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ON SOME IMPULSE CONTROL PROBLEMS WITH CONSTRAINT*

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Abstract. The impulse control of a Markov–Feller process is considered when the impulses are allowed only when a signal arrives. This is referred to as an impulse control problem with constraint. A detailed setting is described, a characterization of the optimal cost is obtained using previous results of the authors on optimal stopping problems with constraint, and an optimal impulse control is identified.

Key words. Markov–Feller processes, information constraints, control by interventions

AMS subject classifications. Primary, 49J40; Secondary, 60J60, 60J75

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1. Introduction.

Impulse control problems were introduced by Bensoussan and Lions [3] in the 1970s, and since then, a considerable literature has been devoted to many aspects and applications of these control problems (e.g., see references in the books by Bensoussan and Lions [4, 5], Bensoussan [2]).

Typically, the state without control follows a stochastic differential equation (SDE) with continuous trajectories, and the control ν is an increasing sequence $\{\vartheta_i\}$ of (stopping) times and a sequence of random variables $\{\xi_i\}$. At time ϑ_i, the control transfers the state immediately from x_{ϑ_i} to ξ_i, with a positive cost per impulse given by $c(x_{\vartheta_i}, \xi_i) > 0$. In between two consecutive (impulse or intervention) times ϑ_i and ϑ_{i+1}, the evolution behaves like the initial SDE, and a nonnegative cost per unit of time $f(x_t) \geq 0$ is applied. Thus, $J_x(\nu)$ denotes the total α-discounted cost on $[0, \infty[$, and the optimal cost function is given by $v(x) = \inf_\nu J_x(\nu)$. Switching control and control by interventions are similar/equivalent problems; e.g., see references in the books by Bensoussan [1, 2], Davis [6], among others.

Usually, the random variable ξ_i should belong to a particular set of "admissible" states (for the current state x_{ϑ_i}), but the (intervention) time ϑ_i is as arbitrary as possible. In our problem presented below, the instants $\{\vartheta_i\}$ are required to satisfy a restriction referred to as "wait for a signal," e.g., an intervention is allowed only at the jump times, an exogenous process, the simplest case being the Poisson process. Over this class of impulse controls, the cost $J_x(\nu)$ is minimized and all previous arguments (of the dynamic programming) should be adapted to this new model.

A key tool to solve an impulse control problem is to regard it as a sequence of optimal stopping time problems. A considerable literature exits on optimal stopping time problems (e.g., see the recent book by Peskir and Shiryaev [19]). However, we have seen an explicit constraint on stopping time problems (i.e., when the process can be stopped) for the first time in Dupuis and Wang [7], and this was extended in several directions in [17], where we have studied a class of optimal stopping problems with a similar (to those mentioned above) type of constraint. For the impulse control with constraint, we obtain a Bellman equation which is not a quasi-variational inequality

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as in the standard case and, an auxiliary impulse control problem, which is essentially in discrete time, is instrumental to solve the initial problem

To the best of our knowledge, the impulse control with constraint in the sense described above has been addressed only in Liang [13], Liang and Wei [14], Wang [22], for diffusion processes and Poisson constraint. In this paper, the settings are such that the initial process $\{x_t : t \geq 0\}$ is a general Markov–Feller process and the times allowed for impulses are the jumps of a (non-necessarily-Poisson) process $\{y_t : t \geq 0\}$, which is not necessarily independent from $\{x_t : t \geq 0\}$.

The paper is organized as follows: section 2 is devoted to the statement of the problem (with notation and assumptions), section 3 addresses the application of the dynamic programming arguments. Next, the Hamilton-Jacobi-Bellman (HJB) equation is solved in section 4 and an optimal control is constructed. Several extensions of the results are discussed in section 5.

2. Statement of the problem. The state (with the exception for the signal) of the dynamical system is a time-homogeneous (right-continuous, left limited) Markov process $\{x_t : t \in [0, \infty[\}$ in a compact metric space E with transition probability function $p(x, t, B)$, i.e.,

$$P\{x_t \in B \,|\, x_s = x\} = p(x, t-s, B) \quad \forall x \in E, \ t > s \geq 0, \ B \in \mathcal{B}(E), \tag{2.1}$$

where $\mathcal{B}(E)$ is the Borel σ-algebra. It is also assumed that $\{x_t : t \geq 0\}$ is a Feller process, i.e., if $\{\Phi(t) : t \geq 0\}$ is its semigroup and $C(E)$ denotes the Banach space (with the sup-norm $\|\cdot\|$) of real-valued continuous functions defined on E then

$$\{\Phi(t) : t \geq 0\} \text{ is a continuous semigroup on } C(E). \tag{2.2}$$

Moreover $A = A_x$ is its infinitesimal generator. This Markov–Feller process is *realized* in a (canonical) filtered probability space $(\Omega, \mathbb{F}, P)$.

An impulse (or intervention) is the action on the evolution of the dynamical system (e.g., at time ϑ) to provoke an instantaneous transition from the state x_ϑ into ξ with ξ in $\Gamma(x_\vartheta)$, a closed subset of E. Actually, to simplify assumptions, it is better to suppose that

$$\Gamma(x) = \Gamma \text{ fixed for every } x \in E \text{ with } \emptyset \neq \Gamma \subset E \text{ closed.} \tag{2.3}$$

This is a sequential-type control, and between two consecutive interventions, the transition probability function (2.1) governs the evolution of the system.

Thus, an arbitrary impulse control is a double sequence $\nu = \{(\vartheta_i, \xi_i) : i \geq 1\}$, where $0 \leq \vartheta_1 \leq \vartheta_2 \leq \cdots$ is an increasing sequence of stopping times satisfying $\vartheta_i \to \infty$, almost surely, and $\{\xi_i : i \geq 1\}$ is a sequence of Γ-valued random variables, such that ξ_i is ϑ_i-adapted.[1] This impulse is implemented only when the time of intervention (or impulse) ϑ_i is finite. Note that, in general, the construction of a suitable probability space, where these impulse controls are "realized" is a hard problem (e.g., see Lepeltier and Marchal [12], Robin [20], among others). Hence, later in this section, we give a quick intuitive idea about this construction, without the complete details.

A process $\{y_t : t \geq 0\}$ with values in the interval $[0, \infty[$ (not necessarily a Markov process by itself) represents the *time elapsed since the last signal*, and, a "signal" arrives at the hitting time of the singleton $\{y = 0\}$. Assume that the couple

[1] In other words, if ϑ_i is relative to the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ then ξ_i is $\mathcal{F}_{\vartheta_i}$-measurable.

$\{(x_t, y_t) : t \geq 0\}$ is a time-homogeneous strong Markov process in $E \times [0, \infty[$, with infinitesimal generator

$$A_{x,y}\varphi(x,y) = A_x\varphi(x,y) + \partial_y\varphi(x,y) + \lambda(x,y)[\varphi(x,0) - \varphi(x,y)], \quad y \geq 0, \tag{2.4}$$

where ∂_y is the partial derivative with respect to y and the intensity function (or jump rate) satisfies

$$\lambda : E \times [0, \infty[\longrightarrow [0, \infty[\text{ is bounded and continuous.} \tag{2.5}$$

It is simple to check that the (complete) state process $\{(x_t, y_t) : t \geq 0\}$ becomes a Markov–Feller process.

Besides a discount factor $\alpha > 0$, there is a running cost (or the cost per unit of time) given by a continuous and bounded function $f \geq 0$, i.e.,

$$\alpha > 0 \quad \text{and} \quad f \colon E \times [0, \infty[\longrightarrow [0, \infty[, \quad f \text{ is continuous,} \tag{2.6}$$

and another function

$$c \colon E \times \Gamma \longrightarrow [c_0, \infty[, \quad c \text{ is continuous and } c_0 > 0, \tag{2.7}$$

representing a cost per impulse. Thus, the cost of an arbitrary impulse control $\nu = \{(\vartheta_i, \xi) : i \geq 1\}$ is given by

$$J_{x,y}(\nu) = \mathbb{E}^{\nu}_{x,y}\left\{\int_0^{\infty} \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t + \sum_{i=0}^{\infty} \mathrm{e}^{-\alpha\vartheta_i} c(x_{\vartheta_i}, \xi_i)\right\}, \tag{2.8}$$

where $\mathbb{E}^{\nu}_{x}$ is the expectation of the process under the impulse control ν with initial conditions $(x_0, y_0) = (x, y)$, and x_{ϑ_i} is the value of the process just before the impulse. But, in our model, not all interventions are permitted; indeed, an intervention at a time $\vartheta \geq 0$ is authorized only when $y_\vartheta = 0$. Hence, an impulse control $\nu = \{(\vartheta_i, \xi_i) : i \geq 1\}$ is called *zero admissible* if $y_{\vartheta_i} = 0$, almost surely, for any $i \geq 1$; while, it is called *admissible* if also the first intervention is strictly positive, i.e., $\vartheta_1 > 0$, almost surely.

Denote by $\mathcal{V}$ (or $\mathcal{V}_0$) the set of admissible (or zero-admissible) impulse controls, all relative to the initial condition $(x_0, y_0) = (x, y)$. Therefore, the optimal cost is defined by

$$v(x,y) = \inf\left\{J_{x,y}(\nu) : \nu \in \mathcal{V}\right\} \quad \forall (x,y) \in E \times [0, \infty[, \tag{2.9}$$

and its associated auxiliary impulse control problem (referred to as the "time-homogeneous" impulse control) has the optimal cost given by

$$v_0(x,y) = \inf\left\{J_{x,y}(\nu) : \nu \in \mathcal{V}_0\right\} \quad \forall (x,y) \in E \times [0, \infty[. \tag{2.10}$$

Actually, the optimal cost $v_0(x, y)$ will be of any use only for $y = 0$.

The aim is to give a characterization of the optimal cost $v(x, y)$ and to construct an optimal (admissible, feedback) impulse control $\hat{\nu}$. The statement of the problem to be solved was presented above, but several details (and specifications on the model) necessary to fully understand the above model are discussed below.

