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ON SOME ERGODIC IMPULSE CONTROL PROBLEMS WITH CONSTRAINT*

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Abstract. This paper studies the impulse control of a general Markov process under the average (or ergodic) cost when the impulse instants are restricted to be the arrival times of an exogenous process, and this restriction is referred to as a constraint. A detailed setting is described, a characterization of the optimal cost is obtained as a solution of an HJB equation, and an optimal impulse control is identified.

Key words. Markov–Feller processes, information constraints, impulse control, control by interventions, ergodic control

AMS subject classifications. Primary, 49J40; Secondary, 60J60, 60J75

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1. Introduction.

Since the introduction of impulse control problems by Bensoussan and Lions in the seventies, many studies have been devoted to various aspects, both theoretical and applied, of this subject (see the references in Bensoussan and Lions [6], Bensoussan [2], Davis [10]). Several of these studies cover the case of the long term average cost or ergodic control (e.g., see Gatarek and Stettner [14] and the references therein). In these works, the impulse control is a sequence of stopping times and random variables acting on the state x_t of the system, and the stopping times can be arbitrary.

In the present paper, we address the ergodic impulse control when the stopping times must satisfy a constraint, namely, the impulses are allowed only at the jump times of a given process y_t, these times representing the arrival of a signal. This type of constraint has been treated in [30] for the case of a discounted cost, using results on the optimal stopping problem with constraints of [29] (which generalizes the results of Dupuis and Wang [11]). To the best of our knowledge, there are only a few references related to impulse control with constraint: Brémaud [7], Liang [24], Liang and Wei [25], and Wang [39]. A different kind of constraint is considered in Costa, Dufour, and Piunovskiy [8], where the constraints are written as infinite-horizon expected discounted costs. Nevertheless, we are not aware of any references in the case of ergodic impulse control with constraint.

In many cases of optimal control, the ergodic control problem is treated via the asymptotic behavior of the discounted problem, often called the vanishing discount approach (e.g., see Bensoussan [3], Gatarek and Stettner [14]). We could use this method for the present problem in the situation where the set of admissible impulses Γ is independent of the state x; however, it seems more difficult to do the same with $\Gamma(x)$. Here, we use a direct approach, based on the study of the ergodic HJB equation. As for the discounted cost, an auxiliary ergodic control problem in discrete time is the basis to solve the original problem. The main results concern the solution of the ergodic HJB equation(s) and the existence of an optimal control.

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The content of the paper is as follows: section 2 includes the statement of the problem, notations, and assumptions; section 3 is devoted to the heuristic derivation of the HJB equations which are solved in section 4; section 5 shows the existence of an optimal control and the characterization of the optimal ergodic cost; section 6 discusses a few extensions.

2. Statement of the problem.

2.1. The uncontrolled process. Let $\{\Omega, \mathcal{F}, \mathcal{F}_t, (x_t, y_t), P_{xy}\}$ be a homogeneous Markov process on $\Omega = D(\mathbb{R}^+; E \times \mathbb{R}^+)$, where $\mathcal{F}_t$ is the universal completion of the canonical σ-algebras and $\mathbb{E}_{xy}$ denotes the expectation with respect to P_{xy}. The x_t component is the process to be controlled by impulses. The y_t component will define the constraint on the controls, namely, the controller may apply an impulse on x_t only at the jumps times of y_t. These jump times represent the arrival times of a signal and, actually, y_t is the elapsed time since the last arrival of a signal. It is assumed that

(2.1)
- E is a compact metric space;
- x_t is a Feller process with semigroup $\Phi(t)$ and infinitesimal generator A_x,

so that $\Phi(t)$ is a continuous semigroup on the space $C(E)$ of real-valued continuous functions on E;

(2.2)
- y_t is a process with values in $\mathbb{R}^+$, which jumps at times $\tau_1, \tau_2, \ldots, \tau_n \to \infty$, and such that $y_{\tau_n} = 0, \quad y_t = t - \tau_n$ for $\tau_n \le t < \tau_{n+1},\ n \ge 1$, and, conditionally to x_t, the intervals between jumps $T_n = \tau_{n+1} - \tau_n$ are independent identically distributed random variables with intensity $\lambda(x, y)$; see [29, 30].

It is also assumed that

(2.3)
- $\lambda(x, y)$ is a nonnegative, continuous, and bounded function defined on $E \times \mathbb{R}^+$ and there exist positive constants, $0 < a_1 < a_2$, such that $a_1 \le \mathbb{E}_{x0}\{\tau_1\} \le a_2$.

For x given, one could consider that the infinitesimal generator of y_t is

$$A_y g = \frac{\partial g}{\partial y} + \lambda(x, y)[g(0) - g(y)]$$

for smooth functions g on $\mathbb{R}^+$.

We will assume that the weak infinitesimal generator of (x_t, y_t) in $C_b(E \times \mathbb{R}^+)$ is

$$A_{xy} g(x, y) = A_x g(x, y) + \frac{\partial g}{\partial y}(x, y) + \lambda(x, y)[g(x, 0) - g(x, y)]. \tag{2.4}$$

In addition, we make the following assumption[1] for x_t: if $\mathcal{B}(E)$ denotes the Borel

[1] (2.6) is sometimes called "Doeblin's condition."

σ-algebra of E, and

$$\text{if } P(x,U)=\mathbb{E}_{x0}\mathbb{1}_U(x_{\tau_1})\ \forall U\in\mathcal{B}(E), \text{ then} \tag{2.5}$$

$$\text{there exists a positive measure } m \text{ on } E \text{ such that}$$
$$0<m(E)\le 1 \ \text{ and } \ P(x,U)\ge m(U)\ \forall U\in\mathcal{B}(E). \tag{2.6}$$

Remark 2.1. (a) We have

$$\mathbb{E}_{x0}\{\tau_1\}=\mathbb{E}_{x0}\Big\{\int_0^\infty t\lambda(x_t,t)\exp\Big(-\int_0^t\lambda(x_s,s)\mathrm{d}s\Big)dt\Big\},$$

so the condition $\mathbb{E}_{x0}\{\tau_1\}\le a_2$ is satisfied if, for instance, $\lambda(x,y)\ge k_0>0$ for $y\ge y_0$, $x\in E$, then $a_2=y_0+1/k_0$. Also if $\lambda(x,y)\le k_1<\infty$ for every $y\ge 0$, and $x\in E$, then $\mathbb{E}_{x0}\{\tau_1\}\ge a_1=1/k_1$.

(b) Note that

$$P(x,U)=\mathbb{E}_{x0}\Big\{\int_0^\infty\lambda(x_t,t)\exp\Big(-\int_0^t\lambda(x_s,s)\mathrm{d}s\Big)\mathbb{1}_U(x_t)\mathrm{d}t\Big\}. \tag{2.7}$$

With (2.7) and assuming the same property λ as in (a) above, one can check that (2.6) is satisfied when the transition probability of x_t has a density with respect to a probability on E satisfying for every $\varepsilon>0$ there exists $k(\varepsilon)$ such that

$$p(x,t,x')\ge k(\varepsilon)>0 \text{ on } \ E\times[\varepsilon,\infty[\times E. \tag{2.8}$$

This is the case, for instance, for periodic diffusion processes (see Bensoussan [3]), and for reflected diffusion processes with jumps; see Garroni and Menaldi [12, 13] (which is also valid for reflected diffusion processes without jumps). ∎

2.2. Assumptions on costs and impulse values. It is assumed that there are a running cost $f(x,y)$ and a cost of impulse $c(x,\xi)$ satisfying

$$\begin{aligned}&f:E\times\mathbb{R}^+\to\mathbb{R}^+ \ \text{ bounded and continuous,}\\ &c:E\times E\to[c_0,+\infty[,\ c_0>0, \ \text{ bounded and continuous.}\end{aligned} \tag{2.9}$$

Moreover, for any $x\in E$, the possible impulses must be in

$$\begin{aligned}&\Gamma(x)=\{\xi\in E:(x,\xi)\in\Gamma\},\\ &\text{where } \Gamma \text{ is a given analytic set in } E\times E,\\ &\text{with the following properties:}\\ &\bullet\ \Gamma(x)\neq\emptyset\ \forall x\in E,\\ &\bullet\ \forall x\in E\ \forall\xi\in\Gamma(x),\ \Gamma(\xi)\subset\Gamma(x),\\ &\bullet\ \forall x\in E\ \forall\xi\in\Gamma(x)\ \forall\xi'\in\Gamma(\xi)\subset\Gamma(x),\\ &\quad c(x,\xi)+c(\xi,\xi')\ge c(x,\xi').\end{aligned} \tag{2.10}$$

Finally, defining the operator M

$$Mg(x)=\inf_{\xi\in\Gamma(x)}\big\{c(x,\xi)+g(\xi)\big\}, \tag{2.11}$$

it is assumed that

$$\begin{aligned}&M \ \text{ maps } \ C(E) \ \text{ into } \ C(E), \quad \text{and}\\ &\text{there exists a measurable selector } \ \hat\xi(x)=\hat\xi(x,g)\\ &\text{realizing the infimum in } \ Mg(x)\ \forall g\in C(E).\end{aligned} \tag{2.12}$$

Remark 2.2. (a) The last property in (2.10) implies that it is not optimal to make simultaneous impulses. (b) Equation (2.12) needs some regularity property of $\Gamma(x)$: e.g., see Davis [10]. (c) It is possible (but not necessary) that x belongs to $\Gamma(x)$, actually, even $\Gamma(x) = E$ for some or every x is allowed. However, an impulse occurs when the system moves from a state x to another state $\xi \neq x$, i.e., it suffices to avoid (or not to allow) impulses that moves x to itself, since they have a higher cost. ◻

2.3. The controlled process. We briefly describe the controlled process. For a detailed construction we refer to Bensoussan and Lions [6] (see also Davis [10], Lepeltier and Marchal [23], Robin [35], Stettner [37]).

