Efficient Conic Decomposition and Projection onto a Cone in a Banach Ordered Space

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Abstract

Consider a Banach space that contains n cones. Decomposition multi-valued mapping describes all decompositions of a given element on addends, such that addend i belongs to the i-th cone.

Decomposition mapping arises in different fields of mathematics and its applications. One of the main area of application is mathematical economics.

This thesis consists of three chapters. The first part of Chapter 1 contains some preliminary results. Properties of decomposition mapping are investigated, and a sublinear function closely related to this mapping is introduced and studied in the rest of this chapter.

In Chapter 2 we study conditions that provide the additivity of the decomposition mapping. For this purpose we introduce and study the Riesz interpolation property and lattice properties of spaces with respect to several preorders. The notion of 2-vector lattice is introduced and studied. Theorems that establish the relationship between the Riesz interpolation property and lattice properties of the dual spaces are given.

In Chapter 3 the notion of the weakly efficient point is introduced (by means of the decomposition mapping). The existence problems are elaborated and some examples are given.

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Contents

Preface

1. The goal of this research is to study general cone decomposition. Let us explain the matter of the problem.

Consider n convex cones $K_1, \ldots, K_n$ in a vector space E with $n \geq 2$. It is possible that some of these cones coincide. Let $L = \sum_{i=1}^{n} K_i$ be the Minkowski sum of these cones. A collection of elements $x_i \in K_i$, $i = 1, \ldots, n$ is called the decomposition of an element $x \in L$ with respect to the collection of cones $(K_i)_{i=1}^{n}$ if $x = x_1 + x_2 + \ldots + x_n$. We are mainly interested in the totality of all possible decompositions for all vectors $x \in L$. In other words we shall study the set-valued mapping σ defined on L by

$$\sigma(x) = \{(x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i = x,\ x_i \in K_i,\ i = 1, \ldots, n\}.$$

The mapping σ is called the *decomposition mapping* with respect to cones $K_1, \ldots, K_n$. We can describe this mapping in the following way. Consider the space E^n and the operator of summation $A : E^n \to E$ defined by

$$A(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i.$$

Let $K = K_1 \times \ldots \times K_n \subset E^n$ and let A_K be the restriction of A to K. Then A_K is the linear operator defined on K and mapping onto $L = \sum_{i=1}^{k} K_i$. It follows directly from the definition of the decomposition mapping that $\sigma(x)$ coincides with the set-valued mapping A_K^{-1} inverse to A_K.

2. Decomposition mapping arises in different fields of mathematics and its applications. The situation when all cones $K_1, \ldots, K_n$ coincide was mainly investigated. We mention here only two fields where the decomposition mapping plays a significant role. One of them is the theory of cones in Banach spaces. Some numerical characteristics of such a cone K can be calculated with the help of decomposition mapping σ defined for the totality of cones $K_1, \ldots, K_n$, where $K_i = K$ for all i (see

[37, 33] for details). One of the important but rarely studied problems, related to cones, is the problem of how to define an operator of the projection of an arbitrary element of a space on a cone [12, 19, 26, 38]. Projections on a system of cones is also of great interest (see [26, 25]). Such projections can be studied by means decomposition mapping with respect to the system under consideration.

The other field of application of the decomposition mapping is mathematical economics. Assume that we have an economy with n agents and m products. Let $E = \mathbb{R}^m$ and K coincide with the cone $\mathbb{R}^m_+ \subset \mathbb{R}^m$ of vectors with nonnegative coordinates. A vector $x = (x^1, \dots, x^m) \in \mathbb{R}^m_+$ describes a certain collection of products (x^j is the quantity of the product j in this collection.) Having vector x, agents need to distribute it between themselves, that is to find vectors $x_1, \dots, x_n \in \mathbb{R}^m_+$ such that $\sum_{i=1}^n x_i = x$. The totality of all such distributions coincides with the set $\sigma(x)$. The decomposition mapping plays an important role in the study of some models of economic equilibrium and economic dynamics (see [24] for details). From an economical point of view it is interesting to consider efficient decompositions of a given element x, that is, decompositions than are better (in a certain sense) that the other decompositions of this element. A cone decomposition theory based on efficiency has been developed by J.E. Martinez Legaz and A. Seeger in [25].

3. We use methods of convex analysis for examination of the decomposition mapping. Let $K_1, \dots, K_n$ be cones in the space E and let σ be a corresponding decomposition mapping. Then the graph

$$\operatorname{gr} \sigma = \{((x_1, \dots, x_n), y) \in E^n \times E : y \in \sigma(x_1, \dots, x_n)\}$$

is a convex cone, hence σ is a convex process (see [30]). In another terminology (see [32, 23, 24]) σ is a superlinear set-valued mapping. The dual theory of superlinear mappings is well developed. We give an explicit description of dual mapping to the decomposition mapping and describe its properties. This approach allows us to discover some interesting properties of the decomposition mapping itself.

4. An important question related to decomposition mapping is to find conditions that guarantee its additivity. In the simplest case when all cones K_i coincide with a cone K, this property is equivalent to the following: the space E with the order relation generated by K possesses the Riesz interpolation property. It is of interest to extend this result to the case of two or more cones. To this end, we introduce a space and objects defined with respect to several cones which can be viewed as generalizations of such classical notions as vector lattice, exact upper and lower bounds, Riesz interpolation property and Riesz decomposition property, Double Partition Lemma etc. On the whole, the problem on additive decomposition can be solved in such spaces. We establish the relationship between Riesz interpolation property with respect to several cones, and lattice properties of the dual space w.r.t. the corresponding dual cones.

The lattices with respect to several cones are quite natural from the point of view of applications to mathematical economics (see **2**). Indeed, it is quite natural to assume that each agent i is interested only in the products with the numbers from a certain subset J_i of the set of induces $\{1, \dots, n\}$. This observation leads to decomposition mapping with respect to a system of cones $K_1, \dots, K_n$, where K_i is a face of the cone $\mathbb{R}^m_+$. It can be shown that decomposition mapping with respect to such systems is additive.

5. Since the decomposition mapping is multi-valued, it is interesting to find some special decompositions of a given element. In particular various kinds of efficient conic decomposition are of certain interest (see [25] where some of these notions are introduced and studied). For certain applications the notion of weakly efficient decomposition is interesting. We introduce this notion and study it. The examination of weakly efficient decomposition can be done by means of the technique of superlinear set-valued mappings. The idea to employ the duality theorems is, of course, not a new one, however, it seems that in our case the use of such a powerful tool as superlinear multi-valued mappings can be very productive. The notion of weak efficiency and the notions of efficiency introduced in [25] are closely related. Existence of a weakly efficient decomposition is an important question. This existence can be guaranteed by the conditions imposed on the cones. The cases where the cones are either solid, or minihedral, or locally compact are discussed here. The last case is the most attractive and interesting from the point of view of nonsmooth analysis, especially from the point of view of quasidifferential calculus [10]. Recall that in this theory the principal tool is the cone of sublinear functions which is, in turn, locally compact.

It follows from the definition of a weakly efficient decomposition that, choosing some set from the dual space, one can get different sets of weakly efficient points. The first idea is to take, as such a set, a weakly* compact set. In the present work this approach is implemented.

6. The thesis is organized as follows. The work consists of Preface, three chapters and Bibliography.

The first part of Chapter 1 contains some preliminary results. Properties of decomposition mapping are investigated, and a sublinear function closely related to this mapping is discussed in the rest of this chapter.

In Chapter 2 we study conditions that provide the additivity of decomposition mapping. We introduce and study the interpolation Riesz property and lattice properties of spaces with respect to several preorders. The notion of a 2-vector lattice is introduced and studied. Theorems that establish a relationship between the interpolation Riesz property and lattice properties of the dual spaces are given.

In Chapter 3 the notion of a weakly efficient point is introduced (by means of decomposition mapping). The existence problems are elaborated and some examples are given.

Chapter 1

Decomposition Mapping

In this chapter some results from the theory of cones, superlinear set-valued mappings and convex analysis (Section 1) are given; the properties of a set-valued decomposition mapping are studied (Section 2), an explicit form of its conjugate mapping is derived, its effective domain is described; and a support function, to a decomposition mapping is examined. (Section 3).

1.1 Preliminaries

1.1.1 Some definitions and results from the theory of cones

Definition 1.1.1. A set K in a vector space E is called *conical*, if $\lambda K \subset K$ for $\lambda > 0$. A convex conical set is called a *cone*.

A conic set K is a cone if and only if

$$x, y \in K, \ \lambda, \mu \in \mathbb{R}_+ \implies \lambda x + \mu y \in K.$$

Definition 1.1.2. A cone K is called *pointed* if $K \bigcap(-K) = \{0\}$, or, equivalently, K does not contain straight lines.

Every cone K in a vector space E induces a *preorder* $(\geq_K)$ on E (i.e., transitive and reflective relation); namely $x \geq_K y$ if and only if $x - y \in K$.

A cone K in a space E is pointed if and only if the relation $\geq_K$ is an *order relation* (i.e., this relation is transitive, reflexive, and antisymmetric).

Let a cone K induce an order relation $\geq_K$ in a vector space E, let $x, y \in E$ and $y \geq_K x$. Then the set

$$\langle x, y\rangle_K := \{u \in E : y \geq_K u \geq_K x\} = (y - K)\bigcap(x + K)$$

is called a *conic interval* (with the endpoints x, y). Let $\Omega \subset X$. An element $u \in E$ such that $x \leq u$ for all $x \in \Omega$ is called an *upper bound* of Ω. An element $v \in E$ such that $x \geq v$ for all $x \in \Omega$ is called a *lower bound* of Ω.

A set $\Omega \in E$ is called *bounded from above (below)*, if this set has an upper bound (lower bound, respectively).

A cone K in a ordered space E is called *a generating cone*, if $K - K = E$.

The following result is well known (see, for example, [36]).

Proposition 1.1.1. *A cone K in a space E is a generating cone, if and only if any finite subset of E is bounded from above or from below.*

Let E be a Banach space. We denote the conjugate to E space by E'. (E' is a Banach space of all linear continuous functions l defined on E with the norm $\|l\| = \sup_{\|x\|\leq 1} l(x)$.)

Let $x \in E$ and $l \in E'$. Then we denote the value of a linear function at the element x by either $l(x)$ or $[l, x]$. This double notation will not lead to misunderstanding.

Let $K \subset E$ be a cone. We denote the conjugate to K cone by K^*. By definition,

$$K^* = \{l \in E' : l(x) \geq 0 \quad \text{for all} \quad x \in K\}.$$

Note the space E is a cone. It is clear that $E^* = \{0\} \neq E'$. Elements of the cone K^* are called positive functions. (In particular, zero is also a positive function.)

Let E be a Banach space with the order relation generated by a cone K. We use the following terminology:

A cone K is called *solid*, if it contains an interior point.

A cone K is called *locally compact*, if K is closed and every closed bounded subset $\Omega \subset K$ is compact. This is equivalent to the following: the set $\{x \in K : \|x\| \leq 1\}$ is compact; K is weakly locally compact if the set $\{x \in K : \|x\| \leq 1\}$ is weakly compact

Now let E be an ordered Banach space. The following two results are well-known.

Theorem 1.1.2. *(V. Klee, see for example [7, 34]) Let E be a Banach space and let a cone K be closed and generating. Then every positive (with respect to K) function on E is continuous.*

Theorem 1.1.3. *(M.Krein-V.Shmulyan, see for example [34].) If, in Banach space E, a cone K is closed and generating, then there exists a number $\gamma > 0$, such that any $x \in E$ can be represented in the form $x = u - v$, where $u, v \in K$, and $\|u\|, \|v\| \leq \gamma\|x\|$.*

This theorem can be generalized in the following way:

Theorem 1.1.4. *(see [2]) In a Banach space E, let two closed cones K_1, K_2 be given, such that $E = K_1 + K_2$. Then there exists a number $\gamma > 0$, such that every $x \in E$ can be represented in the form*

$$x = y_1 + y_2 \quad (y_1 \in K_1, y_2 \in K_2),$$

where $\|y_1\| \leq \gamma\|x\|$, $\|y_2\| \leq \gamma\|x\|$.

We need to extend Theorem 1.1.4 for the case n cones with $n > 2$. We need the notion of an ideally convex set for this. The following definitions and results can be found in [17]

A subset U of a Banach space E is called *ideally convex* if for each bounded sequence $x_n \in U$ and each sequence of positive numbers α_n such that $\sum_n \alpha_n = 1$ it follows that $\sum_n \alpha_n x_n \in U$. It is known that each closed convex set is ideally convex, the sum of a finite number of bounded ideally convex sets is also ideally convex and the intersection of an arbitrary family of ideally convex sets is also ideally convex.

Let U be a convex set. Recall the a point $a \in U$ is called an *algebraic interior* point of U if for each $x \in E$ there exists $\lambda > 0$ such that $a + \lambda x \in U$. Let $0 \in U$. Then 0 is an algebraic interior point of U if and only if $\bigcup_{\lambda > 0} \lambda U = E$.

The following result holds (see [17]).

Theorem 1.1.5. *(E. Lifshitz). Let U be an ideally convex set in a Banach space E. Then each algebraic interior point of the set U is its interior point.*

We now present a generalization of Theorem 1.1.4.

Theorem 1.1.6. *Let E be a Banach space that contain closed cones $K_1, \ldots, K_n$, such that $E = K_1 + \ldots + K_n$. Then there exists a number $\gamma > 0$, such that every $x \in E$ can be represented in the form*

$$x = y_1 + \ldots y_n \qquad (y_i \in K_i, \ i = 1, \ldots, n),$$

where $\|y_i\| \leq \gamma\|x\|$, $i = 1, \ldots, n$.

Proof. Let B the unit ball of E. Consider the sets $U_i = K_i \cap B$, $i = 1, \dots, n$. These sets are closed and convex, hence ideally convex. Since the sets U_i are also bounded it follows that their sum $U = U_1 + \ldots + U_n$ is also ideally convex. Let $x \in E$. Since $E = K_1 + \ldots + K_n$ it follows that there exists $x_1 \in K_1, \dots, x_n \in K_n$ such that $x = \sum_{i=1}^{n} x_i$. Let $m = \max_i \|x_i\|$. Then $x_i \in mU_i$. It follows from this that $\|x\| \leq n \cdot m$, so $x \in n \cdot mU$. We proved that $E = \bigcup_{\lambda > 0} U$. This means that 0 is an algebraic interior point of U. Applying Theorem 1.1.6 we conclude that $0 \in \operatorname{int} U$, therefore there exists $\eta > 0$ such that $\eta B \subset U$. Let $x \in E$, $x \neq 0$. Then $\frac{\eta}{\|x\|} x \eta B \subset U$, hence there exists elements $x_i \in U_i$ such that $x = x_1 + \ldots + x_n$. We have $x_i \in K_i$ and $\|x_i\| \leq 1$. Putting $\gamma = 1/\eta$ we obtain the desired result. □

Now we will discuss some results concerning normal cones.

Definition 1.1.3. A cone K in a Banach space E is called *normal*, if $\inf\{\|x + y\| : x, y \in K,\ \|x\| = \|y\| = 1\} > 0$.

Theorem 1.1.7. *(see for example [34, 36, 16]) Let E be a Banach space with a norm $\|\cdot\|$. Then the following conditions are equivalent:*

1. *K is a normal cone;*
2. *the norm $\|\cdot\|$ is semi-monotone: there exists a constant γ such that inequalities $0 \leq x \leq y$ imply $\|x\| \leq \gamma\|y\|$.*
3. *for each $x \in K$ the conic interval $\langle 0, x \rangle_K$ is bounded in the normed space E (that is $\sup_{y \in K, y \leq x} \|y\| < +\infty$).*

Theorem 1.1.8. *(see for example [34, 36, 16]) If in a Banach space E a cone K is normal, then in E there exists a norm, equivalent to the given one, and monotone on the cone K.*

Theorem 1.1.9. *(M.Krein, see for example [34, 36, 16]) A cone K in a Banach space E is normal if and only if K^* is a generating cone in E^*.*

Theorem 1.1.10. *(T.Ando, see for example [34]) If E is a Banach space with a closed cone K, then the following conditions are equivalent:*

1. *K is a generating cone;*
2. *K^* is a normal cone.*

Let E be a Banach space and $K \subset E$ be a closed cone. We need the following definitions.

1) A function $f \in E'$ is called *uniformly positive* on a closed cone $K \subset E$ if there exists a constant $\delta > 0$ such that $f(x) \geq \delta \|x\|$ for all $x \in K$.

2) A cone K is called *plastered* (see [16]) if there exists a pointed cone $K_1 \subset E$ and a number $\gamma > 0$ such that $x + \gamma \|x\| B \subset K_1$ for each $x \in K \setminus \{0\}$.

Theorem 1.1.11. *(see for example [16]) Let E be a Banach space with the unit ball B and a cone K. Then the following statements are equivalent:*

1. *K is a plastered cone;*
2. *there exists a function $f \in E'$ which is uniformly positive on K ;*
3. *in E there exists a norm, equivalent to the given one and additive on the cone K;*
4. *the cone K has a bounded (with respect to a norm) base Ω (i.e., every $x \in K \setminus \{0\}$ admits the unique representation $x = \alpha y$, where $\alpha > 0$, $y \in \Omega$, and $0 \notin cl\Omega$).*

1.1.2 Sublinear functions and superlinear multi-valued mappings

Let E be a vector space. A function $p : E \to \mathbb{R}_{+\infty} = (-\infty, +\infty]$ is called *sublinear*, if it is subadditive:

$$p(x_1 + x_2) \leq p(x_1) + p(x_2), \qquad x_1, x_2 \in E$$

and positively homogeneous:

$$p(\lambda x) = \lambda p(x), \qquad x \in E, \ \lambda > 0.$$

The function $-\infty$ identically equal to $-\infty$ also will be considered as sublinear.

If p is a sublinear function and $|p(0)| < +\infty$ then $p(0) = 0$. Indeed, it follows from $p(0) = p(2 \cdot 0) = 2p(0)$.

The set $\operatorname{dom} p = \{x \in E : |p(x)| < +\infty\}$ is called the *effective domain* of p.

All results presented in the rest of this subsection are valid for Hausdorff locally convex spaces. Since we will use these results only for sublinear functions and superlinear mappings defined on Banach spaces, we present only versions of these results for Banach spaces here.

Let E be a Banach space. For a sublinear function $p : E \to \mathbb{R}_{+\infty}$ the set

$$\partial p = \{f \in E' \ : \ [f, x] \leq p(x), \ x \in E\}$$

is called the *support set* of p. Let $p : E \to \mathbb{R}_{+\infty} = (-\infty, +\infty]$ be a sublinear function and $x \in \operatorname{dom} p$. The set

$$\partial p(x) = \{f \in \partial p \ : \ [f, x] = p(x), \ x \in E\},$$

is called the *subdifferential* of p at the point x.

It follows from these definitions that $\partial p = \partial p(0)$. Subdifferentials (in particular support sets) of a sublinear functions are w^*-closed convex subsets of E'.

Let $p(x) = -\infty$ for all $x \in E$. Then the support set ∂p is empty. The following result by L.Hermander, (see for example [9, 20, 4]) can be considered as a corollary of Hahn-Banach theorem:

Theorem 1.1.12. *If $p : E \to \mathbb{R}_{+\infty} = (-\infty, +\infty]$ is a sublinear lower semicontinuous function, then the support set ∂p is not empty and*

$$p(x) = \sup\{[f, x] \ : \ f \in \partial p\} \quad (x \in E).$$

A mapping $\psi : p \to \partial p$ is a bijection of the set of all sublinear lower semicontinuous functions on the family of all closed convex subsets of the space E'. This mapping is called the *Minkowski duality* (see [20] for details).

If a sublinear function $p : E \to \mathbb{R}_{+\infty}$ is continuous then this function is either identically $+\infty$ or finite. Later on we will consider only proper sublinear functions, that is functions p such that $\operatorname{dom} p = \{x : p(x) < +\infty\}$ is nonempty. It follows from this that $\operatorname{dom} p = E$ for each continuous sublinear function.

If a sublinear function p is continuous, then its support set ∂p is a w^*-compact set (see [9, 14]). A sublinear function p is continuous if and only if p is bounded, that is

$$\|p\| := \sup_{\|x\| \leq 1} |p(x)| < +\infty$$

The number $\|p\|$ is called the norm of p. We have

$$|p(x) - p(y)| \leq \|p\| \|x - y\|, \qquad x, y \in E.$$

Thus if $\operatorname{dom} p = E$ and p is bounded then p is Lipschitz continuous.

In a Banach space E, let an order relation $\geq$ be introduced by means of a cone K. A function $p : E \to \mathbb{R}_{+\infty}$ is called *monotone*, if the inequality $x \geq y$ $(x, y \in L)$ implies $p(x) \geq p(y)$.

Proposition 1.1.13. *(see for example [32]) Let p be a sublinear, lower semicontinuous function, defined on a Banach space E. Then the monotonicity of p is equivalent to the inclusion $\partial p \subset K^*$.*

The following definitions were introduced in [32].

A subset Ω of a cone K is called *normal* with respect to a cone $L \subset E$, if $\mathrm{cl}\,((\Omega - L) \cap K) = \Omega$. The intersection of all normal sets with respect to a cone L, containing the subset Ω of a cone K, is called the *normal hull* (with respect to L) of Ω and is denoted by $Nh(\Omega)$. If $L = K$ then we omit the words "with respect to L" in the definition of a normal set and the normal hull.

Let E_1, E_2 be Banach spaces.

A multi-valued mapping $\varphi : E_1 \to 2^{E_2}$ is called a *convex process*, if:

1. φ is superadditive, that is $\varphi(x + y) \supset \varphi(x) + \varphi(y)$, $(x, y \in E_1)$;
2. φ is positively homogeneous, i.e. $\varphi(\lambda x) = \lambda\varphi(x)$, $(x \in E_1, \lambda > 0)$;
3. its effective domain is nonempty, i.e. $\mathrm{dom}\,\varphi = \{x \in E_1 \ : \ \varphi(x) \neq \emptyset\} \neq \emptyset$;
4. $0 \in \varphi(0)$.

In other words, φ is a convex process if and only if the graph

$$\mathrm{gr}\,\varphi = \{(x, y) \in E_1 \times E_2 : y \in \varphi(x)\}$$

of the mapping φ is a cone in $E_1 \times E_2$ and $(0, 0) \in \mathrm{gr}\,\varphi$.

Sometimes (see for example [23, 32, 24]) convex processes are called *superlinear multi-valued mappings*. It is more convenient for us to use this terminology.

A set-valued mapping $\varphi : E_1 \to 2^{E_2}$ is called closed if its graph is a closed set. In other words, φ is closed if $(x_k \to x, y_k \in \varphi(x_k), y_k \to y) \implies y \in \varphi(x)$. If φ is a positively homogeneous mapping then $\mathrm{dom}\,\varphi$ is a conic set, if φ is a superlinear mapping then $\mathrm{dom}\,\varphi$ is a cone. Note that the effective domain of a closed mapping is not necessarily closed.

A positively homogeneous mapping φ with $\mathrm{dom}\,\varphi = K$ is called *bounded* if

$$\|\varphi\| := \sup\{\|y\| : y \in \varphi(x),\ x \in K,\ \|x\| \leq 1\} < +\infty.$$

Theorem 1.1.14. *Let $\varphi : E_1 \to 2^{E_2}$ be a closed positively homogeneous mapping, the cone $K := \mathrm{dom}\,\varphi$ be locally compact and $\varphi(0) = \{0\}$, then φ is bounded.*

Proof. Assume that $\|\varphi\| = +\infty$. Then there exist sequences (x_k) and (y_k) such that

$$x_k \in K, \quad \|x_k\| \leq 1, \quad y_k \in \varphi(x_k), \quad \|y_k\| \to +\infty.$$

Let

$$x'_k = \frac{x_k}{\|y_k\|}, \qquad y'_k = \frac{y_k}{\|y_k\|}.$$

Then y'_k is a bounded sequence. Since K is a locally compact cone we can assume without loss of generality that there exists $y' = \lim y'_k$ and $y' \in K$, $\|y'\| = 1$. Since φ is a positively homogeneous mapping, we conclude that $y'_k \in \varphi(x'_k)$. Since $x'_k \to 0$, $y'_k \to y$ and φ is a closed mapping, we have $y' \in \varphi(0)$, which contradicts the equality $\varphi(0) = \{0\}$. □

Proposition 1.1.15. *Let a be a superlinear mapping defined on a cone $L \subset E_1$ and mapping into E_2 with weakly compact images. Let for all $g \in E'_2$ the function p_g defined by $p_g(x) = \sup_{y\in a(x)}[g, y]$ be linear. Then a is an additive mapping: $a(x_1 + x_2) = a(x_1) + a(x_2)$ for all $x_1, x_2 \in L$.*

Proof. . Assume, on the contrary, that there exist vectors $x_1, x_2 \in L$ such that $a(x_1+x_2) \neq a(x_1)+a(x_2)$. Since a is superlinear we have $a(x_1+x_2) \supset a(x_1)+a(x_2)$. Hence there exists $y \in a(x_1 + x_2)$ such that $y \notin a(x_1) + a(x_2)$. The set $a(x_1) + a(x_2)$ is convex and weakly closed. Then there exists $g \in E'$ such that

$$\begin{aligned}
[g, y] &> \sup\{[g, z] : z \in a(x_1) + a(x_2)\} \\
&= \sup\{[g, z] : z \in a(x_1)\} + \sup\{[g, z] : z \in a(x_2)\|\} \\
&= p_g(x_1) + p_g(x_2).
\end{aligned}$$

It follows from this that

$$p_g(x_1 + x_2) = \sup\{[g, z] : z \in a(x_1 + x_2)\} \geq [g, y] > p_g(x_1) + p_g(x_2).$$

This contradicts the linearity of p_g. □

Definition 1.1.4. The multi-valued mapping $\varphi^* : E'_2 \to E'_1$ is called *dual* to a superlinear mapping $\varphi : E_1 \to 2^{E_2}$, if

$$\varphi^*(g) = \{f \in E'_1 : [f, x] \leq [g, y], \ \forall x \in \operatorname{dom}\varphi, \ y \in \varphi(x)\}.$$

The mapping $\varphi' = (\varphi^*)^{-1}$ is called *conjugate* to φ.

The following duality theorem holds:

Theorem 1.1.16. *(see [23, 32]) If φ is a superlinear mapping and the sublinear function $p_g(x) = \inf\{[g, y] : y \in \varphi(x)\}$ is lower semicontinuous for all $g \in E'$, then for all $x \in$ dom $\varphi, g \in$ dom φ^* the following holds*

$$\sup\{[f, x] \ : \ f \in \varphi^*(g)\} = \inf\{[g, y] \ : \ y \in \varphi(x)\}, \quad \partial p_g = \varphi^*(g).$$

1.2 Decomposition mapping and its properties

1.2.1 Decomposition mapping and its conjugate

Let E be a Banach space and let $E^n = E \times E \ldots \times E$ be its cartesian product. We assume that E^n is equipped with the sum-norm: if $X = (x_1, \ldots, x_n) \in E^n$ then $\|X\| = \sum_{i=1}^n \|x_i\|$. By $E', (E^n)'$ we will denote the dual spaces to E and E^n, respectively. Note that $(E^n)' = (E')^n$. For $f \in E$ we have $\|f\| = \sup_{\|x\| \leq 1} |f(x)|$. If $F = (f_1, \ldots, f_n) \in (E^n)'$ we have

$$\|F\| = \max_{i=1,\ldots,n} \|f_i\|.$$

In particular, if $f_1 = \ldots = f_n := f$ then $\|F\| = \|f\|$. In the space E let us consider the totality of convex closed cones $K_1, K_2, \ldots, K_n$, and in the space E^n consider their cartesian product $K = K_1 \times K_2 \times \cdots \times K_n$. The dual cones to $K_1, K_2, \ldots, K_n$ and K will be denoted by $K_1^*, K_2^*, \ldots, K_n^*$ and K^*, respectively. It is clear that $K^* = K_1^* \times K_2^* \times \cdots \times K_n^*$.

We also use the following notation: $L = K_1 + \ldots + K_n$. It is well-known and easy to check that $L^* = \bigcap_{i=1}^n K_i^*$

In the following, let the cone K induce an order in the space E^n.

Definition 1.2.1. A set-valued mapping $\sigma_{K_1,\ldots,K_n} : E \to 2^{E^n}$, defined by

$$\sigma_{K_1,\ldots,K_n}(x) := \begin{cases} \{X = (x_1, \ldots, x_n) \in K \ : \ \sum_{i=1}^n x_i = x\} & x \in L \\ \emptyset & x \notin L \end{cases}$$

is called *decomposition mapping* with respect to cones $K_1, \ldots, K_n$, and the elements of the set $\sigma_{K_1,\ldots,K_n}$ are called the *decompositions* of x.

For the sake of simplicity we denote $\sigma_{K_1,\ldots,K_n}$ by σ if it does not lead to confusion. It is clear that dom $\sigma = L := \sum_{i=1}^n K_i$.

Proposition 1.2.1. *A decomposition mapping $\sigma : E \to 2^{E^n}$ is a closed superlinear mapping.*

Proof. Let $x, y \in$ dom σ and $\lambda > 0$.

a) Let us prove the inclusion $\sigma(x + y) \supset \sigma(x) + \sigma(y)$.

Take an element $Z \in \sigma(x) + \sigma(y)$. Then there exist elements

$$X = (x_1, \ldots, x_n) \in \sigma(x), \quad Y = (y_1, \ldots, y_n) \in \sigma(y)$$

such that $Z = X + Y$, and by the definition of σ we have

$$x_i, y_i \in K_i, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} x_i = x, \ \sum_{i=1}^{n} y_i = y.$$

Since $x_i + y_i \in K_i$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n}(x_i + y_i) = x + y$, then $X + Y \in \sigma(x + y)$.

b) The positive homogeneity can be deduced by the following implications

$$X = (x_1, \ldots, x_n) \in \sigma(\lambda x) \iff x_i \in K_i, i = 1, 2, \ldots, n, \text{ and } \sum_{i=1}^{n} x_i = \lambda x$$

$$\iff \frac{x_i}{\lambda} \in K_i, \ i = 1, 2, \ldots, n, \text{ and } \sum_{i=1}^{n} \frac{x_i}{\lambda} = x \iff \frac{X}{\lambda} \in \sigma(x) \iff X \in \lambda\sigma(x).$$

c) Let (X^k) and (x^k) be sequences in E^n, E respectively, and

$$X^k \to X, \ x^k \to x, \quad \text{and} \quad X^k \in \sigma(x^k), \ k = 1, 2, \ldots .$$

Since K is a closed cone, it follows that the mapping σ is closed. □

Remark 1.2.1. The mapping σ possesses a stronger property than the property to be closed. Indeed if $X^k \to X$ then $x_i^k \to x_i$ for all i and hence $\sum_i^k x \to \sum_i^k x_i$. Thus the following holds: if $X^k \to X$ and $X^k \in \sigma(x^k)$ then there exists $\lim x^k = x$ and $X \in \sigma(x)$.