Note that even if the setting is Markovian, the impulse control problem with optimal cost (2.9) is not homogeneous in time, i.e., the controller should wait for a signal before applying the initial impulse time, and so the time variable t should be included in the analysis, i.e., a cost $v(x, y, t)$ should be defined. However, the auxiliary impulse control problem with optimal cost $v_0(x, y)$ given by (2.10) is a Markovian time-homogeneous (an "almost usual") impulse control problem (except for the constraint of intervening only when the second component of the state vanishes, i.e., when $y = 0$).

2.1. Time elapsed since last signal. In setting up the constraint for a stopping time (or impulse control) problem relative to an initial time-homogeneous strong Markov process $\{x_t : t \geq 0\}$ (i.e., the uncontrolled state of the system), we assume that a stochastic process $\{y_t : t \geq 0\}$ representing the time elapsed since last signal is either given or constructed from an exogenous sequence $\{T_1, T_2, \ldots\}$ of nonnegative independent and identically distributed (IID) random variables with distribution π_0.

If the IID sequence is given a priori then a realization of the stochastic process $\{y_t : t \geq 0\}$ can be defined by induction for an initial condition $y_0 = y$ as follows:

(1) first get a nonnegative random variable T^y independent of $\{T_1, T_2, \ldots\}$ and of the Markov process $\{x_t : t \geq 0\}$ with distribution

$$P\{T^y \in]a, b]\} = P\{T_1 \in]a + y, b + y] \,|\, T_1 \geq y\} = \frac{\pi_0(]a + y, b + y])}{\pi_0(]y, +\infty[)} \tag{2.11}$$

for any $b > a \geq 0$;

(2) and now, define the sequence of signals

$$\tau_0^y = 0, \quad \tau_1^y = T^y, \quad \text{and} \quad \tau_{n+1}^y = \tau_n^y + T_n \quad \forall n \geq 1, \tag{2.12}$$

and the process $\{y_t : t \geq 0\}$ with $y_0 = y$ by the expressions

$$y_t = y_{\tau_{n-1}^y} + t - \tau_{n-1}^y \;\text{ if }\; \tau_{n-1}^y \leq t < \tau_n^y \;\text{ and }\; y_{\tau_n^y} = 0 \;\; \forall n \geq 1. \tag{2.13}$$

For simplicity, assume that the π_0 is supported on the whole $[0, \infty[$ and, so, any non-negative initial values $y_0 = y$ are valid.

In this case $\{y_t : t \geq 0\}$ (by itself) is a time-homogeneous strong Markov with values in $[0, \infty[$. The jumps of the process $\{y_t : t \geq 0\}$ are better understood with an intensity function $\lambda(\cdot)$ of the random variables T_k. Therefore, instead of referring to the common law π_0, it is convenient to assume that a bounded Borel measurable intensity function $y \mapsto \lambda(y)$ exists.

Actually, if the sequence $\{T_1, T_2, \ldots\}$ is only conditionally independent with respect to $\{x_t : t \geq 0\}$ then the (conditional) intensity may also be depending on the variable x, i.e., $\lambda(x, y)$ as in (2.5), and the above construction can be adapted to this situation. The couple $t \mapsto (x_t, y_t)$ is an $E \times [0, \infty[$-valued cad-lag process and, since the jumps have an intensity, we deduce that $P\{\tau_n = t\} = 0$ for every $t \geq 0$ and $n \geq 1$, and therefore, $P\{y_{t-} = y_t\} = 0$ for every $t > 0$. Note that $y_{\tau_n -} = \tau_n - \tau_{n-1} = T_n$ is the arrival time of the n-signal measured from the previous $(n-1)$-signal.

Summing up, in our stopping time (or impulse control) problem with constraint, the couple $\{(x_t, y_t) : t \geq 0\}$ and $\{x_t : t \geq 0\}$ are both time-homogeneous strong Markov processes, but $\{y_t : t \geq 0\}$ alone is not necessarily a Markov process by itself. By taking the image probability, this construction can be moved to the canonical space $D = D([0, \infty[; E \times [0, \infty[)$ under a transition probability $P_{x,y}$, with infinitesimal generator given by (2.4), and the times of jumps (to zero) of the second variable $t \mapsto y_t$ have intensity $t \mapsto \lambda(x_t, y_t)$. The recurrence formula

$$\tau_0 = 0 \quad \text{and} \quad \tau_n = \inf\bigl\{t > \tau_{n-1} : y_t = 0\bigr\}, \quad n \geq 1, \tag{2.14}$$

defines the sequence $\{\tau_n : n = 1, 2, \dots\}$ signals, as a functional of y_t with the initial condition $y_0 = y$.

Furthermore, since $\{x_t : t \geq 0\}$ is a Markov–Feller process, assumption (2.5) implies that the same holds true for the couple, i.e., also

$$(x, y, t) \longmapsto \mathbb{E}_x\{\varphi(x_t, y_t)\} \quad \text{is continuous on } \; E \times [0, \infty[\times[0, \infty[\tag{2.15}$$

for every bounded continuous function φ. This yields

$$\begin{aligned} &(x, y) \mapsto \mathbb{E}_{x,y}\{\mathrm{e}^{-\alpha\tau}\varphi(x_\tau)\} \quad \text{is continuous,} \\ &\quad \tau = \inf\{t > 0 : y_t = 0\}, \quad \alpha \geq 0, \end{aligned} \tag{2.16}$$

for every continuous and bounded function φ.

On the other hand, once the expression of the infinitesimal generator $A_{x,y}$ is known to be given by (2.4), the construction of the corresponding time-homogeneous strong Markov process $\{(x_t, y_t) : t \geq 0\}$ follows, provided $A_{x,y}$ is proved to satisfy sufficient conditions; several results exist in this direction. Actually, in the case of signals given via an IID sequence independent of the $\{x_t : t \geq 0\}$, the expression (2.4) of the joint infinitesimal generator suffices to guarantee the construction of $\{(x_t, y_t) : t \geq 0\}$ without explicitly defining the $\{y_t : t \geq 0\}$, even if its construction itself seems interesting.

2.2. Interventions and costs. Let us first present a quick intuitive idea on the construction of the impulse control in a filtered probability space, without complete details; the reader is referred to Bensoussan and Lions [1, Chapter 6, section 4.2], Davis [6, Chapter 5], Lepeltier and Marchal [12], Robin [20], among others.

For the sake of simplicity, temporarily forget about the constraint, use z or (x, y) indistinctly, and let us describe the actions or steps of interventions on a Markov process $XY = \{z_t = (x_t, y_t) : t \geq 0\}$ with values in (the Polish space) $E \times [0, \infty[$, and transition probability $p(z, t, \cdot)$.

Let us recall that a realization of the time-homogeneous Markov process XY is a family $\{P_z : z \in E \times [0, \infty[\}$ (on which all further constructions are based) of time-invariant probabilities on the canonical space $D = D([0, \infty[, E \times [0, \infty[)$ satisfying $P_z\{z_t \in \cdot \mid z_0 = z\} = p(z, t, \cdot)$, where $z_t(\omega) = \omega(t)$ is the canonical process, and the canonical filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$, $\mathcal{F}_t = \sigma(z_s : 0 \leq s \leq t)$, has been modified to satisfy the usual conditions.

If θ is a finite $\mathbb{F}$-stopping time and ζ is a $\mathcal{F}_\theta$-measurable random variable (with values in $E \times [0, \infty[$) then an initial stochastic condition $z_\theta = \zeta$ is obtained in two steps: (i) a family of probability $P_{s,z}$ on $(D, \mathcal{F}^s_\infty)$ satisfying $P_{s,z}\{z_t \in \cdot\} = p(z, t-s, \cdot)$, and (ii) the substitution $z = \zeta(\omega)$, $s = \theta(\omega)$ into $P_{s,z}$ provides a "regular conditional probability" given $z_\theta = \zeta$, which is only defined on the σ-algebra $\mathcal{F}^\theta_\infty$.

In short, (a) $\mathbb{E}_z$ or P_z is the expectation or probability (defined on $\{\mathcal{F}_t : t \geq 0\}$) with the initial (deterministic) condition $z_0 = z$, and (b) $\mathbb{E}_{\zeta,\theta}$ or $P_{\theta,\zeta}$ is the conditional expectation or a (regular) probability (i.e., first $P_{s,z}$ is constructed for deterministic values and then the substitution $s = \theta(\omega)$, $z = \zeta(\omega)$ is used) with the initial (stochastic) condition $z_\theta = \zeta$ (defined on $\{\mathcal{F}_t : t \geq \theta\}$), all this within the canonical space $D([0, \infty[, E \times [0, \infty[)$, the universally completed filtration $\mathbb{F}$, and the canonical process $(t, \omega) \mapsto z_t = \omega(t)$. It is worth mentioning that both probabilities P_z and $Q_\zeta = P_z(\mathrm{d}\omega)P_{\zeta(\omega),\theta(\omega)}(\cdot)$ are defined on the (universally completed) Borel σ-algebra $\mathcal{F}_\infty$ of the canonical space D, even if P_z and $Q_\zeta = \mathbb{E}_z P_{\theta,\zeta}$ are used on the stochastic interval $[0, \theta]$ and $[\theta, \infty[$, respectively (the notation $\mathbb{E}_z P_{\theta,\zeta}$, instead of

$Q_\zeta = P_z(\mathrm{d}\omega)P_{\theta(\omega),\zeta(\omega)}(\cdot)$, is meant to emphasize the point that Q_ζ is used only for events after θ). Upon some details, it should be clear that these two probabilities P_z and Q_ζ cannot be jointed together in only one probability on D; there is an "imposed" discontinuity at $t = \theta$.