Let us consider $\Omega^\infty = [D(\mathbb{R}^+; E\times\mathbb{R}^+)]^\infty$, and define $\mathcal{F}_t^0 = \mathcal{F}_t$ and $\mathcal{F}_t^{n+1} = \mathcal{F}_t^n \otimes \mathcal{F}_t$ for $n \geq 0$, where $\mathcal{F}_t$ is the universal completion of the canonical filtration as previously.

An *arbitrary impulse control* ν (not necessarily admissible at this stage) is a sequence $(\theta_n, \xi_n)_{n\geq 1}$, where θ_n is a stopping time of $\mathcal{F}_t^{n-1}$, $\theta_n \geq \theta_{n-1}$, and the impulse ξ_n is a $\mathcal{F}_{\theta_n}^{n-1}$ measurable random variable with values in E.

The coordinate in Ω^∞ has the form $(x_t^0, y_t^0, x_t^1, y_t^1, \ldots, x_t^n, y_t^n, \ldots)$, and for any impulse control ν there exists a probability P_{xy}^ν on Ω^∞ such that the evolution of the controlled process (x_t^ν, y_t^ν) is given by the coordinates (x_t^n, y_t^n) of Ω^∞ when $\theta_n \leq t < \theta_{n+1}$, $n \geq 0$ (setting $\theta_0 = 0$), i.e., $(x_t^\nu, y_t^\nu) = (x_t^n, y_t^n)$ for $\theta_n \leq t < \theta_{n+1}$. Note that clearly (x_t^ν, y_t^ν) is defined for any $t \geq 0$, but (x_t^i, y_t^i) is only used for any $t \geq \theta_i$, and $(x_{\theta_i}^{i-1}, y_{\theta_i}^{i-1})$ is the state at time θ_i just before the impulse (or jump) to $(\xi^i, y_{\theta_i}^{i-1}) = (x_{\theta_i}^i, y_{\theta_i}^i)$, as long as $\theta_i < \infty$. For the sake of simplicity, we will not always indicate, in the following, the dependency of (x_t^ν, y_t^ν) with respect to ν. A *Markov impulse control* ν is identified by a closed subset S of $E \times \mathbb{R}^+$ and a Borel measurable function $(x, y) \mapsto \xi(x, y)$ from S into $C = E \times \mathbb{R}^+ \smallsetminus S$, with the following meaning: intervene only when the the process (x_t, y_t) is leaving the continuation region C and then apply an impulse $\xi(x, y)$ while in the stopping region S, moving back the process to the continuation region C, i.e., $\theta_{i+1} = \inf\{t > \theta_i : (x_t^i, y_t^i) \in S\}$, with the convention that $\inf\{\emptyset\} = \infty$ and $\xi_{i+1} = \xi(x_{\theta_{i+1}}^i, y_{\theta_{i+1}}^i)$ for any $i \geq 0$, as long as $\theta_i < \infty$.

Now, the admissible controls are defined as follows, recalling that τ_n are the arrival times of the signal.

DEFINITION 2.3. (i) *A stopping time θ is called "admissible" if almost surely there exists $n = \eta(\omega) \geq 1$ such that $\theta(\omega) = \tau_{\eta(\omega)}(\omega)$ or, equivalently, if θ satisfies $\theta > 0$ and $y_\theta = 0$ a.s.*

(ii) *An impulse control $\nu = \{(\theta_i, \xi_i), i \geq 1\}$ as above is called admissible, if each θ_i is admissible (i.e., $\theta_i > 0$ and $y_{\theta_i} = 0$), and $\xi_i \in \Gamma(x_{\theta_i}^{i-1})$. The set of admissible impulse controls is denoted by $\mathcal{V}$.*

(iii) *If $\theta_1 = 0$ is allowed, then ν is called "zero-admissible." The set of zero-admissible impulse controls is denoted by $\mathcal{V}_0$.*

(iv) *An "admissible Markov" impulse control corresponds to a stopping region $S = S_0 \times \{0\}$ with $S_0 \subset E$, and an impulse function satisfying $\xi(x, 0) = \xi_0(x) \in \Gamma(x)$ for any $x \in S_0$ and, therefore, $\theta_i = \tau_{\eta_i}^i$ and $\eta_{i+1} = \inf\{k > \eta_i : x_{\tau_k^i}^i \in S_0\}$ with $\tau_0^0 = 0$, $\tau_k^i = \inf\{t > \tau_{k-1}^i : y_t^i = 0\}$, for any $k \geq i \geq 1$.* ◻

The discrete time impulse control problem has been considered in Bensoussan [4], Hernández-Lerma and Lasserre [16, 17], and Stettner [36]. As we will see later, it will be useful to consider an auxiliary problem in discrete time for the Markov chain $X_n = x_{\tau_n}$ with the filtration $\mathbb{G} = \{\mathcal{G}_n : n \geq 0\}$, $\mathcal{G}_0 = \mathcal{F}_0$ and $\mathcal{G}_n = \mathcal{F}_{\tau_n}^{n-1}$ for $n \geq 1$. The impulses occur at the stopping times η_k with values in the set $\mathbb{N} = \{0, 1, 2, \ldots\}$

and are related to θ_k by $\eta_i = \inf\{k \geq 1 : \theta_k = \tau_k\}$ for admissible controls $\{\theta_k\}$ and similarly for zero-admissible controls. Thus, we have the following.

DEFINITION 2.4. *If $\nu = \{(\eta_i, \xi_i), i \geq 1\}$ is a sequence of $\mathbb{G}$-stopping times and random variables ξ_i $\mathcal{G}_{\eta_i}$ measurable with $\xi_i \in \Gamma(x_{\tau_{\eta_i}})$, η_i increasing and $\eta_i \to +\infty$ a.s., then ν is referred to as an "admissible discrete time" impulse control if $\eta_1 \geq 1$. If $\eta_i \geq 0$ is allowed, it is referred as an "zero-admissible discrete time" impulse control.* □

One can now define the average cost to be minimized as

$$
\begin{aligned}
&J^T(0,x,y,\nu) = \mathbb{E}^{\nu}_{xy}\Big\{ \int_0^T f(x^{\nu}_s, y^{\nu}_s)\mathrm{d}s + \sum_i \mathbb{1}_{\theta_i \leq T} c(x^{i-1}_{\theta_i}, \xi_i)\Big\}, \\
&J(x,y,\nu) = \liminf_{T\to\infty} \frac{1}{T} J^T(0,x,y,\nu),
\end{aligned} \tag{2.13}
$$

then the problem is to characterize

$$
\mu(x,y) = \inf_{\nu\in\mathcal{V}} J(x,y,\nu). \tag{2.14}
$$

The auxiliary problem is concerned with

$$
\mu_0(x,y) = \inf_{\nu\in\mathcal{V}_0} \tilde{J}(x,y,\nu) \tag{2.15}
$$

with

$$
\tilde{J}(x,y,\nu) = \liminf_{n\to\infty} \frac{1}{\mathbb{E}^{\nu}_{xy}\{\tau_n\}} J^{\tau_n}(0,x,y,\nu), \tag{2.16}
$$

and $J^{\tau_n}(0,x,y,\nu)$ as in (2.13) with $T = \tau_n$.

Remark 2.5. Actually, as seen later, $\mu(x,y) = \mu_0(x,y)$ is a constant. □

3. Dynamic programming. To introduce the HJB equation(s) corresponding to this problem, we consider the dynamic programming argument in a heuristic way. Define the finite horizon cost as

$$
\begin{aligned}
J^T(t,x,y,\nu;h) = \mathbb{E}^{\nu}_{xy}\Big\{ \int_0^{T-t} & f(x^{\nu}_s, y^{\nu}_s)\mathrm{d}s \\
& + \sum_i \mathbb{1}_{\theta_i < T-t} c(x^{i-1}_{\theta_i}, \xi_i) + h(x_{T-t}, y_{T-t})\Big\},
\end{aligned} \tag{3.1}
$$

where h is a bounded terminal cost and the corresponding optimal costs

$$
u^T(t,x,y) = \inf\big\{J^T(t,x,y,\nu) : \nu \in \mathcal{V}\big\} \tag{3.2}
$$

and

$$
u^T_0(t,x,y) = \inf\big\{J^T(t,x,y,\nu) : \nu \in \mathcal{V}_0\big\}. \tag{3.3}
$$

Note that, from the definitions, $u^T(t,x,y) = u^T_0(t,x,y)$ if $y > 0$.

