

The First-fit Chromatic and Achromatic Numbers

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Abstract

This project involved pulling together past work on the achromatic and first-fit chromatic numbers, as well as a description of a proof by Yegnanarayanan et al. Our work includes attempting to find patterns for them in specific classes of graphs and the beginnings of an attempt to prove that for any given a, b, c , such that $2 \leq a \leq b \leq c$, there exists a graph with chromatic number a , first-fit chromatic number b , and achromatic number c .

Executive Summary

This project involved looking at properties of two chromatic invariants of graphs. The first-fit chromatic number is the largest number of colors which may be used in a greedy coloring. More precisely it is defined to be the maximum size of a collection of ordered sets such that no two vertices in the same set are connected and every vertex is connected to an element of all previous sets. The achromatic number is the maximum number of colors which can be used in a proper coloring such that all pairs of colors are adjacent somewhere in the graph. A complete coloring is one which has all pairs of colors adjacent somewhere in the graph.

Tying together what has been done in the past, we read journal articles about the first-fit and achromatic numbers and share some of these results. In the general case, both of these are NP-hard, and they both have been proven to remain so when restricted to some classes of graphs.

We also looked at Nordhaus-Gaddum type results. Much of this was gathering together results in the past. However, we also looked at when the inequalities were at the bound in self-complementary graphs. There was also not progress in this section.

We look at the achromatic number of cycles as well as attempting to look at the achromatic number of other classes of graphs. If k is an odd number of colors then cycles with at least $\binom{k}{2}$ edges can be colored in a complete coloring with k colors. If k is an even number of colors, then cycles with $\frac{k^2}{2}$ or more edges can be colored in a complete coloring with k colors. We also looked some at trees and regular graphs, though there were no results.

We describe a proof by V. Yegnanarayanan, R. Balakrishnan and R. Sampathkumar about how for every $a, b,$ and c such that $2 \leq a \leq b \leq c$, there exists a graph with chromatic number a , achromatic number b , and pseudoachromatic number c . (The pseudoachromatic number requires all pairs of colors to exist adjacent somewhere in the graph but does not require a proper coloring.) This proof functions by constructing a graph with this property [25].

Basing off of that proof, we conjecture that for every $a, b,$ and c such that $2 \leq a \leq b \leq c$ there exists a graph with chromatic number a , first-fit chromatic number b and achromatic number c . We begin an attempt to prove this in this report, giving a graph with a predictably varying first-fit chromatic number that appears like it should have a similarly predictable achromatic number. However, the proof for achromatic number of this family of graphs has not been completed, preventing progress beyond this point.

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Chapter 1

Introduction

1.1 Summary of this Project

This project consisted of looking at the first-fit and achromatic numbers of various graphs and properties of these two parameters. The paper *Inequalities for the First-Fit Chromatic Number*, which included a Nordhaus-Gaddum type result on the first-fit chromatic number, inspired the project. It began with researching Nordhaus-Gaddum type results, before extending to the achromatic number upon following a reference from this paper. This led to comparing the first-fit chromatic number and the achromatic number of graphs, which led to the achromatic number of cycles. After this the project focused on the achromatic number - looking at cycles and trees. Research on the achromatic number led to finding a proof for the existence of graphs with chromatic number a , achromatic number b , and pseudoachromatic number c (for any a, b , and c , when $2 \leq a \leq b \leq c$), which is discussed in section 6.1. This tied the project back to the first-fit chromatic number by using this as inspiration to work on an equivalent proof for the chromatic number, first-fit chromatic number, and achromatic number (increasing in the order given here).

The definitions for the various colorings mentioned are given in 2.1.

1.2 Graph Colorings

Graph coloring problems deal with labeling the vertices (or edges) of a graph in ways that fulfill given constraints. In many of these cases we are dealing with proper colorings - colorings such that no two adjacent vertices receive the the same color. The invariants are often the minimum or maximum number of colors which can be used while fulfilling constraints. For example, the chromatic number is the fewest number of colors which can be used in a proper coloring.

For this we have been primarily working with the first-fit chromatic number and the achromatic number, both of which are properties of graphs which are the largest number of colors usable in a proper coloring of a graph with given

properties. For the first-fit chromatic number every vertex colored color i must be adjacent to all colors less than i . For the achromatic chromatic number, for every pair of colors i and j , a vertex of color i and a vertex of color j must be adjacent somewhere in the graph. We began from a paper on the first-fit chromatic number, chosen because it was an interesting and recent paper. Because both the achromatic number and the first-fit chromatic number are the maximum number of colors able to be used, rather than fewest, some of the methods of approaching the problem we had developed with the first-fit chromatic number also were applicable for finding the achromatic number.

Graph coloring problems can be applied in many situations. Graph coloring problems began from an application: the coloring of a map. Since then, colorings approximating chromatic colorings have been used for solving scheduling and routing problems [17]. Radio frequency assignment similarly is an application of a chromatic coloring [23]. A less direct application of graph coloring is in circuit testing. Create a graph where vertices are the nets of the circuit board. Vertices are adjacent if there is some likelihood of a short between them. This allows the graph to be colored a proper coloring, and pairs of the color classes to be tested for shorts - greatly reducing the number of tests required to check for any shorts on the circuit board [17].

Both the first-fit chromatic and achromatic numbers thus are useful because they are the worst cases of two simple approximation algorithms for the chromatic number. Complete and pseudocomplete colorings are also used in network design - creating clusters such that the clusters are small and they can all directly talk to all other clusters [14].

Chapter 2

Graph Colorings

2.1 Chromatic Definitions

2.1.1 The Chromatic Number

The chromatic number is the most well known property when it comes to graph colorings.

Definition 1. The *chromatic number* is the minimum number of colors which can be used in a proper coloring of a graph. We represent it with $\chi(G)$.

2.1.2 The First-fit Chromatic Number

The first-fit chromatic number, also known as the Grundy number, is the largest number of colors possible in a greedy coloring. We represent it as $\chi_{ff}(G)$

Definition 2. The *first-fit chromatic number*, or the Grundy number, is the maximum size of an ordered collection of independent sets such that no two vertices in the same set are connected, and every vertex is connected to an element of each of the previous sets. Each of these independent sets has a color.

This is equivalent to, the largest number of colors which can be used in a proper coloring such that with the colors in a total ordering, all vertices are connected to a vertex of all colors less than the color of said vertex. This is exhibited in figure 2.1

2.1.3 The Achromatic Number

The achromatic number is another characteristic of the graph corresponding to the maximum number of colors used in a coloring of the graph with a specific constraint. This constraint is that the coloring must be complete.

Definition 3. A *complete* coloring is a proper coloring in which for any two colors i and j , there is a pair of adjacent vertices in the graph, one of which is colored i and one of which is colored j .

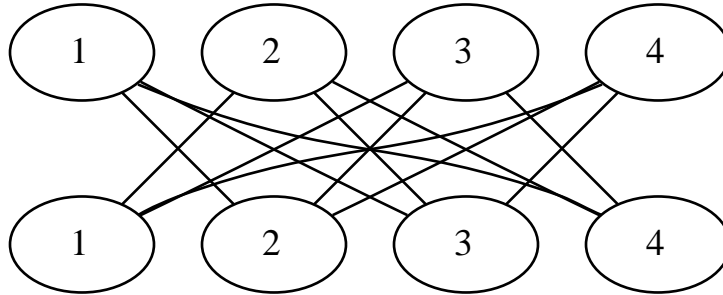


Figure 2.1: The first-fit coloring of an example graph

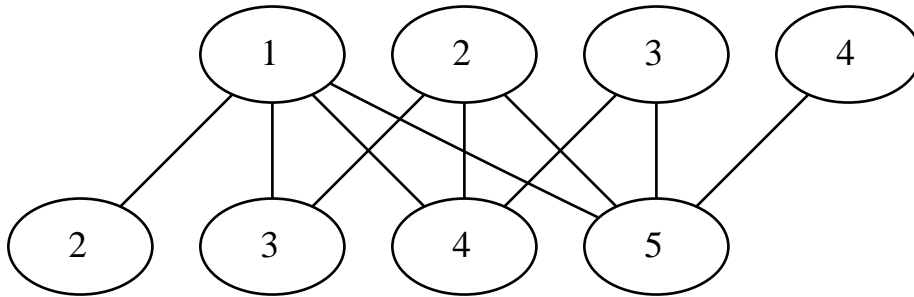


Figure 2.2: An example graph with its achromatic coloring

This allows us to formally define the achromatic number.

Definition 4. The *achromatic number* of a graph G , represented in this paper as $\alpha(G)$ is the maximum number of colors used in a complete coloring of G .

This is exhibited in figure 2.2

2.1.4 The Pseudoachromatic Number

Similar to the achromatic number, but without the requirement of being a proper coloring, there is the pseudoachromatic number.

Like for a complete coloring, we have a definition for a pseudocomplete coloring.

Definition 5. A *pseudocomplete* coloring is a coloring (not necessarily proper) in which any two colors, i and j , are the colors of adjacent vertices somewhere in the graph

Similarly we can define the pseudoachromatic number in terms of a pseudocomplete coloring.

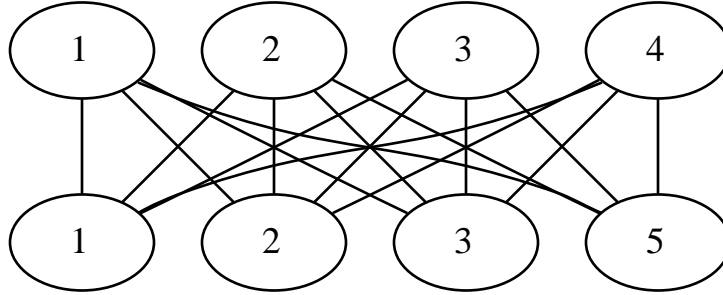


Figure 2.3: An example graph with its pseudoachromatic coloring

Definition 6. The *pseudoachromatic number* of a graph G , represented in this paper as $\psi(G)$ is the maximum number of colors used in a pseudocomplete coloring of G .

