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BKS Theorem and Bell's Theorem in 16 Dimensions

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Abstract

This project gives two new proofs of the Bell-Kochen-Specker (BKS) Theorem for a system of four qubits: A proof based on 11 observables for a four-qubit system and a second proof based on 80 states and 265 orthogonal bases in a 16-dimensional state space derived from the previous observables. These proofs can be converted into proofs of Bell's Theorem by introducing four more qubits that are entangled with the previous qubits in a suitable fashion.

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1 Introduction

1.1 Goal of the Project

The goal of the project is to give several related proofs of the Bell-Kochen-Specker (BKS) Theorem and Bell's Theorem using a system of four qubits. A qubit is any two-state quantum system such as the spin degrees of freedom of an electron or the polarization degrees of freedom of a photon. The BKS Theorem and Bell's Theorem are two basic theorems concerning the interpretation of quantum mechanics. This introduction will begin with a brief explanation, in qualitative terms, of these two important theorems. Then past proofs of the theorems that serve as precursors to the present work will be discussed before an overview of this project, and the motivations behind it, are presented.

1.2 The BKS Theorem and Bell's Theorem

Both the BKS Theorem and Bell's Theorem were proved by John Bell in the years 1964 to 1966 [1, 2, 3]. The BKS Theorem was also proved independently by Kochen and Specker in 1967 [4] and so is currently known by the initials of all of its three authors. The motivation for these theorems came from a paper written by Einstein, Podolsky, and Rosen in 1935 bearing the title, "Is the Quantum Mechanical Description of Reality Complete" [5]? Quantum mechanics holds that the most complete description of a physical system is given by its wavefunction but that even knowledge of the wavefunction does not allow one to make definite predictions about the results of all measurements that can be made on the system. Instead, according to quantum mechanics, one can only predict the probabilities of the various outcomes of a measurement. Einstein found this absence of certainty galling (hence his famous remark, "God does not play dice") and so believed that quantum mechanics could be replaced by a more complete theory.

A more complete theory would supplement the description given by the wavefunction by introducing a set of "hidden variables" that would allow definite predictions to be made in situations where quantum mechanics could only predict probabilities.

The BKS Theorem and Bell's Theorem conclusively demonstrate that the complete theories of the sort that Einstein had hoped for do not actually exist. They do this by presenting physical situations in which a complete theory (interchangeably called a hidden variable theory) and the quantum mechanical theory make very different predictions. In Bell's original scheme, for example, a decaying atom emits a pair of photons (in a singlet state) that fly off in opposite directions. The polarization of each photon is measured along one of several directions and the correlation between the two measurements is observed over a long sequence of runs. Any hidden variable theory, it turns out, predicts that the observed correlations obey a certain inequality, now known as Bell's inequality. The inequality emerges due to the assumption of local realism that lies at the heart of hidden variable theories: Locality asserts that two distant objects cannot have a direct effect upon each other (or, equivalently, that signals cannot travel faster than the speed of light); realism, that particles have a definite character, in this case polarization, prior to measurement and that the act of measurement passively discovers that character. Experimental measurements show that Bell's inequality is violated and, moreover, that it is violated in precisely the manner predicted by quantum mechanics. The experimental proof of Bell's Theorem via this observed violation of Bell's inequality decisively shatters the hope of finding more complete theories.

Experimental work on the proof of Bell's Theorem was carried out by A. Aspect and his collaborators during the 1970s and 1980s [6]. Other groups over the years worked on experimentally verifying Bell's Theorem with increased precision and sophistication, reducing the possibility of experimental loopholes that could make room for new hidden variable theories. This experimental work led to a better understanding of a magical and puzzling phenomenon in the quantum world, one without analog in the macroscopic world. This phenomenon is known as entanglement. Entanglement manifests itself through miraculous correlations between particles in the microscopic world that lead to the observed violation of Bell's inequality. Einstein famously described the idea of entanglement as "spooky actions at a distance."

As recently as 1990, Greenberger, Horne, and Zeilinger (GHZ) [7] exposed a much more stark conflict between local realism and quantum mechanics. They devised an experiment in which local realism predicted one outcome and quantum mechanics predicted the diametrically opposite one. An experiment demonstrating this conflict was subsequently performed. In the long term, the GHZ paper inspired a search for stronger proofs of Bell's Theorem that would avoid the statistical arguments characteristic of earlier proofs. Many stronger inequality-free proofs of Bell's Theorem have subsequently been found. Mermin, for one, has clearly and concisely shown the shocking nature of these inequality-free proofs in the case of two- and three-qubit systems [8]. The proofs presented in this project belong to this class.

Up until now in this discussion Bell's Theorem has been the topic of interest. The difference between this theorem and the related BKS Theorem has not yet been elaborated. The difference between them, in a word, is that they rule out different sorts of hidden variable theories. Suffice it to say that the BKS Theorem rules out so-called noncontextual hidden variable theories while Bell's Theorem rules out local hidden variable theories. What this difference actually means will not be discussed here; the discussion will be postponed until Chapter 2 when the proofs of these theorems using four qubits are presented.

1.3 Precursors to the Project

The original proof of Bell's Theorem, as well as many later proofs, were given using a system of two qubits. A particularly interesting variation of this proof was given by Aravind [9, 10], building on earlier work by Mermin, Peres, and others. This variation was interesting because it provided closely related proofs of the BKS Theorem and Bell's Theorem. Like Bell's original proof the variation used a two-qubit system. Kernaghan and Peres proved the BKS Theorem for a three-qubit system [11]. This project proves both theorems using four qubits. The five-qubit case still awaits future work. Table 1 summarizes the relevant information concerning these n-qubit proofs, starting at $n = 2$ and continuing to $n = 5$.

Qubits	**Dimension**	**Observables**	**States (Bases)**	**Reference**
2	$2^2 = 4$	9	24 (24)	Mermin, Peres, Aravind
3	$2^3 = 8$	10	40 (25)	Kernaghan, Peres
4	$2^4 = 16$	11	80 (265)	Present report
5	$2^5 = 32$	—	—	Future work

Table 1: Relevant information concerning n-qubit proofs

The first column of the table lists the number of qubits and the second the dimension of the state space created by those qubits. (In general, the state space of a system of n qubits has dimension 2^n. In this way, an n-qubit proof can just as well be called, from a different perspective, a 2^n-dimensional proof.) The third column lists the number of observables involved in the BKS proof and the fourth the number of states (and bases formed by those states) involved in the state version of the BKS proof. Each of the above BKS proofs can be converted into a proof of Bell's Theorem by simply doubling the number of qubits (and the dimension of the state space), dividing them into two sets, and exploiting entanglement between the sets. The details apropos observables, states, and bases will be explained in the body of the report.

The last row of the table lists work that is expected to be undertaken in the future. There are certain common trends that have been noticed in the cases that have been studied so far. To what extent the observed trends persist in higher dimensions and to what extent they do not has yet to be studied.

1.4 Motivations Behind the Project

Work on proofs of the BKS Theorem and Bell's Theorem has proceeded for almost half a century, yet it seems that it has not come close to exhausting their content or fully taking advantage of all the riches contained in them. The proofs are interesting at the philosophical and conceptual level because they help illuminate the nature of entanglement, the fundamental mystery at the heart of quantum mechanics, to use the words of Schrödinger.

From a practical point of view, ideas and techniques that first arose in connection with proofs of Bell's Theorem later played a key role in many quantum information protocols such as quantum computation, quantum cryptography, and teleportation, to name a few [12]. The quantum states and bases that arise in the proof of Bell's Theorem in this project have a rich geometric structure, being similar in many respects to the balanced incomplete block designs studied in combinatorics and in statistical decision theory, but with novel features not encountered earlier. It is certainly possible that a study of these geometric structures could lead to applications in fields far removed from quantum mechanics.

More immediately, however, it is hoped that the states and bases that arise here can prove useful in quantum cryptography or other quantum information processing protocols.

1.5 Outline of the Report

The outline of the report is as follows: Chapter 2 gives an observable-based proof of the BKS Theorem for a four-qubit system; then the number of qubits is doubled and entanglement is exploited to prove Bell's Theorem. Prior to both, a "magic" trick is performed demonstrating the truth of the theorems. Chapter 3 gives a state version of the proof of the BKS Theorem by deriving a set of states from the observables of Chapter 2 and showing that they cannot be colored according to a simple set of rules. Chapter 4 ends with some brief concluding remarks concerning the present work and looks ahead to prospective future work. Chapter 5 is an appendix containing a survey of quantum mechanical concepts, mathematical formulae, computer programming code, and any other relevant information that was not included in the main body as it would have been too distracting.

2 Observable-Based Proofs of the BKS Theorem and Bell's Theorem

In this chapter, the Bell-Kochen-Specker (BKS) Theorem will be proved for a four-qubit system and then Bell's Theorem will be proved using an eight-qubit entangled system with two distant observers. The proofs are state-independent in the sense that they hold irrespective of the quantum states of the systems. The BKS Theorem, which asserts that noncontextual hidden variable theories cannot exist in Hilbert spaces of dimension $d > 2$, is first proved by means of a no-coloring argument. Then the related Bell's Theorem is proved in a "double play" that invalidates the existence of local hidden variable theories in quantum mechanics. The truths of these proofs are first communicated by means of a clever "magic" trick before the quantum mechanics of the trick is thoroughly explained to the reader.

2.1 The Trick

Consider an experiment in which a source S emits eight particles, four of which fly off to the left toward an observer, named Alice (A), and the other four to the right toward another observer, Bob (B). Each quartet of particles enters a detector which performs a measurement on them. Figure 1 is a representation of the present source-detectors scheme.

Each detector has a screen that displays the results of the measurement. Each screen is segmented into twenty panels in the form of a five-by-four array, with each panel bearing a number from 1 to 11. There are five switch settings for each detector, with each setting activating the panels in one of the columns of the detector or in its last row. The panels corresponding to each of the switch settings are linked by lines in order to bring out the switch settings more clearly. (The first four unconnected rows

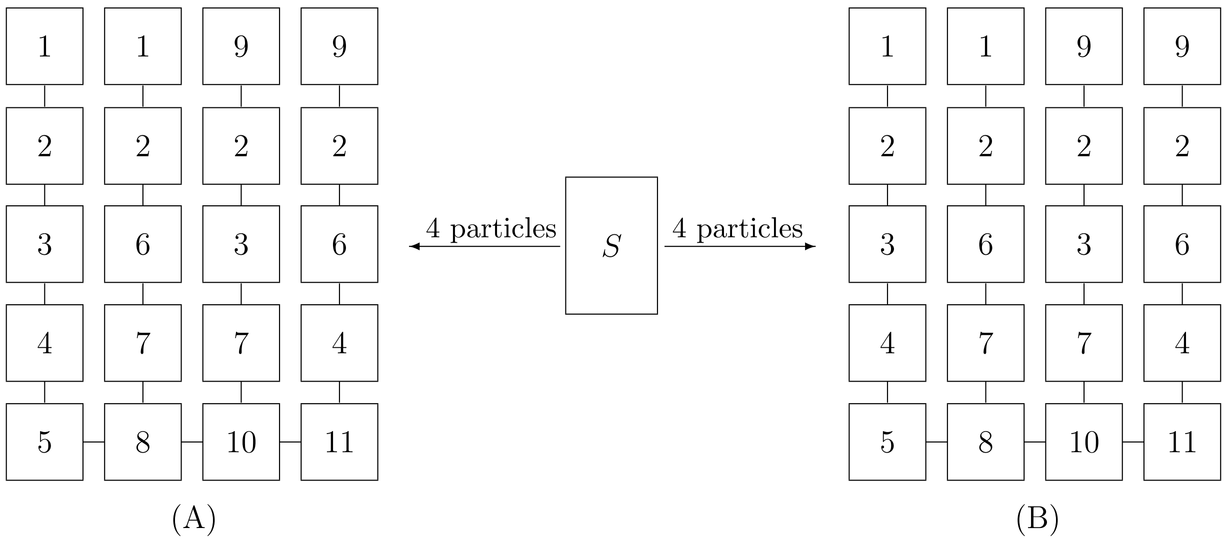

Figure 1: The source-detectors scheme

do not correspond to any switch settings.) When Alice or Bob sets his/her detector's switch to one of these five positions, all the panels of the corresponding column or row light up upon receiving the incoming particles, with each panel lighting up either green or red.