Recall (see Definition 1.1.4) that a set-valued mapping $\varphi^* : (E')^n \to 2^{E'}$ is called dual to $\varphi : E \to 2^{E^n}$, if

$$\varphi^*(G) := \{f \in E' \ : \ [f, x] \le [G, X], \ \forall x, \ X \in \varphi(x)\}.$$

Let us give a proof of the superlinearity of the dual mapping.

Proposition 1.2.2. *Let $\varphi : E \to 2^{E^n}$ be a set-valued mapping. Then the dual mapping φ^* is superlinear.*

Proof. First, let us show the superadditivity of φ^*, i.e.

$$\varphi^*(G_1 + G_2) \supset \varphi^*(G_1) + \varphi^*(G_2),$$

where $G_1, G_2 \in \text{dom } \sigma^*$.

Let $f \in \varphi^*(G_1) + \varphi^*(G_2)$, then there exist elements $f_1, f_2 \in E'$ such that $f_1 \in \varphi^*(G_1)$, $f_2 \in \varphi^*(G_2)$ and $f_1 + f_2 = f$.

The definition of the dual mapping implies

$$[f_1, x] \le [G_1, X], \; [f_2, x] \le [G_2, X] \quad \forall x, X \in \varphi(x).$$

Summing these two inequalities we obtain

$$[f_1 + f_2, x] \le [G_1 + G_2, X],$$

or

$$[f, x] \le [G_1 + G_2, X] \quad \forall x, X \in \varphi(x),$$

i.e. $f \in \varphi^*(G_1 + G_2)$.

Now let $f \in \varphi^*(\lambda G)$, $\lambda > 0$, then

$$[f, x] \le [\lambda G, X],$$

or

$$[f/\lambda, x] \le [G, X] \quad \forall x, X \in \varphi(x).$$

Thus, $f/\lambda \in \varphi^*(G)$ and $\varphi^*(\lambda G) = \lambda \varphi^*(G)$. □

1.2.2 The description of the mapping σ^*

In this subsection we give an explicit description of the mapping σ^* dual to the decomposition mapping $\sigma_{K_1,\dots,K_n} \equiv \sigma$. Put

$$\mathcal{K} = \text{dom } \sigma^*. \tag{1.2.1}$$

It follows from the superlinearity of σ^* that the set $\mathcal{K}$ is a convex cone.

The following theorem allows one to get an explicit form of the mapping σ^* dual to σ.

Theorem 1.2.3. *The equality*

$$\sigma^*(G) = \bigcap_{i=1}^{n} (g_i - K_i^*)$$

holds for all $G = (g_1, \dots, g_n) \in \mathcal{K}$.

Proof. Let $f \in \sigma^*(G)$ $(G \in \mathcal{K})$, then by the definition of σ^* we have

$$[f, x] \leq [G, X] \quad \forall x \in \operatorname{dom} \sigma, \ X \in \sigma(x). \tag{1.2.2}$$

For every $i = 1, 2, \ldots, n$, and any $x_i \in K_i$ put $X_{x_i} = (0, \ldots, 0, x_i, 0, \ldots, 0) \in E^n$. It is clear that $X_{x_i} \in \sigma(x_i)$, and (1.2.2) implies that

$$[f, x_i] \leq [G, X_{x_i}] \quad \forall x_i \in K_i, \ i = 1, 2, \ldots, n,$$

or

$$[f, x_i] \leq [g_i, x_i] \quad \forall x_i \in K_i, \ i = 1, 2, \ldots, n,$$

i.e.

$$[f - g_i, x_i] \leq 0 \quad \forall x_i \in K_i, \ i = 1, 2, \ldots, n.$$

It follows from the definition of the conjugate cone that

$$f - g_i \in -K_i^*,$$

that is $f \in g_i - K_i^*, \ i = 1, 2, \ldots, n.$

This means that

$$f \in \bigcap_{i=1}^{n} (g_i - K_i^*).$$

Conversely, let the last inclusion hold for an element f. Then $g_i - f \in K_i^*, \ i = 1, 2, \ldots, n$, and the definition of the conjugate cone K_i^* implies that for all $x_i \in K_i, \ i = 1, 2, \ldots, n$, we have

$$0 \leq [g_i - f, x_i], \quad \text{or} \quad [f, x_i] \leq [g_i, x_i], \ i = 1, 2, \ldots, n.$$

Summing over i from 1 to n we get

$$\sum_{i=1}^{n} [f, x_i] \leq \sum_{i=1}^{n} [g_i, x_i] \quad \forall x_i \in K_i, \ i = 1, 2, \ldots, n,$$

or

$$[f, \sum_{i=1}^{n} x_i] \leq \sum_{i=1}^{n} [g_i, x_i] \quad \forall x_i \in K_i, \ i = 1, 2, \ldots, n. \tag{1.2.3}$$

Consider now an arbitrary $x \in$ dom σ and $x_i \in K_i, \ i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} x_i = x$. Let $X = (x_1, \ldots, x_n)$. It follows from (1.2.3) that

$$[f, x] \leq [G, X] \quad \forall x, X \in \sigma(x),$$

which is equivalent (by definition) to the fact that $f \in \sigma^*(G)$. □

1.2.3 Domain of the mapping σ^*

It will be shown in this subsection that the cone $\mathcal{K} = \operatorname{dom}\sigma^*$ is the sum of two summands, one of which can be obtained by means of the following assertion.

Proposition 1.2.4. *The equality*

$$K^* = (\sigma^*)^{-1}(0)$$

is valid. (Recall that $K = K_1 \times \ldots \times K_n$.)

Proof. In the view of Theorem 1.2.3 we have

$$G \in (\sigma^*)^{-1}(0) \iff 0 \in \sigma^*(G) \iff 0 \in \bigcap_{i=1}^{n}(g_i - K_i^*)$$

$$\iff 0 \in g_i - K_i^*,\ \forall\, i \iff -g_i \in -K_i^*\ \forall\, i \iff g_i \in K_i^*\ \forall\, i \iff G \in K^*.$$

□

Corollary 1.2.5. *The inclusion $K^* \subset \mathcal{K}$ holds.*

Proof. If $G \in K^*$, then $0 \in \sigma^*(G)$, and therefore $G \in \mathcal{K}$. □

Consider the set

$$M = \{X \in E^n\ :\ \sum_{i=1}^{n} x_i = 0\}.$$

Let M^* be the *orthogonal* subspace to M:

$$M^* = \{G \in (E^n)^*\ :\ [G, X] = 0\quad \forall X \in M\}.$$

(See, for example, [14] where properties of M^* are discussed.) Consider also the diagonal

$$D = \{G = (g, g, \ldots, g)\ :\ g \in E'\}$$

of the space $(E')^n$. It is clear that D is w^*-closed in $(E^n)' = (E')^n$. In the sequel an element $(g, g, \ldots, g) \in D$ will be denoted by $g^\wedge$.

Proposition 1.2.6. *The subspaces M^* and D of the dual space $(E^n)'$ coincide.*

Proof. Let $G = g^\wedge \in D$, then for every $X \in M$ we have

$$[G, X] = \sum_{i=1}^{n}[g, x_i] = [g, \sum_{i=1}^{n} x_i] = 0,$$

i.e. $G \in M^*$, and hence $D \subset M^*$.

Now let us prove the opposite inclusion.

Suppose, there exists an element $\overline{G} \in (E')^n$ such that

$$\overline{G} \in M^* \setminus D.$$

Since D is w^*-closed and convex we can apply the separation theorem which implies the existence of $\overline{X} = (\bar{x}_i) \in E^n$ such that

$$[\overline{G}, \overline{X}] > \sup_{g \in E'} [g^\wedge, \overline{X}] = \sup_{g \in E'} \sum_i [g, \bar{x}_i] = \sum_{g \in E'} [g, \sum_i \bar{x}_i]. \tag{1.2.4}$$

The following cases are possible:

1. if $\overline{X} \in M$, then the right-hand side of the last inequality is equal to zero, and $[\overline{G}, \overline{X}] > 0$. On the other hand, $[\overline{G}, \overline{X}] = 0$, since $\overline{G} \in M^*$;
2. if $\overline{X} \notin M$, then $\sum_i \bar{x}_i \neq 0$ hence

$$\sup_{g \in E'} [g^\wedge, \overline{X}] = +\infty$$

and we have $[\overline{G}, \overline{X}] > +\infty$, therefore the both cases lead us to a contradiction.

□

Remark 1.2.2. The following relation holds for the sets K and M

$$K \bigcap M = \sigma(0).$$

Proposition 1.2.7. *For every $g^\wedge \in M^*$ the equality*

$$\sigma^*(g^\wedge) = g - \bigcap_{i=1}^{n} K_i^*$$

is valid.

Proof. Since the equality

$$[g, x] = [g, \sum_{i=1}^{n} x_i] = \sum_{i=1}^{n} [g, x_i] = [g^\wedge, X]$$

holds for all $x \in \text{dom}\ \sigma$, $X = (x_1, \ldots, x_n) \in \sigma(x)$ and every $g \in E'$, then

$$g \in \sigma^*(g^\wedge),\ \forall g \in E'.$$

From Theorem 1.2.3 it follows that

$$\sigma^*(0) = -\bigcap_{i=1}^{n} K_i^*,$$

then using the superlinearity of the dual mapping σ^* we obtain the relations

$$\sigma^*(g^\wedge) = \sigma^*(g^\wedge + 0) \supset \sigma^*(g^\wedge) + \sigma^*(0) \supset g - \bigcap_{i=1}^{n} K_i^*.$$

These inclusions imply that $\sigma^*(g^\wedge) \neq \emptyset$ for every $g^\wedge \in M^*$. If $f \in \sigma^*(g^\wedge)$ then (see Theorem 1.2.3) $g - f \in K_i^*$, $i = 1, \ldots, n$, and hence

$$g - f \in \bigcap_{i=1}^{n} K_i^* \quad \text{and} \quad f \in g - \bigcap_{i=1}^{n} K_i^*.$$

□

Corollary 1.2.8. *$M^* \subset \mathcal{K}$.*

Indeed, if $G \in M^* = D$ then there exists g such that $G = g^\wedge$. Since $\sigma^*(g^\wedge)$ is nonempty it follows that $G \in \operatorname{dom} \sigma^* = \mathcal{K}$.

Corollary 1.2.9. *If $g^\wedge \in M^*$, $G \in K^*$ then*

$$\sigma^*(g^\wedge + G) = g + \sigma^*(G).$$

Proof. The inclusion $f \in g + \sigma^*(G)$ can be rewritten as

$$f - g \in \bigcap_{i=1}^{n} (g_i - K_i^*).$$

The last inclusion is equivalent to the following

$$f - g \in g_i - K_i^*, \quad \text{or} \quad f \in g + g_i - K_i^*, \; i = 1, 2, \ldots, n,$$

i.e.

$$f \in \bigcap_{i=1}^{n} (g + g_i - K_i^*) = \sigma^*(g^\wedge + G).$$

□

The following theorem provides us with the explicit form of the effective domain of the dual mapping σ^*.

Theorem 1.2.10. *The cone* $\mathcal{K} = dom\,\sigma^*$ *has the form*

$$\mathcal{K} = K^* + M^*.$$

Proof. From Corollaries 1.2.5 and 1.2.8 it follows that $K^* \subset \mathcal{K}$ and $M^* \subset \mathcal{K}$. Since $\mathcal{K}$ is a convex cone, then $K^* + M^* \subset \mathcal{K}$.

Conversely, let an element $G = (g_1, \dots, g_n) \in \mathcal{K}$ and let $f \in \sigma^*(G) = \bigcap_{i=1}^n (g_i - K_i^*)$. Then $f \in g_i - K_i^*$, $i = 1, 2, \dots, n$, hence

$$g_i \in f + K_i^*, \qquad i = 1, \dots, n. \tag{1.2.5}$$

Due to Proposition 1.2.6, an element $f^\wedge = (f, f, \dots, f)$ belongs to M^*. Then (1.2.5) can be expressed in the form $G \in f^\wedge + K^*$, but $f^\wedge \in M^*$ and therefore

$$G \in M^* + K^*.$$

□

Now we will prove several propositions to be used later in this work.

Proposition 1.2.11. *For the inclusion* $f \in \sigma^*(G)$ $(G \in \mathcal{K})$ *to be valid, it is necessary and sufficient that* $G - f^\wedge \in K^*$.

Proof. Due to Theorem 1.2.3 $f \in \sigma^*(G)$ if and only if $f \in \bigcap_{i=1}^n (g_i - K_i^*)$ which is equivalent to $G - f^\wedge \in K^*$. □

Proposition 1.2.12. *Let* $x \in dom\,\sigma := L = \sum_{i=1}^n K_i$ *and let* $G \in \mathcal{K} = dom\,\sigma^*$. *If* $X = (x_1, \dots, x_n) \in \sigma(x)$, $f \in \sigma^*(G)$, *then* $[G - f^\wedge, X] \geq 0$ *and* $[g_i - f, x_i] \geq 0$ *for* $(i = 1, \dots, n)$.

Proof. It follows directly from the equality $[f, x] = [f^\wedge, X]$ $(X \in \sigma(x))$ and the definition of the dual mapping. Indeed, due to this definition we have

$$[G - f^\wedge, X] = [G, X] - [f^\wedge, X] = [G, X] - [f, x] \geq 0.$$

In view of Theorem 1.2.3 we have $f - g_i \in -K_i^*$ for all i. Since $x_i \in K_i$ it follows from this that $[g_i - f, x_i] \geq 0$ for $(i = 1, \dots, n)$. □

Proposition 1.2.13. *If the elements* $X = (x_1, \dots, x_n) \in \sigma(x)$, $f \in \sigma^*(G)$ $(x \in dom\,\sigma,\ G = (g_1, \dots, g_n) \in \mathcal{K})$ *are such that* $[f, x] = [G, X]$ *then*

$$[g_i - f, x_i] = 0 \quad i = 1, \dots, n.$$

Proof. Since $X = (x_1, \ldots, x_n) \in \sigma(x)$ then for $f \in E'$ the following equalities are valid:

$$[f, x] = [f, \sum_{i=1}^{n} x_i] = \sum_{i=1}^{n} [f, x_i] = [f^\wedge, X].$$

We have also $[f, x] = [G, X]$, therefore $[f^\wedge, X] = [G, X]$ or

$$[f^\wedge - G, X] = \sum_{i=1}^{n} [f - g_i, x_i] = 0.$$

Since $f \in \sigma^*(G)$, then Proposition 1.2.12 implies that all terms of the last sum have one and the same sign, then $[f - g_i, x_i] = 0$ $(i = 1, \ldots, n)$. □

Proposition 1.2.14. *If the elements $X \in \sigma(x)$, $f \in \sigma^*(G)$ $(x \in dom\ \sigma,\ G \in \mathcal{K})$ are such that $[f, x] = [G, X]$ and $G = H + g^\wedge$, where $H \in K^*$, $g^\wedge \in M^*$, then*

$$[f - g, x] = [H, X].$$

Proof. From Corollary 1.2.9 it follows that

$$\sigma^*(g^\wedge + H) = g + \sigma^*(H).$$

Since $f \in \sigma^*(g^\wedge + H)$, then $f - g \in \sigma^*(H)$. As for $X \in \sigma(x)$, the equalities

$$[H + g^\wedge, X] = [H, X] + [g^\wedge, X] = [H, X] + [g, x]$$

take place, and $[H + g^\wedge, X] = [f, x]$ then $[f - g, x] = [H, X]$. □

1.2.4 Closedness of $\mathcal{K}$ for $n = 2$

The cone $\mathcal{K}$ is not necessarily closed. We describe conditions which guarantee that $\mathcal{K}$ is closed only for $n = 2$. We need the following Lemma.

Lemma 1.2.15. *Let $n = 2$. Then*

$$\mathcal{K} = \{(h_1, h_2) : h_1 - h_2 \in K_1^* - K_2^*\}.$$

Proof. Let $\mathcal{K}_0 = \{(h_1, h_2) : h_1 - h_2 \in K_1^* - K_2^*\}$. First we show that $\mathcal{K} \subset \mathcal{K}_0$. Let $(h_1, h_2) \in \mathcal{K}$. Since

$$\mathcal{K} = M^* + K^* = D + (K_1^* \times K_2^*)$$

it follows that there exist $f \in E'$ and $l_i \in K_i^*$, $i = 1, 2$ such that $h_1 = f + l_1$, $h_2 = f + l_2$. We have $h_1 - h_2 = l_1 - l_2 \in K_1^* - K_2^*$, hence $(h_1, h_2) \in \mathcal{K}_0$. We have proved that $\mathcal{K} \subset \mathcal{K}_0$. We now prove the opposite inclusion. Let $(h_1, h_2) \in \mathcal{K}_0$. Then there exist $l_1 \in K_1$ and $l_2 \in K_2$ such that $h_1 - h_2 = l_1 - l_2$. Let $f := h_1 - l_1 = h_2 - l_2$. Then $h_1 = f + l_1$, $h_2 = f + l_2$, hence

$$(h_1, h_2) = (f, f) + (l_1, l_2) \in D + (K_1^* \times K_2^*) = \mathcal{K}.$$

□

Theorem 1.2.16. *Let $n = 2$. Then the cone $\mathcal{K}$ is closed if and only if the cone $K_1^* - K_2^*$ is closed.*

Proof. Let $K_1^* - K_2^*$ be closed. Let $(h_1^k, h_2^k) \in \mathcal{K}$, $k = 1, \ldots$ and let $(h_1^k, h_2^k) \to (h_1, h_2)$. It follows from Lemma 1.2.15 that $h_1^k - h_k^2 \in K_1^* - K_2^*$. Hence $\lim_k h_1^k - h_k^2 = h_1 - h_2 \in K_1^* - K_2^*$. Applying again Lemma 1.2.15 we conclude that $(h_1, h_2) \in \mathcal{K}$.

Now assume that $K_1^* - K_2^*$ is not closed. Then we can find a sequence $l^k \in K_1^* - K_2^*$, such that there exists $l := \lim_k l^k$ and $l \notin K_1^* - K_2^*$. Let $g_i^k \in K_i^*$, $i = 1, 2$ be sequences such that $\lim_k g_i^k = 0$. Consider sequences $h_1^k = g_1^k + l^k$ and $h_2^k = g_2^k$, $k = 1, \ldots$. Since $g_1^k - g_2^k \in K_1^* - K_2^*$, $l^k \in K_1^* - K_2^*$ and $K_1^* - K_2^*$ is a cone it follows that

$$h_1^k - h_k^2 = g_1^k - g_2^k + l_k \in K_1^* - K_2^*.$$

Hence $(h_1^k, h_2^k) \in \mathcal{K}_0 = \mathcal{K}$. We have $(h_1^k, h_k^2) \to (l, 0)$. Since $l - 0 = l \notin K_1^* - K_2^*$ it follows that $(l, 0) \notin \mathcal{K}_0 = \mathcal{K}$. Hence $\mathcal{K}$ is not closed. □

1.2.5 Dual to the decomposition mapping in the case when the cone L is normal

In this subsection we consider the dual mapping to the decomposition mapping when the cones $K_1, \ldots, K_n$ are such that their sum $L = K_1 + \ldots + K_n$ is a normal cone (see Definition 1.1.3)

Theorem 1.2.17. *If the cones $K_1, K_2, \ldots, K_n$ in E are such that $\sum_{i=1}^n K_i = L$ is a normal cone, then*

$$K^* + M^* = (E^n)'.$$

Proof. Take an arbitrary element $G = (g_1, \ldots, g_n) \in (E^n)'$. Since L is normal, the Krein Theorem (see 1.1.9) implies that the conjugate cone L^* is a generating cone. Therefore any finite subset of E' is bounded from below. In particular, for the set $\{g_1, \ldots, g_n\} \subset E'$ there exists an element $h \in E'$ such that

$$g_i \geq_{L^*} h, \; i = 1, 2, \ldots, n,$$

however $L^* = \bigcap_{i=1}^n K_i^*$ and hence $g_i - h \in K_i^*$ for all $i = 1, \ldots, n$ which is equivalent to $h \in g_i - K_i^*$, $i = 1, 2, \ldots, n$, or

$$h \in \bigcap_{i=1}^{n} (g_i - K_i^*).$$

In view of Theorem 1.2.3 we have $h \in \sigma^*(G)$. Therefore for every $G = (g_1, \ldots, g_n) \in (E^n)'$ the set $\sigma^*(G) \neq \emptyset$ and $\operatorname{dom} \sigma^* := \mathcal{K} = (E^n)'$, but $\mathcal{K} = K^* + M^*$, which completes the proof. □

Corollary 1.2.18. . *If the cone L is normal then the cone $\mathcal{K}$ is closed.*

Proposition 1.2.19. *If $\sum_{i=1}^n K_i = L$ is a normal cone in E then the decomposition mapping σ is bounded, that is, there exists a constant $C > 0$ such that*

$$\|X\| \leq C\|x\| \quad \text{for each} \quad x \in L \quad \text{and} \quad X \in \sigma(x).$$

Proof. Since L is a normal cone it follows that there exists a constant $m > 0$ such that the inequalities $x \geq_L y \geq_L 0$ imply $\|x\| \geq m\|y\|$. Let $x \in L$ and $X = (x_1, \ldots, x_n) \in \sigma(x)$. For each $j = 1, \ldots, n$ we have $\sum_{i \neq j} x_i \in \sum_{i \neq j} K_i \subset L$, hence $x - x_j \in L$. We also have $x_j \in K_j \subset L$. This means that $x \geq_L x_j \geq_L 0$, hence $\|x\| \geq m\|x_j\|$, $j = 1, \ldots, n$. Since $X = \sum_{j=1}^n \|x_j\|$ we get

$$\|X\| = \sum_{j=1}^{n} \|x_j\| \leq \frac{n}{m}\|x\| = C\|x\|.$$

where $C = n/m$. □

Corollary 1.2.20. *If $\sum_{i=1}^n K_i = L$ is a normal cone in E then $\sigma(0) = 0$.*

1.3 The support function of the decomposition mapping σ

1.3.1 A support function and its property

In this section we will study the properties of the decomposition mapping $\sigma_{K_1,\ldots,K_n} \equiv \sigma$, using the methods of subdifferential calculus.

For every $G \in (E^n)'$ consider the function $p_G : E \to \bar{\mathbb{R}}$ defined by

$$p_G(x) = \inf_{X \in \sigma(x)} [G, X] \qquad (x \in E).$$

(Recall that the infimum of the empty set is equal to $+\infty$, therefore $p_G(x) = +\infty$ for all $x \notin \operatorname{dom}\sigma = L := \sum_{i=1}^n K_i$. Recall also that we assume that $+\infty + (-\infty) = +\infty$.)

The function p_G is called the *support function* of the decomposition mapping σ corresponding to the linear function G.

The function p_G depends not only on G but also on cones $K_1, \dots, K_n$. We now indicate this dependence explicitly:

$$p_G(x) \equiv p_{G,K_1,\dots,K_n}(x) = \inf\left\{\sum_{i=1}^n [g_i, x_i] : \sum_{i=1}^n x_i = x : x_i \in K_i,\ i = 1, \dots, n\right\}.$$

Let

$$q_G(x) \equiv q_{G,K_1,\dots,K_n}(x) = \sup\left\{\sum_{i=1}^n [g_i, x_i] : \sum_{i=1}^n x_i = x : x_i \in K_i,\ i = 1, \dots, n\right\}.$$

Then

$$\begin{aligned} -q_{G,K_1,\dots,K_n}(x) &= -\sup\{\sum_{i=1}^n [g_i, x_i] : \sum_{i=1}^n x_i = x : x_i \in K_i,\ i = 1, \dots, n\} \\ &= \inf\{\sum_{i=1}^n [g_i, -x_i] : \sum_{i=1}^n (-x_i) = -x : -x_i \in -K_i,\ i = 1, \dots, n\} \\ &= p_{G,-K_1,\dots,-K_n}(-x). \end{aligned}$$

Thus

$$q_{G,K_1,\dots,K_n}(x) = -p_{G,-K_1,\dots,-K_n}(-x).$$

It follows from this equality that we do not need to specially study the function q_G; properties of q_G can be easily extracted from the corresponding properties of p_G.

Proposition 1.3.1. *The function p_G is sublinear.*

Proof. Let $x, y \in \operatorname{dom}\sigma$. Then $x + y \in \operatorname{dom}\sigma$ also. Since the mapping σ is superlinear, we have

$$\begin{aligned} p_G(x+y) &= \inf_{Z \in \sigma(x+y)} [G, Z] \le \inf_{Z \in \sigma(x)+\sigma(y)} [G, Z] \\ &= \inf_{X \in \sigma(x), Y \in \sigma(y)} ([G, X] + [G, Y]) \\ &= \inf_{X \in \sigma(x)} \inf_{Y \in \sigma(y)} ([G, X] + [G, Y]) \\ &= \inf_{X \in \sigma(x)} [G, X] + \inf_{Y \in \sigma(y)} [G, Y] = p_G(x) + p_G(y). \end{aligned}$$

If at least one of the elements x, y does not belong to $\operatorname{dom}\sigma$ then $p_G(x) + p_G(y) = +\infty$, so $p_G(x+y) \le p_G(x) + p_G(y)$ in this case as well. Thus p is subadditive. It is easy to check that p is positively homogeneous. □

Assume that $p_G(0) = -\infty$. Then for all $x \in \operatorname{dom}\sigma = \sum_{i=1}^n K_i$ we have

$$p_G(x) = p_G(x + 0) \le p_G(x) + p_G(0) = -\infty$$

so it is important to describe G such that $p_G(0) > -\infty$. Since p_G is positively homogeneous and $0 \in \operatorname{dom} p_G$ it follows that $p_G(0) > -\infty$ which implies $p_G(0) = 0$.

Proposition 1.3.2. *The equality $p_G(0) = 0$ holds if and only if $G \in cl\mathcal{K}$.*

Proof. Since $p_G(0) = \inf_{X \in \sigma(0)}[G, X]$ it follows that $p_G(0) = 0$ if and only if $[G, X] \ge 0$ for all $X \in \sigma(0)$. The set

$$\sigma(0) = \{X = (x_1, \dots, x_n) : \sum_i x_i = 0, x_1 \in K_1, \dots, x_n \in K_n\}$$

coincides with the cone $M \cap K$, hence $p_G(0) = 0$ if and only if $G \in (M \cap K)^*$. However

$$(M \cap K)^* = \operatorname{cl}(M^* + K^*) = \operatorname{cl}(D + K^*) = \operatorname{cl}\mathcal{K}.$$

□

Proposition 1.3.3. *For every $G \in \mathcal{K}$ the equality dom σ = dom p_G holds.*

Proof. Since $G \in \mathcal{K}$ it follows that there exist $f \in E'$ and $l_i \in K_i^*$ such that $G = f^\wedge + (l_1, \dots, l_n)$. Let $x \in \operatorname{dom}\sigma = \sum_{i=1}^n K_i$ and $X = (x_1, \dots, x_n) \in \sigma(x)$ then

$$[G, X] = \sum_{i=1}^n [f, x_i] + \sum_{i=1}^n [l_i, x_i] = [f, x] + \sum_{i=1}^n [l_i, x_i].$$

Note that $[l_i, x_i] \ge 0$ for all i, therefore $[G, X] \ge [f, x]$. Hence

$$p_G(x) = \inf_{X \in \sigma(x)} [G, X] \ge f(x) > -\infty.$$

It is clear that $p_G(x) \le [G, X] < +\infty$. We have proved that $\operatorname{dom}\sigma \subset \operatorname{dom} p_G$. If $x \notin \sum_{i=1}^n K_i = \operatorname{dom}\sigma$ then $p_G(x) = +\infty$ (because the infimum over the empty set is equal to $+\infty$). Hence $\operatorname{dom}\sigma = \operatorname{dom} p_G$. □

1.3.2 Fenchel-Moreau conjugate of the support function

In this subsection we calculate the Fenchel-Moreau conjugate function of the support function p_G. First we prove the following lemma.

Let $\delta(\cdot, \Omega)$ denote the indicator function of the set Ω, i.e.

$$\delta(x, \Omega) = \begin{cases} 0, & x \in \Omega, \\ +\infty, & x \notin \Omega. \end{cases}$$

Note that $\delta(x, \emptyset) = +\infty$ for all x.

Lemma 1.3.4. *Let $p = g + \delta(\cdot, L)$, where $g \in E'$ and $L \subset E$ is a cone. Then the conjugate (in the Fenchel-Moreau sense) function p^* has the form*

$$p^*(h) = \delta(h, g - L^*), \qquad h \in E'.$$

Proof. From the definition of the conjugate function we have

$$\begin{aligned} p^*(h) &= \sup_x\{[h,x] - p(x)\} = \sup_x\{[h-g,x] - \delta(h,L)\} = \sup_{x\in L}[h-g,x] \\ &= \begin{cases} 0, & h-g \in -L^*, \\ +\infty, & h-g \notin -L^*. \end{cases} = \begin{cases} 0, & h \in g - L^*, \\ +\infty, & h \notin g - L^*. \end{cases} = \delta(h, g - L^*). \end{aligned}$$

□

For each element $G = (g_1, \dots, g_n) \in (E^n)'$ consider the set $\sigma^*(G)$.

Theorem 1.3.5. *Let $G = (g_1, \dots, g_n) \in \mathcal{K}$. The conjugate (in the Fenchel-Moreau sense) to p_G function p_G^* has the form*

$$p_G^*(h) = \delta(h, \sigma^*(G)), \qquad h \in E'.$$

Proof. By definition

$$p_G(x) = \inf_{\sum_1^n x_i = x,\ x_i \in K_i,\ i=1,\dots,n} \sum_{i=1}^n [g_i, x_i].$$

Let $p_i = g_i + \delta(\cdot, K_i),\ i = 1, \dots, n$, then

$$p_G(x) = \inf_{\sum_1^n x_i = x,\, x_i \in E} \sum_{i=1}^n p_i(x_i),$$

so the function p_G is the infimal convolution of $p_1, \dots, p_n$:

$$p_G(x) = \bigoplus_{i=1}^n p_i(x).$$

The well-known Young-Fenchel duality theorem implies that

$$p_G^*(h) = \sum_{i=1}^n p_i^*(h),\ h \in E'.$$

But Lemma 1.3.4 yields

$$p_i^*(h) = \delta(h, g_i - K_i^*), \qquad h \in E'.$$

Then taking into account the fact that

$$\sum_{i=1}^{n} \delta(h, g_i - K_i^*) = \delta(h, \bigcap_{i=1}^{n}(g_i - K_i^*)), \qquad h \in E',$$

we have

$$p_G^*(h) = \delta(h, \sigma^*(G)), \qquad h \in E'.$$

□

Remark 1.3.1. If $G \notin \mathcal{K} := \operatorname{dom} \sigma^*$ then $\sigma^*(G) = \emptyset$, hence $p_G^* \equiv +\infty$.

