i, $0 \le i \le r-1$, all graphs G_ℓ have good 2-walks T_ℓ, for $\ell = \{0, 1, ..., i\}$. Then the graph G_{i+1} has a good 2-walk T_{i+1}.*

Proof.

Since G_i is a chordal planar graph with toughness $t(G_i) > \frac{3}{4}$, all simplicial vertices in G_i have degree 2 or 3. Let v be a simplicial vertex in G_i such that all vertices $u_j \in S_i$ are incident with v in G_{i+1}.

<u>**Case 1 :**</u> $d(v) = 2$, $|S_i| = 2$ and $e_{T_i}(v) = 2$.

Let $u_1, u_2 \in S_i$ and suppose first that $N_{G_{i+1}}(u_1) = N_{G_{i+1}}(u_2)$. Due to Lemma 3.3.1, statement (iv)-B, and planarity of G_{i+1}, $d_{G_{i+1}}(u_1) = d_{G_{i+1}}(u_2) = 2$. Let $N_{G_{i+1}}(u_1) = N_{G_{i+1}}(u_2) = X$. Then the graph $G_{i+1} - X$ must have at least three components - a contradiction with the toughness of G_{i+1}. Hence $N_{G_{i+1}}(u_1) \neq N_{G_{i+1}}(u_2)$

Since $e_{T_i}(v) = 2$, we may assume that $v, v^-_{T_i} \in N_{G_{i+1}}(u_1)$ and $v, v^+_{T_i} \in N_{G_{i+1}}(u_2)$. Hence the subgraph $\langle N_{G_{i+1}}(v) \rangle_{G_{i+1}}$ has the structure shown in Fig. 3.2. Then we get T_{i+1} as follows : we remove from T_i the walk $v^-_{T_i}\ v\ v^+_{T_i}$ and replace it with the walk $v^-_{T_i}\ u_1\ v\ u_2\ v^+_{T_i}$.

Clearly the 2-walk T_{i+1} meets the conditions (i), (ii) and (v) of a good 2-walk. Note that $p(u_1) = p(u_2) = v$. Since $e_{T_{i+1}}(u_1) = e_{T_{i+1}}(u_2) = 2$ and $m_{T_{i+1}}(v) = 1$, the 2-walk T_{i+1} also trivially satisfies the conditions (iii) and (iv) of a good 2-walk. Hence T_{i+1} is a good 2-walk in G_{i+1} (see Fig. 3.2).

<u>**Case 2 :**</u> $d(v) = 2$, $|S_i| = 2$ and $e_{T_i}(v) = 1$.

Set $S_i = \{u_1, u_2\}$. As in the proof of Case 1, $N_{G_{i+1}}(u_1) \neq N_{G_{i+1}}(u_2)$. Since $e_{T_i}(v) = 1$, we may assume that $v, v^-_{T_i} \in N_{G_{i+1}}(u_1)$ and $d_{G_{i+1}}(u_2) = 2$. Hence the subgraph $\langle N_{G_{i+1}}(v) \rangle_{G_{i+1}}$ has the structure shown in Fig. 3.2. Then we get T_{i+1} as follows: we remove $v^-_{T_i}\ v\ v^+_{T_i}$ from T_i and replace it with $v^-_{T_i}\ u_1\ v\ u_2\ v\ v^+_{T_i}$.

Clearly, the 2-walk T_{i+1} meets the conditions (i), (ii) and (v) of a good 2-walk. Note that $p(u_1) = p(u_2) = v$. Since $e_{T_{i+1}}(u_1) = 2$ and $d_{G_{i+1}}(u_2) = 2$, the 2-walk T_{i+1} also trivially satisfies the condition (iii) of a good 2-walk. Furthermore, $m_{T_{i+1}}(v) = 2$ but $d_{G_{\varphi(v)}}(v) = 2$ and $|S_{\varphi(v)}| = 2$, therefore T_{i+1} meets the condition (iv) as well. Hence T_{i+1} is a good 2-walk in G_{i+1} (see Fig. 3.2).

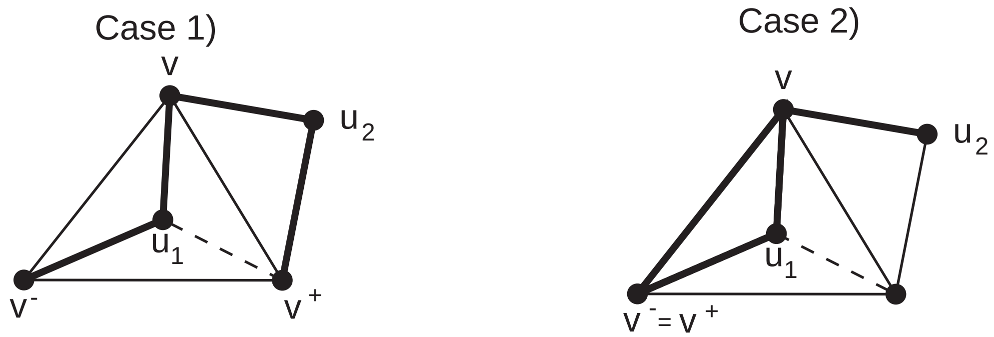

Figure 3.2: Construction of a 2-walk in Cases 1 and 2.

Case 3 : $d(v) = 2$ and $|S_i| \neq 2$.

Then $|S_i| = 1$, otherwise we would get a contradiction with the toughness of G_{i+1}. Then we get a 2-walk T_{i+1} in a similar way as in Case 1 or 2. Observe that if the vertex $u_1 \in S_i$ has degree 3 in G_{i+1}, there always exists a 2-walk T_{i+1} in G_{i+1} such that $e_{T_{i+1}}(u_1) = 2$. Hence, there exists always a good 2-walk T_{i+1}.

Case 4 : $d(v) = 3$ and $|S_i| \leq 3$.

Similarly as in Case 1, for every $u_a \neq u_b$ from the set S_i, $N_{G_{i+1}}(u_a) \neq N_{G_{i+1}}(u_b)$.

Subcase 4.1 : There is at most one vertex $u \in S_i$ such that $\{v^-_{T_i}, v^+_{T_i}\} \cap N_{G_{i+1}}(u) = \emptyset$.

We prove the existence of a good 2-walk T_{i+1} in G_{i+1} separately for $|S_i| = 3$, $|S_i| = 2$ and $|S_i| = 1$.

Subcase 4.1.1 : $|S_i| = 1$

Let $S_i = \{u_1\}$. Note that the vertex u_1 is adjacent in G_{i+1} to v and one or two vertices in $N_{G_i}(v)$.

- u_1 is adjacent to $v_{T_i}^-$ or $v_{T_i}^+$ in G_{i+1}.
  Without loss of generality, we may assume that u_1 is adjacent to $v_{T_i}^-$. Otherwise change the orientation of T_i. We get T_{i+1} as follows: we remove $v_{T_i}^-\ v$ from T_i and we replace it with $v_{T_i}^-\ u_1\ v$. Observe that $e_{T_{i+1}}(u_1) = 2$ and $m_{T_{i+1}}(v) = 1$. Clearly, T_{i+1} is a good 2-walk in G_{i+1}.
- u_1 is not adjacent to $v_{T_i}^-$ and $v_{T_i}^+$ in G_{i+1}
  We get T_{i+1} as follows: we remove $v_{T_i}^-\ v\ v_{T_i}^+$ from T_i and we replace it with $v_{T_i}^-\ v\ u_1\ v\ v_{T_i}^+$. Observe that $e_{T_{i+1}}(u_1) = 1$ and $m_{T_{i+1}}(v) = 2$. Clearly, the 2-walk T_{i+1} meets the conditions (i), (ii), (iv) and (v) of a good 2-walk. If $d_{G_{i+1}}(u_1) = 3$ then $v_{T_i}^- = v_{T_i}^+$. Hence $e_{T_i}(v) = 1$ and T_{i+1} meets the condition (iii) as well.

<u>Subcase 4.1.2 :</u> $|S_i| = 2$

Let $S_i = \{u_1, u_2\}$. Due to the assumption of Subcase 4.1, u_1 or u_2 is adjacent to $v_{T_i}^-$ or $v_{T_i}^+$ in G_{i+1}. Without loss of generality, we may assume that u_1 is adjacent to $v_{T_i}^-$. Otherwise, we change the orientation of T_i. We distinguish two cases.

- u_2 is adjacent in G_{i+1} to $v_{T_i}^+$.
  We get T_{i+1} as follows: we remove $v_{T_i}^-\ v\ v_{T_i}^+$ from T_i and we replace it with $v_{T_i}^-\ u_1\ v\ u_2\ v_{T_i}^+$. Observe that $e_{T_{i+1}}(u_1) = 2$, $e_{T_{i+1}}(u_2) = 2$ and $m_{T_{i+1}}(v) = 1$. Clearly, T_{i+1} is a good 2-walk in G_{i+1}.

- u_2 is not adjacent to $v_{T_i}^+$.
  We may assume that, if u_2 is adjacent to $v_{T_i}^-$ then $d_{G_{i+1}}(u_1) = 3$, otherwise we relabel vertices in S_i. Note that, in this case, both vertices u_1 and u_2 cannot have degree 2 in G_{i+1}, otherwise we would get a contradiction with the toughness of G_{i+1}. We get T_{i+1} as follows: we remove $v_{T_i}^-\ v\ v_{T_i}^+$ from T_i and we replace it with $v_{T_i}^-\ u_1\ v\ u_2\ v\ v_{T_i}^+$. Observe that $e_{T_{i+1}}(u_1) = 2$, $e_{T_{i+1}}(u_2) = 1$ and $m_{T_{i+1}}(v) = 2$. Clearly, the 2-walk T_{i+1} meets the conditions (i), (ii) and (v) of a good 2-walk.

  Note that $d_{G_{i+1}}(u_2) = 3$ only if $v_{T_i}^- = v_{T_i}^+$. Hence $e_{T_i}(v) = 1$ and T_{i+1} meets the condition (iii) of a good 2-walk.

  Recall that $e_{T_{i+1}}(u_1) = 2$ and $m_{T_{i+1}}(p(u_1)) = 2$. Now, we distinguish two cases.

A) $v^-_{T_i} = v^+_{T_i}$.
Observe that $u_1{}^+_{T_{i+1}} = v$ and $u_1{}^-_{T_{i+1}} = v^+_{T_i} = v^-_{T_i}$. Since the edge $v\ v^+_{T_i} \in E(T_{i+1})$, T_{i+1} meets the condition (iv) of a good 2-walk.

B) $v^-_{T_i} \neq v^+_{T_i}$.
Since $d_{G_{i+1}}(u_1) = 3$ and vertex u_2 is not adjacent to $v^+_{T_i}$, $d_{G_{i+1}}(u_2) = 2$. Moreover, $N_{G_{i+1}}(u_2) \subseteq N_{G_{i+1}}(u_1)$. Hence T_{i+1} meets the condition (iv) of a good 2-walk.

Subcase 4.1.3 : $|S_i| = 3$

Let $S_i = \{u_1, u_2, u_3\}$. Then there are no vertices $v_a, v_b \in N_{G_i}(v)$ such that $N_{G_{i+1}}(S_i) = \{v_a, v_b, v\}$ since otherwise the graph $G_{i+1} - \{v, v_a, v_b\}$ has exactly four components - a contradiction with the toughness of G_{i+1}. In other words, it means that $|N_{G_{i+1}}(S_i)| = 4$.

Then we can rename the vertices in S_i such that $v^-_{T_i} \in N_{G_{i+1}}(u_1)$ and $v^+_{T_i} \in N_{G_{i+1}}(u_3)$. Moreover, if $e_{T_i}(v) = 2$, we may assume that if there is a vertex in S_i of degree 2 in G_{i+1} then it is the vertex u_2. If $e_{T_i}(v) = 1$ then we rename vertices in such a way that $d_{G_{i+1}}(u_2) = 3$ if and only if u_2 is not adjacent to $v^-_{T_i} = v^+_{T_i}$ in G_{i+1}.

Then the subgraph $\langle N_{G_{i+1}}(v)\rangle_{G_{i+1}}$ has the structure shown in Fig. 3.3. We get T_{i+1} as follows: we remove $v^-_{T_i}\ v\ v^+_{T_i}$ from T_i and we replace it with $v^-_{T_i}\ u_1\ v\ u_2\ v\ u_3\ v^+_{T_i}$.

Clearly, the 2-walk T_{i+1} meets the conditions (i), (ii) and (v) of a good 2-walk. Note that $p(u_1) = p(u_2) = v$ and $e_{T_{i+1}}(u_2) = 1$. Vertex u_2 has degree 3 in G_{i+1} if and only if either
A) $e_{T_i}(v) = 1$ and u_2 is not adjacent to $p^2(x) = v^-_{T_i} = v^+_{T_i}$ in G_{i+1} or
B) $e_{T_i}(v) = 2$ and $d_{G_{i+1}}(u_1) = d_{G_{i+1}}(u_2) = d_{G_{i+1}}(u_3) = 3$.

Hence the 2-walk T_{i+1} also satisfies the condition (iii) of a good 2-walk. Furthermore, $m_{T_{i+1}}(v) = 2$ but $d_{G_{\varphi(v)}}(v) = 3$ and $|S_{\varphi(v)}| = 3$, therefore T_{i+1} meets (iv) as well. Hence T_{i+1} is a good 2-walk in G_{i+1} (see Fig. 3.3).

Note that we proved a slightly stronger statement. One vertex, let us say u_2, from S_i has $e_{T_{i+1}}(u_2) = 1$. If $e_{T_i}(v) = 2$ and all the vertices in S_i have degree 3 in G_{i+1} then we can choose the vertex u_2 from S_i arbitrarily. Hence we can get three

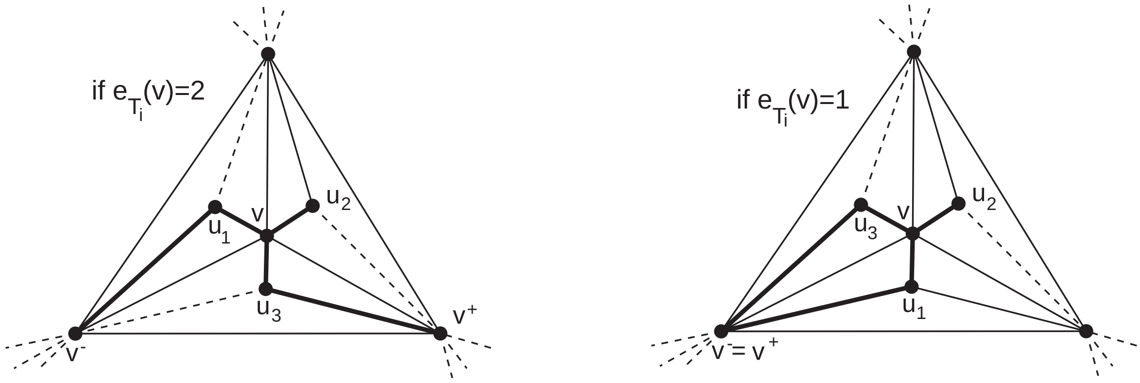

Figure 3.3: Construction of a 2-walk in Case 4.1.3

different good 2-walks in G_{i+1}. Since we use this observation later, we state it as a claim.

Claim 3.3.1 *Under the assumption of Subcase 4.1.3, if* $e_{T_i}(v) = 2$ *and* $d_{G_{i+1}}(u_1) = d_{G_{i+1}}(u_2) = d_{G_{i+1}}(u_3) = 3$, *there exist three different good 2-walks* T_{i+1}, T'_{i+1} *and* $T"_{i+1}$ *in* G_{i+1} *such that* $e_{T_{i+1}}(u_2) = 1$, $e_{T'_{i+1}}(u_1) = 1$ *and* $e_{T"_{i+1}}(u_3) = 1$.

Subcase 4.2 : There are $u_m, u_n \in S_i$ such that $\{v^-_{T_i}, v^+_{T_i}\} \cap N_{G_{i+1}}(u_m) = \emptyset$ and $\{v^-_{T_i}, v^+_{T_i}\} \cap N_{G_{i+1}}(u_n) = \emptyset$.

Then $e_{T_i}(v) = 1$ and the subgraph $\langle N_{G_{i+1}}(v)\rangle_{G_{i+1}}$ has the structure shown in Fig. 3.4. In this case we cannot simply extend the good 2-walk T_i on the new vertices in S_i. We postpone the proof of this subcase. Later, together with Subcase 5.2 and after Subcase 5.1, we show that there exists a good 2-walk T^*_i in G_i, such that $e_{T^*_i}(v) = 2$. Then we transform Subcase 4.2 back to Subcase 4.1.

Case 5 : $d(v) = 3$ and $|S_i| \geq 4$.

Then $|S_i| = 4$, otherwise we would get a contradiction with the toughness of G_{i+1}. As in Case 1, $N_{G_i}(u_a) \neq N_{G_i}(u_b)$, for every $u_a, u_b \in S_i$.

Let S'_i be an arbitrary subset of S_i, such that $|S'_i| = 3$. Then there is no vertex $v' \in N_{G_i}(v)$, such that $\{v'\} \cap N_{G_{i+1}}(S_i) = \emptyset$ since otherwise for the set $X = \{v\} \cup N_{G_i}(v) \setminus \{v'\}$ we have $|X| = 3$ and the graph $G_{i+1} - X$ has exactly

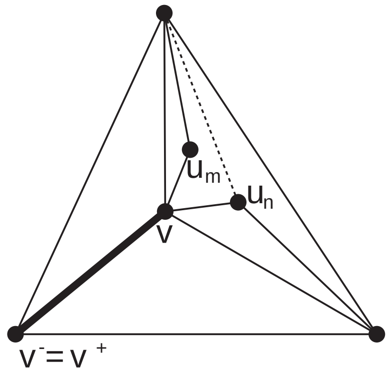

Figure 3.4: Subcase 4.2.

four components - a contradiction with the toughness of G_{i+1}. Then the subgraph $\langle N_{G_{i+1}}(v)\rangle_{G_{i+1}}$ has the structure shown in Fig. 3.5 (up to a symmetry).

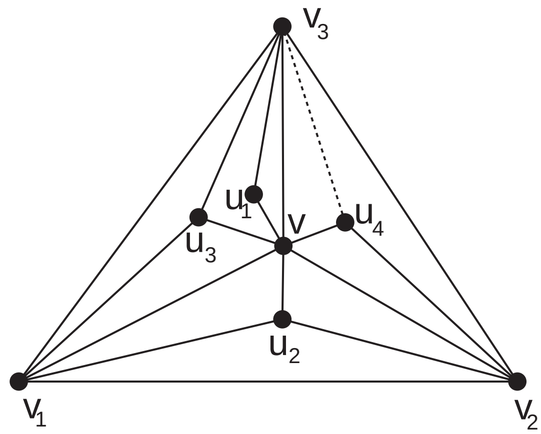

Figure 3.5: Case 5.

<u>Subcase 5.1 :</u> $e_{T_i}(v) = 2$

Let $S_i = \{u_1, u_2, u_3, u_4\}$. We first prove that $m_{T_i}(p(v)) = 1$ or the edge $v^-_{T_i}v^+_{T_i}$ is in the 2-walk T_i in G_i.

Suppose to the contrary that $m_{T_i}(p(v)) = 2$ and $v^-_{T_i}\ v^+_{T_i} \notin E(T_i)$. Due to the properties of a good 2-walk T_i (property (*iv*)), we have the following cases:

A) $d_{G_{\varphi(p(v))}}(p(v)) = 2$ and $|S_{\varphi(p(v))}| = 2$.
Let $X = N_{G_i}(v) \cup \{v\}$. Then $|X| = 4$ and the graph $G_{i+1} - X$ has at least six components, namely, four isolated vertices from S_i, one component with

the other vertex from $S_{\varphi(p(v))}$ and the rest of the graph. But this contradicts the toughness of G_{i+1}.

B) $d_{G_{\varphi(p(v))}}(p(v)) = 3$ and

- $|S_{\varphi(p(v))}| \geq 3$.
  Let $X = N_{G_{\varphi(p(v))}}(p(v)) \cup \{v, p(v)\}$. $|X| = 5$. Then the graph $G_{i+1} - X$ has at least seven components, namely, four isolated vertices from S_i, two components each containing a vertex from $S_{\varphi(p(v))}$ different from v, and the rest of the graph. But this contradicts the toughness of G_{i+1}.
- $|S_{\varphi(p(v))}| = 2$ and there is a vertex $v' \in S_{\varphi(p(v))}$, $v' \neq v$, such that
  - $|S_{\varphi(v')}| = 4$.
    Now let $X = N_{G_{\varphi(p(v))}}(p(v)) \cup \{v, v', p(v)\}$. Then $|X| = 6$ and the graph $G - X$ has at least nine components, namely, four isolated vertices from S_i, four components each containing a vertex from $S_{\varphi(v')}$, and the rest of the graph. But this contradicts the toughness of G_{i+1}.
  - $N_{G_{\varphi(v')}}(v') \subseteq N_{G_i}(v)$.
    Now let $X = N_{G_{i+1}}(v) \cup \{v\}$. Then $|X| = 4$ and the graph $G - X$ has at least six components, namely, four isolated vertices from S_i, one component with v' and the rest of the graph. But this contradicts the toughness of G.