In any run of the experiment, the source emits the particles and Alice and Bob independently and randomly set their switches at any time prior to the particles reaching their detectors and then record the light flashings. It is important to note that the switch selections and detector responses at the two ends take place within a very short time of each other, so short that it would be impossible for a light signal to travel between them in that time. This guarantees that neither Alice's choice of switch setting nor the pattern of flashings she observes on her detector can influence Bob's switch setting or detector flashings in any way (with Bob's choice similarly having no effect at Alice's end). This is done in order to rule out a possible explanation for the magic trick about to be demonstrated.

The magic trick consists of repeating the above experiment a large number of times.

Alice and Bob then find that in each run of the experiment the following two rules are invariably obeyed:

(1) *Parity*. An odd number of panels lights up red for any of the switch settings.

(2) *Correlation*. For any switch setting the similarly-numbered panels that light up at the opposite ends always light up the same color.

What is so magical about the above phenomenon? Since the switch settings and light flashings at Alice's end cannot influence those at Bob's and vice versa, the only explanation for the correlation rule would seem to be that the particles from the source carry instructions to their detectors telling them how to flash when any setting on them is chosen. Moreover, the instructions given to similarly-numbered panels at the two ends must be identical if the panels are always to flash the same colors irrespective of the switch setting used to light them up. In other words, in each run the instructions carried to each detector must make each panel on it flash a perfectly definite color, and the instructions to the two detectors must also be identical.

Now that the correlation rule has been satisfied, one can see how the instructions to a particular detector can be chosen so that the observed parity rule is also satisfied. But it is here that one encounters a problem. Let n_i denote the number of red panels for the ith switch setting ($i = 1, \ldots, 5$) and let $N = \sum_{i=1}^{5} n_i$, the total number of red panels over all switch settings. Then, on the one hand, N must be odd because each of the n_i's must be odd according to the parity rule; on the other hand, N must also be even because each panel occurs in an even number (either twice or four times) of the n_i's. This contradiction shows that instruction sets are impossible and seems to rule out any rational explanation of both the parity and correlation rules. This, in short, constitutes the nature of the magic trick.

2.2 How the Trick is Done

The particles used in the demonstration are actually spin-1/2 particles (or "qubits"). Each panel on the detector represents a particular four-qubit observable. Figure 2 shows the 11 distinct observables represented by the 20 panels.[1] The five switch settings are labeled S_1 to S_5. X and Z are the Pauli spin operators in the x and z directions, respectively: $X \equiv \sigma_x$ and $Z \equiv \sigma_z$. The subscripted numbers indicate the qubits upon which the operators are acting. Thus, Z_1 performs Z on the first qubit entering the detector and leaves the other three unchanged. That is, $Z_1 = Z_1 \otimes I_2 \otimes I_3 \otimes I_4$, where I is of course the identity operator. Similarly, observables such as $ZXZX$ are shorthand for the tensor product $Z_1X_2Z_3X_4$.

It can be shown that the observabes in each column (S_1 to S_4) form a mutually commuting set. Additionally, the observables in the last row S_5 also form a mutually commuting set. Each observable has eigenvalues of only ± 1. Furthermore, the product of the observables in any column is $+I$, whereas the product of the observables in the last row is $-I$.

When any of the settings $S_1, \ldots, S_5$ is chosen on the detector, the detector carries out a measurement of the commuting observables in that column or row on its qubits. For example, if S_1 is chosen, the eigenvalues of the observables Z_1, X_2, Z_3, X_4, and $ZXZX$ are measured simultaneously and exactly because the observables are mutually commuting. After measurement, the panels of the switch settings S_1 to S_4 light up either red or green according to the eigenvalues of the observables: If the eigenvalue is -1, the panel lights up red; if $+1$, it lights up green. An opposite convention exists for switch setting S_5: An eigenvalue of -1 corresponds to a green-flashing panel; an eigenvalue of $+1$ to a red one.

[1] Call a detector with a suitable set of observables on it a "magic rectangle" because it provides for a proof of the BKS Theorem.

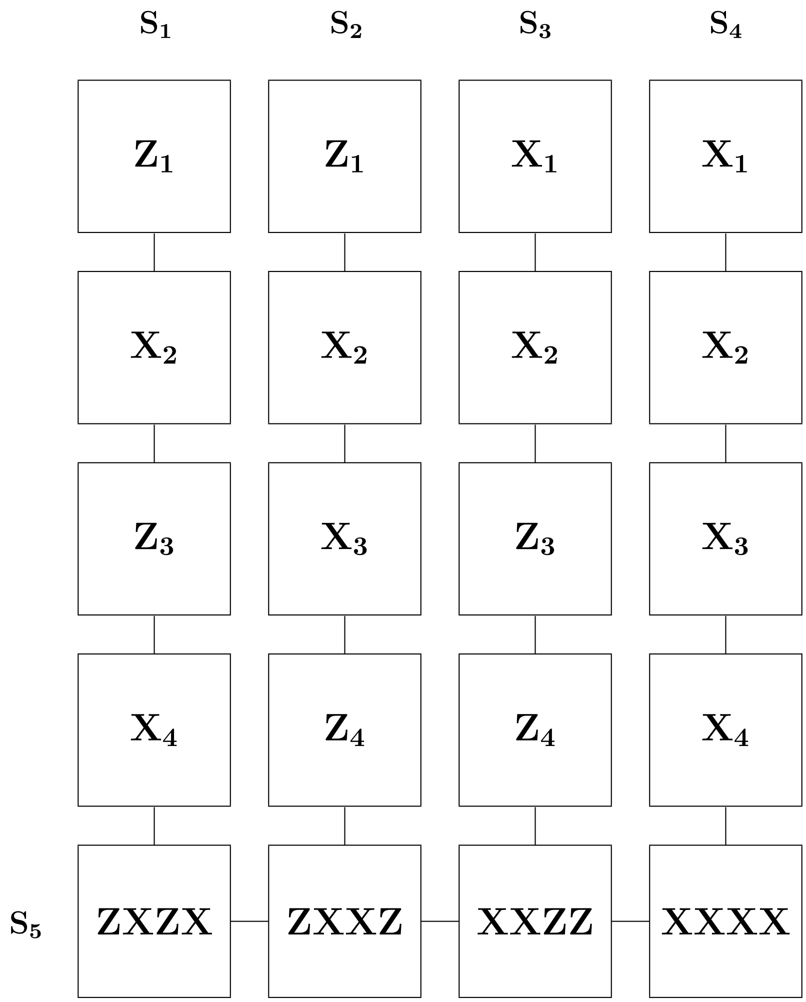

Figure 2: The “magic rectangle” in which each entry is a four-qubit observable

The parity rule summarized in the previous section can now be explained by means of observables and eigenvalues. Because all the observables in any column multiply to give the identity operator, which only has an eigenvalue of +1, it follows that the eigenvalues of the observables individually multiply to give +1. If the only possible values for the eigenvalues are +1 or −1, there must be an even number of eigenvalues of value −1. If the eigenvalue of the last observable in any column is −1 (green), there must be an odd number of −1 eigenvalues (red) among the first four observables in order to have an even number of -1 eigenvalues in total. If, on the other hand, the eigenvalue of the last observable in any column is $+1$ (red), there must be an even number of -1 eigenvalues (red) among the first four observables in order to have an even number of -1 eigenvalues in total. In either case, there ends up being an odd number of red flashes in any column, as the parity rule states.

Alternatively, because the observables in the last row multiply to give minus the identity operator (-1 eigenvalue), there must be an odd number of -1 eigenvalues among these observables. This is the same as saying that there must be an odd number of green flashes in the last row. An odd number of green flashes in turn implies an odd number of red flashes because there are only four panels in the last row. So the parity rule holds in its entirety.

To prove the BKS Theorem, one undertakes the realist's task of assigning definite instruction sets to the qubits telling the detector how to flash and shows that this leads to a contradiction. The task of assigning instruction sets reduces, as already argued, to the task of assigning a definite color to each panel (observable) on the detector. However, this task is seen to be impossible by the argument given in the preceding section, which is reproduced here for convenience. Let n_i denote the number of red panels for the ith switch setting ($i = 1, \ldots, 5$) and let $N = \sum_{i=1}^{5} n_i$, the total number of red panels over all switch settings. Then, on the one hand, N must be odd because each of the n_i's must be odd according to the parity rule; on the other hand, N must also be even because each panel occurs in an even number (either twice or four times) of the n_i's. This contradiction serves to show, as before, that the realist assumption of preexisting definite properties for the qubits is simply untenable.

The principle of noncontextuality was assumed in the above argument because each observable was assigned a definite eigenvalue (color) irrespective of which commuting set of observables it was measured as a part of. Thus the above argument serves to rule out noncontextual hidden variable theories, and so proves the BKS Theorem.

The proof of the BKS Theorem falls short of proving the stronger Bell's Theorem because of this assumption of noncontextuality that went into it. To prove Bell's Theorem one doubles the number of qubits to eight and divides them into two sets. Four qubits fly off toward Alice *and* four toward Bob. The qubits are entangled in the sense that they are all described by a single wavefunction $|\psi\rangle$, the tensor product of four Bell states. One member of each Bell state goes to Alice, the other member to Bob.[2] Because of this entanglement, if Alice measures an observable using one switch setting and Bob measures the same observable using another switch setting, experimentally they always get the same color for that observable. This is the correlation rule of the previous section and is a little more complicated to explain theoretically than the parity rule.[3] The correlation rule, then, truly demonstrates the spookiness of quantum entanglement. It also provides, interestingly enough, the necessary empirical basis for the assumption of noncontextuality present in the BKS Theorem.

If one makes the assumption of locality, which says that distant objects can have no direct effect upon each other, and also subscribes to realism, the belief that observables possess definite eigenvalues even before measurement, then one is forced to explain the puzzling correlation by saying that the particles carry definite instruction sets to their respective detectors telling them how to flash. As already mentioned, designing instruction sets reduces to assigning a definite color to each detector panel in such a way that the parity rule is always satisfied. However, the detector cannot be properly colored in this way, as previously proved. This contradiction leads one to believe that the initial assumption of local realism is false. This is precisely the conclusion of Bell's Theorem. In other words, the theorem, which seems simple but is in fact very deep,

[2]For a more detailed treatment of the entangled wavefunction, see Appendix 5.1.

