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Arbitrage-Free Pricing of XVA for Options in Discrete Time

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Arbitrage-Free Pricing of XVA
for Options in Discrete Time

A Major Qualifying Project
Submitted to the Faculty
Of
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By

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Abstract

The goal of this project is to develop XVA pricing methods for options with discrete time settings. Particularly, this project focuses on risk valuation adjustments pertaining to funding spread and counterparty credit risk, and applies them to the binomial tree model. The final model incorporates both risk valuation adjustments, and numerical examples are provided.

Contents

List of Figures

1 Introduction

A derivative is a financial contracts whose value at settlement depends on the value of an underlying asset. Under the classical models, e.g. Black-Scholes, binomial tree, etc., a set of limiting assumptions are made in order to price derivatives. These assumptions offer ease of calculation, but are not reflective of the complexities of today's financial markets. For example, the framework allows investors to buy and sell any number of stocks, including fractional amounts, and ignores all funding costs. It is also common to assume that financial agents are non-defaultable, which has been disproven by recent events. In financial mathematics, efforts have been made to relax the simplifying assumptions of the classical framework and to propose adjustments to pricing methods that capture and mitigate certain risks. A set of adjustments that seeks to address multiple sources of risk in the market is known as Total Valuation Adjustment pricing (TVA). In the literature, the pricing methods used are referred to as XVA pricing, where 'X' stands for the particular risk that is adjusted for. Modifications to a pricing model that account for risks experienced in a trading environment are referred to as *risk valuation adjustments*.

XVA pricing has been researched more extensively for models in continuous time than models in discrete time. The purpose of this project is to work towards construction of an XVA framework for derivative pricing with discrete time. Particularly, this project incorporates both counterparty credit risk and non-zero funding spread into a binomial tree model.

Funding Spread. An investor often must hedge a derivative with investments in stock, as well as cash that is lent to or borrowed from the treasury. If the hedging portfolio produces the same cash flows as the derivative, it can be used to price it. Traditional models often assume there is one risk-free rate, r, under which all money that passes through the treasury desk accrues. However, the existence of this risk-free rate must be reconsidered, as it is no longer true that a bank funding rate, government rate, or LIBOR can be used in this context. More generally, the rate at which an investor can borrow or lend money can be calculated based on his own credit [1]. This difference in borrowing and lending rates creates challenges in using the replicating portfolio to price a derivative. In this paper, we refer to this difference as *funding spread*.

Credit Risk. In the classical framework, it is also assumed that both parties to a financial derivative contract are non-defaultable. That is, the agreed-upon payments will be made fully at the agreed-upon time. However, it is possible for either party to default on the contract, in which case the non-defaulting party will receive a partial settlement, or zero. As such, it is necessary to consider the possibility of future default when pricing the contract at its outset. We will refer to the risk of default by the counterparty as *credit risk*, and default by the investor as *debit risk*.

The unpredictable nature of derivatives makes credit and debit risk particularly interesting. Unlike other financial contracts with pre-determined payoff structures, such as bonds, in which credit risk is equivalent to lending risk, the value of future payments at the time of default is uncertain for derivatives. For example, the payout of a European call must be at least zero, but has no maximum (given that the stock price at maturity has no limit). This means that an investor who holds a long position in the call will be exposed to an unknown amount of credit risk. Consider derivatives whose payoffs may be either positive or negative

for investors who hold a long position in it. In this case, the exposure to credit and debit risk is uncertain, since at the time of the contract's start, the investor does not know whether he will be the one "owing" or the one "owed" in the case of default. For these reasons, credit risk for a derivative is truly *bilateral* — meaning that each party is exposed to risk from the other [3], [4]. However, this project will only consider *unilateral* credit risk from the point of view of the investor — also referred to in this paper as *counterparty credit risk*.

We begin by exploring basic concepts of no-arbitrage pricing in both complete and incomplete markets. Then, we introduce adjustments to the binomial model based on non-zero funding spread, and prove conditions under which the model is free of arbitrage. Next, we consider the impact of counterparty credit risk on the replicating portfolio and other pricing methods, and formalize these findings into a separate model. Finally, we combine the two sources of risk into a single model, exhibit the results, and briefly discuss areas for further research.

2 Background

2.1 Important Terms and Concepts

- European Call/Put Option – A European option can only be exercised at the date of maturity. For the European call option, the holder may choose to buy an underlying asset at the price agreed in the option or not to exercise at the expiration date. Similarly, the European put option gives the holder the right, but not obligation to sell at the agreed upon price at the maturity date.

- Strike Price – This is the price at which an option may be exercised. For a call option, the strike price indicates how much the security can be bought at, whereas for a put option, it indicates how much the security can be sold at.

- Counterparty – The counterparty is the other party that participates in a transaction.

- Replication/Replicating portfolio – A portfolio that contains shares in tradeable assets, which replicates the cash flows of a different asset or liability at each market state. The replicating portolio can include options.

- Arbitrage – An arbitrage occurs when one starts with no capital, and at some later time, loses no money and has a positive probability of making money. That is, a portfolio with net initial investment of zero — call this Y — would be an arbitrage strategy if at some later time, T,
  1. The probability that $Y_T \geq 0$ is one.
  2. The probability that $Y_T > 0$ is strictly more than zero.

- Hedge – A hedge is an investment that is designed to reduce the risk of unfavorable price movements in a given asset. It will typically take the form of an offsetting position in a related asset.

- Super hedging price – The super hedging price of a portfolio is the smallest amount needed to construct an admissible portfolio such that the future value of the admissible portfolio is no less than the contingent claim.

- Risk-free rate – This rate is the theoretical rate of return on an investment with no risk of financial loss.

- Market Completeness - A complete market is one in which every derivative security can be replicated, with the information available at the current time.

2.2 Fundamental Theorems of Asset Pricing in the Discrete Setting

There are two fundamental theorems for asset pricing in the discrete setting. This section briefly reviews both.

> **The First Fundamental Theorem of Asset Pricing** A discrete model is arbitrage-free if and only if there exists one risk-neutral measure that is equivalent to the physical measure.

> **The Second Fundamental Theorem of Asset Pricing** An arbitrage-free market consisting of stocks and bonds is complete if and only if there exists a unique risk-neutral measure that is equivalent to the physical measure.

The first fundamental theorem enables us to find an alternative measure that agrees with the "real-world", physical measure on impossible events, in order to arrive at an arbitrage-free price for a derivative. Under this risk-neutral measure, the stock price discounted by the risk-free rate is a martingale — meaning, that the conditional expectation of the discounted stock price at a future time is equal to the current stock price, which we will denote as S_t. Formally, we denote the discount factor from time T to time t, $t < T$, as $D_{t,T}$[1]. We denote all information available up to time t under the risk-neutral measure as the sigma-algebra $\mathcal{F}_t$. So[2]:

$$E[D_{t,T}S_T|\mathcal{F}_t] = S_t$$

Under a discrete model, calculation of the risk-neutral measure is relatively simple using the equation above.

Recall that a complete market is one in which a derivative can be replicated at every state. The second fundamental theorem ultimately means that if exact replication is possible, the existence of a unique risk-neutral measure is certain. Pricing by replication as it pertains to the binomial tree model is discussed in the next section.

---

[1]Note that if interest is compounded annually, by risk-free rate r, $D_{0,1} = \frac{1}{1+r}$

[2]Note: by convention, $E[x|\mathcal{F}_0] = E[x]$

2.3   The Binomial Tree Model

The binomial tree model was introduced by Cox, Ross, and Rubinstein as a simple model for option valuation, based on the idea that the underlying asset follows a two-state lattice [5]. We let the underlying asset be a stock, whose initial price is denoted as S_0. The capital gain to the stock over each period can take only one of two values, commonly referred to as the price in 'up-state' and the price in the 'down-state.' The ratio between these prices and S_0 are, repectively, the *up-factor*, u, and the *down-factor*, d. The risk-neutral probability of the stock price reaching the up-state in one period is denoted as p, and the risk-neutral probability of it reaching the down-state is denoted as $1 - p$.
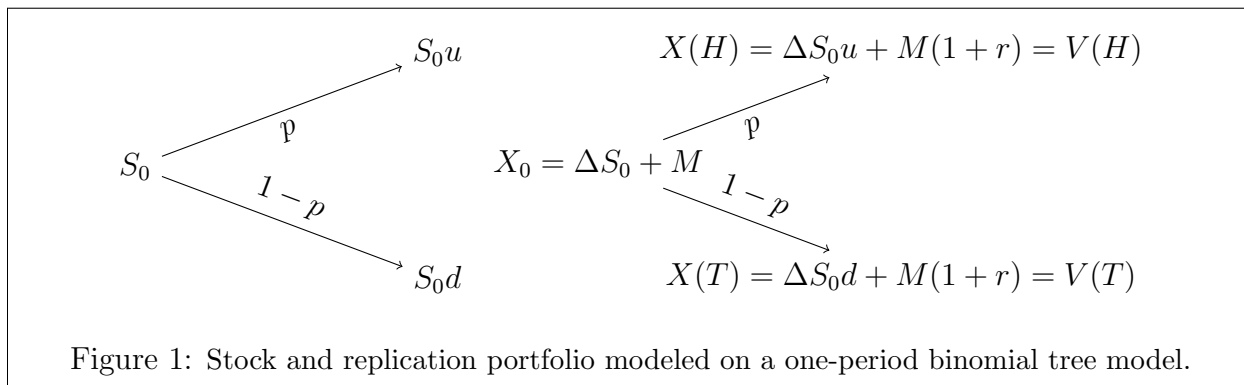

Figure 1: Stock and replication portfolio modeled on a one-period binomial tree model.

Pricing by Replication The model traditionally assumes a market consisting of stocks and bonds. In order to price a derivative, an investor replicates his position in a derivative with shares of stock and bonds (a.k.a. money market account). Let Δ denote the number of shares of stock the investor buys, and M denote the number of shares of money market account the investor lends. Investments into the money market grow at compound interest by the risk-free rate, r, over every period. For this discussion, we will denote the value of the option at the up-state as $V(H)$ and the value of a variable at the down-state as $V(T)$. The same notation is followed for other variables.

Let $X_t = \Delta_t S_t + M_t$ denote the value of the investor's replication portfolio at time t. The aim of the investor is to exactly replicate the cash-flows of the derivative at every market state. (For the following discussion, the absence of a subscript on Δ and M indicates time zero.) Therefore, after one period:

$$X(H) = \Delta S_0 u + M(1 + r) = V(H)$$

$$X(T) = \Delta S_0 d + M(1 + r) = V(T)$$

Thus,

$$\Delta = \frac{V(H) - V(T)}{(u - d)S_0}$$

$$M = \frac{uV(T) - dV(H)}{(u - d)(1 + r)}$$

4

Finally, the price of the derivative, V_0, is equal to the price of the replicating portfolio, $X_0 = \Delta S_0 + M$. The risk-neutral probability of an up-movement is defined such that $p = \frac{(1+r)-d}{u-d}$.