If θ is a stopping time, then

$$
\begin{aligned}
&J^T(t,x,y,\nu;h) \\
&\quad = \mathbb{E}^\nu_{xy}\Big\{\int_0^{\theta\wedge(T-t)} f(x^\nu_s,y^\nu_s)\mathrm{d}s + \sum_i \mathbb{1}_{\theta_i<\theta\wedge(T-t)} c(x^{i-1}_{\theta_i},\xi_i)\Big\} \\
&\qquad + \mathbb{E}^\nu_{xy}\Big\{\int_{\theta\wedge(T-t)}^{(T-t)} f(x^\nu_s,y^\nu_s)\mathrm{d}s \\
&\qquad + \sum_i \mathbb{1}_{\theta\wedge(T-t)\le\theta_i<T-t} c(x^{i-1}_{\theta_i},\xi_i) + h(x_{T-t},y_{T-t})\Big\}.
\end{aligned}
\tag{3.4}
$$

Considering now $u_0^T(t,x,y)$, assuming that one can apply the Markov property and minimizing separately on $[0,\theta[$ and $[\theta,T-t[$, we deduce

$$
\begin{aligned}
u_0^T(t,x,y) = \inf_{\nu\in\mathcal{V}_0} \Big\{ &\mathbb{E}^\nu_{xy}\Big\{\int_0^{\theta\wedge(T-t)} f(x^\nu_s,y^\nu_s)\mathrm{d}s \\
&+ \sum_i \mathbb{1}_{\theta_i<\theta\wedge(T-t)} c(x^{i-1}_{\theta_i},\xi_i)\mathrm{d}s \\
&+ \mathbb{1}_{\theta<T-t} u_0^T(\theta,x_\theta,y_\theta) + \mathbb{1}_{\theta\ge T-t} h(x_{T-t},y_{T-t})\Big\}\Big\}.
\end{aligned}
\tag{3.5}
$$

At time t, if $y=0$, either one applies an impulse, i.e., $\theta_1=0$ or $\theta_1\ge\tau_1$, therefore,

$$
u_0^T(t,x,0) = \min \begin{cases} \inf_{\xi\in\Gamma(x)} \big\{c(x,\xi)+u_0^T(t,\xi,0)\big\} = Mu_0^T(t,x,0), \\ \mathbb{E}_{x0}\Big\{\int_0^{\tau_1\wedge(T-t)} f(x_s,y_s)\mathrm{d}s \\ \qquad + u_0^T(\tau_1\wedge(T-t), x_{\tau_1\wedge(T-t)}, y_{\tau_1\wedge(T-t)})\Big\}. \end{cases}
\tag{3.6}
$$

If $y>0$, no impulse is allowed before τ_1, therefore,

$$
\begin{aligned}
u_0^T(t,x,y) = \mathbb{E}_{xy}\Big\{&\int_0^{\tau_1\wedge(T-t)} f(x_s,y_s)\mathrm{d}s \\
&+ u_0^T(\tau_1\wedge(T-t), x_{\tau_1\wedge(T-t)}, y_{\tau_1\wedge(T-t)})\Big\}.
\end{aligned}
\tag{3.7}
$$

Let us now make the assumption

$$
\begin{aligned}
&\text{there exists a bounded measurable function } w_0(x,y) \\
&\text{and a constant } \mu_0\ge 0 \text{ such that, as } T\to\infty, \\
&\varepsilon(t,x,y,T) := u_0^T(t,x,y) - (T-t)\mu_0 - w_0(x,y) \to 0, \\
&\text{locally uniformly in } (x,y) \text{ for each fixed } t,
\end{aligned}
\tag{3.8}
$$

which is similar to properties showed in Markov decision processes, e.g., Prieto-

Rumeau and Hernández-Lerma [33, Thm 3.6]. Then, using (3.8) in (3.5), one obtains

$$\begin{aligned}&(T-t)\mu_0 + w_0(x,y) + \varepsilon(T)\\&\quad = \inf_{\nu\in\mathcal{V}_0}\bigg\{\mathbb{E}^\nu_{xy}\bigg\{\int_0^{\theta\wedge(T-t)} f(x^\nu_s,y^\nu_s)\mathrm{d}s + \sum_i \mathbb{1}_{\theta_i<\theta\wedge(T-t)}c(x^{i-1}_{\theta_i},\xi_i)\\&\qquad + \mathbb{1}_{\theta<T-t}\big[(T-\theta)\mu_0 + w_0(x_\theta,y_\theta)+\varepsilon(T)\big] + \mathbb{1}_{\theta>T-t}h(x_{T-t},y_{T-t})\bigg\}\bigg\},\end{aligned}$$

and when $T\to\infty$, we deduce (for any θ almost surely finite)

$$\begin{aligned}w_0(x,y) = \inf_{\nu\in\mathcal{V}_0}\bigg\{\mathbb{E}^\nu_{xy}\bigg\{&\int_0^\theta \big[f(x^\nu_s,y^\nu_s)-\mu_0\big]\mathrm{d}s\\&+\sum_i \mathbb{1}_{\theta_i<\theta}c(x^{i-1}_{\theta_i},\xi_i) + \mathbb{1}_{\theta<\infty}w_0(x_\theta,y_\theta)\bigg\}\bigg\}.\end{aligned}$$

Arguing as for (3.6), we get

$$\begin{aligned}w_0(x,0) = \min\bigg\{Mw_0(x,0), \mathbb{E}^\nu_{x0}\bigg\{&\int_0^{\tau_1}[f(x^\nu_s,y^\nu_s)-\mu_0]\mathrm{d}s\\&+\mathbb{1}_{\tau_1<\infty}w_0(x_{\tau_1},0)\bigg\}\bigg\},\end{aligned}$$

and the factor $\mathbb{1}_{\tau_1<\infty}$ is not needed since the assumptions in section 2.1 imply $\mathbb{E}_{x0}\{\tau_1\} < \infty$. Finally, we obtain the HJB equation for (μ_0,w_0),

$$w_0(x,0) = \min\bigg\{Mw_0(x,0), \mathbb{E}_{x0}\bigg\{\int_0^{\tau_1}[f(x_s,y_s)-\mu_0]\mathrm{d}s + w_0(x_{\tau_1},0)\bigg\}\bigg\}, \tag{3.9}$$

$$w_0(x,y) = \mathbb{E}_{xy}\bigg\{\int_0^{\tau_1}[f(x_s,y_s)-\mu_0]\mathrm{d}s + w_0(x_{\tau_1},0)\bigg\}. \tag{3.10}$$

Now, let us apply similar arguments to u^T. Assuming $\theta\ge\tau_1$ in (3.4), one minimizes separately on $[0,\theta[$ and $[\theta,T-t[$. Either θ is an arrival time of the signal and an impulse may be applied or θ is not and no impulse is allowed. In any case, the possible actions on $[\theta,T-t[$ are the same as for the impulse control in $\mathcal{V}_0$. Therefore, we obtain

$$\begin{aligned}u^T(t,x,y) = \inf_{\nu\in\mathcal{V}}\bigg\{\mathbb{E}^\nu_{xy}\bigg\{&\int_0^{\theta\wedge(T-t)} f(x^\nu_s,y^\nu_s)\mathrm{d}s\\&+\sum_i \mathbb{1}_{\theta_i<\theta\wedge(T-t)}c(x^{i-1}_{\theta_i},\xi_i)\\&+\mathbb{1}_{\theta<T-t}u^T_0(\theta,x_\theta,y_\theta) + \mathbb{1}_{\theta>T-t}h(x_{T-t},y_{T-t})\bigg\}\bigg\}.\end{aligned} \tag{3.11}$$

Taking $\theta=\tau_1$ and, since no impulse is allowed before τ_1, this gives

$$\begin{aligned}u^T(t,x,y) = \mathbb{E}_{xy}\bigg\{&\int_0^{\tau_1\wedge(T-t)} f(x_s,y_s)\mathrm{d}s\\&+\mathbb{1}_{\tau_1<T-t}u^T_0(x_{\tau_1},0) + \mathbb{1}_{\tau_1\ge T-t}h(x_{T-t},y_{T-t})\bigg\}.\end{aligned} \tag{3.12}$$

Remark 3.1. Since $u^T(t,x,y) = u_0^T(t,x,y)\ \forall y > 0$ it is sufficient to obtain $u_0^T(t,x,0)$ and $u^T(t,x,0)$ to get the values for $y > 0$. □

Let us assume that (similarly to the case of w_0)

(3.13) there exists a bounded measurable function $w(x,y)$ such that for the same constant $\mu_0 \geq 0$ as in (3.8), $\varepsilon(t,x,y,T) := u^T(t,x,y) - (T-t)\mu_0 - w(x,y) \to 0$, as $T \to \infty$, locally uniformly in (x,y), for each fixed t.