[3]In brief, the total wavefunction must be rewritten as a linear combination of tensor products between the orthonormal states of Alice and Bob's qubits. For more information, see Appendix 5.1.

serves to show that the classical position of local realism is an untenable one.

The existence of local hidden variable theories is thus ruled out by exploiting the features of an entangled system. Because of this, Bell's Theorem is sometimes spoken of as establishing nonlocality, or "spooky actions at a distance," as a fact of nature.

3 State Proof of the BKS Theorem in 16 Dimensions

In the preceding chapter, a proof of the BKS Theorem for a system of four qubits was given by exploiting suitable sets of commuting observables for that system. A different albeit related proof of the BKS Theorem is given in this chapter by deriving a set of 80 states from those sets of observables and showing that they cannot be colored according to a simple set of rules.

3.1 Introductory Remarks

Before proceeding with the proof, it is worthwhile to say something in general about how the BKS Theorem can be proved using a set of quantum states in a Hilbert space. Consider a finite set of states in a Hilbert space of dimension $d > 2$. To each state one wishes to assign a value, either 0 or 1, in such a way that the following two rules always hold:

(1) No two orthogonal states can both be assigned the value 1 *and*

(2) Every member of a set of mutually orthogonal states that span the space cannot be assigned the value 0.

The BKS Theorem asserts that it is *always possible* to find a finite number of states such that all of the states cannot be properly assigned either a 0 or a 1 according to the preceding rules. Kochen and Specker originally proved the theorem using 117 state vectors in a three-dimensional space. Since then, a number of proofs of the theorem have been given in other dimensions [13]. This chapter will give a proof in dimension 16 using 80 suitably chosen states.

In many cases, the states form a certain number of complete bases in such a way that each state occurs in at least one basis with every other state orthogonal to it. One can characterize this situation by saying that one is dealing with a "saturated" configuration.[4] The present 16-dimensional 80-state case is indeed a saturated configuration with each basis, or 16-ad, consisting of 16 members. The occurrence of a saturated configuration is accompanied by a high degree of symmetry in the states which considerably simplifies the BKS proof. Namely, for a saturated configuration, one can combine rules (1) and (2) above in order to restate the BKS Theorem in the following way:

> (**BKS Theorem for Saturated Configurations**) It is impossible to assign a 0 or a 1 to all the states in such a way that each basis formed by the states contains exactly one state with the value 1 and all the others with the value 0.

If one adopts more colorful language, speaking of assigning the value 1 to a state as coloring it green and of assigning the value 0 to it as coloring it red, then a proof of the BKS Theorem in a Hilbert space of d dimensions can be given if one can identify in that space a saturated set of states, each of which cannot be colored green or red in such a way that each basis formed by the states contains exactly one green state and $(d-1)$ red states. This is the way the BKS Theorem will be proved in a Hilbert space of 16 dimensions in this chapter.

The no-coloring proof of the BKS Theorem rules out the existence of hidden variable theories of a particular kind. This can be seen by considering a system in which one wishes to measure the commuting projection operators corresponding to all the states in a given basis. Each measurement returns either a 0 or a 1, the eigenvalues of the projection operator. Because the projection operators in a given basis all commute, it

[4] The 117 vectors used by Kochen and Specker, as well as different sets of vectors used to give other proofs in three dimensions, do not involve saturated configurations of states.

is indeed possible to carry out a precise measurement of all of them in a single experiment without running into the problems posed by the uncertainty principle. Moreover, because the projection operators all sum to unity (as a consequence of the completeness relation), the sum of their measured eigenvalues in any experiment must be 1. This means that only one of the projection operators can return the value 1 and that all of the others must return the value 0.

If one is a realist, like Einstein, and believes that the values returned in an experiment existed before the experiment was carried out, then one is forced to come up with a successful solution to the coloring problem above. An inability to do so therefore demolishes one's position that definite values existed prior to measurement, the major tenet of hidden variable theories.

Note, however, that in the above coloring problem one assigns the same color to a state in every basis in which it occurs. One assumes that the projection operator corresponding to a state will return the same eigenvalue no matter which basis the state is in. Unfortunately, the measurements corresponding to two different bases cannot be carried out at the same time because the projection operators in two different bases do not necessarily commute. If a hidden variable theory assigns the same value to a projection operator irrespective of the basis in which it is measured it is said to be noncontextual. The BKS Theorem thus rules out the existence of noncontextual hidden variable theories. The task of ruling out hidden variable theories that do not rely on the assumption of noncontextuality is the principal task of Bell's Theorem.

3.2 Construction of the 80 States

The "magic rectangle" used to prove the BKS Theorem in the preceding chapter is reproduced in Figure 3. Recall that each entry is a four-qubit observable that acts on the set of either Alice or Bob's qubits. The 80 states presently needed to prove the

BKS Theorem are actually just the eigenstates of the five sets of mutually commuting observables present in the rectangle in the form of the four columns plus the last row. Each set of observables $S_1, \ldots, S_5$ has 16 different mutual eigenstates for a net total of 80 states. For each set, the 16 eigenstates that span the space are linear combinations of the 16 ($= 2^4$) vectors of the computational basis:

$$\begin{aligned} &|0000\rangle, |0001\rangle, |0010\rangle, |0011\rangle, |0100\rangle, |0101\rangle, |0110\rangle, |0111\rangle, \\ &|1000\rangle, |1001\rangle, |1010\rangle, |1011\rangle, |1100\rangle, |1101\rangle, |1110\rangle, |1111\rangle, \end{aligned} \tag{1}$$

where $|0\rangle$ is the "spin up" (eigenvalue $+1$) eigenstate of the Pauli operator Z and $|1\rangle$ is, alternatively, the "spin down" (eigenvalue -1) eigenstate of the operator. As a matter of notation, $|0000\rangle$ is of course equivalent to $|0\rangle|0\rangle|0\rangle|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4$.

3.2.1 Eigenvalue Signatures and State Labels

The 16 eigenstates for each set $S_1, \ldots, S_4$ (the four columns) are uniquely determined by the eigenvalues of the observables for each set. All of these 64 eigenstates can be labeled, therefore, by appropriate eigenvalue signatures. For example, one eigenstate of S_1 might be labeled $(+ + + + +)$ to indicate that all the observables of S_1 have an eigenvalue of $+1$. These eigenvalues uniquely determine the eigenstate (up to a multiplicative factor). However, because the eigenvalues of the observables in any column individually multiply to give $+1$, it is not necessary to list all five eigenvalue signatures. Without any loss of specificity, only four need to be reported because the last one is automatically determined by the previous four, being simply their product. Hence, in the previous example, $(+ + ++)$ would be sufficient, designating the eigenvalues of the first four mutually commuting observables of S_1 respectively.

The remaining 16 eigenstates of set S_5 are exceptional, as expected. They too can be labeled by eigenvalue signatures. Now each label consists of four eigenvalue signatures

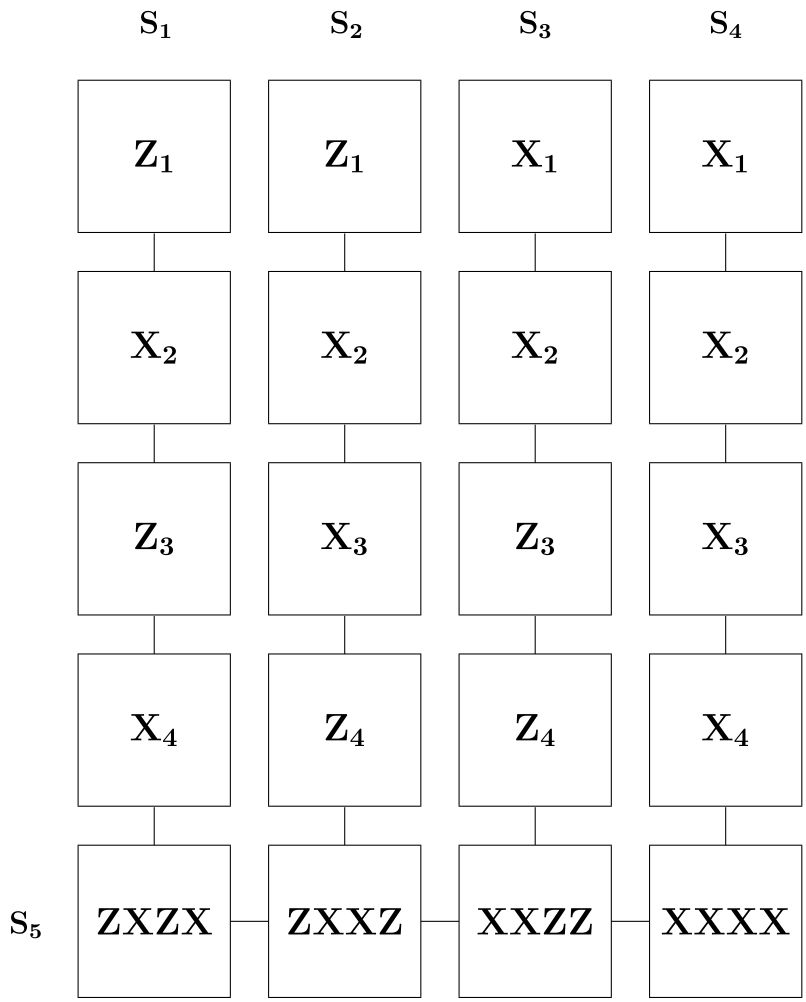

Figure 3: The "magic rectangle" used in the observable-based proof

corresponding to all the observables in the last row. However, because the eigenvalues of the observables in the last row individually multiply to give -1, only eight out of 16 signature combinations are possible. So a degeneracy exists in which two eigenstates are described by one set of eigenvalue signatures.