The existence of a replicating portfolio implies that a derivative can be priced by taking the expectation of discounted payoffs under the risk-neutral measure. Note that we have concluded that the price $V_t = X_t \; \forall t \in [0,T]$ for a derivative with maturity T. Consider a derivative modeled with a one-period binomial tree.

$$E[V_T] = E[X_T] = E[\Delta S_T + M_T] = E[\Delta S_T] + M_0(1+r)^T$$

Suppose we multiply both expectations by the discount factor, $D_{0,T} = \frac{1}{(1+r)^T}$. Then,

$$E\left[\frac{V_T}{(1+r)^T}\right] = E\left[\frac{X_T}{(1+r)^T}\right] = \Delta E\left[\frac{S_T}{(1+r)^T}\right] + M_0$$

Using the fact that the stock price discounted by the risk-free rate is a martingale, we arrive at,

$$E\left[\frac{V_T}{(1+r)^T}\right] = \Delta S_0 + M_0$$

Note that the right-hand side is equivalent to the price of the portfolio, which we have already established is equivalent to the price of the derivative. Thus,

$$V_0 = E\left[\frac{V_T}{(1+r)^T}\right] \tag{1}$$

Similar arguments can be made for a derivative modeled on a multi-period binomial tree. For a multi-period model, we instead take expectations and discount over a single period at a time, starting at maturity.

To outline the methods used to price a derivative on the binomial model, we present a simple example:

Example 1

Suppose a European call option is modeled on a two-period binomial tree. You are given:
  (i) each period is 6 months
  (ii) the current price of a stock is $S_0 = 100$
  (iii) $u = 1.1$, where u is 1 plus the gain on the stock if the price of the stock goes up
  (iv) $d = 0.8$, where d is 1 plus the gain on the stock if the price of the stock goes down
  (v) the risk-free interest rate, compounded annually, is $r = 3\%$
  (vi) the strike price, K, is 100

We will first construct a binomial tree modeling the possible values of S at times 0, 6 months, and 1 year.

$$S_1(HH) = 121$$

$$S_{0.5}(H) = 110$$

$$S_0 = 100$$

$$S_1(HT) = 88$$

$$S_{0.5}(T) = 80$$

$$S_1(TT) = 64$$
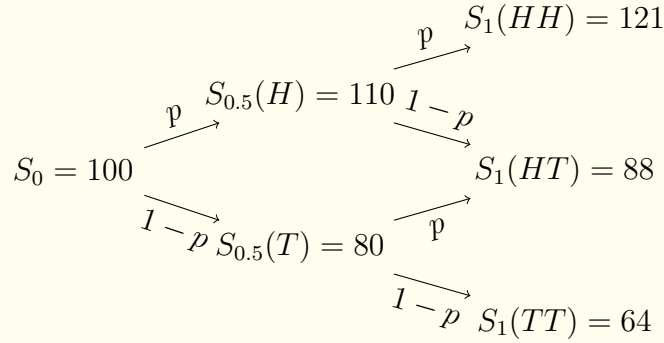
Then, we calculate the payoff of the call at the three possible endpoints — the payoffs resulting from: two downward movements in the stock, two upward movements in the stock, and one upward, one downward movement in the stock.

Recall, the payoff function for a call option is $V_T = (S_T - K)^+$, so:

$V(HH) = 21$, $V(HT) = 0$, $V(TT) = 0$

To find the risk-neutral price of the derivative, we must find the risk-neutral probability of an upward stock movement, p:

$$p = \frac{\frac{1}{D_{0,1}} - d}{u - d} = \frac{(1 + .03)^{\frac{1}{2}} - 0.8}{1.1 - 0.8} \approx 0.7163$$

Next, we move one period to the left on the tree, and calculate the values of the option at the 6-month mark. Since we know the risk-neutral probability of an upward stock movement, we can calculate these prices as conditional expectations at each node:
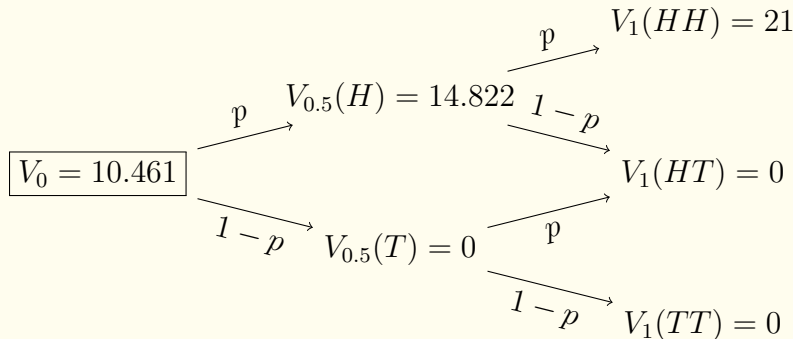
$$V_{0.5}(H) = D_{0.5,1} \cdot E[V_1 | S_{0.5} = 110] = (1.03)^{-\frac{1}{2}} \cdot [p \cdot V(HH) + (1 - p) \cdot V(HT)]$$

$$V_{0.5}(T) = D_{0.5,1} \cdot E[V_1 | S_{0.5} = 80] = (1.03)^{-\frac{1}{2}} \cdot [p \cdot V(HT) + (1 - p) \cdot V(TT)]$$

To arrive at an initial price for time zero, we repeat the same calculation:

$$V_0 = D_{0,0.5} \cdot E[V_{0.5}] = (1.03)^{-\frac{1}{2}} \cdot [p \cdot V_{0.5}(H) + (1 - p) \cdot V_{0.5}(T)]$$

Rounded values are given in the tree below.

$$V_1(HH) = 21$$

$$V_{0.5}(H) = 14.822$$

$$\boxed{V_0 = 10.461}$$

$$V_1(HT) = 0$$

$$V_{0.5}(T) = 0$$

$$V_1(TT) = 0$$

2.4 Pricing in Incomplete Markets

In a traditional binomial tree model, which is a complete market, a derivative can be uniquely priced and replicated. However, in multinomial models, there is insufficient information to find a unique solution for the risk-neutral measure, and the market is incomplete.

2.4.1 Trinomial Model

Consider pricing a European call option on S, modeled with a one-period trinomial tree as follows:

(i) initial stock price is $S_0 = 100$
(ii) the stock price can be one of the set $\{120,110,90\}$ at the end of the period
(iii) the constant risk-free interest rate per period is $r = 4\%$
(iv) the strike price is $K = 105$
(v) the option expires at time $T = 1$
(vi) under risk-neutral probability measure P, let:
    (a) $p_1 = P(S_1 = 120)$
    (b) $p_2 = P(S_1 = 110)$
    (c) $p_3 = P(S_1 = 90)$

To find an arbitrage-free price for this option, one begins by finding the risk-neural measure from the given information. In the tree model, the events at time t are simply the possible values that S_t can take. The events are disjoint — meaning, S_1 cannot be equal to both 120 and 90, for this example. Therefore, we know that the probabilities of these events sum to 1:

$$p_1 + p_2 + p_3 = 1 \tag{2}$$

Additionally, we know that the discounted value of the stock price is a martingale under the risk-neutral measure. $E^P[S_1] = S_0 \cdot (1 + r)$. Thus,

$$E^P[S_1] = 120p_1 + 110p_2 + 90p_3 = 100 \cdot (1 + r) = 104 \tag{3}$$

Treating equations (2) and (3) as a system, we can observe that it is under-informed. There are two equations with three unknowns $\{p_1, p_2, p_3\}$; as such, it is not possible to find a unique solution for P. Since there is no unique solution for the risk-neutral measure, this is not a complete market, and no unique arbitrage-free price can be found.

Instead, we can obtain an interval of values for each p_i. We begin by using equation (2) to solve for one p_i. Then, we substitute this value into (3), solve for p_j where $i \neq j$, and bound that value by $[0,1]$. For example:

$$p_3 = 1 - p_1 - p_2$$

$$30p_1 + 20p_2 = 14 \implies p_1 = \frac{7}{15} - \frac{2p_2}{3} \implies 0 \leq \frac{7}{15} - \frac{2p_2}{3} \leq 1$$

It is clear that from the last inequality, we can conclude $p_2 \in [0, \frac{21}{30}]$. Similarly, we find $p_1 \in [0, \frac{14}{30}], p_3 \in [\frac{3}{10}, \frac{8}{15}]$.

An infinite number of risk-neutral probability measures exist — (one for each possible of $p_1 \in [0, \frac{14}{30}]$, $p_2 \in [0, \frac{21}{30}]$, $p_3 \in [\frac{3}{10}, \frac{8}{15}]$, when chosen consistently). Under any of these measures, the no-arbitrage value of an option is equivalent to the discounted expectation of the payoff, since the payoffs of the derivative are dependent on only the value of the underlying asset, S_t. In this example,

$$C(S, K, T) = E^P \left[\frac{V_1}{(1+r)^1} \right] = \frac{15p_1 + 5p_2 + 0p_3}{1.04}$$

Using this equation, we can parameterize a p_i to find the price bounds for the option. Taking p_2 as our paramater, we arrive at: $C(S, K, T) \in [3.37, 6.73]$.

2.4.2 Replication in an Incomplete Market

The non-uniqueness of price exhibited in the trinomial model example is connected to the existence of a unique replication portfolio and risk-neutral measure. In fact, the formation of an exact replication portfolio is not possible in an incomplete market. For example, in the trinomial model example, in order to replicate V, we must form some portfolio of shares of stock, Δ, and shares in a money market account, M, such that, at time $t = 1$:

$$120\Delta + M \cdot 1.04 = 15$$

$$110\Delta + M \cdot 1.04 = 5$$

$$90\Delta + M \cdot 1.04 = 0$$

This is an over-informed system. As such, it is not possible to find a solution for $\{\Delta, M\}$. A natural choice would be to implement super-hedging. By super-hedging, we ensure that our portfolio outperforms the claim in every terminal market state, i.e.:

$$120\Delta + M \cdot 1.04 \geq 15$$

$$110\Delta + M \cdot 1.04 \geq 5$$

$$90\Delta + M \cdot 1.04 \geq 0$$

To find the initial value of the super-hedging portfolio, $X_0 = \Delta S_0 + M$, we find the lower bound of the interval of X_0's such that the above holds true. In this example, the value is $X_0 = 6.73$, which is the same as the upper price bound that was found in the previous section. Note that with this structure, it is possible to construct a portfolio that exactly replicates one of the three claims at time 1, and exceeds each of the other two claims. If one were to implement strict super-hedging, the inequalities above would be strict, and no state would be exactly replicated.

2.4.3 Binomial Tree with Multiple Assets

Suppose the payoff of an option with maturity time T is $\max[0, Z_T - Y_T - K]$, where Z_T and Y_T are the prices of two different assets at time T, and K is a common strike price. The method for pricing this option depends on whether or not Z_t and Y_t are dependent. For the following discussion, let these assumptions hold true:

Table 1: Joint Probability Distribution between Z_1 and Y_1, with $Z_t \perp Y_t$

	$Y_1 = Y_0 u_Y$	$Y_1 = Y_0 d_Y$
$Z_1 = Z_0 u_Z$	$p_1 \cdot p_3$	$p_1 \cdot p_4$
$Z_1 = Z_0 d_Z$	$p_2 \cdot p_3$	$p_2 \cdot p_4$

(i) the derivative being priced is a European spread call option on Z_t and Y_t

(ii) the values of Z_t and Y_t are modeled using separate 1-period binomial trees

(iii) d_Z, d_Y are the down factors for Z and Y respectively; u_Z, u_Y are the up factors

(iv) risk-free rate, r is a constant

(v) under risk-neutral probability measure P, let:

   (a) $p_1 = P(Z_1 = Z_0 u_Z)$, $p_2 = P(Z_1 = Z_0 d_Z)$, $p_3 = P(Y_1 = Y_0 u_Y)$, $p_4 = P(Y_1 = Y_0 d_Y)$

If Z_t and Y_t are treated as independent, then the joint probability distribution between Z_1 and Y_1 is simply equal to the products of the probabilities, as summarized in Table 1.