Therefore, $m_{T_i}(p(v)) = 1$ or the edge $v^-_{T_i} v^+_{T_i}$ is in the 2-walk T_i in G_i. We obtain a good 2-walk T_{i+1} as follows:

- If $m_{T_i}(p(v)) = 1$ then we choose orientation of T_i, such that $p(v) = v^-_{T_i}$ (see the property of a good 2-walk (ii)). Then we label vertices in S_i such that, if there is a vertex of degree 2 adjacent to $v^-_{T_i}$ in G_{i+1} then we name this vertex u_1, and then we rename the rest of S_i such that $v^-_{T_i} \in N_{G_{i+1}}(u_2)$, $v^+_{T_i} \in N_{G_{i+1}}(u_4)$ and u_3 is the remaining vertex. We may assume that $d_{G_{i+1}}(u_2) = 3$ and $d_{G_{i+1}}(u_4) = 3$.
  If there is no vertex of degree 2 adjacent to $v^-_{T_i}$ in G_{i+1} then we take an arbitrary vertex from S_i of degree 3 in G_{i+1}, which is incident with $v^-_{T_i}$ in G_{i+1}, and we label this vertex u_1. Rename the rest of S_i such that $v^-_{T_i} \in N_{G_{i+1}}(u_2)$,

$v_{T_i}^+ \in N_{G_{i+1}}(u_4)$ and u_3 is the remaining vertex. Note that the degree of u_3 is 2 in G_{i+1}.
Then we get T_{i+1} as follows : we remove $v_{T_i}^- v v_{T_i}^+$ from T_i and we replace it with $v_{T_i}^- u_1 v_{T_i}^- u_2 v u_3 v u_4 v_{T_i}^+$
Since $|S_i| = 4$, T_{i+1} meets the conditions (i), (ii) and (v) of a good 2-walk. Note that $p(u_1) = p(u_2) = v$ and $e_{T_{i+1}}(u_1) = e_{T_{i+1}}(u_3) = 1$. If $d_{G_{i+1}}(u_1) = 2$ then $d_{G_{i+1}}(u_3) = 3$ if and only if $d_{G_{i+1}}(u_2) = d_{G_{i+1}}(u_3) = d_{G_{i+1}}(u_4) = 3$. If $d_{G_{i+1}}(u_1) = 3$ then $d_{G_{i+1}}(u_3) = 2$. Therefore, the 2-walk T_{i+1} satisfies either the condition (iii)-B or the condition (iii)-C of a good 2-walk. Furthermore, $m_{T_{i+1}}(v) = 2$ and $d_{G_{\varphi(v)}}(v) = 3$, but $|S_{\varphi(v)}| = 4$. Therefore T_{i+1} meets (iv) as well. Hence T_{i+1} is a good 2-walk in G_{i+1}.

- If $m_{T_i}(p(v)) = 2$ and the edge $v_{T_i}^- v_{T_i}^+$ is in the 2-walk T_i in G_i then we have the following cases :

  A) If there is a vertex u_a in S_i such that $\{v_{T_i}^-, v_{T_i}^+\} \subset N_{G_{i+1}}(u_a)$ then we relabel vertices in S_i in the following way: $u_1 = u_a$, $v_{T_i}^- \in N_{G_{i+1}}(u_2)$, $v_{T_i}^+ \in N_{G_{i+1}}(u_4)$ and u_3 is the remaining vertex. Clearly, the degree of u_1 is 3 in G_{i+1} and we may assume that $d_{G_{i+1}}(u_3) = 2$. Then we obtain T_{i+1} as follows: we remove $v_{T_i}^- v_{T_i}^+$ and $v_{T_i}^+ v v_{T_i}^-$ from T_i and then we replace it by $v_{T_i}^- u_1 v_{T_i}^+$ and $v_{T_i}^+ u_2 v u_3 v u_4 v_{T_i}^-$
  Clearly, the 2-walk T_{i+1} meets the conditions (i), (ii) (iv) and (v) of a good 2-walk. Note that $e_{T_{i+1}}(u_3) = 1$ but $d_{G_{i+1}}(u_3) = 2$. Therefore the 2-walk T_{i+1} satisfies condition (iii) of a good 2-walk. Hence T_{i+1} is a good 2-walk in G_{i+1}.

  B) If there is no vertex u_a in S_i such that $\{v_{T_i}^-, v_{T_i}^+\} \subset N_{G_{i+1}}(u_a)$, then every vertex in S_i is adjacent to either $v_{T_i}^+$ or $v_{T_i}^-$ in G_{i+1}. Moreover, due to the toughness condition, there are exactly two vertices from S_i adjacent to $v_{T_i}^+$ in G_{i+1} and the other two vertices in S_i are adjacent to $v_{T_i}^-$ in G_{i+1}. Relabel vertices from S_i in the following way: $v_{T_i}^- \in N_{G_{i+1}}(u_1)$, $v_{T_i}^+ \in N_{G_{i+1}}(u_2)$, $v_{T_i}^+ \in N_{G_{i+1}}(u_3)$ and $v_{T_i}^- \in N_{G_{i+1}}(u_4)$. Then we obtain T_{i+1} as follows: we remove $v_{T_i}^- v_{T_i}^+$ and $v_{T_i}^+, v, v_{T_i}^-$ from T_i and then we replace it by $v_{T_i}^- u_1 v u_2 v_{T_i}^+$ and $v_{T_i}^+ u_3 v u_4 v_{T_i}^-$.

Observe that $e_{T_{i+1}}(u_1) = 2$, $e_{T_{i+1}}(u_2) = 2$, $e_{T_{i+1}}(u_3) = 2$, $e_{T_{i+1}}(u_4) = 2$ and $m_{T_{i+1}}(v) = 2$. Since $|S_i| = 4$, the 2-walk T_{i+1} satisfies all the conditions (i)-(v) of a good 2-walk.

See examples of 2-walk T_{i+1} in G_{i+1}, for $m_{T_i}(p(v)) = 1$, in Fig. 3.6 and, for $m_{T_i}(p(v)) = 2$, in Fig. 3.7.

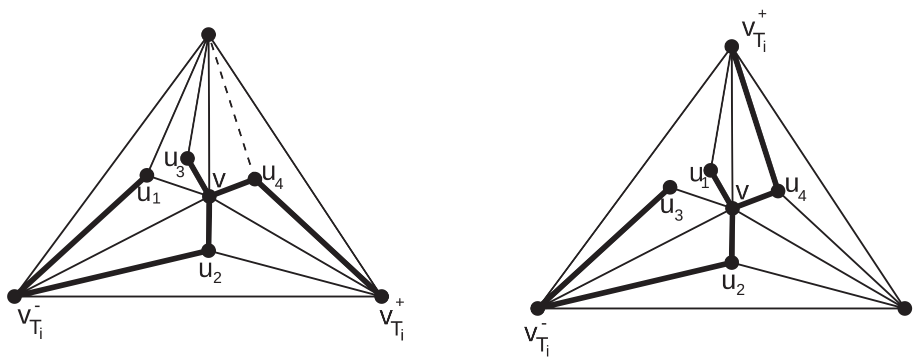

Figure 3.6: Construction of a 2-walk when $|S_i| = 4$ and $m_{T_i}(p(v)) = 1$.

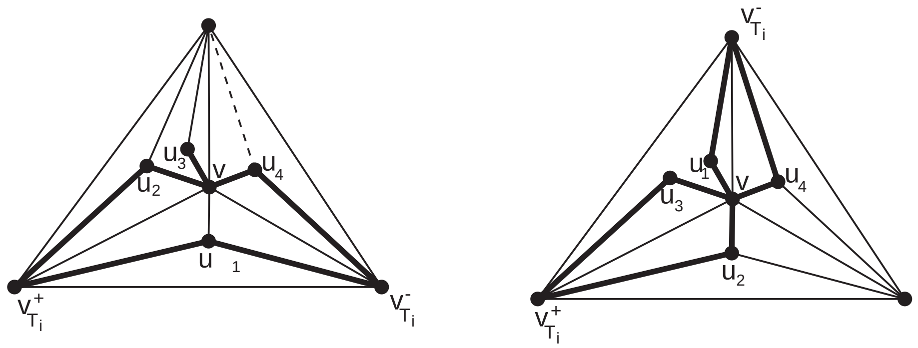

Figure 3.7: Construction of a 2-walk when $|S_i| = 4$ and $m_{T_i}(p(v)) = 2$.

Before we move to another subcase, we summarize when a vertex from S_i, let us say u_3, has $d_{G_{i+1}}(u_3) = 3$ and $e_{T_{i+1}}(u_3) = 1$. It happens only if $m_{T_i}(p(v)) = 1$ in the two following cases.

- There is a vertex $u_1 \in S_i$ of degree 2 in G_{i+1} adjacent to $v^-_{T_i} = p(v)$.
  See that, $e_{T_{i+1}}(u_3) = 1$ if and only if all the vertices in S_i, except for u_1, have the degree 3 in G_{i+1}. But now we can choose the vertex u_3, with $e_{T_{i+1}}(u_3) = 1$,

arbitrarily from $S_i \setminus \{u_1\}$. Hence we can get three different good 2-walks in G_{i+1}. Since we use this observation later, we state it as a claim.

Claim 3.3.2 *Under the assumption of Subcase 5.1, if $m_{T_i}(p(v)) = 1$ and there is a vertex $u_1 \in S_i$ of degree 2 in G_{i+1} adjacent to $v^-_{T_i} = p(v)$ and $d_{G_{i+1}}(u_2) = d_{G_{i+1}}(u_3) = d_{G_{i+1}}(u_4) = 3$, there exist three different good 2-walks T_{i+1}, T'_{i+1} and $T"_{i+1}$ in G_{i+1} such that $e_{T_{i+1}}(u_3) = 1$, $e_{T'_{i+1}}(u_2) = 1$ and $e_{T"_{i+1}}(u_4) = 1$.*

- There is no such a vertex (i.e., vertex from S_i of degree 2 in G_{i+1} and adjacent to $v^-_{T_i} = p(v)$).
  See that, there are two vertices, let us say $u_1, u_2 \in S_i$, adjacent to $v^-_{T_i} = p(v)$ in G_{i+1} and $d_{G_{i+1}}(u_1) = d_{G_{i+1}}(u_2) = 3$. Clearly, either $e_{T_{i+1}}(u_1) = 1$ and $e_{T_{i+1}}(u_2) = 2$, or $e_{T_{i+1}}(u_1) = 2$ and $e_{T_{i+1}}(u_2) = 1$. Hence we can get two different good 2-walks in G_{i+1}. We also use this observation later.

Claim 3.3.3 *Under the assumption of Subcase 5.1, if $m_{T_i}(p(v)) = 1$ and there is no vertex of degree 2 in G_{i+1} from S_i adjacent to $v^-_{T_i} = p(v)$, then there are two vertices $u_1, u_2 \in S_i$, $d_{G_{i+1}}(u_1) = d_{G_{i+1}}(u_2) = 3$, adjacent to $v^-_{T_i} = p(v)$ in G_{i+1}. Then there exist two different good 2-walks T_{i+1} and T'_{i+1} in G_{i+1} such that $e_{T_{i+1}}(u_1) = 1$ and $e_{T'_{i+1}}(u_2) = 1$.*

<u>**Subcase 5.2 :**</u> $e_{T_i}(v) = 1$

In this case we cannot simply extend the good 2-walk T_i on the new vertices in S_i. Due to property (i) of a good 2-walk, we should use vertex v more than twice, which is impossible. So we need to show that there exists a good 2-walk T^*_i in G_i, such that $e_{T^*_i}(v) = 2$. Then we transform Subcase 5.2 to Subcase 5.1. This will be done together with Subcase 4.2.

Claim 3.3.4 *Let W be the class of all good 2-walks in G_i. Under the assumptions of Subcase 4.2 or 5.2 there exists a good 2-walk $T^*_i \in W$, such that $e_{T^*_i}(v) = 2$.*

Proof. Suppose, to the contrary, that for every good 2-walk T_i from W $e_{T_i}(v) = 1$. In the graph G_0, any vertex x has $e_{T_0}(x) = 2$. Thus there is an integer k such that the vertex $p^k(v)$ exist and satisfies

$$e_{T_{\varphi(p^k(v))}}(p^k(v)) = 2.$$

Suppose that the good 2-walk $T_i \in W$ is chosen such that the integer k is smallest possible.

Denote the vertices $p^j(v)$ as w_j, denote the graphs $G_{\varphi(p^j(v))}$ as G'_j, denote the sets $S_{\varphi(p^j(v))}$ as S'_j, and denote the walks $T_{\varphi(p^j(v))}$ as T'_j, for $j = \{1, ..., k\}$.

Due to the property (iii) of a good 2-walk, we have:

$$d_{G'_j}(w_j) = 3, \quad j = \{1, ..., k\}$$

$$e_{T'_j}(w_j) = 1, \quad j = \{1, ..., k-1\}$$

$$w_{j+2} \notin N_{G'_j}(w_j), \quad j = \{1, ..., k-2\}$$

Since $e_{T'_k}(w_k) = 2$ and $e_{T'_{k-1}}(w_{k-1}) = 1$, there are three vertices in S'_k of degree 3 in G'_{k-1}. We will call the path $v, w_1, ..., w_k$ a *critical path* (i.e., a critical path is a path starting at a vertex v, v satisfying the assumption of Subcase 4.2 or 5.2, and ending at a vertex w_k, $e_{T_{\varphi(w_k)}}(w_k) = 2$, where $p(w_i) = w_{i+1}$). Now we consider two cases: A) $|S'_k| = 3$ and B) $|S'_k| = 4$.

A) If $|S'_k| = 3$ then all the vertices in S'_k have degree 3 in G'_{k-1}. Let $S'_k = \{w_{k-1}, w'_{k-1}, w''_{k-1}\}$ and let $N_{G'_k}(w_k) = \{x_1, x_2, x_3\}$, $N_{G'_{k-1}}(w_{k-1}) = \{w_k, x_1, x_2\}$, $N_{G'_{k-1}}(w'_{k-1}) = \{w_k, x_1, x_3\}$, $N_{G'_{k-1}}(w''_{k-1}) = \{w_k, x_2, x_3\}$. Using Claim 3.3.1, we can choose another good 2-walk T^*_{k-1} such that either $e_{T^*_{k-1}}(w'_{k-1}) = 1$ or $e_{T^*_{k-1}}(w''_{k-1}) = 1$. Clearly $e_{T^*_{k-1}}(w_{k-1}) = 2$ and therefore, we cannot obtain any critical path starting at the vertex v in G_{i+1}. Otherwise, such a critical path would end at the vertex w_{k-1} at the latest, which is impossible due to our choice of T_i. We need to show that for such a T^*_{k-1} there exists a good 2-walk T^*_i in G_i, i.e. that we will not get any critical path ending at vertex w_k in some graph G_ℓ, for $1 < \ell \leq i$. Recall that we can choose T^*_{k-1} in G'_{k-1} such that either $e_{T^*_{k-1}}(w'_{k-1}) = 1$ or $e_{T^*_{k-1}}(w''_{k-1}) = 1$.

Assume otherwise, i.e., for both choices of T^*_{k-1} in G'_{k-1} we obtain a critical path ending at the vertex w_k. If $e_{T^*_{k-1}}(w'_{k-1}) = 1$ then we denote this critical path $v', w'_1, ..., w'_a$, where $w'_a = w_k$ and $w'_{a-1} = w'_{k-1}$. If $e_{T^*_{k-1}}(w''_{k-1}) = 1$ then we denote the critical path $v'', w''_1, ..., w''_b$, where $w''_b = w_k$ and $w''_{b-1} = w''_{k-1}$. Observe that all the vertices $v', w'_1, ..., w'_{a-1}$ lie inside the triangle w_k, x_1, x_3 and the vertex v' is adjacent to w'_1, x_1 and x_3 in $G_{\varphi(v')}$. Similarly, all the vertices $v'', w''_1, ..., w'_{b-1}$ lie inside the triangle w_k, x_2, x_3 and the vertex v'' is adjacent to w''_1, x_2 and x_3 in $G_{\varphi(v'')}$. Recall that the original critical path $v, w_1, ..., w_k$ lies inside the triangle w_k, x_1, x_2 and the vertex v is adjacent to w_1, x_1 and x_2 in G_i.

Now we show that G_{i+1} must have toughness $t(G) \leq \frac{3}{4}$. We define a set of vertices X as follows. $X = N_{G'_k}(w_k) \cup \{v, v', v''\}$. If v satisfies the assumptions of Subcase 5.2, add vertex the w_1 into the set X. If v' or v'' satisfy the assumptions of Subcase 5.2, add the vertex w'_1 or w''_1 into the set X. If we remove X from G then the number of components of $G - X$ will be greater than $\frac{4}{3}|X|$, which contradicts the toughness of G (see Fig. 3.8). Hence, we can obtain at most two critical paths ending at the vertex w_k. Therefore we can choose T^*_{k-1} in G'_{k-1} such that in the graph G_i there exists a good 2-walk T^*_i with $e_{T^*_i}(v) = 2$.

B) If $|S'_k| = 4$. Let $N_{G'_k}(w_k) = \{p(w_k), x_2, x_3\}$ and $N_{G'_{k-1}}(w_{k-1}) = \{p(w_k), w_k, x_2\}$. First we show that v satisfies only the assumption of Subcase 4.2.

Assume otherwise, i.e., v satisfies the assumption of Subcase 5.2. We define a set X as follows : $X = N_{G'_k}(w_k) \cup \{w_k, v, w_1\}$. Then $|X| = 6$ because w_k is a simplicial vertex of degree 3 in G'_k. Graph $G_{i+1} - X$ must have at least eight components, namely isolated vertices from S_i, three components, each containing a vertex from S'_k, and the rest of the graph. But this contradicts the toughness assumption.

There are two possible ends of the critical path at vertex w_k. One possible end is that $(w_{k-1})^-_{T'_{k-1}} = w_k$. The second is that $(w_{k-1})^-_{T'_{k-1}} = p(w_k)$ (see Fig. 3.9).

The first case is similar to the case A) (i.e., $|S'_k| = 3$), hence the proof is also similar (just instead of using Claim 3.3.1 we use Claim 3.3.2). Consider the second case. Since $d_{G'_{k-1}}(w_k) = 3$ and $(w_{k-1})^-_{T'_{k-1}} = p(w_k)$ we can use Claim 3.3.3. Therefore, there exists another good 2-walk T^*_{k-1} in G'_{k+1}, different from T'_{k-1}. Clearly

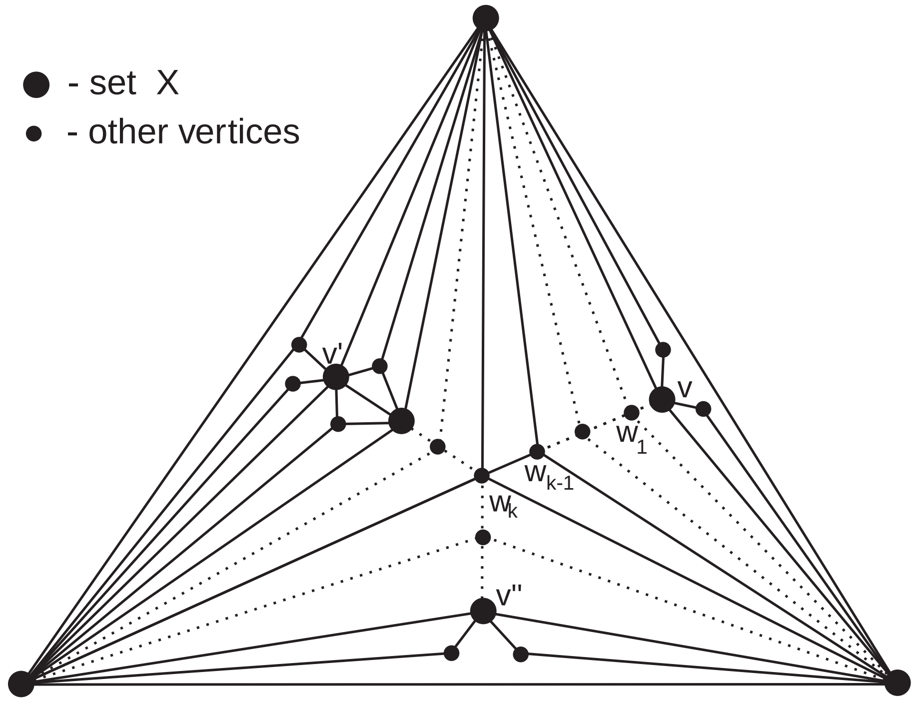

Figure 3.8: Example of three critical paths ending at vertex w_k; Case A).