Then the same arguments give

$$w(x,y) = \inf_{\nu \in \mathcal{V}} \Big\{ \mathbb{E}^\nu_{xy} \Big\{ \int_0^\theta \big[f(x^\nu_s, y^\nu_s) - \mu_0\big] \mathrm{d}s + \sum_i \mathbb{1}_{\theta_i < \theta} c(x^i_{\theta_i -}, \xi_i) + \mathbb{1}_{\theta < \infty} w_0(x_\theta, y_\theta) \Big\} \Big\}, \tag{3.14}$$

and since no control can take place before τ_1, one obtains

$$w(x,y) = \mathbb{E}_{xy} \Big\{ \int_0^{\tau_1} [f(x_s,y_s) - \mu_0] \mathrm{d}s + w_0(x_{\tau_1}, 0) \Big\}. \tag{3.15}$$

4. Solutions of the HJB equations. It is clear from (3.15) that knowledge of $(\mu_0, w_0(x,0))$ will give $w(x,y)$, and also $w_0(x,y)$ for $y > 0$. Therefore the key step is to solve (3.9) for $(\mu_0, w_0(x,0))$. For this purpose, we consider a discrete time HJB equation equivalent to (3.9) as follows: Let us define

$$\ell(x) = \mathbb{E}^\nu_{x0} \Big\{ \int_0^{\tau_1} f(x_s, y_s) \mathrm{d}s \Big\}. \tag{4.1}$$

Recall that under the assumptions of section 2.1, we have

$$0 < a_1 \leq \mathbb{E}_{x0}\{\tau_1\} \leq a_2 \tag{4.2}$$

and, therefore,

$$0 \leq \ell(x) \leq a_2 \|f\|. \tag{4.3}$$

Moreover, $\ell(x)$ is continuous (from the Feller property of x_t and the law of τ_1). From (2.5), define also the operator P on $C(E)$ by

$$Pg(x) = \mathbb{E}_{x0}\{g(x_{\tau_1})\} = \mathbb{E}_{x0}\{g(X_1)\}, \tag{4.4}$$

where X_n is the Markov chain $X_n = x_{\tau_n}$. The Feller property of x_t and the regularity of λ yield

$$Pg(x) \ \text{ maps } \ C(E) \ \text{ into itself} \tag{4.5}$$

and, from (2.12), it is also the case of M, as defined by (2.11). In this section, we denote

$$\tau(x) = \mathbb{E}_{x0}\{\tau_1\}. \tag{4.6}$$

With the previous notation, (3.9) is written as (with $w_0(x) = w_0(x, 0)$)

$$w_0(x) = \min\Big\{ \inf_{\xi\in\Gamma(x)} \{c(x,\xi) + w_0(\xi)\}, \ell(x) - \mu_0\tau(x) + Pw_0(x)\Big\}. \tag{4.7}$$

This expresses the fact that either there is an immediate impulse (when $w_0(x) = Mw_0(x)$) or the chain evolves with the kernel P and a running cost $\ell(x) - \mu_0\tau(x)$ is incurred. We will need the following lemma.

LEMMA 4.1. *Under the assumption* (2.6)*, there exist a positive measure* γ *on* E *and a constant* $0 < \beta < 1$ *such that*

$$P(x, B) \geq \tau(x)\gamma(B) \quad \forall B \in \mathcal{B}(E)\ \forall x \in E, \quad \text{and} \quad \gamma(E) > \frac{1-\beta}{\tau(x)}. \tag{4.8}$$

Proof. Recall that the assumptions on λ imply

$$0 < a_1 \leq \tau(x) \leq a_2.$$

Thus, to satisfy the first inequality in (4.8) it is sufficient to take

$$\gamma(B) = \frac{1}{a_2} m(B) \quad \forall B \in \mathcal{B}(E),$$

where m is the measure in (2.6).

For the second inequality it is sufficient to take $\beta \in]0, 1[$ such that

$$\frac{m(E)}{a_2} > \frac{1-\beta}{a_1} \geq \frac{1-\beta}{\tau(x)} \quad \forall x \in E,$$

i.e., $1 - \beta < m(E)a_1/a_2$. ∎

THEOREM 4.2. *Under the assumptions of section* 2.1*, there exists a solution* (μ_0, w_0) *in* $\mathbb{R}^+ \times C(E)$ *of* (4.7) *and, therefore, of* (3.9).

Proof. For the sake of simplicity, in this proof, we drop the index 0.

We first transform (4.7): The assumptions on $c(x, \xi)$ imply that multiple simultaneous impulses are not optimal (see Remark 2.2), so we restrict the controls to those without multiple simultaneous impulses. Therefore, in (4.7), after an impulse ξ, the chain evolves without control until the next transition, i.e., $w_0(\xi) = \ell(\xi) - \mu\tau(\xi) + Pw(\xi)$. This gives

$$w(x) = \inf_{\xi\in\Gamma(x)\cup\{x\}} \big\{\ell(\xi) + \mathbb{1}_{x\neq\xi} c(x,\xi) - \mu\tau(\xi) + Pw(\xi)\big\}. \tag{4.9}$$

Denote

$$L(x, \xi) = \ell(\xi) + \mathbb{1}_{x\neq\xi} c(x, \xi)$$

and, for $v \in B(E)$,

$$T(x, \xi; v) = L(x, \xi) + Pv(\xi) - \tau(\xi) \int_E v(z)\gamma(\mathrm{d}z)$$

and

$$Rv(x) = \inf_{\xi\in\Gamma(x)\cup\{x\}} T(x, \xi; v).$$

Define also

$$P'(x, \mathrm{d}z) = P(x, \mathrm{d}z) - \tau(x)\gamma(\mathrm{d}z) \quad \forall x \in E,$$

which is a positive measure on E for each x in E, in view of Lemma 4.1. Denote

$$P'(x, v) = \int_E v(z)P'(x, \mathrm{d}z).$$

Letting v_1, v_2 be two bounded measurable functions, we have

$$\big|T(x, \xi; v_1) - T(x, \xi; v_2)\big| = \big|P'(\xi, v_1 - v_2)\big|. \tag{4.10}$$

Moreover, from Lemma 4.1, we have

$$P'(x, E) < \beta \quad \forall x \in E.$$

Therefore from (4.10) we deduce that R is a contraction on $B(E)$ and has a unique fixed point $w = Rw$.

If we define

$$\mu = \int_E w(z)\gamma(\mathrm{d}z),$$

then (μ, w) is a solution of (4.9) and (4.7). Moreover, since

$$Rv(x) = \min\Big\{\ell(x) + P'(x, v), \inf_{\xi\in\Gamma(x)} \big\{\ell(\xi) + c(x, \xi) + P'(\xi, v)\big\}\Big\},$$

it is clear that $Rv \in C(E)$ if $v \in C(E)$ and, therefore, $w \in C(E)$.

In addition, since $\ell(x) \geq 0$ and $c(x, \xi) > 0$, we deduce that the fixed point $w \geq 0$, which also implies $\mu \geq 0$. ∎

As a corollary, $w(x, y)$ in (3.15) is well defined with $w_0(x, 0) = w_0(x)$ from Theorem 4.2.

Remark 4.3. The assumption (4.8) is used in Kurano [21, 22], in the context of semi-Markov decision processes. ∎

Remark 4.4. In the case where $\tau(x)$ is constant, which corresponds to an intensity $\lambda(y)$ independent of x, (4.7) is the HJB equation of a standard discrete time impulse control as studied in Stettner [36] for $\Gamma(x) = \Gamma$ fixed. ∎

5. Existence of an optimal control. We make the following additional assumptions:

(5.1)
for the process (x_t, y_t) as defined in section 2.1,
there exists a unique invariant measure ζ on $E \times \mathbb{R}^+$
and there exists a continuous function $h(x, y)$ such
that, for any stopping time τ with $\mathbb{E}_{xy}\{\tau\} < \infty$,

$$\mathbb{E}_{xy}\big\{h(x_\tau, y_\tau)\big\} = h(x, y) - \mathbb{E}_{xy}\bigg\{\int_0^\tau \big(f(x_t, y_t) - \bar{f}\big)\mathrm{d}t\bigg\},$$

where $\bar{f} = \displaystyle\int_{E\times\mathbb{R}^+} f(x, y)\zeta(\mathrm{d}x, \mathrm{d}y)$.

Note that h plus a constant also satisfies this equation.

For the auxiliary problem, we state the following.