Note that we can also form similar equations to the previous example, for each of our binomial trees - the one modeling Z_t, and the one modeling Y_t. Namely:

$$p_1 + p_2 = 1$$

$$p_3 + p_4 = 1$$

$$Z_0 u_Z (1 - p_1) + Z_0 d_Z p_2 = Z_0 \cdot (1 + r)$$

$$Y_0 u_Y (1 - p_3) + Y_0 d_Y p_4 = Y_0 \cdot (1 + r)$$

Since there are now four equations with four unknowns, we can find a unique solution for $\{p_1, p_2, p_3, p_4\}$, and conclude

$$
\begin{aligned}
C(S, K, T) &= E^P \left[\frac{V_1}{(1+r)^1} \right] \\
&= \frac{\max(0, Z_0 u_Z - Y_0 u_Y - K) \cdot (p_1 p_3)}{(1+r)} + \frac{\max(0, Z_0 u_Z - Y_0 d_Y - K) \cdot (p_1 p_4)}{(1+r)} \\
&+ \frac{\max(0, Z_0 d_Z - Y_0 u_Y - K) \cdot (p_2 p_3)}{(1+r)} + \frac{\max(0, Z_0 d_Z - Y_0 d_Y - K) \cdot (p_2 p_4)}{(1+r)}
\end{aligned}
$$

If the movements of the two underlying assets are dependent, we can no longer use the joint probability distribution described in Table 1. Instead, the probability distribution is redefined as follows:

(a) $q_1 = P((Z_1 = Z_0 u_Z) \cap (Y_1 = Y_0 u_Y))$

(b) $q_2 = P((Z_1 = Z_0 u_Z) \cap (Y_1 = Y_0 d_Y))$

(c) $q_3 = P((Z_1 = Z_0 d_Z) \cap (Y_1 = Y_0 u_Y))$

(d) $q_4 = P((Z_1 = Z_0 d_Z) \cap (Y_1 = Y_0 d_Y))$

(Note that for any probability measure space $(\Omega, \mathcal{F}, Q)$, and disjoint sets $B, C \in \Omega$ such that $B \cup C = \Omega$, $Q(A) = Q(A \cap B) + Q(A \cap C)$ for $A \in \Omega$. Then $Q(X_1 = X_0 u_X) = q_1 + q_2$, etc.) Therefore,

$$q_1 + q_2 + q_3 + q_4 = 1$$
$$Z_0 u_Z(q_1 + q_2) + Z_0 d_Z(q_3 + q_4) = Z_0 \cdot (1 + r)$$
$$Y_0 u_Y(q_2 + q_4) + Y_0 d_Y(q_1 + q_3) = Y_0 \cdot (1 + r)$$

This is now similar to the problem presented by the trinomial model discussed earlier. We have four unknown probabilities, and three equations. Using the same strategy presented with the trinomial example, we can find upper and lower bounds for the price of the spread option, but no unique solution exists.

3 Results

We begin our adjustments to the binomial tree model by exploring the impact of non-equal borrowing and lending rates on the replication portfolio. Then, separately, we develop a model that incorporates counterparty credit risk and introduce a third asset into the replication portfolio to hedge this risk. Finally, we combine these ideas into a single model.

3.1 Model 1: Funding Spread

The traditional framework assumes there is one risk-free interest rate, r, under which cash values are accumulated and discounted. However, it is more practical to suppose that there is a non-zero funding spread of interest rates offered by the treasury desk. Let r_b represent the (constant) borrowing rate, r_l the (constant) lending rate. Note that a negative value in the cash position of the investor's portfolio represents cash borrowed, whereas a positive value is cash lent. In the following discussion, this portfolio is denoted as (l, b, Δ), where l is shares of cash lent, b is shares of cash borrowed, and Δ is shares of stock bought.

In Theorem 1, we introduce necessary conditions on u, d, r_l, and r_b under which one can arrive at an arbitrage-free price for a derivative modeled in a one-period binomial tree.
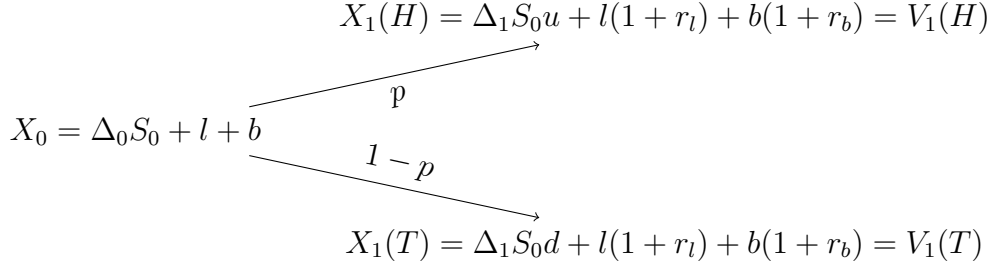
Figure 2: One-period model of length $h = 1$ illustrating a replication portfolio (l, b, Δ). The lent portion of the money market account grows at rate r_l, whereas the borrowed portion of the money market account grows at rate r_b. We assume $r_l \neq r_b$.

Theorem 1. : No-Arbitrage Condition: For a derivative on a stock modeled with a one-period binomial model of length h, let $r_b > 0$ and $r_l > 0$ be the Treasury borrowing and lending rates, respectively, and u, d have their usual definitions. Then no arbitrage is possible if and only if $u > d$, $d < (1 + r_b)^h$, $r_l < r_b$, and $(1 + r_l)^h < u$.

Proof. "$\Leftarrow$:" Suppose an arbitrage strategy $X = (l, b, \Delta)$ exists. Assume $d < (1 + r_b)^h$, $r_l \leq r_b$, and $(1 + r_l)^h < u$. We will prove by contradiction that no such arbitrage strategy exists.

The initial investment $(t = 0)$ for this strategy is:

$$X_0 = l + b + \Delta S_0 = 0 \iff l + b = -\Delta S_0 \tag{4}$$

We can construct a system for the value of the portfolio at time $t = h$. Under the definition of arbitrage, there must be a non-zero probability of making a profit, meaning $X_h > 0$ for one state, and $X_h \geq 0$ for another state at time h. Since u is strictly greater than d, and X is an arbitrage strategy, we can conclude for $\Delta > 0$ that $X_h(H) > 0$, and $X_h(T) \geq 0$, (and the argument for $\Delta < 0$ follows similarly).

$$
\begin{array}{rclcccc}
X_h(H) & = & l(1 + r_l)^h & + & b(1 + r_b)^h & + & \Delta S_0 u & > & 0 \\
X_h(T) & = & l(1 + r_l)^h & + & b(1 + r_b)^h & + & \Delta S_0 d & \geq & 0
\end{array}
$$

Taking the substitution from (4), we find:

$$
\begin{array}{rclcc}
l((1 + r_l)^h - u) & + & b((1 + r_b)^h - u) & > & 0 \\
l((1 + r_l)^h - d) & + & b((1 + r_b)^h - d) & \geq & 0
\end{array}
$$

Note: $b < 0$, and $l > 0$, and, since $r_l \leq r_b$ we can conclude:

$$
\begin{array}{rclclcc}
l((1 + r_l)^h - u) & + & b((1 + r_l)^h - u) & \geq & l((1 + r_l)^h - u) & + & b((1 + r_b)^h - u) & > & 0 \\
l((1 + r_b)^h - d) & + & b((1 + r_b)^h - d) & \geq & l((1 + r_l)^h - d) & + & b((1 + r_b)^h - d) & \geq & 0
\end{array}
$$

From the first inequality:

$$(l + b) \cdot ((1 + r_l)^h - u) > 0 \xRightarrow{u > (1+r_l)^h} (l + b) < 0$$

Similarly, from the second inequality:

$$(l + b) \cdot ((1 + r_b)^h - d) \geq 0 \xRightarrow{d < (1+r_b)^h} (l + b) \geq 0$$

No l, b exist such that $(l + b) < 0$ and $(l + b) \geq 0$. Thus, we arrive at a contradiction.

"$\Rightarrow$" Suppose that $u \leq (1 + r_l)^h$, and that no arbitrage is possible. We will prove by contradiction that $r_l < r_b$, $u > (1 + r_l)^h$, and $d < (1 + r_b)^h$.

Consider the portfolio $Y = (l, b, \Delta) = (S_0, 0, -1)$ Again, we construct a system for the possible values of the portfolio at time h:

$$\begin{array}{rcccl}
Y_h(H) & = & -uS_0 & + & S_0(1 + r_l)^h & = S_0((1 + r_l)^h - u) \\
Y_h(T) & = & -dS_0 & + & S_0(1 + r_l)^h & = S_0((1 + r_l)^h - d)
\end{array}$$

However, since $u \leq (1 + r_l)^h$:

$$\begin{array}{rclcl}
Y_h(H) & = & S_0((1 + r_l)^h - u) & \geq & 0 \\
Y_h(T) & = & S_0((1 + r_l)^h - d) & > & 0
\end{array}$$

Thus, we derive a contradiction.

Now suppose that $d \geq (1 + r_b)^h$.

Consider the portfolio $Z = (l, b, \Delta) = (0, S_0, 1)$. At time h:

$$\begin{array}{rcccl}
Z_h(H) & = & uS_0 & - & S_0(1 + r_b)^h \\
Z_h(T) & = & dS_0 & - & S_0(1 + r_b)^h
\end{array}$$

Since $u > d$ and $d \geq (1 + r_b)^h$, this is equivalent to:

$$\begin{array}{rclcl}
Z_h(H) & = & S_0(u - (1 + r_b)^h) & > & 0 \\
Z_h(T) & = & S_0(d - (1 + r_b)^h) & \geq & 0
\end{array}$$

Z is an arbitrage strategy, which is a contradiction.

Finally, suppose that $r_l > r_b$. Now, we consider the portfolio $W = (l, b, \Delta) = (1, 1, 0)$. The values of this portfolio at time $t = h$ are:

$$W_h(H) = W_h(T) = (1 + r_l)^h - (1 + r_b)^h > 0$$

Again, W is an arbitrage strategy, which contradicts our hypothesis.   QED

3.1.1 Replication with Non-Zero Funding Spread

Without loss of generality, in the following discussion we will suppose that an investor cannot borrow and lend simultaneously to the money market account, since this is a sub-optimal strategy. Because l and b (as defined above) now behave such that $l = 0$ if $b \neq 0$ and vice versa, we revert back to denoting M as the amount invested into the money-market account. We let M accrue at rate $r^* := r_b \mathbb{1}_{\{M<0\}} + r_l \mathbb{1}_{\{M \geq 0\}}$, where $\mathbb{1}$ denotes the indicator function[3].