To define the cost associated with an impulse control, we begin with a sequence $\{\vartheta_i : i \geq 1\}$ of $\mathbb{F}$-stopping times satisfying $\vartheta_1 \leq \vartheta_2 \leq \cdots \leq \vartheta_i \leq \cdots$ and $\vartheta_i \to \infty$, and a sequence $\{\zeta_i : i \geq 1\}$ of random variables such that ζ_i is $\mathcal{F}_{\theta_i}$-measurable, which form an impulse control. Therefore, a sequence of probabilities $\{P_z^{\nu|\vartheta_i} : i \geq 0\}$ can be associated with each impulse control and any initial (deterministic) condition $z_0 = z$ as follows:

Define $P_z^{\nu|\vartheta_0} = P_z$, $\vartheta_0 = 0$, and if $P_z^{\nu|\vartheta_{k-1}}$ (or $\mathbb{E}_z^{\nu|\vartheta_{k-1}}$) has been given then define $P_z^{\nu|\vartheta_k}$ (or $\mathbb{E}_z^{\nu|\vartheta_k}$) on Ω by recurrence,

$$P_z^{\nu|\vartheta_k}(A) = \mathbb{E}_z^{\nu|\vartheta_{k-1}}\big\{P_{\vartheta_k,\zeta_k}\big(A\big)\big\} \quad \forall A \in \mathcal{F}_\infty^{\vartheta_k}, \tag{2.17}$$

i.e., for events after ϑ_k, after k-impulses.

Remark that the law of the canonical process $\{z_t : t \geq 0\}$ changes from $P_z^{\nu|\vartheta_{k-1}}$ to $P_z^{\nu|\vartheta_k}$, e.g., $P_z^{\nu|\vartheta_k}\{z_{\vartheta_k} = \zeta_k\} = 1$ and

$$P_z^{\nu|\vartheta_{k-1}}\{z_{\vartheta_k} \in B \,|\, \mathcal{F}_{\vartheta_{k-1}}\} = p(\zeta_{k-1}, \vartheta_k - \vartheta_{k-1}, B),$$

where $p(z,t,\cdot)$ is the transition probability of the process $Z = XY$.

In our case, only the x-component is affected (i.e., $\zeta_i = (\xi_i, y(\vartheta_i))$ with ξ_i taking values in Γ) and thus, $\nu = \{(\vartheta_i, \xi_i) : i \geq 1\}$ represents an impulse control in this model, the family of probabilities is denoted by $\{P_{x,y}^{\nu|\vartheta_k} : k \geq 0\}$. Note that the canonical process is $\{z_t = (x_t, y_t) : t \geq 0\}$, and $P_{\zeta_k} = P_{\xi_k, y_{\vartheta_k}}$, which means $P_{x,y}$ after the substitution $x = \xi_k(\omega)$ and $y = y_{\vartheta_k(\omega)}(\omega)$. Therefore, within this construction, we have the following.

Definition 2.1. *If $\{\tau_n\}$ is the sequence of signals* (2.14) *then an $\mathbb{F}$-stopping time ϑ is called "admissible" if for almost surely every ω there exists $n = \eta(\omega) \geq 1$ such that $\vartheta(\omega) = \tau_{\eta(\omega)}(\omega)$ or, equivalently, if ϑ almost surely satisfies $\vartheta > 0$ and $y_\vartheta = 0$. Thus, an impulse control $\nu = \{(\vartheta_i, \xi_i) : i \geq 1\}$ is called admissible if each impulse time ϑ_i is an admissible $\mathbb{F}$-stopping time, i.e., $y_{\vartheta_i} = 0$ for every $i \geq 1$, and $\vartheta_1 > 0$, and ξ_i is $\mathcal{F}_{\vartheta_i}$-measurable. If, in addition, $\vartheta_1 = 0$ is allowed then ν is called "zero admissible."* □

Thus, the controller chooses an $\mathbb{F}$-stopping time ϑ_1 and an $\mathcal{F}_{\vartheta_1}$-measurable random variable with values in Γ, so that

$$J_z(\nu|\vartheta_1) = \mathbb{E}_{x,y}^{\nu|\vartheta_0}\left\{\int_0^{\vartheta_1} f(z_t)\mathrm{e}^{-\alpha t}\mathrm{d}t + \mathrm{e}^{-\alpha\vartheta_1}c(x_{\vartheta_1}, \xi_1)\right\}, \quad \vartheta_0 = 0,$$

represents the cost (of interventions) up to the time ϑ_1. The time of intervention is $t = \vartheta_1$ when $\vartheta_1 < \infty$ and no intervention at all when $\vartheta_1 = \infty$. Remark that if $\vartheta_1 = \infty$ then $\mathrm{e}^{-\alpha\vartheta_1}c(x_{\vartheta_1}, \xi_1) = 0$. As a second decision, the controller chooses an $\mathbb{F}$-stopping time ϑ_2 and an $\mathcal{F}_{\vartheta_2}$-measurable random variable ξ_2 with values in Γ, so that

$$J_z(\nu|\vartheta_2) = J_z(\nu|\vartheta_1) + \mathbb{E}_{x,y}^{\nu|\vartheta_1}\bigg\{\int_{\vartheta_1}^{\vartheta_2} f(z_t)\mathrm{e}^{-\alpha t}\mathrm{d}t + \mathrm{e}^{-\alpha\vartheta_2}c(x_{\vartheta_2}, \xi_2)\bigg\}$$

represents the cost (of interventions) up to the time ϑ_2. The time of the second intervention is $t = \vartheta_2$ when $\vartheta_2 < \infty$ and there is no second intervention at all when

$\theta_2 = \infty$. Iterating this procedure, the controller chooses an $\mathbb{F}$-stopping time θ_{k+1} and an $\mathcal{F}_{\theta_{k+1}}$-measurable random variable ξ_{k+1} with values in Γ, so that

$$J_z(\nu|\vartheta_{k+1}) = J_z(\nu|\vartheta_k) + \mathbb{E}^{\nu|\vartheta_k}_{x,y}\left\{\int_{\vartheta_k}^{\vartheta_{k+1}} f(z_t)\mathrm{e}^{-\alpha t}\mathrm{d}t + \mathrm{e}^{-\alpha\vartheta_{k+1}} c(x_{\vartheta_{k+1}}, \xi_{k+1})\right\}, \tag{2.18}$$

represents the cost (of interventions) up to the time ϑ_{k+1}, and the limit $J_z(\nu) = \lim_k J_z(\nu|\vartheta_{k+1})$ is the cost corresponding to the impulse control ν. Note that (2.18) makes sense for $k \geq 0$, after setting $\vartheta_0 = 0$ and $J_z(\nu|\vartheta_0) = 0$. Recall that the expectation $\mathbb{E}^{\nu|\vartheta_k}_{x,y}$ is defined only for events after ϑ_k, and both terms (the integral in t and the other one) are zero on the event $[\vartheta_{k-1} = \infty]$.

Since interventions are allowed only at the time a signal arrives, our impulse control problem with constraint could be considered in a discrete-time setting as follows. Indeed, a realization of the Markov process XY yields also a realization of the Markov chain $\{(x_{\tau_n}, y_{\tau_n}, \tau_n) : n \geq 0\}$ with a filtration $\mathbb{G} = \{\mathcal{G}_n : n \geq 0\}$, $\mathcal{G}_n = \mathcal{F}_{\tau_n}$. This Markov chain takes values in $E \times [0,\infty[\times[0,\infty[$, and τ_n represents the continuous time. Since $y_{\tau_n} = 0$ for every $n \geq 1$, it is convenient to replace y_{τ_n} with y_{τ_n-}, which represents the arrival time of the n signal (measured from the $(n-1)$ signal). The previous construction produces a discrete-time model as in Bensoussan [2] (or Hernández-Lerma and Lasserre [9]), where a Markov chain is controlled by impulses. The discrete-type (or discrete in short) impulses occur at stopping times η_k with values in $\mathbb{N} = \{1, 2, \ldots\}$ (or in $\mathbb{N} = \{0, 1, 2, \ldots\}$ if needed). The relations are $\theta_k = \tau^{k-1}_{\eta_k}$ and $\vartheta_k = \tau^{k-1}_{\eta_k} + \vartheta_{k-1}$ with $\vartheta_1 = \theta_1$. If a translation in time is added then the intermediate variables θ_k are bypassed, and if η_k denotes the translated stopping times, then the relation $\vartheta_k = \tau_{\eta_k}$ is deduced. The sequence of costs is rewritten as $J_{x,y}(\nu|\eta_k)$ instead of $J_{x,y}(\nu|\vartheta_k)$. Remark that there is a one-to-one correspondence between admissible impulse controls $\{(\vartheta_i, \xi_i) : i \geq 1\}$ and admissible discrete impulse controls $\{(\eta_i, \xi_i) : i \geq 1\}$, $\eta_i = \inf\{k \geq 1 : \vartheta_k = \tau_k\}$, and similarly, between zero-admissible impulse controls and all discrete impulse controls. Therefore, with this construction, we have the following.

Definition 2.2. *If $\{\eta_i : i \geq 1\}$ is a sequence of $\mathbb{G}$-stopping times with values in $\mathbb{N} = \{0, 1, 2, \ldots\}$, such that $0 \leq \eta_1 \leq \eta_2 \leq \cdots$ and $\eta_n \to \infty$, almost surely, and $\{\xi_i : i \geq 1\}$ is another sequence of random variables with values in Γ, such that ξ_i is $\mathcal{G}_{\eta_i}$-adapted, then the double sequence $\eta = \{(\eta_i, \xi_i) : i \geq 1)\}$ is referred to as a discrete-time impulse control, which is called admissible when $\eta_1 \geq 1$.* ☐

In general, it is convenient to construct (in some infinite product copy space) a probability $P^\nu_{x,y}$ and a sequence cad-lag processes $\{z^{i,\nu}_t : t \geq 0,\ i = 0, 1, \ldots\}$ to write the cost as

$$J_{x,y}(\nu) = \mathbb{E}^\nu_{x,y}\left\{\int_0^\infty \mathrm{e}^{-\alpha t} f(x^\nu_t, y^\nu_t)\mathrm{d}t + \sum_{i=1}^\infty \mathrm{e}^{-\alpha\vartheta_i} c(x^{i-1,\nu}_{\vartheta_i}, \xi_i)\right\}. \tag{2.19}$$

Note that for diffusion with jumps, this construction is made in the canonical space, without any infinite product copy space.