The 80 eigenstates can be labeled simply by the two-digit numbers $01, 02, \ldots, 79, 80$. Table 2 shows the correspondence between these state labels and the eigenvalue signatures. The 80 eigenstates are divided into five groups: 01–16, 17–32, 33–48, 49–64, and 65–80. Starting from the first set of eigenvalue signatures in each group, the remaining sets of eigenvalue signatures are obtained by means of binary counting with $+$ corresponding to a 0 and $-$ to a 1. Additionally, it is clear that the 80 states are of two types: Type I states (01–64) and Type II states (64–80). This asymmetry has consequences that must be handled later on in proving the theorem.[5]

01	$(++++)$	**17**	$(++++)$	**33**	$(++++)$	**49**	$(++++)$	**65**	$(+++-)$
02	$(+++-)$	**18**	$(+++-)$	**34**	$(+++-)$	**50**	$(+++-)$	**66**	$(+++-)$
03	$(++-+)$	**19**	$(++-+)$	**35**	$(++-+)$	**51**	$(++-+)$	**67**	$(++-+)$
04	$(++--)$	**20**	$(++--)$	**36**	$(++--)$	**52**	$(++--)$	**68**	$(++-+)$
05	$(+-++)$	**21**	$(+-++)$	**37**	$(+-++)$	**53**	$(+-++)$	**69**	$(+-++)$
06	$(+-+-)$	**22**	$(+-+-)$	**38**	$(+-+-)$	**54**	$(+-+-)$	**70**	$(+-++)$
07	$(+--+)$	**23**	$(+--+)$	**39**	$(+--+)$	**55**	$(+--+)$	**71**	$(+---)$
08	$(+---)$	**24**	$(+---)$	**40**	$(+---)$	**56**	$(+---)$	**72**	$(+---)$
09	$(-+++)$	**25**	$(-+++)$	**41**	$(-+++)$	**57**	$(-+++)$	**73**	$(-+++)$
10	$(-++-)$	**26**	$(-++-)$	**42**	$(-++-)$	**58**	$(-++-)$	**74**	$(-+++)$
11	$(-+-+)$	**27**	$(-+-+)$	**43**	$(-+-+)$	**59**	$(-+-+)$	**75**	$(-+--)$
12	$(-+--)$	**28**	$(-+--)$	**44**	$(-+--)$	**60**	$(-+--)$	**76**	$(-+--)$
13	$(--++)$	**29**	$(--++)$	**45**	$(--++)$	**61**	$(--++)$	**77**	$(--+-)$
14	$(--+-)$	**30**	$(--+-)$	**46**	$(--+-)$	**62**	$(--+-)$	**78**	$(--+-)$
15	$(---+)$	**31**	$(---+)$	**47**	$(---+)$	**63**	$(---+)$	**79**	$(---+)$
16	$(----)$	**32**	$(----)$	**48**	$(----)$	**64**	$(----)$	**80**	$(---+)$

Table 2: State labels 01–80 and their corresponding eigenvalue signatures

3.2.2 16-Vector Representation of the States

Any Type I state is trivial to calculate if one knows its set of eigenvalue signatures, the observables to which the eigenvalue signatures refer, and, morever, the spin up and spin down eigenstates of the Z and X operators. The eigenstates of Z are $|0\rangle$ and $|1\rangle$

[5]The 80 states can actually be understood in terms of 64 rank-1 and 8 rank-2 projectors. For more about projectors, see Appendix 5.2.

with eigenvalues $+1$ and -1, respectively:

$$\begin{cases} Z|0\rangle = +|0\rangle, \\ Z|1\rangle = -|1\rangle. \end{cases} \tag{2}$$

The unnormalized eigenstates of X are $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$ with eigenvalues $+1$ and -1, respectively:[6]

$$\begin{cases} X(|0\rangle + |1\rangle) = +(|0\rangle + |1\rangle), \\ X(|0\rangle - |1\rangle) = -(|0\rangle - |1\rangle). \end{cases} \tag{3}$$

Again consider the $(++++)$ eigenstate of S_1, state 01 according to Table 2. Because the first four observables of S_1 are Z_1, X_2, Z_3, and X_4, state 01 is given by, using Equations 2 and 3,

$$01 = |0\rangle_1 \otimes (|0\rangle + |1\rangle)_2 \otimes |0\rangle_3 \otimes (|0\rangle + |1\rangle)_4, \tag{4}$$

which, when expanded, simplifies to

$$01 = |0000\rangle + |0001\rangle + |0100\rangle + |0101\rangle, \tag{5}$$

where the subscript 1234 was dropped from each term on the righthand side for convenience. Furthermore, Equation (5) can be rewritten in a more compact way as a string of 16 numbers representing the coefficients multiplying the computational basis states of Expression (1). Because state 01 is a linear combination of the first, second, fifth, and sixth computational bases with coefficients of one for each, the eigenstate can be

[6] The X operator is analogous to the "bit flip" operator: $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$.

rewritten as

$$01 = 11001100\ 00000000. \tag{6}$$

All of the Type I states can easily be obtained in this simple fashion.

As expected, calculating the Type II states takes a little more work. The observables in the last row, of which the Type II states are eigenstates, are $ZXZX$, $ZXXZ$, $XXZZ$, and $XXXX$, the only possible eigenvalues of which are ± 1. One must first examine how these observables act on an arbitrary 16-vector given by, say, $abcdefgh$ $ijklmnop = a \cdots p$ according to the previously introduced notation. For example, $ZXZX[a \cdots p] = fe\bar{h}\bar{g}ba\bar{d}\bar{c}\ \bar{n}\bar{m}po\bar{j}\bar{i}lk = \pm[abcdefgh\ ijklmnop]$, where $\bar{h}$ is shorthand for $-h$ and so on. If the eigenvalue of $ZXZX$ is $+1$, it must be that, by matching coefficients, $a = f, b = e, c = \bar{h}, d = \bar{g}, i = \bar{n}, j = \bar{m}, k = p$, and $l = o$. If the eigenvalue is -1, eight similar constraints are generated. Now, each Type II state is given by a set of four eigenvalue signatures, each of which imposes constraints that the state must satisfy. For $(+++-)$ signatures, for example, the state must satisfy

$$|+++-\rangle = abb\bar{a}baa\bar{b}\ b\bar{a}\bar{a}\bar{b}a\bar{b}\bar{b}\bar{a}, \tag{7}$$

where a and b are two free parameters, or degrees of freedom. If one lets $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$, two Type II states are generated, states 65 and 66 respectively:

$$\begin{cases} 65 = 100\bar{1}0110\ 0\bar{1}\bar{1}01001\bar{1}, \\ 65 = 0110100\bar{1}\ 100\bar{1}0\bar{1}\bar{1}0. \end{cases} \tag{8}$$

3.2.3 Block Notation

Each of the 80 states can be written in terms of 16 coordinates by performing the calculations of the previous section. To display the results in a neat and compact form, block notation can be employed in which a block of four coordinates of the 16-vectors is abbreviated by a symbol. Forty-one such symbols need to be defined. Table 3 lists 20 of them. The other 21 are just the negation of those 20 plus the zero block $0 = 0000$. (Generally, the negation $\bar{a_1}$ of a_1 is the block of a_1 coordinates multipled by minus one: $\bar{a_1} = \bar{1}\bar{1}00$.)

$a_1 = 1100$	$a_2 = 1\bar{1}00$	$a_3 = 0011$	$a_4 = 001\bar{1}$
$b_1 = 1010$	$b_2 = 0101$	$b_3 = 10\bar{1}0$	$b_4 = 010\bar{1}$
$c_1 = 1000$	$c_2 = 0100$	$c_3 = 0010$	$c_4 = 0001$
$d_1 = 1111$	$d_2 = 1\bar{1}1\bar{1}$	$d_3 = 11\bar{1}\bar{1}$	$d_4 = 1\bar{1}\bar{1}1$
$e_1 = 100\bar{1}$	$e_2 = 0110$	$e_3 = 1001$	$e_4 = 01\bar{1}0$

Table 3: Notation for four-coordinate blocks

This block notation allows the 80 states to be conveniently summarized per Table 4. These states are of course unnormalized.

01	a_1a_100	17	b_1b_100	33	$c_1c_1c_1c_1$	49	$d_1d_1d_1d_1$	65	$e_1e_2\bar{e_2}e_1$
02	a_2a_200	18	b_2b_200	34	$c_2c_2c_2c_2$	50	$d_2d_2d_2d_2$	66	$e_2e_1e_1\bar{e_2}$
03	a_3a_300	19	b_3b_300	35	$c_3c_3c_3c_3$	51	$d_3d_3d_3d_3$	67	$e_1e_2e_2\bar{e_1}$
04	a_4a_400	20	b_4b_400	36	$c_4c_4c_4c_4$	52	$d_4d_4d_4d_4$	68	$e_2e_1\bar{e_1}e_2$
05	$a_1\bar{a_1}00$	21	$b_1\bar{b_1}00$	37	$c_1\bar{c_1}c_1\bar{c_1}$	53	$d_1\bar{d_1}d_1\bar{d_1}$	69	$e_3e_4\bar{e_4}e_3$
06	$a_2\bar{a_2}00$	22	$b_2\bar{b_2}00$	38	$c_2\bar{c_2}c_2\bar{c_2}$	54	$d_2\bar{d_2}d_2\bar{d_2}$	70	$e_4e_3e_3\bar{e_4}$
07	$a_3\bar{a_3}00$	23	$b_3\bar{b_3}00$	39	$c_3\bar{c_3}c_3\bar{c_3}$	55	$d_3\bar{d_3}d_3\bar{d_3}$	71	$e_3e_4e_4\bar{e_3}$
08	$a_4\bar{a_4}00$	24	$b_4\bar{b_4}00$	40	$c_4\bar{c_4}c_4\bar{c_4}$	56	$d_4\bar{d_4}d_4\bar{d_4}$	72	$e_4e_3\bar{e_3}e_4$
09	$00a_1a_1$	25	$00b_1b_1$	41	$c_1c_1\bar{c_1}\bar{c_1}$	57	$d_1d_1\bar{d_1}\bar{d_1}$	73	$e_3\bar{e_4}e_4e_3$
10	$00a_2a_2$	26	$00b_2b_2$	42	$c_2c_2\bar{c_2}\bar{c_2}$	58	$d_2d_2\bar{d_2}\bar{d_2}$	74	$e_4\bar{e_3}\bar{e_3}\bar{e_4}$
11	$00a_3a_3$	27	$00b_3b_3$	43	$c_3c_3\bar{c_3}\bar{c_3}$	59	$d_3d_3\bar{d_3}\bar{d_3}$	75	$e_3\bar{e_4}\bar{e_4}\bar{e_3}$
12	$00a_4a_4$	28	$00b_4b_4$	44	$c_4c_4\bar{c_4}\bar{c_4}$	60	$d_4d_4\bar{d_4}\bar{d_4}$	76	$e_4\bar{e_3}e_3e_4$
13	$00a_1\bar{a_1}$	29	$00b_1\bar{b_1}$	45	$c_1\bar{c_1}\bar{c_1}c_1$	61	$d_1\bar{d_1}\bar{d_1}d_1$	77	$e_1\bar{e_2}e_2e_1$
14	$00a_2\bar{a_2}$	30	$00b_2\bar{b_2}$	46	$c_2\bar{c_2}\bar{c_2}c_2$	62	$d_2\bar{d_2}\bar{d_2}d_2$	78	$e_2\bar{e_1}\bar{e_1}\bar{e_2}$
15	$00a_3\bar{a_3}$	31	$00b_3\bar{b_3}$	47	$c_3\bar{c_3}\bar{c_3}c_3$	63	$d_3\bar{d_3}\bar{d_3}d_3$	79	$e_1\bar{e_2}\bar{e_2}\bar{e_1}$
16	$00a_4\bar{a_4}$	32	$00b_4\bar{b_4}$	48	$c_4\bar{c_4}\bar{c_4}c_4$	64	$d_4\bar{d_4}\bar{d_4}d_4$	80	$e_2\bar{e_1}e_1e_2$

Table 4: Block notation for the 80 states

3.3 Construction of the 265 Bases (16-ads)

A basis for this 16-dimensional 80-state configuration is a group of 16 mutually orthogonal states called a 16-ad. In order to determine these 16-ads, a few remarks are in order. First, there are three possible values for the magnitude of the inner product between any normalized Type I state and any other normalized state: $1/2$, $\sqrt{2}/4$, and 0. Table 5 is a histogram showing how many states (of either type) have those inner product magnitudes between themselves and any Type I state. It is immediately evident that there are 59 states (51 Type I + 8 Type II) orthogonal to any Type I state.