THEOREM 5.1. *Under the assumptions of* section 2.1, *and* (5.1),

$$\mu_0 = \inf_{\nu \in \mathcal{V}_0} \tilde{J}(x, 0, \nu), \tag{5.2}$$

where μ_0 *is given by* Theorem 4.2 *and* $\tilde{J}$ *is defined by* (2.16). *Moreover, there exists an optimal feedback control.*

Proof. Let us first show that

$$\tilde{J}(x, y, 0) = \bar{f}, \tag{5.3}$$

where $\nu = 0$ means "no control," that is,

$$\tilde{J}(x, y, 0) = \liminf_{n \to \infty} \frac{1}{\mathbb{E}_{xy}\{\tau_n\}} \mathbb{E}_{xy}\bigg\{ \int_0^{\tau_n} f(x_t, y_t) \mathrm{d}t \bigg\}.$$

Indeed, from (5.1), we have

$$\mathbb{E}_{xy}\big\{h(x_{\tau_n}, y_{\tau_n})\big\} = h(x, y) - \mathbb{E}_{xy}\bigg\{ \int_0^{\tau_n} \big[f(x_t, y_t) - \bar{f}\big] \mathrm{d}t \bigg\} \tag{5.4}$$

(note that $\mathbb{E}\{\tau_n\} < \infty$ and $y_{\tau_n} = 0$), which gives

$$\frac{1}{\mathbb{E}_{xy}\{\tau_n\}} \mathbb{E}_{xy}\bigg\{ \int_0^{\tau_n} f(x_t, y_t) \mathrm{d}t \bigg\} = \bar{f} + \frac{1}{\mathbb{E}_{xy}\{\tau_n\}} \mathbb{E}_{xy}\big\{h(x, y) - h(x_{\tau_n}, y_{\tau_n})\big\}.$$

Taking the limit when $n \to \infty$, since $y_{\tau_n} = 0$ and $h(x, 0)$ is bounded (since h is continuous and E compact), (5.3) is obtained.

As a consequence, one can restrict the set of controls to those such that

$$\tilde{J}(x, y, \nu) \le \tilde{J}(x, y, 0) = \bar{f}. \tag{5.5}$$

Next let us show that

$$\mu_0 \le \tilde{J}(x, 0, \nu) \quad \forall v \in \mathcal{V}_0. \tag{5.6}$$

Rewrite the HJB equation (4.7) as

$$w_0(x) = \min\big\{M w_0(x), L(x) + P w_0(x)\big\}$$

with $L(x) = \ell(x) - \tau(x)\mu_0$.

From this equation, for any discrete impulse control $\{(\eta_i, \xi_i) : i \ge 1\}$, we deduce

$$w_0(x, 0) \le \mathbb{E}_{x0}^{\nu}\bigg\{ \sum_{i=0}^{n-1} L(X_i) + \sum_j \mathbb{1}_{\eta_j \le n} c(X_{\eta_j}, \xi_j) + w_0(X_n) \bigg\},$$

where, here, X_i denotes the controlled discrete time process.

Denoting $\nu = \{(\theta_i, \xi_i) : i \ge 1\}$ as the impulse control corresponding to $\{(\eta_i, \xi_i) : i \ge 1\}$, i.e., with $\theta_i = \tau_{\eta_i}$, we obtain

$$w_0(x) \le \mathbb{E}_{x0}^{\nu}\bigg\{ \int_0^{\tau_n} \big[f(x_t, y_t) - \mu_0\big] \mathrm{d}t + \sum_j \mathbb{1}_{\theta_j \le \tau_n} c(x_{\theta_j}^{j-1}, \xi_j) + w_0(x_{\tau_n}) \bigg\}$$

and, therefore,

$$\mu_0 \le \frac{1}{\mathbb{E}^{\nu}_{x0}\{\tau_n\}}\mathbb{E}^{\nu}_{x0}\bigg\{\int_0^{\tau_n} f(x_t,y_t)\mathrm{d}t + \sum_j \mathbb{1}_{\theta_j\le\tau_n} c(x^{j-1}_{\theta_j},\xi_j) + w_0(x_{\tau_n}) - w_0(x)\bigg\},$$

and since w is bounded and $\mathbb{E}^{\nu}_{x0}\{\tau_n\}\to\infty$, we obtain (5.6).

Therefore, with (5.5),

$$\mu_0 \le \inf_{\nu\in\mathcal{V}_0}\big\{\tilde J(x,0,\nu)\big\} \le \tilde J(x,0,0) = \bar f.$$

Consequently, if $\mu_0=\bar f$ then

$$\mu_0 = \inf_{\nu\in\mathcal{V}_0}\big\{\tilde J(x,0,\nu)\big\} = \tilde J(x,0,0),$$

and "do nothing" (i.e., the control without any impulse) is optimal.

Let us now consider the case $\mu_0<\bar f$. Using (5.4) for $n=1$, the HJB equation (3.9) for (μ_0,w_0) can be rewritten, dropping again the index 0 for simplicity and with $\psi=Mw$,

$$(w-h)(x) = \min\big\{(\psi-h)(x), (\bar f-\mu)\mathbb{E}_{x0}\{\tau_1\} + \mathbb{E}_{x0}\{(w-h)(x_{\tau_1})\}\big\},$$

where, here, $h=h(x,0)$. Moreover, since h is defined up to an additive constant and is bounded, one can assume $h\le 0$, and rewrite it as

$$\tilde w(x) = \min\big\{\tilde\psi(x), \tilde\ell(x) + P\tilde w(x)\big\} \tag{5.7}$$

with $\tilde w = w-h$, $\tilde\psi=\psi-h$, and $\tilde\ell(x) = (\bar f-\mu)\mathbb{E}_{x0}\{\tau_1\}$.

For the Markov chain $X_n=x_{\tau_n}$, this (5.7) is the HJB equation of a stopping time problem as studied in Bensoussan [4, Chapter 7, pp. 67–77], with the conditions stated herein, namely, $\tilde\psi\ge 0$, $\ell(x)\ge\ell_0>0$. Therefore, we have

$$\tilde w(x) = \inf\Bigg\{\mathbb{E}_x\bigg\{\sum_{j=1}^{\eta-1}\tilde\ell(X_j) + \tilde\psi(X_\eta)\bigg\}\Bigg\},$$

where the infimum is taken over the $\mathbb{G}$-stopping times η (with values in $\mathbb{N}$), and there is an optimal control $\hat\eta$ given by

$$\hat\eta = \inf\big\{n\ge 0 : \tilde w(X_n) = \tilde\psi(X_n)\big\}$$

with $\mathbb{E}_x\{\hat\eta\}<\infty$.

Going back to w, the same result holds, with the same $\hat\eta$, namely,

$$w(x) = \inf_{\eta}\Bigg\{\mathbb{E}_x\bigg\{\sum_{j=1}^{\eta-1}\ell(X_j) + \psi(X_\eta)\bigg\}\Bigg\},$$

with

$$\ell(x) = \mathbb{E}_{x0}\int_0^{\tau_1}[f(x_t,y_t)-\mu]\mathrm{d}t$$

and

$$w(x) = \inf_{\eta}\Bigg\{\mathbb{E}_x\bigg\{\sum_{j=1}^{\eta-1}\ell(X_j) + Mw(X_\eta)\bigg\}\Bigg\}.$$

By the standard results on the optimal stopping time problems,

$$M_n = \mathbb{E}_x \bigg\{ \sum_{j=1}^{n-1} \ell(X_j) + w(X_n) \bigg\} \quad \text{is a } \mathcal{G}_n \text{ submartingale}$$

and

$$\hat{M}_n = \mathbb{E}_x \bigg\{ \sum_{j=1}^{(\eta-1)\wedge(n-1)} \ell(X_j) + w(X_{\eta\wedge n}) \bigg\} \quad \text{is a } \mathcal{G}_n \text{ martingale.}$$

From this, for the zero-admissible (discrete) impulse control $\nu = \{(\eta_i, \xi_i) : i \geq 1\}$ one can compute

$$w(x) \leq \mathbb{E}^{\nu}_{x0} \bigg\{ \sum_{i=1}^{n-1} \ell(X_i) + \sum_i \mathbb{1}_{\eta_i \leq n} c(X_{\eta_i}, \xi_i) + w(X_n) \bigg\}$$

which means

$$w(x) \leq \mathbb{E}^{\nu}_{x0} \bigg\{ \int_0^{\tau_n} \big[f(x^{\nu}_t, y^{\nu}_t) - \mu \big] \mathrm{d}t + \sum_i \mathbb{1}_{\theta_{\eta_i} \leq \tau_n} c(x^{i-1}_{\theta_{\eta_i}}, \xi_i) + w(x^{n-1}_{\tau_n}) \bigg\}$$

and, therefore, since w is bounded and $\mathbb{E}_{x0}\{\tau_n)\} \to \infty$ we deduce

$$\mu \leq \tilde{J}(x, 0, \nu) \quad \forall \nu \in \mathcal{V}_0. \tag{5.8}$$

Then, defining $\hat{\nu} = \{(\hat{\eta}_i, \hat{\xi}_i) : i \geq 1\}$ by

$$\hat{\eta}_i = \inf \big\{ n \geq \hat{\eta}_{i-1} : w(X_n) = Mw(X_n) \big\}, \quad \hat{\xi}_i = \hat{\xi}(X_{\hat{\eta}_i}), \quad i \geq 1,$$

where $\hat{\eta}_0 = 0$ and $\hat{\xi}(x)$ is a Borel measurable selector realizing the infimum in $Mw(x)$, we obtain the equality in (5.8) using the martingale property of $\tilde{M}_n$. ∎

Remark 5.2. (1) When $\lambda(x, y) = \lambda(y)$, y_t is independent of x_t and under the assumptions of section 2.1, y_t has a unique invariant measure given by

$$\zeta_1(B) = \frac{1}{\mathbb{E}\{\tau\}} \mathbb{E} \bigg\{ \int_0^{\tau} \mathbb{1}_B(y_t) \mathrm{d}t \bigg\} \quad \forall B \in \mathcal{B}(\mathbb{R}^+)$$
$$\text{with} \quad \tau = \inf \big\{ t \geq 0 : y_t = 0 \big\},$$

(e.g., see Davis [10, pp. 130–131]); actually, in this case, y_t is a simple example of a piecewise deterministic Markov process. Therefore, if x_t has a unique invariant probability ζ_2 on E, then the couple (x_t, y_t) has the invariant probability $\zeta = \zeta_2 \otimes \zeta_1$.