3. Dynamic programming (DP). The impulse control problem has been defined above, but some more details are necessary before applying the DP principle. As discussed earlier, two models are presented with state (x, y) and time t:

(a) The initial control problem with optimal cost $v(x, y)$ given by (2.9), where the constraint "wait to intervene until a signal arrives" has been translated as "intervene only when $y = 0$ and $t > 0$."

(b) The auxiliary problem with optimal cost $v_0(x, y)$ given by (2.10), where the constraint wait to intervene until a signal arrives has been translated as "intervene only when $y = 0$."

In this section, we will use $f(x)$ instead of $f(x, y)$ to shorten the writing, and the expression (2.19) for the cost function. First let us comment on these two descriptions:

(1) *Time homogeneous.* The second model (b) is homogeneous in time, but the first one is not. This means that to properly use the DP arguments, the cost $v(x, y)$ should include the time variable as part of the state, i.e., to define a cost $v(x, y, s)$, where the evolution begins at time $t = s$, with the constraint as in model (a), i.e., $v(x, y, 0) = v(x, y)$ and for $s > 0$,

$$J_{x,y,s}(\nu) = \mathbb{E}^{\nu}_{x,y}\left\{\int_0^{\infty} \mathrm{e}^{-\alpha(t-s)} f(x^{\nu}_{s+t})\mathrm{d}t + \sum_i \mathrm{e}^{-\alpha(\vartheta_i - s)} c(x^{i-1,\nu}_{s+\vartheta_i}, \xi_i)\right\},$$
$$v(x, y, s) = \inf\big\{J_{x,y,s}(\nu) : \nu \text{ any admissible impulse control}\big\},$$

where now admissible means "intervene only when $y = 0$ and $s \neq 0$." Since all data are time homogeneous, the equality

$$v(x, y, s) = \mathrm{e}^{-\alpha s} v(x, y) \quad \forall x \in E \ \ \forall y, s \geq 0, \tag{3.1}$$

holds true.

(2) *Multiple impulses.* Another point to clarify is the possibility of making several impulses at the time, i.e., if "intervene" means "stop and restart" the dynamic of the system then, upon arrival of a signal, the controller may stop and restart multiple times (say n times, each impulse from x_i to $\xi_i = x_{i+1}$, $i = 1, \ldots, n$, with a cost $c(x_i, \xi_i)$, where the state begins at $x_1 = x$ and ends at x_{n+1}). If no other assumption is made, multiple impulses are not excluded from the optimal decision. For the sake of simplicity, we assume in the following that

$$c(x, \xi_1) + c(\xi_1, \xi_2) \geq c(x, \xi_2) \quad \forall x \in E, \ \xi_1, \xi_2 \in \Gamma. \tag{3.2}$$

The expression

$$\varphi(x) \mapsto (M\varphi)(x) = \inf_{\xi \in \Gamma}\big\{\varphi(\xi) + c(x, \xi)\big\} \tag{3.3}$$

defines the impulse operator M.

Now, by comparing the constraints in models (a) and (b), it is clear that there is no difference when the state variable $y > 0$, and there are possible impulses at the initial time when $y = 0$, i.e., any control in (b) can be expressed as an initial impulse followed by a control in (a). Therefore, directly from the definitions, (2.9) and (2.10) of the optimal costs follow the relations

$$\begin{aligned} &v(x, y) \geq v_0(x, y) \quad \forall x \in E,\ y \geq 0, \ \text{ with} = \text{if } \ y > 0, \\ &v_0(x, 0) = \min\left\{v(x, 0),\ \inf_{\xi \in \Gamma}\{v(\xi, 0) + c(x, \xi)\}\right\} \quad \forall x \in E, \end{aligned} \tag{3.4}$$

hold, provided assumption (3.2) is enforced.

3.1. Weak DP equations. If $\theta \geq 0$ is a time at which the evolution of the system is "stopped and restarted" then all interventions at times $\vartheta_i < \theta$ will be applied within the time interval $[0, \theta[$, and the remaining impulses are left for the time interval $[\theta, \infty[$. For a given impulse control ν, this is written as $\nu_{[0,\theta[}$ and $\nu_{[\theta,\infty[}$.

If $\tau_{i-1} < \theta < \tau_i$ for some $i \geq 1$, then the class of impulse controls acting on the time interval $[\theta, \infty[$ is the same for both models (since $y_\theta \neq 0$). However, if $\theta = \tau_i$ for some $i \geq 1$, then the class of impulse controls on model (b) is unchanged, and that of model (a) becomes those of model (b).

Using the expression of the cost $J_{x,y}(\nu)$ given by (2.19), we have

$$J_{x,y}(\nu) = \mathbb{E}^\nu_{x,y} \left\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x^\nu_t) \mathrm{d}t + \sum_i \mathrm{e}^{-\alpha\vartheta_i} c\left(x^{i-1,\nu}_{\vartheta_i}, \xi_i\right) \mathbb{1}_{\{\vartheta_i < \theta\}} \right\} + \mathbb{E}^\nu_{x,y} \left\{ \int_\theta^\infty \mathrm{e}^{-\alpha t} f(x^\nu_t) \mathrm{d}t + \sum_i \mathrm{e}^{-\alpha\vartheta_i} c\left(x^{i-1,\nu}_{\vartheta_i}, \xi_i\right) \mathbb{1}_{\{\vartheta_i \geq \theta\}} \right\}.$$

Now, assuming the impulse control is a Markovian feedback and the controlled process satisfies the strong Markov property, we have the equality

$$\begin{aligned} &\mathbb{E}^\nu_{x,y} \left\{ \int_\theta^\infty \mathrm{e}^{-\alpha t} f(x^\nu_t) \mathrm{d}t + \sum_i \mathrm{e}^{-\alpha\vartheta_i} c\left(x^{i-1,\nu}_{\vartheta_i}, \xi_i\right) \mathbb{1}_{\{\vartheta_i \geq \theta\}} \right\} \\ &= \mathbb{E}^\nu_{x,y} \left\{ \mathbb{E}^\nu_{x,y} \left\{ \int_\theta^\infty \mathrm{e}^{-\alpha t} f(x^\nu_t) \mathrm{d}t + \sum_i \mathrm{e}^{-\alpha\vartheta_i} c\left(x^{i-1,\nu}_{\vartheta_i}, \xi_i\right) \mathbb{1}_{\{\vartheta_i \geq \theta\}} \mid \mathcal{F}_\theta \right\} \right\} \\ &= \mathbb{E}^\nu_{x,y} \left\{ \mathrm{e}^{-\alpha\theta} J_{x^\nu_\theta, y^\nu_\theta}(\nu_{[\theta,\infty[}) \right\}, \end{aligned}$$

where $\nu_{[\theta,\infty[}$ means the impulses after θ, and we obtain

$$J_{xy}(\nu) = \mathbb{E}^\nu_{x,y} \left\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x^\nu_t) \mathrm{d}t + \sum_i \mathrm{e}^{-\alpha\vartheta_i} c\left(x^{i-1,\nu}_{\vartheta_i}, \xi_i\right) \mathbb{1}_{\{\vartheta_i < \theta\}} \right\} + \mathbb{E}^\nu_{x,y} \left\{ \mathrm{e}^{-\alpha\theta} J_{x^\nu_\theta, y^\nu_\theta}(\nu_{[\theta,\infty[}) \right\}.$$

Hence, minimizing first on $\nu_{[\theta,\infty[}$ and then on $\nu_{[0,\theta[}$, the so-called weak dynamic programming equation (wDPE) for v is obtained, namely,

$$u(x, y) = \inf_\nu \mathbb{E}^\nu_{x,y} \left\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x^\nu_t) \mathrm{d}t + \sum_i \mathrm{e}^{-\alpha\vartheta_i} c\left(x^{i-1,\nu}_{\vartheta_i}, \xi_i\right) \mathbb{1}_{\vartheta_i < \theta} + u\left(x^{i_\theta - 1,\nu}_\theta, y_\theta, \theta\right) \right\},$$

where $i_\theta = \sup\{i : \vartheta_i < \theta\}$ with $i_\theta = 1$ if $\vartheta_i \geq \theta$ for every $i \geq 1$. Next, since the first intervention on $[\theta, \infty[$ must satisfy $\vartheta_1 \geq \tau_1 > 0$, any zero-admissible control can be applied on $[\theta, \infty[$ and, therefore, the previous equality becomes

$$u(x, y) = \inf_\nu \mathbb{E}^\nu_{x,y} \left\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x^\nu_t) \mathrm{d}t + \sum_i \mathrm{e}^{-\alpha\vartheta_i} c\left(x^{i-1,\nu}_{\vartheta_i}, \xi_i\right) \mathbb{1}_{\vartheta_i < \theta} + \mathrm{e}^{-\alpha\theta} u_0\left(x^{i_\theta - 1,\nu}_\theta, y_\theta\right) \right\}, \quad \theta \geq \tau_1, \tag{3.5}$$

which together with

$$u(x,y) = \inf_{\nu} \mathbb{E}^{\nu}_{x,y} \left\{ \int_0^{\theta} \mathrm{e}^{-\alpha t} f(x^{\nu}_t)\mathrm{d}t + \mathrm{e}^{-\alpha\theta} u\left(x^{\nu}_{\theta}, y_{\theta}\right) \right\}, \quad \theta < \tau_1,$$

is the wDPE.