Inner Product Magnitude	$1/2$	$\sqrt{2}/4$	0
Number of States	12	8	59
Type I	12	0	51
Type II	0	8	8

Table 5: Inner product histogram for any Type I state

Table 6 presents the same data, this time for any normalized Type II state. Thirty-two Type I states have an inner product magnitude of $\sqrt{2}/4$ between themselves and any Type II state, while 47 states (32 Type I + 15 Type II) are orthogonal to any Type II state.[7]

Inner Product Magnitude	$\sqrt{2}/4$	0
Number of States	32	47
Type I	32	32
Type II	0	15

Table 6: Inner product histogram for any Type II state

Knowing the orthogonalities for any Type I and Type II state, one can work out all

[7] The data in Tables 5 and 6 were obtained by the Maple program `innerproducts.mw` in Appendix 5.3. However, it is possible to determine the orthogonalities for any Type I and Type II state simply by inspection.

the bases formed by the 80 states. There turns out to be 265 bases in total. In Appendix 5.4 can be found a list of all the bases in lexicographic order, meaning the state numbers in each basis increase from left to right and the bases are ordered sequentially according to their leading entries. Each basis is on a separate line preceded by a three-digit number (basis label) $001, 002, \ldots, 264, 265$. Like the 80 states, the 265 bases formed by them can also be divided into two groups: A group of 256 Type A bases involving only Type I states and a group of 9 Type B bases involving, either partially or wholly, Type II states.

3.4 Quantum Block Design (QBD)

A compact way of characterizing the 256 Type A bases is through the notion of a Quantum Block Design (QBD). It is unnecessary to characterize the remaining 9 Type B bases by a QBD since there is a small number of them. For the Type A bases which are comprised of only Type I states, however, a QBD is a good way of conveying a sense of the structure of the bases without having to look directly at them. It does this by defining several parameters:[8]

(1) Let v be the total number of Type I states. That is, $v = 64$.

(2) Suppose each Type I state occurs in r of the Type A bases. If b is the total number of Type A bases and k is the maximum number of mutually orthogonal states in the space, it follows that $vr = bk$. In the present case, $r = 64$, $b = 256$, and $k = 16$.

(3) Consider the companions of any Type I state in all the Type A bases in which it occurs. Suppose that, among the companions, n_1 states each occur once in those

[8]Several of the parameters of the QBD were determined by the Maple program `quantumblockdesign.mw` in Appendix 5.5.

bases, n_2 states each occur twice, and so on. In general, then, n_k states each occur k times in those bases in which the Type I state occurs. Then it must be true that $1 \cdot n_1 + 2 \cdot n_2 + \cdots + k \cdot n_k = (k-1)r$. In the present case, $n_{16} = 45$, $n_{32} = 3$, $n_{48} = 3$, and all the other n_k's are zero. That is, 45 states each occur 16 times, 3 states each occur 32 times, and 3 states each occur 48 times.

The parameters of the QBD can be written neatly in one line as

$$[\ v, r;\ b, k \mid (1, n_1), \ldots, (k, n_k)\ ], \tag{9}$$

which in the present case becomes

$$[\ 64, 64;\ 256, 16 \mid (16, 45), (32, 3), (48, 3)\ ]. \tag{10}$$

The remaining 9 Type B bases are bases 065, 066, 205, 206, 253, 254, 263, 264, and 265, the elements of which are shown in Table 7. (Interestingly, they all come in pairs except basis 265.) They play a key role in the no-coloring proof that follows.

065:	[01, 04, 06, 07, 10, 11, 13, 16, 73, 74, 75, 76, 77, 78, 79, 80]
066:	[02, 03, 05, 08, 09, 12, 14, 15, 65, 66, 67, 68, 69, 70, 71, 72]
205:	[17, 20, 22, 23, 26, 27, 29, 32, 69, 70, 71, 72, 77, 78, 79, 80]
206:	[18, 19, 21, 24, 25, 28, 30, 31, 65, 66, 67, 68, 73, 74, 75, 76]
253:	[33, 36, 38, 39, 42, 43, 45, 48, 67, 68, 71, 72, 75, 76, 79, 80]
254:	[34, 35, 37, 40, 41, 44, 46, 47, 65, 66, 69, 70, 73, 74, 77, 78]
263:	[49, 52 ,54, 55, 58, 59, 61, 64, 65, 66, 71, 72, 75, 76, 77, 78]
264:	[50, 51, 53, 56, 57, 60, 62, 63, 67, 68, 69, 70, 73, 74, 79, 80]
265:	[65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80]

Table 7: The 9 Type B bases formed by the 80 states

3.5 The No-Coloring Proof of the BKS Theorem

The stage has now been set for a no-coloring proof of the BKS Theorem in 16 dimensions using the 80 states as the saturated set. It must be shown that it is impossible to color each of the states either green or red in such a way that each basis (or 16-ad) contains exactly one green state and 15 red states. The proof can be given using the technique of *reductio ad absurdum* (proof by contradiction): One assumes that a coloring is possible and then proceeds to show that this leads to a contradiction; hence the initial assumption must be false and the BKS Theorem is thereby proved.

Assume, then, that a satisfactory coloring of the 80 states exists. Let the four "defining" 16-ads be those 16-ads consisting solely of Type I states 01–16, 17–32, 33–48, and 49–64, respectively. These are the Type A 16-ads 001, 178, 245, and 262. It must be the case that if a satisfactory coloring exists each of these defining 16-ads is comprised of exactly one green state and 15 red states. Then it follows that among all the 64 Type I states only four are colored green while the remaining 60 are colored red. With the Type I states colored in this way, assume that all the Type A 16-ads formed by them are satisfactorily colored. Inspection of the 9 Type B 16-ads of Table 7 shows that the states 01–16 are distributed among the first two Type B 16-ads 065 and 066, 17–32 among the next two 16-ads 205 and 206, 33–48 among 253 and 254, and 49–64 among 263 and 264. The four green states present in the defining 16-ads can fall, then, within these eight Type B 16-ads in $2^4 = 16$ ways. That is, one green state can be in either 16-ad 065 or 066, another green state in either 205 or 206, and so on.

Each of these 16 possibilities can be identified by listing the Type B 16-ads in which the green states occur in terms of a string of four numbers, each of which can be either 1 or 2. The first number in the string is 1 if the first green state is in 16-ad 065; it is 2 if the first green state is in 16-ad 066. The second number in the string is 1 or 2 if the second green state is in 205 or 206, respectively. The same is true for the other two numbers in the string. For each possibility one must investigate to see if the coloring of the Type I states given by the possibility leads to a satisfactory coloring of the 16 Type II states and, consequently, to a satisfactory coloring of the 9 Type B 16-ads. One quick way to see that a satisfactory coloring of the Type II states and Type B 16-ads is impossible is to see if the coloring of the Type I states ultimately forces all the states in the last 16-ad 265 to be colored red. If 16-ad 265 can be satisfactorily colored, on the other hand, then all the 265 16-ads can be satisfactorily colored. Table 8 lists all the 16 possibilities as four-number strings and indicates whether or not each possibility enables satisfactory coloring of the 265 16-ads.[9]

Green States	**Satisfactory Coloring?**	**Green States**	**Satisfactory Coloring?**
1111	NO	2111	YES
1112	YES	2112	NO
1121	YES	2121	NO
1122	NO	2122	YES
1211	YES	2211	NO
1212	NO	2212	YES
1221	NO	2221	YES
1222	YES	2222	NO

Table 8: The 16 possibilities satisfactorily coloring the 265 16-ads

As an example, consider possibility 1111, which is shorthand for saying that the green Type I states are in 16-ads 065, 205, 253, and 263. According to Table 7, coloring one Type I state green in each of these 16-ads forces, if a satisfactory coloring exists, all Type II states to be colored red. This in turn causes 16-ad 265 to be colored entirely red, making a satisfactory coloring impossible. Similarly, possibility 1112 only forces states 67–80 to be colored red. This leaves 65 and 66 free to be colored differently, either red or green, in 16-ad 265. Hence, 1112 leads to a satisfactory coloring of all the

[9]The yes-no answers of Table 8 were determined by the Maple program `satisfactorycolorings.mw` in Appendix 5.6.

16-ads.

To complete the proof, then, it is sufficient to show that *all* satisfactory colorings of the Type I states 01–64 and their associated Type A 16-ads involve coloring states green in one of the possibilities that lead to an unsatisfactory coloring of all the 16-ads. In other words, it must be shown that all satisfactory colorings of the Type I states are of the kind of possibilities that answered "NO" in Table 8.

One can carry out a systematic computer search of all possible satisfactory colorings of the Type I states to arrive at 128 different possibilities, as shown in Table 9. In the table, each possibility is expressed in terms of the four Type I states which are colored green. Of course, all the other Type I states are colored red. It can be shown, moreover, that each possibility, or quartet, is exactly of the kind of possibilities that lead to an unsatisfactory coloring. Each quartet of green Type I states falls among the Type B 16-ads in such a way that its four-number string always contains an even number of 1's and an even number of 2's. There are eight of these kinds of strings: 1111, 1122, 1212, 1221, 2112, 2121, 2211, and 2222.[10] According to Table 8, these possibilities are precisely the ones that do not lead to a satisfactory coloring of the 16-ads! No matter what satisfactory coloring of the Type I states one makes it always turns out that the 265 16-ads cannot be colored according to the rule that in each 16-ad one state is colored green and the other 15 red. Hence the no-coloring proof of the BKS Theorem in 16 dimensions using a saturated set of 80 states is complete.

[10]The systematic computer search to determine the 128 quartets and their corresponding four-number strings was performed in the Maple program `quartets.mw` in Appendix 5.7.