This is exhibited in figure 2.3

2.2 The Relationship Between the Various Colorings

Claim 1. $\chi_{ff}(G) \geq \chi(G)$

Proof. The chromatic number is the fewest number of colors which can be used in a proper coloring. The first-fit chromatic number must be a proper coloring, thus $\chi_{ff}(G) \geq \chi(G)$ because the existence of a proper coloring using fewer colors than the chromatic number is a contradiction. \square

Claim 2. $\alpha(G) \geq \chi_{ff}(G)$

Proof. For a first-fit coloring all vertices are connected to a vertex of all previous colors. This means that for every color, every vertex of that color is connected to a vertex of all previous colors. This means that for every pair of colors, there exists some adjacent vertices colored those colors in any first-fit coloring. Thus $\alpha(G) \geq \chi_{ff}(G)$. \square

Claim 3. $\psi(G) \geq \alpha(G)$

Proof. Both achromatic and pseudoachromatic colorings require that all pairs of colors are adjacent somewhere in the graph. As pseudocomplete is a strictly weaker constraint than complete, all complete graphs are pseudocomplete and $\psi(G) \geq \alpha(G)$. \square

Claim 4. *All chromatic colorings are complete.*

Proof. Assume there is a chromatic coloring which is not complete.

Because this coloring is not complete, there is some pair of colors, i and j , which are never adjacent. Because these are never adjacent, changing the color of all vertices colored j to color i , still is a legal coloring, as this cannot force two vertices of the same color to be adjacent.

However, this coloring uses one fewer color than the initial coloring, making the initial coloring not a chromatic coloring. This is a contradiction.

Therefore all chromatic colorings are complete. \square

2.3 Necessary Constraints

In order to fulfill the constraints required by these colorings, there are necessary but not sufficient conditions.

Number of edges and the achromatic number The achromatic number requires all pairs of colors to be adjacent somewhere in the graph. For k colors the graph must contain $\binom{k}{2}$ edges in order for all pairs to be possible. Thus the achromatic number can be at most be the maximum k such that $\binom{k}{2} \geq |E(G)|$.

Max degree and the achromatic number In order for a graph to have achromatic number k , there must be enough connectivity for all colors to be adjacent to a vertex of all other colors. By the pigeonhole principle, with n vertices and k colors, there must be some color, i , with no more than $\lfloor \frac{n}{k} \rfloor$ vertices colored i . These vertices cannot each be adjacent to more than Δ other vertices, and in order for all color pairs to exist, this must be enough vertices for an adjacency to all other colors. This gives, in the general case, the lower bound,

$$\lfloor \frac{n}{k} \rfloor \cdot \Delta \geq k - 1$$

. In the case of a regular graph, this claim is strengthened as all vertices have the same degree.

Chapter 3

Literature Review

3.1 Complexity Classes

It is relevant to know what the complexity classes of problems being worked on are. Here we include when the achromatic number and first-fit chromatic numbers have been determined to be NP-hard or not.

3.1.1 Achromatic Number

Finding the achromatic number of a graph is NP-hard; determining whether it is greater than some number is NP-complete. This was proven in [24] using minimum edge dominating sets. In this paper, they also prove that even when limited to the complements of bipartite graphs, it is an NP-hard problem [24].

Determining whether the achromatic number of a graph is at least some number, k , remains NP-complete for many classes of graphs. When restricted to bipartite graphs it remains NP-complete [11]. Restricting further, it remains NP-complete when restricted to only trees [6]. Showing the NP-completeness when restricted to trees involved reducing to the harmonious number problem, the fewest number of colors required if every pair of colors is adjacent at most once. According to [4], few problems on unlabeled graphs remain NP-complete when limited to cographs and interval graphs, though the achromatic number problem is one which remains NP-complete.

However, there are also classes of graphs for which finding the achromatic number is not NP-hard. While the achromatic number of trees is NP-hard, the achromatic number of bounded degree trees is not. Similarly, the achromatic number of bounded degree forests is polynomial solvable [7]. It is also not an NP-hard problem on complements of trees; the achromatic number can be reduced to the edge dominating problem on the complement of the graph, and this problem has a linear time solution for trees [24].

3.1.2 First-fit Chromatic Number

For any given graph, G integer k , whether $\chi_{ff}(G) \geq k$ has a polynomial time solution. However, the first-fit chromatic number is in general an NP-hard problem. It remains NP-hard when limited to the complements of bipartite graphs, as shown in the same paper. Similar to the achromatic number, these results were found by reduction to the edge dominating number of their complements [26].

However, restricting to certain classes of graphs can cause the first-fit chromatic number to be polynomial solvable. [1] provides a proof that P_4 -reducible, extended P_4 -reducible, P_4 -sparse, extended P_4 -sparse, P_4 -extendable, P_4 -lite, P_4 -tidy, P_4 -laden, and extended P_4 -laden graphs can have their first-fit chromatic number found in polynomial time [1].

3.2 Literature Review of the Achromatic Number

The achromatic number is known for a variety of families of graphs. One paper covers the achromatic number of wheel graphs, gear graphs, and other central graphs [21]. Another covers the central graphs of banana graph, helm graph, and web graph [22].

Details about the achromatic number are also known. For an m -dimensional hypercube it is known that there exist constants c_1 and c_2 independent of m such that $c_1\sqrt{m2^{m-1}} \leq \psi(G) \leq c_2(\sqrt{m2^{m-1}})$, however it is not known what these constants are [19].

Looking at unions of cycles, if there are k cycles c_1 through c_k with lengths l_1 through l_k , then a cycle of length $p = \sum_{i=1}^k l_i$ has the same achromatic number as $\cup_{i=1}^k c_i$, if $k \leq \sqrt{\frac{p}{2}}$. That is the achromatic number of a cycle is the same as the achromatic number of a union of cycles with the same size if the number of cycles in the latter graph is less than the $\sqrt{\frac{p}{2}}$. In some cases, there is not a limit on the number of cycles on the graph. Cycles of length $3k$ always have the same achromatic number as k 3-cycles, while cycles of $4k$ always have the same achromatic number as k 4-cycles [15].

There are similar results on paths. The achromatic number of a disjoint union of paths of k paths with lengths a_1 through a_k is the largest number such that $\sum_{i=1}^k a_i \geq \binom{n}{2} + f(k, n)$ where $f(k, n) = 0$ if n is odd, $f(k, n) = 0$ if n is even and $k \geq \frac{n}{2}$, and $\frac{n}{2} - k$ if n is even and $k < \frac{n}{2}$. [16]

There is another paper that proves that there is only a finite number of irreducible graphs with any given achromatic number. It also describes all graphs with achromatic number less than 4 [13].

While it had been conjectured that for any tree, the achromatic number and the pseudoachromatic number were the same, this has been proven false [9].

Because finding the achromatic number is NP-complete, there are approximation algorithms. It has been proven that the achromatic number of a graph can be approximated within $O(n/\sqrt{\log_2(n)})$ where n is the number of vertices

[8]. This paper also provides approximation algorithms for a general graph, trees, and graphs with large girth [8]. Similarly, there is an approximation for bipartite graphs [14].

3.3 Literature Review on the First-fit Chromatic Number

Less work has been done on the first-fit chromatic number than the achromatic number, and for it most of the work has been under the other name, Grundy number.

Take a list of all the vertices in the graph, and then color them in order such that all vertices must be a previous color if this is a proper coloring. Also, the color for v_i for any i must be the color which results in the fewest colors used when there are multiple options for the color of v_i . The maximum number of number of colors used when varied over the orderings of the vertices is called the ochromatic number. This has been shown to be equivalent to the first-fit chromatic number [10].

A new variant, the partial Grundy coloring, is also defined in [10]. This varies from the first fit in that not every vertex must be a Grundy vertex - or fulfill the constraint we have been calling the first-fit constraint. Instead every color class must have one Grundy vertex [10].

There are also results on the products of graphs and their first-fit chromatic numbers. This includes inequalities as well as two cases where the the first-fit chromatic number of the product of two graphs is known. When G is a tree or when $\chi_{ff}(G) = \Delta(G) + 1$, then $\chi_{ff}(G[H]) = \chi_{ff}(G) \times \chi_{ff}(H)$ [2].

Because finding the first-fit chromatic number is NP-hard, people approximate it. Some of these approximations are for specific classes of graphs. A linear time algorithm for the partial Grundy number on trees is provided in [20].

Chapter 4

Nordhaus-Gaddum type results

In the beginning of this project we researched what has been done in the past for Nordhaus-Gaddum type results as well as looking at graphs and their complements.

4.1 Our Attempts

Part of what was done for this project was looking at graphs for nordhaus-gaddum type results. Attempts were made for finding a nordhaus-gaddum type result for the achromatic number before finding the result in a book. Similarly attempting to combine the chromatic number, first-fit chromatic number, and achromatic number by looking at $A(G) + B(G^c)$ did not lead to any progress, and later results were found by looking through papers.

One thing we did was looked at self-complementary graphs to see when they reached the bounds for the nordhaus-gaddum type inequalities. However, as this was done before figuring out how to use the computer to help look, it was generally done by hand and thus only on small graphs. For graphs on 8 or fewer vertices, only one of $\chi(G) + \chi(G^c)$ and $\chi_{ff}(G) + \chi_{ff}(G^c)$ could possibly be reached because of parity.

Both of the self-complementary graphs of order 5 reach the bound for the Nordhaus-Gaddum inequality, $\chi(G) + \chi(\bar{G}) = 6 = n + 1$. However, neither of these reach the bound for the first-fit chromatic number. This would be impossible because with 5 vertices, the bound is $n + 2$ or 7, which is not divisible by 2.

We also looked at two of order 8. Both of these fulfilled the bound for the first-fit chromatic number, $\chi_{ff}(G) + \chi_{ff}(G^c) = 10 = 8 + 2$, and were one below on the Nordhaus-Gaddum inequality.

These were also found for graphs with greater order using a computer, however the results were unfortunately lost in a hard drive failure.

4.2 Past Work with Nordhaus-Gaddum Type Results

Nordhaus and Gaddum proved in 1956 that for a graph G with n vertices $\lceil 2\sqrt{n} \rceil \leq \chi(G) + \chi(G^c) \leq n + 1$. Results of this sort are referred to as Nordhaus-Gaddum type results.