$e_{T^*_{k-1}}(w_{k-1}) = 2$ and therefore, we cannot obtain any critical path starting at the vertex v in G_{i+1}. Otherwise, such a critical path would end at the vertex w_{k-1} at the latest, which is impossible due to our choice of T_i. We need to show that for such T^*_{k-1} there exists a good 2-walk T^*_i in G_i, i.e., that we will not get any critical path ending at vertex w_k in some graph G_ℓ, for $1 < \ell \leq i$.

Assume otherwise, i.e., for the good 2-walk T^*_{k-1} in G'_{k-1} we obtain a critical path ending at w_k in the graph G_ℓ. Let $v', w'_1, ..., w'_a$ be this critical path, where $w'_a = w_k$, and $w'_{a-1} = w'_{k-1}$. Similarly as for v, v' satisfies only the assumption of Subcase 4.2. Observe that all the vertices $v', w'_1, ..., w'_{a-1}$ lie inside the triangle $w_k, p(w_k), x_3$ and the vertex v' is adjacent to w'_1, w_k and x_3 in $G_{\varphi(v')}$. Recall that the original critical path $v, w_1, ..., w_k$ lies inside the triangle $w_k, p(w_k), x_2$ and the vertex v is adjacent to w_1, w_k and x_2 in G_i.

Then the graph $G - \{v', w_k, x_3\}$ has four components, namely two isolated vertices from the set S_ℓ, one isolated vertex from the set S'_k and the rest of the graph - a contradiction with the toughness assumption (see Fig. 3.10). Hence, we can not

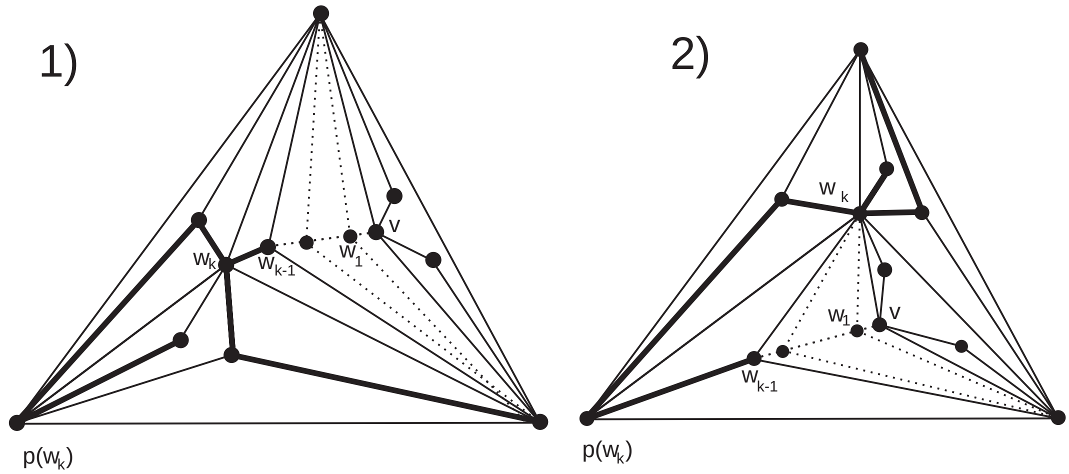

Figure 3.9: Examples of two different critical paths in Case B).

obtain a critical path ending at the vertex w_k. Therefore, for the good 2-walk T^*_{k-1} in G'_{k-1}, there exists a good 2-walk T^*_i in G_i with $e_{T^*_i}(v) = 2$. ■

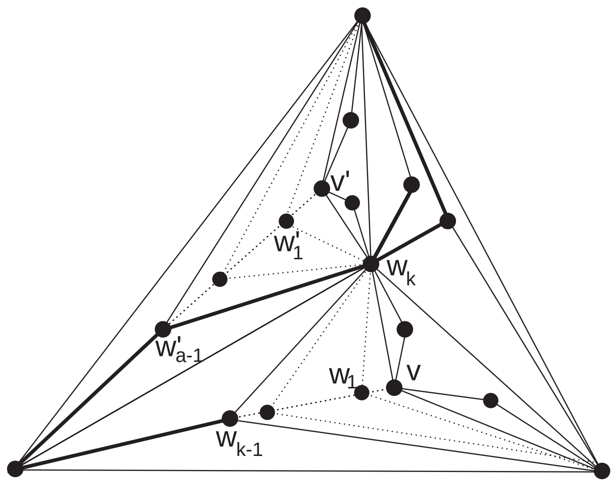

Figure 3.10: Two critical paths ending at vertex w_k; Case B).

At this stage we have finished the proof of Subcases 4.2 and Subcase 5.2. It follows that we have finished the proof of Lemma 3.3.3, since we have discussed all possible sets S_i. ■

Since the graph K_3 has a 2-walk, proof of Theorem 3.3.1 follows immediately from Lemma 3.3.3.

3.4 2-walks in K_4-minor free graphs

K_4-minor free graphs form an important subclass of planar graphs (recall that a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor). Fundamental structural properties of K_4-minor free graphs have been established first by G. A. Dirac in [33]. An alternative characterization of K_4-minor free graphs involves the notion of a 'tree-width', a notion well-studied both in structural graph theory as well as in theoretical computer science [15]. A graph G is K_4-minor free if and only if its tree-width is at most 2. A tree-width of a graph can be described using the notion of tree decompositions, introduced in Chapter 1, or using the notion of k-trees. Recall that the class of k-trees can be defined recursively as follows: a complete graph K_{k+1} of order $k+1$ is a k-tree, and if G is a k-tree then a graph obtained from G and K_{k+1} by identifying k vertices contained in a complete subgraph of G and K_{k+1} is also a k-tree. Hence, 2-trees are obtained from triangles by identifying pairs of edges. A graph G is K_4-minor free if and only if it is a subgraph of a 2-tree. In fact, chordal 2-connected K_4-minor free graphs are precisely 2-trees. Finally, K_4-minor free graphs are also related to series-parallel graphs which we introduce later: every 2-connected K_4-minor free graph is series-parallel and every block of a K_4-minor free graph is series-parallel.

Our main result is that every K_4-minor free graph which is more than $\frac{4}{7}$-tough has a 2-walk. On the other hand, we construct a $\frac{4}{7}$-tough K_4-minor free graph with no 2-walk. The graph that we construct is a 2-tree, i.e., it is also chordal. Hence, our bound is also the best possible for K_4-minor free chordal graphs, the class of graphs that coincide with chordal planar graphs G with $\omega(G) \leq 3$. Let us finally remark that it is not hard to generalize our construction to produce an infinite family of $\frac{4}{7}$-tough chordal K_4-minor free graphs with no 2-walk. These results were found jointly with Z. Dvořák and D. Král'.

We deal with K_4-minor free graphs which are more than $\frac{4}{7}$-tough. Since each $\frac{4}{7}$-tough graph is 2-connected, all graphs that we consider are series-parallel graphs. The class of *series-parallel graphs* can be obtained by the following construction based on *blocks with poles*. The simplest series-parallel block is an edge and its two end-vertices are its poles. If G and H are two blocks with poles v_1 and v_2 and w_1 and w_2, the graph obtained by identifying the poles v_2 and w_1, such that v_1 and

w_2 are its new poles, is the block obtained by a *serial join* of G and H. The graph obtained from G and H by identifying the poles v_1 and w_1 and the poles v_2 and w_2 is the block obtained by a *parallel join* of G and H. All blocks obtained by a series of serial and parallel joins from single edges form the class of series-parallel graphs. In the rest of this section, we also refer to blocks used in the construction of series-parallel graphs as *series-parallel blocks* in order to avoid confusion with 2-edge-connected subgraphs that are also called blocks (although we do not use this term in the alternative meaning at all). Vertices of a series-parallel block distinct from the poles are called *inner* vertices.

An important notion used in our proofs is the notion of an 'A-bridge'. If $A \subseteq V(G)$ then an *A-bridge* of G is a maximal subgraph of G such that any two vertices of it are joined by a path with all inner vertices distinct from those contained in A. The vertices of an A-bridge contained in the set A are called *attachments* and its other vertices are *inner vertices.* A simplest A-bridge is an edge with both end-vertices contained in A; an A-bridge with internal vertices is said to be *non-trivial.* Hence, $\tau(A)$ is equal to the number of non-trivial A-bridges.

Let us now state a simple structural lemma concerning series-parallel blocks.

★ **Lemma 3.4.1** *Let G be a series-parallel block with poles v_1 and v_2. If G is not a single edge then there exists an inner vertex v_0 such that each $\{v_1, v_2, v_0\}$-bridge has exactly two attachments, and there are a $\{v_1, v_2, v_0\}$-bridge with the attachments v_1 and v_0 and a $\{v_1, v_2, v_0\}$-bridge with the attachments v_2 and v_0.*

Proof. We proceed by induction on the number of inner vertices of a series-parallel block. If G is obtained by a serial join of two blocks, set v_0 to be the pole of the two blocks that was identified. If G is obtained by a parallel join of two blocks, at least one of the two blocks is not a single edge (we deal with simple graphs only) and this block contains a vertex v_0 with the properties described in the statement of the lemma. Since the other block used in the parallel join is a $\{v_1, v_2, v_0\}$-bridge with attachments v_1 and v_2, all the $\{v_1, v_2, v_0\}$-bridges have two attachments. ■

We also need to introduce the notion of a 'proper' series-parallel block. Let G be a series-parallel graph, let H be one of the blocks obtained in the construction of G, and let v_1 and v_2 be the poles of H. We say that H is a *proper* block if H has only one $\{v_1, v_2\}$-bridge but G has at least one non-trivial $\{v_1, v_2\}$-bridge different from H. Note that being a proper block is a property that depends not only on the block H but also on G.

Next we introduce notation that we use in the proof of our main result. We show that a proper block of a $\frac{4}{7}$-tough series-parallel graph contains 2-walks of certain specific types unless it contains one of the obvious obstacles for their existence. The considered types of 2-walks are called green, red, blue, black and grey. Similarly, the obstacles are called green, red and blue.

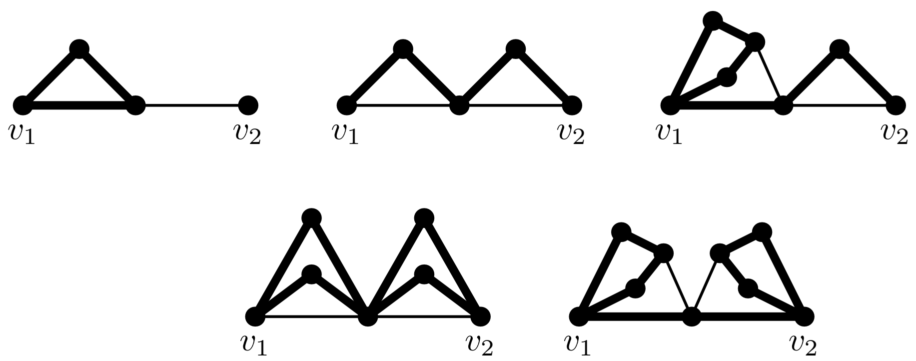

Figure 3.11: Examples of green, red, blue, black and grey walks (in this order). The green and blue walks are from vertex v_1.

We start by introducing the types of 2-walks. Let G be a proper series-parallel block of a $\frac{4}{7}$-tough series-parallel graph and let v_1 and v_2 be its poles. Examples of all the introduced types of walks can be found in Figure 3.11. A *green walk from* v_i is a closed walk that starts and ends at v_i, visits each inner vertex of G once or twice, and does not visit any of the poles, except at the beginning and the end of the walk. A *red walk* is an open walk that starts at v_1, ends at v_2, visits each inner vertex of G once or twice and does not visit any of the poles except at the beginning and the end of the walk. A *blue walk from* v_i is an open walk that starts at v_i, ends at the other pole of G, visits each inner vertex of G once or twice, visits v_i at most twice but it visits the other pole only at the end of the walk. A *black walk* is a closed walk that visits both v_1 and v_2 once and each inner vertex of G once or twice. Note that, alternatively, a black walk can be viewed also as a collection of two open walks, each starting at v_1 and ending at v_2, that visit together all the inner vertices of G once or twice. Finally, a *grey walk* is an open walk that starts at v_1, ends at v_2 and visits each vertex of G once or twice.

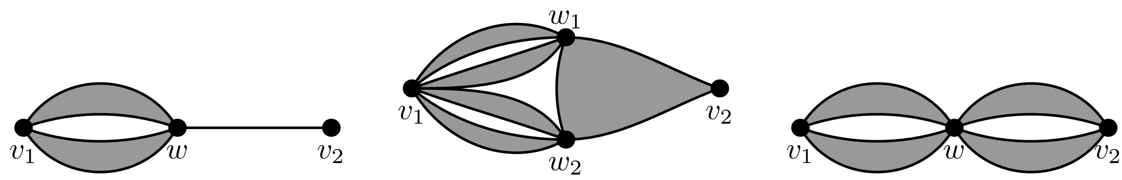

Figure 3.12: Green and blue obstacles at vertex v_1 and a red obstacle (depicted in this order).

We now describe some obvious obstacles for the existence of the described types of walks. It turns out that these obstacles, under the assumption that G is more than $\frac{4}{7}$-tough, are the only ones that can preclude the existence of a particular type of a walk. We say that G contains a *green obstacle at* v_i if there exists an inner vertex w such that there are two non-trivial $\{v_1, v_2, w\}$-bridges with the attachments v_i and w (see Fig. 3.12). Clearly, if G contains a green obstacle at v_i, G cannot contain a green walk from the other pole: indeed, such a walk must enter each of the two bridges from w, since it must avoid the vertex v_i, and thus the vertex w would be visited three times—for the first time before tracing the first of the bridges, for the second time after tracing the first and before tracing the second bridge, and for the third time after tracing the second bridge.

We say that G contains a *blue obstacle at* v_i (see Fig. 3.12) if there exist inner vertices w_1 and w_2 such that there are two non-trivial $\{v_1, v_2, w_1, w_2\}$-bridges with the attachments v_i and w_1, two non-trivial $\{v_1, v_2, w_1, w_2\}$-bridges with the attachments v_i and w_2, and a non-trivial $\{v_1, v_2, w_1, w_2\}$-bridge with the attachments v_{3-i}, w_1 and w_2. If G contains a blue obstacle at v_i, then G cannot contain a blue walk from v_{3-i}: such a blue walk must enter or exit one of the two $\{v_1, v_2, w_1, w_2\}$-bridges with the attachments v_i and w_1 through v_i (otherwise, w_1 would be visited three times), and similarly one of the bridges with the attachments v_i and w_2 must enter or exit through v_i (otherwise, w_2 would be visited three times). Hence, the vertex v_i would be visited twice and thus there is no blue walk from v_{3-i}. Note that we do not need the $\{v_1, v_2, w_1, w_2\}$-bridge with the attachments v_{3-i}, w_1 and w_2

to be non-trivial in order to prevent the existence of a blue walk, however, in our considerations, the bridge will always be non-trivial.

Finally, we say that G contains a *red obstacle* if there exists an inner vertex w such that there are two $\{v_1, v_2, w\}$-bridges with the attachments v_1 and w, and two $\{v_1, v_2, w\}$-bridges with the attachments v_2 and w (see Fig. 3.12). If G contains a red obstacle then it cannot contain a red walk—indeed, such a walk can enter only one of the two $\{v_1, v_2, w\}$-bridges with the attachments v_1 and w from the vertex v_1, and thus it must enter and exit the other bridge through w. Similarly, one of the two $\{v_1, v_2, w\}$-bridges with the attachments v_2 and w is entered and exited through w. Then, w is visited three times—we conclude that there is no red walk. Similarly, the presence of a blue obstacle at any of the two poles prevents the existence of a red walk.

We now state five lemmas on the existence of each type of a walk. These lemmas will be proven in the next section.

★ **Lemma 3.4.2** *Let G be a proper series-parallel block of a $\frac{4}{7}$-tough series-parallel graph and let v_1 and v_2 be its poles. If G does not contain a green obstacle at the pole v_2 then G contains a green walk from v_1. Analogously, if G does not contain a green obstacle at the pole v_1, then G contains a green walk from v_2.*

★ **Lemma 3.4.3** *Let G be a proper series-parallel block of a $\frac{4}{7}$-tough series-parallel graph and let v_1 and v_2 be its poles. If G contains neither a red obstacle nor a blue obstacle at the pole v_1 or v_2 then G contains a red walk.*

★ **Lemma 3.4.4** *Let G be a proper series-parallel block of a $\frac{4}{7}$-tough series-parallel graph and let v_1 and v_2 be its poles. If G does not contain a blue obstacle at the pole v_1 then G contains a blue walk from v_2. Analogously, if G does not contain a blue obstacle at the pole v_2 then G contains a blue walk from v_1.*

★ **Lemma 3.4.5** *Every proper series-parallel block G of a $\frac{4}{7}$-tough series-parallel G contains a black walk.*

★ **Lemma 3.4.6** *Every proper series-parallel block G of a $\frac{4}{7}$-tough series-parallel G contains a grey walk.*

Before proceeding with the proofs of Lemmas 3.4.2–3.4.6, let us derive the main result assuming we have already proved the lemmas.

★ **Theorem 3.4.1** *If G is a K_4-free minor graph that is more than $\frac{4}{7}$-tough then G has a 2-walk.*

Proof. Since G is more than $\frac{1}{2}$-tough, then G is 2-connected and thus series-parallel. If G has less than four vertices then it is either a single vertex, an edge or a triangle and the statement of the theorem readily follows. We assume in the rest that G has at least four vertices. Since G is 2-connected, it is obtained by a parallel join of series-parallel blocks $B_1, \ldots, B_k$ with poles v_1 and v_2. Without loss of generality, we can assume that each B_i is either an edge or a series-parallel block obtained by a serial join. Since G is $\frac{4}{7}$-tough, at most three of the blocks $B_1, \ldots, B_k$ are non-trivial.

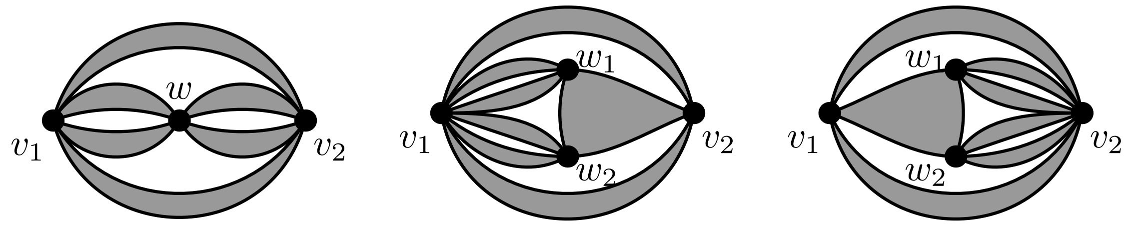

Figure 3.13: Configurations from the proof of Theorem 3.4.1 in case of three non-trivial series-parallel blocks.

If there are three non-trivial blocks B_1, B_2 and B_3 then neither of them contains a red or a blue obstacle at v_1 or v_2. If B_1 contained a red obstacle (with a vertex w as in the definition) then G would have six non-trivial $\{v_1, v_2, w\}$-bridges which is impossible because of the toughness assumption (see Fig. 3.13). If B_2 contained a blue obstacle at v_1 (with vertices w_1 and w_2 as in the definition) then G would have seven non-trivial $\{v_1, v_2, w_1, w_2\}$-bridges which is also impossible because of the toughness assumption. The case that B_2 contained a blue obstacle at v_2 is symmetric. We conclude that each B_i, $i = 1, 2, 3$, contains a red walk by Lemma 3.4.3

(note that all the blocks B_i are proper). The red walks of B_1 and B_2 and the black walk of B_3 (which exists by Lemma 3.4.5) combine to a 2-walk of G.

If there are exactly two non-trivial blocks B_1 and B_2 then each of them is proper and thus contains a black walk by Lemma 3.4.5. The two black walks combine to a 2-walk of G.

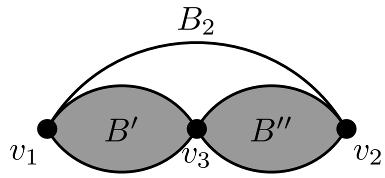

Figure 3.14: Illustration of the notation used in the proof of Theorem 3.4.1 in the case of a single non-trivial series-parallel block.