Corollary 1.3.6. *The second conjugate function p_G^{**} has the form*

$$p_G^{**}(x) = \sup_{h \in \sigma^*(G)} [h, x], \qquad x \in E$$

for every $G \in \mathcal{K}$.

Proof. Applying Theorem 1.3.5 we conclude that

$$p_G^{**}(x) = \sup_{h \in E'} ([h, x] - \delta(h, \sigma^*(G)) = \sup_{h \in \sigma^*(G)} [h, x].$$

□

Corollary 1.3.7. *Let function p_G be lower semicontinuous. Then the equality $p_G^{**} = p_G$ is valid.*

Indeed, $f^{**} = f$ for all lower semicontinuous convex functions.

Corollary 1.3.8. *Let $G \in (cl\mathcal{K}) \setminus \mathcal{K}$. Then p_G is not lower semicontinuous.*

Proof. Since $G \notin \mathcal{K}$ it follows that $p_G^* = +\infty$. Hence $p_G^{**} = -\infty$. If p_G is lower semicontinuous then $p_G(x) = p_G^{**}(x) = -\infty$ for all x. However, in view of Proposition 1.3.2 we have $p_G(0) = 0$. □

Corollary 1.3.9. *Let function p_G be lower semicontinuous. Then $\partial p_G = \sigma^*(G)$.*

Proof. The set $\sigma^*(G)$ is w^*-closed and convex. Applying Corollary 1.3.6 and Corollary 1.3.7 we conclude that $p_G(x) = \sup_{h \in \sigma^*(G)} [h, x]$. The desired result follows now from the Minkowski duality. □

1.3.3 Lower semicontinuity of support functions

We now describe some cases where the function p_G is continuous (hence, lower semicontinuous) for all $G \in \mathcal{K}$.

Theorem 1.3.10. *Let cones $K_1, \dots K_n$ in a Banach space E be such that $E = K := \sum_{i=1}^n K_i$. Then the function p_G $(G \in \mathcal{K})$ is continuous.*

Proof. From Theorem 1.1.6 it follows that in the Banach space $E = K_1 + \dots + K_n$ there exists a number $\gamma > 0$ such that every element $x \in E$ can be represented in the form

$$x = \sum_{i=1}^n y_i \quad (y_i \in K_i, i = 1, \dots, n)$$

where $\|y_i\| \le \gamma \|x\|$, $i = 1, \dots, n$. We have

$$(y_1, \dots y_n) \in \sigma(x) = \{X = (x_1, \dots, x_n) \in K_1 \times K_2 \ : \ x = x_1 + \dots + x_i\}$$

and therefore for $G = (g_1, \dots, g_i) \in \mathcal{K}$ we have

$$\begin{aligned} p_G(x) &= \inf_{X \in \sigma(x)} [G, X] = \inf_{(x_1,\dots,x_n) \in \sigma(x)} \sum_{i=1}^n [g_i, x_i] \\ &\le \sum_{i=1}^n [g_i, y_i] \le \sum_{i=1}^n \|g_i\| \|y_i\| \le \gamma \sum_{i=1}^n \|g_i\| \|x\| \le C\|x\|, \end{aligned}$$

where $C = \gamma \sum_{i=1}^n \|g_i\|$. We proved that the sublinear function p_G is bounded on E and therefore it is continuous. □

Theorem 1.3.11. *Let the the space E be reflexive and let the cone $L = \sum_{i=1}^n K_i$ be normal. Then the function p_G is lower semicontinuous for all $G \in \mathcal{K}$.*

Proof. Since the cone L is normal it follows that (see Theorem 1.2.17) $\mathcal{K} = (E^n)'$ and (see Proposition 1.2.19) the mapping σ is bounded. Let $x \in L$ and let r be a number such that $\|x\| < r$. Let $B = \{x' \in E : \|x'\| \le r\}$. Then the set $\sigma(B)$ is contained in the ball $B_1 = \{X \in E^n : \|X\| \le r\|\sigma\|\}$. The set $B \times B_1$ is weakly compact and the mapping σ is weakly closed. Hence this mapping is weakly upper semicontinuous on B. We will now show that the function p_G is weakly lower semicontinuous at x. Indeed, let

$$\lambda < p_G(x) = \inf_{X \in \sigma(x)} G(X).$$

Consider the set $A = \{Y \in E^n : [G, Y] > \lambda\}$. Then the set $\sigma(x)$ is contained in the open set A. Since σ is weakly upper semicontinuous then there exists a weak neighborhood V of x such that $\sigma(V) \subset A$. If $y \in V$ then $p_G(y) = \inf_{Y \in \sigma(y)}[G, Y] \geq \lambda$. Hence p_G is weakly lower semicontinuous. Since p_G is convex, this function is also strongly lower continuous. □

Some results from the theory of Banach lattices are used in the next theorem. The short description of properties of lattices that we use can be found in the next chapter (see Subsection 2.1.1).

Theorem 1.3.12. *Assume that $K_1 = \ldots = K_n := L$ where L is a cone such that the ordered space E with the order relation $\geq$ generated by L is a vector lattice. Assume also that the norm in E is monotone: $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. Then the function p_G is lower semicontinuous on L for each $G \in \mathcal{K}$.*

Proof. We will prove that the restriction of p_G on L is a continuous function on L. Since $p_G(x) = +\infty$ for $x \notin L$, this continuity implies lower semicontinuity of p_G on E.

Let $x \in L$ and $(x_1, \ldots, x_n) \in \sigma(x)$. Since $x_i \in K_i = L$, $i = 1, \ldots, n$, then

$$x = \sum_{j=1}^{n} x_j \geq_L x_i \geq_L 0, \qquad i = 1, 2, \ldots, n.$$

Since the norm is monotone it follows that $\|x\| \geq \|x_i\|$, $i = 1, \ldots, n$.

From Theorem 1.2.10 it follows that every element $G = (g_1, \ldots, g_n) \in \mathcal{K}$ can be represented in the form $G = H + g^\wedge$, where $H = (h_1, \ldots, h_n) \in L^* \times \cdots \times L^*$, $g^\wedge \in M^*$, i.e. there exist $h_i \in L^*$, $i = 1, \ldots, n$ and $g \in E^*$ such that

$$g_i = h_i + g, \; i = 1, 2, \ldots, n.$$

Let $x \in L$. Since for $X = (x_1, \ldots, x_n) \in \sigma(x)$ we have

$$G(X) = \sum_{i=1}^{n} h_i(x_i) + \sum_{i=1}^{n} g(x_i) = \sum_{i=1}^{n} h_i(x_i) + [g, x],$$

we conclude that

$$p_G(x) = [g, x] + \left(\inf_{\sum_1^n x_i = x, \; x_i \in L, \; i=1,\ldots,n} \sum_{i=1}^{n} [h_i, x_i] \right) = [g, x] + q(x)$$

where

$$q(x) = \inf_{\sum_1^n x_i = x, \; x_i \in L, \; i=1,\ldots,n} \sum_{i=1}^{n} [h_i, x_i], \qquad x \in L \tag{1.3.1}$$

Since the function $x \mapsto [g, x]$ is continuous, we need only to prove that q is continuous. It easy to check that q is a sublinear function. We now show that q is bounded on L. Indeed let $x \in L$. Then

$$\begin{aligned} q(x) &= \inf_{\sum_1^n x_i = x,\ x_i \in L,\ i=1,\ldots,n} \sum_{i=1}^n [h_i, x_i] \\ &\leq \inf_{\sum_1^n x_i = x,\ x_i \in L,\ i=1,\ldots,n} \sum_{i=1}^n \|h_i\| \|x_i\| \\ &\leq \sum_{i=1}^n \|h_i\| \|x\| = \Gamma \|x\|, \end{aligned}$$

where $\Gamma = \sum_{i=1}^n \|h_i\|$.

We now show that the function q is increasing, that is $x \geq_L y \geq_L 0$ implies $q(x) \geq q(y)$. Indeed let $x, y \in L$ and $x - y := z \in L$. Let $X = (x_1, \ldots, x_n) \in \sigma(x)$. Since $x = y + z$ with $y, z \in L$ and $x = (x_1, \ldots, x_n)$ it follows from the Double Partition Lemma (see Section 3.1.1) that there exists $y_1, \ldots, y_n \in L$ and $z_1, \ldots, z_n \in L$ such that $x_i = y_i + z_i$, $i = 1, \ldots, n$. Let $Y = (y_1, \ldots, y_n)$. Then $Y \in \sigma(y)$ and $Y \leq X$ in the sense that $y_i \leq x_i$ for all $i = 1, \ldots, n$. Since $h_i \in L^*$ we have $[h_i, x_i] \geq [h_i, y_i]$ for all i. This implies $\sum_i [h_i, x_i] \geq \sum_i [h_i y_i]$. Thus for each $X \in \sigma(x)$ we found $Y \in \sigma(y)$ such that $[H, X] \geq [H, Y]$. It follows from this inequality that $q(x) \geq q(y)$. We have proved that q is increasing.

We now consider the function $\bar{q}$ defined on the entire space E by the formula:

$$\bar{q}(x) = q(x^+), \quad \text{where} \quad x^+ = \sup(x, 0).$$

Let us identify some properties of $\bar{q}$.

1) $\bar{q}(x) = q(x)$ for $x \in L$. Indeed, $x^+ = x$ for $x \in L$.

2) $\bar{q}$ is a subaddive function. Indeed, we have for $x, y \in E$: $(x + y)^+ \leq x^+ + y^+$. Since q is increasing and sublinear, it follows that

$$\bar{q}(x + y) = q(x + y)^+) \leq q(x^+ + y^+) \leq q(x^+) + q(y^+) = \bar{q}(x) + \bar{q}(y).$$

3) $\bar{q}$ is a positively homogeneous function. This is clear.

Thus, $\bar{q}$ is a sublinear extension of q. We also have for $x \in E$:

$$\bar{q}(x) = q(x^+) \leq \Gamma \|x_+\| \leq \Gamma \|x\|.$$

Thus the sublinear function $\bar{q}$ defined on E is bounded. Hence this function is continuous. Since $\bar{q}(x) = q(x)$ for $x \in L$ it follows that q is continuous on L. □

1.3.4 Infimum of a sublinear function over the $\sigma(x)$

Let E be a Banach space with cones $K_1, \ldots, K_n$. Assume that the space E^n is equipped with the order relation generated by the cone $K = K_1 \times \ldots \times K_n$.

Let $Q : E^n \to \mathbb{R}$ be some sublinear, monotone, continuous function, defined on the space E^n (dom $Q = E^n$). Consider a function s_Q defined on E by:

$$s_Q(x) = \inf_{X \in \sigma(x)} Q(X) \quad x \in E.$$

We need the following Minimax Theorem (see for example [29]).

Theorem 1.3.13. *Let (E, F) be a pair of vector spaces with a coupling function $(e, f) \mapsto [e, f]$ $(e \in E, f \in F)$. Assume that E and F are equipped with weak topologies $\sigma(E, F)$ and $\sigma(F, E)$ respectively. Let $\Omega \subset F$ and $\Sigma \subset E$ be convex sets and the set Ω is $\sigma(F, E)$-compact. Then*

$$\inf_{e \in \Sigma} \sup_{f \in \Omega} [f, e] = \sup_{f \in \Omega} \inf_{e \in \Sigma} [f, e].$$

Note that if E is a Banach space and $F = E'$ then $\sigma(F, E)$ coincides with weak*-topology.

Lemma 1.3.14. *Let $Q : E^n \to \mathbb{R}$ be a sublinear continuous function. The equality*

$$s_Q(x) = \sup_{G \in \partial Q} p_G(x)$$

holds, where ∂Q is the support set of the function Q.

Proof. Since Q is a sublinear continuous function, then the support set ∂Q is w^*-compact and the Hörmander Theorem (see 1.1.12) yields

$$Q(X) = \max_{G \in \partial Q} [G, X], \ X \in E.$$

Hence,

$$s_Q(x) := \inf_{X \in \sigma(x)} Q(X) = \inf_{X \in \sigma(x)} \max_{G \in \partial Q} [G, X].$$

Since for any $x \in E$ the set $\sigma(x)$ is convex and the set ∂Q is convex and w^*-compact, we can use Theorem 1.3.13:

$$\inf_{X \in \sigma(x)} \sup_{G \in \partial Q} [Q, X] = \sup_{G \in \partial Q} \inf_{X \in \sigma(x)} [G, X] = \sup_{G \in \partial Q} p_G(x).$$

□

Proposition 1.3.15. *Let Q be a sublinear continuous monotone function. Assume that functions p_G are lower semicontinuous for all $G \in K^*$. Then s_Q is a lower semicontinuous function.*

Proof. In view of Lemma 1.3.14 we have $s_Q(x) = \sup_{G \in \partial Q} p_G(x)$. Now we note, that since Q is monotone, then the inclusion $\partial Q \subset K^*$ holds (see Proposition 1.1.13). Since p_G is lower semicontinuous for $G \in K^*$ it follows that s_Q is also lower semicontinuous as the supremum of lower semicontinuous functions. □

Theorem 1.3.16. *The support set of the function s_Q has the form*

$$\partial s_Q = cl \sigma^*(\partial Q),$$

where cl denotes the closure of a set the in weak- topology.*

Proof. It was shown previously that

$$s_Q(x) = \sup_{G \in \partial Q} p_G(x).$$

Using subdifferential calculus and Corollary 1.3.9, we get

$$\partial s_Q = \partial s_Q(0) = \mathrm{cl}\,\mathrm{co} \bigcup_{G \in \partial Q} \partial p_G(0) = \mathrm{cl}\,\mathrm{co} \bigcup_{G \in \partial Q} \sigma^*(G) = \mathrm{cl}\,\mathrm{co}\; \sigma^*(\partial Q).$$

Since the mapping σ^* is superlinear and the set ∂Q is convex it follows that $\sigma^*(\partial Q)$ is convex, hence $\mathrm{co}\, \sigma^*(\partial Q) = \sigma^*(\partial Q)$. □

Chapter 2

The additivity of the decomposition mapping and lattices with respect to several preorders

In this chapter we study conditions that provide the additivity of the decomposition mapping σ. In order to give a description of these conditions we need to extend many notions of the theory of ordered space for spaces that are equipped with several preorders. We also describe a sublinear single-valued operator, acting on an ordered Banach space, such that the conjugate set-valued mapping to this operator coincides with σ (Section 2). Kantorovich - Riesz type theorems are proved. Examples are given.

2.1 The additivity of the decomposition mapping

2.1.1 Preliminaries

We need some definitions and results from the theory of Banach lattices.

Consider a set E equipped with an order relation $\geq$. An element u is called the *infimum* of a set $\Omega \subset E$ ($u = \inf \Omega$) if

- $u \leq x$ for each $x \in \Omega$;
- if $z \leq x$ for all $x \in \Omega$ then $z \leq u$.

An element v is called the *supremum* of $\Omega \subset E$ $(v = \sup \Omega)$ if

- $v \geq x$ for each $x \in \Omega$;
- if $z \geq x$ for all $x \in \Omega$ then $z \geq v$.

Definition 2.1.1. An ordered vector space E, such that for any finite set Ω there exists $\sup \Omega$ and $\inf \Omega$ is called a *vector lattice.* A cone $K \subset E$ is called *minihedral* if E with the order relation $\geq_K$ generated by K is a vector lattice.

It is easy to check that this definition is equivalent to the following one:

Definition 2.1.2. An ordered vector space E, such that for any set $\{x, y\} \subset E$ consisting of two elements there exists $\sup(x, y)$ and $\inf(x, y)$ is called a vector lattice.

If $x \in E$, where E is a vector lattice then the element $|x| = \sup(x, -x)$ is called the modulus of x. It is easy to check that $|x| = x^+ + x_-$ where $x^+ = \sup(x, 0)$, $x_- = -\inf(x, 0)$. Elements x^+ and x_- are called the *positive* and *negative* parts of x, respectively.

A vector lattice, such that its every nonempty subset bounded from above has the supremum, is called a *K-space* (or *Kantorovich space*) or *Riesz space*

Let E be an ordered Banach space with the order relation $\geq_K$, introduced by a cone K.

Definition 2.1.3. We say that an ordered vector space E possesses the *Riesz interpolation property,* if for every four elements a_1, a_2, b_1, b_2, satisfying the inequality $b_i \geq_K a_j$ $(i, j = 1, 2)$, there exists an "intermediate" element $c \in E$, such that $b_i \geq_K c \geq_K a_j$ $(i, j = 1, 2)$.

It is easy to see that this definition is equivalent to the following

Definition 2.1.4. We say that an ordered vector space E possesses the Riesz interpolation property if for all finite sets $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ such that $x_i \leq y_j$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$ there exists $z \in Z$ such that $x_i \leq z \leq y_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

It is easy to show, that every vector lattice possesses the interpolation property, for example, we can consider an intermediate element $z = \inf(y_1, \ldots, y_m)$ or $z = \sup(x_1, \ldots, x_n)$.

We need also two more definitions. Both of them can be expressed in terms of arbitrary finite sets and sets that consists of two elements only. For the sake of simplicity we consider only sets that consists of two elements.

Definition 2.1.5. We say that an ordered vector space E possesses the *Riesz decomposition property* if for every four elements x_1, x_2, y_1, y_2, satisfying the inequality $x_1 + x_2 \leq y_1 + y_2$ and for every z such that $x_1 + x_2 \leq z \leq y_1 + y_2$, there exist elements z_1, z_2, such that $z = z_1 + z_2$ and $x_1 \leq z_1 \leq y_1$ and $x_2 \leq z_2 \leq y_2$.

Definition 2.1.6. We say that the *double partition lemma* holds in an ordered vector space E if the relations

$$x = x_1 + x_2 \ (x_1 \geq 0,\ x_2 \geq 0), \qquad x = y + z \ \ (y \geq 0,\ z \geq 0)$$

implies the existence of positive elements y_1, z_1, y_2, z_2 such that

$$x_1 = y_1 + z_1, \ \ x_2 = y_2 + z_2, \qquad y = y_1 + y_2, \ \ z = z_1 + z_2.$$

It is known that the following assertions are equivalent (see, for example, [35]) :

- A space E possesses the Riesz interpolation property;
- A space E possesses the Riesz decomposition property;
- The double partition lemma holds in E;

The Riesz interpolation property is closely related to the minihedrality of the conjugate cone. The following results hold (see for example, [16, 36]:

Theorem 2.1.1. *Let E be an ordered Banach space and let the cone K of positive elements be a generating and normal cone. Then in the dual space E' the Riesz interpolation property is equivalent to the minihedrality of a cone K^*.*

Theorem 2.1.2. *(L.Kantorovich-F.Riesz) If an ordered Banach space E possesses the Riesz interpolation property, and the cone K of positive elements is a closed generating and normal cone, then the dual space E' is a K-space.*

A proof can be found in [36].

2.1.2 Riesz interpolation property in a space with two cones

Consider an ordered Banach space with the cone of positive elements K. Consider now the family of cones $K_1, \ldots K_n$ with an arbitrary $n > 1$ where $K_i = K$ for each $i = 1, \ldots, n$. It can be shown that the decomposition mapping $\sigma_{K_1,\ldots,K_n}$ is additive if and only if the space (E, K) possesses the Riesz interpolation property. (See Theorem 2.1.3, where a more general result is proved.) Our goal is to generalize this result for the case of different cones $K_1, \ldots, K_n$. For this purpose we need to generalize the notions of vector lattice and Riesz interpolation property for a space with different cones. In the classical situation where a cone K can be repeated n times with an arbitrary n we have different equivalent definitions of vector lattice (see Definitions 2.1.1 and 2.1.2, respectively). One of them is given in terms of arbitrary finite sets and the other one in terms of sets that contain only two elements. If we have different cones $K_1, \dots, K_n$ then the situation is different: we can consider the supremum and the infimum only finite sets that contain exactly n elements with the given n. A similar remark can be made with respect to the Riesz interpolation property, the Riesz decomposition property and the double partition lemma.

We will start with the Riesz interpolation property.

Let pointed cones $K_1, \dots, K_n$ in a vector space E be given. Each of them induces its own order relation $\geq_i$ $(i = 1, \dots, n)$ on E . The space E with cones $K_1, \dots K_n$ is denoted by $E = (E; K_1, \dots, K_n)$.

Remark 2.1.1. If the cones $K_1, \dots, K_n$ coincide and are equal to a cone K, we will use either notation $(E, K_1, \dots K_n)$ with $K_i = K$, $i = 1, \dots, n$ or notation (E, K) (if the latter is used, it is assumed that the number n is known).

For the sake of simplicity we consider the case $n = 2$. Then we show how the definitions and results obtained can be extended for an arbitrary n.

We introduce the following definition.

Definition 2.1.7. Consider a space $(E; K_1, K_2)$ and let $L = K_1 + K_2$. We say that $(E; K_1, K_2)$ possesses the Riesz interpolation property if for for every four elements $x_1, x_2, y_2, y_2 \in E$, satisfying the inequalities

$$y_1 \geq_{K_1} x_1, \quad y_2 \geq_{K_2} x_2, \quad y_1 \geq_L x_2, \quad y_2 \geq_L x_1, \tag{2.1.1}$$

there exists an "intermediate" element $c \in E$ such that

$$y_1 \geq_{K_1} c \geq_{K_1} x_1, \quad \text{and} \quad y_2 \geq_{K_2} c \geq_{K_2} x_2, \tag{2.1.2}$$

We will also call this property *the Riesz interpolation property in E with respect to cones* K_1, K_2.

Remark 2.1.2. It follows from (2.1.2) that $y_1 \geq_L c \geq_L x_2$ and $y_2 \geq_L c \geq_L x_1$. Indeed, if there exists an element $c \in E$ such that

$$y_1 \geq_{K_1} c \geq_{K_1} x_1 \quad \text{and} \quad y_2 \geq_{K_2} c \geq_{K_2} x_2,$$

then $c - x_1 \in K_1 \subset L$, $y_2 - c \in K_2 \subset L$.

Since $c - x_2 \in K_2 \subset L$, $y_1 - c \in K_1 \subset L$ then $x_2 \leq_L c$ and $y_1 \geq_L c$.

Note that

$$K_1 + K_1 = K_1, \quad K_2 + K_2 = K_2, \quad K_1 + K_2 = L, \quad K_2 + K_1 = L.$$

Hence (2.1.1) can be expressed in the form

$$y_j - x_i \in K_i + K_j$$

We will use the definition of an interval $\langle x, y\rangle_H$ with respect to a cone $H \subset E$. Recall that

$$\langle x, y\rangle_H = (x + H) \bigcap (y - H), \qquad (x, y \in E, \ y \geq_H x).$$

We can express Definition 2.1.7 in terms of intervals: if x_1, x_2, y_1, y_2 are four elements such that $y_j - x_i \in K_i + K_j, \ i, j = 1, 2$, then

$$\langle x_1, y_1\rangle_{K_1} \cap \langle x_2, y_2\rangle_{K_2} \neq \emptyset. \tag{2.1.3}$$

It follows from (2.1.3) and Remark 2.1.2 that

$$\bigcap_{i,j=1,2} \langle x_i, y_j\rangle_{K_i+K_j} \neq \emptyset.$$

Remark 2.1.3. To check the Riesz interpolation property with respect to the cones K_1, K_2 in the space $E = (E; K_1, K_2)$ it is sufficient to verify that an intermediate element exists under the additional hypothesis: $x_1, x_2 \in L$. Indeed, assume that the Riesz interpolation property holds for all four-tips $\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2$ such that $\tilde{y}_j - tx_i \in K_i + K_j$ and $\tilde{x}_1, \tilde{x}_2 \in L$. Let $x_i, y_j \in E, \ i, j = 1, 2$ and $y_j - x_i \in K_i + K_j$ $(i, j = 1, 2)$. Let $z = x_1 + x_2 - y_1$. Consider four elements $\tilde{x}_1 = x_1 - z$, $\tilde{x}_2 = x_2 - z$, $\tilde{y}_1 = y_1 - z$, $ty_2 = y_2 - z$. We have

$$tx_1 := x_1 - z = y_1 - x_2 \in L, \quad \tilde{x}_2 := x_2 - z = y_1 - x_1 \in K_1 \subset L.$$

Therefore the Riesz interpolation property holds for elements $x_i - z, \ y_j - z$ $(i, j = 1, 2)$ so an element $\tilde{c}$ exists such that

$$\tilde{y}_1 \geq_{K_1} \tilde{c} \geq_{K_1} \tilde{x}_1, \quad \text{and} \quad \tilde{y}_2 \geq_{K_2} \tilde{c} \geq_{K_2} \tilde{x}_2.$$

Let $c = \tilde{c} + z$. Then

$$y_1 \geq_{K_1} c \geq_{K_1} x_1, \quad \text{and} \quad y_2 \geq_{K_2} c \geq_{K_2} x_2.$$

We have proved that the Riesz interpolation property holds in $(E; K_1, K_2)$.

2.1.3 Riesz decomposition property and double partition lemma in a space with two cones

Definition 2.1.8. We say that the space $E = (E; K_1, K_2)$ possesses the Riesz decomposition property if

$$\langle x_1 + x_2, y_1 + y_2\rangle_{K_1+K_2} = \langle x_1, y_1\rangle_{K_1} + \langle x_2, y_2\rangle_{K_2}$$

for all $x_1, y_1 \in K_1,\ x_2, y_2 \in K_2$ such that $y_1 \geq_{K_1} x_1,\ y_2 \geq_{K_2} x_2$.

Consider a space $(E; K_1, K_2)$. Let $x_1, y_1 \in K_1, x_2, y_2 \in K_2$ and $y_1 \geq_{K_1} x_1$, $y_2 \geq_{K_2} x_2$. Then

$$\langle x_1 + x_2, y_1 + y_2\rangle_{K_1+K_2} \supset \langle x_1, y_1\rangle_{K_1} + \langle x_2, y_2\rangle_{K_2}. \tag{2.1.4}$$

Indeed if $x_1 \leq_{K_1} z_1 \leq_{K_1} y_1$ and $x_2 \leq_{K_2} z_2 \leq_{K_2} y_2$ then $z_1 - x_1 \in K_1, y_1 - z_1 \in K_1$, $z_2 - x_2 \in K_2$, $y_2 - z_2 \in K_2$, hence

$$x_1 + x_2 \leq_{K_1+K_2} z_1 + z_2 \leq_{K_1+K_2} y_1 + y_2.$$

In view of (2.1.4), the Riesz decomposition property is equivalent to the following:

$$\langle x_1 + x_2, y_1 + y_2\rangle_{K_1+K_2} \subset \langle x_1, y_1\rangle_{K_1} + \langle x_2, y_2\rangle_{K_2}.$$

This means that each element z such that

$$x_1 + x_2 \leq_{K_1+K_2} z \leq_{K_1+K_2} y_1 + y_2$$

can be represented as the sum $z = z_1 + z_2$ with

$$x_1 \leq_{K_1} z_1 \leq_{K_1} y_1 \quad \text{and} \quad x_2 \leq_{K_2} z_2 \leq_{K_2} y_2.$$

Thus if $K_1 = K_2 := K$ and K induces an order relation in E then Definition 2.1.8 coincides with Definition 2.1.5.

Remark 2.1.4. The Riesz decomposition property with respect to cones K_1, K_2 is equivalent to the fact that the equality

$$\langle 0, x + y\rangle_{K_1+K_2} = \langle 0, x\rangle_{K_1} + \langle 0, y\rangle_{K_2}$$

holds for all $x \in K_1,\ y \in K_2$.

Indeed, let we know that for each $y_1 \in K_1$ and $y_2 \in K_2$ and z such that

$$0 \leq_{K_1+K_2} z \leq_{K_1+K_2} y_1 + y_2$$

there exists $z_1 \in K_1$ and $z_2 \in K_2$ such that

$$z_1 + z_2 = z \quad \text{and} \quad z_1 \leq_{K_1} y_1, \;\; z_2 \leq_{K_2} y_2 \tag{2.1.5}$$

Let $x_1, y_1 \in K_1$, $x_2, y_2 \in K_2$ and $y_1 \geq_{K_1} x_1$, $y_2 \geq_{K_2} x_2$. Let

$$x_1 + x_2 \leq_{K_1+K_2} z \leq_{K_1+K_2} y_1 + y_2.$$

Then

$$0 \leq_{K_1+K_2} z - (x_1 + x_2) \leq_{K_1+K_2} (y_1 + y_2) - (x_1 + x_2).$$

Using (2.1.5) with respect to $y_1 - x_1$ and $y_2 - x_2$ we can conclude that required elements z_1 and z_2 exist.

Consider space $(E; K_1, K_2)$ with two cones K_1 and K_2. Consider two arbitrary elements $y_1, z_1 \in K_1$ and two arbitrary elements $y_2, z_2 \in K_2$. Let

$$x_1 = y_1 + z_1, \;\; x_2 = y_2 + z_2 \quad \text{and} \quad y = y_1 + y_2, \;\; z = z_1 + z_2. \tag{2.1.6}$$

and $x = x_1 + x_2$. Then $x_1 \in K_1$, $x_2 \in K_2$ and $x \in L$. We can also represent x as the sum of two elements from L: $x = y + z$. The reverse assertion will be called the double partition lemma.

Definition 2.1.9. We say that the *double partition lemma* holds in the space $E = (E; K_1, K_2)$, if the reverse assertion holds: if for an element $x \in L$ the following equalities hold:

$$x = x_1 + x_2, \quad \text{where} \quad x_1 \in K_1, x_2 \in K_2$$

and

$$x = y + z, \quad \text{where} \quad y, z \in L,$$

then elements $y_1, z_1 \in K_1$, $y_2, z_2 \in K_2$ exist such that each x_i $(i = 1, 2)$ can be represented in the form $x_i = y_i + z_i$ and also $y = y_1 + y_2$ and $z = z_1 + z_2$.

Remark 2.1.5. Let $K_1 = K_2 := K$. Then the Riesz interpolation property holds in the space $(E; K_1, K_2)$ if the ordered space possesses the "classical" Riesz interpolation property. The same conclusion can be made with respect to the Riesz decomposition property and the double partition lemma.

2.1.4 Additivity of the decomposition mapping

The decomposition mapping $\sigma_{K_1,K_2} = \sigma : E \to 2^{E^2}$ with respect to cones K_1 and K_2 in the space $E = (E; K_1, K_2)$ is expressed in the following way:

$$\sigma(x) = \{X = (x_1, x_2) \in K_1 \times K_2 : x_1 + x_2 = x\} \quad (x \in E).$$

Recall that $\operatorname{dom}\sigma = L := K_1 + K_2$. We are interested in conditions that guarantee the additivity of the decomposition mapping.

The following theorem claims that all above definitions are equivalent and that each of them is equivalent to the required additivity.