This project began with understanding the paper, [12]. This paper included Nordhaus-Gaddum type results for the first-fit chromatic number. It includes a general Nordhaus-Gaddum type bound depending on the number of vertices. If $8 \geq n \geq 3$, then $\chi_{ff}(G) + \chi_{ff}(G^c) \leq n + 2$, if $n = 9$ then $\chi_{ff}(G) + \chi_{ff}(G^c) \leq n + 3$, and if $n \geq 10$ then $\chi_{ff}(G) + \chi_{ff}(G^c) \leq \lceil \frac{5n+2}{4} \rceil$ [12].

The first-fit chromatic number of families of graphs lead to some other Nordhaus-Gaddum type results. If G is a bipartite graph then $\chi_{ff}(G) + \chi_{ff}(G^c) \leq n + 2$ [12]. If you define an almost regular graph to be a graph which has the degree of vertices vary by one, then for a regular or almost regular graph, G , $\chi_{ff}(G) + \chi_{ff}(G^c) \leq n + 2$ [26]. This same paper provides a Nordhaus-Gaddum type result on forests. For a forest, G with n vertices and k components, $\chi_{ff}(G) + \chi_{ff}(G^c) \leq n - k + 3$ [26].

For a general simple graph, G , with n vertices, there is also Nordhaus-Gaddum type inequalities for the achromatic number and the pseudoachromatic number.

$$\alpha(G) + \alpha(G^c) \leq \lceil \frac{4n}{3} \rceil$$

$$\psi(G) + \psi(G^c) \leq \lceil \frac{4n}{3} \rceil$$

For all both of these, the bound is exact for every n [5].

Combining multiple of the invariants also has results.

$$\alpha(G) + \psi(G^c) \leq \lceil \frac{4n}{3} \rceil$$

$$\chi(G) + \psi(G^c) \leq \lceil n + 1 \rceil$$

There are also results for extremal graphs where $A(G)B(G^c)$ is minimum where A and B are each pair of the chromatic number, achromatic number, and pseudoachromatic numbers. Similarly there are some cases characterized for minimum $A(G) + B(G^c)$ [3].

Chapter 5

Classes of graphs

5.1 Cycles

This section describes the relationship between the achromatic number and the number of vertices in a cycle. This work was a combined effort of Andrew Marut, Ethan Thompson and I.

Before proving this, the original pattern was found by using a prolog program to determine the achromatic number of small cycles with $\binom{k}{2}$ edges. The code is in appendix B.

5.1.1 Odd Number of Colors

We now find the achromatic number of a cycle with $\binom{k}{2}$ edges and an odd number of colors.

Claim 5. *A cycle with $\binom{k}{2}$ edges with odd k has enough connectivity to be colored with k colors.*

Proof. As this is a cycle, $n = e = \binom{k}{2}$, and $\Delta = 2$. Substituting into the inequality gives us

$$\begin{aligned} \lfloor \frac{\binom{k}{2}}{k} \rfloor \cdot 2 &= k - 1 \\ &= \lfloor \frac{\frac{k!}{2(k-2)!}}{k} \rfloor \\ &= \lfloor \frac{(k-1)!}{2(k-2)!} \rfloor \cdot 2 \\ &= \lfloor \frac{k-1}{2} \rfloor \cdot 2 \end{aligned}$$

If k is odd, then $k - 1$ is even, so

$$\frac{k-1}{2} \cdot 2 = k - 1$$

$$k - 1 \geq k - 1 \quad \square$$

Thus there is enough connectivity for a cycle with $\binom{k}{2}$ edges to have achromatic number k .

Upper Bound

Using the inequality discussed above, $\lfloor \frac{n}{k} \rfloor * \Delta \geq k - 1$ we can algebraically show that a cycle with $\binom{k}{2}$ edges (and therefore vertices), fulfills this constraint without having enough edges for redundancy.

Claim 6. *A cycle with $\binom{k}{2}$ edges with odd k has achromatic number at most k*

Proof. $\binom{k+1}{2} > \binom{k}{2}$ and there are only $\binom{k}{2}$ edges in the graph. Therefore the achromatic number cannot be greater than k . \square

Lower Bound

We have an upper bound on the achromatic number of $C_{\binom{k}{2}}$ of k , because there is not enough edges for all color pairs given $k + 1$ colors. This means, if we can show that the achromatic number has a lower bound of k , then we have proven the achromatic number of $C_{\binom{k}{2}}$ is exactly k .

We can represent all color pairs with a complete graph, with each vertex associated with one color. If we can use this, to construct a cycle with all color pairs, we have shown that it is possible to color vertices such that all $\binom{k}{2}$ color pairs are present over the $\binom{k}{2}$ edges.

Claim 7. *A cycle with $\binom{k}{2}$ edges with odd k has a complete coloring with k colors, thus the achromatic number is k .*

Proof. We can represent all color pairs with k colors with K_k , with all vertices corresponding to a color. Because it is a complete graph, all vertices are adjacent to all other vertices, and every pair of colors is present.

If we traverse an Eulerian circuit of K_k , where each vertex corresponds to a color, then we have covered every pair of colors. Because k is odd, every vertex has an even degree and the graph is connected, therefore the graph is Eulerian.

The complete graph K_k has $\binom{k}{2}$ edges. We can construct a coloring of $C_{\binom{k}{2}}$ by coloring each vertex the color of the corresponding vertex in K_k while traversing the Eulerian circuit on K_k .

This leads to a complete coloring with k colors on $C_{\binom{k}{2}}$.

We have shown a lower bound on the achromatic number of $C_{\binom{k}{2}}$ which matches the upper bound. Therefore the achromatic number of $C_{\binom{k}{2}}$, with odd k , is k . \square

5.1.2 Even Number of Colors

Not Enough Edges

Using the inequality discussed above, $\lfloor \frac{n}{k} \rfloor * \Delta \geq k - 1$ we can algebraically show that a cycle with $\binom{k}{2}$ edges (and therefore vertices), does not fulfill this constraint without having enough edges for redundancy.

Claim 8. *A cycle with $\binom{k}{2}$ colors with even k does not have a complete coloring with k colors.*

Proof. As this is a cycle, $n = e = \binom{k}{2}$, and $\Delta = 2$. Substituting into the inequality gives us

$$\begin{aligned} \lfloor \frac{\binom{k}{2}}{k} \rfloor \cdot 2 &= k - 1 \\ &= \lfloor \frac{\frac{k!}{2(k-2)!}}{k} \rfloor \\ &= \lfloor \frac{(k-1)!}{2(k-2)!} \rfloor \cdot 2 \\ &= \lfloor \frac{k-1}{2} \rfloor \cdot 2 \end{aligned}$$

If k is even, then $k - 1$ is odd, so

$$\begin{aligned} \frac{k-2}{2} \cdot 2 &= k - 2 \\ k - 2 &\not\geq k - 1 \end{aligned} \quad \square$$

k=4

We can begin by looking at the case where $k = 4$. With 4 colors all colors must be adjacent to at least 3 different vertices. Because this is a 2-regular graph, this means that all colors must have two vertices colored that color. C_8 does have a complete coloring with 4 colors shown in figure 5.1.

In order to have enough connectivity, we needed to add $2 = 4/2 = k/2$ redundant edges.

Lower Bound

Like in the $k = 4$ case, in the general even case we have k colors, all of which need to be adjacent to $k - 1$ colors. This means that each color must have $\lceil \frac{k-1}{2} \rceil = \frac{k}{2}$ vertices colored that color. With k colors and $\frac{k}{2}$ vertices per color, there is a total of $\frac{k^2}{2}$ vertices (and edges). This is $\frac{k}{2}$ more edges than the minimum number of edges.

With even k and a cycle, $\binom{k}{2} + \frac{k}{2} = \frac{k^2}{2}$ edges is a lower bound on the minimum required for achromatic number k .

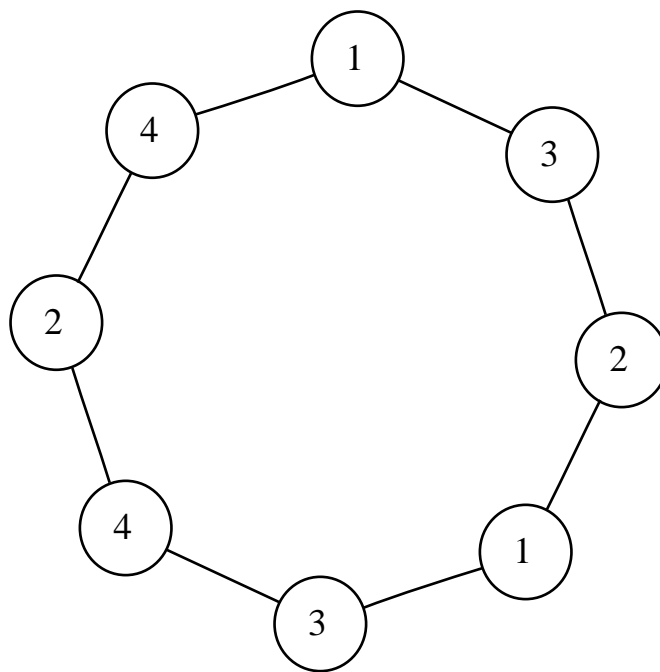


Figure 5.1: C_8 with a complete coloring of 4 colors

Upper Bound

We can represent all color pairs with a complete graph, with each vertex associated with one color and use this as a beginning for our construction.

Claim 9. *With even k and a cycle with $\frac{k^2}{2}$ edges, you can color this cycle in a complete coloring with k colors, thus the achromatic number is k .*

Proof. We can represent all color pairs with k colors with K_k , with all vertices corresponding to a color. Because it is a complete graph, all vertices are adjacent to all other vertices, and every pair of colors is present.

If we traverse an Eulerian circuit of K_k then we have covered every pair of colors. However, k is even, so every vertex has odd degree, and we do not have an Eulerian circuit.

We have an even number of vertices, so we can add edges connecting pairs of vertices until all vertices have an even degree. This adds $\frac{k}{2}$ edges.

The multigraph with order $\binom{k}{2}$ with K_k as a subgraph, and all vertices with degree k , has an Eulerian circuit.

This graph has $\binom{k}{2} + \frac{k}{2} = \frac{k^2}{2}$ edges. We can construct a coloring of $C_{\frac{k^2}{2}}$ by coloring each vertex the color of the corresponding vertex in our multigraph while traversing the Eulerian circuit on the multigraph.

This leads to a complete coloring with k colors on $C_{\frac{k^2}{2}}$.