In a market with non-equal borrowing and lending rates, we cannot price by replication in the usual way. To construct a replicating portfolio for a long position in V, denoted as X_V, one would utilize either a borrowing or lending position in M, depending on the behavior of the derivative. The replicating portfolio for the short position, X_{-V}, must take the opposite money-market account strategy to X_V. Without the presence of a risk-free rate, the prices of X_V and $-X_{-V}$ are no longer equal. Therefore, the no-arbitrage price for the derivative is not a single point, but is an interval of values.

In a one-period binomial tree model of length h, the initial value of the replicating portfolio X_V can be found as follows:

$$X_V = \Delta_V S_0 + M_V$$
$$\Delta_V = \frac{V_h(H) - V_h(T)}{S_0(u - d)}$$
$$M_V = \frac{u \cdot V_h(T) - d \cdot V_h(H)}{(u - d)(1 + r_V^*)}$$

Consider similar equations for X_{-V}:

$$X_{-V} = \Delta_{-V} S_0 + M_{-V}$$
$$\Delta_{-V} = \frac{-V_h(H) + V_h(T)}{S_0(u - d)}$$
$$M_{-V} = \frac{-u \cdot V_h(T) + d \cdot V_h(H)}{(u - d)(1 + r_{-V}^*)}$$

We can come to the following five conclusions directly. These conclusions are used to construct replicating portfolios under non-zero funding spreads.

(a) $\Delta_{-V} = \Delta_V$
(b) $u \cdot V_h(T) - d \cdot V_h(H) > 0 \Rightarrow M_V > 0$
(c) $u \cdot V_h(T) - d \cdot V_h(H) < 0 \Rightarrow M_V < 0$
(d) $r_V^* = r_b \mathbb{1}_{\{M_V<0\}} + r_l \mathbb{1}_{\{M_V \geq 0\}}$
(e) $r_{-V}^* = r_b \mathbb{1}_{\{M_V \geq 0\}} + r_l \mathbb{1}_{\{M_V < 0\}}$

3

$$\mathbb{1}_{\{M<0\}} = \begin{cases} 1, & \text{if } M < 0 \\ 0, & \text{else} \end{cases}$$

$$\mathbb{1}_{\{M \geq 0\}} = \begin{cases} 1, & \text{if } M \geq 0 \\ 0, & \text{else} \end{cases}$$

For the following discussion, we denote the price of the replicating portfolio for a long position in V as $\Phi(X_V)$. Similarly, let $\Phi(X_{-V})$ denote the price of the replicating portfolio for a short position in V. Denote $\Phi(X_V^*)$ as the minimum price of a strictly super-hedging portfolio for a long position in V. Similarly, denote $\Phi(X_{-V}^*)$ as the minimum price of a strictly super-hedging portfolio for a short position in V. All prices are set at the same time t.

Theorem 2. Let V be a European-type derivative modeled on a one-period binomial tree with non-zero funding spread. Assume the no-arbitrage conditions from Theorem 1. Then, any arbitrage-free price of the derivative at time t, V_t, satisfies:

$$\max\{-\Phi(X_{-V}^*), -\Phi(X_{-V})\} \leq V_t \leq \min\{\Phi(X_V^*), \Phi(X_V)\}$$

That is, the prices within this interval are arbitrage-free, and only those outside of the interval produce arbitrage opportunities. If $\max\{-\Phi(X_{-V}^*), -\Phi(X_{-V})\} = -\Phi(X_{-V}^*)$, then the interval is open on the left: $V_t > -\Phi(X_{-V}^*)$. If $\min\{\Phi(X_V^*), \Phi(X_V)\} = \Phi(X_V^*)$, then the interval is open on the right: $V_t < \Phi(X_V^*)$.

Proof. The proof for this theorem will be split into two parts. *First*: to show that the largest no-arbitrage price of a derivative is equal to the minimum between the price to replicate it and the price to super-replicate it. *Second*: to show that the smallest no-arbitrage price is equal to the maximum between the inverse of the price to replicate and super-replicate short positions in the derivative. *Third*: to show that the prices between the largest and smallest price are arbitrage-free.

Part 1. <u>Show $V_t \leq \min\{\Phi(X_V), \Phi(X_V^*)\}$</u>

Suppose $\min\{\Phi(X_V^*), \Phi(X_V)\} = \Phi(X_V)$, and $V_t > \Phi(X_V)$.
Consider the strategy that takes a short position in V_t and a long position in X_V. Denote this strategy as Y. The initial cash-flow for Y is:

$$Y_0 = -\Phi(X_V) + V_t > 0$$

Then, at time h, the value of portfolio Y would be:

$$Y_h(H) = V_h(H) - V_h(H) = 0$$

$$Y_h(T) = V_h(T) - V_h(T) = 0$$

Since the initial profit still stands, Y is an arbitrage strategy. Therefore, $V_t \leq \Phi(X_V)$

Suppose $\min\{\Phi(X_V^*), \Phi(X_V)\} = \Phi(X_V^*)$, and $V_t \geq \Phi(X_V^*)$.
Then one can choose to buy X_V^* and sell the derivative for V_t, for an initial cash flow of:

$$Y_0 = -\Phi(X_V^*) + V_t > 0$$

Then, at time h, the value of portfolio Y would be:

$$Y_h(H) = X_h^*(H) - V_h(H) > V_h(H) - V_h(H) = 0$$

$$Y_h(T) = X_h^*(T) - V_h(T) > V_h(T) - V_h(T) = 0$$

Since S_V uses strict super-replication, a positive profit is guaranteed at both states. (Note: if X_V^* used traditional super-replication, then X_V^* would exactly replicate V at one state, and super-replicate it at the other, so a positive profit would still be guaranteed.) So, Y is an arbitrage strategy.

Additionally, since X_V^* is strict, its price can be expressed by:

$$\Phi(X_V^*) := \inf\{\Phi(Z)|Z_h(H) > V_h(H) \text{ and } Z_h(T) > V_h(T)\}$$

where Z is a portfolio comprised of stock and money-market investments. As such, V_t cannot equal $\Phi(X_V^*)$.

Thus $V_t < \Phi(X_V^*)$

Part 2. <u>Show $V_t \geq \max\{-\Phi(X_{-V}), -\Phi(X_{-V}^*)\}$</u>

Suppose $\max\{-\Phi(S_{-V}), -\Phi(X_{-V})\} = -\Phi(X_{-V}^*)$, and $V_t < -\Phi(X_{-V}^*) \iff -V_t > \Phi(X_{-V}^*)$.

Then one can buy X_{-V}^* and sell $-V_t$ (equivalent to buying the derivative for V_t) for an initial profit of:

$$Y_0 = -V_t - \Phi(X_{-V}^*) > 0$$

Then, at time h, the value of the portfolio Y would be:

$$Y_h(H) > -V(H) + V(H) = 0$$

$$Y_h(T) > -V(T) + V(T) = 0$$

So, Y is an arbitrage strategy, which is a contradiction.Therefore, $V_t > -\Phi(X_{-V}^*)$

Now, suppose $\max\{-\Phi(X_{-V}^*), -\Phi(X_{-V})\} = -\Phi(X_{-V})$, and $V_t < -\Phi(X_{-V}) \iff -V_t > \Phi(X_{-V})$.

Then one can buy X_{-V} and buy the derivative for V_t for an initial profit of:

$$Y_0 = -V_t - \Phi(X_{-V}) > 0$$

Then, at time h, the value of the portfolio Y would be:

$$Y_h(H) = V(H) - V(H) = 0$$

$$Y_h(T) = V(T) - V(T) = 0$$

So, Y is an arbitrage strategy, which is a contradiction. Therefore, $V_t \geq -\Phi(X_{-V})$

Part 3. Show any arbitrage-free price V_t satisfies

$$\max\{-\Phi(X^*_{-V}), -\Phi(X_{-V})\} \le V_t \le \min\{\Phi(X^*_V), \Phi(X_V)\}$$

Suppose the hypothesis is true. Consider a strategy $Y_t = -aV_t + \Delta_t S_t + M_t$, which we claim to be a selling arbitrage strategy, where a is the number of shares of V_t bought. That is, the initial cash-flow at time t is

$$aV_t - \Delta_t S_t - M_t > 0,$$

for some arbitrary $\{\Delta_t, M_t\}$ such that at time T, there is positive profit, and at the other state, there is no loss. Without loss of generality, we let $Y(H) > 0$ and $Y(T) \ge 0$. (This choice is inconsequential.):

$$Y(H) = -aV(H) + \Delta_t S_t u + M_t(1 + r^*)^{(T-t)} \ge 0$$

$$Y(T) = -aV(T) + \Delta_t S_t d + M_t(1 + r^*)^{(T-t)} > 0,$$

for $r^* = r_b$ or $r^* = r_l$. From this, we conclude:

$$aV(H) \le \Delta_t S_t u + M_t(1 + r^*)^{(T-t)}$$

$$aV(T) < \Delta_t S_t d + M_t(1 + r^*)^{(T-t)}$$

However, this implies that $a\Phi(X^*_V) \le \Delta_t S_t + M_t$. From our initial cash flow, this implies,

$$\Phi(X^*_V) < V_t,$$

which is a contradiction. So, Y is not an arbitrage strategy.

Now suppose that $Y_t = aV_t - \Delta_t S_t - M_t$ is a buying arbitrage strategy. Then, the initial cash flow at time t is

$$-aV_t + \Delta_t S_t + M_t > 0,$$

for some arbitrary $\{\Delta_t, M_t\}$ such that at time T, there is positive profit, and at the other state, there is no loss. Without loss of generality, we let $Y(T) > 0$ and $Y(H) \ge 0$. (This choice is inconsequential.):

$$Y(H) = aV(H) - \Delta_t S_t u - M_t(1 + r^*)^{(T-t)} \ge 0$$

$$Y(T) = aV(T) - \Delta_t S_t d - M_t(1 + r^*)^{(T-t)} > 0,$$

for $r^* = r_b$ or $r^* = r_l$. From this, we conclude:

$$-aV(H) \le -\Delta_t S_t u - M_t(1 + r^*)^{(T-t)}$$

$$-aV(T) < -\Delta_t S_t d - M_t(1 + r^*)^{(T-t)}$$

However, this implies that $a\Phi(X^*_{-V}) \le -\Delta_t S_t - M_t$. From our initial cash flow, this implies,

$$-\Phi(X^*_{-V}) > V_t,$$

which is a contradiction. So, Y is not an arbitrage strategy.

<div align="right">QED</div>

3.1.2 Remark About Super-replication in Theorem 2

One is tempted to assume that the parameters of the binomial model would be chosen such that u and d do not lie between the returns on borrowing and lending, i.e. $d < 1 < (1+r_l)^h < (1+r_b)^h < u$. However, this is not always true. In fact, it has even been the case that depositors to government banks have been *charged* interest for short-term investments — meaning that r_l can actually be negative [2].