Theorem 2.1.3. *The followings statements are equivalent:*

1. *the space $E = (E; K_1, K_2)$ possesses the Riesz interpolation property;*
2. *the space $E = (E; K_1, K_2)$ possesses the Riesz decomposition property;*
3. *the double partition Lemma takes place in the space $E = (E; K_1, K_2)$;*
4. *the decomposition mapping $\sigma_{K_1,K_2} = \sigma : E \to 2^{E^2}$ is additive, i.e. if $x, y \in L$ then $\sigma(x+y) = \sigma(x) + \sigma(y)$. (Here $L = K_1 + K_2$.)*

Proof. $1 \implies 2$. In view of Remark 2.1.4 it is enough to show that

$$\langle 0, x_1 + x_2 \rangle_{K_1+K_2} = \langle 0, x_1 \rangle_{K_1} + \langle 0, x_2 \rangle_{K_2}.$$

Let $x_1 \in K_1,\ x_2 \in K_2$ and $y \in L$ and let $x_1 + x_2 \geq_L y$. We can express these conditions in the following way:

$$y \geq_{K_1} y - x_1, \quad x_2 \geq_{K_2} 0, \quad y \geq_L 0, \quad x_2 \geq_L y - x_1.$$

Let us apply the Riesz interpolation property to these inequalities, and find an intermediate element, i.e. an element $c \in E$ such that

$$y \geq_{K_1} c \geq_{K_1} y - x_1, \quad x_2 \geq_{K_2} c \geq_{K_2} 0. \tag{2.1.7}$$

Let $y_1 = y - c$ and $y_2 = c$. Then (2.1.7) yields

$$y_1 \in K_1, \quad y_2 \in K_2, \quad x_1 \geq_{K_1} y_1, \quad x_2 \geq_{K_2} y_2.$$

We have also $y = y_1 + y_2$, i.e. y_1 and y_2 form the required decomposition and

$$y_1 \in \langle 0, x_1 \rangle_{K_1}, \quad y_2 \in \langle 0, x_2 \rangle_{K_2}.$$

$2 \implies 3$. Let an element $x \in E$ be such that $x = x_1 + x_2$, where $x_1 \in K_1,\ x_2 \in K_2$ and $x = y + z$, where $y, z \in L$. Then $x_1 + x_2 \geq_L y \geq_L 0$. By the Riesz decomposition property elements $y_1 \in K_1,\ y_2 \in K_2$ exist such that

$$x_1 \geq_{K_1} y_1, \; x_2 \geq_{K_2} y_2, \; y = y_1 + y_2.$$

Let $z_1 = x_1 - y_1,\ z_2 = x_2 - y_2$. We have $z_1 \in K_1,\ z_2 \in K_2,\ x_1 = y_1 + z_1,\ x_2 = y_2 + z_2$ and

$$z_1 + z_2 = x_1 + x_2 - (y_1 + y_2) = x - y = z.$$

Therefore the elements y_1, y_2, z_1, z_2 are as desired.

$3 \implies 4$. Let $y, z \in L$. Since the decomposition mapping σ is superlinear, then $\sigma(y + z) \supset \sigma(y) + \sigma(z)$. Let us prove the opposite inclusion. Let $X = (x_1, x_2) \in \sigma(y + z)$, then by the definition of the mapping σ we have

$$x_1 \in K_1,\ x_2 \in K_2 \quad \text{and} \quad x_1 + x_2 = y + z.$$

In view of the double partition Lemma there exist elements $y_1, z_1 \in K_1,\ y_2, z_2 \in K_2$, such that every x_i $(i = 1, 2)$ can be represented in the form $x_1 = y_1 + z_1,\ x_2 = y_2 + z_2$ and $y = y_1 + y_2,\ z = z_1 + z_2$. It means that

$$Y = (y_1, y_2) \in \sigma(y), \quad Z = (z_1, z_2) \in \sigma(z)$$

and $X = Y + Z$, i.e. $X \in \sigma(y) + \sigma(z)$.

$4 \implies 3$. It can be proved by an argument similar to that in the proof of $3 \implies 4$.

$3 \implies 1$. Let elements $a_1, a_2, b_1, b_2 \in E$ satisfy the inequalities

$$b_1 \geq_{K_1} a_1,\ b_1 \geq_L a_2,\ b_2 \geq_{K_2} a_2,\ b_2 \geq_L a_1.$$

Let $u_1 = b_1 - a_1 \in K_1,\ u_2 = b_2 - a_2 \in K_2,\ v_1 = b_2 - a_1 \in L,\ v_2 = b_1 - a_2 \in L$. Then $u_1 + u_2 = v_1 + v_2$. From the double partition Lemma it follows that elements $y_1, z_1 \in K_1$ and $y_2, z_2 \in K_2$ exist such that

$$u_1 = y_1 + z_1 \in K_1,\ u_2 = y_2 + z_2 \in K_2 \quad \text{and} \quad v_1 = y_1 + y_2 \in L,\ v_2 = z_1 + z_2 \in L.$$

The element $c = a_1 + y_1$ is an intermediate between a_i and b_j $(i, j = 1, 2)$. Indeed, $u_1 = b_1 - a_1 \geq_{K_1} y_1$ yields $b_1 \geq_{K_1} a_1 + y_1 \geq_{K_1} a_1$,and $v_1 = b_2 - a_1 \geq_L y_1$ implies that $b_2 \geq_L a_1 + y_1 \geq_L a_1$. Since the equality $v_1 = y_1 + y_2$ yields $b_2 - y_2 = a_1 + y_1$, then from $u_2 = b_2 - a_2 \geq_L y_2$ and $u_1 = b_1 - a_1 \geq_L y_1$ we obtain $b_1 \geq_L a_1 + y_1 = b_2 - y_2 \geq_L a_2$. Finally, the inequality $u_2 = b_2 - a_2 \geq_{K_2} y_2$ yields $b_2 \geq_{K_2} b_2 - y_2 = a_1 + y_1 \geq_{K_2} a_2$. □

2.1.5 Examples

First we will give an example of cones such that the decomposition mapping σ is nonadditive.

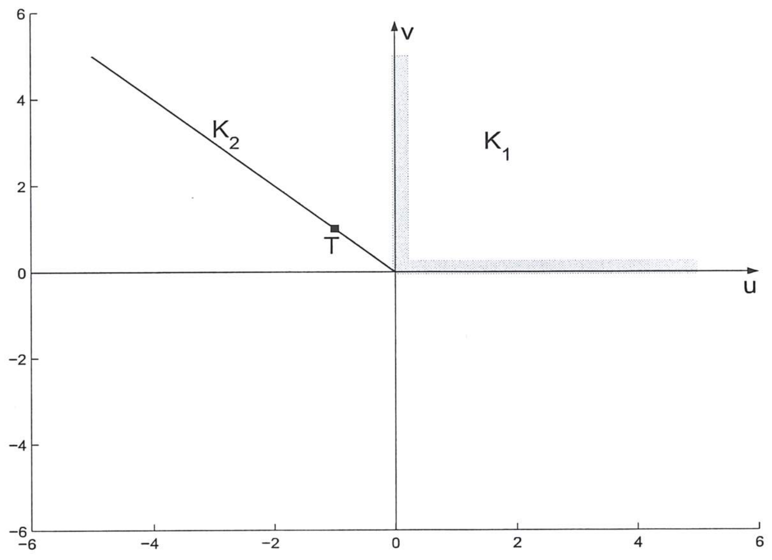

Figure 2.1: $E = (\mathbb{R}^2; K_1, K_2)$

Example 2.1.1. Let the following cones be given in the space $E = \mathbb{R}^2$: the positive orthant and the ray passing through the point $T = (-1, 1) \in \mathbb{R}^2$ (see Fig. 2.1), i.e.

$$K_1 = \{X = (u, v) \in \mathbb{R}^2 \ : \ u \geq 0, v \geq 0\},$$

$$K_2 = \{X = (u, v) \in \mathbb{R}^2 \ : \ u = -\lambda, v = \lambda, \lambda \geq 0\}.$$

Let $x = (1, 0) \in K_1$, $y = (-1, 1) \in K_2$. Then $x + y = (0, 1)$. We have

$$\sigma(x) = \{X = (x_1, x_2) : x_1 \in K_1,\ x_2 \in K_2,\ x_1 + x_2 = (1, 0)\}.$$

Let $(x_1, x_2) \in \sigma(x)$ with $x_1 = (x_1^1, x_1^2)$, $x_2 = (x_2^1, x_2^2)$. Then

$$x_1^1 \geq 0,\ x_1^2 \geq 0, \quad x_1^1 + x_2^1 = 1,\ x_1^2 + x_2^2 = 0;$$

$$x_2^1 = -\lambda, \quad x_2^2 = \lambda \ \text{ where } \ \lambda \geq 0.$$

Using these relations we can easily obtain that

$$x_2^1 = x_2^2 = x_1^2 = 0, \qquad x_1^1 = 1.$$

Therefore $x_1 = x$, $x_2 = 0$. Thus

$$\sigma(x) = \{(x, 0)\}. \tag{2.1.8}$$

Let us calculate $\sigma(y)$. We have

$$\sigma(y) = \{Y = (y_1, y_2) : y_1 \in K_1,\ y_2 \in K_2,\ \ y_1 + y_2 = (-1, 1)\}.$$

Let $(y_1, y_2) \in \sigma(y)$ with $y_1 = (y_1^1, y_1^2)$, $y_2 = (y_2^1, y_2^2)$. Then

$$y_1^1 \geq 0,\ y_1^2 \geq 0, \quad y_1^1 + y_2^1 = -1,\ y_1^2 + y_2^2 = 1;$$

$$y_2^1 = -\lambda, \quad y_2^2 = \lambda \quad \text{where} \quad \lambda \geq 0.$$

An easy calculation shows that $y_1 = 0$, $y_2 = y$, hence

$$\sigma(y) = \{(0, y)\}. \tag{2.1.9}$$

Let $z = (0, 1) = x + y$. Let us calculate $\sigma(z)$. We have

$$\sigma(z) = \{Z = (z_1, z_2) : z_1 \in K_1,\ z_2 \in K_2,\ \ z_1 + z_2 = (0, 1)\}.$$

Let $(z_1, z_2) \in \sigma(z)$ with $z_1 = (z_1^1, z_1^2)$, $z_2 = (z_2^1, z_2^2)$. Then

$$z_1^1 \geq 0,\ z_1^2 \geq 0, \quad z_1^1 + z_2^1 = 0,\ z_1^2 + z_2^2 = 1;$$

$$z_2^1 = -\lambda, \quad z_2^2 = \lambda \quad \text{where} \quad \lambda \geq 0,\ \ z_1^1 - \lambda = 0,\ z_1^2 + \lambda = 1.$$

It follows from this that $z_1^1 + z_1^2 = 1$. It is easy to see that vectors z_1, z_2 for which the mentioned conditions hold have the form

$$z_1 = (\alpha, 1 - \alpha),\ z_2 = (-\alpha, \alpha) \quad \text{with} \quad \alpha \in [0, 1]. \tag{2.1.10}$$

On the other hand each pair (z_1, z_2) such that (2.1.10) holds belongs to $\sigma(z)$. Thus

$$\sigma(z) = \{Z = ((\alpha, 1 - \alpha), (-\alpha, \alpha)) : \alpha \in [0, 1]\}.$$

Alternatively it follows from (2.1.8) and (2.1.9) that

$$\begin{aligned} \sigma(x) + \sigma(y) &= \{(x, 0) + (0, y)\} = \{(1, 0) + (-1, 1)\} = \{(0, 1)\} \\ &= \{(\alpha, 1 - \alpha), (-\alpha, \alpha)) : \alpha = 0)\}. \end{aligned}$$

Thus $\sigma(z) \neq \sigma(x) + \sigma(y)$.

Consider one more example.

Example 2.1.2. Again let us take space $E = \mathbb{R}^2$. Let $K_1 = \mathbb{R}^2_+$ be the positive orthant and let $K_2 = \{X = (u, v) \in \mathbb{R}^2 : u = 0, v = \lambda \geq 0\}$ be the ray passing through the point $T = (0, 1) \in \mathbb{R}^2$, i.e. (see Fig. 2.2). Note that $K_1 + K_2 = K_1$.

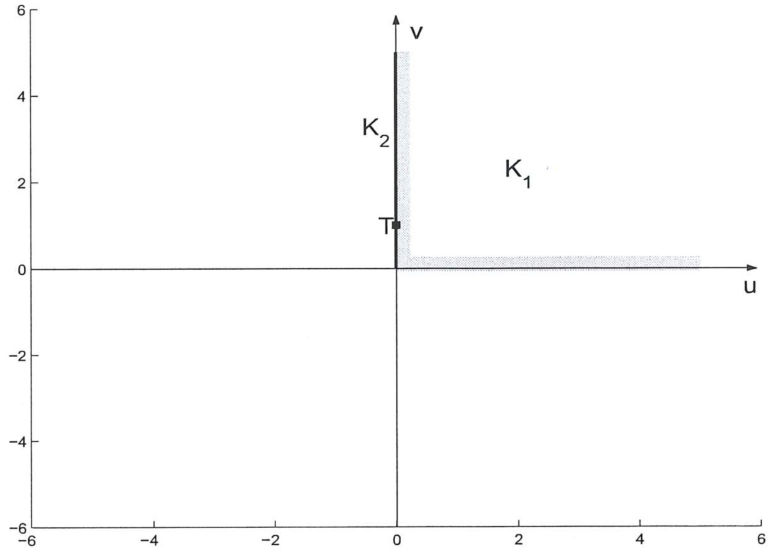

Figure 2.2: $E = (\mathbb{R}^2; K_1, K_2)$

Let $y \in K_2$. Then

$$\langle 0, y\rangle_{K_2} = \{x' \in K_1 : 0 \leq x' \leq y\} = \langle 0, y\rangle_{K_1}.$$

Note that for each $u, v \in \mathbb{R}^n_+ = K_1$ it holds that:

$$\langle 0, u\rangle_{K_1} + \langle 0, v\rangle_{K_1} = \langle 0, u+v\rangle_{K_1}.$$

Hence for $x \in K_1$ and $y \in K_2$ we have

$$\langle 0, x\rangle_{K_1} + \langle 0, y\rangle_{K_2} = \langle 0, x\rangle_{K_1} + \langle 0, y\rangle_{K_1} = \langle 0, y\rangle_{K_1+K_2}.$$

Thus the Riesz decomposition property holds in the space $(E; K_1, K_2)$.

Let K_1 be a cone and K_2 be a subcone of K_1. Recall, that K_2 is called a *face* of K_1, if the inclusions $x, y \in K_1$ and $x + y \in K_2$ imply $x, y \in K_2$.

In Example 2.1.2 the cone K_1 and its face K_2 were considered. As it will be shown in the following theorems, this situation can be considered in a more general case.

Theorem 2.1.4. *Assume that the double partition Lemma holds in the space $E = (E, K_1)$ and let a cone K_2 be a face of the cone K_1. Then the double partition Lemma is valid in the space $E = (E; K_1, K_2)$.*

Proof. Let $z_1 + z_2 = x + y$, where $x, y \in K_1 + K_2$, $z_1 \in K_1$, $z_2 \in K_2$. Since the double partition Lemma holds in the space $E = (E; K_1)$, there exist elements $x_1, x_2, y_1, y_2 \in K_1$ such that

$$z_1 = x_1 + y_1, \quad z_2 = x_2 + y_2, \quad x = x_1 + x_2, \quad y = y_1 + y_2.$$

As $x_2, y_2 \in K_1$, $z_2 = x_2 + y_2 \in K_2$ and the cone K_2 is a face of the cone K_1, then $x_2, y_2 \in K_2$, i.e. the double partition Lemma holds in the space $E = (E; K_1; K_2)$ with respect to the cones K_1 and K_2. □

Remark 2.1.6. Let the double partition lemma be valid in the space $(E; K_1, K_2)$ where K_2 is a face of K_1. We could not derive from this that this lemma holds in (E, K_1), hence the assertions: "the double partition lemma holds in $(E; K_1)$" and "the double partition lemma holds in $(E; K_1, K_2)$ are not equivalent".

Theorem 2.1.5. *Let the space $E = (E; H)$ possess the Riesz interpolation property, and let cones K_1, K_2 be faces of the cone H. Then the space $E = (E; K_1, K_2)$ possesses the Riesz interpolation property.*

Proof. Let $L = K_1 + K_2$ and elements $x_1, x_2, y_1, y_2 \in E$ satisfy the following relations:

$$y_1 \geq_{K_1} x_1, \quad y_1 \geq_L x_2, \quad y_2 \geq_{K_2} x_2, \quad y_2 \geq_L x_1.$$

Since $K_1, K_2, L \subset H$ then $y_i \geq_H x_j$, $i, j = 1, 2$.

As the space $E = (E; H)$ possesses the Riesz interpolation property, then there exists an element $c \in E$ such that

$$y_i \geq_H c \geq_H x_j, \ i, j = 1, 2,$$

i.e.

$$y_1 - c \in H, \quad y_2 - c \in H, \quad c - x_1 \in H, \quad c - x_2 \in H.$$

It follows from the inequality $y_1 \geq_{K_1} x_1$ that

$$y_1 - x_1 = (y_1 - c) + (c - x_1) \in K_1.$$

In the same manner the inequality $y_2 \geq_{K_2} x_2$ implies

$$y_2 - x_2 = (y_2 - c) + (c - x_2) \in K_2.$$

As K_1, K_2 are faces of the cone H, we have

$$y_1 - c, \ c - x_1 \in K_1, \quad y_2 - c, \ c - x_2 \in K_2,$$

i.e. $y_1 \geq_{K_1} c \geq_{K_1} x_1$, $y_2 \geq_{K_2} c \geq_{K_2} x_2$. Therefore, the space $E = (E; K_1, K_2)$ possesses the Riesz interpolation property. □

The definitions and results presented above can be easily extended to the case where the number of cones is greater than two. We will consider this only for the Riesz decomposition property. This property in the space $E = (E; K_1, \dots, K_n)$ can be expressed in the following form: if $x_i \in K_i$ $(i = 1, \dots, n)$ then

$$\langle 0, x_1 + x_2 + \dots + x_n \rangle_{K_1+K_2+\dots+K_n} = \langle 0, x_1 \rangle_{K_1} + \langle 0, x_2 \rangle_{K_2} + \dots + \langle 0, x_n \rangle_{K_n}.$$

Lemma 2.1.6. *Let the Riesz decomposition property hold for the space $(E; K_1, \dots, K_{n-1})$ with the cones $K_1, \dots K_{n-1}$. Let $K^{(1)} = K_1 + \dots + K_{n-1}$ and let the Riesz decomposition property hold for the space $(E, K^{(1)}, K_n)$ with the cones $K^{(1)}, K_n$. Then this property also holds for the space $(E; K_1, \dots, K_n)$ with the cones $K_1, \dots K_n$.*

Proof. We have for an arbitrary $x_i \in K_i$, $i = 1, \dots, n-1$:

$$\langle 0, x_1 + x_2 + \dots + x_{n-1} \rangle_{K_1+K_2+\dots+K_{n-1}} = \langle 0, x_1 \rangle_{K_1} + \langle 0, x_2 \rangle_{K_2} + \dots + \langle 0, x_{n-1} \rangle_{K_{n-1}}$$

and we also have for $y \in K^{(1)}$ and $x_n \in K_n$:

$$\langle 0, y + x_n \rangle_{K^{(1)}+K_n} = \langle 0, y \rangle_{K^{(1)}} + \langle 0, x_n \rangle_{K^n}.$$

Let $y = x_1 + x_2 + \dots + x_{n-1} \in K^{(1)}$. Since

$$K^{(1)} + K_n = K_1 + \dots + K_{n-1} + K_n$$

it follows that

$$\langle 0, y + x_n \rangle_{K^{(1)}+K_n} = \langle 0, x_1 + x_2 + \dots + x_n \rangle_{K_1+K_2+\dots+K_n}$$

and

$$\begin{aligned}\langle 0, y \rangle_{K^{(1)}} + \langle 0, x_n \rangle_{K_n} &= \langle 0, x_1 + x_2 + \dots + x_{n-1} \rangle_{K_1+K_2+\dots+K_{n-1}} + \langle 0, x_n \rangle_{K_n} \\ &= \langle 0, x_1 \rangle_{K_1} + \langle 0, x_2 \rangle_{K_2} + \dots + \langle 0, x_{n-1} \rangle_{K_{n-1}} + \langle 0, x_n \rangle_{K_n}.\end{aligned}$$

Thus the result follows. □

Using this lemma and induction we can easily extend all results that known for the Riesz decomposition property for the case of two cones, to the case of n cones. Definition of the Riesz interpolation property can be extended to the case of n-cones in a similar manner. We can also define in a similar way what it means for the double partition lemma to hold with respect to n cones and define the additivity of the decomposition mapping in this situation. Using induction it is easy to extend all results that were proved in this section for the case of two cones to the case of n cones.

2.2 A vector lattice with respect to several preorders

2.2.1 Supremum and Infimum in a space with several cones

Let cones $K_1, \dots, K_n$ be given in a vector space E. Let us introduce a partial order $\geq_{K_i}$, $i = 1, \dots, n$ on E by means of these cones putting for each i : $x \geq_{K_i} y$ or $y \leq_{K_i} x$, if $x - y \in K_i$ $(x, y \in E)$. As usual we denote this space by $E = (E; K_1, \dots, K_n)$.

Let us introduce the notions of supremum and infimum in the space $E = (E; K_1, \dots, K_n)$. We will need these notions only for sets of n elements so we give corresponding definitions only for such subsets of E.

Let $\{x_1, \dots, x_n\} \subset E = (E; K_1, \dots, K_n)$.

Definition 2.2.1. An element $u \in E = (E; K_1, \dots, K_n)$ is called an *infimum* of the set $\{x_1, \dots, x_n\}$ with respect to $K_1, \dots, K_n$, if

(i) $x_i \geq_{K_i} u$ for every $i = 1, 2, \dots, n$;

(ii) if an element $z \in E$ is such that $x_i \geq_{K_i} z$ for every $i = 1, 2, \dots, n$, then $u \geq_{K_i} z$, $i = 1, 2, \dots, n$.

We will denote an element with properties (i) and (ii) by $u = \mathrm{Inf}\{x_1, \dots, x_n\}$.

A supremum is defined in a similar way.

Definition 2.2.2. An element $v \in E = (E; K_1, \dots, K_n)$ is called a *supremum* of the set $\{x_1, \dots, x_n\}$ with respect to $K_1, \dots, K_n$, if

(i) $v \geq_{K_i} x_i$ for every $i = 1, 2, \dots, n$;

(ii) if an element $z \in E$ is such that $z \geq_{K_i} x_i$ for every $i = 1, 2, \dots, n$, then $z \geq_{K_i} v$, $i = 1, 2, \dots, n$.

We will denote an element with properties (i) and (ii) by $v = \mathrm{Sup}\{x_1, \dots, x_n\}$.

Remark 2.2.1. The suggested notation is not well defined since we use the same symbol for different elements. However this will not lead to misunderstanding; also starting from the page after next we will consider only cones $K_1, \dots, K_n$ such that each finite set $\{x_1, \dots, x_n\}$ can have no more than one $\mathrm{Inf}\{x_1, \dots, x_n\}$ and one $\mathrm{Sup}\{x_1, \dots, x_n\}$.

Remark 2.2.2. If all $K_1, \dots, K_n$ coincide, then the definitions of Inf and Sup coincide with the definitions of ordinary inf and sup for n elements.

Let us study the properties of these new objects.

Proposition 2.2.1. *Let*

$$(\bigcap_{i=1}^{n} K_i) \cap (-\bigcap_{i=1}^{n} K_i) = \{0\}. \tag{2.2.1}$$

Then each set $\{x_1, \dots, x_n\}$ cannot have more than one infimum and supremum with respect to $(K_1, \dots, K_n)$.

Proof. Assume that elements u and $u' \neq u$ are infimums of a set $x_1, \dots, x_n$ with respect to $K_1, \dots, K_n$. Since $x_i \geq_{K_i} u$ for every i and $u' = \operatorname{Inf}(x_1, \dots, x_n)$ we conclude that $u' \geq_{K_i} u$ for all i. Hence $u' - u \in K_i$ for all i. This means that $u' - u \in \bigcap_{i=1,\dots,n} K_i$. The same argument shows that $u - u' \in \bigcap_{i=1,\dots,n} K_i$, i.e. $u' - u \in -\bigcap_{i=1,\dots,n} K_i$. Since $(\bigcap_{i=1}^{n} K_i) \cap (-\bigcap K_i)_{i=1}^{n} = \{0\}$ it follows that $u = u'$. The same argument shows that a set $(x_1, \dots, x_n)$ cannot have more than one supremum with respect to $(K_1, \dots, K_n)$. □

The following example shows that if condition (2.2.1) does not hold then a set of n elements can have more than one infimum.

Example 2.2.1. Let $E = \mathbb{R}^3$ and

$$K_1 = \{x = (x^1, x^2, x^3) \in E : x^1 \geq 0\}, \qquad K_2 = \{x = (x^1, x^2, x^3) \in E : x^2 \geq 0\}.$$

Let $x_1 = (x_1^1, x_1^2, x_1^3)$ and $x_2 = (x_2^1, x_2^2, x_2^3)$ be a set of two elements. Let

$$u^1 := \min\{x_1^1, x_2^1\} \qquad u^2 := \min\{x_1^2, x_2^2\}.$$

Consider an element $u = (u^1, u^2, u^3)$, where u^3 is an arbitrary number. Then u is an infimum of (x_1, x_2) with respect to (K_1, K_2). Indeed, $x_1 \geq_{K_1} u$ and $x_2 \geq_{K_2} u$. It is clear that $x_i \geq_{K_i} z$ implies $u \geq_{K_i} z$, $i = 1, 2$. Thus the set (x_1, x_2) has an infinite set of infimums.

In the rest of this chapter we always assume that we consider infimum and supremum only with respect to a system $(K_1, \dots, K_n)$ of cones such that (2.2.1) holds:

$$(\bigcap_{i=1}^{n} K_i) \cap (-\bigcap K_i)_{i=1}^{n} = \{0\}.$$

We will now examine some simple properties of the Infimum and Supremum.

Theorem 2.2.2. *Let $x_i, y_i \in E = (E; K_1, \dots, K_n)$, $i = 1, 2, \dots, n$ and let there exist $Inf\{x_1; \dots; x_n\}$, $Sup\{x_1; \dots; x_n\}$, $Inf\{y_1; \dots; y_n\}$, $Sup\{y_1; \dots; y_n\}$ with respect to cones $K_1, \dots, K_n$.*

Then the following assertions are valid:

1. $Sup\{x_1; \dots; x_n\} \geq_{K_i} Inf\{x_1; \dots; x_n\}$, $(i = 1, 2, \dots, n)$;
2. *there exist* $Sup\{-x_1; \dots; -x_n\}$ *and* $Inf\{-x_1; \dots; -x_n\}$ *and*
$$Inf\{x_1; \dots; x_n\} = -Sup\{-x_1; \dots; -x_n\},$$
$$Sup\{x_1; \dots; x_n\} = -Inf\{-x_1; \dots; -x_n\};$$
3. *for every* $z \in E$ *there exist* $Sup\{x_1 + z; \dots; x_n + z\}$ *and* $Inf\{x_1 + z; \dots; x_n + z\}$ *and*
$$Inf\{x_1; \dots; x_n\} + z = Inf\{x_1 + z; \dots; x_n + z\},$$
$$Sup\{x_1; \dots; x_n\} + z = Sup\{x_1 + z; \dots; x_n + z\};$$
4. *for every* $\lambda > 0$ *there exist* $Inf\{\lambda x_1; \dots; \lambda x_n\}$ *and* $Sup\{\lambda x_1; \dots; \lambda x_n\}$ *and*
$$\lambda Inf\{x_1; \dots; x_n\} = Inf\{\lambda x_1; \dots; \lambda x_n\},$$
$$\lambda Sup\{x_1; \dots; x_n\} = Sup\{\lambda x_1; \dots; \lambda x_n\};$$
5. *for every* $\lambda \leq 0$ *there exist* $Sup\{\lambda x_1; \dots; \lambda x_n\}$ *and* $Inf\{\lambda x_1; \dots; \lambda x_n\}$ *and*
$$\lambda Inf\{x_1; \dots; x_n\} = Sup\{\lambda x_1; \dots; \lambda x_n\},$$
$$\lambda Sup\{x_1; \dots; x_n\} = Inf\{\lambda x_1; \dots; \lambda x_n\};$$
6. *if* $x_i \geq_{K_i} y_i$, $i = 1, 2, \dots, n$, *then*
$$Inf\{x_1; \dots; x_n\} \geq_{K_i} Inf\{y_1; \dots; y_n\},\ i = 1, 2, \dots, n,$$
$$Sup\{x_1; \dots; x_n\} \geq_{K_i} Sup\{y_1; \dots; y_n\},\ i = 1, 2, \dots, n.$$

Proof. Let $u = \mathrm{Inf}\{x_1; \dots; x_n\}$, $v = \mathrm{Sup}\{x_1; \dots; x_n\}$

1. The assertion follows immediately from the definitions:
$$\mathrm{Sup}\{x_1; \dots; x_n\} \geq_{K_i} x_i \geq_{K_i} \mathrm{Inf}\{x_1; \dots; x_m\},\ (i = 1, 2, \dots, n).$$

2. Since $x_i \geq_{K_i} u$, then $-u \geq_{K_i} -x_i$, $i = 1, \ldots, n$;

   Let an element $t \in E$ be such that

$$t \geq_{K_i} -x_i \quad \text{or} \quad x_i \geq_{K_i} -t, \ i = 1, 2, \ldots, n.$$

   From the definition of u we have $u \geq_{K_i} -t$ or $t \geq_{K_i} -u$, $i = 1, 2, \ldots, n$, i.e.

$$-u = \text{Sup}\{-x_1; \ldots; -x_n\}.$$

   In a similar way it can be shown that

$$\text{Sup}\{x_1; \ldots; x_n\} = -\text{Inf}\{-x_1; \ldots; -x_n\}.$$

3. Let $z \in E$. Then the inequality $x_i \geq_{K_i} u$ yields $x_i + z \geq_{K_i} u + z$, $i = 1, 2, \ldots, n$. If an element $v \in E$ is such that $x_i + z \geq_{K_i} v$ or $x_i \geq_{K_i} v - z$, $i = 1, 2, \ldots, n$, then the definition of u implies that $u \geq_{K_i} v - z$ or $u + z \geq_{K_i} v$, $i = 1, 2, \ldots, n$ i.e. $u + z = \text{Inf}\{x_1 + z; \ldots; x_n + z\}$.

   The second equality can be proved in a similar way.

4. The case when $\lambda = 0$ is obvious. Consider the case $\lambda > 0$.

   From $x_i \geq_{K_i} u$ we have $\lambda x_i \geq_{K_i} \lambda u$, $i = 1, 2, \ldots, n$. If an element $z \in E$ is such that $\lambda x_i \geq_{K_i} z$, $i = 1, 2, \ldots, n$ then $x_i \geq_{K_i} z/\lambda$, $i = 1, 2, \ldots, n$, and hence $u \geq_{K_i} z/\lambda$ or $\lambda u \geq_{K_i} z$, $i = 1, 2, \ldots, n$.