We have shown an upper bound on the number of edges in a graph with achromatic number k which matches the lower bound. Therefore $C_{\frac{k^2}{2}}$ is the smallest cycle with achromatic number k for even k . \square

5.2 Regular graphs

While $\lfloor \frac{n}{k} \rfloor * \Delta \geq k - 1$ and $e \geq \binom{k}{2}$ are necessary and sufficient in the case of a cycle, these are not sufficient in the case of a general regular graph.

$K_{3,3}$ is a counterexample. $K_{3,3}$ has 6 vertices and 9 edges. If we attempted a complete coloring with 4 colors, the necessary constraints hold -

$$9 > \binom{k}{2} = \binom{4}{2} = 6$$

and

$$3 = 4 - 1 = k - 1 \leq \lfloor \frac{n}{k} \rfloor * \Delta = \lfloor \frac{6}{4} \rfloor * 3 = 3$$

However, $K_{3,3}$ does not have a complete coloring on 4 colors.

Claim 10. *$K_{3,3}$ does not have a complete coloring on 4 colors.*

Proof. For a complete coloring, all pairs of colors must appear adjacent somewhere in the graph. This means that there must be a vertex in the first independent set colored 1 and one in the second independent set colored 2. If there was to be a vertex colored a third color, it must fall into one of these two sets

because they form a partition. Because it is complete bipartite, this vertex is adjacent to one of the two previous vertices, of color i . This means we do not currently have a vertex colored 3, and a vertex colored j , adjacent at this point in time. If we are to color a new vertex j in order to fulfill this constraint, it must fall into the set the vertex colored 3 is not in. However, any vertex in the other set is also adjacent to a vertex already colored j . Because two adjacent vertices cannot be colored the same color, there is no complete coloring with 3 or more colors on a complete bipartite graph.

$K_{3,3}$ is a complete bipartite graph, therefore has achromatic number 2. \square

Thus, we know that these constraints are not sufficient for all m -regular graphs for arbitrary m .

5.3 Achromatic Number of Trees

At one point during the project we looked at trees with $\binom{k}{2}$ edges which had the achromatic number k to see if we could find any patterns. While after this we found previous work done on the achromatic number and trees as well as the fact that the achromatic number of a tree is an NP-complete problem, this work is included. There was no progress made, but the simple things which were looked at are included in this section. The comparisons made were between the degree sequences of the different trees and the achromatic number in the cases where there were $\binom{k}{2}$ edges and the achromatic number was k .

To begin this process, we looked at the achromatic number of all non-isomorphic trees of size 6. After this we looked at all non-isomorphic trees of size 10. These graphs were generated using the program nauty, which is discussed in appendix A. These graphs were ran through prolog program written for this project, which determined the achromatic number. This code is in B.

5.3.1 Trees with Size Six

The path on 7 vertices does not have a complete coloring with 4 colors. We know this because

$$\lfloor \frac{n}{k} \rfloor * \Delta = \lfloor \frac{7}{4} \rfloor * 2 = 1 * 2 = 2 \not\geq 4 - 1 = k - 1$$

This tree has degree sequence 2,2,2,2,2,1,1.

There are three non-isomorphic graphs with degree sequence 3,2,2,2,1,1,1. They all have achromatic number 4.

There are two non-isomorphic graphs with degree sequence 3,3,2,1,1,1,1. These are pictured in figure 5.2

The graph with the two vertices of degree three adjacent has a complete coloring with 4 colors. However the one with them not adjacent, does not. In that case, the vertex of degree two is adjacent to both vertices of degree three, and the set of vertices with the same color as the vertex of degree three, is not connected to three different vertices.

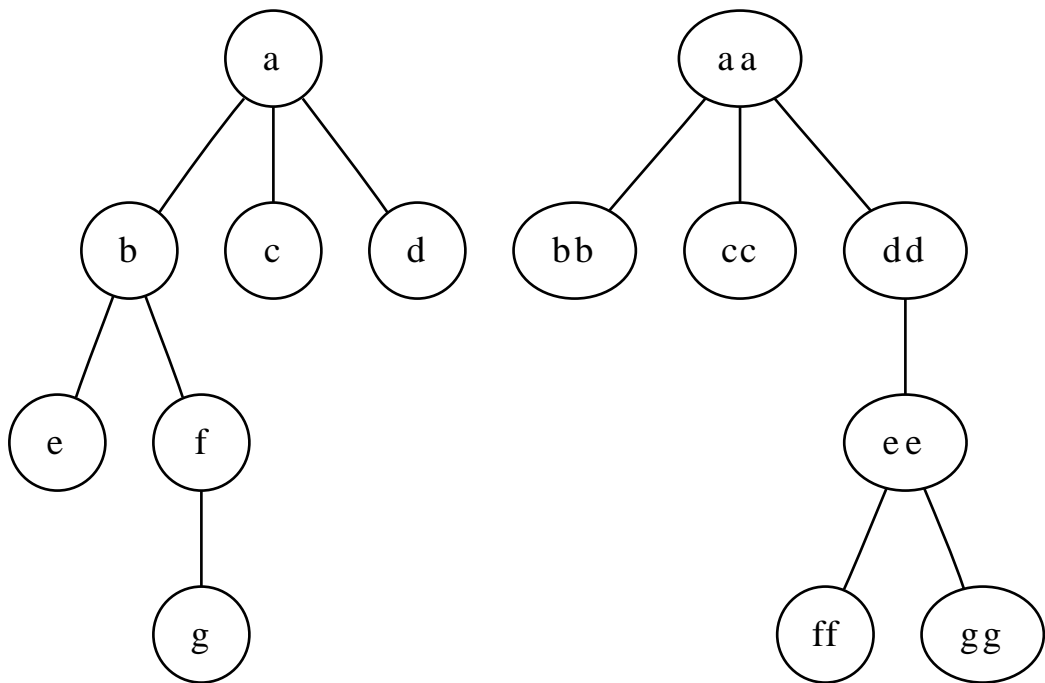


Figure 5.2: The two trees with degree sequence 3,3,2,1,1,1,1

degree sequence	$\alpha(G)$
2,2,1,1	3
2,2,2,2,2,1	3
3,2,2,2,1,1,1	4
3,3,2,1,1,1,1	3 or 4
4,2,2,1,1,1,1	3
5,2,1,1,1,1,1	3
6,1,1,1,1,1,1	2

Table 5.1: Degree Sequences and corresponding achromatic numbers for trees.

The other trees, those with maximum degree 4, 5, or 6, cannot have achromatic number 4. This is because there must be a repeated pair of colors - there is a vertex adjacent to greater than 3 vertices, and there are not more than 3 colors for it to be paired with.

The only tree of size $\binom{3}{2} = 3$ is the path on 4 vertices. The degree sequence is 2, 2, 1, 1 and the achromatic number is 3, with the coloring 1-2-3-1.

The list of the trees we currently have with their pseudochromatic numbers are in table 5.1.

One immediate observation is that if $\binom{k}{2} = e$, and $\Delta(G) > (k - 1)$ then the achromatic number cannot be k , because of repeated pairs of colors. This is true in trees or in other graphs.

Another immediate observation is that even in the case of only trees, the achromatic number cannot be determined from only the degree sequence.

5.3.2 Trees with Size Ten

To try to determine whether we could find any patterns as to when the degree sequence, we found the achromatic number of trees of size 10 (removing the cases where the achromatic number was necessarily less than 5).

With this size graph, we are looking at the case where $\binom{k}{2} = e$ and $\alpha(G) = k$, with $k = 5$.

We know that if $\Delta(G) > (k - 1)$, then the achromatic number cannot be k . This means we don't have to check the cases with $\Delta(G) > 4$

All of the trees of order 11 with max degree less than 5 were generated with the nauty command `"/geng -c -t -f -b -D4 11 — ./pickg -g0"`. What this does is described in appendix A.

There are 159 trees of order 11 with max degree less than 5. The achromatic numbers of these 159 graphs were found with a prolog program, and 129 of the graphs fulfilled the bound. Only three degree sequences uniquely determined the achromatic number and in two of these cases the degree sequence uniquely determined a tree.

Chapter 6

Existence of graphs

6.1 Chromatic, Achromatic, and Pseudoachromatic

In this section we discuss a proof in *On the existence of graphs with prescribed coloring parameters* [25]. Their proof is described in detail.

The proof begins by constructing a family of graphs with predictable chromatic, achromatic, and pseudoachromatic numbers. The pseudoachromatic number can be increased without increasing the other two numbers, and the achromatic number can be increased without increasing the chromatic number.

Theorem 1. *For any a, b, c such that $2 \leq a \leq b \leq c$, there exists a graph with chromatic number a , achromatic number b , and pseudoachromatic number c .*

Proof. Construct a graph, call it $G_{m,n}$ which is a bipartite graph with a complete bipartite subgraph as well as a subgraph of which the achromatic number is dependent on the number of vertices.

This graph has bipartition (A,B) where A is $\{u, u_1, \dots, u_m\} \cup \{y_1, \dots, y_n\}$ and B is $\{v, v_1, \dots, v_m\} \cup \{x_1, \dots, x_n\}$.

$E(G) = \{(u_i, x_j) | 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{(v_i, y_j) | 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{(y_i, x_j) | 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{(u_i, v_j) | 1 \leq i \leq j \leq m\} \cup \{(u, v_m), (v, y_n)\}$

This graph is pictured in figure 6.1 with the double line meaning that the two sets of vertices are joined - that is that every vertex in one subset is adjacent to every vertex in the other subset.

Chromatic number

Because this graph is bipartite the chromatic number is 2.

Achromatic number

The achromatic number of this graph is $m+3$.