Similarly, for the optimal cost v_0 given by (2.9), adding the time as an extra state variable is not necessary and the wDPE reads as

$$\begin{aligned} u_0(x,y) = \inf_{\nu} \mathbb{E}^{\nu}_{x,y} \bigg\{ & \int_0^{\theta} \mathrm{e}^{-\alpha t} f(x^{\nu}_t)\mathrm{d}t + \sum_i \mathrm{e}^{-\alpha\vartheta_i} c\left(x^{i-1,\nu}_{\vartheta_i}, \xi_i\right) \mathbb{1}_{\vartheta_i < \theta} \\ & + \mathrm{e}^{-\alpha\theta} u_0\left(x^{i_\theta - 1,\nu}_{\theta}, y_{\theta}\right) \bigg\} \end{aligned} \tag{3.6}$$

for any stopping time $\theta \geq 0$.

3.2. HJB equations. In view of the first relation (3.4), the interest of the optimal cost v_0 is limited to the initial condition $(x_0, y_0) = (x, 0)$.

Therefore, since an impulse (at time $t = 0$) is allowed within the first period $[0, \tau_1[$, take $\theta = \tau_1$ in the wDPE equation (3.6) to deduce

$$\begin{aligned} u_0(x,0) = \min \bigg\{ & \inf_{\xi_1 \in \Gamma} \{u_0(\xi_1, 0) + c(x, \xi_1)\}, \\ & \mathbb{E}_{x,0} \left\{ \int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau_1} u_0(x_{\tau_1}, 0) \right\} \bigg\} \end{aligned}$$

or, equivalently,

$$u_0(x,0) = \min\{M u_0(x,0), R u_0(x,0)\} \quad \forall x \in E, \tag{3.7}$$

where M is given by (3.3) and

$$\varphi(x) \mapsto (R\varphi)(x,0) = \mathbb{E}_{x,0} \left\{ \int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau_1} \varphi(x_{\tau_1}) \right\} \tag{3.8}$$

with $\mathbb{E}_{x,0}$ referring to the initial condition $x_0 = x$ and $y_0 = 0$ relative to the Markov process $\{(x_t, y_t) : t \geq 0\}$.

Now, for any initial condition $(x_0, y_0) = (x, y)$, take $\theta = \tau_1$ (this first signal may depend on y) in the wDPE equation (3.5) to get the equality

$$u(x,y) = \mathbb{E}_{x,y} \left\{ \int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau_1} u_0(x_{\tau_1}, 0) \right\}, \tag{3.9}$$

which yields the values of $u(x,y)$ for $y \geq 0$, once $u_0(x_{\tau_1}, 0)$ is known. Together with the first equality (3.4), this also provide the values of $u_0(x,y)$, for $y > 0$.

To write an equation with u alone, the relation (3.4) in (3.9) gives

$$\begin{aligned} u(x,y) = \mathbb{E}_{x,y} \bigg\{ & \int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t \\ & + \mathrm{e}^{-\alpha\tau_1} \min\left\{ u(x_{\tau_1}, 0), \inf_{\xi_1 \in \Gamma} \{u(\xi_1, 0) + c(x_{\tau_1}, \xi_1)\} \right\} \bigg\} \end{aligned}$$

or, equivalently,

$$u(x,y) = \big(R(\min\{u(\cdot,0),(Mu(\cdot,0))\}\big)(x,y) \quad \forall x,y, \tag{3.10}$$

with the same operators M and R, given by (3.3) and (3.8), but now R is used for (x,y), and $\mathbb{E}_{x,y}$ is referring to the initial condition $x_0 = x$ and $y_0 = y$ relative to the Markov process $\{(x_t,y_t) : t \geq 0\}$.

Finally, consider (3.9) with $y = 0$, taking the minimum value with $(Mu(\cdot,0))(x)$, and using the relation (3.4),

$$\begin{aligned} u_0(x,0) &= \min\big\{u(x,0),(Mu(\cdot,0))(x)\big\} \\ &= \min\left\{(Mu(\cdot,0))(x),\, \mathbb{E}_{x,0}\left\{\int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau_1} u_0(x_{\tau_1},0)\right\}\right\} \end{aligned}$$

is obtained.

3.3. Comments on settings and proofs. As mentioned earlier, because of the sequential aspect of the impulse control policies, the discrete DP is a good choice to deal with constraint impulse control problems of this type. Therefore, a discrete (or sequential) setting would be as follows: begin with the initial state $(x_0,y_0) = (x,y)$ and within a period $n \geq 1$, the time goes from τ_{n-1} to τ_n and the state moves from $x_{\tau_{n-1}}$ to x_{τ_n}, $y_{\tau_n} = 0$, and the running cost for the given period is

$$\mathbb{E}^{\nu}_{x,y}\left\{\int_{\tau_{n-1}}^{\tau_n} \mathrm{e}^{-\alpha t} f(x^{\nu}_t)\mathrm{d}t\right\}.$$

If an intervention is decided, then an impulse appears only at the beginning of a period, with a cost

$$\mathbb{E}^{\nu}_{x,y}\left\{\mathrm{e}^{-\alpha\tau_{n-1}} c\left(x^{n-2,\nu}_{\tau_{n-1}},\xi_{n-1}\right)\right\}, \quad n \geq 2.$$

The difference is that, within the initial period $[\tau_{n-1},\tau_n[$, $n = 1$, the model (b) allows intervention and model (a) does not allow it (and the cost per impulse is adjusted accordingly). Thus, adding over all periods, $J_{x,y}(\nu)$ is the total cost for an impulse control $\nu = \{(\eta_i,\xi_i) : i \geq 1\}$, and $v(x,y)$ and $v_0(x,y)$ are the optimal costs.

Recall that u or u_0 replace the optimal cost v or v_0 in the HJB equations to indicate the "formal approach." Now, analyzing each period, the controller may intervene or not, with a corresponding cost. This yields directly the HJB equation (3.7) for u_0. Because on the first period there is no intervention for the cost v, (3.9) is obtained. The difference between both models can be expressed with the equality (3.4), i.e.,

$$v_0(x,0) = \min\big\{Mv(x,0)(x), v(x,0)\big\} \quad \forall x \in E,$$

which can be used in (3.9) to deduce the HJB equation (3.10) for u.

To actually prove the wDPE, note that, because the cost within the time interval $[\theta,\infty[$ is always larger than $\mathrm{e}^{-\alpha\theta} u_0(x_{\theta-},y_\theta)$, the inequalities

$$\begin{aligned} u(x,y) \geq \inf_{\nu} \mathbb{E}^{\nu}_{x,y}\bigg\{ &\int_0^{\theta} \mathrm{e}^{-\alpha t} f(x^{\nu}_t)\mathrm{d}t + \sum_i \mathrm{e}^{-\alpha\vartheta_i} c\left(x^{i-1,\nu}_{\vartheta_i},\xi_i\right) \mathbb{1}_{\vartheta_i<\theta} \\ &+ \mathrm{e}^{-\alpha\theta} u_0\left(x^{i_\theta-1,\nu}_{\theta},y_\theta\right)\bigg\} \end{aligned}$$

and

$$u_0(x,y) \geq \inf_{\nu} \mathbb{E}^{\nu}_{x,y}\bigg\{ \int_0^{\theta} \mathrm{e}^{-\alpha t} f(x_t^{\nu})\mathrm{d}t + \sum_i \mathrm{e}^{-\alpha\vartheta_i} c\left(x_{\vartheta_i}^{i-1,\nu}, \xi_i\right) \mathbb{1}_{\vartheta_i<\theta} + \mathrm{e}^{-\alpha\theta} u_0\left(x_{\theta}^{i_\theta-1,\nu}, y_\theta\right)\bigg\},$$

are immediately deduced. However the converse inequalities are more delicate; this involves either using an ε-optimal (or optimal) impulse control or properly splitting the action of an impulse control into the two time intervals $[0,\theta[$ and $[\theta,\infty[$. Certainly, either way is doable, but perhaps a little tedious. In our case, since a strong version of the DP (or HJB) equation will be proved, this verification is not necessary.

Remark that if τ is given by (2.16) then

$$\mathbb{E}^{\nu}_{x,y}\bigg\{ \int_0^{\tau} \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t = \mathbb{E}^{\nu}_{x}\left\{ \int_0^{\infty} \exp\left(-\int_0^t \lambda(x_s, y+s)\mathrm{d}s\right) \mathrm{e}^{-\alpha t} f(x_t, y+t)\mathrm{d}t\right\}$$

and

$$\mathbb{E}^{\nu}_{x,y}\Big\{\mathrm{e}^{-\alpha\tau}\varphi(x_\tau,y_\tau)\Big\} = \mathbb{E}^{\nu}_{x}\left\{ \int_0^{\infty} \mathrm{e}^{-\alpha t} \exp\left(-\int_0^t \lambda(x_s, y+s)\mathrm{d}s\right) \lambda(x_t,y+t)\varphi(x_t,0)\mathrm{d}t\right\}$$

can be used to deduce the following assertion.

If φ is a continuous and bounded function and

$$w(x,y) = (R\varphi(\cdot,0))(x,y)$$

with R given by (3.8), then the function w belongs to the domain in $C_b(E\times[0,\infty[)$ of the infinitesimal generator of the Markov process $\{(x_t, y+t) : t\geq 0\}$, and

$$\begin{aligned} &- A_x w(x,y) - \frac{\partial w(x,y)}{\partial y} + \lambda(x,y)w(x,y) + \alpha w(x,y) \\ &\quad = f(x,y) + \lambda(x,y)\varphi(x,0) \quad \forall x,y, \end{aligned} \tag{3.11}$$

which can be used as an alternative definition of the operator R given by (3.8). Therefore, the HJB equation (3.10) can be written as

$$\begin{aligned} &- A_x u(x,y) - \frac{\partial u(x,y)}{\partial y} + \lambda(x,y)u(x,y) + \alpha u(x,y) \\ &\quad = f(x,y) + \lambda(x,y)\min\big\{u(x,0),\, (Mu(\cdot,0))(x)\big\} \quad \forall x,y, \end{aligned}$$

or, equivalently,

$$\begin{aligned} &- A_{x,y}u(x,y) + \alpha u(x,y) + \lambda(x,y)\big[u(x,0) - (Mu(\cdot,0))(x)\big]^+ \\ &\quad = f(x,y) \quad \forall (x,y)\in E\times[0,\infty[. \end{aligned} \tag{3.12}$$

All this cannot be applied to the HJB equation (3.7), but the equality (3.4) expresses $u_0(x,0)$ in term of $u(x,0)$.