   Therefore $\lambda u = \text{Inf}\{\lambda x_1; \ldots; \lambda x_n\}$.

   The second equality can be proved by similar reasonings.

5. Let $\lambda \leq 0$. Applying 4. we conclude that

$$\text{Inf}\{|\lambda| x_1; \ldots; |\lambda| x_n\} = |\lambda| u, \ \text{Sup}\{|\lambda| x_1; \ldots; |\lambda| x_n\} = |\lambda| v.$$

   From 2. it follows that

$$\text{Sup}\{\lambda x_1; \ldots; \lambda x_n\} = -\text{Inf}\{|\lambda| x_1; \ldots; |\lambda| x_n\} = -|\lambda| u = \lambda u,$$

$$\text{Inf}\{\lambda x_1; \ldots; \lambda x_n\} = -\text{Sup}\{|\lambda| x_1; \ldots; |\lambda| x_n\} = -|\lambda| v = \lambda v.$$

6. Since $y_i \geq_{K_i} \text{Inf}\{y_1; \ldots; y_n\}$, $i = 1, \ldots, n$, then $x_i \geq_{K_i} \text{Inf}\{y_1; \ldots; y_n\}$, $i = 1, 2, \ldots, n$. The definition of the element u yields

$$u \geq_{K_i} \text{Inf}\{y_1; \ldots; y_n\}, \ i = 1, 2, \ldots, n.$$

   The opposite equality can be obtained in a similar way.

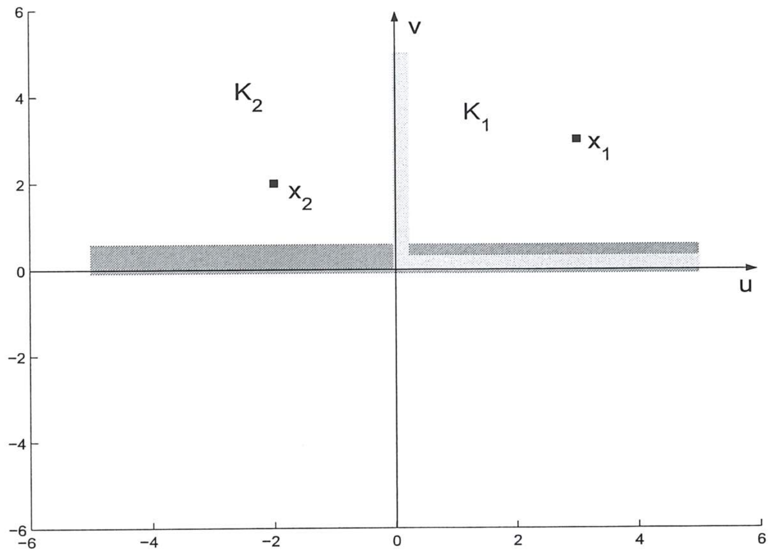

Figure 2.3: $E = (\mathbb{R}^2; K_1, K_2)$

□

In general, the operation Inf and Sup do not commute in the sense that $\mathrm{Inf}(x_1, x_2)$ is not necessarily equal to $\mathrm{Inf}(x_2, x_1)$ and $\mathrm{Sup}(x_1, x_2)$ is not necessarily equal to $\mathrm{Sup}(x_2, x_1)$. Consider an example

Example 2.2.2. In the cartesian plane $E = \mathbb{R}^2$ take the positive orthant $K_1 = \mathbb{R}^2_+$ and the upper half-plane

$$K_2 = \{(u, v) \in \mathbb{R}^2 \ : \ v \in \mathbb{R}_+\}$$

(see Fig. 2.3).

Consider the space $(E; K_1, K_2)$. Let $x_1 = (3, 3)$, $x_2 = (-2, 2) \in E$. Then $U := \mathrm{Inf}\{x_1; x_2\} = (3, 2)$ (see Fig.2.4).

Indeed, $x_1 - U = (0, 1) \in K_1$, $x_2 - U = (-5, 1) \in K_2$, so $x_1 \geq_{K_1} U$, $x_2 \geq_{K_2} U$.

Let an element $z = (z_1, z_2) \in E$ be such that $x_1 \geq_{K_1} z$ and $x_2 \geq_{K_2} z$, i.e.

$$3 \geq z_1, \quad 3 \geq z_2, \quad 2 \geq z_2.$$

This yields $3 \geq z_1$, $2 \geq z_2$, i.e. $U \geq_{K_1} z$ and $U \geq_{K_2} z$. So then $U = \mathrm{Inf}\{x_1; x_2\}$.

We will now show that $\mathrm{Inf}\{x_2; x_1\} = x_2$ (see Fig.2.5).

Indeed, $x_1 - x_2 = (5, 1) \in K_1$. Thus $x_1 \geq_{K_1} x_2$. We also have $x_2 \geq_{K_2} x_2$. The desired equality easily follows from this.

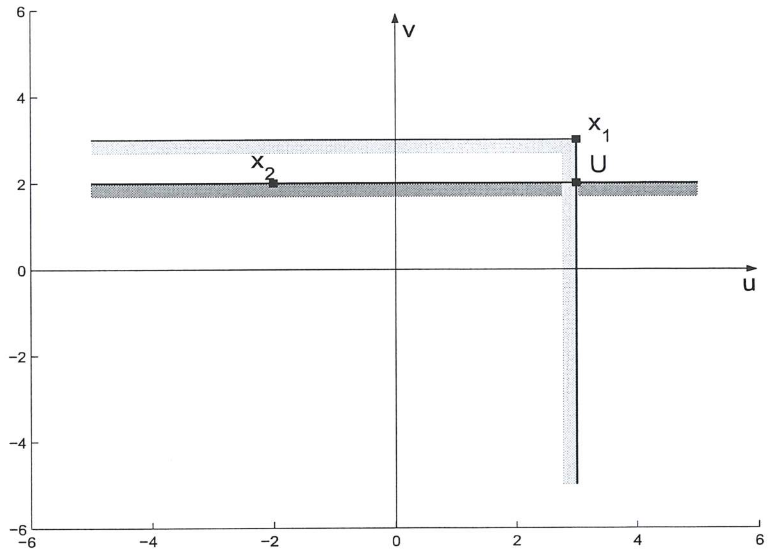

Figure 2.4: $u = \text{Inf}\{x_1; x_2\}$

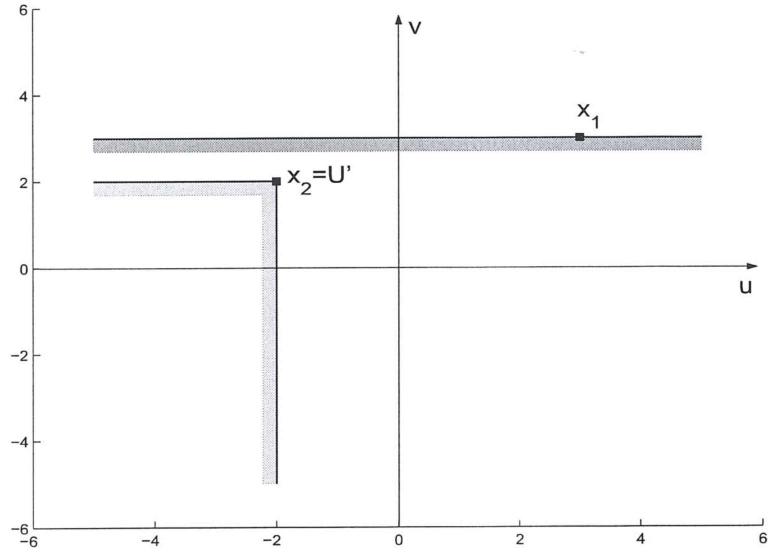

Figure 2.5: $U' = \text{Inf}\{x_2; x_1\}$

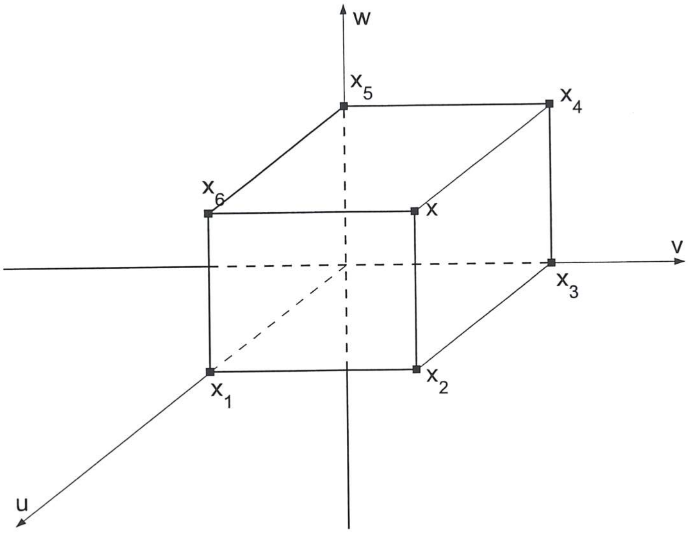

Figure 2.6: $E = (\mathbb{R}^3; \mathbb{R}^3_+)$, $< 0, x >_{\mathbb{R}^3_+}$

Therefore this simple example shows us that

$$\text{Inf}\{x_1; x_2\} \neq \text{Inf}\{x_2; x_1\}.$$

The operation Inf and Sup with respect to a system of cones can be useful for the description of some objects. We now present an interesting example. Consider a space (E, K_1) where $E = \mathbb{R}^n$ and $K_1 = \mathbb{R}^n_+$. Let $x \in \mathbb{R}^n_{++} = \text{int}\,\mathbb{R}^n_+$. Consider the conic segment $\langle 0, x\rangle_{K_1}$. This is a parallelepiped with 2^n vertices. One of these vertices is zero and one more of the vertices is x. We cannot describe other vertices of $\langle 0, x\rangle$ in terms of the order relation generated by the cone K_1.

We will now show that Inf operation allows, by choosing appropriate cones to "catch" other vertices of the parallelepiped $\langle 0, x\rangle$. Moreover for each of the vertices x_j there exists a cone H_j such that $x_j = \text{Inf}(x, 0)$ with respect to the pair of cones (K_1, H_j).

Example 2.2.3. Let $E = \mathbb{R}^n$ be the Euclidian space and $K_1 = \mathbb{R}^n_+$ be the positive orthant. Let $x = (x^1, \ldots, x^n) \in E$ be an element with positive coordinates: $x^i > 0$, $(i = 1, 2, \ldots, n)$. Then the set

$$\langle 0, x\rangle_{K_1} = \{y \in \mathbb{R}^n \ : \ x \geq_{K_1} y \geq_{K_1} 0\}$$

is an n-dimensional parallelepiped (see Fig. 2.6).

Let $k = 2^n$ be the number of vertices of $\langle 0, x\rangle_{K_1}$ and let $x_j = (x^1_j, \ldots, x^n_j)$ $(j = 1, 2, \ldots, k)$ be these vertices .

Let us introduce the index sets $I = \{1, 2, \ldots, n\}$ and let

$$I_j = \{i \in I \ : \ x_j^i = 0\}, \ j = 1, 2, \ldots, k.$$

Observe, that if $i \notin I_j$ then

$$x^i = x_j^i \quad (j \in I).$$

Consider the cone:

$$H_j = \{(y^1, \ldots, y^n) \in \mathbb{R}^n \ : \ y^i \in \mathbb{R}_+, \quad i \in I_j\}.$$

The following assertion holds:

Proposition 2.2.3. *The vertex x_j $(j = 1, \ldots, k)$ of the parallelepiped $\langle 0, x\rangle_{K_1}$ can be calculated as Inf$\{x; 0\}$ in the space $(E; K_1, H_j)$ $j = 1, \ldots, k$.*

Proof. Since $x_j = (x_j^1, \ldots, x_j^n) \in \langle 0, x\rangle_{K_1}$ $(j = 1, 2, \ldots, k)$ it follows that $x \geq_{K_1} x_j$, $j = 1, 2, \ldots, k$. From the construction of the set I_j and the cone H_j it is easy to see that $-x_j \in H_j$, i.e. $0 \geq_{H_j} x_j$.

Now let an element $z = (z^1, \ldots, z^n) \in E$ be such that

$$x \geq_{K_1} z, \qquad 0 \geq_{H_j} z.$$

Then $x^i \geq z^i$, $i \in I$ and $z^i \in -\mathbb{R}_+$, $i \in I_j$. Since $x^i = x_j^i$ for $i \notin I_j$ and $x_j^i \geq 0$ $(i \in I)$ then $x_j^i \geq z^i$ $(i \in N)$, i.e. $x_j \geq_{K_1} z$.

As $x_j^i - z^i \geq 0$ $(i \in I_j)$, then

$$x_j - z = (x_j - z, x_j - z, \ldots, x_j - z) \in H_j.$$

Thus we have proved that $x_j \geq_{K_1} z$, $x_j \geq_{H_j} z$. This means that $x_j = \mathrm{Inf}\{x; 0\}$ (with respect to the pair of cones K_1, H_j). □

In the following, unless otherwise indicated, we will consider the case where the number of cones is equal to two.

Let cones K_1 and K_2 be given in a space E. Let us introduce new definitions.

Definition 2.2.3. We say that a pair of cones K_1 and K_2 *generates* a space E, if $E = K_1 - K_2$.

It is clear that $E = K_1 - K_2$ if and only if $E = K_2 - K_1$.

Definition 2.2.4. A set $\Omega \subset E = (E; K_1, K_2)$ is called *bounded from above (below)* if an element $u \in E$ exists such that $u \geq_{K_i} x$ ($x \geq_{K_i} u$, respectively), $i = 1, 2$ for all $x \in \Omega$.

Observe that the following simple proposition holds.

Proposition 2.2.4. *1) If for each $x \in E$ the two-element subset $\{0, x\}$ is bounded from below then a pair of cones K_1 and $-K_2$ generates the space E.*

2) If for each $x \in E$ the subset $\{0, x\}$ is bounded from above then a pair of cones K_1, K_2 generates the space E.

Proof. 1) If the two-element set $\{x, 0\}$ is bounded from below, then there exists $u \in E$ such that

$$x \geq_{K_1} u, \qquad 0 \geq_{K_2} u,$$

i.e. $x - u \in K_1,\ u \in -K_2$.

Then the element x can be represented in the form $x = (x - u) + u \in K_1 + K_2 = K_1 - (-K_2)$ and since x is an arbitrary element, we obtain $E = K_1 - (-K_2)$.

2) Let $\{x, 0\}$ be bounded from above, then there exists $u \in E$ such that

$$u \geq_{K_1} x, \qquad u \geq_{K_2} 0,$$

i.e. $x - u \in -K_1,\ u \in K_2$.

Then the element x can be represented in the form $x = x - u + u \in -K_1 + K_2$ and since x is an arbitrary element, we obtain $E = K_2 - K_1$. □

Proposition 2.2.5. *Assume that the cone $H := K_1 \cap K_2$ is generating. Then for each $x, y \in E$ the set $\{x, y\}$ is bounded from above and from below.*

Proof. Let $x, y \in E$. Since $E = H - H$ it follows that there exists $x_1, y_1 \in H$, $x_2, y_2 \in H$ such that $x = x_1 - x_2$, $y = y_1 - y_2$. This means that $x \leq_H x_1$, $y \leq_H y_1$. We have $x \leq_H x_1 \leq_H x_1 + y_1$ and $y \leq_H y_1 \leq_H x_1 + y_1$. Since $K_1 \supset H$, $K_2 \supset H$ it follows that $x \leq_{K_1} x_1 + y_1$, $y \leq_{K_2} x_1 + y_1$. Thus $\{x, y\}$ is bounded from above. A similar argument shows that this set is bounded from below. □

2.2.2 2-vector lattices

Definition 2.2.5. A space $E = (E; K_1, K_2)$ is called a *2-lower (upper) vector semilattice*, if for any two elements $x_1, x_2 \in E$ there exists $\mathrm{Inf}\{x_1, x_2\}$ ($\mathrm{Sup}\{x_1, x_2\}$, respectively) in the space $E = (E; K_1, K_2)$.

Definition 2.2.6. A space $E = (E; K_1, K_2)$ is called a *2-vector lattice*, if for any two elements $x_1, x_2 \in\subset E$ there exist $\text{Inf}\{x_1, x_2\}$ and $\text{Sup}\{x_1, x_2\}$ in the space $E = (E; K_1, K_2)$.

The following proposition contains an example of a 2-vector lattice.

Proposition 2.2.6. *Let* $E = \mathbb{R}^n$, $K_1 = \mathbb{R}^n_+$, $K_2 = \{x \in \mathbb{R}^n : x^1 \geq 0\}$. *Then* $(E; K_1, K_2)$ *is a 2-vector lattice.*

Proof. Let $x \in \mathbb{R}^n, y \in \mathbb{R}^n$. First we will show that there exists $\text{Sup}(x, y)$. Consider separately the two cases.

1) $x^1 \geq y^1$. Then $x \geq_{K_1} x$ and $x \geq_{K_2} y$. It easily follows from this that $x = \text{Sup}(x, y)$.

2) $x^1 \leq y^1$. Let $u = (y^1, x^2, \dots, x^n)$. Then $u \geq_{K_1} x$ and $u \geq_{K_2} y$. It is easy to establish that $z \geq_{K_1} x$, $z \geq_{K_2} y$ implies $z \geq_{K_1} u$, $z \geq_{K_2} u$. Hence $u = \text{Sup}(x, y)$.

Now we establish that $\text{Inf}(x, y)$ exists.

1) Let $x^1 \leq y^1$. Since $x \leq_{K_1} x$ and $x \leq_{K_2} y$ it follows that $x = \text{Inf}(x, y)$.

2) Let $x^1 \geq y^1$. Let $v = (y^1, x^2, \dots x^n)$. Then $x \leq_{K_1} v$, $y^1 \leq_{K_2} v$. It is easy to establish that $v = \text{Inf}(x, y)$. □

Example 2.2.4. The space (E, K_1, K_2) with $E = \mathbb{R}^2, K_1 = -\mathbb{R}^2_+$ and $K_2 = \{x = (x^1, x^2) : x^2 \leq 0\}$ is depicted on Fig. 2.7. It is clear that $\text{Sup}(x, y)$ in (E, K_1, K_2) coincides with the $\text{Inf}(x, y)$ in $(E, -K_1, -K_2)$; here $-K_1 = \mathbb{R}^2_+$, $-K_2 = \{x = (x^1, x^2) : x^2 \geq 0\}$. Let $x = (4, -2)$, $y = (2, 1)$. (See Fig 2.7.) Then $V = \text{Sup}(x, y) = (2, -2)$ (see Fig. 2.8) and $U = \text{Inf}(x, y) = x = (4, -2)$ (see Fig. 2.9).

We will now present more complicated examples of a 2-vector lattice.

Proposition 2.2.7. *Let* (S, Σ, μ) *be a measure space and* $E = L^p(S, \Sigma, \mu)$ *with* $0 \leq p \leq +\infty$. *Assume that E is equipped with the natural order relation* $(x \geq y \iff x(s) \geq y(s)$ *a.e.)* . *Let* K_1 *be the cone of nonnegative on S functions* $x \in E$. *Let* $B \in \Sigma$ *and* $K_2 = \{x \in E : x(s) \geq 0,\ s \in B\}$ *be the cone of nonnegative on B functions. Then*

1) *the space* (E, K_1, K_2) *is a 2-vector lattice; if* $x, y \in E$ *then* $Sup(x, y) = v$ *and* $Inf(x, y) = u$, *where*

$$v(s) = \begin{cases} \sup(x(s), y(s)) & s \in B \\ x(s) & s \in S \setminus B; \end{cases} \tag{2.2.2}$$

$$u(s) = \begin{cases} \inf(x(s), y(s)) & s \in B \\ x(s) & s \in S \setminus B. \end{cases} \tag{2.2.3}$$

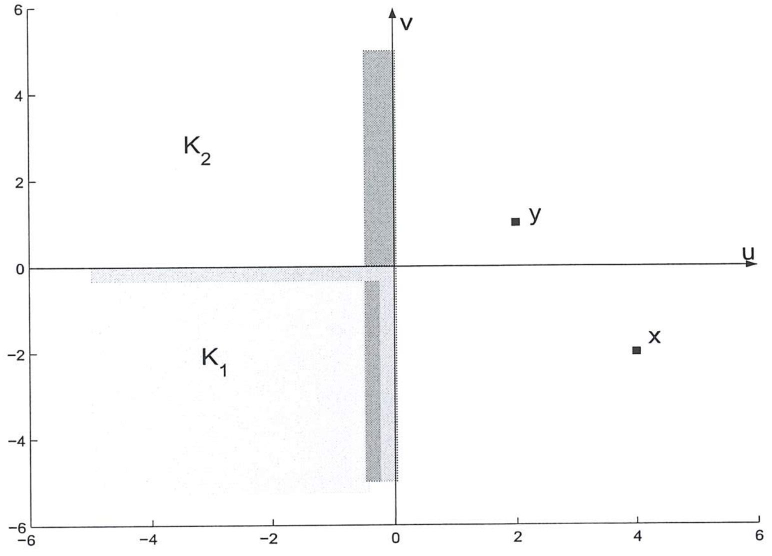

Figure 2.7: $E = (\mathbb{R}^2; K_1, K_2)$

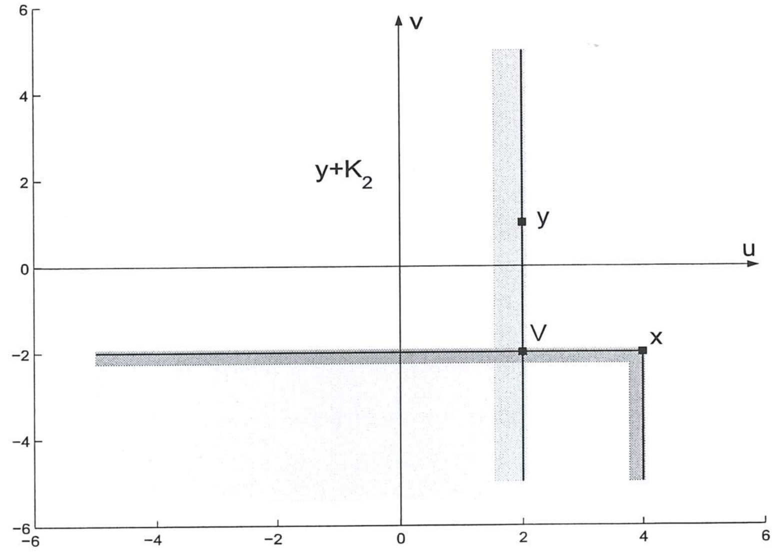

Figure 2.8: $V = \text{Sup}\{x; y\}$

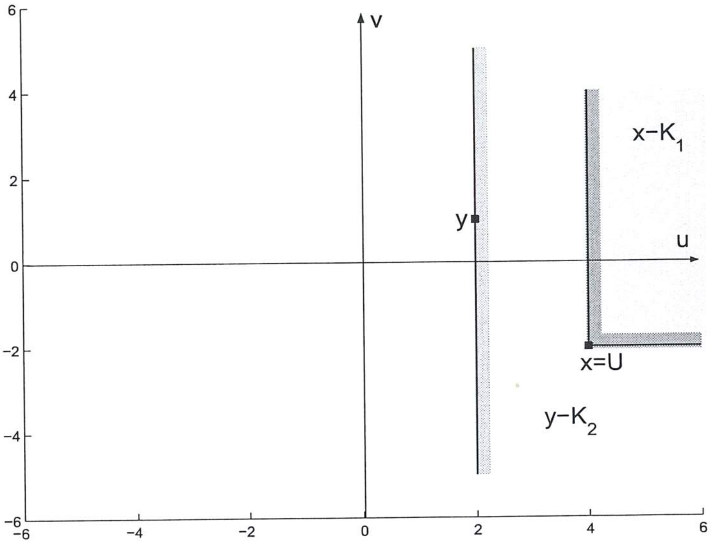

Figure 2.9: $U = \text{Inf}\{x; y\}$

2) the space (E, K_2, K_1) *is a 2-vector lattice; if* $x, y \in E$ *then* $Sup(x, y) = v'$ *and* $Inf(x, y) = u'$, *where*

$$v'(s) = \begin{cases} \sup(x(s), y(s)) & s \in B \\ y(s) & s \in S \setminus B; \end{cases} \tag{2.2.4}$$

$$u'(s) = \begin{cases} \inf(x(s), y(s)) & s \in B \\ y(s) & s \in S \setminus B. \end{cases} \tag{2.2.5}$$

Proof. 1) Let $x, y \in E$. We will prove that v defined by (2.2.2) coincides with $\text{Sup}(x, y)$ in (E, K_1, K_2). First we will show that $v \geq_{K_1} x$. Indeed, $v(s) \geq x(s)$ for $s \in B$ and $v(s) = x(s)$ for $s \in S \setminus B$, hence $v \geq_{K_1} x$. Since $v(s) \geq y(s)$ for $s \in B$, it follows that $v \geq_{K_2} y$. Now let $z \geq_{K_1} x$ and $z \geq_{K_2} y$. Then $z(s) \geq x(s)$ for all $s \in S$ and $z(s) \geq y(s)$ for $s \in B$, hence $z \geq_{K_1} v$ and $z \geq_{K_2} y$.

The same argument shows that the function u defined by (2.2.3) is equal to $\text{Inf}(x, y)$ in (E, K_1, K_2).

2) Let $x, y \in E$ and let v' be defined by (2.2.4). Then $v'(s) \geq x(s)$ for $s \in B$ and $v'(s) \geq y(s)$ for all $s \in S$, hence $v' \geq_{K_2} x$ and $v' \geq_{K_1} y$. It is easy to check that $(z \geq_{K_2} x, z \geq_{K_1} y) \implies (z \geq_{K_2} v',\ z \geq_{K_1})$, so $v' = \text{Sup}(x, y)$ in (E, K_2, K_1). The same argument shows that $u' = \text{Inf}(x, y)$ in (E, K_2, K_1). □

Proposition 2.2.8. *Let (S, Σ, μ) be a measure space and $E = L^p(S, \Sigma, \mu)$ with $0 \leq p \leq +\infty$. Let $B_1 \in \Sigma$ and $B_2 = S \setminus B_1$. Consider the cones*

$$K_1 = \{x \in E : x(s) \geq 0,\ s \in B_1\}, \quad K_2 = \{x \in E : x(s) \geq 0,\ s \in B_2\}.$$

Then $(E; K_1, K_2)$ is 2 vector lattice and for each $x, y \in E$ we have

$$Sup(x, y) = Inf(x, y) = \begin{cases} x(s) & s \in B_1 \\ y(s) & s \in B_2. \end{cases}$$

Proof. The proof follows immediately from the definitions of Sup and Inf. □

It follows from Proposition 2.2.8 that in 2- vector lattices the equality $\mathrm{Inf}(x, y) = \mathrm{Sup}(x, y)$ can be valid for $x \neq y$. Of course this is impossible in classical lattices.

Theorem 2.2.9. *Let $E = (E; K_1, K_2)$ be a 2-vector lattice. Then for any $x_1, x_2 \in E$ the equalities*

$$x_1 + x_2 = Inf\{x_1; x_2\} + Sup\{x_2; x_1\} = Inf\{x_2; x_1\} + Sup\{x_1; x_2\} \tag{2.2.6}$$

hold.

Proof. Let $x_1, x_2 \in E$, then Item 3. of Theorem 2.2.2 yields

$$\mathrm{Sup}\{x_2; x_1\} - x_1 - x_2 = \mathrm{Sup}\{x_2 - x_1 - x_2; x_1 - x_1 - x_2\} = \mathrm{Sup}\{-x_1; -x_2\}.$$

Item 2. of the same theorem implies that

$$\mathrm{Sup}\{-x_1; -x_2\} = -\mathrm{Inf}\{x_1; x_2\}.$$

Hence, $x_1 + x_2 = \mathrm{Inf}\{x_1; x_2\} + \mathrm{Sup}\{x_2; x_1\}$.

Similarly, since

$$\mathrm{Sup}\{x_1; x_2\} - x_2 - x_1 = \mathrm{Sup}\{x_1 - x_2 - x_1; x_2 - x_2 - x_1\} = \mathrm{Sup}\{-x_2; -x_1\},$$

and

$$\mathrm{Sup}\{-x_2; -x_1\} = -\mathrm{Inf}\{x_2; x_1\},$$

then

$$x_1 + x_2 = \mathrm{Inf}\{x_2; x_1\} + \mathrm{Sup}\{x_1; x_2\}.$$

□

Let a space $E = (E; K_1, K_2)$ be a 2-vector lattice.

Now let us introduce the following definitions.

Definition 2.2.7. The elements

$$x'_+ = \mathrm{Sup}\{0; x\}, \quad x'_- = -\mathrm{Inf}\{x; 0\}$$

are called *the positive and the negative parts* of an element $x \in E = (E; K_1, K_2)$ with respect to a pair of cones (K_1, K_2).

It follows from the definition of Sup and Inf that $x'_+ \geq_{K_1} 0$ and $-x'_- \leq_{K_2} 0$, hence $x'_+ \in K_1$ and $x'_- \in K_2$.

Definition 2.2.8. The elements

$$x''_+ = \mathrm{Sup}\{x; 0\} \in K_2, \quad x''_- = -\mathrm{Inf}\{0; x\} \in K_1$$

are called *the positive and the negative parts* of an element $x \in E = (E; K_1, K_2)$ with respect to a pair of cones (K_2, K_1).

Put

$$|x|' = x'_+ + x'_-, \quad |x|'' = x''_+ + x''_-.$$

We have $|x|' \in L$, $|x|'' \in L$, where $L = K_1 + K_2$.

2.2.3 Absolute value in 2-vector lattices

Definition 2.2.9. The quantity

$$|x| = \frac{|x|' + |x|''}{2}$$

is called *the absolute value* of an element $x \in E = (E; K_1, K_2)$ in a 2-vector lattice.

These new notions will be illustrated by the following examples.

Example 2.2.5. Let $E = \mathbb{R}^2$ be the cartesian plane, $K_1 = \mathbb{R}^2_+$ be the positive orthant, and $K_2 = \{x = (x^1, x^2) \in \mathbb{R}^2 \ : \ x^1 \in \mathbb{R}_+\}$ be the right half-plane (see Fig. 2.10).