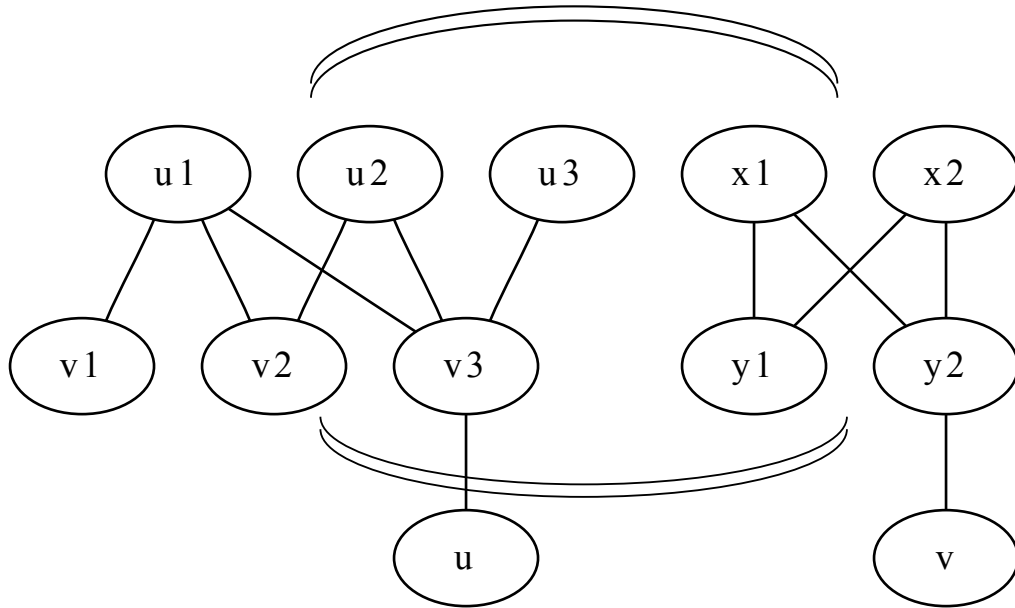


Figure 6.1: A picture of the bipartite graph

Lower Bound

$$\alpha(G) \geq m + 3$$

Color the graph the following:

- For all i , color y_i c_{m+2} and x_i c_{m+3} .
- For $1 \leq i \leq m$ color u_i c_i and v_i c_{i+1} .
- Color u c_{m+2} .
- Color v c_1 .

As an example $G_{3,2}$ is shown in figure 6.2.

This construction gives a complete coloring with $m + 3$ colors, and thus the achromatic number must be at least $m + 3$.

Upper Bound $\alpha(G) \leq m + 3$

Assume there is a complete coloring with $m + 4$ colors. Let f be this coloring.

X cannot contain vertices of more than two different colors. As every vertex of X is connected to every vertex of Y and every vertex of U , any color which appears in X cannot appear in U or Y . As X is an independent set, and is not adjacent to anything in $V \cup u \cup v$, the only edges which satisfy the complete coloring constraint for the colors in X , must have both vertices in $V \cup u \cup v$. (v_m, u) can satisfy the adjacency requirement of two colors, allowing X to contain

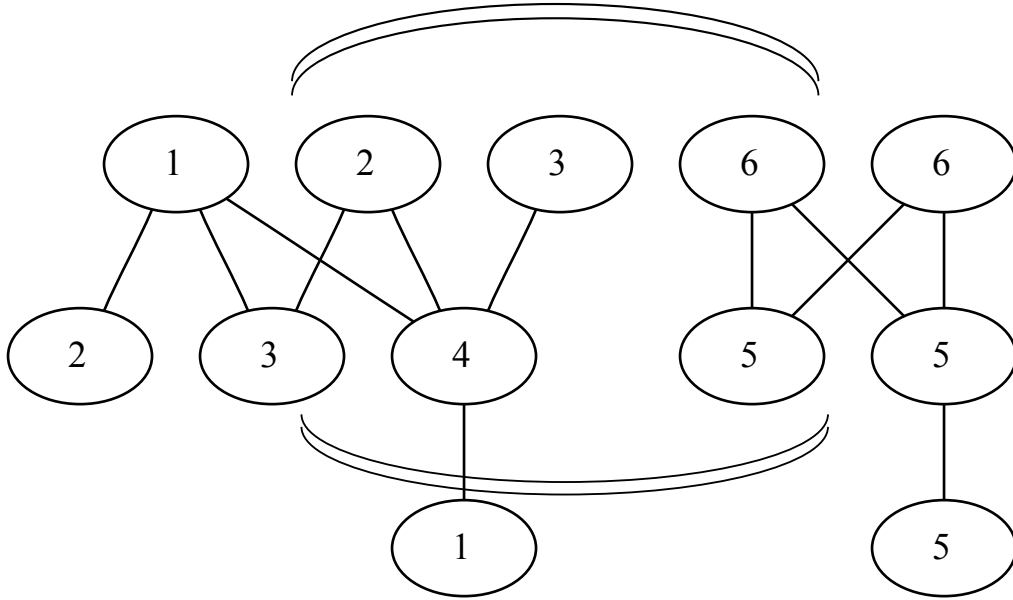


Figure 6.2: A small graph showing a complete coloring with $m+3$ colors.

vertices of two separate colors. However, $V \cup u \cup v \cup X$ contains no other pairs of adjacent vertices, and thus cannot fulfill the complete coloring constraint. Thus X cannot contain vertices of more than two different colors.

The same logic holds for Y , with $U \cup u \cup v$ in the place of $V \cup u \cup v$. (y_n, v) allows one pair of colors, but more than two colors in Y contradicts it being a complete coloring.

We now have four cases, both X and Y containing one color, both X and Y containing two colors, X containing one while Y contains two, and Y containing one while X contains two.

Case 1: X and Y both receive one color.

Call the color received by X c_1 and the color received by Y c_2 . If $f^* = f - \{c_1, c_2\}$ then $|f^*| \geq m + 2$. This means that there must be at least two colors, c_a, c_b that are not received by V , as $|V| = m$. This means that the two colors are only possibly received by $U \cup \{u, v\}$, which is an independent set of vertices. This means that the coloring cannot be complete as there cannot be an edge between vertices colored c_a and c_b . Thus the graph cannot have X and Y both receive one color and $\alpha(G_{m,n}) \geq m + 4$.

Case 2: Y receives two colors while X receives one.

The color of X must be different than the two in Y in order for it to be a coloring.

Call the color received by X c_1 and the two received by Y c_2 and c_3 . Without loss of generality assume $f(y_n) = c_2$ and $f(v) = c_3$. If $f^* = f - \{c_1, c_2, c_3\}$ then $|f^*| \geq m + 1$. This means there must be a color c_a , that is not received by V. This color could only be represented in $U \cup \{u\}$. However no vertex in, $U \cup \{u\}$ is adjacent to any vertex in $Y \cup \{v\}$ and c_2 and c_3 cannot be used in X or U, as it would not be a legal coloring. This means that any vertices colored c_a are not adjacent to either c_2 or c_3 , making the coloring not complete. This is a contradiction and the graph cannot have X receive one color and Y receive two and $\alpha(G_{m,n}) \geq m + 4$.

Case 3: Y receives one color while X receives two.

We can apply the same logic as in case two, when Y receives one color and X receives two.

Call the color received by Y c_1 and the two received by X c_2 and c_3 . Without loss of generality assume $f(v_m) = c_2$ and $f(u) = c_3$. If $f^* = f - \{c_1, c_2, c_3\}$ then $|f^*| \geq m + 1$ This means there must be a color c_a that is not received by U. This color can only be represented in $V \cup \{v\} - \{v_m\}$. However, no vertex of $V \cup \{v\}$ is adjacent to any vertex in X. This means that there is no edge between a vertex of color c_a and one of c_3 , making the coloring not complete.

Case 4: X and Y both receive two colors.

We can one again apply the same logic as in cases two and three.

Let X receive colors c_1 and c_2 and Y receive colors c_3 and c_4 . Without loss of generality assume $f(v_m) = c_1, f(u) = c_2, f(y_n) = c_3$ and $f(v) = c_4$. If $f^* = f - \{c_1, c_2, c_3, c_4\}$ then $|f^*| \geq m$. This means there must be a color c_a that is not received by $V - \{v_m\}$ as $|V - \{v_m\}| = m - 1$. v_m cannot be this color because it has already been colored c_1 . This means there must be a vertex of this color in U. However, no vertices of U are adjacent to any vertices in Y, so there is not any edge between a vertex of color c_a and a vertex of color c_3 . Thus the coloring is not complete.

Because we have proven $G_{m,n}$ cannot be colored in a complete coloring with $m + 4$ colors we have shown that $\alpha(G_{m,n}) \leq m + 3$.

Because $\alpha(G_{m,n}) \geq m + 3$ and $\alpha(G_{m,n}) \leq m + 3$, $\alpha(G_{m,n}) = m + 3$.

Pseudoachromatic number

$$\Psi(G_{m,n}) = m + n + 2$$

Lower Bound

Color the graph the following:

- For $1 \leq i \leq m$, color u_i c_i .
- For $1 \leq i \leq m$ color v_i c_{i+1} .

- For $1 \leq j \leq n$ color x_i c_{m+j+1} .
- For $1 \leq j \leq n$ color y_i c_{m+j+2} .
- Color u c_{m+2}
- Color v c_1

This is a pseudocomplete coloring with $m + n + 2$ colors, thus $\psi(G_{m,n}) \geq m + n + 2$.

Upper Bound

$G_{m,n}$ is a subgraph of $K_{m+n+1,m+n+1}$.

Let f be a pseudocomplete coloring of $K_{a,a}$. Assume $|f(V(K_{a,a}))| \geq a + 2$. This means there exists two colors, call them c_1 and c_2 , which are both not represented in one part (an independent set containing a vertices) of the graph. This means they both must be represented in the other part. However, as it is a bipartite graph, this second half is also an independent set. Thus there are no two vertices colored c_1 and c_2 that are adjacent. This is a contradiction to it being a pseudocomplete coloring and thus $\psi(K_{a,a}) < a + 2$.

This means $\psi(K_{m+n+1,m+n+1}) \leq m + n + 2$. The pseudoachromatic number cannot be increased by adding edges and $G_{m,n}$ is a subgraph so $\psi(G_{m,n}) \leq m + n + 2$.

Because $\psi(G_{m,n}) \leq m + n + 2$ and $\psi(G_{m,n}) \geq m + n + 2$,

$$\psi(G_{m,n}) = m + n + 2$$

Completing the proof

The graphs with chromatic number a , achromatic number b , and pseudoachromatic number c are now constructed.

Case 1: $a = b$

If the chromatic number and achromatic number are equal then the graph K_{a-2} joined with $K_{c-a+1,c-a+1}$ has chromatic number a , achromatic number b , pseudoachromatic number c .