We begin by observing that when $u < (1+r_b)^h$, or $d > (1+r_l)^h$, it is less expensive to super-replicate a derivative using only shares of stock, than it is to super-replicate by investing in both stock and money-market. Take, for example, the case where $d > (1+r_l)^h$. Then, if one were to construct a super-hedging portfolio for V, $X_V^* = \Delta^* S_0 + M^*$, one could use any Δ^* and M^* strategies that would outperform the derivative. However, the investor would not choose to lend money to the money-market account, since the return per dollar would be less than the return per share of stock (i.e. it is more expensive to lend money than to take the equivalent long position on stock). Suppose $M^* = 1$ dollar. Then that dollar will be worth $(1+r_l)^h$ dollars at time h, whereas with $\Delta^* = \frac{1}{S_0}$ shares, that investment will be worth either u or d dollars at time h — both of which are larger than $(1+r_l)^h$. The same argument follows for the case when $u < (1+r_b)^h$; in this case, the investor is looking to minimize the return on what he has borrowed, since a higher return only increases the amount that he must repay later. Therefore, it is always optimal to super-replicate by using only investments in stock when $u < (1+r_b)^h$, or $d > (1+r_l)^h$.

We observe that it is sometimes possible to replicate $-V$ for a V that pays non-negative values, starting from a positive value. Then, when $\Phi(X_{-V})$ is negated to find a suitable price for V, the outcome is a negative value. This creates an arbitrage opportunity, since the investor has been paid to construct the portfolio, and is again paid a non-negative amount from the payoff of the derivative. However, when one constructs a super-hedging portfolio for $-V$ using only shares in stock, the price is negative — as expected — and $-\Phi(X_{-V}^*)$ is positive, and arbitrage-free. See the following example for details.

Example 2

Consider a derivative modeled on a one-period binomial tree which distributes payoffs of 13.4 in the down state and 11.8 in the up state at time 1, with the following specifications: $S_0 = 100$, $u = 1.015$, $d = 1.014$, $r_b = 3\%$, $r_l = 1\%$.
We begin with finding the replicating portfolio for $-V$.

$$\Delta_{-V} = \frac{-V(H) + V(T)}{S_0(u-d)} = \frac{-11.8 + 13.4}{100(1.015 - 1.014)} = 16$$

$$M_{-V} = \frac{-uV(T) + dV(H)}{(u-d)(1 + r_{-V}^*)} = \frac{-1.015 \cdot 13.4 + 1.014 \cdot 11.8}{(1.015 - 1.014) \cdot (1 + .03)} \approx -1588.16$$

$$X_{-V} = \Delta_{-V} S_0 + M_{-V} \approx 11.85$$

So, $-\Phi(X_{-V}) = -11.85$

Next, we find the price of the super-hedging portfolio for $-V$.

$$X^*(H) = \Delta^* S_0 u + 0 > -11.8$$

$$X^*(T) = \Delta^* S_0 d + 0 > -13.4$$

$$\Rightarrow \Delta^* = \max\{\frac{-11.8}{1.015 \cdot 100}, \frac{-13.4}{1.014 \cdot 100}\} \approx -.116$$

$$\Phi(X^*_{-V}) \approx -.116 * 100 = -11.6$$

So, $-\Phi(X^*_{-V}) = 11.6$

Finally, $\max\{-\Phi(X^*_{-V}), -\Phi(X_{-V})\} = \boxed{11.6}$

3.2 Model 2: Counterparty Credit Risk

Recall that without any additional information, the value of a derivative is dependent only on the value of the underlying asset. However, it is prudent to consider an adjustment to the model based on the possibility that the counterparty may default, which would result in partial (or zero) payment. This is referred to as *unilateral Credit Valuation Adjustment (CVA)* from the point of view of the investor. Of course, in a more realistic scenario, either party may default, but we presume there is only one credit-risky party. For this model, we restore the assumption that there exists some risk-free interest rate, r.

To hedge against this new risk, we must introduce a third asset into the replication portfolio: a corporate bond issued by the counterparty. This bond earns some rate, r_c, which differs from the risk-free interest rate. Since the bond is issued by the counterparty,:

  (i) each share of the bond is worth zero in the case of counterparty default, OR

  (ii) each share of the bond is worth $(1 + r_c)^t$ at time t if the counterparty does not default.

The introduction of this third asset effectively hedges against the counterparty default risk. From the point of view of the investor, the bond may be used to mitigate the negative impact of defaulting on the contract.

Next, we will discuss our attempts at modeling the derivative and replication portfolio under these new conditions.

3.2.1 Unilateral CVA: The $1 + \epsilon$ Model

For this model, we treat counterparty default and the determination of the derivative payoff as two non-simultaneous events. Instead, we assume that default can occur shortly after the moment that the underlying asset is valuated. This means that for a derivative modeled with a one-period binomial tree of length h, the information that is known at time h includes:

  • the price of the underlying asset, S_h, and money-market account
  • the full exercise value of the derivative, V_h

whereas the information available at time $h + \epsilon$, for small $\epsilon > 0$ includes:

  • whether or not the counterparty has defaulted
  • the realized exercise value of the derivative (some portion, $\alpha \in [0, 1]$ of the full exercise value)

For simplicity, we discuss a model with length $h = 1$; i.e., the $1 + \epsilon$ Model. We also assume the recovery rate, α, is zero.

We do not allow the investor to trade in the counterparty bond during the first period. Instead, the initial portfolio includes investments in only the stock and money-market account, and $X_0 = \Delta_0 S_0 + M_0$.

At time 1, the investor knows the value of the underlying asset. Based on that knowledge, he allocates funds away from the money market account or stock and into the bond. We denote the amount invested in the bond at the up and down state as B_1^H and B_1^T, respectively. Thus, the value of the portfolio (in the up-state and down-state) before and after rebalancing can be expressed by the following:

$$X_1(H) = \Delta_0 S_0 u + M_0(1 + r) \overset{rebalance}{=} \Delta_1^H S_0 u + M_1^H + B_1^H \tag{5}$$

$$X_1(T) = \Delta_0 S_0 d + M_0(1 + r) \overset{rebalance}{=} \Delta_1^T S_0 d + M_1^T + B_1^T \tag{6}$$

One is not concerned with exact replication at time 1. Instead, exact replication occurs only at time $1 + \epsilon$ at the four terminal nodes. During the second period, from $t = 1$ to $t = 1 + \epsilon$, the stock and money-market account values remain constant. Together, they essentially act as a cash account.

We now consider replication at the four terminal nodes. Let $_D X_{1+\epsilon}(H)$, $_D X_{1+\epsilon}(T)$ represent the value of the portfolio in the case of the counterparty's default, given that the stock moves up and down, respectively. Then we can express the value of the portfolio at the four terminal nodes as follows:

$$X_{1+\epsilon}(H) = \Delta_1^H S_0 u + M_1^H + B_1^H(1 + r_c)^\epsilon = V_1(H) \tag{7}$$

$$_D X_{1+\epsilon}(H) = \Delta_1^H S_0 u + M_1^H = \alpha V_1(H) = 0 \tag{8}$$

$$X_{1+\epsilon}(T) = \Delta_1^T S_0 d + M_1^T + B_1^T(1 + r_c)^\epsilon = V_1(T) \tag{9}$$

$$_D X_{1+\epsilon}(T) = \Delta_1^T S_0 d + M_1^T = \alpha V_1(T) = 0 \tag{10}$$

See Figure 3 of a $(1 + \epsilon)$-step tree for illustration.

We begin by treating equations (7),(8) as a system, and equations (9),(10) as a separate system. This allows us to solve for B_1^H and B_1^T.

$$B_1^H = \frac{V_1(H)}{(1+r_c)^\epsilon}, B_1^T = \frac{V_1(T)}{(1+r_c)^\epsilon}$$

Additionally, we can set limitations for the choices of $\Delta_1^H, M_1^H, \Delta_1^T, M_1^T$. From equations (7) - (10), we can conclude:

$$\Delta_1^H S_0 u + M_1^H = 0 = \Delta_1^T S_0 d + M_1^T$$

Note that Δ and M sum to zero and therefore can be disregarded. The aim is to find values for Δ_0 and M_0 in order to find the price of the replicating portfolio. Finally, we use (5),(6) to arrive at:

$$\Rightarrow M_0 = \frac{1}{(1+r_c)^\epsilon} \cdot \frac{uV_1(T)-dV_1(H)}{(u-d)(1+r)}, \Delta_0 = \frac{1}{(1+r_c)^\epsilon} \cdot \frac{V_1(H)-V_1(T)}{S_0(u-d)},$$

and the price of V at $t = 0$ is $X_0 = \Delta_0 S_0 + M_0$.

Figure 3: Illustration of the $1 + \epsilon$ Model. Note that for derivatives with $T > 1$, the tree would extend only from the terminal nodes in which no default occurred. In this diagram, the branches in green would be extended, and those in red would terminate.
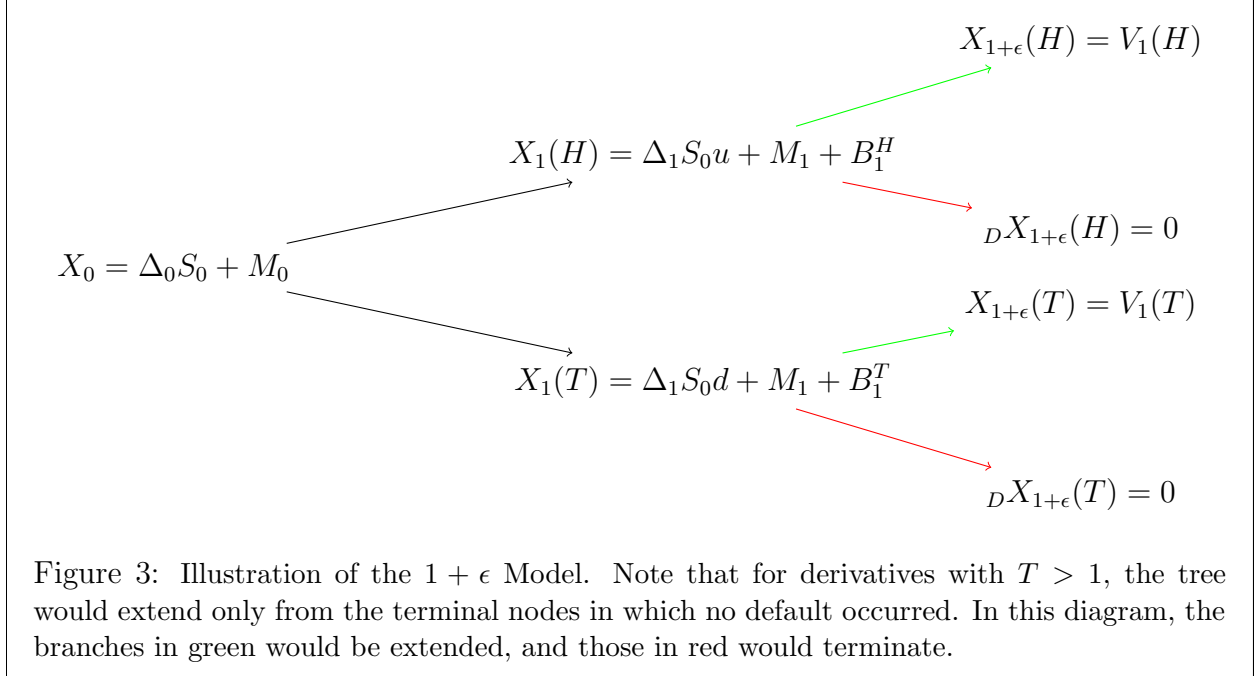
3.2.2 Extension to Multi-Period Model

Assumptions

We maintain the same assumptions regarding the risk-free interest rate r and the risk-neutral probability of a stock price increase p. We extend the $1 + \epsilon$ model into a multi-period model, and generalize it to allow for longer "ϵ" periods. We also relax the assumption of a recovery rate, α, of zero.