Consider the point $x = (-2, -2) \in E$. It can be shown (compare with the proof of Proposition 2.2.6) that (see Figs. 2.11, 2.12, 2.13, 2.14)

$$U = \mathrm{Inf}\{x; 0\} = (-2, -2) \text{ and } x'_- = -U = (2, 2); \quad x'_+ = \mathrm{Sup}\{0; x\} = (0, 0);$$

$$U' = \mathrm{Inf}\{0; x\} = (-2, 0) \text{ and } x''_- = -U' = (2, 0); \quad x''_+ = \mathrm{Sup}\{x; 0\} = (0, -2).$$

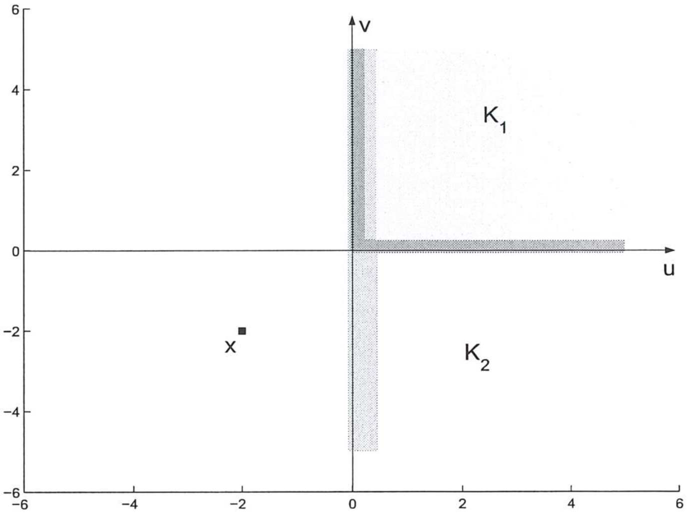

Figure 2.10: $x \in E = (\mathbb{R}^2; K_1, K_2)$

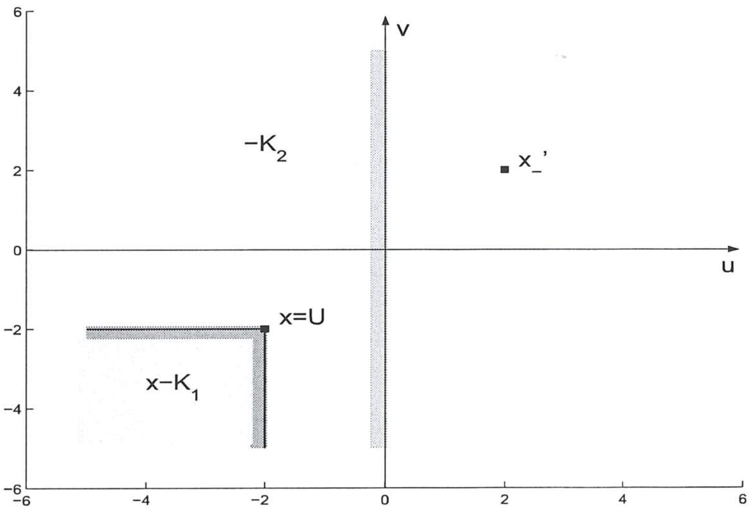

Figure 2.11: $U = \operatorname{Inf}\{x, 0\} \in E,\ x'_{-} \in K_2$

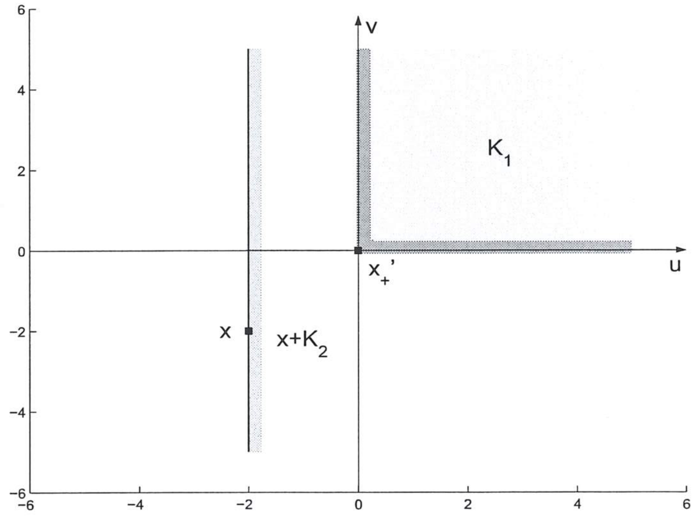

Figure 2.12: $x'_+ = \mathrm{Sup}\{0; x\} \in K_1$

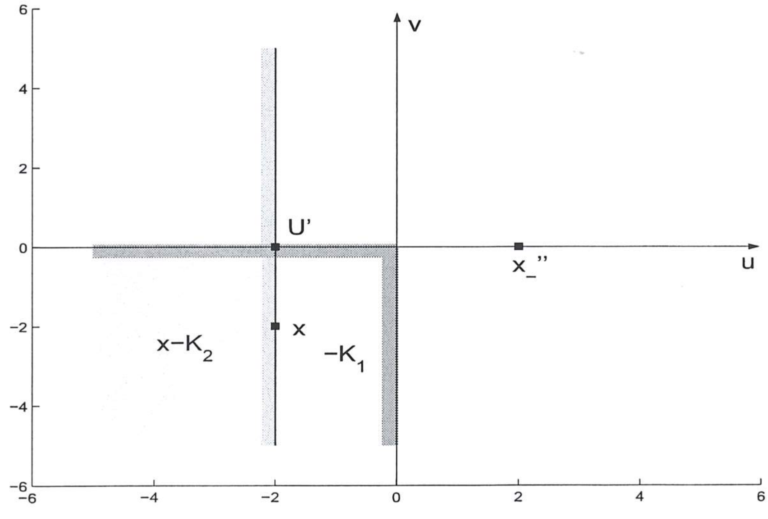

Figure 2.13: $U' = \mathrm{Inf}\{0, x\} \in E,\ x''_- \in K_1$

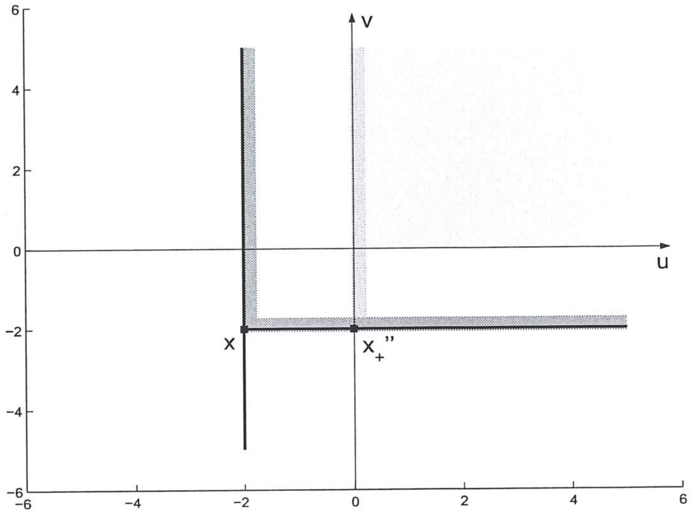

Figure 2.14: $x''_+ = \text{Sup}\{x; 0\} \in K_2$

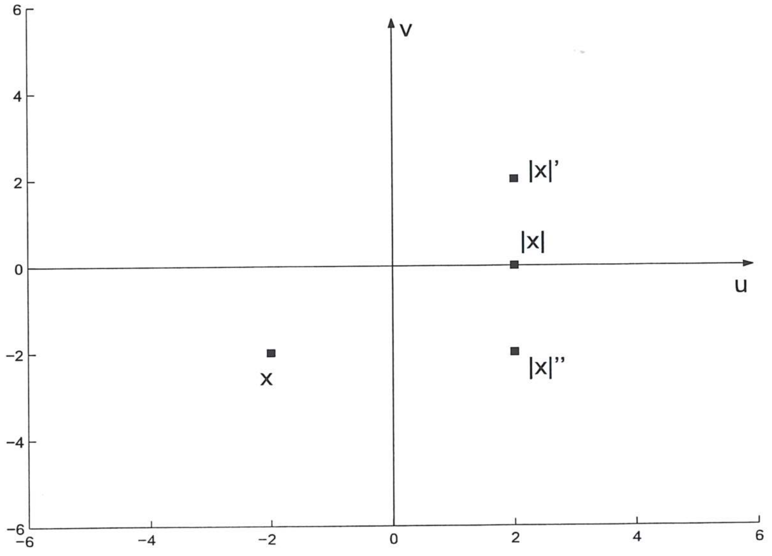

Figure 2.15: $x,\ |x|',\ |x|'' \in E$

We have $|x|' = x_+ + x_- = (2,2)$, $|x|'' = x''_+ + x''_- = (2,-2)$

$$|x| = \frac{(|x|' + |x|'')}{2} = (2,0) \quad \text{(see Fig. 2.15)}.$$

Example 2.2.6. Let (S, Σ, μ) be a measure space and let $E = L^p(S, \Sigma, \mu)$. Consider the space (E, K_1, K_2) where $K_1 = \{x \in E : x(s) \geq 0, \text{a.e } s \in S\}$, $K_2 = \{x \in E : x(s) \geq 0 : x(s) \geq 0, \text{a.e. } s \in B\}$, where $B \in \Sigma$. Let $u \in E$. Then

$$|u|(s) = \begin{cases} |u(s)| & s \in B \\ 0 & s \notin B. \end{cases}$$

Indeed, applying Proposition 2.2.7, we have

$$u'_+(s) := \text{Sup}(0, u) = \begin{cases} \sup(0, u(s)) & s \in B \\ 0 & s \notin B. \end{cases}$$

$$u'_-(s) : -\text{Inf}(u, 0) = \begin{cases} -\inf(u(s), 0) & s \in B \\ -u(s) & s \notin B. \end{cases}$$

Therefore

$$|u|'(s) := u'_+(s) + u'_-(s) = \begin{cases} |u(s)| & s \in B \\ -u(s) & s \notin B. \end{cases}$$

$$u''_+(s) := \text{Sup}(u, 0) = \begin{cases} \sup(u(s), 0) & s \in B \\ u(s) & s \notin B. \end{cases}$$

$$u''_-(s) := \text{Inf}(0, u) = \begin{cases} -\inf(u(s), 0) & s \in B \\ 0 & s \notin B. \end{cases}$$

Therefore

$$|u|''(s) =:= u''_+(s) + u''_-(s) = \begin{cases} |u(s)| & s \in B \\ 0 & s \notin B. \end{cases}$$

Finally we have

$$|u|(s) : \frac{1}{2}(|u|' + |u|'') = \begin{cases} |u(s)| & s \in B \\ 0 & s \notin B. \end{cases}$$

Example 2.2.7. Let $E = L^p(S, \Sigma, \mu)$, $B_1 \in \Sigma, B_2 = S \setminus \Sigma$, $K_1 = \{x \in E : x(s) \geq 0,\ s \in B_1\}$, $K_2 = \{x \in E : x(s) \geq 0,\ s \in B_2\}$. Then (see Proposition 2.2.8) we have for $x, y \in E$:

$$\text{Sup}(x,y)(s) = \text{Inf}(x,y)(s) = \begin{cases} x(s) & s \in B_1 \\ y(s) & s \in B_2. \end{cases}$$

It follows from this that

$$x'_+(s) = \mathrm{Sup}(0,x)(s) = \begin{cases} 0 & s \in B_1 \\ x(s) & s \in B_2. \end{cases}$$

$$x'_-(s) = -\mathrm{Inf}(x,0)(s) = \begin{cases} -x(s) & s \in B_1 \\ 0 & s \in B_2. \end{cases}$$

$$x''_+(s) = \mathrm{Sup}(x,0)(s) = \begin{cases} x(s) & s \in B_1 \\ 0 & s \in B_2. \end{cases}$$

$$x''_-(s) = -\mathrm{Inf}(0,x)(s) = \begin{cases} 0 & s \in B_1 \\ -x(s) & s \in B_2. \end{cases}$$

Then

$$|x|'(s) = x'_+(s) + x'_-(s) = \begin{cases} -x(s) & s \in B_1 \\ x(s) & s \in B_2 \end{cases}$$

$$|x|''(s) = x''_+(s) + x''_-(s) = \begin{cases} x(s) & s \in B_1 \\ -x(s) & s \in B_2 \end{cases}$$

Hence

$$|x| = 0 \quad \text{for all} \quad x \in E.$$

Now let us study the properties of these new objects.

Theorem 2.2.10. *Let $E = (E; K_1, K_2)$ be a 2-vector lattice, and let $x, y \in E$. Then*

1. $x = x'_+ - x'_- = x''_+ - x''_-$;
2. $|x|' = Sup\{-x; x\}$, $|x|'' = Sup\{x; -x\}$,

$$|x| = Sup\{0; x\} + Sup\{0; -x\} = Sup\{x; 0\} + Sup\{-x; 0\} \tag{2.2.7}$$

and $|x| \geq_{K_i} 0$, $i = 1, 2$;

3. $|-x| = |x|$.

If at least one of the cones K_1, K_2 is a pointed cone then

4. $Inf\{x'_-; x'_+\} = 0$, $Inf\{x''_-; x''_+\} = 0$;
5. $|x|' = Sup\{x'_-; x'_+\}$, $|x|'' = Sup\{x''_+; x''_-\}$,

If both K_1 and K_2 are pointed then

6. $|x| = 0$ *if and only if* $x = 0$;

Proof. 1. By substituting $x_2 = 0$ in (2.2.6), we obtain the required result.

2. Taking into account Item 1. of the current theorem and Item 2. of Theorem 2.2.2 we have

$$|x|' = x'_+ + x'_- = (x'_+ - x'_-) + (x'_- + x'_-) = x + 2x'_- =$$
$$= x - \text{Inf}\{2x; 0\} = x + \text{Sup}\{-2x; 0\} = \text{Sup}\{-x; x\}.$$

Similarly

$$|x|'' = x''_+ + x''_- = (x''_+ - x''_-) + (x''_- + x''_-) = x + 2x''_- =$$
$$= x - \text{Inf}\{0; 2x\} = x + \text{Sup}\{0; -2x\} = \text{Sup}\{x; -x\}.$$

Thus

$$\begin{aligned} |x| &= \frac{|x|' + |x|''}{2} = \frac{\text{Sup}\{-x; x\} + \text{Sup}\{x; -x\}}{2} \\ &= \frac{(\text{Sup}\{-x; x\} + x) + (-x + \text{Sup}\{x; -x\})}{2} = \frac{\text{Sup}\{0; 2x\} + \text{Sup}\{0; -2x\}}{2} \\ &= \text{Sup}\{0; x\} + \text{Sup}\{0; -x\} \geq_{K_1} 0, \end{aligned}$$

and

$$\begin{aligned} |x| &= \frac{(\text{Sup}\{-x; x\} - x) + (x + \text{Sup}\{x; -x\})}{2} \\ &= \frac{\text{Sup}\{-2x; 0\} + \text{Sup}\{2x; 0\}}{2} \\ &= \text{Sup}\{-x; 0\} + \text{Sup}\{x; 0\} \geq_{K_2} 0. \end{aligned}$$

3. It is obvious.

4. Let $u = \text{Inf}\{x'_+; x'_-\}$. Since $x'_+ \geq_{K_1} 0$, $x'_- \geq_{K_2} 0$, then $u \geq_{K_i} 0$, $i = 1, 2$. Let $z_1 = x'_+ - u$, $z_2 = x'_- - u$. It follows from the definition of Inf that $z_1 \geq_{K_1} 0$ and $z_2 \geq_{K_2} 0$. Item 1. of the current Theorem yields $x = z_1 - z_2$, and therefore $z_1 \geq_{K_2} x$. Since also $z_1 \geq_{K_1} 0$ we have $z_1 \geq_{K_i} \text{Sup}\{0; x\} = x'_+$, $i = 1, 2$.

   The latter inequality implies that $u = x'_+ - z_1 \in -K_i$, $i = 1, 2$. Since at least one of the cones K_1 and K_2 is pointed it follows that $u = 0$.

   The second equality can be deduced by similar reasoning.

5. Theorem 2.2.9 and Item 4. of the current theorem yield

$$|x|' = x'_+ + x'_- = \text{Sup}\{x'_-; x'_+\} + \text{Inf}\{x'_+; x'_-\} = \text{Sup}\{x'_-; x'_+\},$$
$$|x|'' = x''_+ + x''_- = \text{Sup}\{x''_+; x''_-\} + \text{Inf}\{x''_+; x''_-\} = \text{Sup}\{x''_+; x''_-\}.$$

6. Let $|x| = 0$. Applying (2.2.7) we have $\text{Sup}\{x;0\} + \text{Sup}\{-x;0\} = 0$, therefore $\text{Inf}(x,0) = -\text{Sup}\{-x;0\} = \text{Sup}\{x;0\}$. We have

$$x \geq_{K_1} \text{Inf}\{x;0\} = \text{Sup}\{x;0\} \geq_{K_1} x.$$

Since K_1 is a pointed cone, then $x = \text{Inf}\{x;0\} = \text{Sup}\{x;0\}$. It follows from this that $x \leq_{K_2} 0$ and $x \geq_{K_2} 0$. Since K_2 is a pointed cone, we have $x = 0$. The proof of assertion $x = 0 \implies |x| = 0$ is trivial.

□

Proposition 2.2.11. *Consider a space (E, K_1, K_2) such that the cone $H = K_1 \cap K_2$ is a generating cone. Assume that for each $h_1, h_2 \in H$ there exists Inf(h_1, h_2) and Sup(h_1, h_2) in the space of (E, K_1, K_2). Then (E, K_1, K_2) is a 2-vector lattice.*

Proof. Let $x, y \in E$. Since H is a generating cone, then (see Proposition 1.1.1) the set $\{x, y\} \subset (E, H)$ is bounded from below, i.e. there exists an element $z \in E$ such that $x, y \geq_H z$. This means that $x - z \in H$, $y - z \in H$ so there exists $u = \text{Inf}(x - z, y - z)$ in the space (E, K_1, K_2) . The result follows now from Theorem 2.2.2, Item 3. A similar argument shows that there exists $\text{Sup}(x, y)$. □

Now we consider Inf and Sup in a 2-vector lattice $E = (E; K_1, K_2)$ as operators acting from the space E^2 to the space E.

We need the following definitions. Let G be a vector space. An operator $A : G \to (E, K_1, K_2)$ is called *sublinear* if A is positively homogeneous ($A(\lambda x) = \lambda A(x)$ for all $x \in G$ and $\lambda > 0$) and subadditive: for each $x_1, x_2 \in G$ it holds:

$$A(x_1 + x_2) \leq_{K_i} A(x_1) + A(x_2), \qquad i = 1, 2.$$

An operator $A : G \to (E, K_1, K_2)$ is called *superlinear* if A is positively homogeneous and superadditive: if for each $x_1, x_2 \in G$ it holds:

$$A(x_1 + x_2) \geq_{K_i} A(x_1) + A(x_2), \qquad i = 1, 2.$$

Theorem 2.2.12. *Consider operators $P : E^2 \to E$ and $Q : E^2 \to E$, where*

$$P(X) = \mathit{Inf}\{x_1; x_2\}, \; Q(X) = \mathit{Sup}\{x_1; x_2\}, \text{ where } X = (x_1, x_2) \in E^2.$$

Then P is a superlinear operator and Q is a sublinear one.

Proof. Both operators P and Q are positively homogeneous (see Theorem 2.2.2, Item 4.). So we need only to prove that P is superadditive and Q is subaddive.

We will start with P. Let $X^1 = (x_1^1, x_2^1) \in E^2$, $X^2 = (x_1^2, x_2^2) \in E^2$. Then

$$P(X^1) = \mathrm{Inf}\{x_1^1; x_2^1\}, \quad P(X^2) = \mathrm{Inf}\{x_1^2; x_2^2\},$$

$$P(X^1 + X^2) = \mathrm{Inf}\{x_1^1 + x_1^2; x_2^1 + x_2^2\}.$$

By the definition of Inf we have

$$x_1^1 \geq_{K_1} P(X^1), \quad x_2^1 \geq_{K_2} P(X^1), \quad x_1^2 \geq_{K_1} P(X^2), \quad x_2^2 \geq_{K_2} P(X^2).$$

Therefore

$$x_1^2 + x_1^2 \geq_{K_1} P(X^1) + P(X^2), \; x_2^1 + x_2^2 \geq_{K_2} P(X^1) + P(X^2).$$

From the definition of Inf, we obtain

$$\mathrm{Inf}\{x_1^1 + x_1^2; x_2^1 + x_2^2\} \geq_{K_i} P(X^1) + P(X^2), \; i = 1, 2,$$

or

$$P(X^1 + X^2) \geq_{K_i} P(X^1) + P(X^2), \; i = 1, 2.$$

Consider now the operator Q.

Let

$$Q(X^1) = \mathrm{Sup}\{x_1^1; x_2^1\}, \qquad Q(X^2) = \mathrm{Sup}\{x_1^2; x_2^2\},$$

$$Q(X^1 + X^2) = \mathrm{Sup}\{x_1^1 + x_1^2; x_2^1 + x_2^2\}.$$

Then

$$Q(X^1) \geq_{K_1} x_1^1, \quad Q(X^1) \geq_{K_2} x_2^1 \quad \text{and} \quad Q(X^2) \geq_{K_1} x_1^2, \quad Q(X^2) \geq_{K_2} x_2^2$$

and

$$Q(X^1) + Q(X^2) \geq_{K_1} x_1^1 + x_1^2, \qquad Q(X^1) + Q(X^2) \geq_{K_2} x_2^1 + x_2^2.$$

Thus

$$Q(X^1) + Q(X^2) \geq_{K_i} \mathrm{Sup}\{x_1^1 + x_1^2; x_2^1 + x_2^2\} = Q(X^1 + X^2), \; i = 1, 2.$$

□

The following theorem states that x'_+, x'_-, x''_+, x''_- are sublinear projections onto the corresponding cones. First, we will prove the following lemma.

Lemma 2.2.13. *Let $E = (E; K_1, K_2)$ be a 2-vector lattice with the pointed cones K_1 and K_2. Then for every $x \in E$ in the relations*

$$Sup\{0; Sup\{0; x\}\} = Sup\{0; x\}, \quad Sup\{Sup\{x; 0\}; 0\} = Sup\{x; 0\}$$

and

$$Inf\{0; Inf\{0; x\}\} = Inf\{0; x\}, \quad Inf\{Inf\{x; 0\}; 0\} = Inf\{x; 0\}$$

are valid.

Proof. We will prove only the first equality. Other assertions can be proved by similar reasoning. Let $U = \text{Sup}\{0; x\}$ and $V = \text{Sup}\{0; U\}$. We have $U \geq 0_{K_1}$, $U \geq_{K_2} U$, hence $U \geq_{K_i} \text{Sup}(0, U) = V$, $i = 1, 2$.

Conversely, $V \geq_{K_1} 0$, $V \geq_{K_2} U$ yield $V \geq_{K_i} \text{Sup}\{0; U\} \geq_{K_2} U$. Since K_2 is a pointed cone then $U \geq_{K_2} V$ and $V \geq_{K_2} U$ imply $U = V$. □

An operator $A : E \to E$ where E is called a *projector* if $A^2 = A$.

Theorem 2.2.14. *Let a space $E = (E; K_1, K_2)$ be a 2-vector lattice. Assume that the cones K_1 and K_2 are pointed. Consider the operators T'_+, T'_-, T''_+, T''_- defined on E by*

$$T'_+(x) = x'_+, \quad T'_-(x) = x'_-, \quad T''_+(x) = x''_+, \quad T''_-(x) = x''_-.$$

Then these operators, acting from E to E are sublinear projectors, besides $T'_+(E) \subset K_1$, $T''_-(E) \subset K_1$ and $T'_-(E) \subset K_1, T''_+(E) \subset K_2$.

Proof. Let $x \in E = (E; K_1, K_2)$. Let

$$T'_+(x) = x'_+, \quad T'_-(x) = x'_-, \quad T''_+(x) = x''_+, \quad T''_-(x) = x''_-.$$

Consider the vectors $Y_x = (x, 0) \in E^2$, $Z_x = (0, x) \in E^2$. Then

$$T'_+(x) = Q(Z_x), \quad T'_-(x) = -P(Y_x), \quad T''_+(x) = Q(Y_x), \quad T''_-(x) = -P(Z_x),$$

where the operators P and Q are the same as in Theorem 2.2.12. Then Theorem 2.2.12 implies the sublinearity of T'_+, T'_-, T''_+, T''_-.

The definitions of x'_+, x'_-, x''_+, x''_- yield

$$T'_+(x), \ T''_-(x) \in K_1 \quad \text{and} \quad T'_-(x), \ T''_+(x) \in K_2$$

for all $x \in E$.

Finally, let us show that $(T'_+)^2 = T'_+$, $(T'_-)^2 = T'_-$, $(T''_+)^2 = T''_+$, $(T''_-)^2 = T''_-$.

It can easily be obtained by means of Lemma 2.2.13:

$$(T'_+)^2(x) = T'_+(T'_+(x)) = T'_+(x'_+) = \text{Sup}\{0; \text{Sup}\{0; x\}\} = \text{Sup}\{0; x\} = T'_+(x),$$

where $x \in E$.

By acting analogously with T'_-, T''_+, T''_- the required assertion can be proved. □

2.2.4 Kantorovich -Riesz type theorems

Let $E = (E; K_1, K_2)$ be a space with two cones K_1, K_2. Consider the space $E' = (E'; K_1^*, K_1^*)$ with the cones K_1^*, K_2^*, where E' is the dual space to E and K_i^* are the conjugate cones to K_i $(i = 1, 2)$. We consider the relation between the Riesz interpolation property in $E = (E; K_1, K_2)$ and the property of $E' = (E'; K_1^*, K_1^*)$ to be a 2-vector lattice. As above, let

$$\sigma \equiv \sigma_{K_1,K_2}(x) = \{X = (x_1, x_2) \in K_1 \times K_2 \ : \ x_1 + x_2 = x\} \quad (x \in K_1 + K_2),$$

be the decomposition mapping with respect to the cones K_1, K_2 and let

$$p_G(x) = \inf_{Y \in \sigma_{K_1,K_2}(x)} [G, Y] \qquad (x \in E, \ G \in \operatorname{dom} \sigma^*)$$

be the support function of σ corresponding to a linear function G (p_G was defined and studied in Section 1.3).

First, we will prove the following assertion.

Proposition 2.2.15. *Let cones K_1, K_2 be given in the space E and let $L = K_1 + K_2$. If the decomposition mapping $\sigma \equiv \sigma_{K_1,K_2} : E \to 2^{E^2}$ is additive on the cone L, then p_G is a positive additive on L function for every $G \in K^* = K_1^* \times K_2^*$.*

Proof. Let $G \in K^*$ and $x, y \in L$. Since σ is additive, then $\sigma(x + y) = \sigma(x) + \sigma(y)$. Thus,

$$p_G(x + y) = \inf_{Z \in \sigma(x+y)} [G, Z] = \inf_{Z \in \sigma(x)+\sigma(y)} [G, Z] =$$

$$= \inf_{Z' \in \sigma(x), \ Z'' \in \sigma(y)} [G, Z' + Z''] = \inf_{Z' \in \sigma(x)} [G, Z'] + \inf_{Z'' \in \sigma(y)} [G, Z''] =$$

$$= p_G(x) + p_G(y).$$

We proved that p_G is additive on L. Now let us show that p_G $(G \in K^*)$ is positive on the cone L, i.e. if $x \in L$ then $p_G(x) \geq 0$. Indeed, it follows from the fact that $\sigma(x) \subset K = K_1 \times K_2$ and $G \in K^* = K_1^* \times K_2^*$. □

Proposition 2.2.16. *Assume that the cone $L = K_1 + K_2$ from Proposition 2.2.15 is generating and closed. Consider a function l_G define on E by*

$$l_G(x) = p_G(x_1) - p_G(x_2), \qquad x = x_1 - x_2, \quad x_1, x_2 \in L. \tag{2.2.8}$$

Then l_G is well defined and $l_G \in E'$.

Proof. First we show that l_G is well-defined. Let $x = x_1 - x_2 = y_1 - y_2$. Since $x_1 + y_2 = y_1 + x_2$ and p_G is additive it follows that $p_G(x_1) + p_G(y_2) = p_G(y_1) + p_G(x_2)$, therefore

$$p_G(x_1) - p_G(y_2) = p_G(y_1) - p_G(y_2).$$

This means that the number $l_G(x)$ does not depend on the presentation of x as the difference of two elements from L. It is clear that l_G is an additive function. Since p_G is sublinear it follows that p_G is positive homogeneous. Let $x = x_1 - x_2$. Then $-x = x_2 - x_1$, hence $l_G(-x) = p_G(x_2) - p_G(x_1) = -l_G(x)$. Thus p_G is homogeneous. Since the cone L is generating and closed it follows (see Theorem 1.1.2) that each positive on L linear function is continuous, hence $l_G \in E'$. □

Let

$$q_G(x) = \sup_{Y \in \sigma_{K_1,K_2}(x)} [G, Y] \qquad (x \in E,\ G \in \operatorname{dom} \sigma^*). \tag{2.2.9}$$

The links between q_G and p_G were discussed at the beginning of Section 1.3. Assume that the mapping σ is additive. Then the function q_G is additive. Assume that the cone L is generating and closed. Then the function

$$m_G(x) = q_G(x_1) - q_G(x_2), \qquad x = x_1 - x_2 \tag{2.2.10}$$

is well defined. This function is a linear continuous function defined on E. These results can be proved in the same manner as the corresponding results for the function p_G.

The following statement is a version of Theorem 2.1.2 (L. V. Kantorovich-F. Riesz) for spaces with two cones.

Theorem 2.2.17. *Let E be a Banach ordered space with the closed cones K_1, K_2 and let the cone $L = K_1 + K_2$ be closed and normal. If the space $E = (E; K_1, K_2)$ possesses the Riesz interpolation property with respect to the cones K_1, K_2 then the dual space $E' = (E'; K_1^*, K_2^*)$ is a 2-vector lattice with respect to the conjugate cones K_1^*, K_2^*.*

Proof. Since $L = K_1 + K_2$ it follows that $L^* = K_1^* \cap K_2^*$. Since L is normal it follows (see Theorem 1.1.9) that L^* is a generating cone. In view of Proposition 2.2.11 it is enough to show that $\operatorname{Inf}(g_1, g_2)$ and $\operatorname{Sup}(g_1, g_2)$ exist for elements $G = (g_1, g_2)$ with $g_1, g_2 \in L^*$. We will prove only the existence of $\operatorname{Inf}(x_1, x_2)$. The existence of $\operatorname{Sup}(x_1, x_2)$ can be proved by a similar argument.

Theorem 2.1.3 shows that the Riesz interpolation property with respect to the cones K_1, K_2 in the space $E = (E; K_1, K_2)$ is equivalent to the additivity of the decomposition mapping, so applying Proposition 2.2.16 we conclude that the function l_G defined by (2.2.8) is a positive linear continuous function.

We will prove that $l_G = \mathrm{Inf}\{g_1; g_2\} \in E' = (E'; K_1^*, K_2^*)$.