Joining a bipartite graph to K_{a-2} forces the chromatic number to increase by 2, because of the $a - 2$ vertices being adjacent to both independent sets and the two independent sets are adjacent. Thus the chromatic number is a .

Joining a complete bipartite graph to K_{a-2} can only increase the achromatic number by 2, because adding a 3rd color forces it to not be a proper coloring. Thus the achromatic number is a which is the same as b .

$K_{c-a+1,c-a+1}$ has pseudoachromatic number $c - a + 2$ as proven above. Joining on $a - 2$ more vertices each with a new color causes the pseudoachromatic number to be $c - a + 2 + (a - 2) = c - a + a - 2 + 2 = c$.

Case 2: $b = a + 1$

If $a = 2$ then use the complete bipartite with $c - 1$ vertices in the two independent sets minus one edge.

It is bipartite so $a = 2$.

Removing one edge allows you to add one color for the achromatic coloring. The achromatic number cannot be 4, because besides those two disconnected vertices, the rest of the graph can only have achromatic number 2 and those two are not connected. However coloring them both 3, and coloring all other vertices on one of the remaining independent sets 1 and the other independent set 2 gives a complete coloring with 3 colors. Thus $b = 3$.

We know the pseudoachromatic cannot be greater than $c = c - 1 + 1$ for reasons described above. We can color it by rearranging the graph so that the two vertices without an edge connecting them are on the end. Call the two independent sets A and B . Color a_i color i , for $1 \leq i \leq c - 3$ color b_i color $i+1$, color b_{c-2} color c , and color b_{c-1} color $c - 1$. This is a pseudocomplete coloring with c colors. Thus the pseudoachromatic color is c .

If $a > 2$ then the graph is a vertex joined to $K_{c-a, c-a}$ joined to K_{a-2} joined to a vertex.

The chromatic number is a because of the reasoning given in case 1.

The achromatic number is $b = a + 1$. Joining the complete bipartite graph to the complete graph can only increase the achromatic number by 2 (as described above). However, joining one vertex to the complete bipartite and another one to the complete graph allows both of those to be colored one extra color which is adjacent to all vertices.

The pseudoachromatic number is c . The pseudoachromatic number of the complete bipartite graph is $c - a + 1$ as proved above. Joining this to K_{a-2} adds $a - 2$ more colors exactly. This leads to $c - a + 1 + a - 2 = c - 1$ colors. The vertex joined to the complete bipartite graph can be colored a previously unused color and the vertex joined to the complete graph must be colored the same color in order to have all pairs of colors used. Thus the pseudoachromatic number is c .

Case 3: $b \geq a + 2$

If $b \geq a + 2$ then join K_{a-2} to $G_{b-a-1, c-b+1}$ for the graph with chromatic number a , achromatic number b , and pseudoachromatic number c .

The achromatic and pseudoachromatic numbers cannot increase by more than $a - 2$ between $G_{b-a-1, c-b+1}$ and that graph with K_{a-2} joined, because all edges added include one vertex of a color not used in $G_{b-a-1, c-b+1}$. However, they increase by 1 because these $a - 2$ colors are adjacent to all vertices.

The chromatic number is a because all vertices in K_{a-2} are adjacent to all vertices in two other independent sets $U \cup Y \cup \{u\}$ and $V \cup X \cup \{v\}$. These sets have adjacencies between them, disallowing for them to be colored the same color. Thus we need $a - 2 + 2 = a$ colors to have a proper coloring.

The achromatic number is b because $a - 2 + (b - a - 1 + 3) = a - a + 3 - 3 + b = b$.

The pseudoachromatic number is c because $a-2+(b-a-1+(c-b+1)+2) = a-a+2-2+b-b+1-1+c = c$ \square

6.2 The First-fit Chromatic Number for $G_{m,n}$

In the section above we describe a family of graphs constructed in [25] used to prove that for any a , b , and c such that $2 \leq a \leq b \leq c$ there exists a graph such that $\chi(G) = a$, $\alpha(G) = b$, $\psi(G) = c$. In this section we determine the first-fit chromatic number of this family of graphs is 3.

Lemma 1. *Neither U nor V can contain both a vertex of color 2 and a vertex with color greater than 2*

Proof. Let c_a be some color greater than 2.

Assume 2 and c_a are on the same side. Without loss of generality let this side be V .

Let v_i be a vertex colored c_a . Because c_a is greater than 2, it must be connected to some vertex of color 2. This vertex of color 2 cannot be in Y because all vertices of Y are adjacent to all vertices of V and by our assumptions V contains a vertex colored 2, which would cause this to not be a proper coloring. This means, if v_i is colored c_a then for some $j \leq i$, u_j must be colored 2.

Because there is some vertex colored 2 in V , there must be a vertex v_k such that $k < j$ colored 2. This is because if $k \geq j$ then 2 is adjacent to 2 and we do not have a proper coloring.

Because this is a first-fit coloring v_k must be adjacent to a vertex of color 1. Y cannot contain a vertex of 1 because if Y contained one, then X and V could not contain any vertices of color 1. This means that U would not be adjacent to any vertices of color 1, but u_j must be adjacent to a vertex of color 1.

This means that there must be a vertex u_l such that $l \leq k$ that is colored 1.

However u_j must also be adjacent to a vertex of color 1. This adjacent vertex cannot be in X , because then the vertex u_l which is colored 1, is adjacent to another vertex colored 1, and we do not have a proper coloring.

If the vertex colored 1 and adjacent to u_l is in V , then it must be v_h for some $h \geq j$. But in that case u_l and v_m both colored 1, are adjacent, because v_h is adjacent to all u_x such that $x \leq h$ and $l \leq k \leq j \leq h$, making $l \leq h$.

Thus we have a contradiction and V cannot contain vertices colored both c_a and 2. \square

Lemma 2. *There is no first-fit coloring of $G_{m,n}$ such that $U \cup V \cup \{u\}$ receives 4 colors*

Proof. $\{u\}$ is irrelevant for this coloring. It cannot be colored greater than 2 because of having degree 1.

Case 1: If it is colored 2, then v_m must be colored 1, which forces none of $U \cup Y$ to contain a vertex colored one, making all vertices of $V \cup X$ be colored 1. This means all vertices of U are colored 2, as they cannot be adjacent to anything colored 2. Similarly all vertices of Y are colored 2, because only y_n has any non-colored vertices, and it cannot be anything greater than 1 (if it is a 2, then y_n is colored 1 and this is not a proper coloring).

Case 2: If u is colored 1 and v_m is not adjacent to any other vertices colored 1, then $U \cup Y$ cannot contain any vertices colored 1. This means X must contain only vertices colored 1. Similarly $V - \{v_m\}$ must all be colored 1 as they cannot be adjacent to something colored 1. Because U is only adjacent to 1 vertex which has not been predetermined in color and all of U is adjacent to that vertex, all of U must either be colored 2 or 3.

Subcase 2.1: If U is colored 3, then v_m must be colored 2. This causes Y to all be forced to be colored 3 as it is not adjacent to any 3s and they are all adjacent to both 1 and 2 (and v cannot be colored 3 because of its degree).

Subcase 2.2: If U is all colored 2, then v_m cannot be 2, which makes Y not adjacent to any vertices colored 2, forcing all of it to be colored 2 (if v is colored 2, then y_n is colored 1, which would make this not a proper coloring). Because v_m is the only uncolored vertex and it is not adjacent to any vertices colored 3, it cannot be colored 4. However it is adjacent to both vertices colored 1 and vertices colored 2, making it colored 3.

If we have a vertex colored 4, it must be adjacent to both a vertex colored 2 and a vertex colored 3.

Because the coloring of u is irrelevant, we can say without loss of generality that the vertex colored 4 is in U .

From Lemma 1, we know that the vertex colored 3 and the vertex colored 2 cannot both be in V .

Case 1: If the vertex colored 3 is in V

Because no vertex colored 2 is in V , there must be a vertex colored 2 in X .

However, the vertex colored 3 in V must be adjacent to some vertex colored 2. V is only adjacent to vertices in $U \cup Y$ (though not all of them), so some vertex in $U \cup Y$ must be colored 2. However all vertices in X are adjacent to all vertices in $U \cup Y$. As a vertex in X is already required to be colored 2, this forces two vertices of color 2 to be adjacent, making it not a proper coloring.

Case 2: If the vertex colored 2 is in V

Because no vertex colored 3 is in V there must be a vertex colored 3 in X .

This vertex colored 3 must be adjacent to some vertex colored 2, meaning there must be some vertex colored 2 in $U \cup Y$.

If the vertex colored 2 is in Y then we do not have a proper coloring, as all vertices of V and Y are adjacent.

This means the vertex colored 2 must be in U . However, there is a vertex colored 4 in U and by Lemma 1 we have a contradiction.

All cases lead to a contradiction so there cannot be a complete coloring such that $U \cup V \cup \{u\}$ receives 4 colors. \square

Lemma 3. *The first fit chromatic number of $G_{m,n}$ is 3*

Proof. In figure 6.3 we color $G_{3,2}$ with 3 colors. This can be generalized as a coloring for $G_{m,n}$

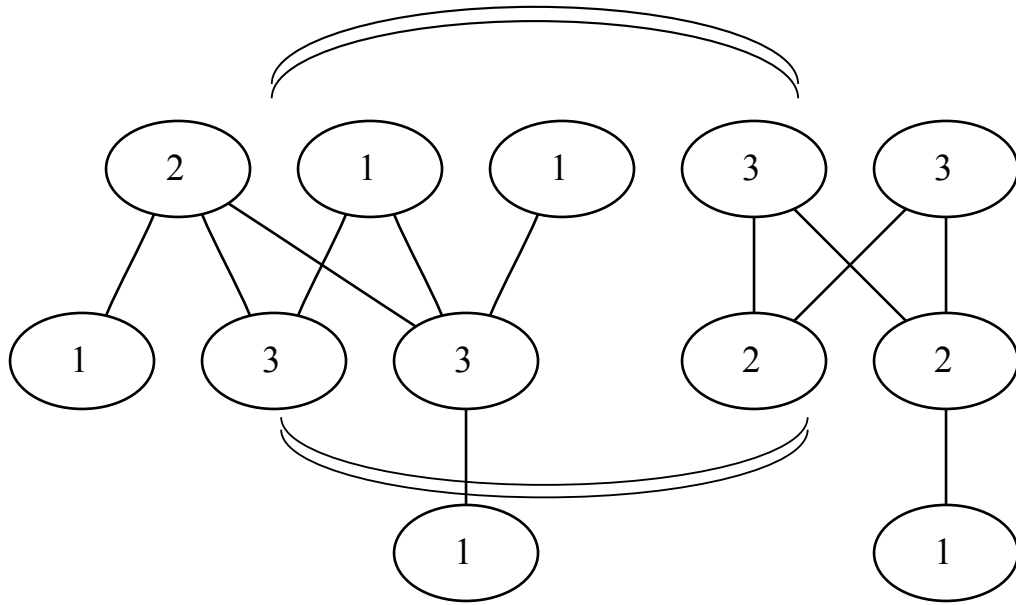


Figure 6.3: A coloring fulfilling the first-fit constraint of $G_{m,n}$

Color $G_{m,n}$ the following:

- u_1 is colored 2.
- v_1 is colored 1.
- for $2 \leq i \leq m$ color u_i 1.
- for $2 \leq i \leq m$ color v_i 3.
- for $1 \leq j \leq n$ color x_j 3.
- for $1 \leq j \leq n$ color y_j 2.