1	1, 17, 33, 49	33	5, 21, 37, 53	65	9, 25, 33, 49	97	13, 29, 37, 53
2	1, 17, 41, 57	34	5, 21, 45, 61	66	9, 25, 41, 57	98	13, 29, 45, 61
3	1, 18, 34, 49	35	5, 22, 38, 53	67	9, 26, 34, 49	99	13, 30, 38, 53
4	1, 18, 42, 57	36	5, 22, 46, 61	68	9, 26, 42, 57	100	13, 30, 46, 61
5	1, 19, 33, 51	37	5, 23, 37, 55	69	9, 27, 33, 51	101	13, 31, 37, 55
6	1, 19, 41, 59	38	5, 23, 45, 63	70	9, 27, 41, 59	102	13, 31, 45, 63
7	1, 20, 34, 51	39	5, 24, 38, 55	71	9, 28, 34, 51	103	13, 32, 38, 55
8	1, 20, 42, 59	40	5, 24, 46, 63	72	9, 28, 42, 59	104	13, 32, 46, 63
9	2, 17, 33, 50	41	6, 21, 37, 54	73	10, 25, 33, 50	105	14, 29, 37, 54
10	2, 17, 41, 58	42	6, 21, 45, 62	74	10, 25, 41, 58	106	14, 29, 45, 62
11	2, 18, 34, 50	43	6, 22, 38, 54	75	10, 26, 34, 50	107	14, 30, 38, 54
12	2, 18, 42, 58	44	6, 22, 46, 62	76	10, 26, 42, 58	108	14, 30, 46, 62
13	2, 19, 33, 52	45	6, 23, 37, 56	77	10, 27, 33, 52	109	14, 31, 37, 56
14	2, 19, 41, 60	46	6, 23, 45, 64	78	10, 27, 41, 60	110	14, 31, 45, 64
15	2, 20, 34, 52	47	6, 24, 38, 56	79	10, 28, 34, 52	111	14, 32, 38, 56
16	2, 20, 42, 60	48	6, 24, 46, 64	80	10, 28, 42, 60	112	14, 32, 46, 64
17	3, 17, 35, 49	49	7, 21, 39, 53	81	11, 25, 35, 49	113	15, 29, 39, 53
18	3, 17, 43, 57	50	7, 21, 47, 61	82	11, 25, 43, 57	114	15, 29, 47, 61
19	3, 18, 36, 49	51	7, 22, 40, 53	83	11, 26, 36, 49	115	15, 30, 40, 53
20	3, 18, 44, 57	52	7, 22, 48, 61	84	11, 26, 44, 57	116	15, 30, 48, 61
21	3, 19, 35, 51	53	7, 23, 39, 55	85	11, 27, 35, 51	117	15, 31, 39, 55
22	3, 19, 43, 59	54	7, 23, 47, 63	86	11, 27, 43, 59	118	15, 31, 47, 63
23	3, 20, 36, 51	55	7, 24, 40, 55	87	11, 28, 36, 51	119	15, 32, 40, 55
24	3, 20, 44, 59	56	7, 24, 48, 63	88	11, 28, 44, 59	120	15, 32, 48, 63
25	4, 17, 35, 50	57	8, 21, 39, 54	89	12, 25, 35, 50	121	16, 29, 39, 54
26	4, 17, 43, 58	58	8, 21, 47, 62	90	12, 25, 43, 58	122	16, 29, 47, 62
27	4, 18, 36, 50	59	8, 22, 40, 54	91	12, 26, 36, 50	123	16, 30, 40, 54
28	4, 18, 44, 58	60	8, 22, 48, 62	92	12, 26, 44, 58	124	16, 30, 48, 62
29	4, 19, 35, 52	61	8, 23, 39, 56	93	12, 27, 35, 52	125	16, 31, 39, 56
30	4, 19, 43, 60	62	8, 23, 47, 64	94	12, 27, 43, 60	126	16, 31, 47, 64
31	4, 20, 36, 52	63	8, 24, 40, 56	95	12, 28, 36, 52	127	16, 32, 40, 56
32	4, 20, 44, 60	64	8, 24, 48, 64	96	12, 28, 44, 60	128	16, 32, 48, 64

Table 9: 128 quartets of green Type I states

4 Conclusion

4.1 Looking Back

Building upon earlier work by Mermin, Kernaghan, Peres, Aravind, and others, this project investigated proofs of the BKS Theorem and Bell's Theorem using a system of four qubits in a 16-dimensional Hilbert space.

A magic trick, new in a line of tricks given by Mermin and Aravind in smaller dimensions in the past, demonstrated to a general audience the spookiness of quantum mechanics that the theorems quite dramatically expose. There were 11 unique observables that were required to perform the trick, which interestingly matched well with previous results that showed a total of 9 and 10 observables were necessary in the two- and three-qubit systems, respectively. An experiment which exploits the technology of quantum gates and circuits can actually be performed to verify the theoretical results presented in the trick, the only difficulty being the fabrication of the last row of the detector which consists of all non-trivial four-qubit observables. The fact that the proofs presented in Chapter 2 were state-independent gives more flexibility to a potential experiment.

In the states-based proof of the BKS Theorem, the 80 eigenstates (or, alternatively, the 72 rank-1 and rank-2 projectors) of the observables, along with their 265 bases, were computed. The set of 80 states was in fact a saturated set of states that could not be properly colored: It was shown, via a lengthly yet sound argument, that each state could not be colored green or red in such a way that each basis formed by the states contained exactly one green state and 15 red states. This was a global proof of the theorem because every basis formed by the states was used to establish the proof. The search to reduce the number of states needed ended fruitlessly, although previous work had indeed discovered so-called quantum kaleidoscopes, or different subsets of states that provide for proofs of the BKS Theorem. Previous two- and three-qubits proofs, instead of using the total number of bases possible, used only 9 and 11 bases with 18 and 36 projectors, respectively. On the other hand, similar to prior research, the ratio of the number of projectors to the dimension of the space (n/d) was found to be 4.5.

4.2 Looking Forward

Research investigating the relationship between the states and bases obtained here and applications to quantum cryptography was beyond the scope of this project but could be examined in the future. The advantage of quantum cryptography is that the presence of an eavesdropper, Eve, is easily detectable because a measurement on a transmitted state alters the state. One of the entangled states obtained in this project can be sent to both Alice and Bob and they can individually measure the state using one of the bases. The basis Alice uses can be different from the basis Bob uses. After a large number of runs Alice and Bob can compare their results and keep those in which they used bases that had states in common. In this way they can build quantum code words (keys).

Usually in science one tackles a simpler case of a particular problem before looking at a more advanced case. Having learned much from the four-qubit proofs, it is hoped that the five-qubit two-observer system can be studied in the near future. (It has a state space of 32 dimensions!) It should be both interesting and illuminating.

5 Appendices

5.1 Details of the Wavefunction

In the four-qubit two-detector scheme, a possible normalized wavefunction is a pair of Bell states:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{12} + |11\rangle_{12}) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{34} + |11\rangle_{34}), \tag{11}$$

where $|0\rangle_i$ and $|1\rangle_i$ are "spin up" and "spin down" eigenstates, respectively, of the Pauli operator σ_z for qubits $i = 1, \ldots, 4$. Qubits 1 and 3 would go to Alice, qubits 2 and 4 to Bob.

In general, the so-called Bell basis consists of the following four states:

$$\begin{cases} |\phi^+\rangle = |00\rangle + |11\rangle, \\ |\phi^-\rangle = |00\rangle - |11\rangle, \\ |\psi^+\rangle = |01\rangle + |10\rangle, \\ |\psi^-\rangle = |01\rangle - |10\rangle. \end{cases} \tag{12}$$

The orthonormal basis in the joint space of qubits 1 and 3 can be written as

$$|\psi_i\rangle = a_i|00\rangle_{13} + b_i|01\rangle_{13} + c_i|10\rangle_{13} + d_i|11\rangle_{13}, \tag{13}$$

where $i = 1, \ldots, 4$ and $a_i, \ldots, d_i$ are complex numbers. The orthonormal basis for qubits 2 and 4 is then given by

$$|\phi_i\rangle = a_i^*|00\rangle_{24} + b_i^*|01\rangle_{24} + c_i^*|10\rangle_{24} + d_i^*|11\rangle_{24}, \tag{14}$$

where, again, $i = 1, \ldots, 4$. It follows that the total wavefunction can be rewritten as

$$|\psi\rangle = \frac{1}{2}\sum_{i=1}^{4} |\psi_i\rangle|\phi_i\rangle. \tag{15}$$

Generalization of the preceding formulae to six- and eight-qubit systems is achieved in a natural way.

5.2 The Projection Operator

Suppose one has a system prepared as a superposition of n orthonormal quantum states. The total wavefunction is then

$$|\psi_{total}\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + \cdots + c_n|\psi_n\rangle. \tag{16}$$

If one wanted to know the projection of the system onto a certain state $|\psi_m\rangle$ (one of the n orthonormal states), one can define the projection operator as

$$P_1 = |\psi_m\rangle\langle\psi_m|, \tag{17}$$

the outer product of $|\psi_m\rangle$ with itself. The operator P_1 is called a rank-1 projection operator, or projector, because it determines the projection onto only one state (hence the subscript 1 in P_1). It follows that

$$P_1|\psi_{total}\rangle = |\psi_m\rangle\langle\psi_m|[c_1|\psi_1\rangle + c_2|\psi_2\rangle + \cdots + c_m|\psi_m\rangle + \cdots + c_n|\psi_n\rangle] = c_m|\psi_m\rangle \tag{18}$$

because $\langle\psi_l|\psi_{l'}\rangle = \delta_{ll'}$. In other words, after the measurement P_1 on the system, $|\psi_{total}\rangle$ will collapse into $c_m|\psi_m\rangle$ with $\langle P_1\rangle = |c_m|^2$.

If, however, the system is prepared as a single (pure) quantum state, say $|\psi_n\rangle$, then

$$P_1|\psi_n\rangle = \begin{cases} 1 \cdot |\psi_n\rangle \text{ if } n = m, \\ 0 \cdot |\psi_n\rangle \text{ if } n \neq m. \end{cases} \tag{19}$$

These are just two eigenvalue equations. Either the system is in state m, in whice case the eigenvalue of P_1 is one, or the system is not in state m (perhaps in some state orthogonal to m), in which case the eigenvalue is zero. In other words, one is asking the system a yes-no question: The system answers "yes" if it is in the state about which one is inquiring; it answers "no" if it is not.

The projection operator is also Hermitian (${P_1}^\dagger = P_1$) and idempotent (${P_1}^2 = P_1$). Moreover, any P_{1i} and P_{1j} commute ($[P_{1i}, P_{1j}] = 0$). One can also define higher-order projectors: The rank-n projector would be given by

$$P_n = \sum_{i=1}^{n} |\psi_i\rangle\langle\psi_i|. \tag{20}$$

For example, a rank-2 projector might be written as $P_2 = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$.

According to quantum mechanics, one can measure operators simultaneously and exactly if all the operators commute with each other. If, for example, P_{ψ_1} and P_{ψ_2} are two projection operators corresponding to states $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively, then simultaneous and exact measurements of these two operators can be performed if $[P_{\psi_1}, P_{\psi_2}] = P_{\psi_1}P_{\psi_2} - P_{\psi_2}P_{\psi_1} = |\psi_1\rangle\langle\psi_1|\psi_2\rangle\langle\psi_2| - |\psi_2\rangle\langle\psi_2|\psi_1\rangle\langle\psi_1| = 0$. This is true if and only if $\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle = 0$, that is, if and only if $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal.

A particular nth-dimensional basis consists of n orthogonal states $\{\psi_i\}$, $i = 1, \ldots, n$. Therefore all the projection operators $\{P_{\psi_i}\}$, $i = 1, \ldots, n$, corresponding to the orthogonal states of the basis commute. If one prepares a system in one of the n orthogonal states in the basis, say $|\phi\rangle$, then simultaneously measuring the set $\{P_{\psi_i}\}$ on the state

$|\phi\rangle$ yields one and only one eigenvalue of value one and $(n-1)$ eigenvalues of value zero. The system answers "yes" to one state, "no" to the others. This is true in particular because of the completeness relation

$$\sum_{i=1}^{n} |\psi_i\rangle\langle\psi_i| = I, \tag{21}$$

where I is the identity operator. Because I only has an eigenvalue of one, the sum of the eigenvalues of the projection operators must be one. If all the projection operators only have eigenvalues of zero and one, one of them must return a one, the others zero.

Both hidden variable theories and quantum mechanics are consistent with the above rules concerning simultaneous measurements. Hidden variable theories make the claim that one can actually know ahead of time what set of eigenvalues will be returned following the $\{P_{\psi_i}\}$ measurements on $|\phi\rangle$. Quantum mechanics does not make that claim. Showing that assigning definite eigenvalues to every state leads to a violation of the above rules is how the BKS Theorem is proved in the body of the report.