The last case is that there is a single non-trivial block B_1. Note that we cannot apply Lemma 3.4.5, since B_1 is not a proper block. In this case, $k = 2$ and B_2 is a single edge. The block B_1 was obtained by a serial join of two blocks B' and B'' (see Figure 3.14). Since G has at least four vertices, one of the blocks B' and B'' is non-trivial, say B' is a non-trivial series-parallel block. Let v_3 be the common pole of B' and B''. Observe now that the graph G can also be obtained in the following way: perform the serial join of B'' and B_2 identifying the vertex v_2 and let B_0 be the obtained block with poles v_1 and v_3. G is then obtained by the parallel join of B_0 and B'. Since both B_0 and B' are non-trivial, we can now proceed as in the case of two or three non-trivial blocks which we have analyzed before and conclude that G has a 2-walk. ■

We prove Lemmas 3.4.2–3.4.6 together, by induction on the number of their vertices. In the proof, we use the induction assumption that all the five lemmas have been established for all proper blocks with fewer vertices.

Proof of Lemmas 3.4.2–3.4.6. If G is a single edge or a two-edge path, the statements of all the lemmas clearly hold. In the rest, we assume that G contains at least two inner vertices. Let v_0 be the vertex of G as described in Lemma 3.4.1. Let $A_1, \ldots, A_k$ be the $\{v_1, v_2, v_0\}$-bridges with the attachments v_1 and v_0, and $B_1, \ldots, B_\ell$ the $\{v_1, v_2, v_0\}$-bridges with the attachments v_2 and v_0. Note that since G is proper, any bridge with the attachments v_1 and v_2 must be a single edge and a walk does not have to trace the bridge (the bridge does not have any inner vertices). Hence, we can assume that there is no such bridge at all.

At most two of the bridges $A_1, \ldots, A_k$ are non-trivial; otherwise, the original graph contains four non-trivial $\{v_1, v_0\}$-bridges (at least three bridges A_i and a bridge containing v_2) and the entire graph is at most $\frac{1}{2}$-tough, contradicting the assumption. If $k \geq 2$ then we can assume that all the bridges $A_1, \ldots, A_k$ are non-trivial, since the trivial bridges do not have to be traced by a walk and we can remove them from the list. Hence, it is enough to consider the following three cases:

- $k = 1$ and A_1 is a bridge formed by a single edge.
- $k = 1$ and A_1 is a non-trivial bridge.
- $k = 2$ and both A_1 and A_2 are non-trivial bridges.

Similarly, only the following three cases need to be considered regarding the bridges $B_1, \ldots, B_\ell$:

- $\ell = 1$ and B_1 is a bridge formed by a single edge.
- $\ell = 1$ and B_1 is a non-trivial bridge.
- $\ell = 2$ and both B_1 and B_2 are non-trivial bridges.

Also note that each non-trivial bridge A_i is a non-trivial series-parallel block with the poles v_1 and v_0, and each non-trivial bridge B_i is a non-trivial series-parallel block with the poles v_2 and v_0.

We now prove several technical claims concerning the existence of certain walks in the blocks A_i and B_i that we later use to construct the desired walks in G.

Claim 3.4.1 *If $k = 2$ then each of the blocks A_1 and A_2 has a red walk. Analogously, if $\ell = 2$ then each of the blocks B_1 and B_2 has a red walk.*

By symmetry, we need only focus on the case that $k = 2$ and show that A_1 has a red walk. By the induction assumption, it is enough to show that A_1 does not contain a red obstacle or a blue obstacle at v_1 or v_0: if A_1 contained a red obstacle then the entire graph would contain six non-trivial $\{v_1, v_0, w\}$-bridges, contradicting

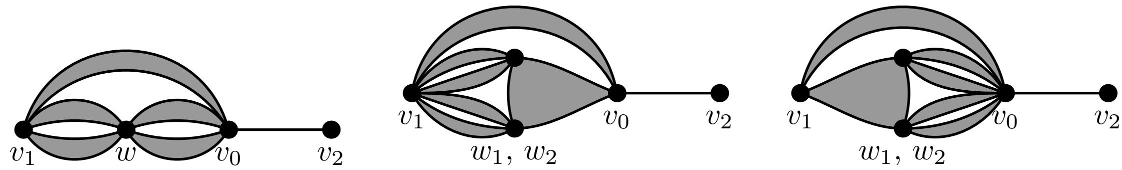

Figure 3.15: Possible configurations if $k = 2$ and A_1 does not contain a red walk.

the assumption that the graph is more than $\frac{4}{7}$-tough (see Fig. 3.15). If A_1 contained a blue obstacle then the entire graph would contain seven non-trivial $\{v_1, v_0, w_1, w_2\}$-bridges, also contradicting the assumption that the graph is more than $\frac{4}{7}$-tough.

Claim 3.4.2 *If $k = 2$ and G does not contain a blue obstacle at v_1 then A_1 or A_2 contains a green walk from v_0. Analogously, if $\ell = 2$ and G does not contain a blue obstacle at v_2 then B_1 or B_2 contains a green walk from v_0.*

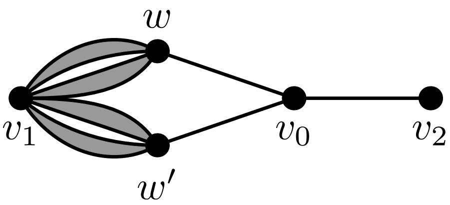

Figure 3.16: The configuration if $k = 2$ and both A_1 and A_2 contain green obstacles at v_1.

If both A_1 and A_2 contained green obstacles at v_1 with vertices w_1 and w_2 then G would contain a blue obstacle at v_1 with w_1 and w_2. Note that the $\{v_1, v_2, w_1, w_2\}$-bridge with the attachments v_2, w_1 and w_2 is non-trivial since it contains v_0 (see Fig. 3.16).

Claim 3.4.3 *If $k = \ell = 2$ then A_1 or A_2 contains a green walk from v_1. Analogously, if $k = \ell = 2$ then B_1 or B_2 contains a green walk from v_2.*

If both A_1 and A_2 contained a green obstacle at v_0, say with vertices w_1 and w_2 then G would contain seven non-trivial $\{v_0, v_2, w_1, w_2\}$-bridges contradicting our assumption that G is more than $\frac{4}{7}$-tough (see Fig. 3.17). The claim now follows

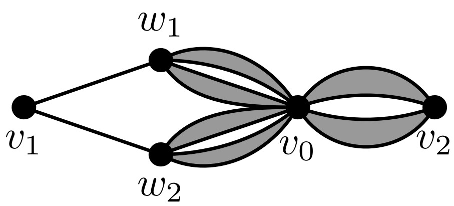

Figure 3.17: The configuration if $k = \ell = 2$ and both A_1 and A_2 contain green obstacles at v_0.

from the induction assumption. Analogously, B_1 or B_2 contains a green walk from v_0.

Claim 3.4.4 *A_1 or B_1 contains neither a blue obstacle at v_0 nor a red obstacle.*

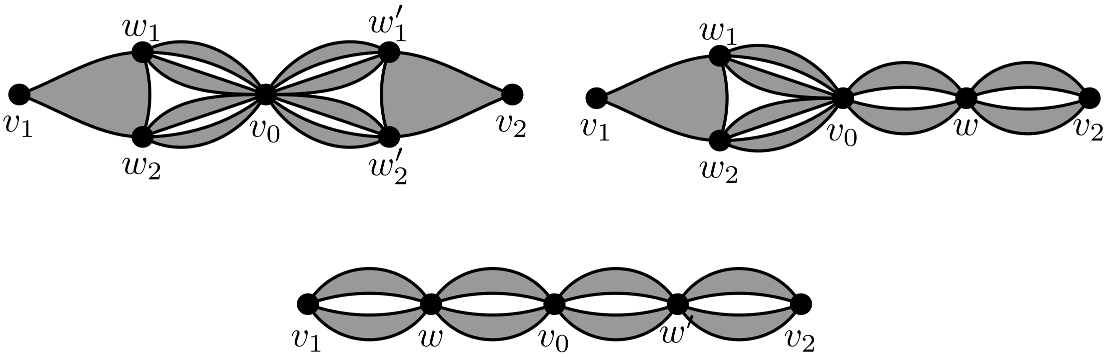

Figure 3.18: Possible configurations if both A_1 and B_1 contain a blue obstacle at v_0 or a red obstacle.

If both A_1 and B_1 contained blue obstacles at v_0 with vertices w_1 and w_2, and w_1' and w_2', respectively then there would be nine non-trivial $\{v_0, w_1, w_2, w_1', w_2'\}$-bridges (see Fig. 3.18). Hence, the graph would be at most $\frac{5}{9}$-tough contradicting the assumption that it is more than $\frac{4}{7}$-tough.

If A_1 contained a blue obstacle at v_0 (with vertices w_1 and w_2) and B_1 a red obstacle (with a vertex w) then the whole graph would have nine non-trivial $\{v_0, v_2, w, w_1, w_2\}$-bridges. Again, this is excluded by the assumption. The case that A_1 contained a red obstacle and B_1 a blue one is symmetric.

If A_1 contained a red obstacle (with a vertex w) and B_1 also contained a red obstacle (with a vertex w') then the whole graph would have nine non-trivial $\{v_0, v_1, v_2, w, w'\}$-bridges which is impossible by the assumption. The statement of the claim now readily follows.

Claim 3.4.5 *If $\ell = 2$ then A_1 contains neither a blue obstacle at v_0 nor a red obstacle. Analogously, if $k = 2$ then B_1 contains neither a blue obstacle at v_0 nor a red obstacle.*

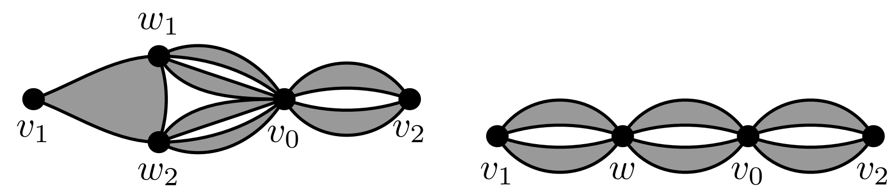

Figure 3.19: Possible configurations if A_1 contains a blue obstacle at v_0 or a red obstacle, and $\ell = 2$.

If A_1 contained a blue obstacle at v_0 then there would be seven non-trivial $\{w_1, w_2, v_0, v_2\}$-bridges contradicting the assumption that the graph is more than $\frac{4}{7}$-tough (see also Fig. 3.19). If A_1 contained a red obstacle then there would be seven non-trivial $\{v_1, w, v_0, v_2\}$-bridges. The case of $k = 2$ is symmetric.

We are now ready to construct the desired types of walks in G.

Proof of Lemma 3.4.2. *If G does not contain a green obstacle at v_2 then it has a green walk from v_1.*

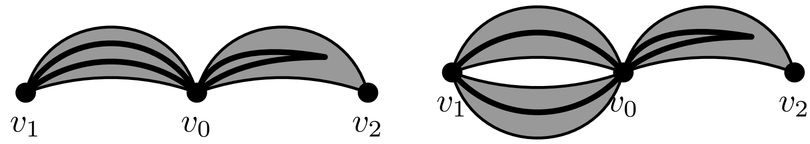

Figure 3.20: Green walks (drawn in bold) constructed in the proof of Lemma 3.4.2.

Note that $\ell = 1$, otherwise, G would contain a green obstacle with $w = v_0$. In addition, B_1 does not contain a green obstacle at v_2 since such an obstacle would also be a green obstacle of G. Hence B_1 has a green walk from v_0 by the induction. If $k = 1$, the green walk of B_1 can be combined with a black walk of A_1 to a green walk of G. If $k = 2$, the green walk of B_1 can be combined with two red walks of A_1 and A_2 (which exist by Claim 3.4.1) to a green walk of G. We conclude that G has a green walk from v_1 unless it has a green obstacle at v_2. The reader can check Figure 3.20 for the illustration of the proof of this claim.

Proof of Lemma 3.4.3. *If G contains neither a red obstacle nor a blue obstacle at v_1 or v_2 then G has a red walk.*

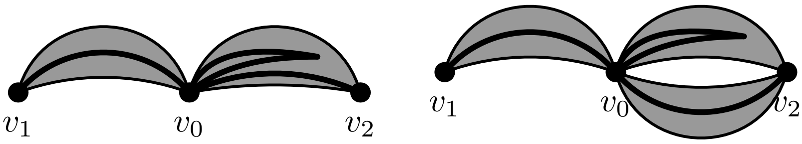

Figure 3.21: Red walks (drawn in bold) constructed in the proof of Lemma 3.4.3.

Since G does not have a red obstacle, $k = 1$ or $\ell = 1$. By symmetry, we can assume that $k = 1$. Let us first consider the case $\ell = 1$. By Claim 3.4.4, the assumptions of the claim and the induction, A_1 or B_1 contains a red walk. By symmetry, let us say that A_1 has a red walk. Since G does not contain a blue obstacle at v_2, B_1 does not contain it either and thus B_1 has a blue walk from v_0. The red walk of A_1 and the blue walk of B_1 combine to a red walk of G (see Fig. 3.21).

If $\ell = 2$ then B_1 or B_2 contains a green walk from v_0 by Claim 3.4.2. By symmetry, we assume that B_1 has a green walk from v_0. By Claim 3.4.5 and the induction, A_1 has a red walk. The red walk of A_1, the green walk of B_1 and a red walk of B_2 (which exists by Claim 3.4.1) can be combined to a red walk of G (see also Fig. 3.21).

Proof of Lemma 3.4.4. *If G does not contain a blue obstacle at v_2 then it has a blue walk from v_1.*

Assume first that $k = \ell = 1$. By Claim 3.4.4, A_1 or B_1 contains neither a red obstacle nor a blue obstacle at v_0. If A_1 has this property then A_1 contains a blue walk from v_1 by the induction and B_1 has a blue walk from v_0 (otherwise, a blue

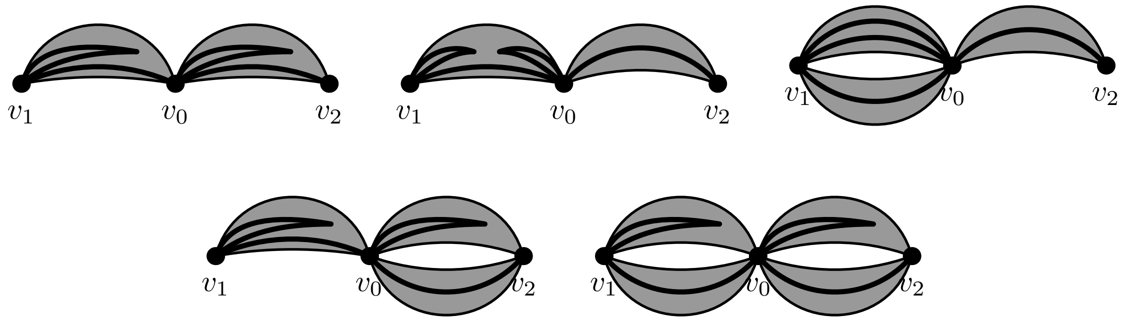

Figure 3.22: Blue walks (drawn in bold) constructed in the proof of Lemma 3.4.4.

obstacle at v_2 of B_1 would also be a blue obstacle at v_2 of G). The two blue walks combine to a blue walk of G from v_1. If B_1 contains neither a red obstacle nor a blue obstacle at v_0, B_1 has a red walk by the induction. In addition, A_1 has a grey walk by the induction. The grey and the red walks combine to a blue walk from v_1 (see Fig. 3.22).

If $k = 2$ and $\ell = 1$ then B_1 has a red walk by Claim 3.4.5 and the induction. By Claim 3.4.1, both A_1 and A_2 have red walks, and by the induction, they have black walks, too. The black walk of A_1 and the red walks of A_2 and B_1 combine to a blue walk of G from v_1.

If $\ell = 2$ then B_1 or B_2 contains a green walk from v_0 by Claim 3.4.2. Assume that B_1 does (the other case is symmetric). By Claim 3.4.1, B_2 contains a red walk. If $k = 1$, a blue walk of A_1 from v_1 which exists by the induction and Claim 3.4.5, combines with the green walk of B_1 and the red walk of B_2 to a blue walk of G from v_1. If $k = 2$, A_1 or A_2 has a green walk from v_1 by Claim 3.4.3. By the symmetry, we can assume that A_1 has a green walk from v_1. Since A_2 has a red walk by Claim 3.4.1, the green walks of A_1 and B_1 and the red walks of A_2 and B_2 combine to a blue walk of G from v_1 (see Fig. 3.22).

Proof of Lemma 3.4.5. *G has a black walk.*

The black walk of G is comprised of a black walk of A_1 if $k = 1$ or two red walks of A_1 and A_2 if $k = 2$ (such red walks exist by Claim 3.4.1), and of a black walk of B_1 if $\ell = 1$ or two red walks of B_1 and B_2 if $\ell = 2$ (see Fig. 3.23).

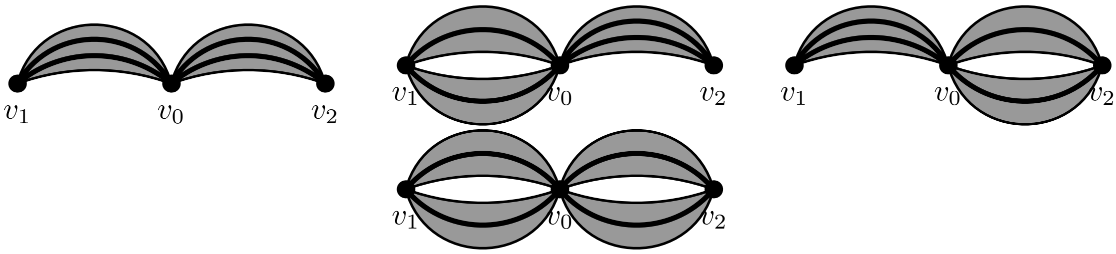

Figure 3.23: Black walks (drawn in bold) constructed in the proof of Lemma 3.4.5.

Proof of Lemma 3.4.6. *G has a grey walk.*

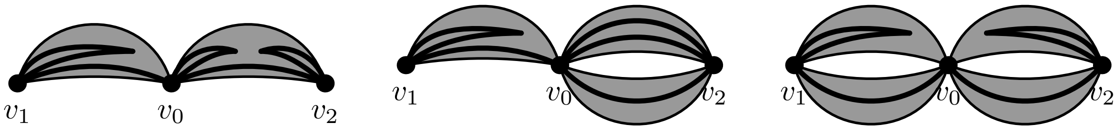

Figure 3.24: Grey walks (drawn in bold) constructed in the proof of Lemma 3.4.6.

Assume first that $k = 1$ and $\ell = 1$. By Claim 3.4.4, A_1 or B_1 does not contain a blue obstacle at v_0, say A_1 does not. By the induction, A_1 has a blue walk from v_1 and B_1 has a grey walk. The two walks combine to a grey walk of G (see Fig. 3.24).

Next, we consider the case $k = 1$ and $\ell = 2$. By Claim 3.4.5, A_1 does not contain a blue obstacle at v_0. Hence, it has a blue walk from v_1 by the induction. Both B_1 and B_2 have black walks by the induction and red walks by Claim 3.4.1. The blue walk of A_1 from v_1, a black walk of B_1 and a red walk of B_2 combine to a grey walk of G. The case $k = 2$ and $\ell = 1$ is symmetric.

The final case that we need to consider is that $k = \ell = 2$. By Claim 3.4.1, A_1, A_2, B_1 and B_2 have red walks. By Claim 3.4.3, A_1 or A_2 has a green walk from v_1, say A_1 does. Similarly, we can suppose that B_1 has a green walk from v_2. The green walks of A_1 and B_1 and the red walks of A_2 and B_2 combine to a grey walk of G. ■

3.5 A $\frac{4}{7}$-tough graph with no 2-walk

In this section, we construct a $\frac{4}{7}$-tough 2-tree with no 2-walk. We start by introducing two series-parallel blocks, depicted in Fig. 3.25 and 3.26. The two blocks have the common property that any 2-walk tracing them must contain an inner edge incident with y, as stated in the next two lemmas (an edge of a block is *inner* if it does not join its poles).

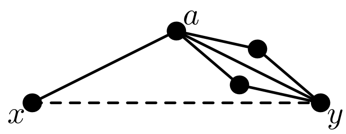

Figure 3.25: A series-parallel block with poles x and y such that any 2-walk must contain an inner edge incident with y.

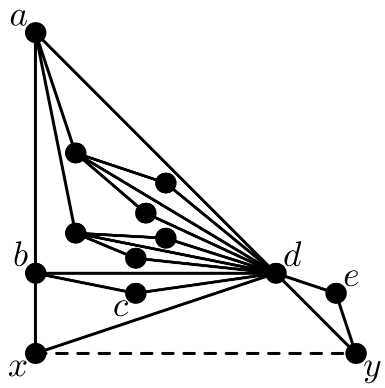

Figure 3.26: A series-parallel block with poles x and y such that any 2-walk must contain an inner edge incident with y.