4. Solving the HJB equation. In order to solve the HJB equations, we need some results on the optimal stopping problems with constraint.

4.1. Stopping time with constraint. For convenience let us adapt the results in our previous paper [17] (which were presented as reward problems with only a terminal reward and without the x-dependency of the intensity λ) to the current situation.

The assumptions on $\{x_t, y_t : t \geq 0\}$, α, f are those of section 2, and we add a positive terminal cost ψ in $C_b(E \times [0, \infty[)$.

The cost function is

$$J_{x,y}(\theta, \psi) = \mathbb{E}_{x,y} \left\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x_t, y_t) \mathrm{d}t + \mathrm{e}^{-\alpha\theta} \psi(x_\theta, y_\theta) \right\}$$

with an optimal cost

$$u(x, y) = \inf \big\{ J_{x,y}(\theta, \psi) : \ \theta > 0, \ y_\theta = 0 \big\}, \tag{4.1}$$

i.e., θ is any admissible stopping time, and an auxiliary optimal cost is defined as

$$u_0(x, y) = \inf \big\{ J_{x,y}(\theta, \psi) : \ y_\theta = 0 \big\}, \tag{4.2}$$

which forms a homogeneous Markovian model. Moreover, if τ is the first signal, i.e.,

$$\tau = \inf \big\{ t > 0 : y_t = 0 \big\}$$

then

$$u_0(x, y) = \min \left\{ \psi, \mathbb{E}_{x,y} \left\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t, y_t) \mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau, y_\tau) \right\} \right\} \tag{4.3}$$

and

$$u(x, y) = \mathbb{E}_{x,y} \left\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t, y_t) \mathrm{d}t + \mathrm{e}^{-\alpha\tau} \min\{\psi, u\}(x_\tau, y_\tau) \right\}, \tag{4.4}$$

are the corresponding HJB equations, and both problems are related by the equation

$$u(x, y) = \mathbb{E}_{x,y} \left\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t, y_t) \mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau, y_\tau) \right\}. \tag{4.5}$$

Note that $y_\tau = 0$.

THEOREM 4.1. *Let us assume* (2.2), (2.5), (2.6), *and* $\psi \geq 0$ *in* $C_b(E \times [0, \infty[)$. *Then the variational inequality (VI)* (4.3) *and* (4.4) *each has a unique solution in* $C_b(E \times [0, \infty[)$, *which are the optimal costs* (4.1) *and* (4.2), *respectively. Moreover, the first exit time from the continuation region is optimal, i.e., the discrete stopping times*

$$\begin{aligned} \hat\theta &= \inf \big\{ t > 0 : u(x_t, y_t) \leq \psi(x_t, y_t), \ y_t = 0 \big\}, \\ \hat\theta_0 &= \inf \big\{ t \geq 0 : u_0(x_t, y_t) = \psi(x_t, y_t), \ y_t = 0 \big\} \end{aligned} \tag{4.6}$$

are optimal, namely, $u(x, y) = J_{x,y}(\hat\theta, \psi)$ *and* $u_0(x, y) = J_{x,y}(\hat\theta_0, \psi)$. *Furthermore, the relation* (4.5) *holds.* ◻

Remark that $u_0 = \min\{u, \psi\}$ may not belong to the domain $D(A_{x,y}) \subset C_b(E \times [0, \infty[)$ of the infinitesimal generator $A_{x,y}$ of the Markov process $\{(x_t, y_t) : t \geq 0\}$. However, the optimal cost u given by (4.1) belongs to $D(A_{x,y})$ and

$$-A_{x,y}u(x,y) + \alpha u(x,y) + \lambda(x,y)[u(x,0) - \psi(x,y)]^+ = f(x,y) \tag{4.7}$$

for any (x, y) in $E \times [0, \infty[$. Also recall that several extensions are possible, in particular, the use of data with polynomial growth (instead of bounded).

There are some references regarding the stopping time problem with Poisson constraint (e.g., Dupuis and Wang [7], Lempa [11], Liang and Wei [14]), while there are many more about the usual or standard stopping times problem (e.g., the books by Bensoussan and Lions [4], Peskir and Shiryaev [19], among several others books and papers).

4.2. Existence and uniqueness. If $D(A)$ is the domain of the infinitesimal generator in $C(E)$ of the semigroup $\{\Phi(t) : t \geq 0\}$ corresponding to the initial Markov process $\{x_t : t \geq 0\}$, and

$$u^0(x) = \mathbb{E}_x \int_0^\infty \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t \quad \forall x \in E, \tag{4.8}$$

then u^0 is the unique solution in $D(A) \subset C(E)$ of the equation $-Au^0 + \alpha u^0 = f$. This function u^0 is the cost of no intervention, i.e., when the controller chooses not to apply any impulse to the system. Since all costs are supposed nonnegative, the interval

$$C(u^0) = \big\{\varphi \in C(E) : 0 \leq \varphi(x) \leq u^0(x),\ \forall x \in E\big\} \tag{4.9}$$

contains the optimal costs v and v_0 given by (2.9) and (2.10).

Consider the HJB equation (3.7), i.e., denoting $u_0(x) = u_0(x, 0)$,

$$\begin{aligned} u_0(x) = \min\Big\{ &\inf_{\xi\in\Gamma}\{u_0(\xi) + c(x,\xi)\}, \\ &\mathbb{E}_{x,0}\Big\{\int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau_1} u_0(x_{\tau_1})\Big\}\Big\}, \end{aligned} \tag{4.10}$$

and recall the impulse operator M and the (almost resolvent) operator R (which used the running cost f), given by (3.3) and (3.8), to write (4.10) as $u_0 = \min\{Mu_0, Ru_0\}$, and (4.8) as the solution of the equation $u^0 = Ru^0$.

Also consider the scheme $u_0^n = \min\{Mu_0^{n-1}, Ru_0^n\}$ with $u_0^0 = u^0$, i.e.,

$$\begin{aligned} u_0^n(x) = \min\Big\{ &\inf_{\xi\in\Gamma}\big\{u_0^{n-1}(\xi) + c(x,\xi)\big\}, \\ &\mathbb{E}_{x,0}\Big\{\int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau_1} u_0^n(x_{\tau_1})\Big\}\Big\} \end{aligned} \tag{4.11}$$

for any $n \geq 1$. As seen in the previous paper [17] (see also section 4.1), this equation has a unique solution in $C(E)$, which is the optimal cost of a stopping time problem with constraint, i.e., $u_0^n = v_0^n(\cdot, 0)$ with

$$v_0^n(x,y) = \inf_\theta \mathbb{E}_{x,y} \left\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t + \mathrm{e}^{-\alpha\theta} M v_0^{n-1}(x_\theta, 0) \right\}, \tag{4.12}$$

where the minimization is over all zero-admissible stopping times θ, i.e., $\theta = \tau_\eta$ for some discrete stopping time $\eta \geq 0$.

THEOREM 4.2. *Under assumptions* (2.2)*,* (2.5)*,* (2.6)*,* (2.7)*, and* (3.2)*, the monotone decreasing sequence* $\{u_0^n : n \geq 1\}$ *defined by* (4.11) *converges to* u_0*, and there are two constants* $C > 0$ *and* ρ *in* $]0,1[$ *such that*

$$0 \leq u_0^n(x) - u_0(x) \leq C\rho^n \quad \forall x \in E,\ n \geq 1. \tag{4.13}$$

Moreover, the HJB equation (4.10) *has one and only one solution* u_0 *within the interval* $C(u^0)$ *given by* (4.8)*, and the representation*

$$u_0(x) = \inf_\theta \mathbb{E}_{x,0} \left\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t + \mathrm{e}^{-\alpha\theta} M u_0(x_\theta) \right\} \tag{4.14}$$

holds true, where the minimization is over all zero-admissible stopping times θ*, i.e.,* $\theta = \tau_\eta$ *for some discrete stopping time* $\eta \geq 0$*.*

Proof. Using section 4.1, from the scheme (4.11), we deduce that $\{u_0^n : n \geq 1\}$ is a monotone decreasing sequence of nonnegative functions and, hence, the limit $\lim_n u_0^n(x) = u_0(x)$ exists for every x in E. Since the operator M maps $C(E)$ into itself, for any v in $C(E)$ the VI

$$w(x) = \min \left\{ Mv(x),\, \mathbb{E}_{x,0} \left\{ \int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t + \mathrm{e}^{-\alpha \tau} w(x_{\tau_1}) \right\} \right\},$$

has a unique solution w in $C(E)$, which is the optimal cost of the stopping time problem with constraint (with the stopping cost $\psi = Mv$), namely,

$$w(x) = \inf_\theta \mathbb{E}_{x,0} \left\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t + \mathrm{e}^{-\alpha\theta} M v(x_\theta, 0) \right\},$$

where the minimization is over all zero-admissible stopping times θ, i.e., $\theta = \tau_\eta$ for some discrete stopping time $\eta \geq 0$. This defines the nonlinear operator $v \mapsto w = T(v)$.