5.3 Inner Product Magnitudes

What follows is the Maple program `innerproducts.mw` that calculates the inner product magnitudes between any normalized Type I or Type II state and any other normalized state and prints them to the screen.

```
// Initialization
restart:
with(linalg):


// List of the 80 states as vectors W(1) to W(80)
W(1):=([1,1,0,0,1,1,0,0,0,0,0,0,0,0,0,0]): ...
```

```
W(80):=([0,1,1,0,-1,0,0,1,1,0,0,-1,0,1,1,0]):

// Create a zero vector of 16 components for each state
for a from 1 by 1 to 80 do
A(a):=zerovector(16):od:

// Normalization of the 80 states
for a from 1 by 1 to 80 do
summ:=0:
for m from 1 by 1 to 16 do
summ:=summ+(W(a)[m])^2:od:
normconst:=sqrt(summ):
for n from 1 by 1 to 16 do
A(a)[n]:=(W(a)[n])/normconst:
od:od:

// Inner product magnitudes for state 1 (Type I) calculated and printed
a:=1:
for b from 1 by 1 to 80 do
if b <> a then z1:=evalm(A(a) &* A(b)):print(b,abs(z1)):fi:od:

// Inner product magnitudes for state 65 (Type II) calculated and printed
a:=65:
for b from 1 by 1 to 80 do
if b <> a then z2:=evalm(A(a) &* A(b)):print(b,abs(z2)):fi:od:
```

5.4 List of the 265 Bases

The following is a list of all the 265 bases formed by the 80 states:

001: [01, 02, 03, 04, 05, 06, 07, 08, 09, 10, 11, 12, 13, 14, 15, 16]

002: [01, 02, 03, 04, 05, 06, 07, 08, 09, 10, 11, 12, 29, 30, 31, 32]

003: [01, 02, 03, 04, 05, 06, 07, 08, 13, 14, 15, 16, 25, 26, 27, 28]

004: [01, 02, 03, 04, 05, 06, 07, 08, 25, 26, 27, 28, 29, 30, 31, 32]

005: [01, 02, 03, 04, 05, 06, 09, 10, 11, 12, 13, 14, 39, 40, 47, 48]

006: [01, 02, 03, 04, 05, 06, 13, 14, 25, 26, 27, 28, 39, 40, 47, 48]

007: [01, 02, 03, 04, 05, 07, 09, 10, 11, 12, 13, 15, 54, 56, 62, 64]

008: [01, 02, 03, 04, 05, 07, 13, 15, 25, 26, 27, 28, 54, 56, 62, 64]

009: [01, 02, 03, 04, 06, 08, 09, 10, 11, 12, 14, 16, 53, 55, 61, 63]

010: [01, 02, 03, 04, 06, 08, 14, 16, 25, 26, 27, 28, 53, 55, 61, 63]

011: [01, 02, 03, 04, 07, 08, 09, 10, 11, 12, 15, 16, 37, 38, 45, 46]

012: [01, 02, 03, 04, 07, 08, 15, 16, 25, 26, 27, 28, 37, 38, 45, 46]

013: [01, 02, 03, 04, 09, 10, 11, 12, 13, 14, 15, 16, 21, 22, 23, 24]

014: [01, 02, 03, 04, 09, 10, 11, 12, 21, 22, 23, 24, 29, 30, 31, 32]

015: [01, 02, 03, 04, 09, 10, 11, 12, 21, 22, 29, 30, 55, 56, 63, 64]

016: [01, 02, 03, 04, 09, 10, 11, 12, 21, 23, 29, 31, 38, 40, 46, 48]

017: [01, 02, 03, 04, 09, 10, 11, 12, 22, 24, 30, 32, 37, 39, 45, 47]

018: [01, 02, 03, 04, 09, 10, 11, 12, 23, 24, 31, 32, 53, 54, 61, 62]

019: [01, 02, 03, 04, 09, 10, 11, 12, 37, 38, 39, 40, 45, 46, 47, 48]

020: [01, 02, 03, 04, 09, 10, 11, 12, 37, 38, 39, 40, 61, 62, 63, 64]

021: [01, 02, 03, 04, 09, 10, 11, 12, 45, 46, 47, 48, 53, 54, 55, 56]

022: [01, 02, 03, 04, 09, 10, 11, 12, 53, 54, 55, 56, 61, 62, 63, 64]

023: [01, 02, 03, 04, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 27, 28]

024: [01, 02, 03, 04, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]

025: [01, 02, 03, 04, 21, 22, 25, 26, 27, 28, 29, 30, 55, 56, 63, 64]

026: [01, 02, 03, 04, 21, 23, 25, 26, 27, 28, 29, 31, 38, 40, 46, 48]

027: [01, 02, 03, 04, 22, 24, 25, 26, 27, 28, 30, 32, 37, 39, 45, 47]

028: [01, 02, 03, 04, 23, 24, 25, 26, 27, 28, 31, 32, 53, 54, 61, 62]

029: [01, 02, 03, 04, 25, 26, 27, 28, 37, 38, 39, 40, 45, 46, 47, 48]

030: [01, 02, 03, 04, 25, 26, 27, 28, 37, 38, 39, 40, 61, 62, 63, 64]

031: [01, 02, 03, 04, 25, 26, 27, 28, 45, 46, 47, 48, 53, 54, 55, 56]

032: [01, 02, 03, 04, 25, 26, 27, 28, 53, 54, 55, 56, 61, 62, 63, 64]

033: [01, 02, 05, 06, 07, 08, 09, 10, 13, 14, 15, 16, 35, 36, 43, 44]

034: [01, 02, 05, 06, 07, 08, 09, 10, 29, 30, 31, 32, 35, 36, 43, 44]

035: [01, 02, 05, 06, 09, 10, 13, 14, 35, 36, 39, 40, 43, 44, 47, 48]

036: [01, 02, 05, 07, 09, 10, 13, 15, 35, 36, 43, 44, 54, 56, 62, 64]

037: [01, 02, 06, 08, 09, 10, 14, 16, 35, 36, 43, 44, 53, 55, 61, 63]

038: [01, 02, 07, 08, 09, 10, 15, 16, 35, 36, 37, 38, 43, 44, 45, 46]

039: [01, 02, 09, 10, 13, 14, 15, 16, 21, 22, 23, 24, 35, 36, 43, 44]

040: [01, 02, 09, 10, 21, 22, 23, 24, 29, 30, 31, 32, 35, 36, 43, 44]

041: [01, 02, 09, 10, 21, 22, 29, 30, 35, 36, 43, 44, 55, 56, 63, 64]

042: [01, 02, 09, 10, 21, 23, 29, 31, 35, 36, 38, 40, 43, 44, 46, 48]

043: [01, 02, 09, 10, 22, 24, 30, 32, 35, 36, 37, 39, 43, 44, 45, 47]

044: [01, 02, 09, 10, 23, 24, 31, 32, 35, 36, 43, 44, 53, 54, 61, 62]

045: [01, 02, 09, 10, 35, 36, 37, 38, 39, 40, 43, 44, 45, 46, 47, 48]

046: [01, 02, 09, 10, 35, 36, 37, 38, 39, 40, 43, 44, 61, 62, 63, 64]

047: [01, 02, 09, 10, 35, 36, 43, 44, 45, 46, 47, 48, 53, 54, 55, 56]

048: [01, 02, 09, 10, 35, 36, 43, 44, 53, 54, 55, 56, 61, 62, 63, 64]

049: [01, 03, 05, 06, 07, 08, 09, 11, 13, 14, 15, 16, 50, 52, 58, 60]

050: [01, 03, 05, 06, 07, 08, 09, 11, 29, 30, 31, 32, 50, 52, 58, 60]

051: [01, 03, 05, 06, 09, 11, 13, 14, 39, 40, 47, 48, 50, 52, 58, 60]

052: [01, 03, 05, 07, 09, 11, 13, 15, 50, 52, 54, 56, 58, 60, 62, 64]

053: [01, 03, 06, 08, 09, 11, 14, 16, 50, 52, 53, 55, 58, 60, 61, 63]

054: [01, 03, 07, 08, 09, 11, 15, 16, 37, 38, 45, 46, 50, 52, 58, 60]

055: [01, 03, 09, 11, 13, 14, 15, 16, 21, 22, 23, 24, 50, 52, 58, 60]

056: [01, 03, 09, 11, 21, 22, 23, 24, 29, 30, 31, 32, 50, 52, 58, 60]

057: [01, 03, 09, 11, 21, 22, 29, 30, 50, 52, 55, 56, 58, 60, 63, 64]

058: [01, 03, 09, 11, 21, 23, 29, 31, 38, 40, 46, 48, 50, 52, 58, 60]

059: [01, 03, 09, 11, 22, 24, 30, 32, 37, 39, 45, 47, 50, 52, 58, 60]

060: [01, 03, 09, 11, 23, 24, 31, 32, 50, 52, 53, 54, 58, 60, 61, 62]

061: [01, 03, 09, 11, 37, 38, 39, 40, 45, 46, 47, 48, 50, 52, 58, 60]

062: [01, 03, 09, 11, 37, 38, 39, 40, 50, 52, 58, 60, 61, 62, 63, 64]

063: [01, 03, 09, 11, 45, 46, 47, 48, 50, 52, 53, 54, 55, 56, 58, 60]

064: [01, 03, 09, 11, 50, 52, 53, 54, 55, 56, 58, 60, 61, 62, 63, 64]

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069: [02, 04, 05, 06, 10, 12, 13, 14, 39, 40, 47, 48, 49, 51, 57, 59]

070: [02, 04, 05, 07, 10, 12, 13, 15, 49, 51, 54, 56, 57, 59, 62, 64]

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072: [02, 04, 07, 08, 10, 12, 15, 16, 37, 38, 45, 46, 49, 51, 57, 59]

073: [02, 04, 10, 12, 13, 14, 15, 16, 21, 22, 23, 24, 49, 51, 57, 59]

074: [02, 04, 10, 12, 21, 22, 23, 24, 29, 30, 31, 32, 49, 51, 57, 59]

075: [02, 04, 10, 12, 21, 22, 29, 30, 49, 51, 55, 56, 57, 59, 63, 64]

076: [02, 04, 10, 12, 21, 23, 29, 31, 38, 40, 46, 48, 49, 51, 57, 59]

077: [02, 04, 10, 12, 22, 24, 30, 32, 37, 39, 45, 47, 49, 51, 57, 59]

078: [02, 04, 10, 12, 23, 24, 31, 32, 49, 51, 53, 54, 57, 59, 61, 62]

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086: [03, 04, 05, 07, 11, 12, 13, 15, 33, 34, 41, 42, 54, 56, 62, 64]

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106: [05, 06, 07, 08, 13, 14, 15, 16, 33, 34, 35, 36, 41, 42, 43, 44]

107: [05, 06, 07, 08, 13, 14, 15, 16, 33, 34, 35, 36, 57, 58, 59, 60]

108: [05, 06, 07, 08, 13, 14, 15, 16, 41, 42, 43, 44, 49, 50, 51, 52]

109: [05, 06, 07, 08, 13, 14, 15, 16, 49, 50, 51, 52, 57, 58, 59, 60]

110: [05, 06, 07, 08, 17, 18, 19, 20, 25, 26, 27, 28, 29, 30, 31, 32]

111: [05, 06, 07, 08, 17, 18, 25, 26, 29, 30, 31, 32, 51, 52, 59, 60]