Evidently for all $x_1 \in K_1$, $x_2 \in K_2$ the following inequalities hold:

$$l_G(x_1) = p_G(x_1) \leq g_1(x_1), \quad l_G(x_2) = p_G(x_2) \leq g_2(x_2).$$

By the definition of the conjugate cone we obtain

$$l_G \leq_{K_1^*} g_1, \quad l_G \leq_{K_2^*} g_2.$$

Let an element $h \in E'$ be such that

$$h \leq_{K_1^*} g_1, \quad h \leq_{K_2^*} g_2.$$

Let $x \in L$ and elements $x_1 \in K_1$ and $x_2 \in K_2$ be such that $x = x_1 + x_2$. (In other words, $(x_1, x_2) \in \sigma_{K_1,K_2)}(x)$.) Since $x_1 \in K_1$ and $x_2 \in K_2$ we have

$$[h, x_1] \leq [g_1, x_1], \qquad [h, x_2] \leq [g_2, x_2].$$

Hence

$$[h, x] \leq [g_1, x_1] + [g_2, x_2] \quad \text{for all} \quad (x_1, x_2) \in \sigma_{K_1,K_2}(x).$$

This yields

$$[h, x] \leq \inf_{(x_1,x_2)\in\sigma_{K_1,K_2}(x)} \{[g_1, x_1] + [g_2, x_2]\} = p_G(x) = l_G(x), \quad (x \in L).$$

Therefore $[h, x] \leq [l_G, x]$, $(x \in L)$, that is $h \leq_{L^*} l_G$. Since $L^* = K_1^* \bigcap K_2^*$ it follows that

$$h \leq_{K_1^*} f, \quad h \leq_{K_2^*} f.$$

This means that l_G is the infimum of the elements g_1, g_2 with respect to the cones K_1^*, K_2^*. We have proved that for each $g_1, g_2 \in L^*$ the infimum with respect to K_1^*, K_2^* exists.

We now turn to the supremum $\mathrm{Sup}(g_1, g_2)$. The existence of $\mathrm{Sup}(g_1, g_2)$ can be proved by the same argument using functions q_G defined by (2.2.9) instead of p_G and functions m_G defined by (2.2.10) instead of l_G. □

In relation with this theorem we consider Examples 2.1.2 and 2.2.2 that were discussed above.

Example 2.2.8. Let $E = \mathbb{R}^2$ be the cartesian plane, $K_1 = \mathbb{R}^2_+$ be the positive orthant and $K_2 = \{\lambda U\}_{\lambda \geq 0}$ be the ray passing through the point $U = (0, 1)$ (see Fig. 2.16).

Then $K_1^* = \mathbb{R}^2_+$ and $K_2^* = \{(u, v) \in \mathbb{R}^2 \ : \ v \in \mathbb{R}_+\}$ are conjugate cones to K_1 and K_2 respectively.

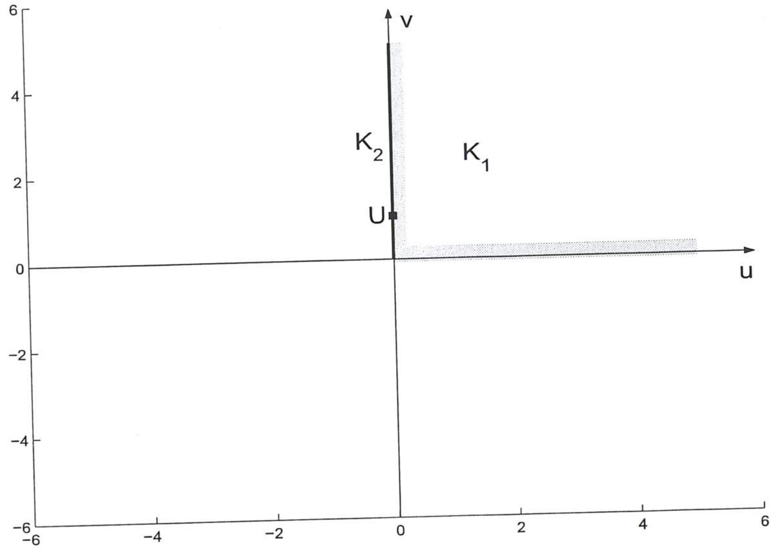

Figure 2.16: $E = (\mathbb{R}^2; K_1, K_2)$

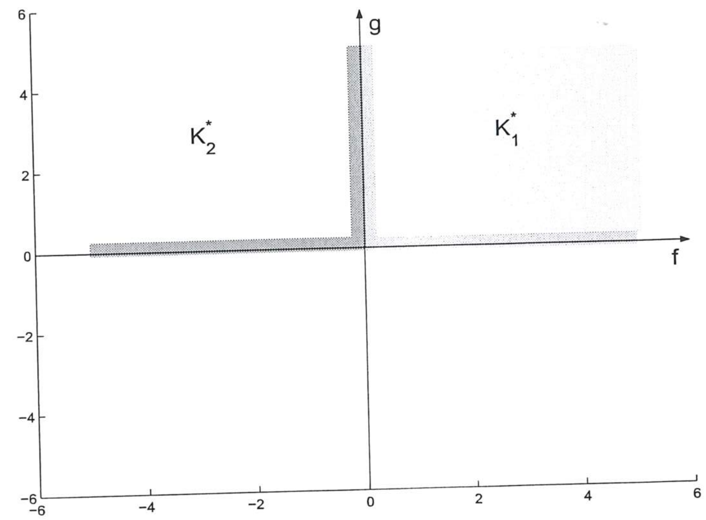

Figure 2.17: $E' = (E'; K_1^*, K_2^*)$

It is evident that the space E and the cones $K_1, K_2, L = K_1 + K_2$ satisfy all the hypotheses of Theorem 2.2.17 (see Fig. 2.17) and the Riesz interpolation property takes place in the space $E = (\mathbb{R}^2; K_1, K_2)$ with respect to the cones K_1 and K_2 (see Example 2.1.2). Then Theorem 2.2.17 implies that the dual space E' is a 2-vector lattice with respect to the positive orthant K_1^* and the half-plane K_2^* which is confirmed by Example 2.2.2 stated at the beginning of this section.

It is interesting to find conditions that guarantee that the inverse to the statement in Theorem 2.2.17 holds. We will demonstrate that this statement is valid if E is a reflexive space. Actually we will prove the following stronger result.

Theorem 2.2.18. *Let $E = (E; K_1, K_2)$ be a reflexive Banach space with cones K_1 and K_2. Assume that the cone $L = K_1 + K_2$ is closed, normal and generating. Assume that the space $(E'; K_1^*, K_2^*)$ is a 2-vector lower semilattice. Then (E, K_1, K_2) possesses the Riesz interpolation property.*

Proof. For the sake of definiteness we assume that $(E'; K_1^*, K_2^*)$ is a 2-vector lower semilattice. We will check that the decomposition mapping $\sigma \equiv \sigma_{K_1,K_2}$ is additive, this implies the Riesz interpolation property. We will show that for all $G = (g_1, g_2) \in (E')^2$ the support function p_G of the decomposition mapping $\sigma \equiv \sigma_{K_1,K_2}$ coincides with the restriction of a certain linear function on L. Recall that p_G is sublinear and (see Theorem 1.3.11) is lower semicontinuous for all $G \in (E^2)'$.

Let $U = \{h \in E' : h \leq_{K_1^*} g_1,\ h \leq_{K_2^*} g_2\}$. We have for each $h \in U$, $x \in L$ and $X = (x_1, x_2) \in \sigma(x)$:

$$[h, x_1] \leq [g_1, x_1], \qquad ([h, x_2] \leq [g_2, x_2]).$$

Therefore

$$[h, x] = [h, x_1] + [h, x_2] \leq \inf_{X=(x_1,x_2)\in\sigma(x)} ([g_1, x_1] + [g_2, x_2]) = p_G(x).$$

We have demonstrated that $h \in \partial p_G$, so $U \subset \partial p_G$.

Now let $h \in \partial p_G$ and let $x_1 \in K_1$. Then

$$[h, x_1] \leq p_G(x_1) = \inf_{X=(x_1',x_2')\in\sigma(x_1)} [g_1, x_1'] + [g_2, x_2'] \leq [g_1, x_1] + [g_2, 0] = [g_1, x_1].$$

Thus $h \leq_{K_1^*} g_1$. In the same manner we can show that $h \leq_{K_2^*} g_2$. It follows from this that $\partial p_G \subset U$.

Thus we have proved that $\partial p_G = U$. Let $h_G = \mathrm{Inf}(g_1, g_2)$. Then $h_g \in U$ and $h_G \geq_{K_1^*} h$, $h_G \geq_{K_2^*} h$ for all $h \in U$. Since p_G is lower semicontinuous we have

$$p_G(x) = \sup_{h\in\partial p_G} [h, x] = \sup_{h\in U} [h, x].$$

Since $h_G \geq_{K_i^*} h$ for all $h \in U$ we have that $h_G(x_i) \geq h(x_i)$ for all $h \in U$ and $x_i \in K_i$, $(i = 1, 2)$ hence

$$p_G(x_1) = \sup_{h \in U}[h, x_1] = [h_G, x_1], \qquad x \in K_1$$

and

$$p_G(x_2) = \sup_{h \in U}[h, x_2] = [h_G, x_2], \qquad x \in K_2.$$

Now let $x \in L$ and $X = (x_1, x_2) \in \sigma(x)$. Since p_G is sublinear and $h_G \in \partial p_G$ we have

$$[h_G, x] \leq p_G(x) \leq p_G(x_1) + p_G(x_2) = [h_G, x_1] + [h_G, x_2] = [h_G, x].$$

Thus $p_G(x) = [h_G, x]$ for all $x \in L$. Hence we can consider p_G as the restriction of a function $h_G \in E'$ to the cone L.

Applying Proposition 1.2.19 we conclude that the decomposition mapping is bounded, therefore sets $\sigma(x)$ are bounded for all $x \in K$. Since the space E is reflexive it follows that these sets are weakly compact. We now can apply Proposition 1.1.15 that show that σ is an additive mapping. □

A similar result can be proved for 2-vector upper semilattices.

Theorem 2.2.19. *Let $E = (E; K_1, K_2)$ be a reflexive Banach space with cones K_1 and K_2. Assume that the cone $L = K_1 + K_2$ is closed, normal and generating. Assume that the space $(E'; K_1^*, K_2^*)$ is a 2-vector upper semilattice. Then (E, K_1, K_2) possesses the Riesz interpolation property.*

The proof is similar to that of Theorem 2.2.18. We need to consider the superlinear function q_G, where $g_G(x) = \sup_{X=(x_1,x_2)\in\sigma(x)}([g_1, x_1] + [g_2, x_2]$ and repeat the proof of Theorem 2.2.18 with obvious changes.

Corollary 2.2.20. *Let $E = (E; K_1, K_2)$ be a reflexive Banach space with cones K_1 and K_2. Assume that the cone $L = K_1 + K_2$ is closed, normal and generating. If the space $(E'; K_1^*, K_2^*)$ is either a 2-vector lower semilattice or 2-vector upper semilattice then this space is a vector lattice.*

Indeed, applying either Theorem 2.2.18 or Theorem 2.2.19 we conclude that $(E; K_1, K_2)$ possesses Riesz interpolation property. Combining this with Theorem 2.2.17 we obtain the desired result.

Chapter 3

Weakly-efficient Decompositions

3.1 Preliminaries

Efficient decomposition of an element with respect to a given collection of cones is important for many applications. Such decompositions often arise in mathematical economics (see [24]). Many examples of efficient decomposition are given in the paper [25] by J.E. Martinez Legaz and A. Seeger. The authors of this seminal paper proposed to use several definitions of efficiency. They are as follows.

Let $(E; K_1, K_2)$ be a Banach space with closed cones K_1 and K_2. Consider the decomposition mapping σ_{K_1,K_2} defined by K_1, K_2.

Definition 3.1.1. A decomposition $(\overline{x}_1, \overline{x}_2) \in \sigma(x)$ $(x \in L := K_1 + K_2)$ is called *efficient* if the pair $(\overline{x}_1, \overline{x}_2)$ is an efficient point of the set $\sigma(x)$ with respect to the partial order induced by the cone $K_1 \times K_2$, i.e. if $(x_1, x_2) \in \sigma(x)$ and $(\overline{x}_1, \overline{x}_2) - (x_1, x_2) \in K_1 \times K_2$, then

$$(x_1, x_2) - (\overline{x}_1, \overline{x}_2) \in K_1 \times K_2.$$

If K_1, K_2 are pointed cones then $(\overline{x}_1, \overline{x}_2) - (x_1, x_2) \in K_1 \times K_2$ implies $(x_1, x_2) = (\bar{x}_1, \bar{x}_2)$.

Definition 3.1.2. A decomposition $(\overline{x}_1, \overline{x}_2) \in \sigma(x)$ $(x \in K_1 + K_2)$ is called *ideal* if it satisfies the inclusion

$$(x_1, x_2) - (\overline{x}_1, \overline{x}_2) \in K_1 \times K_2 \quad \forall (x_1, x_2) \in \sigma(x).$$

Definition 3.1.3. A decomposition $(\overline{x}_1, \overline{x}_2) \in \sigma(x)$ $(x \in K_1 + K_2)$ is called *efficient in the i-th component* $(i = 1, 2)$ if the inclusions $(x_1, x_2) \in \sigma(x)$ and $\overline{x}_i - x_i \in K_i$ imply $x_i - \overline{x}_i \in K_i$ $(i = 1, 2)$.

The problem of conical decomposition and its efficiency first rose in the theory of vector lattices (see [15] and references therein). In fact, if a space E is a vector lattice with a cone H which induces an order, then for any element $x \in E$ the following equality takes place

$$x = x \vee 0 + x \wedge 0,$$

where $x \vee 0 = \sup(x, 0) \in H$, $x \wedge 0 = \inf(x, 0) \in -H$. Thus, as we see here, there is a decomposition of the element $x \in E$ into the elements from the cones H and $-H$. The pair $(x \vee 0, x \wedge 0) \in H \times (-H)$ is called a *lattice decomposition* [15, 34]. Furthermore, it is well-known that if there exist elements $y \in H$, $z \in -H$ such that

$$x = y + z,$$

then $y \geq_H x \vee 0$, $z \geq_H x \wedge 0$.

In terms of the above definitions, this fact means that the pair $(x \vee 0, x \wedge 0) \in \sigma(x)$ is an ideal efficient decomposition of the element x with respect to the pair of cones $(H, -H)$. It is easy to show that (see [25]) the lattice decomposition $(x \vee 0, x \wedge 0)$ is the unique efficient decomposition of the element x.

Furthermore, let us note the following J.-J.Moreau theorem on the orthogonal decomposition [26, 25], which is one of the most brilliant result in this field.

Here and in the following $[\cdot, \cdot]$ is the scalar product in a Hilbert space $E = (E, [\cdot, \cdot])$ and $\|\cdot\| = \sqrt{[\cdot, \cdot]}$ is the norm in E.

Theorem 3.1.1. *Let $(E, [\cdot, \cdot])$ be a Hilbert space, $K_1 = K \in E$ be a cone, and $K_2 = -K^*$ be its polar. Let x, π_1 and π_2 be three elements of E. Then the following statements are equivalent:*

a) elements π_1, π_2 are the projections of x onto K_1 and K_2, respectively, i.e.

$$\pi_i = argmin\{\|x - y_i\| \ : \ y_i \in K_i\}, \quad i = 1, 2;$$

b) $x = \pi_1 + \pi_2$, $\pi_1 \in K_1$, $\pi_2 \in K_2$ and $[\pi_1, \pi_2] = 0$,

It can be shown (see [25]) that the orthogonal Moreau decomposition $(\pi_1, \pi_2) \in \sigma(x)$ is componentwise efficient.

The following recent result from [25] should be mentioned here:

Theorem 3.1.2. *Let K_1 and K_2 be two closed cones in a Hilbert space $(E, [\cdot, \cdot])$ and let $x \in L := K_1 + K_2$. Consider an arbitrary pair $(x_1, x_2) \in \sigma(x)$, where $\sigma \equiv \sigma_{K_1,K_2}$. Then the following two statements are equivalent:*

a) elements π_1, π_2 are the projections of x_1 and x_2 on the sets $K_1 \cap (x - K_2)$ and $K_2 \cap (x - K_1)$, respectively, i.e. π_1 and π_2 are minimizers of the problems

$$\|x_1 - y_1\| \to \min, \qquad \|x_2 - y_2\| \to \min,$$
$$y_1 \in K_1 \cap (x - K_2). \qquad y_2 \in K_2 \cap (x - K_1).$$

b) the pair (π_1, π_2) is the unique maximizer of the problem

$$[x_1, y_1] + [x_2, y_2] + [y_1, y_2] \to \max$$
$$(y_1, y_2) \in \sigma(x).$$

Furthermore, under the following additional assumption imposed on the cones K_1, K_2 and the elements x_1, x_2:

$$[x_1, x_2] \leq 0 \quad \textit{for all} \quad x_1 \in K_1, \ x_2 \in K_2\}$$
$$x_1 - x_2 \in -K_1^* \quad \textit{or} \quad x_2 - x_1 \in -K_2^*,$$

the pair (π_1, π_2), defined by the above equivalent conditions, is an efficient decomposition of the element x.

The existence of efficient decompositions (in various senses) has been studied by many authors (see, for example, [6, 13, 21]).

In the next section we will introduce the notion of weakly-efficient decomposition, which is different from notions of efficiency defined in [25]. We will examine the properties of weakly-efficient decomposition using the techniques developed in the previous chapters; a relation to the efficient decomposition will be shown and examples given (Section 3.2); the case of a weakly-efficient decomposition with respect to the support set of some sublinear function with given properties will be studied separately (Section 3.3).

3.2 A weakly-efficient decomposition

As was mentioned, in [25] the definition of an efficient decomposition was given, where the order was chosen as the efficiency criterion. Here a new definition will be proposed, with the duality involved, and the technique of superlinear multi-valued mappings will be used.

Let E be a Banach space. Consider cones $K_1, \ldots, K_n$ in E and their Cartesian product $K = K_1 \times \cdots \times K_n$ in E^n. As before, $\sigma = \sigma_{K_1,\ldots,K_n} : E \to 2^{E^n}$ is the decomposition mapping with respect to the cones $K_1, \ldots, K_n$ i.e.

$$\sigma(x) = \{X = (x_1, \ldots, x_n) \in K = K_1 \times \cdots \times K_n \ \sum_{i=1}^{n} x_i = x\} \quad (x \in L := \sum_i K_i).$$

Let $\mathcal{D}$ be a subset of the set $\mathcal{K} = \text{dom } \sigma^*$, i.e. $\mathcal{D} \subset \mathcal{K}$.

Definition 3.2.1. A point $X \in \sigma(x)$ where $x \in \text{dom } \sigma$ is called a *weakly efficient decomposition* of x with respect to a given subset $\mathcal{D}$, if there exist functions $G \in \mathcal{D}$ and $f \in \sigma^*(G)$ such that $[f, x] = [G, X]$.

A quadruple of vectors (x, X, f, G) in this definition is called *compatible* with respect to the mapping σ.

In the following we will see that the set $\mathcal{D} \subset \mathcal{K}$ plays an important role in the definition of a weakly-efficient decomposition. Theorem 1.2.10 implies the equality $\mathcal{K} = K^* + M^*$, where $M^* = D := \{G = (g, \dots, g) : g \in E'\}$ and we can observe the following facts related to the set M^*.

Proposition 3.2.1. *If $\mathcal{D} = M^*$ then any decomposition $X \in \sigma(x)$ is a weakly-efficient decomposition with respect to $\mathcal{D}$.*

Proof. It is a consequence of the fact that for all $x \in \text{dom } \sigma$, $X \in \sigma(x)$ and any $g^\wedge \in \mathcal{D} = M^*$ Proposition 1.2.7 yields

$$g \in \sigma^*(g^\wedge) \quad \text{and} \quad [g, x] = [g^\wedge, X].$$

□

Proposition 3.2.2. *Let $\mathcal{D} \subset K^*$. Let $X \in \sigma(x)$, $f \in \sigma^*(G)$ ($x \in dom\ \sigma := L$, $G \in \mathcal{K}$) and $G = H + g^\wedge \in \mathcal{K}$ where $H \in K^*$, $g^\wedge \in M^*$. Then, if the quadruple (x, X, f, G) is compatible, then the quadruple $(x, X, f - g, H)$ is also compatible with respect to the mapping σ.*

Proof. This result is a corollary of Proposition 1.2.14. □

These two assertions allow us to consider only the case $\mathcal{D} \subset K^*$.

Since we are interested in a decomposition of any element of the space E into the cones, we will consider the cones $K_1, K_2, \dots, K_n$ such that

$$\sum_{i=1}^{n} K_i = E, \quad \text{i.e.} \quad L := \text{dom } \sigma = E.$$

Then Theorem 1.3.10 implies that for all $G \in \mathcal{K}$ the sublinear function

$$p_G(x) = \inf\{[G, X] \ : \ X \in \sigma(x)\}, \quad (x \in E)$$

is continuous. Recall that a cone K is called solid if $\text{int } K \neq \emptyset$. The following theorem establishes the relation between the efficient and weakly-efficient decompositions.

Theorem 3.2.3. *Let $K := K_1 \times \cdots \times K_n$ be a solid cone in the space E^n, then if a decomposition $X \in \sigma(x)$ $(x \in E)$ is efficient, then it is weakly-efficient, if $\mathcal{D} = K^*$.*

Proof. Let $x \in E$. As $X \in \sigma(x)$ is an efficient point with respect to the solid cone K, then

$$\operatorname{int}(X - K) \bigcap \sigma(x) = \emptyset.$$

Since K is a solid cone, by means of the separation theorem, we will find a function $G \in E'$ such that

$$\sup_{Z \in X - K} [G, Z] \leq \inf_{Y \in \sigma(x)} [G, Y] \tag{3.2.1}$$

which implies the following two inequalities:

$$[G, X] \leq \inf_{Y \in \sigma(x)} [G, Y] = p_G(x) \tag{3.2.2}$$

and

$$\sup_{Y \in -K} [G, Y] < +\infty.$$

The latter yields $G \in K^*$.

Since

$$X \in \sigma(x), \quad \text{and} \quad p_G(x) = \inf_{Y \in \sigma(x)} [G, Y]$$

it follows that $[G, X] \geq p_G(x)$. Combining this inequality with (3.2.2) we get

$$[G, X] = p_G(x). \tag{3.2.3}$$

Since p_G is continuous, then the corollaries of Theorem 1.3.5 yield

$$p_G(x) = \sup_{f \in \sigma^*(G)} [f, x].$$

In view of Corollary 1.3.9 we have $\sigma^*(G) = \partial p_G$. Since p_G is a continuous function, it follows that $\sigma^*(G)$ is a nonempty and weakly$*$ compact set, therefore

$$p_G(x) = \max_{f \in \sigma^*(G)} [f, x].$$

Let a function $f \in \sigma^*(G)$ be such that $[f, x] = p_G(x)$.

The equality

$$[f, x] = [G, X]$$

takes place, and hence the quadruple of vectors (x, X, f, G) is compatible with respect to the mapping σ. So, it is proved that the decomposition $X \in \sigma(x)$, $(x \in E)$ is weakly efficient if $\mathcal{D} = K^*$. □

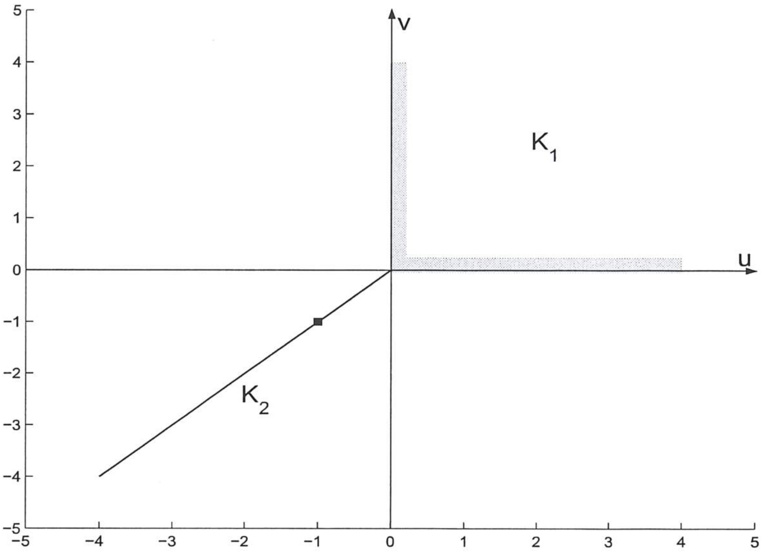

Figure 3.1: $E = \mathbb{R}^2$

The following examples imply that the problem of the existence of a weakly-efficient decomposition with respect to $\mathcal{D} = K^*$ can be reduced to the problem of finding the min of the function p_G, $(G \in \mathcal{D})$.

First we introduce the following definition. Consider a Banach space $(E; H, Q)$ with closed cones H and Q. The set $H \bigcap (x - Q)$ is called the *positive germ* of an element x on the cone $(-Q)$ (see [20]). We can use the positive germ for calculation of the support function p_G of the decomposition mapping $\sigma_{H,Q} \equiv \sigma$. Indeed

$$\begin{aligned} p_G(x) &= \inf_{x_1+x_2=x,\ x_1\in H, x_2\in Q} \{[g_1, x_1] + [g_2, x_2]\} \\ &= [g_2, x] + \inf_{x_1 \in H,\ x - x_1 \in Q} [g_1 - g_2, x_1] \\ &= [g_2, x] + \inf_{x_1 \in H \cap (x-Q)} [g_1 - g_2, x_1]. \end{aligned}$$

This formula will be used later.

Example 3.2.1. Consider the plane $E = \mathbb{R}^2$ and take the positive orthant and the ray, passing through the point $t = (-1, -1)$ as the cones, i.e. $K_1 = \mathbb{R}^2_+$, $K_2 = \{\lambda t\}_{\lambda \geq 0}$ (see Fig. 3.1).

Then the dual cones are (see Fig. 3.2) $K_1^* = \mathbb{R}^2_+$ and

$$K_2^* = \{(f, g) \in \mathbb{R}^2 \ : \ [(f, g), (-1, -1)] \geq 0\} = \{(f, g) \in \mathbb{R}^2 \ : \ f \leq -g\}.$$

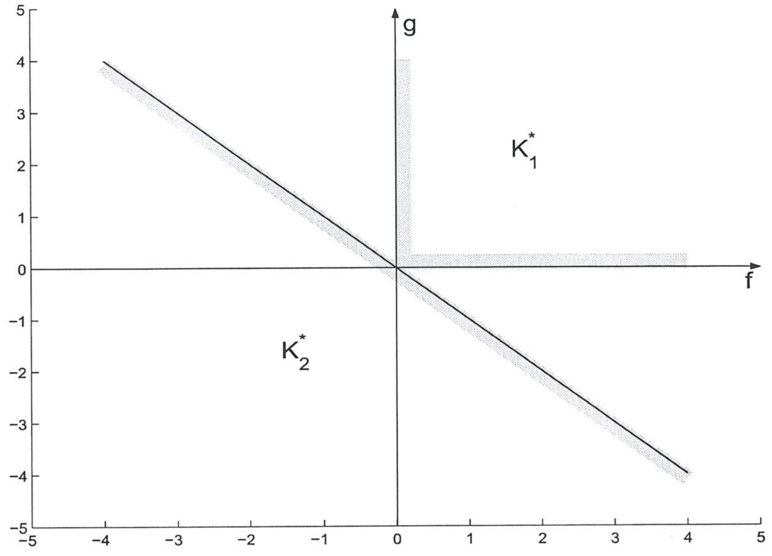

Figure 3.2: $E' = \mathbb{R}^2$

Let $x^0 = (-2, 3) \in \mathbb{R}^2$, $g_1 = (4, 2)$, $g_2 = (-2, -3)$ and $G = (g_1, g_2) \in \mathcal{D} = K_1^* \times K_2^*$. The decomposition mapping $\sigma_{K_1,K_2} \equiv \sigma$ is expressed in the following way:

$$\sigma(x^0) = \{(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \ : \ x_i = (u^i, v^i),\ i = 1, 2,\ u^1 \geq 0,\ v^1 \geq 0,$$
$$u^2 = -\lambda,\ v^2 = -\lambda,\ \lambda \geq 0,\ u^1 + u^2 = -2,\ v^1 + v^2 = 3\}.$$

Let us represent $p_G(x^0)$ in the form:

$$\begin{aligned} p_G(x^0) &= \inf_{x_1 + x_2 = x^0,\ x_1 \in K_1,\ x_2 \in K_2} \{[g_1, x_1] + [g_2, x_2]\} \\ &= \inf_{x_1 \in K_1, x^0 - x_1 \in K_2} \{[g_1, x_1] + [g_2, x^0 - x_1]\} \\ &= [g_2, x^0] + \inf_{x_1 \in K_1 \cap (x^0 - K_2)} [g_1 - g_2, x_1]. \end{aligned}$$

The set over which the last inf is taken, is defined as follows (see Fig. 3.3):

$$K_1 \bigcap (x^0 - K_2) = \{(u, v) \in \mathbb{R}^2 \ : \ u, v \geq 0,\ u - v = -5\}.$$

An easy calculation shows that:

$$\overline{x}_1 := \operatorname{argmin}\{[g_1 - g_2, x_1] \ : \ x_1 \in K_1 \bigcap (x^0 - K_2)\} = (0, 5).$$

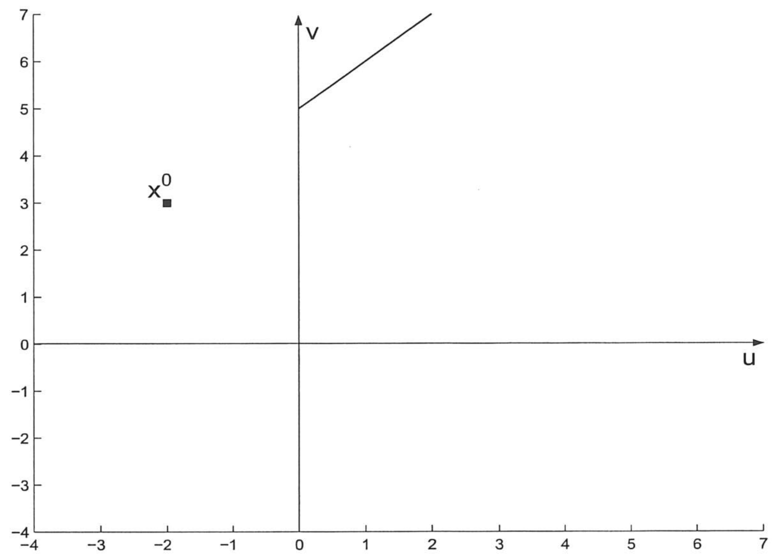

Figure 3.3: $K_1 \bigcap (x^0 - K_2)$

Then $\overline{x}_2 = x - \overline{x}_1 = (-2, -2)$, $\overline{X} = (\overline{x}_1, \overline{x}_2) \in \sigma(x)$ and

$$p_G(x^0) = 20.$$

We have (see Fig. 3.4):

$$\sigma^*(G) = (g_1 - K_1^*) \bigcap (g_2 - K_2^*) = \{(f, g) \in \mathbb{R}^2 \ : \ 4 \geq f, \ 2 \geq g, \ -5 \leq f + g\}.$$

It is easy to show that the linear function $f \mapsto [f, x^0] = -2f^1 + 3f^2$ attains its maximal value on the set $\sigma^*(G)$ at the point $\overline{f} = (-7, 2)$. Therefore, it is shown that the quadruple of vectors $(x^0, \overline{X}, \overline{f}, G)$ is compatible with respect to the mapping σ.