We suppose there are $2N$ total periods, which consist of N "trade periods" in which both stock price and money-market accounts can be traded, and N "default periods" in which the counterparty may default. The counterparty cannot default during a trade period, and the stock and money market accounts remain constant during a default period.

- The length of each trade period is denoted by h.
- The length of each default period is denoted by g.
- For an option with maturity date T, $g + h = \frac{T}{N}$. Note that g does not necessarily have to be "small" as compared to h. This differs from the "$1+\epsilon$" model, which assumed $\epsilon > 0$ to be small.
- We express time as the sum of "a" trading periods and "b" default periods, i.e.:
  $t = ah + bg$, for $a, b \in \{0, 1, ...N\}$, $a = b$, or $a = b + 1$.
- Each trade period must directly precede a default period, and each default period must directly follow a trade period.

Now, we consider the new risk-neutral measure. Let the filtration containing market information (stock and money-market account values) be denoted as $\mathbb{H} := \{\mathcal{H}_t\}$, the filtration containing default information be denoted as $\mathbb{G} := \{\mathcal{G}_t\}$, with $\mathbb{H}$ and $\mathbb{G}$ independent. Note that since stock and cash is constant in the default period, $S_t \perp \mathbb{G}$, and $M_t \perp \mathbb{G}$, $0 \le t \le T$.

Define $\mathcal{F}_t = \mathcal{H}_t \bigvee \mathcal{G}_t$, which contains information on both stock movements and defaults which have occurred up to time t.

Now, we define the risk-neutral probability of the counterparty's default, denoted as q. This value can be thought of as a default intensity, for which the probability of default is depends on the length of time. Suppose that the risky bond issued by the counterparty earns rate r_c, and is modeled by a one-period tree of length g. At time g, each dollar invested in the bond will be worth either $(1+r_c)^g$ with probability $1-q$, or 0 with probability q. We assume that during this period, the bond earns the risk-free rate. However, recall that the money-market account remains constant during the default period. Therefore, the risk-free rate we use is zero. Thus, in this period under the risk-neutral measure:

$$1(1+0)^g = (1+r_c)^g \cdot (1-q) + 0 \cdot q \iff q = 1 - (1+r_c)^{-g}$$

From this calculation, we can surmise that instances of default, and q itself, are independent of the market information in the filtration $\mathcal{H}_t$. Thus, to verify that the value of the derivative is a martingale under $\mathbb{F}$, it suffices to show that the discounted stock price is a martingale. Since $S_t \perp \mathbb{G}$, we know that

$$E[S_T D_{t,T}|\mathcal{F}_t] = E[S_T D_{t,T}|\mathcal{G}_t \vee \mathcal{H}_t] = E[S_T D_{t,T}|\mathcal{H}_t] = S_t$$

So, discounted future stock price is a martingale under the probability measure space $(\Omega, \mathcal{F}, P)$ — the risk-neutral measure. Therefore, the price of the derivative can be found using the discounted expected payoffs under $\mathbb{F}$. Generalizing, we say that at time, $t = ah + bg$, the price of the derivative, C_t, is:

$$C_t = E[V_T D_{t,T}|\mathcal{F}_t]$$

$$= (D_{0,h})^{N-a} \cdot [\alpha q + (1-q)]^{N-b} \cdot \sum_{i=0}^{N-a} \binom{N-a}{i} p^i (1-p)^{N-i} V(T, S_t u^i d^{N-a-i})$$

Note that $N - a$ and $N - b$ represent the number of trading periods and default periods left before expiration. In the equation above, the discounting factors during default periods are ignored, because the risk-free rate during these periods is zero. The second term, $[\alpha q + (1-q)]^{N-b}$, is arrived at by calculation of the expected value of the derivative taken prior to every default period. The summation is the typical form for the expected value of a random variable that is distributed binomially $(V_T \sim B(N-a, p))$. [4]

3.2.3   Replicating Portfolio

Taking the four-period model depicted in Figure 4 as an example, we can replicate the option using three assets, as before. The replicating portfolio reflects movements from two trading periods, as well as two default periods. We begin finding portfolio values (Δ, M, B), by looking at the two-equation systems at the terminal nodes:

$$X_{2h+2g}^{HH} = \Delta_{2h+g}^{HH} S_0 u^2 + M_{2h+g}^{HH} + B_{2h+g}^{HH}(1+r_c)^g = V(HH) \tag{11}$$

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[4]By convention, $\binom{0}{0} = 1$, and $\binom{1}{0} = 0$.
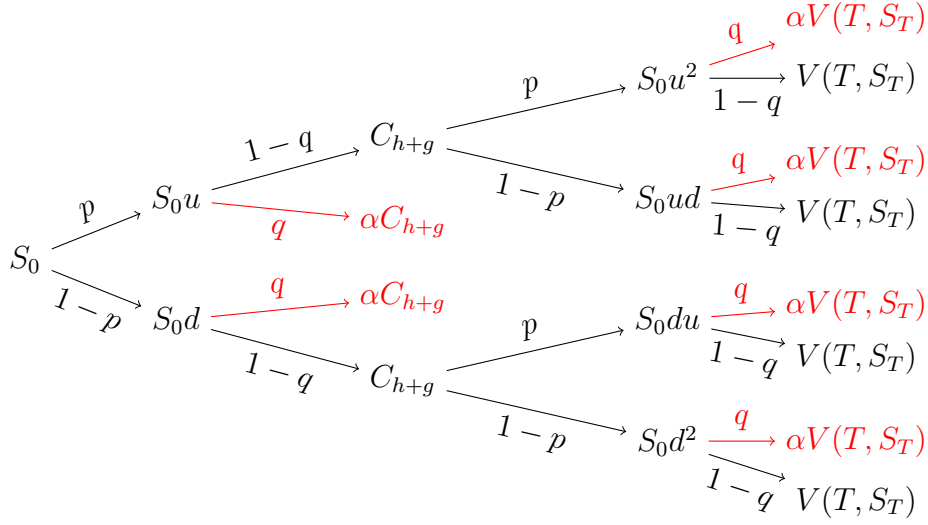
Figure 4: Illustration of stock movements and payoffs to the investor in a 4-period model. Values in red occur as the result of counterparty default.

$$_D X^{HH}_{2h+2g} = \Delta^{HH}_{2h+g} S_0 u^2 + M^{HH}_{2h+g} + 0 = \alpha V(HH)$$

$$X^{HT}_{2h+2g} = \Delta^{HT}_{2h+g} S_0 ud + M^{HT}_{2h+g} + B^{HT}_{2h+g}(1+r_c)^g = V(HT) \tag{12}$$

$$_D X^{HT}_{2h+2g} = \Delta^{HT}_{2h+g} S_0 ud + M^{HT}_{2h+g} + 0 = \alpha V(HT)$$

$$X^{TH}_{2h+2g} = \Delta^{TH}_{2h+g} S_0 du + M^{TH}_{2h+g} + B^{TH}_{2h+g}(1+r_c)^g = V(TH) \tag{13}$$

$$_D X^{TH}_{2h+2g} = \Delta^{TH}_{2h+g} S_0 du + M^{TH}_{2h+g} + 0 = \alpha V(TH)$$

$$X^{TT}_{2h+2g} = \Delta^{TT}_{2h+g} S_0 d^2 + M^{TT}_{2h+g} + B^{TT}_{2h+g}(1+r_c)^g = V(TT) \tag{14}$$

$$_D X^{TT}_{2h+2g} = \Delta^{TT}_{2h+g} S_0 d^2 + M^{TT}_{2h+g} + 0 = \alpha V(TT)$$

Notice that these portfolios are not rebalanced before termination.
From these systems, we find B_{2h+g} values. For events, $\omega_i \in \{H, T\}$:

$$B^{\omega_1 \omega_2}_{2h+g} = \frac{1-\alpha}{(1+r_c)^g} V(\omega_1 \omega_2) = \frac{1-\alpha}{(1+r_c)^g} C_{2h+2g}$$

Continuing, we find that any portfolio chosen such that

$$\Delta^{\omega_1 \omega_2}_{2h+g} S(\omega_1 \omega_2) + M^{\omega_1 \omega_2}_{2h+g} = \alpha C_{2h+2g}$$

is a replication portfolio for the terminal nodes.

Therefore, we move backward by one default period to find the values of Δ_{h+g}, M_{h+g}, and B_{h+g}. To do so, we set the value of X_{2h+g} to the expected value of future payoffs, discounted

by one default period (since this is equivalent to the price of the derivative, C_{2h+g}). Note that before rebalancing, the portfolio uses Δ, M, and B from time $t = h + g$.

$$X_{2h+g}^{HH} = \Delta_{h+g}^H S_0 u^2 + M_{h+g}^H (1+r)^h + B_{h+g}^H \overset{rebalance}{=} \Delta_{2h+g}^{HH} S_0 u^2 + M_{2h+g}^{HH} + B_{2h+g}^{HH} = E^Q[V(HH)]$$

$$X_{2h+g}^{HT} = \Delta_{h+g}^H S_0 ud + M_{h+g}^H (1+r)^h + B_{h+g}^H \overset{rebalance}{=} \Delta_{2h+g}^{HT} S_0 ud + M_{2h+g}^{HT} + B_{2h+g}^{HT} = E^Q[V(HT)]$$

We evaluate the expectations and simplify to the system below, (and do the same for portfolio values under events $\{TH\}$ and $\{TT\}$).

$$X_{2h+g}^{HH} = \Delta_{h+g}^H S_0 u^2 + M_{h+g}^H (1+r)^h + B_{h+g}^H = (\alpha q + (1-q))V(HH)$$
$$X_{2h+g}^{HT} = \Delta_{h+g}^H S_0 ud + M_{h+g}^H (1+r)^h + B_{h+g}^H = (\alpha q + (1-q))V(HT)$$

Recall that $B_{h+g}^H = 0$. Thus, we arrive at generalized formulas for $\Delta_{ah+bg}^{\omega_1 \omega_2 ... \omega_a}$, $M_{ah+bg}^{\omega_1 \omega_2 ... \omega_a}$, and $B_{ah+bg}^{\omega_1 \omega_2 ... \omega_a}$, as follows:

$$\Delta_{ah+bg}^{\omega_1 \omega_2 ... \omega_a} S_{ah+bg} + M_{ah+bg}^{\omega_1 \omega_2 ... \omega_a} = \begin{cases} \alpha C_{ah+(b+1)g}, & \text{if } a \neq b \\ C_{(a+1)h+bg}, & \text{if } a = b \end{cases}$$

$$B_{ah+bg}^{\omega_1 \omega_2 ... \omega_a} = \begin{cases} \frac{1-\alpha}{(1+r_c)^g} C_{ah+(b+1)g}, & \text{if } a \neq b \\ 0, & \text{if } a = b \end{cases}$$

Above, we use S_t as shorthand for $S(\omega_1 \omega_2 ... \omega_a)$. Note that $a \neq b \iff$ 'at the beginning of default periods', and $a = b \iff$ 'during trade periods'.