(2) If $f(x, y) = f(x)$ then it is sufficient to assume that Poisson's equation for x_t alone, i.e., $-A_x h(x) = f(x) - \bar{f}$, has a continuous solution.

(3) In the general case (namely, $\lambda(x, y)$ and $f(x, y)$), it seems necessary to have an explicit knowledge of x_t to verify directly assumption (5.1), which is clearly satisfied if there exists a continuous bounded solution of Poisson's equation $-A_{xy} h(x, y) = f(x, y) - \bar{f}$, but this assumption is too restrictive in our case because of y_t. This would not be the case if the signal y_t belongs to a bounded interval $[0, b]$ instead of the whole $\mathbb{R}^+$, however, some new difficulties arrive with the corresponding infinitesimal generator A_{xy}. ∎

COROLLARY 5.3. *Under the assumptions of* section 2.1 *and* (5.1),

$$\mu_0 = \inf_{\nu \in \mathcal{V}} \tilde{J}(x, y, \nu), \tag{5.9}$$

where μ_0 *is given by* Theorem 4.2 *and* $\tilde{J}$ *is defined by* (2.16).

Proof. Recall the definition of $w_0(x, y)$ in (3.10):

$$w_0(x, y) = \mathbb{E}_{xy}\Big\{ \int_0^{\tau_1} \big[f(x_t, y_t) - \mu_0\big]\mathrm{d}t + w_0(x_{\tau_1}, 0)\Big\},$$

where $w_0(x, 0) = w_0(x)$ is (with μ_0) the solution obtained in Theorem 4.2.

If $\nu = \{(\tau_i, \xi_i) : i \ge 1\}$ is an admissible impulse control, then $\theta_1 \ge \tau_1$. Therefore, θ_1 can be written as $\theta_1 = \tau_1 + \tilde{\theta}_1$, where $\tilde{\theta}_1$ is a stopping time with respect to $\mathcal{F}_{\tau_1+t}$, and similarly with $\tau_2 = \tau_1 + \tilde{\tau}_2$. From the properties of w_0, we have

$$w_0(x_{\tau_1}) \le \mathbb{E}^{\nu}_{x_{\tau_1}0}\Big\{ \int_0^{\tilde{\tau}_2} \big[f(x_t, y_t) - \mu_0\big]\mathrm{d}t + \mathbb{1}_{\tilde{\theta}_1 \le \tilde{\tau}_2} c(x^0_{\tilde{\theta}_1}, \xi_1) + w_0(x_{\tilde{\tau}_2}, 0)\Big\},$$

which gives

$$w_0(x, y) \le \mathbb{E}^{\nu}_{xy}\Big\{ \int_0^{\tau_1} \big[f(x_t, y_t) - \mu_0\big]\mathrm{d}t + \int_{\tau_1}^{\tau_2} \big[f(x_t, y_t) - \mu_0\big]\mathrm{d}t$$
$$+ \mathbb{1}_{\theta_1 \le \tau_2} c(x^0_{\theta_1}, \xi_1) + w_0(x_{\tau_2}, 0)\Big\}$$

or, equivalently,

$$w_0(x, y) \le \mathbb{E}^{\nu}_{xy}\Big\{ \int_0^{\tau_2} \big[f(x_t, y_t) - \mu_0\big]\mathrm{d}t + \mathbb{1}_{\theta_1 \le \tau_2} c(x^0_{\theta_1}, \xi_1) + w_0(x_{\tau_2}, 0)\Big\}.$$

Iterating this argument we obtain

$$\mu_0 \le \tilde{J}(x, y, \nu) \quad \forall \nu \in \mathcal{V}.$$

Using the optimal control defined in Theorem 5.1, we get the equality for the control $\hat{\nu}_1$ translated by τ_1, i.e., with $\hat{\theta}_i = \tau_1 + \tilde{\tau}_{\hat{\eta}_i}$, $i \ge 1$. ☐

Remark 5.4. As mentioned in Arapostathis et al. [1, p. 287], our definition, either (2.13) or (2.16), of our cost with "lim inf" gives a rather "optimistic" measure of performance; however, the inequality just before (5.8) shows that essentially there are no changes if lim inf is replaced by lim sup in the definition, either (2.13) or (2.16), of the cost, either $J(x, y, \nu)$ or $\tilde{J}(x, y, \nu)$. Moreover, the minimization with either lim inf or lim sup yields the same value $\mu = \mu_0$. ☐

PROPOSITION 5.5. *If* (μ_0, w_0) *is a solution of* (3.9) *provided by* Theorem 4.2*, then* $w(x, y)$*, defined by* (3.15)*, is the solution of the equation*

$$-A_{xy}w(x) + \lambda(x, y)\big[w(x, 0) - Mw(x, 0)\big]^+ = f(x, y) - \mu_0. \tag{5.10}$$

Proof. The first step is to show the following lemma.

LEMMA 5.6. *Under the assumptions of section* 2.1*, we have*

$$w_0(x) = \min\big\{w(x, 0), Mw(x, 0)\big\}. \tag{5.11}$$

Proof of lemma. Note that

$$w(x,0) = \mathbb{E}_{x0}\bigg\{\int_0^{\tau_1}\big[f(x_t,y_t)-\mu_0\big]\mathrm{d}t + w_0(x_{\tau_1})\bigg\} := Tw_0(x).$$

Therefore

$$\min\{Mw_0(x), w(x,0)\} = \min\{Mw_0(x), Tw_0(x)\} = w_0,$$

where the last equality comes from (3.9) for (μ_0, w_0).

So the point is now to show

$$Mw_0(x) = Mw(x,0). \tag{5.12}$$

Observe also that, by definition,

$$w_0(x) \le Tw_0(x) = w(x,0) \tag{5.13}$$

and, therefore,

$$Mw_0(x) \le Mw(x,0). \tag{5.14}$$

For a fixed x, let $\hat{\xi}$ satisfy the infimum in $Mw_0(x)$, i.e., $Mw_0(x) = c(x,\hat{\xi}) + w_0(\hat{\xi})$. Let us check that $\hat{\xi}$ is also a minimizer for $Mw(x,0)$. Indeed, the equality

$$w_0(x) = \min\{Mw_0(x), w(x,0)\}$$

yields

$$Mw_0(x) = c(x,\hat{\xi}) + \min\{w(\hat{\xi},0), Mw_0(\hat{\xi})\}.$$

Now, if $Mw_0(\hat{\xi}) < w(\hat{\xi},0)$, then

$$Mw_0(x) = c(x,\hat{\xi}) + Mw_0(\hat{\xi}) = c(x,\hat{\xi}) + c(\hat{\xi},\xi') + w_0(\xi',0),$$

where ξ' realizes the infimum in $Mw_0(\hat{\xi})$.

However, if we assume that the strict inequality holds in assumption (2.10) on c, then one gets

$$Mw_0(x) > c(x,\xi') + w_0(\xi',0) \ge Mw_0(x),$$

which is impossible. Therefore,

$$Mw_0(\hat{\xi}) \ge w(\hat{\xi},0)$$

and

$$Mw_0(x) = c(x,\hat{\xi}) + w(\hat{\xi},0) \ge Mw(x,0).$$

Coming back to the assumption (2.10) on c, let us replace c with $c_\varepsilon(x,\xi) = \mathbb{1}_{x\neq\xi}\varepsilon + c(x,\xi)$ (where the strict inequality holds in assumption (2.10) on c_ε), and as $\varepsilon \to 0$, we also get $Mw_0(x) \ge Mw(x,0)$. Hence, this together with (5.14) gives (5.12) and, therefore, (5.11). ∎

Continuing with the proof, the equality (5.11) obtained in the above Lemma 5.6 in (3.15) gives

$$w(x,y) = \mathbb{E}_{xy}\bigg\{\int_0^{\tau_1}\big[f(x_t,y_t)-\mu_0\big]\mathrm{d}t + \min\big\{w(x_{\tau_1},0), Mw(x_{\tau_1},0)\big\}\bigg\}. \tag{5.15}$$

Note that, $y_t = y + t$ on $[0,\tau_1[$ under P_{xy}.