- u is colored 1.
- v is colored 1.

This is a coloring fulfilling the first-fit constraint using 3 colors.

Thus we now need to give an upper bound of 3 for the first-fit chromatic number.

We know from lemma 2 that the vertices $U \cup V \cup \{u\}$ cannot include a vertex colored 4.

v cannot have color 4, because it has degree 1. Thus we have two cases, when X contains a vertex of color 4 and when Y contains a vertex of color 4.

Case 1: Y contains a vertex of color 4.

From lemma 1 we know that V does not contain vertices with color 2 and vertices with color 3.

If V contains a vertex with color 3, then $U \cup Y$ must contain a vertex colored 2. However, every vertex in $U \cup Y$ is adjacent to all vertices in X , and some vertex in X must have color 2 for some vertex in Y to have color 4. u being colored 2 cannot allow y_n to be colored 4 without X containing a vertex of color 4, because then u would not be adjacent to any vertex of color 1.

If V contains a vertex of color 2, then U can either contain a vertex of color 3, or contain no vertices of color 3.

If U contains a vertex of color 3, then X cannot contain any vertices of color 3 because all vertices of X are adjacent to all vertices of U . Similarly V cannot contain a vertex of color 3 because by lemma 1 it cannot contain a vertex of color 2 and a vertex of color 3. Because v cannot be colored 3 by its degree, no vertex in Y can be adjacent to any vertices of color 3, and thus cannot be color 4.

If U does not contain any vertices of color 3, then the vertices of U only receive colors 1 and 2.

Subcase: If U only receives 1

In order for Y to contain a vertex colored 4, this vertex must be adjacent to some vertex colored 3.

However V cannot have a vertex colored 3 because of our assumption that V contains a vertex colored 2. This means X must contain a vertex colored 3.

In order for this vertex to be colored 3 it must be adjacent to a vertex colored 2. X is only adjacent to vertices in $Y \cup U$.

We have assumed that U has no vertices colored 2 for this case. Thus Y must have a vertex colored 2. However all vertices in Y are adjacent to all vertices in X , so this is not a proper coloring.

Subcase: If U contains a vertex colored 2

V contains a vertex colored 2 which must be adjacent to a vertex colored 1. This vertex could either be in U or in Y .

v_m is adjacent to all vertices of U as well as all vertices of Y . This means that v_m is adjacent to a vertex with color 2 and a vertex with color 1, and thus cannot be colored either color 2 or color 1.

However, by lemma 1, V cannot contain a vertex colored 2 and a vertex colored any color greater than 2. Thus there is a contradiction.

Case 2: X contains a vertex colored 4.

We can apply the same logic as in the previous case to show that X cannot contain a vertex colored 4.

Thus $G_{m,n}$ cannot have a first-fit coloring with 4 or greater colors. As we have shown both a lower and upper bound of 3, $\chi_{ff}(G_{m,n}) = 3$. \square

6.3 Chromatic, First-fit Chromatic, and Achromatic Numbers

The following section is work which I have done inspired by the proof in section 6.1. It proves for any a, b, c such that $2 \leq a \leq b \leq c$ there is a graph with chromatic number a , achromatic number b and pseudoachromatic number c . In this section we begin an attempt to prove that for any a, b, c such that $2 \leq a \leq b \leq c$ there is a graph with chromatic number a , first-fit chromatic number b and achromatic number c .

Conjecture 1. *For any a, b, c such that $2 \leq a \leq b \leq c$, there exists a graph with chromatic number a , first-fit chromatic number b , and achromatic number c .*

We begin by constructing a family of bipartite graphs which we can vary first-fit chromatic number and achromatic number with, while keeping a constant chromatic number. Figure 6.4 includes an example graph from this family.

Chromatic Number:

Because this graph is bipartite the chromatic number is 2.

First-fit Chromatic Number

For a lower bound we can color

Lower Bound

- u_1 is colored 2.
- v_1 is colored 1.
- x_1 is colored 3.
- y_1 is colored 2.

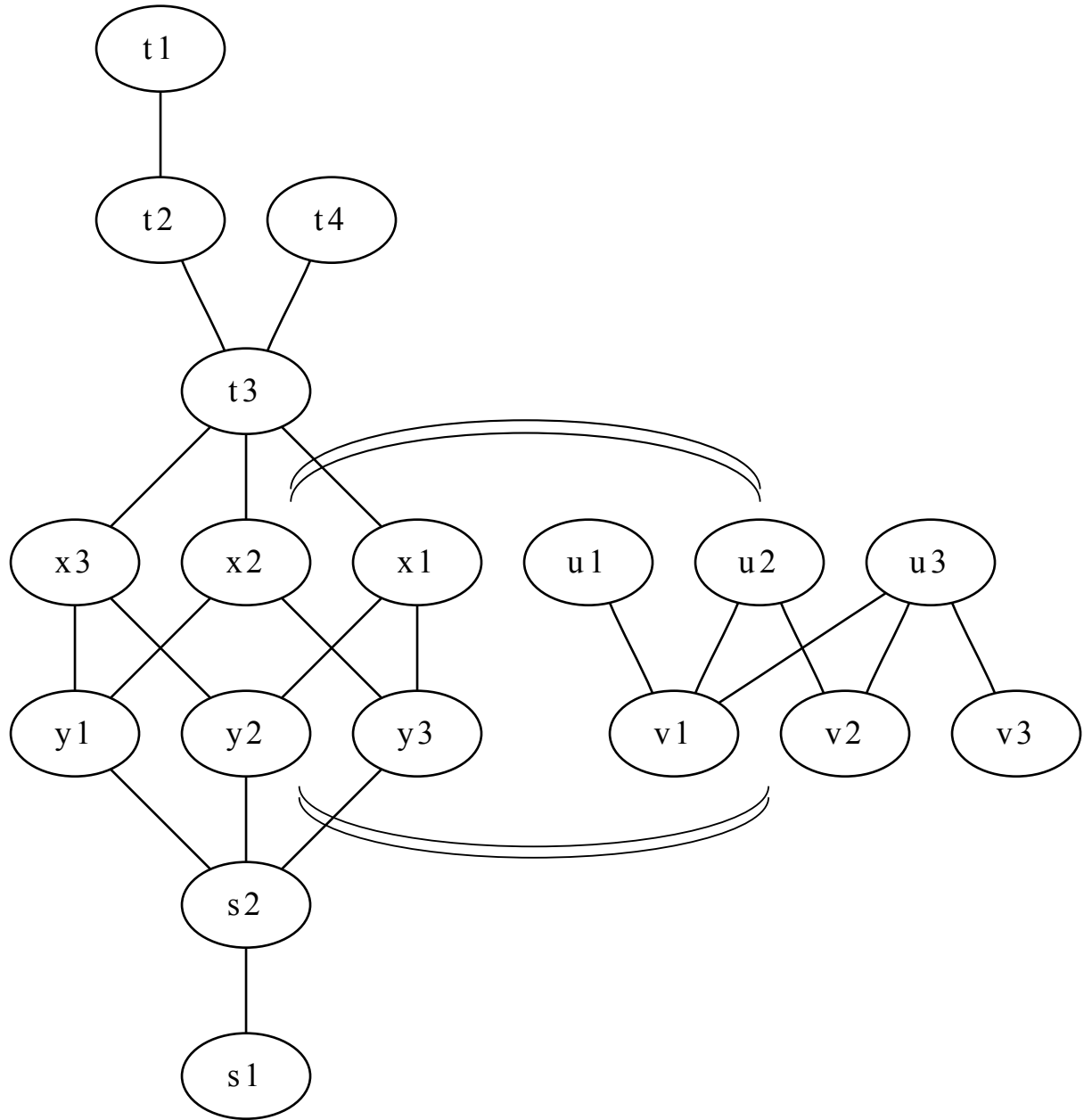


Figure 6.4: The graph used for proving the existence of a graph with chromatic number a , first-fit chromatic number b and achromatic number c .

- for $2 \leq i \leq m$ color u_i 1.
- for $2 \leq i \leq m$ color v_i 3.
- for $2 \leq j \leq n$ color x_j $j + 3$.
- for $2 \leq j \leq n$ color y_j $j + 3$.
- t_1 is colored 1.
- t_2 is colored 2.
- t_3 is colored 3.
- t_4 is colored 1.
- s_1 is colored 1.
- s_2 is colored 2.

This is a coloring which uses $n + 3$ colors.

Upper Bound

Assume that this graph can be colored with $n + 4$ colors.

As neither lemma 1 nor lemma 2 relies on what the subgraph containing only the vertices $U \cup V$ is joined to is complete bipartite, only bipartite, these two lemmas still hold.

This means that we know that $U \cup V$ can only receive 3 colors, and that neither U nor V receives both 2 and 3.

Without loss of generality, let's say that V receives 3. This means that U receives 2.

In order for Y to receive anything greater than a 2 it must be adjacent to a 2. X cannot receive 2 because all vertices in X are adjacent to all vertices in U and that places a 2 adjacent to a 2. V cannot receive 2 because it has received 3 and by lemma 1 V cannot receive both 2 and 3. This means that for Y to receive anything greater than a 2 s_2 must receive 2 and thus s_1 must receive 1 for the 2 to be adjacent to a 1.