Let us consider the price of a derivative modeled on a 2N-period model, at time $t = Nh + (N-1)g$ — just before the final default period. In the previous model, we calculated the price of the derivative using expectations discounted by the risk-free rate. Denote this value as C_t. Recall,

$$C_t = (\alpha q + (1-q)) \cdot V(\omega_1 \omega_2 ... \omega_a) \tag{15}$$

Now, we take the value of the replicating portfolio at t, denoted by $X_t^{\omega_1 \omega_2 ... \omega_a}$. Based on the generalized formulas in the previous discussion, we find:

$$X_t^{\omega_1 \omega_2 ... \omega_a} = \Delta_t^{\omega_1 \omega_2 ... \omega_a} S_t + M_t^{\omega_1 \omega_2 ... \omega_a} + B_t^{\omega_1 \omega_2 ... \omega_a}$$
$$= \left[\frac{1-\alpha}{(1+r_c)^g} + \alpha \right] \cdot V(\omega_1 \omega_2 ... \omega_a) \tag{16}$$

Compare (15) to (16), and find that:

$$\alpha q + (1-q) = \frac{1-\alpha}{(1+r_c)^g} + \alpha$$

We will denote this constant as λ. We will use this constant to eliminate the dependence on the risk-neutral measure, and continue our discussion with differential borrowing and lending rates.

3.3 Model 3: Funding Spread and Unilateral CVA

In our third model, we combine Model 1 and Model 2, in order to incorporate both funding spread and unilateral CVA. This is relatively simple, since we have decided to divide our previous model into trade periods and default periods.

For the price of the derivative prior to default periods, we can calculate the price as a single point, since the differential rates have no effect during these periods. The price at time $t = nh + (n-1)g$ is simply,

$$C_{nh+(n-1)g}^{\omega_1,\ldots,\omega_n} = \lambda \cdot C_{nh+ng}^{\omega_1,\ldots,\omega_n}$$

However, when moving backward over a trade period, the borrowing and lending rates do affect the price of the derivative. The counterparty default risk, on the other hand, has no effect. Therefore, Theorem 2 can be applied directly. For the price of a derivative at time $t = nh + ng$, we denote the lower bound with $L_{nh+ng}^{\omega_n}$, and the upper bound with $U_{nh+ng}^{\omega_n}$. Note that both L and U are dependent on the two possible values of $C_{(n+1)h+ng}^{\omega_n}$. This does not matter for calculations just prior to the last trading period, since the price of the derivative at the terminal nodes are the realized exercised values. However, this does become important for further recursive calculations.

$$L_{nh+ng}^{\omega_n} = \max\{-\Phi(X^*_{-C_{(n+1)h+ng}^{\omega_n}}), -\Phi(X_{-C_{(n+1)h+ng}^{\omega_n}})\}$$

$$U_{nh+ng}^{\omega_n} = \min\{\Phi(X^*_{C_{(n+1)h+ng}^{\omega_n}}), \Phi(X_{C_{(n+1)h+ng}^{\omega_n}})\}$$

$$C_{nh+ng}^{\omega_n} \in [L_{nh+ng}^{\omega_n}, U_{nh+ng}^{\omega_n}]$$

See Figure 5 for illustration.

Once two $C_{nh+ng}^{\omega_n}$ have been chosen, (one from the up state, and one from the down state), the process of multiplying by lambda (to calculate the price prior to the previous default period), and finding the arbitrage-free interval (over the previous trading period) is repeated. For each pair of C_t^H, C_t^T, there is a single arbitrage-free interval $[L_0, U_0]$.

However, the purpose of the model is to find an arbitrage-free price range for the derivative itself — not specific to prices chosen at a future date. The arbitrage-free interval of prices for the derivative as a whole can be captured by taking the lower bound to be the minimum possible value of L_0, and the upper bound to be the maximum possible value of U_0. This interval may be open, closed, or half-open dependent on the parameters passed into the model. The exact mapping between the intervals for the two prices at time $nh + ng$ and the boundary points of the interval two periods back is not yet known.

4 Numerical Implementation

The final model is quite simple to implement. In order to verify that the model produces reasonable price ranges, we developed a spreadsheet to run 4-period examples. What follows is an outline of some results from this spreadsheet.

Specifications
The derivatives being considered are puts and calls.
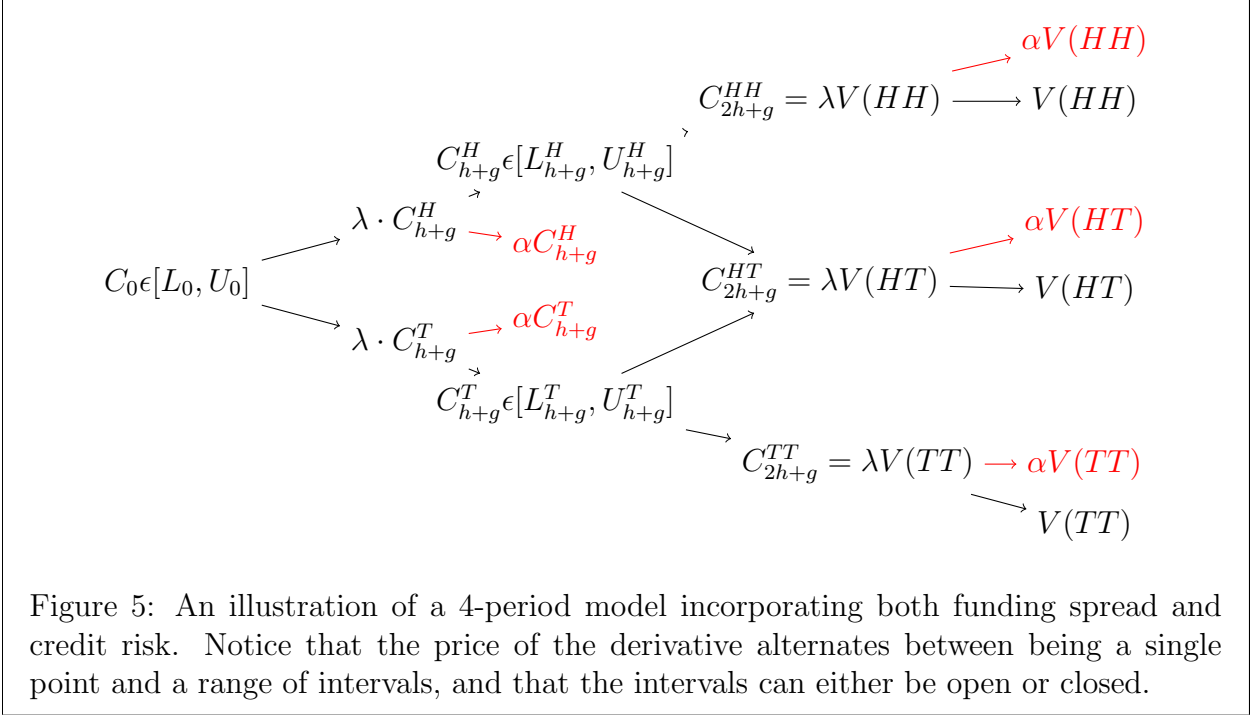
Figure 5: An illustration of a 4-period model incorporating both funding spread and credit risk. Notice that the price of the derivative alternates between being a single point and a range of intervals, and that the intervals can either be open or closed.

- For simplicity of calculation, we let each trade period and each default period be of length $h = g = 1$.
- We first calculate the ranges $[L_2^H, U_2^H]$, $[L_2^T, U_2^T]$.
- We then choose 100 stratified C_2^H's and C_2^T's from each range to simulate model outcomes under different combinations of $\{C_2^T, C_2^H\}$.
- We store the resulting lower price bounds, L_0, in a 100x100 matrix, and do the same for the upper price bounds, U_0.
- To present the results from the model, we represent these matrices as planes in three-dimensional space, where:
    - The X-axis and Y-axis correspond to values of C_2^T and C_2^H, respectively.
    - The Z-axis corresponds to both the lower and upper bounds, with the lower bounds in the lower plane, and the upper bounds in the upper plane.
- The baseline parameters for the model are the following, and should be assumed unless otherwise stated:
    - $\alpha = 0.30$, $r_c = 0.05$, $r_b = 0.03$, $r_l = 0.01$, $S_0 = 100$, $K = 90$

The model was first implemented without using super-hedging. Meaning, the upper and lower bounds were chosen such that:

$$V_t \in [-\Phi(X_{-V}), \Phi(X_V)]$$

When the model incorporates super-hedging, the arbitrage-free price intervals typically become more narrow, and do not include negative values. The difference in results between implementation of only replication and incorporation of super-hedging is particularly noticeable when the values of u or d are set to lie between the rates of gain from lending and borrowing. In some situations, the price to super-hedge the long position in V was much

less expensive than the price to exactly hedge it; so, the model excluded these high positive values as upper bounds as well. See the figures below for comparisons between these two methods.

First is a comparison of the outputs for a call option which is modeled on a tree with $u = 1.025$ and $d = 1.015$ (Figure 6). In this case, both factors are between the borrowing and lending rate. In turn, this causes a peak to form in the lower bounds and a valley to form in the upper bounds. In Figure 6, both the valley in the upper plane and the peak in the lower plane run nearly parallel to the horizon line from the viewer's perspective.
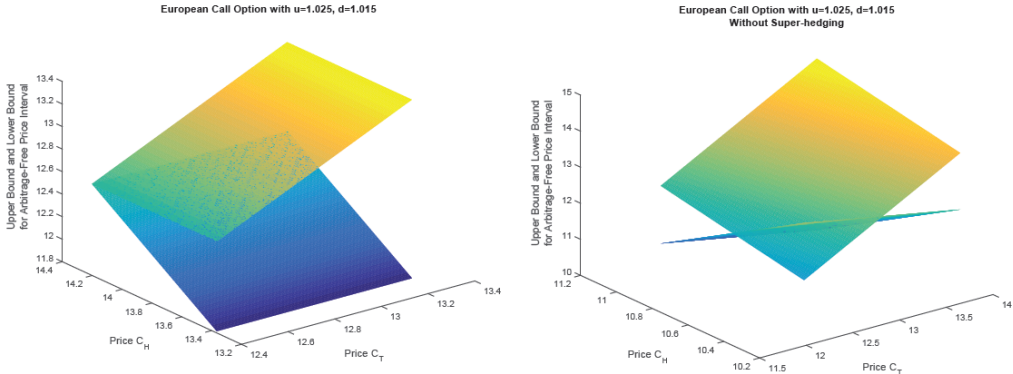
Figure 6: Rendering of model outputs with and without super-replication for a European Call Option.