Using the law of τ_1, one can write

$$w(x,y) = \mathbb{E}_{xy}\Big\{ \int_0^\infty \lambda(x_t, y+t) \exp\Big(- \int_0^t \lambda(x_r, y+r)\mathrm{d}r \Big) \mathrm{d}t \times \int_0^t \big[f(x_s, y+s) - \mu_0\big]\mathrm{d}s \Big\} + \mathbb{E}_{xy}\Big\{ \int_0^\infty \lambda(x_t, y+t) \times \exp\Big(- \int_0^t \lambda(x_r, y+r)\mathrm{d}r \Big) \min\Big\{ w(x_t, 0), Mw(x_t, 0)\Big\}\mathrm{d}t \Big\} = (1) + (2).$$

After integrating by parts, the first term (1) becomes

$$(1) = \mathbb{E}_{xy}\Big\{ \int_0^\infty \exp\Big(- \int_0^t \lambda(x_r, y+r)\mathrm{d}r \Big) \big[f(x_t, y+t) - \mu_0\big]\mathrm{d}t \Big\},$$

therefore,

$$w(x,y) = \mathbb{E}_{xy}\Big\{ \int_0^\infty \exp\Big(- \int_0^t \lambda(x_r, y+r)\mathrm{d}r \Big) \times \Big[f(x_t, y+t) - \mu_0 + \lambda(x_t, y+t) \min\big\{ w(x_t, 0), Mw(x_t, 0)\big\}\Big]\mathrm{d}t \Big\}.$$

Since

$$\min\big\{ w(x,0), Mw(x,0)\big\} = w(x,0) - \big[w(x,0) - Mw(x,0)\big]^+,$$

one gets

$$w(x,y) = \mathbb{E}_{xy}\Big\{ \int_0^\infty \exp\Big(- \int_0^t \lambda(x_r, y+r)\mathrm{d}r \Big) \varphi(x_t, y+t)\mathrm{d}t \Big\}, \tag{5.16}$$

with

$$\varphi(x,y) = f(x,y) - \mu_0 + \lambda(x,y)w(x,0) - \lambda(x,y)\big[w(x,0) - Mw(x,0)\big]^+. \tag{5.17}$$

It is clear that $(x_t, y+t)$ is a homogeneous Markov process and we can consider probabilities $\tilde{P}_{xy}$ such that

$$\tilde{\mathbb{E}}_{xy}\big\{ g(x_t, y_t)\big\} = \Phi(t) g(x, y+t)$$

and write (5.16) as

$$w(x,y) = \tilde{\mathbb{E}}_{xy}\Big\{ \int_0^\infty \exp\Big(- \int_0^t \lambda(x_r, y_r)\mathrm{d}r \Big) \varphi(x_t, y_t)\mathrm{d}t \Big\}. \tag{5.18}$$

By the Markov property, (5.18) gives

$$w(x,y) = \tilde{\mathbb{E}}_{xy}\Big\{ \int_0^t \exp\Big(- \int_0^s \lambda(x_r, y_r)\mathrm{d}r \Big) \varphi(x_s, y_s)\mathrm{d}s \Big\} + \tilde{\mathbb{E}}_{xy}\Big\{ \exp\Big(- \int_0^t \lambda(x_r, y_r)\mathrm{d}r \Big) w(x_t, y_t) \Big\}. \tag{5.19}$$

From (5.19) one can check that the process

(5.20)
$$M_t = \int_0^t \exp\Big(- \int_0^s \lambda(x_r, y_r)\mathrm{d}r \Big) \varphi(x_s, y_s)\mathrm{d}s + \exp\Big(- \int_0^t \lambda(x_r, y_r)\mathrm{d}r \Big) w(x_t, y_t)$$

is a martingale for the probability $\tilde{P}_{xy}$.

Then we use the following lemma, which is a slight modification of Lemma 3.3 in Bensoussan and Lions [5, p. 354] so we skip its proof.

LEMMA 5.7. *Let ψ_t, ζ_t, and v_t be bounded adapted processes. If*

$$M_t = \int_0^t \psi_s \mathrm{d}s + \zeta_t \quad \forall t \geq 0$$

is a martingale, then the process

$$\rho_t = \zeta_t \exp\Big(\int_0^t v_s \mathrm{d}s \Big) + \int_0^t (\psi_s - v_s \zeta_s) \exp\Big(\int_0^s v_r \mathrm{d}r \Big) \mathrm{d}s$$

is also a martingale. □

Now, let us apply the previous lemma to

$$\zeta_t = \exp\Big(- \int_0^t \lambda(x_r, y_r)\mathrm{d}r \Big) w(x_t, y_t),$$

$$\psi_t = \lambda(x_t, y_t) \exp\Big(- \int_0^t \lambda(x_r, y_r)\mathrm{d}r \Big) \varphi(x_t, y_t)$$

with $v_t = \lambda(x_t, y_t) - \alpha$, $\alpha > 0$.

From (5.20) we deduce that

$$\rho(t) = \mathrm{e}^{-\alpha t} w(x_t, y_t) + \int_0^t \mathrm{e}^{-\alpha s} \big[\lambda(x_s, y_s)\varphi(x_s, y_s) + \big(\alpha - \lambda(x_s, y_s)\big) w(x_s, y_s)\big] \mathrm{d}s$$

is a martingale and, therefore,

$$w(x, y) = \tilde{\mathbb{E}}_{xy} \Big\{ \mathrm{e}^{-\alpha t} w(x_t, y_t) + \int_0^t \mathrm{e}^{-\alpha s} \big[\lambda(x_s, y_s)\varphi(x_s, y_s) + \big(\alpha - \lambda(x_s, y_s)\big) w(x_s, y_s)\big] \mathrm{d}s \Big\}$$

and

(5.21)
$$w(x, y) = \tilde{\mathbb{E}}_{xy} \Big\{ \int_0^{+\infty} \mathrm{e}^{-\alpha t} \big[f(x_t, y_t) - \mu_0 + \lambda(x_s, y_s)\big(w(x_t, 0) - w(x_t, y_t)\big) + \alpha w(x_t, y_t) - \lambda(x_s, y_s)\big(w(x_s, 0) - M w(x_t, 0)\big)^+ \big] \mathrm{d}t \Big\}$$

with the definition of φ in (5.17).

But since w is a bounded and continuous function, as well as f, the expression (5.21) gives the resolvent of process $(x_t, y + t)$, the generator of which is

$$A_x + \frac{\partial}{\partial y},$$

therefore, we can write

$$(5.22)\qquad -A_x w(x,y) - \frac{\partial w(x,y)}{\partial y} + \alpha w(x,y) = f(x,y) - \mu_0 + \alpha w(x,y) + \lambda(x,y)\big[w(x,0) - w(x,y)\big] - \lambda(x,y)\big[w(x,0) - Mw(x,0)\big]^+.$$

Rearranging the terms, and recalling the expression (2.4) for A_{xy}, we deduce (5.10). □

We can now state the following theorem.

THEOREM 5.8. *Under the assumptions of* section 2.1, *and* (5.1), *we have*

$$(5.23)\qquad \mu_0 = \inf_{\nu\in\mathcal{V}} J(x,y,\nu) = J(x,y,\hat{\nu}),$$

where μ_0 *is given by* Theorem 4.2 *and* J *is defined by* (2.13), *and* $\hat{\nu}$ *is defined as in* Corollary 5.3, *i.e.,* $\hat{\nu} = \{\hat{\theta}_i, \hat{\xi}_i : i \ge 1\}$ *is the admissible Markov impulse control corresponding to the stopping region* $S_0 = \{x \in E : w_0(x) = Mw_0(x)\}$ *and impulse function* $\xi_0(x) = \hat{\xi}(x)$, *which is a Borel measurable optimal selector of* $Mw_0(x)$*; see Definition* 2.3 *and assumption* (2.12).

Proof. From (5.22), we have

$$(5.24)\qquad -A_{xy} w(x,y) + \alpha w(x,y) = f(x,y) - \mu_0 + \alpha w(x,y) - \lambda(x,y)\big[w(x,0) - Mw(x,0)\big]^+$$

which implies, in particular, that

$$M_T^\alpha = \int_0^T \big(f(x_t,y_t) - \mu_0 + \alpha w(x_t,y_t)\big)\mathrm{e}^{-\alpha t}\mathrm{d}t + w(x_T,y_T)\mathrm{e}^{-\alpha T}$$

is a submartingale.

Since w is bounded, one can let $\alpha \to 0$ in M_T^α to deduce that

$$M_T = \int_0^T \big(f(x_t,y_t) - \mu_0\big)\mathrm{d}t + w(x_T,y_T)$$

is a submartingale.