From the definition of T, it can be shown that $v \mapsto T(v)$ is a nondecreasing and concave mapping from $C(E)$ into itself, i.e.,

$$\begin{gathered} u \leq v \ \text{ implies } \ T(u) \leq T(v), \\ \gamma T(u) + (1-\gamma) T(v) \leq T(\gamma u + (1-\gamma) v) \quad \forall \gamma \in [0,1]. \end{gathered} \tag{4.15}$$

The next point is to check that

$$\begin{gathered} 0 \leq v \leq u^0 \ \text{ implies } \ 0 \leq T(v) \leq u^0, \\ \exists\, r \in]0,1[\ \text{ such that } \ r u^0 \leq T(0). \end{gathered} \tag{4.16}$$

Indeed, looking at $T(v)$ as the optimal cost, it is clear that T maps the interval $C(u^0)$ into itself. Next, from (4.8) it follows that u^0 is bounded and, hence, the assumption (2.7) implies that there exists r in $]0,1[$ such that $ru^0(x) \leq c_0 \leq M(0)(x)$ for every x in E. Also, the strong Markov property and $f \geq 0$ give

$$\begin{aligned} ru^0(x) &= \mathbb{E}_{x,0}\left\{\int_0^\theta \mathrm{e}^{-\alpha t} r f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau_1} r u^0(x_\theta)\right\} \\ &\leq \mathbb{E}_{x,0}\left\{\int_0^\theta \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\theta} M(0)(x_\theta)\right\} \end{aligned}$$

for any zero-admissible stopping time θ. This implies $ru^0 \leq T(0)$.

Now, we are ready to implement the arguments of Hanouzet and Joly [8]. For the sake of completeness, details on those arguments are given below. First, the second part of (4.16) together with (4.15) yield

$$\begin{aligned} &\text{if } \ v, \tilde{v} \in C(u^0), \ \gamma \in [0,1], \ \text{ and } \ v - \tilde{v} \leq \gamma v, \\ &\text{then } \ T(v) - T(\tilde{v}) \leq \gamma(1-r)T(v) \end{aligned} \tag{4.17}$$

with the same r as in (4.16). Indeed, $\gamma 0 + (1-\gamma)v \leq \tilde{v}$ implies

$$T(\tilde{v}) \geq T(\gamma 0 + (1-\gamma)v) \geq \gamma T(0) + (1-\gamma)T(v)$$

and so

$$T(v) - T(\tilde{v}) \leq \gamma T(v) - \gamma T(0) \leq \gamma T(v) - \gamma r u^0 \leq \gamma(1-r)T(v).$$

Then, we iterate (4.17) with $u_0^{n+1} = T(u_0^n)$ and $u_0^0 = u^0$ as follows: since $T(u_0^0) \leq u_0^0 = u^0$ and $u_0^0 \geq u_0^1 \geq 0$, we have $0 \leq u_0^0 - u_0^1 \leq u_0^0$ for $n = 1$. By means of (4.17) with $\gamma = 1$, we get $0 \leq u_0^1 - u_0^2 \leq (1-r)u_0^1$; for $n = 2$, by induction, from (4.17) with $\gamma = (1-r)^{n-1}$ we deduce

$$0 \leq u_0^n - u_0^{n+1} \leq (1-r)^n u_0^n, \quad n = 0, 1, \ldots,$$

which gives

$$0 \leq u_0^n - u_0 \leq (1-r)^n \frac{\|u_0\|}{r}, \quad n = 0, 1, \ldots.$$

Hence, the limit u_0 as $n \to \infty$ exits uniformly, and the estimate (4.13) follows. Finally, take $n \to \infty$ in (4.11) and (4.12) to obtain (4.10) and (4.14). ∎

Since $u_0(x, 0)$ is known, in view of (3.9), i.e.,

$$u(x,y) = \mathbb{E}_{x,y}\left\{\int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau_1} u_0(x_{\tau_1}, 0)\right\},$$

the function $u(x, y)$ can be obtained. Nevertheless, it may be convenient to state directly a result for u similar to Theorem 4.2, even if almost the same arguments are used.

THEOREM 4.3. *Under assumptions* (2.2), (2.5), (2.6), (2.7), *and* (3.2), *the monotone decreasing sequence* $\{u^n : n \geq 1\}$ *defined by* (4.23) *converges to* u, *and there are two constants* $C > 0$ *and* ρ *in* $]0,1[$ *such that*

$$0 \leq u^n(x) - u(x) \leq C\rho^n \quad \forall x \in E, \ n \geq 1. \tag{4.18}$$

Moreover, the HJB equation (4.22) *has one and only one solution* u *within the interval* $C(u^0)$ *given by* (4.8)*, and then the representation*

$$u(x) = \inf_{\theta} \mathbb{E}_{x,0} \left\{ \int_0^{\theta} \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t + \mathrm{e}^{-\alpha\theta} M u(x_\theta) \right\} \tag{4.19}$$

holds true, where the minimization is over all admissible stopping times θ*, i.e.,* $\theta = \tau_\eta$ *for some discrete stopping time* $\eta \geq 1$*. Furthermore,* u^n *and* u *belong to the domain* $D(A_{x,y}) \subset C_b(E \times [0, \infty[)$ *of the infinitesimal generator* $A_{x,y}$*,*

$$\begin{aligned} &- A_{x,y} u(x,y) + \alpha u(x,y) + \lambda(x,y) \big[u(x,0) - (Mu(\cdot,0))(x) \big]^+ \\ &\quad = f(x) \quad \forall (x,y) \in E \times [0,\infty[, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} &- A_{x,y} u^n(x,y) + \alpha u(x,y) + \lambda(x,y) \big[u^n(x,0) - (Mu^{n-1}(\cdot,0))(x) \big]^+ \\ &\quad = f(x) \quad \forall (x,y) \in E \times [0,\infty[\ \ \forall n \geq 1, \end{aligned} \tag{4.21}$$

hold true.

Proof. Let us give only some comments and arguments used in this proof. Consider the HJB equation (3.10), i.e.,

$$\begin{aligned} u(x,y) = \mathbb{E}_{x,y} \bigg\{ & \int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t \\ & + \mathrm{e}^{-\alpha \tau_1} \min \left\{ u(x_{\tau_1}, 0),\ \inf_{\xi \in \Gamma} \{ u(\xi, 0) + c(x_{\tau_1}, \xi) \} \right\} \bigg\}, \end{aligned} \tag{4.22}$$

and recall the impulse operator M and the operator R, given by (3.3) and (3.8), to write (4.22) as (3.12), i.e., (4.20). This yields the scheme

$$u^n = R\left(\min\left\{ u^n(\cdot,0), Mu^{n-1}(\cdot,0) \right\} \right)$$

with u^0 given by (4.8), i.e.,

$$\begin{aligned} u^n(x,y) = \mathbb{E}_{x,y} \bigg\{ & \int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t \\ & + \mathrm{e}^{-\alpha \tau_1} \min \left\{ u^n(x_{\tau_1}, 0),\ \inf_{\xi \in \Gamma} \left\{ u^{n-1}(\xi, 0) + c(x_{\tau_1}, \xi) \right\} \right\} \bigg\} \end{aligned} \tag{4.23}$$

or, equivalently, (4.21). As seen in the previous paper [17] (see also section 4.1), this equation has a unique solution in $C(E)$, which is the optimal cost of a stopping time problem with constraint, i.e., $u^n = v^n$ with

$$v^n(x,y) = \inf_{\theta} \mathbb{E}_{x,y} \left\{ \int_0^{\theta} \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t + \mathrm{e}^{-\alpha\theta} M v^{n-1}(x_\theta, 0) \right\},$$

where the minimization is over all admissible stopping times θ, i.e., $\theta = \tau_\eta$ for some discrete stopping time $\eta \geq 1$. ∎

THEOREM 4.4. *Under assumptions* (2.2), (2.6), (2.7), (2.5), *and* (3.2), *the unique solution of the HJB equation* (4.22) *is the optimal cost* (2.9), *i.e.,*

$$u(x,y) = \inf \big\{ J_{x,y}(\nu) : \nu \text{ any admissible impulse control} \big\} \tag{4.24}$$

or

$$u(x,y) = \inf \big\{ J_{x,y}(\eta) : \eta \text{ any discrete impulse control with } \eta_1 \geq 1 \big\}$$

for every (x,y) *in* $E \times [0,\infty[$.

Proof. First, let us show that

$$u^n(x,y) = \inf \big\{ J_{x,y}(\nu) : \nu \in S_n \big\}, \quad n \geq 0, \tag{4.25}$$

where S_n denotes the set of admissible impulse controls with at most n interventions.

Since $u^0(x,y)$ is defined by (4.8), it is clearly the cost without any intervention. Then using (4.23) and section 4.1 on the optimal stopping with constraint, we have

$$u^1(x,y) = \inf_{\theta} \mathbb{E}_{x,y} \left\{ \int_0^{\theta} \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t + \mathrm{e}^{-\alpha \theta} M u^0(x_\theta, 0) \right\}, \tag{4.26}$$

where θ is any admissible stopping time. For an admissible pair (ϑ_1, ξ_1), the equality (4.26) gives

$$u^1(x,y) \leq \mathbb{E}_{x,y} \left\{ \int_0^{\vartheta_1} \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t + \mathrm{e}^{-\alpha \vartheta_1} c(x_{\vartheta_1}, \xi_1) + \mathrm{e}^{-\alpha \vartheta} u^0(\xi_1) \right\}$$

and from the definition of u^0, we get

$$u^1(x,y) \leq \mathbb{E}_{x,y} \left\{ \int_0^{\infty} \mathrm{e}^{-\alpha t} f(x_t) \mathrm{d}t + \mathrm{e}^{-\alpha \vartheta_1} c(x_{\vartheta_1}, \xi_1) \right\} = J_{x,y}(\nu) \quad \forall \nu \in S_1.$$

Moreover, since the optimal stopping problem with constraint (4.26) has an optimal solution $\hat{\theta}$, and taking $\hat{\xi}$ minimizing $\xi \to [c(x_{\hat{\theta}}, \xi) + u^0(\xi, 0)]$, we have $u^1(x,y) = J(\hat{\nu})$ for a $\hat{\nu}$ in S_1. Therefore the equality (4.25) is proved for $n = 1$. Certainly, this argument can be iterated to complete the proof of (4.25)

To show the validity of (4.24), we start with $u^n(x,y) \leq J_{x,y}(\nu, n)$ for ν in S_n, where $J_{x,y}(\nu, n)$ is the cost for ν in S_n, to obtain $u(x,y) = \lim_n u^n(x,y) \leq \lim_n J_{x,y}(\nu, n) = J_{x,y}(\nu)$, as $n \to \infty$ for any admissible impulse control ν.