★ **Lemma 3.5.1** *Let G be the series-parallel block with poles x and y depicted in Fig. 3.25. Any 2-walk contains at least one inner edge of G incident with y.*

Proof. If no inner edge of G incident with y is contained in a 2-walk then the 2-walk must come to a from x, then visit both the common neighbours of a and y and return to x. However, a would be visited three times in this way. The statement of the lemma now follows. ■

★ **Lemma 3.5.2** *Let G be the series-parallel block with poles x and y depicted in Figure 3.26. Any 2-walk contains at least one inner edge of G incident with y.*

Proof. Let us consider a 2-walk that contains neither the edge dy nor the edge ey. Then the 2-walk enters and leaves the vertex e through the edge de. By Lemma 3.5.1, d is incident with at least one edge contained in each of the two copies of the block depicted in Figure 3.25 pasted along the edge ad. Since the 2-walk visits d at most twice, the 2-walk cannot use the edge bd or the edge cd. Hence, the 2-walk enters the block through the vertex x, it comes from x to b, visits c, continues to a (in order to reach d), and eventually returns from a to b and leaves the block through x. However, in this way, the 2-walk visits b three times, which is impossible. We conclude that every 2-walk contains at least one inner edge of G incident with y. ■

The graph that we present as an example of a $\frac{4}{7}$-tough series-parallel graph with no 2-walk is depicted in Fig. 3.27. It is easy to verify that the graph is not only series-parallel, but it is in fact a 2-tree. Also note that the graph contains three copies of the block from Fig. 3.25 and two copies of the block from Fig. 3.26. Let us first argue that it has no 2-walk.

★ **Lemma 3.5.3** *The 2-tree depicted in Fig. 3.27 has no 2-walk.*

Proof. By Lemmas 3.5.1 and 3.5.2, every 2-walk of the graph contains five edges incident with the vertex a. However, such a 2-walk must visit a at least three times, which is impossible. ■

Next, we argue that the graph depicted in Fig. 3.27 is $\frac{4}{7}$-tough.

★ **Theorem 3.5.1** *The 2-tree depicted in Fig. 3.27 is an example of a $\frac{4}{7}$-tough 2-tree with no 2-walk.*

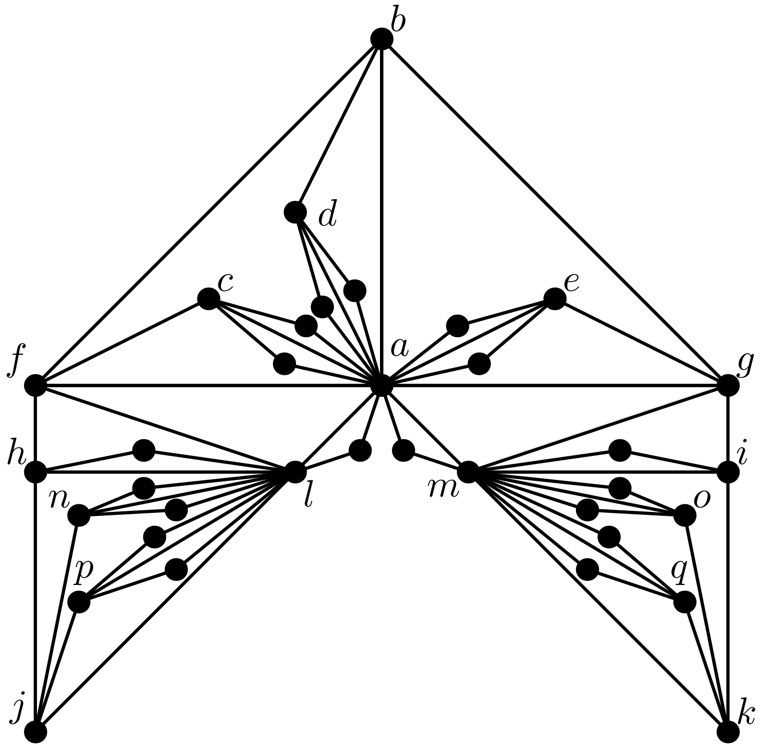

Figure 3.27: A $\frac{4}{7}$-tough 2-tree with no 2-walk.

Proof. By Lemma 3.5.3, it is enough to show that the graph is $\frac{4}{7}$-tough. The number of non-trivial A-bridges for $A = \{h, l, n, p\}$ is seven and thus the graph is at most $\frac{4}{7}$-tough. In the rest, we show that the graph is $\frac{4}{7}$-tough.

Assume that G is less than $\frac{4}{7}$-tough, and let A be a non-empty inclusion-wise minimal set of vertices such that $|A|/\tau(A) < \frac{4}{7}$. For any proper subset B of A, we can infer the following from the choice of A:

$$\begin{aligned}
\frac{|A|}{\tau(A)} &< \frac{|A| - |B|}{\tau(A \setminus B)} \\
\tau(A)|B| &< |A| \left(\tau(A) - \tau(A \setminus B)\right) \\
\frac{\tau(A)}{|A|}\,|B| &< \tau(A) - \tau(A \setminus B) \\
\frac{7|B|}{4} &< \tau(A) - \tau(A \setminus B)
\end{aligned}$$

Hence, if $|B| = 1$, $\tau(A) \geq \tau(A \setminus B) + 2$. In particular, each vertex of A is an attachment of at least three non-trivial A-bridges and thus A contains no vertices of degree two. Similarly, every pair of vertices of A is incident with five non-trivial A-bridges (unless $|A| = 2$), every triple with seven such bridges (unless $|A| = 3$) and every quadruple with nine such bridges (unless $|A| = 4$). In our further considerations, we will argue that A does not contain certain subsets B based on the number of A-bridges incident with the vertices B and implicitly assume that B is a proper subset of A; the case $B = A$ will not be explicitly analyzed but it is easy to check that our arguments extend to such case, too.

Let $B = A \cap \{j, h, l, n, p\}$. If $j \in B$ then neither h, n nor p can be contained in B (there would not be three non-trivial A-bridges incident with them). On the other hand, l is contained in B, since otherwise j would not be incident with at least three non-trivial A-bridges. However, the pair j and l is now incident with at most four non-trivial A-bridges: those containing h, n, p and the common neighbour of a and l. Since this is impossible, we infer that $j \notin B$.

Assume that $l \in B$. If $B = \{h, l, n, p\}$ then the quadruple h, l, n and p is incident with at most eight non-trivial A-bridges, which is impossible. If $B = \{h, l, n\}$ then the triple h, l and n is incident with at most six non-trivial A-bridges, which is also impossible. Similarly, $B \neq \{h, l, p\}$. If $B = \{l, n, p\}$, the triple l, n and p is incident with at most six non-trivial A-bridges, which is impossible. If $B = \{l, n\}$, the pair l and n is incident with at most four non-trivial A-bridges, which is also impossible. Similarly, $B \neq \{l, p\}$. If $B = \{h, l\}$ then the pair h and l is incident with at most four non-trivial A-bridges, which is impossible, too. Hence, $B = \{l\}$ and l is incident with at most two non-trivial A-bridges, which is impossible as well. We eventually conclude that $l \notin B$. Hence, neither n nor p are contained in A (they cannot be incident with three non-trivial A-bridges if $l \notin B$). The only two cases that remain are $B = \{h\}$ and $B = \emptyset$. Since the former case is excluded (h would be incident with a single non-trivial A-bridge), we infer that $B = \emptyset$. Analogously, it holds that $A \cap \{i, k, m, o, q\} = \emptyset$ and thus that $A \subseteq \{a, b, c, d, e, f, g\}$.

Assume first that $b \in A$. Since b must be incident with at least three non-trivial A-bridges, a must also be contained in A (otherwise, f and g cannot be in different A-bridges), and $f \notin A$, $g \notin A$, and $d \notin A$. Let $\alpha = |A \cap \{c, e\}|$. There are $3 + 2\alpha$

non-trivial A-bridges, and $|A|/\tau(A) = (2+\alpha)/(3+2\alpha) \geq 4/7$. We conclude that $b \notin A$.

Assume now that $f \in A$. Since f must be incident with at least three non-trivial A-bridges, $a \in A$ and $c \notin A$. If g is also contained in A, let $\alpha = |A \cap \{d\}|$ (note that $e \notin A$ in this case). It is easy to derive that $|A|/\tau(A) = (3+\alpha)/(5+2\alpha) \geq \frac{4}{7}$. If $g \notin A$, let $\alpha = |A \cap \{d, e\}|$. We derive that $|A|/\tau(A) = (2+\alpha)/(3+2\alpha) \geq \frac{4}{7}$. We eventually conclude that $f \notin A$. By symmetry, $g \notin A$. We can now conclude that $A \subseteq \{a, c, d, e\}$.

If $a \notin A$ then none of the vertices c, d or e can be incident with three non-trivial A-bridges. Hence, $a \in A$. Let $\alpha = |A \cap \{c, d, e\}|$. Since there are $1+2\alpha$ non-trivial A-bridges, we have that $|A|/\tau(A) = (1+\alpha)/(1+2\alpha) \geq \frac{4}{7}$. We can now conclude that there is no set A with $|A|/\tau(A) < \frac{4}{7}$ and the graph is $\frac{4}{7}$-tough. ■

Chapter 4

r-trestles

4.1 Introduction

As already mentioned, there are several ways how to generalize the concept of hamiltonicity. The last generalization of the concept of hamiltonicity we deal with in this thesis is an 'r-trestle'. An *r-trestle* is a 2-connected graph with maximum degree Δ at most equal to r. We say that a graph G has an r-trestle if G contains a spanning subgraph which is an r-trestle. The concept of an r-trestle is a generalization of a hamiltonian cycle, while a 2-trestle in a graph is exactly a hamiltonian cycle.

Spanning subgraphs with bounded degree have been studied deeply, for example, k-spanning trees, spiders, etc. But, surprisingly, very few researchers have worked on 2-connected spanning subgraphs with bounded degree. As far as we know, there are only three papers dealing with trestles (see next section).

In this largely unexplored area we obtain one new result. We prove that every $K_{1,r}$-free graph has an r-trestle. Moreover, we present graphs that show that our result is sharp. We also state some corollaries of this results. We will finish this chapter with several open problems.

4.2 Literature review

Before proceeding with known results concerning r-trestles, we recall the definition of an r-trestle. An *r-trestle* is a 2-connected graph with maximum degree at most r. We say that a graph G has an r-trestle if G contains a spanning subgraph which is an r-trestle. From this definition, it is immediate that if a graph has an r-trestle then it also has an $(r+1)$-trestle, since an r-trestle is itself an $(r+1)$-trestle, by definition. Obviously, the converse is not true. For an example of a graph that has a 3-trestle but not a 2-trestle, see Fig 4.1.

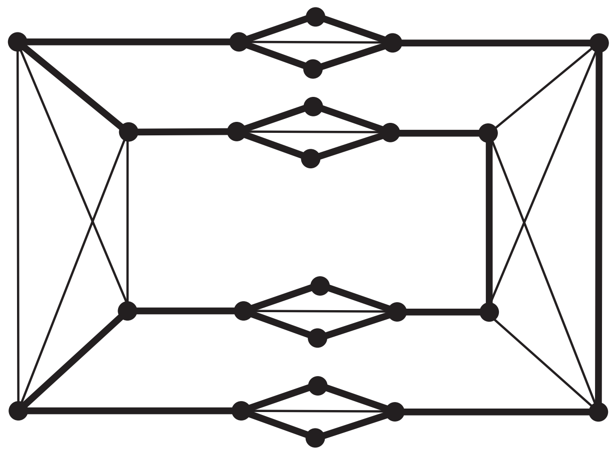

Figure 4.1: A non-hamiltonian graph with a 3-trestle (in bold).

Another good example is the Tutte's graph (Fig 2.13). The Tutte's graph is a 3-connected planar cubic graph without a hamiltonian cycle. Clearly, this graph has a 3-trestle (since the graph is cubic, the graph itself is a 3-trestle).

Similarly to the result from Jackson and Wormald for k-walks, Voss and Tkáč in [89] found a corresponding necessary condition for the existence of an r-trestle in a graph.

Theorem 4.2.1 [89] *No graph G with toughness $t(G) < \frac{2}{r}$ (where the integer r is greater than one) has an r-trestle.*

Proof. Let G be a graph with toughness $t(G) < \frac{2}{r}$ where the integer r is greater than one. Suppose that G has an r-trestle T. Since $t(G) < \frac{2}{r}$, there exists a subset S_0 of the vertex set of G ($S_0 \subset V(G)$) with the following property.

$$\frac{|S_0|}{\omega(G - S_0)} = t(G) < \frac{2}{r}$$

So G contains a vertex set S_0 such that

$$2\omega(G - S_0) > k|S_0|.$$

If G has an r-trestle T then $S_0 \subset V(G) = V(T)$ and so every vertex from S_0 has in T degree at most r. Since T is 2-connected, every component of $G - S_0$ is adjacent with at least two vertices from S_0. This means that the following inequality holds.

$$2\omega(G - S_0) \leq k|S_0|$$

But this contradicts the earlier inequality. ■

It seems that there is a relationship between toughness and the existence of an r-trestle in a graph. Tkáč and Voss generalized the Chvátal's conjecture for trestles.

Conjecture 4.2.1 [89] *For every integer r greater than one, there is a real number $t_r > 0$ such that every t_r-tough graph has an r-trestle.*

In contrast with k-walks, this conjecture is still open for all r greater than one. A *polyhedral graph* is a 3-connected planar graph and a *k-separator* is a vertex cut-set of size exactly k. In the same paper, [89] Tkáč and Voss also proved the existence of a 3-trestle in a polyhedral graph in which each of the separator-hypergraphs $\mathcal{H}(G)$ has at most one cycle. For a detailed definition and explanation, see [89]. Just briefly, for each polyhedral graph G, the separator-hypergraph can be constructed as follows: $V(G) = V(\mathcal{H}(G))$ and the edges of $\mathcal{H}(G)$ are the 3-separators. The authors also showed that this result is sharp. They found polyhedral graphs with more than one cycle in their separator-hypergraph, which have no 3-trestle.

Clearly, a polyhedral graph does not have to have a 2-trestle (hamiltonian cycle). For example, see the Tutte's graph in Fig 2.13. In [4] Barnette extended this result and showed that there is a polyhedral graph with no 5-trestle. On the other hand, in [46] Gao proved that every 3-connected graph on the plane, projective plane, torus and Klein bottle has a 6-trestle (for definitions of projective plane, torus and Klein bottle, see [32]).

Since r-trestles in graphs can be viewed as a generalization of hamiltonian cycles in graphs, we can also raise the question of the existence of degree conditions implying the existence of r-trestles. A minimum-degree condition for the existence of an r-trestle has been recently proved by Jendrol', Ryjáček and Schiermeyer [59].

The last stated, but for us the most interesting, result is due to Ryjáček and Tkáč. In 2004 they proved the following theorem.

Theorem 4.2.2 [85] *Every 2-connected claw-free graph has a 3-trestle.*

Recall that a claw is the graph $K_{1,3}$. It can be easily observed that every connected $K_{1,2}$-free graph is a complete graph and therefore hamiltonian (i.e., it has a 2-trestle). A natural question that arises immediately is, whether in Theorem 4.2.2, 'claw' can be replaced by '$K_{1,r}$-free' and '3-trestle' replaced by 'r-trestle'. This was conjectured by Ryjáček and Tkáč.

Conjecture 4.2.2 [85] *Every 2-connected $K_{1,r}$-free graph has an r-trestle.*

As our main result in this chapter we prove that Conjecture 4.2.2 holds. This result was found jointly with R. Kužel.

4.3 Trestles in $K_{1,r}$-free graphs

Before we start with the proof, we state some useful definitions. A 2-connected graph G is *edge-minimal 2-connected* if the removal of any of its edges decreases its connectivity. This means that after removing an arbitrary edge from an edge-minimal 2-connected graph, the graph becomes only 1-connected. This immediately implies the existence of at least one cutvertex in the graph. A *block* B of a graph G is a maximal 2-connected subgraph of G, or an edge $uv \in E(G)$ with $d_G(u) = 1$, or a bridge. An edge is called a *chord* of a graph G if the edge connects two non-adjacent vertices on a cycle in the graph G. A graph G having no chord in any of its cycles is called a *chord-free* graph. As foreshadowed in the introduction of this chapter, we next prove the following theorem.

★ **Theorem 4.3.1** *Every 2-connected $K_{1,r}$-free graph has an r-trestle.*

Before we give the proof of Theorem 4.3.1, we present some useful lemmas that will be used later in the proof.

★ **Lemma 4.3.1** *Let G be an edge-minimal 2-connected graph with at least four vertices. Then*

(i) *G is chord-free (G has no chord).*
(ii) *G is triangle-free (G does not contain a cycle C_3).*
(iii) *any vertex $u \in V(G)$ with $d_G(u) = r$ is a center of an induced $K_{1,r}$ in G.*

Proof.

(i) Suppose to the contrary that G has a chord e. Since e is a chord, the graph $G - e$ is also 2-connected. Therefore, G is not edge-minimal.
(ii) Assume G has a triangle. Then, clearly, one of its edges must be a chord, which is impossible due to (i).
(iii) Let u be a vertex of degree r that is not a center of an induced $K_{1,r}$. Then there exist vertices u_1, $u_2 \in N(u)_G$ which are adjacent. Obviously, vertices u, u_1 and u_2 form a triangle, which is impossible due to (ii).

■

★ **Lemma 4.3.2** *Let G be a 2-connected edge-minimal graph and let $e = uv \in E(G)$. Then*

(*i*) *there is a unique block $B_{e,v} \subset G - e$ and a unique cutvertex z of $G - e$ such that $v, z \in V(B_{e,v})$.*

(*ii*) *all the vertices in $N_G(v)$ except u are in $V(B_{e,v})$.*

Proof.

(*i*) Clearly, there exists such a block in $G - e$. Since vertex v is not a cutvertex of $G - e$, v lies only in one block of $G - e$.

(*ii*) Assume otherwise. Then there is a neighbour of v, let us say w, $w \neq u$, and $w \notin B_{e,v}$. Since the graph G is 2-connected, there exist a path P from w to z not using any of the vertices from $V(B_{e,v})$, except z. Then $B_{e,v} \cup P \cup vw$ is a 2-connected subgraph of G. But this contradicts with the maximality of $B_{e,v}$

■

Proof of Theorem 4.3.1

Since a connected $K_{1,2}$-free graph is complete, hence hamiltonian, we may assume that $r \geq 3$. We use a proof by contradiction. Suppose that we have a 2-connected $K_{1,r}$-free graph G without an r-trestle. Choose a subgraph $T \subset G$ such that T is 2-connected, with maximum degree $\Delta(T) \leq r$ and

1. $|V(T)|$ is maximal,
2. subject to 1, $|E(T)|$ is minimal.

According to our assumptions, T is not a spanning subgraph of G. Therefore, there must be at least one vertex which is in G but not in T. Since G is 2-connected, there are two vertices $x, y \in V(T)$ and an x, y-path P such that $P \subset (G \backslash T) \cup \{x, y\}$. Now, from all the possible subgraphs T choose one with

3. $d_T(x) + d_T(y)$ is minimal.

We may assume that $d_T(x) \geq d_T(y)$, otherwise we relabel the vertices x and y. Clearly, $d_T(x) = r$, since otherwise $T \cup P$ spans more vertices in G, which is impossible due to our choice of T.

The vertex x is not adjacent to y in T. Otherwise $T - xy + P$ is a 2-connected subgraph of G of maximum degree r that spans more vertices than T.

Let x' denote the neighbour of x on P and x_i, $i = 1, ..., r$, denote the neighbours of x in T. Since G is $K_{1,r}$-free and T is triangle-free, there must be an edge $e \in E(G) \setminus E(T)$ connecting two neighbours of x. If $e = x'x_i$, for some $i \in \{1, ..., r\}$, then $T' = T - xx_i + e + xx'$ is a 2-connected subgraph of G with $\Delta(T') \leq r$ which spans more vertices than T. Again, this cannot happen. Without loss of generality, we may assume that $e = x_1x_2$.

Using Lemma 4.3.2, we have two blocks $B \subset T - xx_1$ and $B' \subset T - xx_2$ and two cutvertices z, z', such that $x_1, z \in V(B)$ and $x_2, z' \in V(B')$. Moreover, from the statement (ii) we immediately see that the block B is connected to the rest of the T only by vertices x_1 (i.e., by edge xx_1) and z and the block B' is connected to the rest of the T only by vertices x_2 (i.e., by edge xx_2) and z'. Note that, $V(B) \neq V(B')$, otherwise the vertex x would be a cutvertex of T. Since B and B' are blocks (i.e., maximal 2-connected subgraphs or edge), $|V(B) \cap V(B')| \leq 1$. If there is a vertex in the intersection of B and B' then $V(B \cap B') = \{z = z'\}$. Hence, without loss of generality, suppose that either $y \notin V(B)$ or $y = z = z'$.