112: [05, 06, 07, 08, 17, 19, 25, 27, 29, 30, 31, 32, 34, 36, 42, 44]

113: [05, 06, 07, 08, 18, 20, 26, 28, 29, 30, 31, 32, 33, 35, 41, 43]

114: [05, 06, 07, 08, 19, 20, 27, 28, 29, 30, 31, 32, 49, 50, 57, 58]

115: [05, 06, 07, 08, 29, 30, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44]

116: [05, 06, 07, 08, 29, 30, 31, 32, 33, 34, 35, 36, 57, 58, 59, 60]

117: [05, 06, 07, 08, 29, 30, 31, 32, 41, 42, 43, 44, 49, 50, 51, 52]

118: [05, 06, 07, 08, 29, 30, 31, 32, 49, 50, 51, 52, 57, 58, 59, 60]

119: [05, 06, 09, 10, 11, 12, 13, 14, 17, 18, 19, 20, 39, 40, 47, 48]

120: [05, 06, 13, 14, 17, 18, 19, 20, 25, 26, 27, 28, 39, 40, 47, 48]

121: [05, 06, 13, 14, 17, 18, 25, 26, 39, 40, 47, 48, 51, 52, 59, 60]

122: [05, 06, 13, 14, 17, 19, 25, 27, 34, 36, 39, 40, 42, 44, 47, 48]

123: [05, 06, 13, 14, 18, 20, 26, 28, 33, 35, 39, 40, 41, 43, 47, 48]

124: [05, 06, 13, 14, 19, 20, 27, 28, 39, 40, 47, 48, 49, 50, 57, 58]

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127: [05, 06, 13, 14, 39, 40, 41, 42, 43, 44, 47, 48, 49, 50, 51, 52]

128: [05, 06, 13, 14, 39, 40, 47, 48, 49, 50, 51, 52, 57, 58, 59, 60]

129: [05, 07, 09, 10, 11, 12, 13, 15, 17, 18, 19, 20, 54, 56, 62, 64]

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133: [05, 07, 13, 15, 18, 20, 26, 28, 33, 35, 41, 43, 54, 56, 62, 64]

134: [05, 07, 13, 15, 19, 20, 27, 28, 49, 50, 54, 56, 57, 58, 62, 64]

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137: [05, 07, 13, 15, 41, 42, 43, 44, 49, 50, 51, 52, 54, 56, 62, 64]

138: [05, 07, 13, 15, 49, 50, 51, 52, 54, 56, 57, 58, 59, 60, 62, 64]

139: [06, 08, 09, 10, 11, 12, 14, 16, 17, 18, 19, 20, 53, 55, 61, 63]

140: [06, 08, 14, 16, 17, 18, 19, 20, 25, 26, 27, 28, 53, 55, 61, 63]

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144: [06, 08, 14, 16, 19, 20, 27, 28, 49, 50, 53, 55, 57, 58, 61, 63]

145: [06, 08, 14, 16, 33, 34, 35, 36, 41, 42, 43, 44, 53, 55, 61, 63]

146: [06, 08, 14, 16, 33, 34, 35, 36, 53, 55, 57, 58, 59, 60, 61, 63]

147: [06, 08, 14, 16, 41, 42, 43, 44, 49, 50, 51, 52, 53, 55, 61, 63]

148: [06, 08, 14, 16, 49, 50, 51, 52, 53, 55, 57, 58, 59, 60, 61, 63]

149: [07, 08, 09, 10, 11, 12, 15, 16, 17, 18, 19, 20, 37, 38, 45, 46]

150: [07, 08, 15, 16, 17, 18, 19, 20, 25, 26, 27, 28, 37, 38, 45, 46]

151: [07, 08, 15, 16, 17, 18, 25, 26, 37, 38, 45, 46, 51, 52, 59, 60]

152: [07, 08, 15, 16, 17, 19, 25, 27, 34, 36, 37, 38, 42, 44, 45, 46]

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156: [07, 08, 15, 16, 33, 34, 35, 36, 37, 38, 45, 46, 57, 58, 59, 60]

157: [07, 08, 15, 16, 37, 38, 41, 42, 43, 44, 45, 46, 49, 50, 51, 52]

158: [07, 08, 15, 16, 37, 38, 45, 46, 49, 50, 51, 52, 57, 58, 59, 60]

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234: [21, 23, 29, 31, 33, 34, 35, 36, 38, 40, 46, 48, 57, 58, 59, 60]

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253: [33, 36, 38, 39, 42, 43, 45, 48, 67, 68, 71, 72, 75, 76, 79, 80]

254: [34, 35, 37, 40, 41, 44, 46, 47, 65, 66, 69, 70, 73, 74, 77, 78]

255: [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52]

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265: [65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80]

5.5 Parameters of the Quantum Block Design (QBD)

What follows is the Maple program `quantumblockdesign.mw` that calculates the value of r and of all the n_k's for the Type A bases and prints them to the screen.

```
// Initialization
restart:
with(linalg):

// List of the 265 bases as vectors W(1) to W(265)
W(1):=([1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16]): ...
W(265):=([65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80]):

// Calculation of r for the Type I states
for i from 1 by 1 to 64 do
num:=0:
for j from 1 by 1 to 265 do
if j <> 65 and j <> 66 and j <> 205 and j <> 206 and
   j <> 253 and j <> 254 and j <> 263 and j <> 264 and j <> 265 then
```

```
for k from 1 by 1 to 16 do
if (W(j)[k]) = i then num:=num+1:fi:od:
fi:od:
print(i,num):od:

// Create a vector with 64 elements of value zero
A:=zerovector(64):

// Determine the frequency of occurrence of the companions of any
   Type I state in all the Type A bases in which it occurs
for j from 1 by 1 to 265 do
if j <> 65 and j <> 66 and j <> 205 and j <> 206 and
   j <> 253 and j <> 254 and j <> 263 and j <> 264 and j <> 265 then
for k from 1 by 1 to 16 do
if (W(j)[k]) = 1 then
for l from 1 by 1 to 16 do
z:=W(j)[l] : if z <> 1 then A[z]:=A[z]+1:fi:od:
fi:od:fi:od:

// Determine the largest frequency of occurrence
z:=0:
for m from 1 by 1 to 64 do
if A[m] > z then z:=A[m]:fi:od:

// Print the n_k's to the screen
for n from 1 by 1 to z do
```

```
num:=0:
for o from 1 by 1 to 64 do
if A[o] = n then num:=num+1:fi:od:
if num <> 0 then print(num,"states occur",n,"times"):fi:od:
```

5.6 Satisfactory Colorings

What follows is the Maple program `satisfactorycolorings.mw` that finds out whether or not each possibility of the green Type I states falling among the Type B 16-ads enables satisfactory coloring of the 265 16-ads. It prints each possibility as a four-number string alongside a "YES" for satisfactory coloring or a "NO" for unsatisfactory coloring.

```
// Initialization
restart:
with(linalg):

// List of the 265 16-ads as vectors W(1) to W(265)
W(1):=([1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16]): ...
W(265):=([65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80]):

// Begin the loop to determine all 16 possibilities of strings abcd
for a from 1 by 1 to 2 do
for b from 1 by 1 to 2 do
for c from 1 by 1 to 2 do
for d from 1 by 1 to 2 do

// Assign a value of zero to each Type II state by default
```

```
for n from 64 by 1 to 80 do
L(n):=0:od:

// Assign a value of one (red) to all Type II states in a 16-ad with a
   green state already in it
if a = 1 then
for n from 9 by 1 to 16 do
L(W(65)[n]):=1:od:fi:
if a = 2 then
for n from 9 by 1 to 16 do
L(W(66)[n]):=1:od:fi:

if b = 1 then
for n from 9 by 1 to 16 do
L(W(205)[n]):=1:od:fi:
if b = 2 then
for n from 9 by 1 to 16 do
L(W(206)[n]):=1:od:fi:

if c = 1 then
for n from 9 by 1 to 16 do
L(W(253)[n]):=1:od:fi:
if c = 2 then
for n from 9 by 1 to 16 do
L(W(254)[n]):=1:od:fi:
```

```
if d = 1 then

for n from 9 by 1 to 16 do

L(W(263)[n]):=1:od:fi:

if d = 2 then

for n from 9 by 1 to 16 do

L(W(264)[n]):=1:od:fi:


// Add all the values assigned to the Type II states in 16-ad 265

total:=0:

for n from 1 by 1 to 16 do

total:=total+L(W(265)[n]):od:


// If the total is 16, 265 is entirely red (unsatisfactory coloring);

   otherwise a satisfactory coloring exists

if total = 16 then print (a,b,c,d,‘NO‘):fi:

if total <> 16 then print (a,b,c,d,‘YES‘):fi:


od:od:od:od:
```

5.7 Calculation of the 128 Quartets

What follows is the Maple program `quartets.mw` that calculates all the 128 different ways of properly coloring the Type I states and their associated Type A 16-ads. It displays each possibility on a single line in terms of the four Type I states that are colored green. It also determines and prints the four-number string showing how each quartet falls among the Type B 16-ads.

```
// Initialization
restart:
with(linalg):

// List of the 265 16-ads as vectors W(1) to W(265)
W(1):=([1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16]): ...
W(265):=([65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80]):

// List of the 80 states as vectors X(1) to X(80)
X(1):=([1,1,0,0,1,1,0,0,0,0,0,0,0,0,0,0]): ...
X(80):=([0,1,1,0,-1,0,0,1,1,0,0,-1,0,1,1,0]):

// Counter for the number of quartets
num:=1:

// a, b, c, and d are the Type I states colored green
for a from 1 by 1 to 16 do
for b from 17 by 1 to 32 do
for c from 33 by 1 to 48 do
for d from 49 by 1 to 64 do

// Counter for the number of Type A 16-ads properly colored
state:=0:

// Coloring the Type I states: -1 is green; +1 is red
for e from 1 by 1 to 64 do
```

```
if e <> a and e <> b and e <> c and e <> d then Z(e):=1 else Z(e):=-1:
fi:od:


// Count how many Type A 16-ads are properly colored by adding up the
   values assigned to each state in a 16-ad and making sure the total
   is 14 (15 red + 1 green)
for f from 1 by 1 to 265 do
if f <> 65 and f <> 66 and f <> 205 and f <> 206 and
   f <> 253 and f <> 254 and f <> 263 and f <> 264 and f <> 265 then
addition:=0:
for g from 1 by 1 to 16 do
addition:=addition+Z(W(f)[g]):od:
if addition = 14 then state:=state+1:fi:
fi:od:


// If all the Type A 16-ads are properly colored, determine how each
   of the Type I states falls among the Type B 16-ads in terms of a
   four-number string and then print the quartet number, the four Type I
   states colored green, and the four-number string
if state = 256 then


for l from 1 by 1 to 16 do
if W(65)[l] = a then o1:=1:fi:
if W(66)[l] = a then o1:=2:fi:
if W(205)[l] = b then o2:=1:fi:
if W(206)[l] = b then o2:=2:fi:
```

```
if W(253)[l] = c then o3:=1:fi:

if W(254)[l] = c then o3:=2:fi:

if W(263)[l] = d then o4:=1:fi:

if W(264)[l] = d then o4:=2:fi:

od:


print(num,a,b,c,d,‘   ‘,o1,o2,o3,o4):num:=num+1:fi:


od:od:od:od:
```

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