For an arbitrary impulse control $\nu = \{(\theta_i,\xi_i) : i \ge 1\}$ in $\mathcal{V}$, we have

$$w(x,y) \le \mathbb{E}_{xy}^\nu \int_0^{\theta_1\wedge T} \big(f(x_t,y_t) - \mu_0\big)\mathrm{d}t + w(x_{\theta_1\wedge T}, y_{\theta_1\wedge T}),$$

i.e.,

$$w(x,y) \le \mathbb{E}_{xy}^\nu \int_0^T \big(f(x_t,y_t) - \mu_0\big)\mathrm{d}t - \mathbb{1}_{\theta_1\le T}\int_{\theta_1}^T \big(f(x_t,y_t) - \mu_0\big)\mathrm{d}t + \mathbb{1}_{\theta_1\le T} w(x_{\theta_1},0) + \mathbb{1}_{\theta_1 > T} w(x_T,y_T).$$

Moreover, note that even if we do not have, in general, $w \le Mw$, we do have

$$w(x_{\theta_1},0) \le c(x_{\theta_1},\xi_1) + w(\xi_1,0)$$

at the times of the impulses.

So we have

$$w(x,y) \le \mathbb{E}^{\nu}_{xy}\Big\{ \int_0^T \Big(f(x_t,y_t)-\mu_0\Big)\mathrm{d}t + \mathbb{1}_{\theta_1\le T}c(x_{\theta_1},\xi_1) + \mathbb{1}_{\theta_1\le T}w(\xi_1,0) - \mathbb{1}_{\theta_1\le T}\int_{\theta_1}^T \big(f(x_t,y_t)-\mu_0\big)\mathrm{d}t + w(x_T,y_T) - \mathbb{1}_{\theta_1\le T}w(x_T,y_T)\Big\}$$

and since (by the submartingale property)

$$\mathbb{E}^{\nu}_{xy}\Big\{\mathbb{1}_{\theta_1\le T}\Big[w(\xi_1,0) - \int_{\theta_1}^T \big(f(x_t,y_t)-\mu_0\big)\mathrm{d}t - w(x_T,y_T)\Big]\Big\} \le 0,$$

we obtain

$$w(x,y) \le \mathbb{E}^{\nu}_{xy}\Big\{ \int_0^T \big(f(x_t,y_t)-\mu_0\big)\mathrm{d}t + \mathbb{1}_{\theta_1\le T}c(x_{\theta_1},\xi_1) + w(x_T,y_T)\Big\};$$

iterating this argument, we deduce

$$w(x,y) \le \mathbb{E}^{\nu}_{xy}\Big\{ \int_0^T \big(f(x_t,y_t)-\mu_0\big)\mathrm{d}t + \sum_{i=1}^{\infty}\mathbb{1}_{\theta_i\le T}c(x^{i-1}_{\theta_i},\xi_i) + w(x_T,y_T)\Big\},$$

which implies that

$$\mu_0 \le J(x,y,\nu) \quad \forall \nu \in \mathcal{V}.$$

Next using the control $\hat{\nu}$ as defined in Corollary 5.3, we also have $\mu_0 = J(x,y,\hat{\nu})$, and equality (5.23) follows. ∎

6. Extensions. For instances, we reconsider our assumptions on the space E and on the signal process.

6.1. Locally compact. When E is locally compact, we take $C(E) = C_b(E)$, and we replace (2.1) by the assumption $\Phi(t)C_0 \subset C_0$, C_0 being the space of functions vanishing at infinity (from Palczewski and Stettner [32, Corollary 2.2], then $\Phi(t)C \subset C$ and $\lim_{t\to 0}\Phi(t)g(x) = g(x)$ uniformly on every compact of E).

The rest of the assumptions of sections 2.1 and 2.2 remain the same. Then there is no difficulty in extending the result of section 4, but the assumption (5.1) is no longer sufficient to extend the results of section 5. One possible additional assumption is to require that $h(x,0)$ be bounded. This is the case when the Poisson equation $-A_{xy}h = f - \bar{f}$ has a bounded solution (see Stettner [38] for conditions giving this property), but the extension of the results under more general assumptions would require further work. Note that in the particular case when λ is independent of x and f depends only on x, we can adapt the proof of Theorem 5.1 by using only $h(x)$ defined as the solution of the discrete time Poisson equation $-(P-I)h = \ell - \bar{\ell}$, which has a bounded solution under (2.6) with $\bar{\ell}$ being the integral of ℓ with respect to the invariant measure of P, which is (in this case) the same as the invariant measure of $\Phi(t)$.

Let us mention that the assumption (2.6) is also relatively restrictive when E is locally compact; actually, considering, for instance, a nondegenerate diffusion process in $\mathbb{R}^d$, one cannot assume (2.8). A less restrictive assumption is to replace (2.6) by

$$\begin{aligned} &\text{there exist a recurrent set } K\text{, a positive}\\ &\text{measure } m\text{, and } 0<\alpha<1 \text{ such that}\\ &P(x,B) \ge \alpha \mathbb{1}_K(x)m(B) \quad \forall B \in \mathcal{B}(E)\\ &\text{with } 0 < m(K) \le 1. \end{aligned} \tag{6.1}$$

As shown in Höpfner and Löcherbach [18], if, for instance, x_t is a diffusion process which is Harris recurrent, then, in particular, there exists K, α', m as in (6.1) such that

$$k_1 \int_0^\infty \mathrm{e}^{-k_1 t} \big(\Phi(t) \mathbb{1}_B\big)(x) \mathrm{d}t \geq \alpha' \mathbb{1}_K(x) m(B),$$

from which we deduce

$$P(x, B) \geq \frac{k_0}{k_1} \alpha' \mathbb{1}_K(x) m(B);$$

this is (6.1) with $\alpha = k_0 \alpha'/k_1$.

Note that, for example, nondegenerate diffusions for which there exists an adequate Liapunov function are Harris recurrent; see Löecherbach [26] for details. Now, to obtain a solution of (4.9), one can adapt the results of Luque-Vásquez and Hernández-Lerma [27], which has an ergodicity assumption of the type (6.1). Then, the results of section 5 can be extended with (6.1), at least when $h(x, 0)$ is bounded.

6.2. Infinite dimension. If $\{x_t : t \geq 0\}$ takes values in an infinite dimensional space, then the prototype could be given by a stochastic partial differential equation, where E is a Banach or Hilbert space (e.g., Menaldi and Sritharan [31]). Several techniques are available to treat optimal stopping and impulse control problems in this context (e.g., see [28], Priola [34], and the discussion in [29, section 5.1.2]). Moreover, see the book by Da Prato and Zabczyk [9] for some results of ergodicity in infinite dimensions.

Actually, this includes a weak C_b-semigroup, but calculations are harder, even under suitable assumptions. However, further works are needed to fully analyze and to really include infinite dimensions.

6.3. Other signal processes. The signal admit several generalizations, e.g., sticky signal, i.e., when the process y_t may remain for a positive time at 0 so that the controller has a positive continuous-time interval where impulses can be applied. Indeed, this can regarded as signals "on/off" by simply assuming that the process y_t takes values in $\mathbb{R}$ and only while $y \leq 0$ are impulses allowed or admissible.

Even more general is the case where the process giving the signals is a semi-Markov process (which cover both the independent and identically distributed case and the pure jump Markov processes) conditioned to the initial Markov process x_t. Assume that $\{y_t^1 : t \geq 0\}$ is a semi-Markov process with values in a space E_1 (with the discrete topology and the Borel σ-algebra) and $\{y_t = (y_t^1, y_t^2) : t \geq 0\}$ is the appropriated Markov process where $\{y_t^2 : t \geq 0\}$ is the elapsed time since the last jump of $\{y_t^1 : t \geq 0\}$; e.g., see Davis [10, Appendix, pp. 256–279], Gikhman and Skorokhod [15, section III.3, pp. 226–249], Jacod [19], and Robin [35], among others. If we are given $\lambda : E_1 \times [0, \infty[\longrightarrow [0, \infty[$ satisfying

$$0 \leq \lambda(x, y^1, y^2) \leq M, \qquad (x, y_1, y_2) \mapsto \lambda(y^1, y^2) \quad \text{continuous}$$

and a transition probability $q(x, y^1, y^2, \Gamma)$ with

$$(x, y^2) \mapsto \int_{E_1} q(x, y^1, y^2, \mathrm{d}z) \varphi(z) \quad \text{continuous},$$

for every φ bounded measurable on E_1, one can show that $\{y_t : t \geq 0\}$ can be constructed as a Markov process with infinitesimal generator (for a given x in E)

$$A_y(x) f(y^1, y^2) = \frac{\partial f(y^1, y^2)}{\partial y^2} + \lambda(x, y^1, y^2) \Big[\int_{E_1} q(x, y^1, y^2, \mathrm{d}z) f(z, 0) - f(y^1, y^2) \Big],$$

and $A_{xy} = A_x + A_y$ is the infinitesimal generator of the Markov–Feller process (x_t, y_t). In this case, a given (proper or not) subset D of the space E_1 determines whether or not impulses are allowed. Calculations are complicated, but perhaps not insurmountable. These types of models are particular cases of general hybrid control models, which are considered in more details in a coming book by Jasso-Fuentes, Menaldi, and Robin [20].

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