We define three sequences of vertices $U = (u_0, u_1, ..., u_k)$, $V = (v_1, ..., v_k)$ and $C = (c_1, ..., c_k)$ and a sequence of sets of vertices $\mathbb{B} = (B_0, ..., B_k)$. Let $u_0 = x$, $u_1 = x_1$ and $v_1 = x_2$ and let c_1 be the cutvertex of $T - u_0u_1$ such that $c_1 \in B_1$. Set $B_0 = V(G)$ and $B_1 = V(B)$. If $d_T(u_i) = r$, for $i \geq 1$, then, using Lemma 4.3.1 and the fact that G is $K_{1,r}$-free, there is an edge w_1w_2 in $\langle N(u_i)_T \rangle_G$ which is not in T. Without loss of generality, assume that $w_1 \neq u_{i-1}$. Now, let $u_{i+1} = w_1$, $v_{i+1} = w_2$ and B_{i+1} be the set of the vertices in the block of $T - u_iu_{i+1}$ such that $u_{i+1} \in B_{i+1}$. Let c_{i+1} be the cutvertex of $T - u_iu_{i+1}$, such that $c_{i+1} \in B_{i+1}$. If $d_T(u_i) < r$ then set $k = i$ (i.e., we stop defining the sequences).

Examining the properties of these sequences, we can state the following: for any i, $0 \leq i \leq k-1$, $u_iu_{i+1} \in E(T)$, and for any j, $1 \leq j \leq k$, $u_jv_j \in E(G)$ and $u_jv_j \notin E(T)$.

Claim 4.3.1 *For every i, $1 \leq i \leq k-1$, vertex u_{i+1} is not a cutvertex of $T - u_{i-1}u_i$ (i.e., $u_{i+1} \neq c_i$).*

Proof. Assume to the contrary that the vertex u_{i+1} is a cutvertex of $T - u_{i-1}u_i$. Then the block $\langle B_i \rangle_T$ is connected to the rest of the T only by vertices u_i (i.e., by edge $u_{i-1}u_i$) and u_{i+1}. Since T is 2-connected, there is a path P' between u_{i+1} and u_i outside of the block $\langle B_i \rangle_T$ (i.e., not using any edge from $\langle B_i \rangle_T$). Since $d_{\langle B_i \rangle_T} \geq 2$, there is a path P'' between u_i and u_{i+1} in $\langle B_i \rangle_T$ avoiding the edge u_iu_{i+1}. Then $P' \cup P''$ is a cycle in T and the edge u_iu_{i+1} is a chord, which is a contradiction with the edge-minimality of T. ■

Consequently, for any i, $0 \leq i \leq k-1$, vertex $u_{i+1} \in B_i$. Therefore, $B_{i+1} \subsetneq B_i$ because $u_i \notin B_{i+1}$. So, for any j, $i \leq j \leq k$, $u_j \in B_i$ (see Fig. 4.2).

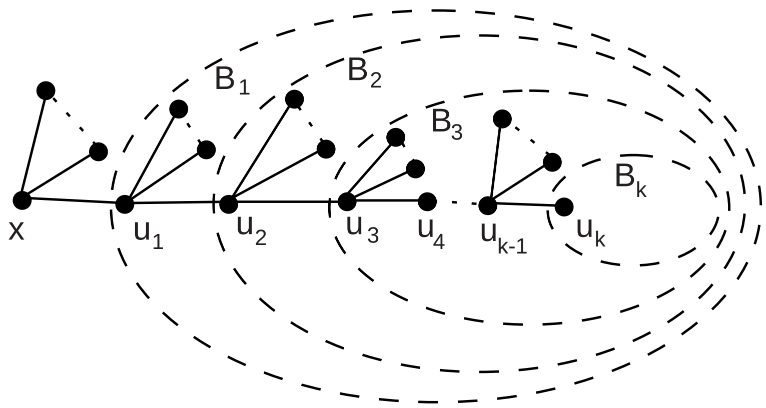

Figure 4.2: Structure of sets B_i

Therefore, vertices in U induce a path in T and $d_T(u_k) < r$.

Now we prove by induction that in G there exists a 2-connected subgraph T^* of G with $\Delta(T^*) \leq r$ such that $|V(T^*)| = |V(T)|$, $|E(T^*)| = |E(T)|$, $d_{T^*}(y) = d_T(y)$ and $d_{T^*}(x) < d_T(x)$.

Let $T_k = T$. For every i, $k \geq i \geq 1$, we define T_{i-1} as follows. $V(T_{i-1}) = V(T_i)$ and $E(T_{i-1}) = E(T_i) - v_i u_{i-1} + u_i v_i$.

Claim 4.3.2 *For every i, $k \geq i \geq 0$, T_i is a 2-connected subgraph of G with $\Delta(T_i) \leq r$ and $|V(T_i)| = |V(T)|$, $|E(T_i)| = |E(T)|$, $d_{T_i}(y) = d_T(y)$ and $d_{T_i}(u_i) < r$. Moreover, all the sets $B_0, ..., B_{i-1}$ induce 2-connected subgraphs in T_i and if $i \neq k$ then either the set B_i or $B_i \setminus \{u_i\}$ induces a 2-connected subgraph in T_i.*

Proof. The claim holds for $i = k$ because $T = T_k$ and either B_k induces a 2-connected subgraphs or $\langle B_k \rangle_{T_k}$ is an edge. Assume that the claim holds for $j = i$, $k \geq j \geq 2$. We prove that it holds also for $i = j-1$. According to our assumption, T_j is a 2-connected subgraph of G with $\Delta(T_j) \leq r$, such that $d_{T_j}(u_j) < r$, and the sets $B_1, ..., B_{j-1}$ induce 2-connected subgraphs in T_j and either the set B_j or $B_j \setminus \{u_j\}$ induces 2-connected subgraph in T_j. We know that $T_{j-1} = T_j - v_j u_{j-1} + u_j v_j$. Therefore, $|V(T_{j-1})| = |V(T_j)|$, $|E(T_{j-1})| = |E(T_j)|$. Since the vertex y is not in the sequence U, $d_{T_{j-1}}(y) = d_{T_j}(y)$. It is clear that $d_{T_{j-1}}(u_j) = r$ and $d_{T_{j-1}}(u_{j-1}) < r$, hence $\Delta(T_{j-1}) \leq r$. Now it remains to prove the 2-connectivity.

Due to the induction hypothesis, the set B_j induces a 2-connected subgraph in T_j. Clearly, the subgraphs $\langle B_j \rangle_{T_j}$ and $\langle B_j \rangle_T$ are different but, apart from this subgraph $\langle B_j \rangle_{T_j}$, the graphs T_j and T might differ only in one edge. In T, the edges connecting $\langle B_j \rangle_T$ to T are only the edges going from c_j plus the edge $u_{j-1}u_j$. In T_j, there might be instead of this edge $u_{j-1}u_j$ the edge $u_{j-1}u_{j+1}$. This happens only if $v_{j+1} = u_{j-1}$. We have the following two cases.

(i) $v_j \neq u_{j-2}$. (see Fig. 4.3)

We also distinguish two subcases.

A) B_j induces a 2-connected subgraph in T_j.

If $r > 3$ then by switching the edges $v_j u_{j-1}$ and $u_j v_j$ and knowing that $u_j \neq c_j$ (Claim 4.3.1), it is obvious that $\langle B_{j-1} \rangle_{T_{j-1}}$ remains 2-connected (see Fig. 4.3). If $r = 3$ then by switching the edges $v_j u_{j-1}$ and $u_j v_j$ and knowing that $u_j \neq c_j$ (Claim 4.3.1), $\langle B_{j-1} \setminus \{u_{j-1}\} \rangle_{T_{j-1}}$ is 2-connected subgraph of T_{j-1} and $d_{\langle B_{j-1} \rangle_{T_{j-1}}}(u_{j-1}) = 1$.

Then, all $\langle B_0 \rangle_{T_{j-1}}, ..., \langle B_{j-2} \rangle_{T_{j-1}}$ are also 2-connected subgraphs of T_{j-1}.

B) $B_j \setminus \{u_j\}$ induces a 2-connected subgraph in T_j.
Observe that $d_{\langle B_j \rangle_{T_j}}(u_j) = 1$ and $d_{\langle B_{j-1} \rangle_{T_j}}(u_j) = 2$ and $r = 3$. Therefore, using the same arguments as before, $\langle B_{j-1} \setminus \{u_{j-1}\} \rangle_{T_{j-1}}$ is a 2-connected subgraph of T_{j-1} and all $\langle B_0 \rangle_{T_{j-1}}, ..., \langle B_{j-2} \rangle_{T_{j-1}}$ are also 2-connected subgraphs of T_{j-1}.

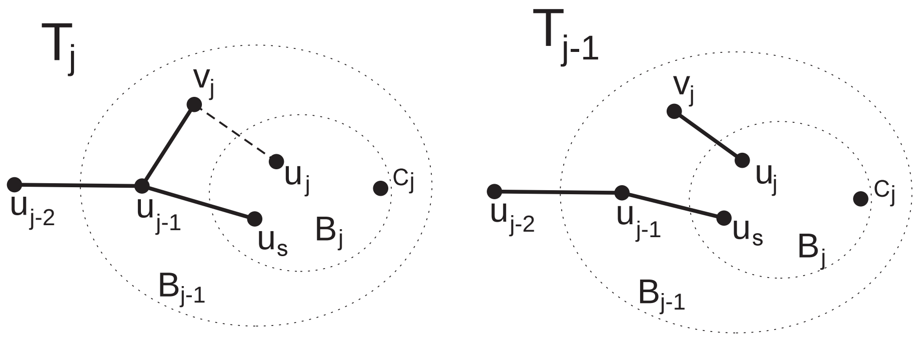

Figure 4.3: 2-connectivity in Case (i).

(ii) $v_j = u_{j-2}$. (see Fig. 4.4)
A) B_j induces a 2-connected subgraph in T_j.
Then, by switching the edges $v_j u_{j-1}$ and $u_j v_j$, $\langle B_{j-1} \rangle_{T_j} \simeq \langle B_{j-1} \rangle_{T_{j-1}}$ therefore B_{j-1} induces a 2-connected subgraph in T_{j-1} as well. Recall that c_{j-1} is the cutvertex of $T - u_{j-2}u_{j-1}$, such that $c_{j-1} \in B_{j-1}$. Since B_j induces a 2-connected subgraph of T_j, vertex c_{j-1} is also the cutvertex of $T_j - u_{j-2}u_{j-1}$. The graphs T_j and T_{j-1} differ only in one edge. In T_j, the edges connecting $\langle B_{j-1} \rangle_{T_j}$ to T_j are only the edges going from c_{j-1} plus the edge $u_{j-2}u_{j-1}$. In T_{j-1}, there is instead of this edge $u_{j-2}u_{j-1}$ the edge $u_{j-2}u_j$. Using Claim 4.3.1, we know that $u_j \neq c_{j-1}$. Therefore, the set B_{j-2} induces a 2-connected subgraph of T_{j-1} (see Fig. 4.4).
Then, all $\langle B_0 \rangle_{T_{j-1}}, ..., \langle B_{j-3} \rangle_{T_{j-1}}$ are also 2-connected subgraphs of T_{j-1}.

B) $B_j \setminus \{u_j\}$ induces a 2-connected subgraph in T_j.
Observe that $d_{\langle B_j \rangle_{T_j}}(u_j) = 1$ and $d_{\langle B_{j-1} \rangle_{T_j}}(u_j) = 2$ and $r = 3$. Using the same arguments as before, $\langle B_{j-1} \rangle_{T_{j-1}}$ and $\langle B_{j-2} \rangle_{T_{j-1}}$ are 2-connected subgraphs of T_{j-1} and all $\langle B_0 \rangle_{T_{j-1}}, ..., \langle B_{j-3} \rangle_{T_{j-1}}$ are also 2-connected subgraphs of T_{j-1}.

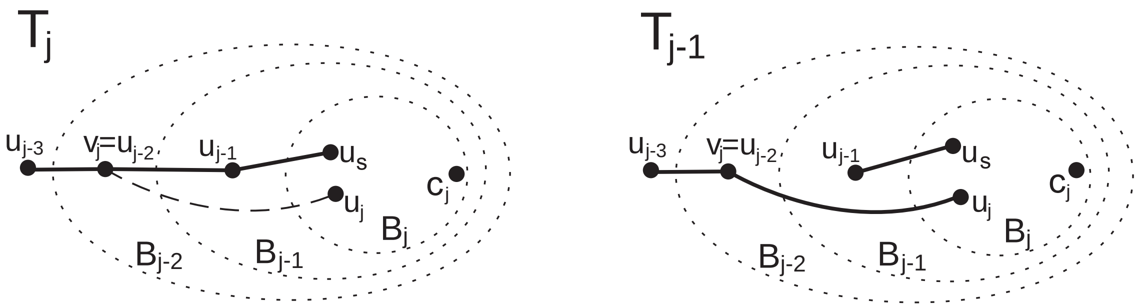

Figure 4.4: 2-connectivity in Case (ii)

■

We have proved that T_0 is 2-connected subgraph of G with $\Delta(T_0) \leq r$, and $|V(T)| = |V(T_0)|$, $|E(T)| = |E(T_0)|$, $d_T(y) = d_{T_0}(y)$ and $d_{T_0}(x) < r$. But this is a contradiction with the choice of T. Thus, Theorem 4.3.1 is proved.

■

One might ask, whether Theorem 4.3.1 can be improved. By improving we mean "is it possible to prove that every 2-connected and $K_{1,r}$-free graph has an $(r-1)$-trestle ?" The answer is "no".

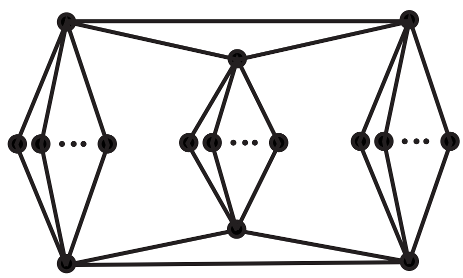

Figure 4.5: Sharpness example.

Figure 4.5 shows that the result of Theorem 4.3.1 cannot be improved. It can be easily seen that the example shows a $K_{1,r}$-free graph having an r-trestle but no $(r-1)$-trestle, for $r \geq 3$. Furthermore, every partial example can be extended to an infinite class of non-isomorphic graphs having the same properties. The general construction of these graphs is the following.

Take two copies of an arbitrary clique with odd number of vertices. Take a vertex u from the first clique and the corresponding vertex u' from the second clique. Connect u and u' with $r-2$ disjoint paths of arbitrary lengths greater then one. Do this for every vertex in the first clique.

Now we show that this graph has an r-trestle but no $(r-1)$-trestle. Clearly, all the paths between the two cliques must be in the trestle. Every vertex from the first or second clique already has degree $r-2$. To connect these paths in a 2-connected subgraph we have to use some edges from the cliques. Since the size of the cliques is odd, we cannot use a perfect matching in the first or second clique. Therefore, at least one vertex in the first clique and one vertex in the second clique must be adjacent to at least two vertices in the same clique in a trestle, and hence, have a degree at least r. Clearly, r is enough to form a 2-connected subgraph. Therefore, this graph has an r-trestle but no $(r-1)$-trestle.

★ **Corollary 4.3.1** *The result of Theorem 4.3.1 is sharp.*

Let us say a few words about $K_{1,r}$-free graphs to conclude this chapter. For a fixed r, $K_{1,r}$-free graphs form a special subclass of graphs. But if we consider r not as a fixed constant but as a variable then every graph is $K_{1,r}$-free for some r. Clearly, if Δ is the maximum degree in a graph G then G is definitely $K_{1,\Delta+1}$-free. We can also consider the local independence number α'. Recall that the independence number $\alpha(G)$ is the size of a largest independent set of vertices in G. *Local independence number* $\alpha'(G)$ is defined as follows: $\alpha'(G) = \max_{v\in V(G)} \alpha(\langle N(v)\rangle)$. Hence, $\alpha'(G)$ is the maximum number of elements in a largest independent set in the neighbourhood of any vertex. Considering this fact, every graph is $K_{1,\alpha'+1}$-free and, due to our result, every 2-connected graph has an $(\alpha'+1)$-trestle.

Let us mention once more a result of Kaiser, Kužel, Li and Wang [63].

Theorem 4.3.2 [63] *Every bridgeless graph of maximum degree* Δ *admits a* $\lceil(\Delta+1)/2\rceil$*-walk.*

Combining Theorem 4.3.2 and our new result (Theorem 4.3.1), we get the following corollary.

★ **Corollary 4.3.2** *Every 2-connected $K_{1,r}$-free graphs has a $\lceil (r+1)/2 \rceil$-walk.*

Thus, we have improved Theorem 3.2.8 from [58] of Jackson and Wormald, for $j = 2$.

Chapter 5

Moore graphs and digraphs

5.1 Introduction

The topology of a network can be modeled by a graph, either directed or undirected, depending on the application. The design of large interconnection networks has become of growing interest due to recent advances in very large scale integrated technology. Theoretic research includes constructing optimal designs for network topology. The degree of a vertex corresponds to a constraint on the limited number of connections from a node in a network. Diameter of a graph or digraph is related the maximum data communication delay in the network.

What is then the largest number of nodes in a network with a limited degree and diameter? In graph theory terms, we get the well known degree/diameter problem:

Given natural numbers Δ and D, find the largest possible number of vertices $n_{\Delta,D}$ in a graph of maximum degree Δ and diameter at most D.

This is the undirected version of the degree/diameter problem. The statement of the directed version of the problem differs only in that 'degree' is replaced by 'out-degree'. Therefore, the degree/diameter problem for directed graphs is:

Given natural numbers d and k, find the largest possible number of vertices $n_{d,k}$ in a digraph of maximum out-degree d and diameter at most k.

Research activities related to the degree/diameter problem can be divided into two main areas. In the first area, researchers focus on the proofs of nonexistence of graphs or digraphs of order close to the general upper bounds, known as the Moore bounds. In the second area, researchers deal with constructions of large graphs and digraphs, improving lower bounds of $n_{\Delta,D}$ (respectively, $n_{d,k}$). In this thesis we focus only on the proofs of nonexistence of graphs and digraphs of order close to the Moore bound. For completeness, in Section 5.2 we give a short overview of undirected Moore graphs and graphs close to the Moore bound. However, our focus will be on the directed case. We briefly mention some interesting results and open problems for Moore digraphs and digraph close to the Moore bound (Section 5.3). Then in Section 5.4 we present new properties of almost Moore digraphs with selfrepeats from which we deduce the nonexistence of almost Moore digraphs for certain values of k and d.

5.2 Moore graphs

As mentioned before, there is a natural upper bound on the largest possible order $n_{\Delta,D}$ of a graph G of maximum degree Δ and diameter D. Let v be a vertex of the graph G. The number of vertices at distance 1 is at most Δ. The number of vertices at distance 2 is at most $\Delta(\Delta-1)^1$. Generally, the number of vertices at distance k is at most $\Delta(\Delta-1)^{k-1}$ (see Fig. 5.1).

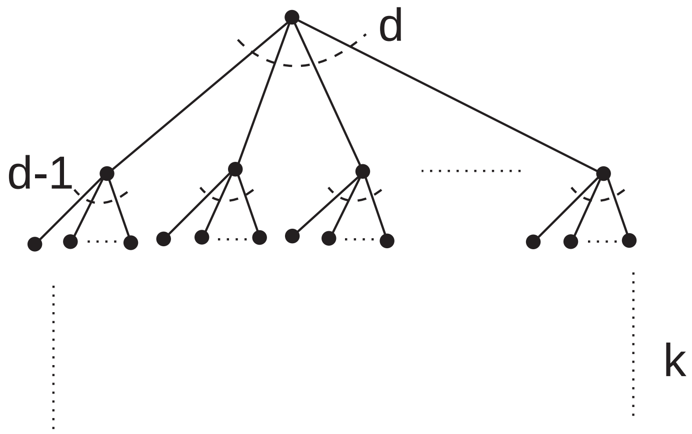

Figure 5.1: Moore bound for graphs.

Summing all these numbers of vertices in distance zero up to D, we get

$$n_{\Delta,D} \leq \sum_{i=0}^{D} 1 + \Delta + \Delta(\Delta-1) + ... + \Delta(\Delta-1)^{D-1}.$$

The right-hand side of the inequality is called the *Moore bound* and is denoted by $M_{\Delta,D}$. A graph with order $M_{\Delta,D}$ is called a *Moore graph*. Clearly, such a graph has to be regular of degree Δ.

The study of Moore graphs began with the paper [56] by Hoffman and Singleton. In this pioneering paper [56], the authors focused on Moore graphs of diameter 2 and 3. In the case of diameter $D = 2$, they proved that Moore graphs exist only for $\Delta = 2, 3, 7$ and possibly 57. They also proved that all known Moore graphs are unique for the values of Δ and D. The proofs use eigenvalues and eigenvectors of the adjacency matrix of graphs.

Damerell in [31] proved that there are no Moore graphs for $\Delta \geq 3$ and $D \geq 3$. An independent proof of this result was also given by Bannai and Ito in [2]. Summarizing the known results concerning the existence of Moore graphs, we get the following.