Similarly in order for X to receive anything greater than a 3, it must be adjacent to a 3. Y cannot receive 3 because all vertices in Y are adjacent to all vertices in V and that places a 3 adjacent to a 3. U cannot receive a 3 because it has received 2 and by lemma 1 U cannot receive both 2 and 3. This means that for X to receive anything greater than a 3 t_3 must receive 3. This means that t_4 is colored 1, t_2 is colored 2, and t_1 is colored 1.

This means that either X or Y must receive color $n + 4$. Without loss of generality let's assume that Y receives it.

Y is adjacent to all vertices in V , s_2 and all but one vertex in X .

V cannot receive more than 2 colors because of the combination of lemmas 1 and 2.

As one vertex can only receive one color and s_2 is one vertex, s_2 receives one color.

In X , Y is adjacent to $n - 1$ vertices. These can receive at most $n - 1$ colors as no vertex can receive more than one color.

If we sum up the number of colors that Y can be adjacent to we get $n - 1$ colors from X , two colors from V , and one color from s_2 and $2+1+(n-1) = n+2$.

However a vertex colored $n + 4$ must be adjacent to vertices colored all previous colors, of which there are $n + 3$. This is not adjacent to vertices of $n + 3$ different colors and thus is not a first-fit coloring.

This is a contradiction, and thus the graph cannot be colored fulfilling the first-fit constraint with $n + 4$ colors. Thus the first-fit number is $n + 3$.

Achromatic Number

We are attempting to find the achromatic number of this family of graphs. The following are the current bounds that we have found.

A complete coloring with $m + n + 2$ colors is possible. This can be done with

- for $1 \leq i \leq m$ color u_i i .
- for $1 \leq i \leq m$ color v_i $i + 1$.
- for $1 \leq j \leq n$ color x_j $j + m + 1$.
- for $2 \leq j \leq n - 1$ color y_j $j + m + 1$.
- y_n is colored $m + n + 2$
- t_1 is colored 1.
- t_2 is colored $m + n + 2$.
- t_3 is colored 1.
- t_4 is colored 2.
- s_1 is colored 1.
- s_2 is colored $m + n + 1$.

We also know that the achromatic number of this graph is no greater than $m + n + 4$.

$\psi(K_{m+n+3, m+n+3}) \leq m + n + 4$, as proven in section 6.1. This graph, G , is a subset of $K_{m+n+3, m+n+3}$ which means that $\psi(G) \leq m + n + 4$. The achromatic number is less than or equal to the pseudoachromatic number so $\alpha(G) \leq m + n + 4$.

Appendix A

Nauty

Nauty is a program used for computing automorphism groups of graphs. [18] One use is generating all non-isomorphic graphs with given properties.

For this project I needed to generate graphs with specific properties in order to find the achromatic and first-fit numbers of this graph. The most important command was `geng` - generate graph. I also used `pickg` - select graphs with given properties.

When generating trees of order 11 I used

```
./geng -c -t -f -b -D4 11 | ./pickg -g0
```

The command `geng` generates all non-isomorphic graphs of a given class, (-c specifies only connected graphs, -t specifies triangle-free graphs, -f specifies 4-cycle free graphs, -D4 specifies the maximum degree to be no greater than 4, 11 is the number of vertices in the graph).

The command `pickg` selects graphs according to the properties, -g0 specifies acyclic graphs. While `pickg` could have been passed all graphs on 11 vertices, generating the subset of the graphs of order 11 was more efficient.

Appendix B

Prolog Code for Finding the Achromatic Number

The following code was written to find the achromatic number by searching for colorings with a given number of colors until it cannot find one, and then subtracting one. Some checks were included, such that it would not try to search for a coloring when we know there cannot be a complete coloring - these include making sure that there is enough edges for every pair to be possible as well as making sure there is enough connectivity that vertices would not necessarily need to be multiple colors.

As this was originally written to find the achromatic number of cycles, there are also ways to generate cycles with a given number of vertices.

This was written by Stephanie Fuller with help from Jonathan Gibbons, as it was written in the process of learning prolog.

```
/*E is the set of edges in the graph,  
N is the number of vertices in the graph */
```

```
alist(List, Key, Value) :-  
    member([Key,Value],List).
```

```
alist_push(List, Key, Value, Newlist) :-  
    Newlist = [[Key,Value]|List].
```

```
adjacent(A,B,E) :-  
    member([A,B],E);  
    member([B,A],E).
```

```
color(Node,Color,Coloring) :-  
    alist(Coloring,Node,Color).
```

```
conflict(Coloring,E) :-
```



```

adjacent(X,Y,E),
color(X,Color,Coloring),
color(Y,Color,Coloring).

color_node(Node, Colors, Coloring, NewColoring,E) :-
    alist_push(Coloring,Node,C,NewColoring),
    member(C,Colors),
    \+ conflict(NewColoring,E).

color_from_pair(ColorPair, Color):-
    ColorPair = [_ ,Color].

next_possible_colors(Colors,Coloring,[NewColor|UsedColors]) :-
    maplist(color_from_pair,Coloring,UsedColorBag),
    sort(UsedColorBag,UsedColors),
    length(UsedColors,MaxColorIndex),
    NextColorIndex is MaxColorIndex + 1,
    nth1(NextColorIndex,Colors,NewColor).

next_possible_colors(Colors,Coloring,UsedColors) :-
    maplist(color_from_pair,Coloring,UsedColorBag),
    sort(UsedColorBag,UsedColors),
    length(UsedColors,MaxColorIndex),
    length(Colors,MaxColorIndex).

color_nodes([Node | Nodes], Colors, Coloring, FinalColoring,E) :-
    next_possible_colors(Colors, Coloring, NextColors),
    color_node(Node, NextColors, Coloring, NewColoring,E),
    color_nodes(Nodes, Colors, NewColoring, FinalColoring,E).

color_nodes([], _, Coloring, FinalColoring, _) :-
    FinalColoring = Coloring.

all_unordered_pairs_with(A, [B|R],T) :-
    T=[[A,B]|C],
    all_unordered_pairs_with(A,R,C).
all_unordered_pairs_with(_, [], []).

all_unordered_pairs([A|R],T) :-
    all_unordered_pairs_with(A,R,U),
    all_unordered_pairs(R,V),
    append(U,V,T).
all_unordered_pairs([], []).

```

```

nodes([[A,B]|Rest],Nodes):-
    nodes(Rest,RestList),
    merge_set([A],RestList,Y),
    merge_set([B],Y,Nodes).
nodes([],[]).

complete(Colors,Coloring,E):-
    nodes(E,Nodes),
    complete(Nodes,Colors,Coloring,E).

complete(Nodes, Colors, Coloring,E) :-
    color_nodes(Nodes, Colors, [], Coloring,E),
    all_unordered_pairs(Colors, Color_pairs),
    check_color_pairs(Color_pairs, Coloring,E).

check_color_pairs([[C1, C2]| OtherPairs], Coloring,E) :-
    adjacent(N1, N2,E),
    color(N1, C1, Coloring),
    color(N2, C2, Coloring),
    check_color_pairs(OtherPairs, Coloring,E).
check_color_pairs([],_,_).

cycle(N,E) :-
    E = [[1,N]|RestEdges],
    chain(N,RestEdges).

chain(N,E) :-
    chain(1,N,E).
chain(I,N,E) :-
    I < N,
    J is I+1,
    E=[[I,J]|RestEdges],
    chain(J,N,RestEdges);
    I == N,
    E=[].

color_list(NumColors,L) :-
    color_list(1,NumColors,L).
color_list(I,NumColors,L) :-
    I < NumColors,
    L=[I|RestColors],
    J is I+1,
    color_list(J,NumColors,RestColors);
    I = NumColors,
    L = [I].

```

```

factorial(N,Answer):-
    factorial(N,1,Answer).

/* n!/k!*/
factorial(N,K,Answer):-
    N =\= K,
    M is N -1,
    factorial(M,K,X),
    Answer is N * X.
factorial(K,K,Answer):-
    Answer is 1.

choose(N,K,Answer):-
    M is N-K,
    factorial(N,M,Top),
    factorial(K,Bot),
    Answer is Top/Bot.

mdegree(Edges, Node, Answer):-
    degree(Node, Edges, Answer).

degree(Node, [[A,B] | RestEdges], Answer):-
    Node == A,
    degree(Node, RestEdges, Count),
    Answer is Count + 1;
    Node == B,
    degree(Node, RestEdges, Count),
    Answer is Count + 1;
    Node =\= A,
    Node =\= B,
    degree(Node, RestEdges, Answer).
degree(_, [], 0).

max_degree(E, Answer):-
    nodes(E, Nodes),
    max_degree(Nodes, E, Answer).

max_degree([A | RestNodes], E, Answer):-
    degree(A, E, Degree1),
    max_degree(RestNodes, E, Degree2),
    Answer is max(Degree1, Degree2).
max_degree([], _, 0).

checks(NumColors, E):-
    length(E, NumEdges),

```

```

choose(NumColors,2,Answer),
NumEdges >=Answer,
max_degree(E,MaxDegree),
nodes(E,Nodes),
length(Nodes,NumNodes),
X is floor(NumNodes / NumColors) * MaxDegree +1,
X >= NumColors.

achromatic(E,FinalNum):-
    achromatic(2,E,FinalNum).

achromatic(NumColors,E,FinalNum):-
    checks(NumColors,E),
    color_list(NumColors,L),
    complete(L,_,E),
    NewNum is NumColors+1,
    !,
    achromatic(NewNum,E,FinalNum).

achromatic(NumColors,_,FinalNum):-
    FinalNum is NumColors-1.

log_term(File, Term):-
    open(File, append, Stream, []),
    write_term(Stream, Term, []),
    nl(Stream),
    close(Stream).

do_stuff(File,[G|Graphs]):-
    nodes(G,N),
    maplist(mdegree(G),N, D),
    msort(D, Ds),
    reverse(Ds, Dsk),
    achromatic(G,Psc),
    Res=[edges(G), degree_sequence(Dsk),achromatic_number(Psc)],
    log_term(File,Res),
    do_stuff(File,Graphs).

do_stuff(_, []).

```

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