Example 3.2.2. Again, consider the cartesian plane $E = \mathbb{R}^2$ with the cones

$$K_1 = \{(u, v) \in \mathbb{R}^2 \ : \ v \geq u, \ v \geq -u\},$$

$$K_2 = \{(u, v) \in \mathbb{R}^2 \ : \ v \leq 2u, \ u \leq 0\},$$

$$K_3 = \{(u, v) \in \mathbb{R}^2 \ : \ v = -\frac{1}{3}u, \ v \leq 0\} = \{\lambda(3, -1)\}_{\lambda \geq 0}$$

(see Fig. 3.5).

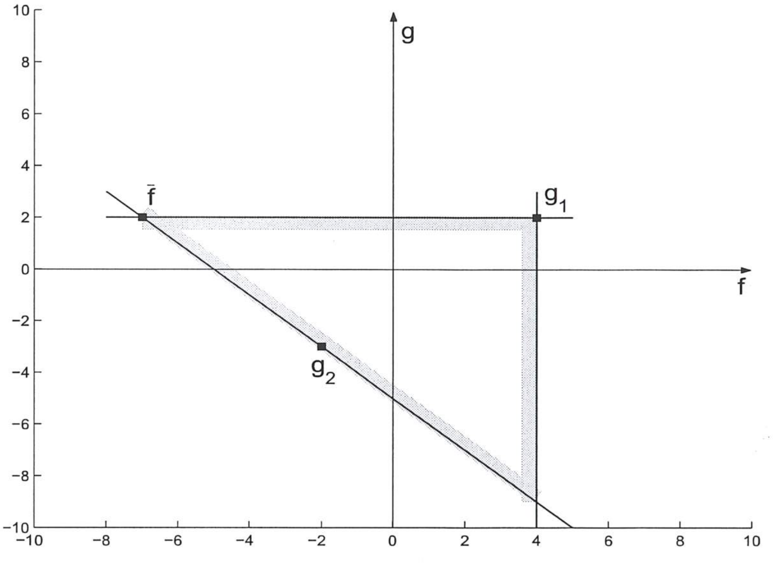

Figure 3.4: $\sigma^*(G)$

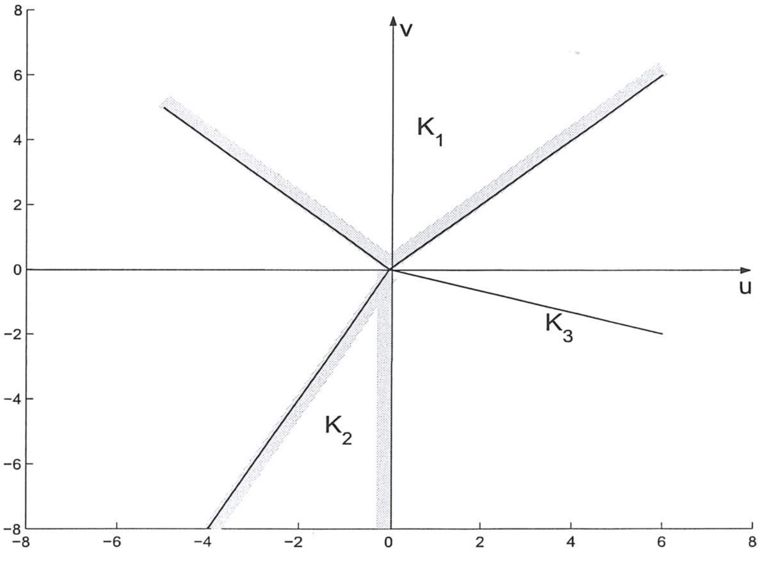

Figure 3.5: $E = \mathbb{R}^2$

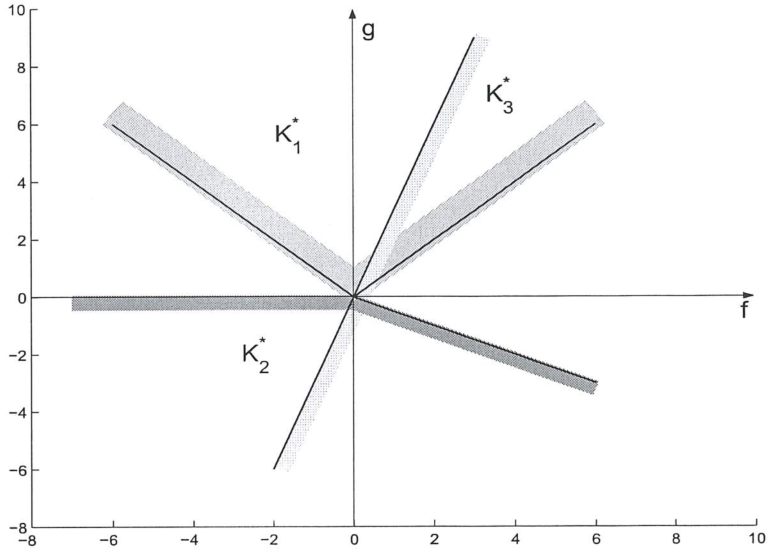

Figure 3.6: $E' = \mathbb{R}^2$

The conjugate cones have the form (see Fig. 3.6):

$$K_1^* = \{(f,g) \in \mathbb{R}^2 \ : \ g \geq f, \ g \geq -f\},$$

$$K_2 = \{(f,g) \in \mathbb{R}^2 \ : \ g \leq -\frac{1}{2}f, \ g \leq 0\},$$

$$K_3^* = \{(f,g) \in \mathbb{R}^2 \ : \ 3f \geq g\}.$$

Take the point $x^0 = (-5,2) \in \mathbb{R}^2$ and the vectors $g_1 = (0,4) \in K_1^*$, $g_2 = (-2,4) \in K_2^*$ and $g_3 = (2,-2) \in K_3^*$, i.e. $G = (g_1, g_2, g_3) \in \mathcal{D} = K_1^* \times K_2^* \times K_3^*$.

Let us calculate the set $\sigma_{K_1,K_2,K_3}(x^0) \equiv \sigma(x^0)$. We have

$$\begin{aligned} \sigma(x^0) \ = \ & \{(x_1,x_2,x_3) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \ : \ x_i = (u^i, v^i), \ i = 1,2,3, \\ & v^1 \geq u^1, v^1 \geq -u^1, \ v^2 \leq 2u^2, \ v^2 \leq 0, \ u^3 = 3\lambda, \ v^3 = -\lambda, \ \lambda \geq 0, \\ & u^1 + u^2 + u^3 = -5, \ v^1 + v^2 + v^3 = 2\}. \end{aligned}$$

Let us represent the function p_G in the form

$$p_G(x^0) = \inf_{x_1+x_2+x_3=x^0, \ x_i \in K_i, \ i=1,2,3} \{[g_1,x_1] + [g_2,x_2] + [g_3,x_3]\} =$$

$$[g_2, x^0] + \inf_{x_3 \in K_3} [g_3 - g_2, x_3] + \inf_{x_3 \in K_3, \ x_1 \in K_1 \cap (x^0 - x_3 - K_2)} [g_1 - g_2, x_1].$$

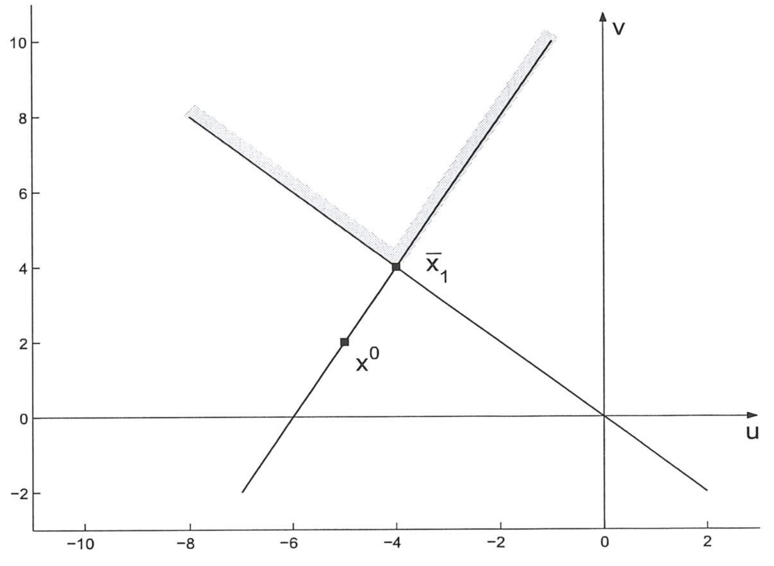

Figure 3.7: $K_1 \bigcap (x^0 - K_2 - K_3)$

For each $x_3 = \lambda(3, -1) \in K_3$ $(\lambda \geq 0)$, the set $K_1 \bigcap (x^0 - x_3 - K_2)$ can be explicitly expressed (see Fig. 3.7) as

$$K_1 \bigcap (x^0 - x_3 - K_2) = \{(u, v) \in \mathbb{R}^2 \ : \ v \geq -u, \ v \geq 2u + 12 + 7\lambda\}$$

.

By substituting the values of $g_1, g_2,$ and g_3 in the preceding expression of the function p_G we have

$$p_G(x^0) = [(-2, -4), (-5, 2)] + \inf_{\lambda \geq 0} [(4, 2), (3\lambda, -\lambda)] +$$

$$\inf_{\lambda \geq 0} \inf_{v \geq -u, \ v \geq 2u+12+7\lambda} [(2, 8), (u, v)] =$$

$$2 + 0 + \inf_{\lambda \geq 0} \inf_{v \geq -u, \ v \geq 2u+12+7\lambda} \{2\lambda + 8v\}.$$

It can be easily verified that both the last infimums are attained at

$$\overline{\lambda} = 0, \ \overline{x}_1 = (\overline{u}, \overline{v}) = (-4, 4) \in K_1, \ \text{ i.e. } p_G(x^0) = 26 \text{ and}$$

$$\overline{x}_3 = \overline{\lambda}(3, -1) = 0, \ \overline{x}_2 = x - \overline{x}_1 - \overline{x}_3 = (-1, -2).$$

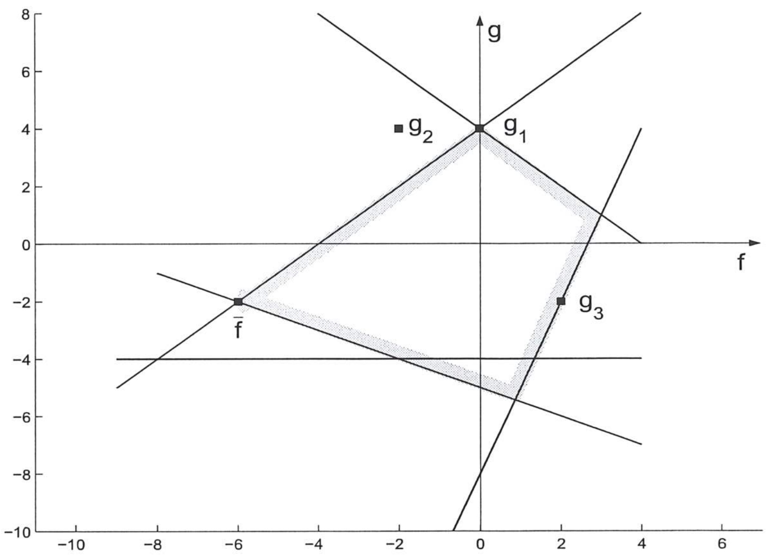

Figure 3.8: $\sigma^*(G)$

The conjugate mapping with respect to the cones K_1^*, K_2^*, and K_3^* at a point $G = (g_1, g_2, g_3) \in K_1^* \times K_2^* \times K_3^*$ has the form (see Fig. 3.8):

$$\begin{aligned}\sigma^*(G) &= (g_1 - K_1^*)\bigcap(g_2 - K_2^*)\\ &= \{(f,g) \in \mathbb{R}^2 \ : \ g \le -f+4,\ g \le f+4,\\ &\quad g \ge 3f-8, g \ge -4,\ g \ge -\frac{1}{2}f - 5\}.\end{aligned}$$

It is easy to show that the linear function $f \mapsto [f, x^0] = -5f^1 + 2f^2$ attains its maximal value on the set $\sigma^*(G)$ at the point $\overline{f} = (-6, -2)$. Hence, it is shown that the quadruple of vectors $(x^0, \overline{X}, \overline{f}, G)$ is compatible with respect to the mapping σ.

A weakly efficient decomposition is not always efficient. However, in some special cases we can say that the efficient and weakly efficient decompositions coincide. We consider one of such cases.

Consider a closed solid cone H in the space E and let a point $-\overline{x} \in \mathrm{int}H$. Let us introduce the following cones:

$$\Lambda = \{\lambda\overline{x}\}_{\lambda \ge 0} \quad \text{and} \quad \mathcal{D}_{\overline{x}} = \{G = (g_1, g_2) \in H^* \times L^* \ : \ [g_2 - g_1, \overline{x}] \ge 0\}.$$

Theorem 3.2.4. *For any $x \in E$ in the set $\sigma(x) = \sigma_{H,\Lambda}(x)$ there exists a unique weakly efficient decomposition with respect to the set $\mathcal{D} = \mathcal{D}_{\overline{x}}$, which is an efficient one.*

Proof. Since $(-H)$ is a solid cone and $\overline{x} \in \operatorname{int}(-H)$, then $E = H + \Lambda$. Therefore for every $x \in E$ there exists $\lambda \geq 0$ such that $x \in \lambda\overline{x} + H$. It follows from this that the set $\Omega_{x,\overline{x}} = \{\lambda \in \mathbb{R}_+ : \lambda\overline{x} \in x - H\}$ is nonempty, hence $\overline{\lambda} = \inf\{\lambda \ : \ \lambda \in \Omega_{x,\overline{x}}\} < +\infty$.

Let $G = (g_1, g_2) \in \mathcal{D}_{\overline{x}}$, i.e. $[g_2 - g_1, \overline{x}] \geq 0$, and let $x \in E$. Then

$$p_G(x) = \inf_{x_1 + x_2 = x,\ x_1 \in H,\ x_2 \in \Lambda} \{[g_1, x_1] + [g_2, x_2]\} =$$

$$[g_1, x] + \inf_{x_2 \in \Lambda,\ x - x_2 \in H} [g_2 - g_1, x_2] = [g_1, x] + \inf_{\lambda \in \Omega_{x,\overline{x}}} [g_2 - g_1, \lambda\overline{x}] =$$

$$[g_1, x] + [g_2 - g_1, \overline{\lambda}\overline{x}] = [g_1, \overline{y}] + [g_2, \overline{\lambda}\overline{x}],$$

where $\overline{y} = x - \overline{\lambda}\overline{x} \in H$.

As the sublinear function p_G is continuous and $\operatorname{dom} p_G = E$ then the support set $\partial p_G = \sigma^*(G)$ is a nonempty weakly compact set, and there exists $\overline{f} \in \sigma^*(G)$ such that

$$p_G(x) = \max_{f \in \sigma^*(G)} [f, x] = [\overline{f}, x].$$

Therefore

$$[g_1, \overline{y}] + [g_2, \overline{\lambda}\overline{x}] = [\overline{f}, x] \quad (x \in E).$$

And hence the decomposition $\overline{X} = (\overline{y}, \overline{\lambda}\overline{x}) \in \sigma_{H,\Lambda}(x)$ is weakly-efficient with respect to $\mathcal{D}_{\overline{x}}$. (Indeed, $G \in \mathcal{D}_{\overline{x}}$, $\overline{f} \in \sigma^*(G)$ and $[G, \overline{X}] = [\overline{f}, x]$.)

Let a pair $(z_1, z_2) \in \sigma_{H,L}(x)$ $(x \in E)$ be such that

$$(\overline{y}, \overline{\lambda}\overline{x}) - (z_1, z_2) \in H \times \Lambda.$$

Since $z_2 \in \Lambda$ there exists $\widetilde{\lambda} \geq 0$ such that $z_2 = \widetilde{\lambda}\overline{x}$.

The fact that $\overline{\lambda}\overline{x} - \widetilde{\lambda}\overline{x} \in \Lambda$ implies $\overline{\lambda} \geq \widetilde{\lambda}$. Then the definition of $\overline{\lambda}$ yields $\overline{\lambda} = \widetilde{\lambda}$. So, $\overline{y} = z_1$ and $(z_1, z_2) - (\overline{y}, \overline{\lambda}\overline{x}) \in H \times \Lambda$, and the proof is complete. □

Let a vector lattice (E, H) be given with the order $\geq_H$ induced by the cone H. As it was mentioned in Section 3.1, for each $x \in E = H - H$ the lattice decomposition $\widetilde{X} = (x \vee 0, x \wedge 0) \in H \times (-H)$, where $x \vee 0 = \sup(x, 0)$, $x \wedge 0 = \inf(x, 0)$ is the unique efficient decomposition of the element x. Furthermore, it is an ideal decomposition, i.e. for all $(x_1, x_2) \in H \times (-H)$ such that $x_1 + x_2 = x$ the inclusion

$$(x_1, x_2) - (x \vee 0, x \wedge 0) \in H \times (-H)$$

holds.

Theorem 3.2.5. *In a vector lattice (E, H) the decomposition $(x \vee 0, x \wedge 0) \in \sigma(x) = \sigma_{H,-H}(x)$ is weakly-efficient with respect to $\mathcal{D} = H^* \times (-H^*)$ for any $x \in E$, furthermore, the relation*

$$p_G(x) = [g_1, x \vee 0] + [g_2, x \wedge 0]$$

is valid for any $G = (g_1, g_2) \in H^* \times (-H^*)$.

Proof. Let $G = (g_1, g_2) \in H^* \times (-H^*)$, then $g_1 - g_2 \in H^*$ and the following equalities hold:

$$\begin{aligned} p_G(x) &= \inf_{x_1+x_2=x,\ x_1 \in H,\ x_2 \in -H} \{[g_1, x_1] + [g_2, x_2]\} \\ &= \inf_{x_1 \in H,\ x-x_1 \in -H} \{[g_1, x_1] + [g_2, x - x_1]\} \\ &= [g_2, x] + \inf_{x_1 \in H, x-x_1 \in -H} [g_1 - g_2, x_1] \\ &= [g_2, x] + \inf_{x_1 \geq_H 0,\ x_1 \geq_H x} [g_1 - g_2, x_1] \\ &= [g_2, x] + \inf_{x_1 \geq_H x \vee 0} [g_1 - g_2, x_1] \\ &= [g_2, x] + [g_1 - g_2, x \vee 0] = [g_1, x \vee 0] + [g_2, x - x \vee 0] \\ &= [g_1, x \vee 0] + [g_2, x \wedge 0]. \end{aligned}$$

Let $x \in E$. As the sublinear function p_G is continuous then the support set $\partial p_G = \sigma^*(G)$ is a nonempty weakly compact set, and there exists $\tilde{f} \in \sigma^*(G)$ such that

$$p_G(x) = \max_{f \in \sigma^*(G)} [f, x] = [\tilde{f}, x].$$

Let $\widetilde{X} = (x \vee 0, x \wedge 0)$. Then the quadruple of vectors $(x, \widetilde{X}, \tilde{f}, G)$ is compatible with respect to the mapping σ and hence the decomposition $\widetilde{X} = (x \vee 0, x \wedge 0)$ is weakly-efficient with respect to $\mathcal{D} = H^* \times (-H^*)$. □

Theorem 3.2.6. *Let closed cones K_1 and K_2 be given in a reflexive space E and let $x \in E$ be a point such that the positive germ $K_1 \cap (x - K_2)$ is bounded. Then there exists a weakly-efficient decomposition with respect to an arbitrary nonempty subset $\mathcal{D}$ of $K_1^* \times K_2^*$ in $\sigma_{K_1,K_2}(x)$.*

Proof. Let $G = (g_1, g_2) \in \mathcal{D}$. We have

$$p_G(x) = \inf_{x_1+x_2=x,\ x_1 \in K_1, x_2 \in K_2} \{[g_1, x_1] + [g_2, x_2]\} =$$

$$= [g_2, x] + \inf_{x_1 \in K_1,\ x-x_1 \in K_2} [g_1 - g_2, x_1] = [g_2, x] + \inf_{x_1 \in K_1 \cap (x-K_2)} [g_1 - g_2, x_1].$$

Since E is reflexive and the set $K_1 \cap (x - K_2)$ is bounded it follows that this set is weakly compact, hence there exists $\bar{x}_1 \in H \cap (x - K_2)$ such that

$$\inf_{x_1 \in K_1 \cap (x - K_2)} [g_1 - g_2, x_1] = [g_1 - g_2, \bar{x}_1].$$

Let $\bar{x}_2 = x - \bar{x}_1$. Then

$$p_G(x) = [g_2, x] + \inf_{x_1 \in K_1 \cap (x-K_2)} [g_1 - g_2, x_1] = [g_2, x] + [g_1 - g_2, \bar{x}_1] = [g_1, \bar{x}_1] + [g_2, \bar{x}_2].$$

As the sublinear function p_G is continuous, then the support set $\partial p_G = \sigma^*(G)$ is a nonempty weakly compact set, and hence for the given x there exists $\overline{f} \in \sigma^*(G)$ such that

$$p_G(x) = \max_{f \in \sigma^*(G)} [f, x] = [\overline{f}, x], \quad (x \in E).$$

Then

$$[g_1, \bar{x}_1] + [g_2, \bar{x}_2] = [\overline{f}, x].$$

And hence the decomposition $\bar{X} = (\bar{x}_1, \bar{x}_2) \in \sigma_{K_1,K_2}(x)$ is weakly-efficient with respect to $\mathcal{D}$, since $G \in \mathcal{D}$, $\overline{f} \in \sigma^*(G)$ and $[G, \bar{X}] = [\overline{f}, x]$. □

Remark 3.2.1. The result of Theorem 3.2.6 holds if we assume that H is a weakly locally compact cone in an arbitrary (not necessarily reflexive) Banach space E.

3.3 A weakly-efficient decomposition with respect to the support set

In this section we will continue to study a weakly-efficient decomposition in the case where the set $\mathcal{D} \subset \mathcal{K}$ is w*-compact. Then this set coincides with the support set of a sublinear function Q, namely $Q(x) = \max\{[g, x] : g \in \mathcal{D}\}$. It is convenient to use the function Q later.

Let $E = (E : K_1, \ldots, K_n)$ be a Banach space. Assume that the space $(E^n)'$ is equipped by the order relation $\geq_K$ generated by the cone $K = K_1^* \times \ldots \times K_n^*$.

The following theorem provides us a method for finding a weakly-efficient decomposition with respect to the support set of a sublinear function (the existence of min on σ is assumed).

Theorem 3.3.1. *Let a continuous sublinear function $Q : E^n \to \mathbb{R}$ be monotone. Let the decomposition mapping σ be bounded. Let $\bar{x} \in E$ be an element such that argmin$\{Q(X) : X \in \sigma(\bar{x})\} \neq \emptyset$. Then any decomposition $\bar{X} \in$ argmin$\{Q(X) : X \in \sigma(\bar{x})\}$ is a weakly efficient decomposition with respect to $\mathcal{D} = \partial Q$.*

Proof. Consider the function

$$s_Q(x) = \inf_{X \in \sigma(x)} Q(X), \qquad x \in E.$$

Let $\overline{X} \in \operatorname{argmin}\{Q(X) \; : \; X \in \sigma(\bar{x})\}$, i.e. $Q(\overline{X}) = s_Q(\bar{x})$.

As the function Q satisfies the assumptions of Lemma 1.3.14, then

$$s_Q(x) = \sup_{G \in \partial Q} p_G(x), \; x \in E.$$

Here the support set $\partial Q \subset K^*$, due to the monotonicity of Q.

Since the function Q is sublinear and continuous then ∂Q is a nonempty and weakly compact set, therefore there exists a vector $\overline{G} \in \partial Q$ such that $s_Q(\bar{x}) = p_{\overline{G}}(\bar{x})$.

As $\overline{X} \in \operatorname{argmin}\{Q(X) \; : \; X \in \sigma(\bar{x})\}$ and

$$p_{\overline{G}}(\bar{x}) = \inf_{X \in \sigma(\bar{x})} [\overline{G}, X],$$

then

$$[\overline{G}, \overline{X}] = p_{\overline{G}}(\bar{x}).$$

Since the sublinear function Q is continuous it follows tat this function is bounded. The boundness of Q and σ implies the boundness (hence, continuity) of the function $p_{\overline{G}}$. Then Corollaries 1.3.6 and 1.3.7 imply

$$p_{\overline{G}}(\bar{x}) = \sup_{f \in \sigma^*(\overline{G})} [f, \bar{x}].$$

It also follows from the boundness of $p_{\overline{G}}$ that $\sigma^*(\overline{G})$ is a nonempty weakly compact set, then there exists a vector $\overline{f} \in \sigma^*(\overline{G})$ such that $p_{\overline{G}}(\bar{x}) = [\overline{f}, \bar{x}]$.

As a result for the vector $\bar{x}$ and the vector $\overline{X} \in \sigma(\bar{x})$ we have found two elements $\overline{G} \in \mathcal{D} = \partial Q$ and $\overline{f} \in \sigma^*(\overline{G})$ such that

$$[\overline{f}, \bar{x}] = [\overline{G}, \overline{X}],$$

and the weak efficiency of the decomposition $\overline{X} \in \sigma(\bar{x})$ is proved. □

From the theorem it follows, that the problem of the existence of a weakly-efficient decomposition with respect to the support set of a sublinear monotone function is closely related to the problem of the existence of min of this function on the set $\sigma(x)$, $x \in E$.

3.4 Efficiency in 2-vector lattices

In this section we will give a brief discussion on ideal decomposition (see Definition 3.1.2) in a 2-vector lattice $E = (E; K_1, K_2)$. The decomposition mapping $\sigma : E \to 2^{E^2}$ with respect to the pair of cones $(K_1, -K_2)$ is expressed in the form:

$$\sigma_{K_1,-K_2}(x) = \{X = (x_1, x_2) \in K_1 \times (-K_2) \ : \ x_1 + x_2 = x\} \ (x \in E),$$

and with respect to the pair of cones $(K_2, -K_1)$ it can be written in the form:

$$\sigma_{K_2,-K_1}(x) = \{X = (x_1, x_2) \in K_2 \times (-K_1) \ : \ x_1 + x_2 = x\} \ (x \in E).$$

Theorem 3.4.1. *Let $E = (E; K_1, K_2)$ be a 2-vector lattice, and $x \in E$, then*

1. *if $x = y - z$ where $y \in K_1$, $z \in K_2$, then $y \geq_{K_i} x'_+$, $z \geq_{K_i} x'_-$, $i = 1, 2$, i.e. the pair $(x'_+, -x'_-) \in \sigma_{K_1,-K_2}(x)$ is an ideal decomposition with respect to the cone $K_1 \times (-K_2)$.*

2. *if $x = z - y$ where $z \in K_2$, $y \in K_1$, then $z \geq_{K_i} x''_+$, $y \geq_{K_i} x''_-$, $i = 1, 2$, i.e. the pair $(x''_+, -x''_-) \in \sigma_{K_2,-K_1}(x)$ is an ideal decomposition with respect to the cone $K_2 \times (-K_1)$.*

Proof. 1. Since $y \geq_{K_1} 0$, $z = y - z \geq_{K_2} 0$, or $y \geq_{K_1} 0$, $y \geq_{K_2} x$ then the definition of x'_+ yields $y \geq_{K_i} x'_+$, $i = 1, 2$. Since $0 \geq_{K_1} -y = -x - z$, $z \geq_{K_2} 0$ or $z \geq_{K_1} -x$, $z \geq_{K_2} 0$, then

$$z \geq_{K_i} \mathrm{Sup}\{-x; 0\} = -\mathrm{Inf}\{x; 0\} = x'_-, \ i = 1, 2.$$

2. The relations $z \geq_{K_1} x$, $z \geq_{K_2} 0$ imply $z \geq_{K_i} x''_+$, while the relations $y \geq_{K_1} 0$ and $y \geq_{K_2} -x$ imply that $y \geq_{K_i} \mathrm{Sup}\{0; -x\} = -\mathrm{Inf}\{0; x\} = x''_-$, $i = 1, 2$.

□

Theorem 3.4.2. *Let the function p_G defined on E by*

$$p_G(x) = \inf_{y \in \sigma_{K_1,-K_2}(x)} [G, Y], \qquad x \in E$$

be continuous for every $G \in K_1^* \times K_2^*$. Then the decomposition $X' = (x'_+, -x'_-) \in \sigma_{K_1,-K_2}(x)$ is weakly-efficient in a 2-vector lattice $E = (E; K_1, K_2)$ with respect to the set $\mathcal{D}' = K_1^* \times (-K_2^*)$

Proof. Let $x \in E = K_1 - K_2,\ G = (g_1, g_2) \in K_1^* \times (-K_2^*)$, then $g_1 - g_2 \in K_1^* + K_2^*$ and the following equalities hold with respect to the function p_G:

$$p_G(x) = \inf_{Y=(y_1,y_2)\in\sigma_{K_1,-K_2}(x)} [G, Y] =$$

$$= \inf_{y_1+y_2=x,\ y_1\in K_1,\ y_2\in -K_2} \{[g_1, y_1] + [g_2, y_2]\} =$$

$$= [g_2, x] + \inf_{y_1\in K_1,\ x-y_1\in -K_2} [g_1 - g_2, y_1] =$$

$$= [g_2, x] + \inf_{y_1\geq_{K_1} 0,\ y_1\geq_{K_2} x} [g_1 - g_2, y_1] =$$

$$= [g_2, x] + \inf_{y_1\geq_{K_i} x'_+,\ i=1,2} [g_1 - g_2, y_1] =$$

$$= [g_2, x] + [g_1 - g_2, x'_+] = [g_1, x'_+] + [g_2, -x'_-].$$

Since the sublinear function p_G is continuous and $\operatorname{dom} p_G = E$ $(G \in K_1^* \times (-K_2^*))$ then the support set $\partial p_G = \sigma^*_{K_1,-K_2}(G)$ is a nonempty weakly compact set, and

$$p_G(x) = \max_{f\in\sigma^*_{K_1,-K_2}(G)} [f, x] = [f', x]\ (x \in E).$$

Thus the quadruple (x, X', f', G) is compatible with respect to the mapping σ and therefore the decomposition $X' = (x'_+, -x'_-) \in \sigma_{K_1,-K_2}(x)$ is weakly-efficient with respect to $\mathcal{D}' = K_1^* \times (-K_2^*)$. □

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