Additionally, by incorporating super-hedging into the model, the ranges of arbitrage-free prices can sometimes narrow or widen. Using the same example as pictured in Figure 6, we can illustrate these changes in Figure 7.
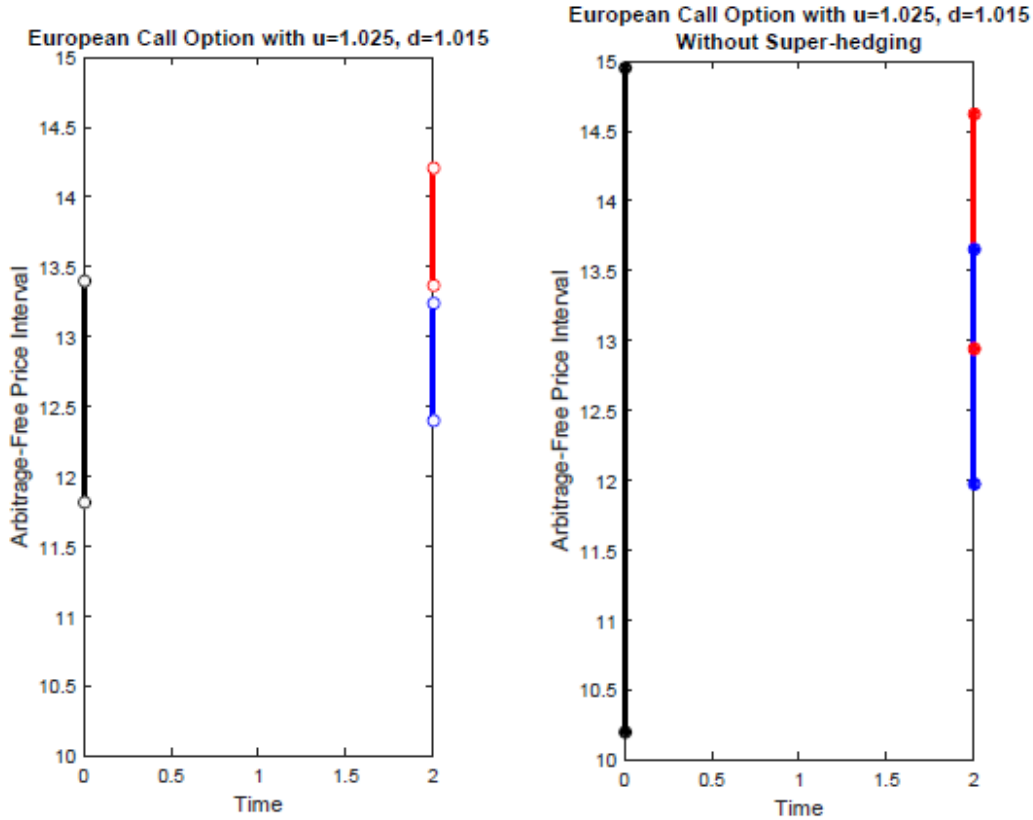
Figure 7: Rendering of price intervals with and without super-replication for a European Call Option.

In these two graphs, the arbitrage-free intervals are pictured as vertical lines at times 0 and 2. (Nothing is pictured at time 1, since any single point within the intervals at time 2 are arbitrage-free.) An open circle indicates that the interval is open at that end because the boundary point was produced by super-replication. The blue line represents the interval for C_2^H and the red line represents the interval for C_2^T. Notice that for this option, the price intervals no longer overlap at time 2, and narrow considerably at both time 0 and time 2 after implementing super-replication.

We notice a different problem with the put option. Figure 8 depicts a put option modeled with $u = 1.02$ and $d = 0.7$. Without using super-replication, the initial price bounds can be negative. By using super-replication, though, the lower bounds are near zero, but strictly positive. The interval for the initial price for the derivative is open on the left.
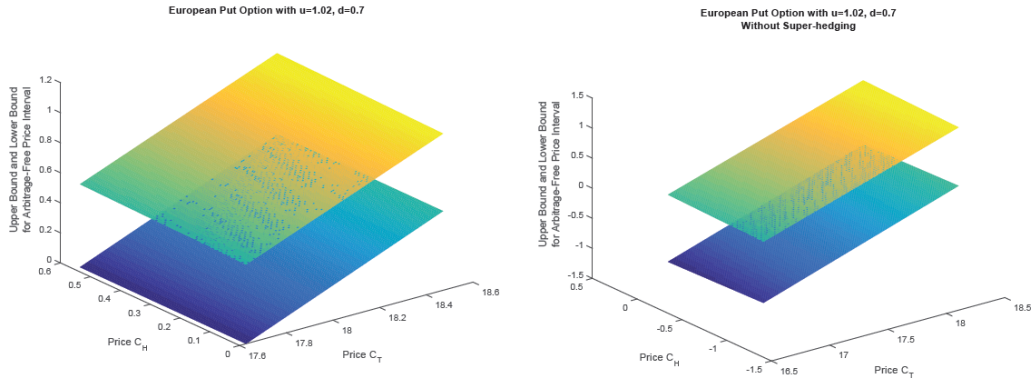
Figure 8: Rendering of model outputs with and without super-replication for a European Put option.

Again, note how the arbitrage-free price intervals narrow and do not produce negative results when super-hedging is used for the put, in Figure 9.
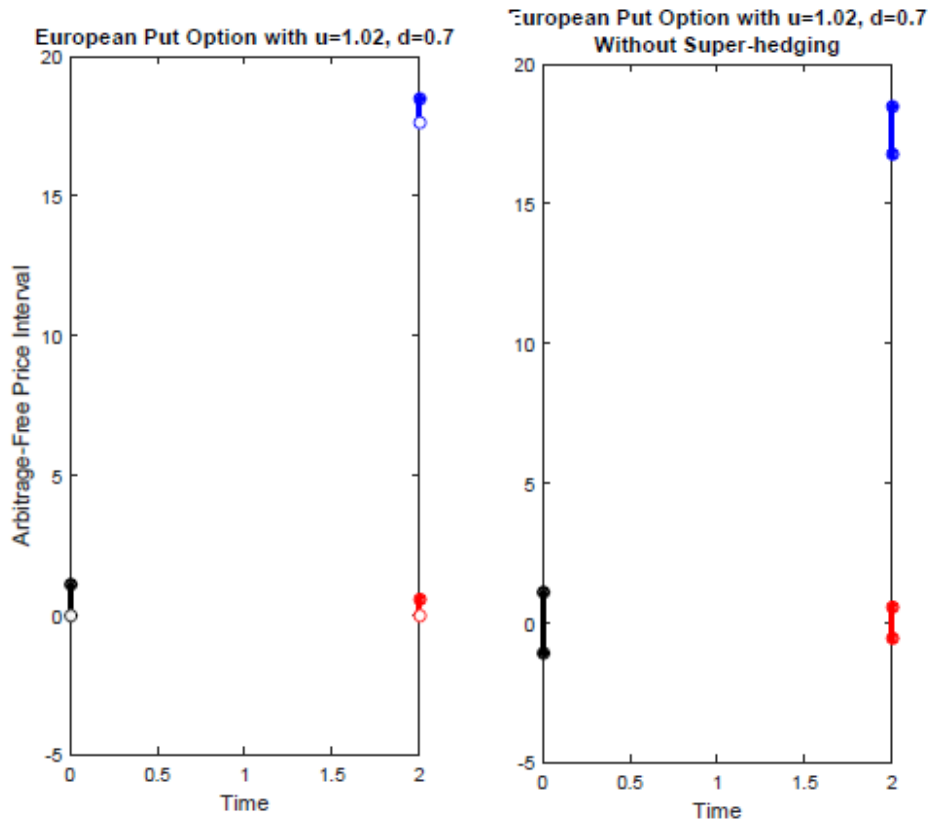
Figure 9: Rendering of price intervals with and without super-replication for a European Put option.

28

5 Recommendations for Further Research

As financial market risks change, the necessity for a comprehensive XVA framework for derivative valuation increases. Though strides are being made in the continuous time environment, a framework for discrete time is lacking. The research we have done does not suffice for a comprehensive framework, as it does not account for a multitude of risks that are typically considered, and makes many simplifying assumptions.

One particular area this model could improve is incorporating bilateral CVA and collateralization. Some XVA frameworks consider valuation adjustments based on the institution's or investor's possibility of default (DVA) counter-intuitive and exclude it [3]. However, since derivatives are typically valued by a third party, and are inherently bilateral in nature, such an adjustment is suitable. Similarly, collateral is often posted in two-party agreements in the form of cash. This collateral, like other cash assets, can be invested and earns a particular rate. Collateral can be introduced as another asset in the replicating portfolio.

Additionally, since our research always takes the point of view of the investor, it remains to be seen how funding spread is affected by a third-party valuation. Particularly, the counterparty will be charged different borrowing and lending rates based on their credit, which results in differing prices between the two parties. These differences must be reconciled and formalized mathematically.

The Binomial Asset Pricing Model can be related to the Black-Scholes framework by allowing the number of periods between $t = 0$ and $t = T$ to approach infinity. The binomial model approaches a lognormal distribution for a stock under appropriate scaling. Our model differs, however, since each trading period is followed by a default period. The length of each period in relation to each other will likely have some effect when the number of periods approaches infinity, since default intensity and the constant λ both depend on the length of the default period. The scaling of this model in such a way is an important area of research.

Of course, this model can be improved upon by incorporating more risk valuation adjustments, and eliminating certain simplifying assumptions, such as zero trading cost and the ability to trade any number of stock. However, this model does provide a foundation for research on the XVA framework with discrete time setting.

6 Appendix: Spreadsheet used for Numerical Implementation

Figure 10: This figure contains the inputs, outputs, final no-arbitrage price interval, and the no-arbitrage price intervals for C_2^T and C_2^H.

Figure 11: This figure displays a small portion a big, colored rectangle matrix of lower bound values with low prices colored green, high prices colored red, and prices in between colored yellow to orange.

Figure 12: This figure displays a small portion a larger matrix containing upper bound values with low prices colored green, high prices colored red, and prices in between colored yellow to orange.

References

[1] Bichuch, M., Capponi, A. and Sturm, S., 2015. *Arbitrage-Free Pricing of XVA–Part I: Framework and Explicit Examples.* `arXiv:1501.05893`.

[2] Kennedy, S., Randow, J., 2016. *Negative Interest Rates.*
`http://www.bloombergview.com/quicktake/negative-interest-rates`

[3] Ernst & Young. 2014. *Credit Valuation Adjustments for Derivative Contracts.*
`http://www.ey.com/Publication/vwLUAssets/EY-credit-valuation-`
`adjustments-for-derivative-contracts/$FILE/EY-Applying-FV-April-2014.pdf`

[4] Gregory, Jon. 2012. *Counterparty Credit Risk and Credit Value Adjustment: A Continuing Challenge for Global Financial Markets.* Hoboken, NJ: Wiley, p.16-18.

[5] Cox, J., Ross, S., Rubinstein, M., 1979, *Option Pricing: a Simplified Approach,*
`http://fisher.osu.edu/ fellingham.1/seminar/CRR79.pdf`