- Moore graphs of diameter $D = 1$ and degree $\Delta \geq 1$ are the complete graphs $K_{\Delta+1}$.
- Moore graphs of degree $\Delta = 2$ and diameter $D \geq 2$ are the cycles of odd length, C_{2D+1}.
- Moore graphs of diameter $D = 2$ and degree $\Delta \geq 3$ are the Petersen graph (see Fig. 5.2) of degree $\Delta = 3$, and the Hoffman-Singleton graph (see Fig. 5.3) of degree $\Delta = 7$.
- The existence of Moore graphs of diameter $D = 2$ and degree $\Delta = 57$ is still open.
- There are no Moore graphs for $\Delta \geq 3$ and $D \geq 3$.

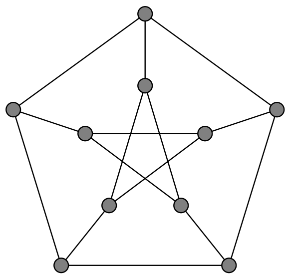

Figure 5.2: Petersen graph.

Since Moore graphs exist only in trivial cases, apart from the Petersen and Hoffman-Singleton graphs, the study of the existence of large graphs of given diameter and maximum degree focuses on graphs whose order is "close" to the Moore bound, that is, graphs of order $M_{\Delta,D} - \delta$, for δ small. The parameter δ is called the *defect*. For convenience, by a (Δ, D)*-graph* we will understand any graph of maximum degree Δ and of diameter at most D. If such a graph has order $M_{\Delta,D} - \delta$ then it will be referred to as a (Δ, D)*-graph of defect* δ.

In 1980 Erdős, Fajtlowitcz and Hoffman in [39] proved that, apart from the cycle C_4, there are no graphs of degree Δ, diameter 2 and defect 1. This was generalized

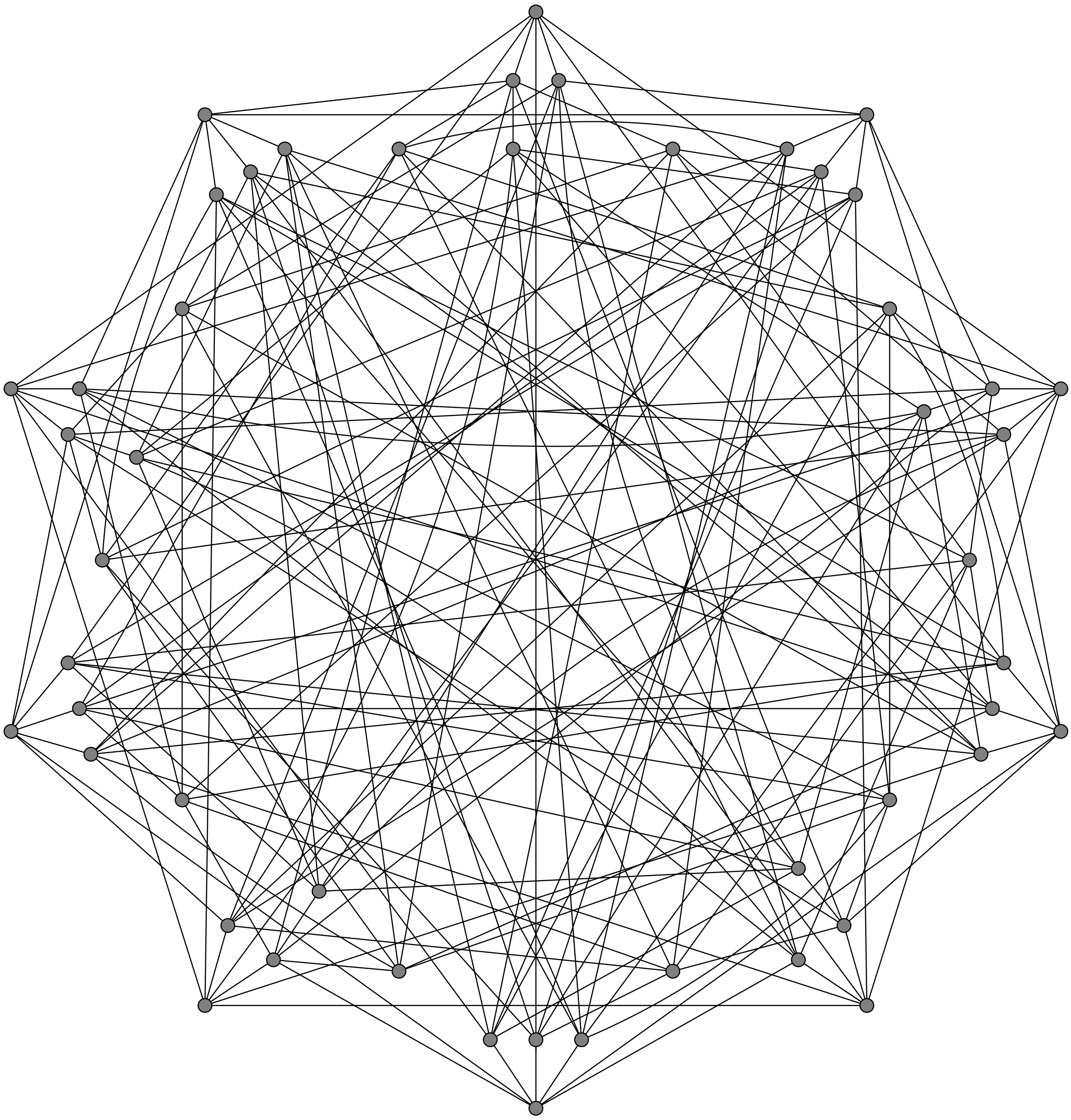

Figure 5.3: Hoffman-Singleton graph [88].

by Bannai and Ito in [3] and also by Kurosawa and Tsujii in [68], to all diameters. Therefore, the only graphs of order $M_{\Delta,D} - 1$ are the cycles C_{2D}.

Let us now focus on graphs with defect $\delta = 2$. It is obvious that if $\Delta = 2$ then the (Δ, D)-graphs of defect 2 are the cycles C_{2D-1}. For $\Delta \geq 3$, only five (Δ, D)-graphs of defect 2 are known at present: Two $(3, 2)$-graphs of order 8, a $(4, 2)$-graph of order 15, a $(5, 2)$-graph of order 24, and a $(3, 3)$-graph of order 20.

In 1992 Jorgensen in [61] proved that a graph with maximum degree 3 and diameter $D \geq 4$ cannot have defect 2, and Miller and Simanjuntak in [74] proved that a graph with maximum degree 4 and diameter $D \geq 3$ also cannot have defect 2. In general, very little is known about Moore graphs with defect greater than or equal to 3.

5.3 Moore digraphs

The notation of diameter and maximum out-degree used in a digraph is usually different from the notation of the corresponding terms in undirected graphs. For digraphs, diameter is usually denoted by k and maximum out-degree d. The natural upper bound on the largest possible order $n_{d,k}$ of a digraph G of maximum out-degree d and diameter k is similar to, but not the same as in the undirected version. Let v be a vertex of a digraph G. The number of vertices at distance 1 is d. The number of vertices at distance 2 is d^2. Generally, the number of vertices at distance ℓ is d^ℓ (see next Fig. 5.4).

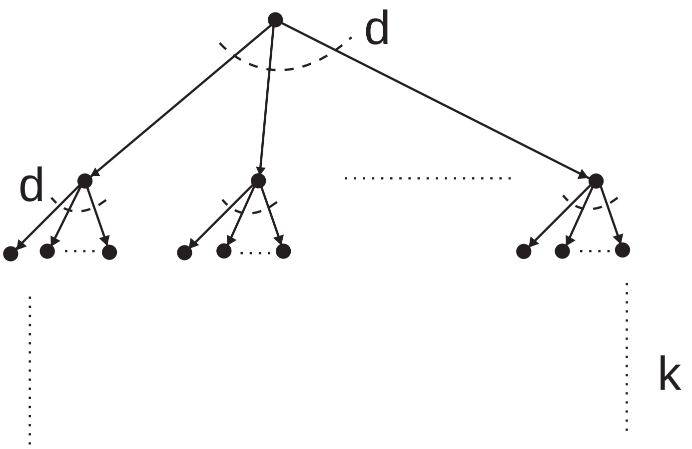

Figure 5.4: Moore bound for digraphs.

Summing all these numbers of vertices in distance zero up to k, we get

$$n_{d,k} \leq \sum_{i=0}^{k} 1 + d + d^2 + ... + d^k.$$

The right-hand side of the inequality is called the *directed Moore bound* and is denoted by $M_{d,k}$. A digraph with order $M_{d,k}$ is called a *Moore digraph*. Clearly, such a digraph has to be out-regular of out-degree d.

In 1974 Plesník and Znám in [81] proved that Moore digraphs exist only in the trivial cases when $d = 1$ (directed cycles of length $k + 1$, C_{k+1}) or $k = 1$ (complete digraphs of order $d+1$, K_{d+1}). Six years later, in 1980, Bridges and Toueg presented in [21] a short and very elegant proof, which we state in this thesis.

Theorem 5.3.1 [81] *Moore digraphs exist only for $d = 1$ or $k = 1$.*

Proof [21]. Let G be a Moore digraph with out-degree d and diameter k. Clearly, there are no cycles of length less than $k + 1$, and for any two vertices u, $v \in G$, there exists a unique directed path from u to v of length less than $k + 1$. Let A be an adjacency matrix of G. Then we we have the following equation for G.

$$I + A + A^2 + ... + A^k = J,$$

where matrix I is the identity matrix and J the matrix with all its entries equal to one. Then the eigenvalues of A are d and some of the roots of

$$1 + x + x^2 + ...x^k = 0.$$

The roots of this polynomial are the roots of $x^{k+1} = 1$. Let $x_1, x_2, ..., x_{n-1}$ be the eigenvalues of A, other than d. Since G has no cycle of length less than $k + 1$, we have

$$trace(A^p) = 0,\ p = 1, 2, ..., k.$$

Hence

$$d^p + \sum_{j=1}^{n-1} x_j^p = 0,\ j = 1, 2, ..., k.$$

Since all the eigenvalues $x_1, x_2, ..., x_{n-1}$ lie on a cycle in the complex plane and their sum is an integer (see previous equation, for $p = 1$), we have that for arbitrary eigenvalue x_i, there exists an eigenvalue x_j such that either $x_i = -x_j$ or $x_i = \overline{x}_j$. Using this fact and the fact that $\overline{x}_i = x_i^k$, we have

$$-d = \sum_{j=1}^{n-1} x_j = \sum_{j=1}^{n-1} \overline{x}_j = \sum_{j=1}^{n-1} x_j^k = -d^k.$$

Thus $d = d^k$, which is true only if $d = 1$ or $k = 1$. ■

Similarly to the Moore graphs, Moore digraphs exist only in trivial cases. Therefore, the study of the existence of large digraphs of given diameter and maximum out-degree focuses on digraphs whose order is 'close' to the Moore bound, that is, digraphs of order $M_{d,k} - \delta$, for δ small. For convenience, by a (d,k)*-digraph* we will understand any digraph of maximum out-degree d and of diameter at most k. If such a digraph has order $M_{d,k} - \delta$ then it will be referred to as a (d,k)*-digraph of defect* δ. Moore digraphs with defect one are also called *almost Moore digraphs*.

A digraph of maximum out-degree $d \geq 2$, diameter k, and order $n = M_{d,k} - \delta$ must be out-regular for small values of δ (more precisely, if $\delta < M_{d,k-1}$). On the other hand, to prove the in-regularity (regularity of in-degree) of digraphs is not so easy. Surprisingly, there exist digraphs of out-degree d and diameter k with defect only two or three, in which not all vertices have the same in-degree. These digraphs are out-regular but not in-regular. For example, when $d = 2$, $k = 2$, $n = 5$ (defect 2), there are 9 non-isomorphic digraphs. Five of these digraphs are diregular (see Fig. 5.5) and 4 are non-diregular (see Fig. 5.6).

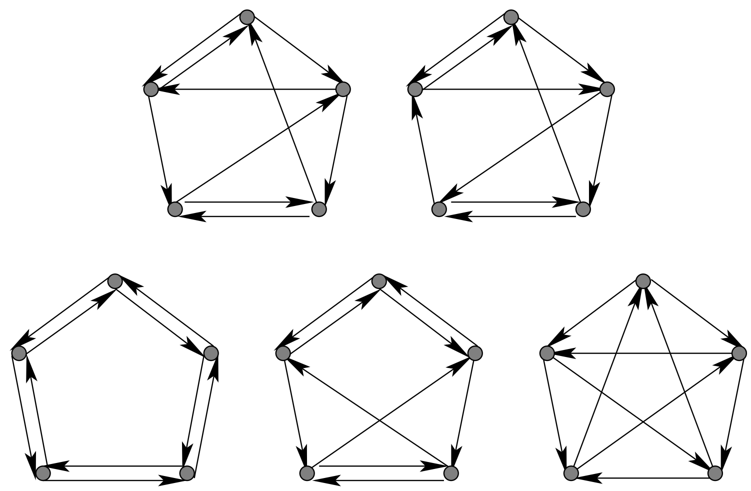

Figure 5.5: Five non-isomorphic diregular digraphs of order $M_{2,2} - 2$ [77].

It is interesting to note that there are more diregular digraphs than non-diregular ones for the parameters $n = 5$, $d = 2$, $k = 2$. But for $n = 11$, $d = 3$, $k = 2$, (defect also two) the situation is quite different. There are at least four non-isomorphic non-diregular digraphs [88] but only one diregular digraph [5].

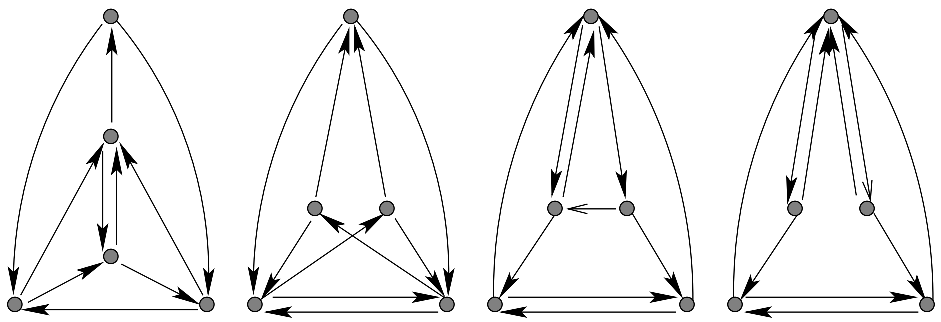

Figure 5.6: Four non-isomorphic non-diregular digraphs of order $M_{2,2} - 2$ [77].

The diregularity of almost Moore digraphs was proved by Miller, Gimbert, Širáň and Slamin in [73], followed by a result of Miller and Slamin in [75] proving that every digraph of defect 2, maximum out-degree 2 and diameter $k \geq 3$ is diregular. Miller and Slamin also conjecture that all defect 2 digraphs of maximum out-degree $d \geq 2$ and diameter $k \geq 3$ are diregular. The question of diregularity or otherwise of digraphs with defect greater than 2 is completely open.

In contrast with Moore graphs with defect one (which do not exist apart from the trivial cases), for diameter $k = 2$, line digraphs of complete digraphs are examples of almost Moore digraphs for any $d \geq 2$. Interestingly, for out-degree $d = 2$, there are two other non-isomorphic diregular digraphs of order $M_{2,2} - 1$ (see Fig 5.7).

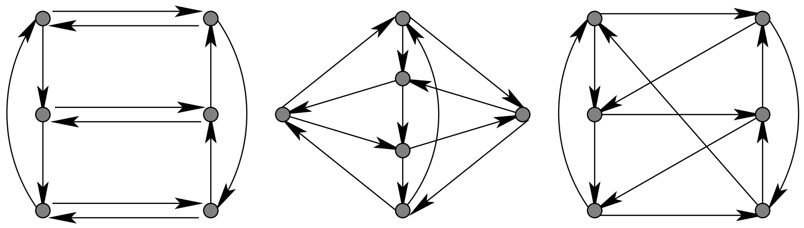

Figure 5.7: Three non-isomorphic diregular digraphs of order $M_{2,2} - 1$ [77].

In 2001, Gimbert [47] completely solved the problem of classification of almost Moore digraphs of diameter 2 for out-degree $d \geq 3$, proving that line digraphs of complete digraphs are the only almost Moore digraphs.

On the other hand, Miller and Fris [72] proved that there are no almost Moore digraphs of maximum out-degree 2 for any $k \geq 3$. Result of Baskoro, Miller, Širáň and Sutton [9] shows that there are no almost Moore digraphs of out-degree 3 for any $k \geq 3$.

The existence of digraphs of defect 2 is widely open. The only result in this direction is due to Miller and Širáň [76]. They proved that digraphs of defect 2 do not exist for out-degree $d = 2$ and all $k \geq 3$. For the remaining values of $k \geq 3$ and $d \geq 3$, the existence of digraphs of defect 2 remains completely open.

5.4 Almost Moore digraphs

As mentioned before, the diregularity of almost Moore digraphs has been proven by Miller, Gimbert, Širáň and Slamin in [73]. Since the order of a (d, k)-digraph G is one less than the Moore bound then, for every vertex $u \in V(G)$, there exists exactly one vertex $v \in V(G)$ such that there are two walks of length $\leq k$ from u to v. Such a vertex is called the *repeat* of u, denoted by $r(u)$. In case $r(u) = u$, vertex u is called a *selfrepeat* (the two walks, in this case, have lengths 0 and k). Let S be a set of vertices of an almost Moore digraph. By $r(S)$ we denote a set of all repeats of the vertices in S. Baskoro, Miller and Plesník [7] showed that the function r is an *automorphism* on $V(G)$.

For any integer $p \geq 1$, the repeat function defines a composition $r^p(v) = r(r^{p-1}(v))$ with $r^0(v) = v$. Then, for every vertex v of G, there exists a smallest natural number $\omega(v)$, called the *order* of v, such that $r^{\omega(v)}(v) = v$. Hence, $\omega(v)$ is the length of the permutation cycle containing v.

Let v be a vertex of an almost Moore digraph G. For integer i, let $N^i(v)$ be the set of vertices at distance i from v, if $i \geq 0$, and at distance i to v, if $i < 0$. Hence, for $i = 0$ and $i = \pm 1$, we have, in particular, $N^0(v) = \{v\}$, $N^1(v) = N^+(v)$ is the sets of *out-neighbourhood* of v, and $N^{-1}(v) = N^-(v)$ is the *in-neighbourhood* of v. The *least common multiple* of m and n is denoted by $\text{lcm}(m, n)$.

Thanks to the diregularity, all almost Moore digraphs have the following property, proven by Baskoro, Miller, Plesník and Znám [8], which was originally proved only for diregular digraphs.

Theorem 5.4.1 [8] *$N^+(r(v)) = r(N^+(v))$ and $N^-(r(v)) = r(N^-(v))$ for any vertex v of an almost Moore digraph.*

This theorem is known as the Neighbourhood Theorem. In 1998 Baskoro, Miller and Plesník proved interesting structural property of almost Moore digraphs with selfrepeats [7]. We include here a shorter and more readable version of the proof.

Theorem 5.4.2 [7] *An almost Moore digraph contains either no selfrepeats or exactly k selfrepeats, for $k \geq 3$.*

Proof. Let G be an almost Moore digraph. If there is a selfrepeat in G then G contains a cycle C_k and all the vertices on the C_k are selfrepeats. Next, assume that there are more than k selfrepeats in G. Clearly, every selfrepeat lies on exactly one k-cycle. Let u be a selfrepeat on a k-cycle C and let x be a selfrepeat not in C. There is a path of length $\leq k$ from u to x; denote this path P. Recall that $r(u) = u$ and $r(x) = x$ and the function r is an automorphism on $V(G)$, i.e., $r(G) = G$. Let P' be a path from $r(u)$ to $r(x)$ in $r(G)$. Since $r(u) = u$ and $r(x) = x$, $P = P'$. Otherwise, there would be two paths from u to x in G, which would make vertex x the repeat of u. Since the defect is one, this is impossible. Hence $P' = r(P) = P$ and all the vertices on P are selfrepeats. This means that any two selfrepeats are connected by a path consisting only of selfrepeats.

Then some selfrepeat v on C has at least two selfrepeats in $N^+(v)$, one on C and one on P. Using Neighbourhood Theorem, there are also at least two vertices z_1 and z_2, $z_1 \in C$, $z_1 \notin C$ in $N^+(z)$, where z is a predecessor of u in C. Using arcs of C and zz_2, we obtain a path of length k from u to z_2. Clearly, this path is unique, otherwise z_2 would be a repeat of u. Since u is an arbitrary selfrepeat in G, we conclude that in such a digraph, for arbitrary selfrepeat u, there exists a selfrepeat z_1 such that distance $d(u, z_2) = k$.

Consider the induced subdigraph $G[V_1]$, where V_1 is the set of all selfrepeats in G. Then the diameter of $G[V_1]$ is k. Using Neighbourhood Theorem, we see that $G[V_1]$ is also diregular of degree d_1. Observe that $G[V_1]$ is an almost Moore digraph of diameter k, $k \geq 3$ and degree d_1, $d_1 \geq 2$ where every vertex is a selfrepeat. Applying the method of Bridges and Toueg [21], it was found that such digraphs cannot exist [8]. Therefore, G can only have either 0 or k selfrepeats. ■

In this section, we show that almost Moore digraphs do not exist for infinite number of values of k for $d = 4, 5$ and 6. To show this, we use the structure of vertex orders in almost Moore digraphs. Baskoro, Cholily and Miller [6] found an explicit formula for enumerating vertices of all orders present in an almost Moore digraph based on the given information of the repeat structure of out-neighbours of any one selfrepeat.

Let v_0 be a selfrepeat of G and let $N^+(v_0)$ consist of permutation cycles with lengths $1 = s_0, s_1, s_2, \ldots, s_t$ and multiplicities $1 = m_0, m_1, m_2, \ldots, m_t$. It is then clear that $d = 1 + \sum_{i=1}^{t} m_i s_i$. We denote by S_1 the set of all vertex orders of $N^+(v_0)$. Thus, $S_1 = \{s_0, s_1, \ldots, s_t\}$. Next, we define a set, denoted by S_2, as the set of all $\text{lcm}(s_i, s_j)$, where $s_i, s_j \in S_1$ and $\text{lcm}(s_i, s_j) \notin S_1$. Later, if $S_2 \neq \emptyset$ then the i^{th} element of S_2 will be denoted by $s_{2,i}$. In general, we can continue to define the set S_m, $3 \leq m \leq k$, as the set of the least common multiples of any m vertex orders of S_1 but including only those least common multiples which are not already members of any S_i, $i < m$.

Theorem 5.4.3 [6] *Let G be an almost Moore digraph with a selfrepeat v_0. If $u \in N^i(v_0)$, $2 \leq i \leq k$, then $\omega(u) \in \bigcup_{j=1}^{i} S_j$.*

Let G be an almost Moore digraph (defect 1) with $d \geq 4$ and $k \geq 3$. We classify the vertices of G with respect to their repeats, more precisely, with respect to the position of the repeat in the digraph G. We have the following three types of vertices:

- Type 0 : $r(u) = u$ (selfrepeat)
- Type I : $r(u)$ is not adjacent to u
- Type II : $r(u)$ is adjacent to u

Let x be a vertex of type II. Suppose $N^-(x) = \{v_1, ..., v_d\}$ and $N^+(x) = \{u_1, ..., u_d\}$. Clearly, there is at least one and at most two vertices in $N^-(x)$ at distance $k-1$ from every vertex in $N^+(x)$, i.e., $\exists v_i \in N^{k-1}(u_j)$, for $1 \leq j \leq k$. Recall that one vertex from $N^+(x)$ is the repeat of x. Since the defect is only one, there are two vertices, say u_1 and u_2, such that $r(x) \in N^{k-1}(u_1)$ and also $r(x) \in N^{k-1}(u_2)$. We distinguish between two subcases.

- IIa : $\exists$ vertex v_j from $N^-(x)$ such that either $v_j \in N^{k-1}(u_1)$ or $v_j \in N^{k-1}(u_2)$. Note that if $v_j \in N^{k-1}(u_1)$ then $r(u_1) = x$; otherwise $r(u_2) = x$.

- IIb : $\nexists$ such vertex, i. e. $\exists$ a vertex u_i, $u_i \neq u_1$, $u_i \neq u_2$, such that $r(u_i) = x$.

Observe that in the out-neighbourhood and the in-neighbourhood of a vertex of type II, there is also at least one vertex of type II. For details, see Fig. 5.8.

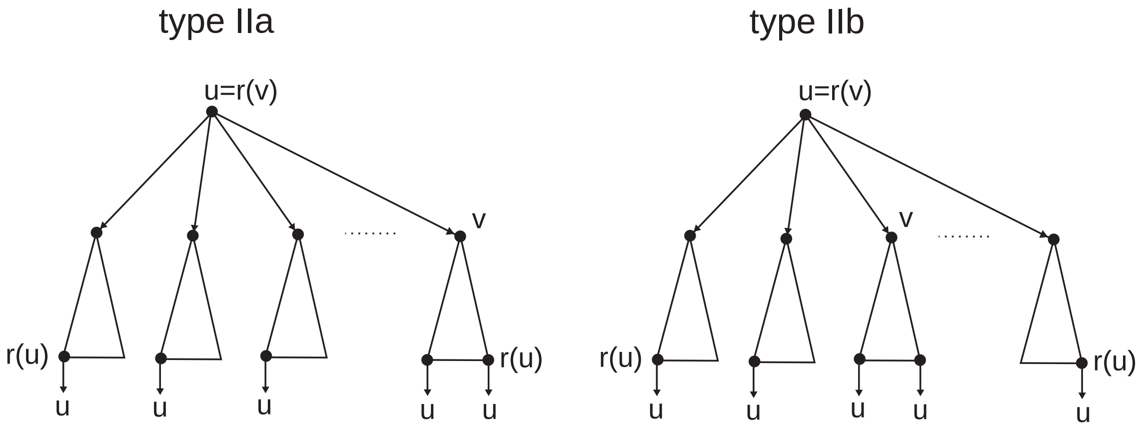

Figure 5.8: Repeat configuration - types IIa and IIb

★ **Lemma 5.4.1** *Let G be an almost Moore digraph. Then in the out-neighbourhood and in-neighbourhood of a vertex of type 0 there is one vertex of type 0 and $(d-1)$ vertices of type I.*

Proof. Since there are exactly k selfrepeats forming C_k, every selfrepeat has exactly one selfrepeat as an out-neighbour and one as an in-neighbour. It remains to prove that there is no vertex of type II in the out-neighbourhood and in-neighbourhood of a vertex of type 0. We state the proof only for the out-neighbourhood. The case for the in-neighbourhood is almost the same as for the out-neighbourhood.

Let u be a selfrepeat and v be a vertex of type II. Assume to the contrary that v is in the out-neighbourhood of u. Using the Neighbourhood Theorem, we get $r(v) \in N^-(u)$. Since v is a vertex of type II, there is an edge $r(v)v$. Thus there are two paths uv and $ur(v)v$ from u to v, a contradiction. ■

★ **Lemma 5.4.2** *Let G be an almost Moore digraph. Then vertices of type IIa form a cycle in G. Moreover, vertices of type IIb form a cycle in G, as well.*

Proof. Let v be a vertex of type II. First we prove that in the out-neighbourhood of v in G, there is exactly one vertex of type II.

It is clear that in the out-neighbourhood of v, there is at least one vertex of type II. Assume that there are two vertices, a and b, of type II. From the Neighbourhood Theorem, $r(a), r(b) \in N^-(r(v))$. Therefore, there are two paths $P_1 = r(v)va$ and $P_2 = r(v)r(a)a$ from $r(v)$ to a. Thus $v = r(a)$. For the same reason, $v = r(b)$. Then vertex v is a repeat twice. Since in almost Moore graphs every vertex is a repeat exactly once, we have a contradiction.

Clearly, also in the in-neighbourhood of v in G, there is exactly one vertex of type II in $N_G^-(v)$. Hence, vertices of type II form a cycle in G. It is easy to see that vertices of type IIb form a cycle of length greater than $k + 1$.

Since an almost Moore digraph G is diregular, then the map that assigns to each vertex $v \in V(G)$ its repeat $r(v)$ is an automorphism of G and so, the type of the repeat of a vertex v must be the same as the type of v. Using the fact that for vertex v of type II, the repeat of v is adjacent to v, we conclude that a vertex of type IIa is adjacent to exactly one vertex of the same type. This is immediately true also for vertices of type IIb.

■

Therefore, vertices of type II form a cycle $\overrightarrow{C_\ell}$. The permutation cycle $\overleftarrow{C_\ell}$ of these vertices is the same cycle but only with the opposite direction.

★ **Corollary 5.4.1** *Vertices of type II form a cycle C_ℓ of length ℓ and a permutation cycle of length ℓ, where ℓ is greater than or equal to $k + 1$.*

In an almost Moore digraph there are exactly k selfrepeats on C_k or none. Assume that there are no vertices of type II. Then, apart from the C_k, every edge lies on exactly one cycle of length $k+1$. Therefore, nd (no selfrepeat) or $nd-k$ (k selfrepeatss) must be divisible by $k+1$.

Assume that there are no vertices of type IIb. Vertices of type IIa form C_{k+1} and therefore the number of vertices of type IIa must be divisible by $k+1$. For every vertex of type IIa, there is exactly one out-going edge which is exactly in two cycles C_{k+1} and all the other out-going edges are each exactly in one C_{k+1}. Then we have the same divisibility condition as before. Again, nd (no selfrepeat) or $nd-k$ (k selfrepeat) must be divisible by $k+1$.

Let G be an almost Moore digraph, with $d=4$ and $k \geq 3$, containing selfrepeats. Making use of Theorem 5.4.3, the length of all permutation cycles of all the vertices in G, apart from selfrepeats, is three. Since the length of a permutation cycle of vertices of type II is at least $k+1$, there are no vertices of type II. In that case the divisibility condition must hold.

★ **Corollary 5.4.2** *Let G be an almost Moore digraph, with $d=4$ and $k \geq 3$, containing selfrepeats. If such a digraph exists then $4^{k+2}-13$ must be divisible by $3(k+1)$.*

Using computer, we verified that the divisibility conditions for $d=4$ fails for values of k from 3 up to 100000.

★ **Corollary 5.4.3** *Almost Moore digraphs with selfrepeats do not exist for $d=4$ and $k=\{3,...,100000\}$*

Let G be an almost Moore digraph, with $d=5$ and $k \geq 3$, containing selfrepeats. Theorem 5.4.3 implies that the length of all permutation cycles of all the vertices in G, apart from the selfrepeats, is two or four. Since the length of a permutation cycle of vertices of type IIb is at least $k+2$, there are no vertices of type IIb. In that case the divisibility condition must hold.

★ **Corollary 5.4.4** *Let G be an almost Moore digraph, with $d=5$ and $k \geq 3$, containing selfrepeats. If such a digraph exists, $5^{k+2}-21$ must be divisible by*

$4(k+1)$.

Using computer, we verified that the divisibility conditions for $d = 5$ fails for values of k from 3 up to 100000 except for $k = 3$, 25 and 387.

★ **Corollary 5.4.5** *Almost Moore digraphs with selfrepeats do not exist for* $d = 5$ *and* $k = \{3, ..., 100000\} \setminus \{3, 25, 387\}$

More generally, let W be the set of all possible lengths of permutation cycles in the neighbourhood of a selfrepeat. Define an integer ℓ as follows: $\ell = \max_{U \subseteq W}$ lcm (U). Recall that the length of the permutation cycle of a vertex of type IIb is greater than $k+1$. So, if $\ell \leq k+1$ then there are no vertices of type IIb. Hence $nd - k$ must be divisible by $k+1$. The following lemmas show when $nd-k$ and nd are divisible by $k+1$.

★ **Lemma 5.4.3** *Let* n *be the order of an almost Moore digraph. Let* $nd - k$ *be divisible by* $k+1$. *If* $k+1$ *is a prime number then* $d(d-1) \equiv k(d-1)$ *mod* $k+1$.

Proof. In an almost Moore digraphs the number of vertices is $n = d + d^2 + d^3 + ... + d^k$. We have the following equation.

$$nd \equiv k \bmod k+1$$

$$d^2 + d^3 + d^4 + ... + d^{k+1} \equiv k \bmod k+1$$

Now we use Fermat's theorem, i.e., $c^p \equiv c \bmod p$ if p is prime number. We obtain

$$d + d^2 + d^3 + d^4 + ... + d^k \equiv k \bmod k+1$$

$$d\frac{d^{k+1}-1}{d-1} \equiv k \bmod k+1$$

$$d(d-1) \equiv k(d-1) \bmod k+1.$$

■

★ **Lemma 5.4.4** *Let nd be divisible by $k+1$. If $k+1$ is a prime number then $d(d-1) \equiv 0 \mod k+1$.*

Proof. Similar to the proof of the previous lemma. ■

To conclude this chapter, we state three structural lemmas for almost Moore digraphs with respect to the types of vertices.

★ **Lemma 5.4.5** *Let G be an almost Moore digraph of out-degree 4 and diameter k. Let u and v be any two vertices different from the selfrepeats in G. Then $|N^+(u) \cap N^+(v)| \leq 1$.*

Proof. Assume to the contrary that $|N^+(u) \cap N^+(v)| > 1$.

Let $N^+(u) = \{u_1, u_2, u_3, u_4\}$. Then we have the following three cases.

Case 1 : $|N^+(u) \cap N^+(v)| = 4$. Then $T_k^+(u) = T_k^+(v)$. Since $u \in T_k^+(v)$, then $u \in T_k^+(u)$. Therefore, u must be a selfrepeat, a contradiction.

Case 2 : $|N^+(u) \cap N^+(v)| = 3$. We may assume that $v \in T_{k-1}^+(u_4)$. Then (v, u_4) is not in $E(G)$. Otherwise u is a selfrepeat. Then $N^+(v) = \{u_1, u_2, u_3, x\}$.

Let z be a vertex in $T_{k-1}^+(u)$ such that (z, v) is an edge in G. Let z_1, z_2 and z_3 be the out-neighbours of z different from v. Then there must be a path P of length k from v to z. P cannot be shorter, otherwise v would be a selfrepeat. Vertex x must lie on P between v and z, otherwise we could not reach z. There must be also paths Q_1, Q_2 and Q_3 from v to z_1, z_2 and z_3. Moreover, for the same reason, vertex x must lie on Q_1 between v and z_1, similarly x on Q_2 and x on Q_3. Since z is at distance k from v, paths Q_1, Q_2 and Q_3 do not contain vertex z. We have the following paths : $vxP'z$, $vxQ_1'z_1$, $vxQ_2'z_2$ and $vxQ_3'z_3$. Therefore, there are two different paths from x to z_i, for $i = \{1, 2, 3\}$, namely, $xPzz_1$ and xQ_iz_i. But this is a contradiction, since vertex v cannot have three repeats.

Case 3 : $|N^+(u) \cap N^+(v)| = 2$. It is clear that $v \in T_k^+(u)$. We may assume that $v \in T_{k-1}^+(u_4)$. Then $vu_4 \notin E(G)$. Otherwise u is a selfrepeat. We may assume that $N^+(v) = \{u_1, u_2, y, x\}$. Let $y \neq r(v)$. Then vertex y is either in $T_{k-1}^+(u_3)$ or in $T_{k-1}^+(u_4)$. Let z be a vertex in $T_{k-1}^+(u)$ such that zy is an edge in G. Let z_1, z_2 and z_3 be the out-neighbours of z different from v.

We have to reach vertex z from y. Therefore, there must be a path P of length k from v to z. P cannot be shorter, otherwise y would be a repeat of v. Vertex x must lie on P between v and z, otherwise we could not reach z. There must be also paths Q_1, Q_2 and Q_3 from v to z_1, z_2 and z_3. Moreover, for the same reason, vertex x must lie on Q_1 between v and z_1, similarly x on Q_2 and x on Q_3. Since z is at distance k from v, paths Q_1, Q_2 and Q_3 do not contain vertex z. We have the following paths : $vxP'z$, $vxQ_1'z_1$, $vxQ_2'z_2$ and $vxQ_3'z_3$. Therefore, there are two different paths from x to z_i for $i = \{1, 2, 3\}$, namely, $xPzz_1$ and x, Q_i, z_i. But this is a contradiction, since vertex v cannot have three repeats. ■

★ **Lemma 5.4.6** *Let $\omega(u)$ be the repeat order of u and D be the distance $d(u, r(u))$. Then $\omega D \geq k$.*

Proof. Assume to the contrary that $\omega D \leq k$. Then $r^\omega(u) \in T_k^+(u)$. But $r^\omega(u) = u$, hence u must be a selfrepeat, a contradiction. ■

★ **Lemma 5.4.7** *Let v be a selfrepeat and x be a vertex of type II in an almost Moore digraph. Then the distance $d(v, x) = d(x, v) = k$.*

Proof. Assume to the contrary that $d(v, x) < k$. Let P be a shortest path from v to x of length ℓ, $\ell < k$. Similarly, there is also path P' from $r(v) = v$ to $r(x)$ of length ℓ. Recall that edge $(r(x), x) \in E(G)$. Clearly, $r(x) \notin P$, otherwise the length of P would be $\ell - 1$. Hence $P \neq P'$. Then, there are two paths of length less than or equal k from v to x, in particular, P and $P'r(x)x$. But this is a contradiction because x cannot be a repeat of v.

Now assume to the contrary that $d(x, v) < k$. Let P be a shortest path from x to v of length $\ell < k$. Similarly, there is also a path P' from $r(x)$ to $r(v) = v$ of length ℓ. Recall that the edge $(r(x), x) \in E(G)$. Clearly, $r(x) \notin P$, otherwise the length of P would be $\ell - 1$. Hence $P \neq P'$. Then, there are two paths of length less than or equal to k from $r(x)$ to v, in particular, P' and $r(x)xP$. But this is a contradiction because v cannot be a repeat of $r(x)$. ∎

★ **Lemma 5.4.8** *Let u be a vertex of type II in an almost Moore digraph on a cycle C containing only vertices of type II. Then the distance between u and another vertex of type II, not in C, is exactly k.*

Proof. Assume to the contrary that there is a vertex v of type II, such that $d(u, v) < k$ and $v \notin C$. Let P be a shortest path from u to v of length ℓ, $\ell < k$. Let u' be the first vertex on P, such that the out-neighbour u'' of u' on P is a vertex which type is not II. Note that, since every vertex of type II is adjacent to exactly one vertex of type II, such a vertex u' must exist. Due to Lemma 5.4.7, u'' is a vertex of type I. Let v' be the first vertex on P, such that the in-neighbour v'' of v' on P is a vertex which is of type other than II. Note that path u'', P, v'' contains only vertices of type I.

Recall that $r(G) = G$. Hence there is a path Q from $r(u')$ to $r(v')$. Clearly, Q also contains, apart from $r(u')$ and $r(v')$, only vertices of type I. Moreover, the lengths of $u'Pv'$ and Q are equal. Recall that edges $r(u')u'$, $r(v')v' \in E(G)$. Then $u' \notin Q$ and $r(v') \notin P$. Then there are two different paths of length less than or equal k from $r(u')$ to v', in particular, $r(u')u'Pv'$ and $r(u')Qr(v')v'$. But this is a contradiction because $r(u')$ cannot be a repeat of v'. ∎

We conjecture that vertices of type II cannot exist in an almost Moore digraph.

Chapter 6

Conclusion

The main contribution of this thesis are in investigating the existence of generalized hamiltonian cycles in a graph and in finding properties and existence of almost Moore digraphs. There are two main sections of original contributions in this thesis. First, in Chapter 3 and 4, we focused on generalized hamiltonian cycles, in Chapter 5, we deal with Moore digraph, in particular, with almost Moore digraphs.

In Chapter 3 we obtained new results concerning k-walks and, in particular, 2-walks. We proved the existence of a 2-walk in chordal planar graphs with toughness greater then $\frac{3}{4}$. We were able also to decrease the toughness condition to $\frac{3}{5}$. Unfortunately, we proved the stronger statement only with the help of a computer. Moreover, we found the toughness threshold for the existence of 2-walks in K_4-minor free graphs, proving that every K_4-minor free graphs with toughness greater than $\frac{4}{7}$ has a 2-walk. We also found a chordal graph, which is K_4-minor free, with toughness exactly $\frac{4}{7}$ without a 2-walk. Moreover this graph is chordal also. We state the following problem.

Problem 6.0.1. *What is the toughness threshold for the existence of a 2-walk in chordal planar graphs?*

In Chapter 4 we proved that every $K_{1,r}$-free graph has an r-trestle. Moreover, we presented graphs that show that our result is sharp, so the result cannot be improved. Motivated by a result of Oberly and Sumner [78], and a recent result of Kužel [69] we propose the following conjecture.

Conjecture 6.0.1 *Every connected, locally connected graph $K_{1,r}$-free graph has an $(r-1)$-trestle.*

In Chapter 5 we focused on almost Moore digraphs. We obtained some new properties of almost Moore digraphs with selfrepeats. From these properties we were able to prove the nonexistence of almost Moore digraphs for infinitely many values of k and d. We still believe that almost Moore digraphs do not exist for $d \geq 3$ and $k \geq 3$ and this is our last conjecture.

Conjecture 6.0.2 *Almost Moore digraphs do not exist for $d \geq 3$ and $k \geq 3$.*

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