Since $S_n \subset \mathcal{V}$ we have

$$u^n(x,y) = \inf_{\nu \in S_n} \big\{ J_{x,y}(\nu) \big\} \geq \inf_{\nu \in \mathcal{V}} \big\{ J_{x,y}(\nu) \big\},$$

and as $n \to \infty$, we get

$$u(x,y) = \lim_n u^n(x,y) \geq \inf_{\nu \in \mathcal{V}} \big\{ J_{x,y}(\nu) \big\},$$

which completes the argument to prove the equality (4.24). ∎

THEOREM 4.5. *Under assumptions* (2.2), (2.6), (2.7), (2.5), *and* (3.2), *the first exit time of the continuation region provides an optimal admissible impulse control.*

Proof. First, if u is the optimal cost then (1) the continuation region $[u < Mu]$ is defined as all x in E such that $u(x,y) < Mu(x,0)$, (2) the optimal jump-to is a Borel minimizer $\hat{\xi}(x)$ of $Mu(x,0)$, i.e., $x \mapsto \hat{\xi}(x)$ is a Borel functions from E into Γ and $c(x,\hat{\xi}(x)) + u(\hat{\xi}(x),0) = Mu(x,0)$ for every x in E., and (3) the first exit time of $[u < Mu]$ is defined as

$$\hat{\theta}(x,y,s) = \inf\big\{t > s : u(x_{t-s}, y_{t-s}) = Mu(x_{t-s},0), \quad y_{t-s} = 0\big\},$$

and $\hat{\theta}(x,y,s) = \infty$ if $u(x_{t-s}, y_{t-s}) < Mu(x_{t-s},0)$ for every $t > s$ such that $y_t = 0$. Note that he Markov process $t \mapsto (x_{t-s}, y_{t-s})$ for $t \geq s$, represents the initial condition $(x_s, y_s) = (x,y)$ as discussed in section 2.2. Moreover, the continuity ensures that

$$u\left(x_{\hat{\theta}(x,y,s)-s}, 0\right) = c\left(x_{\hat{\theta}(x,y,s)-s}, \hat{\xi}(x_{\hat{\theta}(x,y,s)-s})\right) + u\left(\hat{\xi}(x_{\hat{\theta}(x,y,s)-s}), 0\right)$$

if $\hat{\theta}(x,y,s) < \infty$.

Therefore, the evolution under the above feedback and initial conditions (x,y) is as follows:

(1) first, $\vartheta_1 = \hat{\theta}(x,y,0)$ and $\xi_1 = \hat{\xi}(x_{\vartheta_1})$ when $\vartheta_1 < \infty$ (we may use an isolated "coffin" state ∂ to set $x_\infty = \partial$ and $\hat{\xi}(\partial) = \partial$);

(2) next $\vartheta_{k+1} = \hat{\theta}(\xi_k, 0, \vartheta_k)$ for any $k \geq 1$.

This produces an admissible impulse control $\hat{\nu} = \{(\vartheta_k, \xi_k) : k \geq 1\}$.

Recall the representation (4.19), namely,

$$u(x,y) = \inf_{\theta} \mathbb{E}_{x,y}\left\{\int_0^\theta \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha t} Mu(x_\theta, 0)\right\},$$

where the minimization is over all admissible stopping times θ, with the initial condition $(x_0, y_0) = (x,y)$. Moreover, if $u(x,y,s) = \mathrm{e}^{-\alpha s} u(x,y)$ then the homogeneity in time yields

$$u(x,y,s) = \inf_{\theta \geq s} \mathbb{E}_{x,y,s}\left\{\int_s^\theta \mathrm{e}^{-\alpha(t-s)} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha(\theta - s)} Mu(x_\theta, 0)\right\} \tag{4.27}$$

under the initial condition $(x_s, y_s) = (x,y)$.

For the initial condition $(x_0, y_0) = (x,y)$, from (4.27) with $s = 0$ we get

$$\begin{aligned} u(x,y) &= \mathbb{E}_{x,y}\left\{\int_0^{\vartheta_1} \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\vartheta_1} c(x_{\vartheta_1}, \xi_1) + \mathrm{e}^{-\alpha\vartheta_1} u(\xi_1, 0)\right\} \\ &= J_{x,y}(\hat{\nu}|\vartheta_1) + \mathbb{E}_{x,y}^{\hat{\nu}|\vartheta_1}\left\{\mathrm{e}^{-\alpha\vartheta_1} u(x_{\vartheta_1}, 0)\right\}, \end{aligned}$$

since $x_{\vartheta_1} = \xi_1$ under $P_{x,y}^{\hat{\nu}|\vartheta_1}$. Similarly, using (4.27) with $s = \vartheta_1$,

$$\begin{aligned} &\mathbb{E}_{x,y}^{\hat{\nu}|\vartheta_1}\left\{\mathrm{e}^{-\alpha\vartheta_1} u(x_{\vartheta_1}, 0)\right\} \\ &\quad = \mathbb{E}_{x,y}^{\hat{\nu}|\vartheta_1}\left\{u(x_{\vartheta_1}, 0, \vartheta_1)\right\} \\ &\quad = \mathrm{e}^{-\alpha\vartheta_1} \mathbb{E}_{x,y}^{\hat{\nu}|\vartheta_1}\left\{\int_{\vartheta_1}^{\vartheta_2} \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\vartheta_2} c(x_{\vartheta_1}, \xi_2) + \mathrm{e}^{-\alpha\vartheta_2} u(\xi_2, 0)\right\}, \end{aligned}$$

and this gives

$$u(x,y) = J_{x,y}(\hat{\nu}|\vartheta_2) + \mathbb{E}_{x,y}^{\hat{\nu}|\vartheta_2}\big\{\mathrm{e}^{-\alpha\vartheta_2}u(x_{\vartheta_2},0)\big\}$$

after recalling (2.18), i.e.,

$$J_z(\nu|\vartheta_{k+1}) = J_z(\nu|\vartheta_k) + \mathbb{E}_{x,y}^{\nu|\vartheta_k}\bigg\{\int_{\vartheta_k}^{\vartheta_{k+1}} f(x_t)\mathrm{e}^{-\alpha t}\mathrm{d}t + \mathrm{e}^{-\alpha\vartheta_{k+1}}c(x_{\vartheta_{k+1}},\xi_{k+1})\bigg\},$$

which represents the cost (of interventions) up to the time ϑ_{k+1}, and makes sense for $k \geq 0$, with $\vartheta_0 = 0$, and $J_z(\nu|\vartheta_0) = 0$.

Thus, iterate this argument to deduce

$$u(x,y) = J_{x,y}(\hat{\nu}|\vartheta_k) + \mathbb{E}_{x,y}^{\hat{\nu}|\vartheta_k}\big\{\mathrm{e}^{-\alpha\vartheta_k}u(x_{\vartheta_k},0)\big\} \quad \forall k \geq 1,$$

and as $n \to \infty$,

$$u(x,y) \geq \lim_k J_{x,y}(\hat{\nu}|\vartheta_k) = J_{x,y}(\hat{\nu}).$$

Since $u(x,y)$ is the optimal cost, we have

$$u(x,y) = J_{x,y}(\hat{\nu}) \quad \text{and} \quad \lim_n \mathbb{E}_{x,y}^{\hat{\nu}|\vartheta_k}\big\{\mathrm{e}^{-\alpha\vartheta_k}u(x_{\vartheta_k},0)\big\} = 0,$$

proving the optimality of the admissible impulse control $\hat{\nu}$. ∎

5. Extensions.

Some possible extensions are discussed below, without full details and only with precise indications. A full analysis would take much more space.

5.1. Other impulse control problems.

Let us consider the "variable" case of $\Gamma(x)$,

$$\Gamma : E \mapsto 2^E \quad \text{with } \Gamma(x) \text{ closed } \forall x \in E,$$

where the condition (3.2) becomes

$$c(x,\xi_1) + c(\xi_1,\xi_2) \geq c(x,\xi_2) \quad \forall x \in E,\ \xi_1 \in \Gamma(x),\ \xi_2 \in \Gamma(x) \cap \Gamma(\xi_1).$$

There are several convenient settings (e.g., Bensoussan and Lions [5], Davis [6, Chapter 5, pp.186–255], Seydel [21], among others), for the multivalued mapping Γ, but it always seems necessary to ensure that the impulse intervention operator M,

$$M\varphi(x) = \inf\big\{\varphi(x) + c(x,\xi) : \xi \in \Gamma(x)\big\} \quad \forall x \in E,$$

possessed the properties

$$\begin{cases} \text{(a)} \quad M \text{ maps continuously } C_b(E) \text{ into itself,} \\ \text{(b)} \quad \text{there exists a Borel measurable minimizer for } M, \end{cases} \tag{5.1}$$

where a minimizer means a Borel function $\hat{\xi}\colon E \times C_b(E) \to E$ satisfying

$$\hat{\xi}(x,\varphi) \in \Gamma(x), \quad M\varphi(x) = \varphi(x) + c(x,\hat{\xi}(x,\varphi)) \quad \forall x,\varphi.$$

Sometimes, the space $C_b(E)$ is replaced by another suitable space, e.g., Borel and bounded functions $B(E)$ or continuous and bounded with a polynomial weight. Certainly, some modifications are necessary to include the variable y used in the constrained models under consideration.

In inventory models, the impulse intervention operator M may take the form

$$M\varphi(x) = \inf\big\{\varphi(x+\xi) + c(x,\xi) : \xi \geq 0\big\} \quad \forall x \in \mathbb{R}^d,$$

where ξ is the jump (the order placed) instead of the jump-to as described earlier. Clearly, in this case, the multivalued function is given by $\Gamma(x) = \{\xi \in \mathbb{R}^d : \xi \geq x\}$.

Switching problems with constraint (or more general hybrid models; e.g., see Menaldi [16]) can be studied with almost the same technique. In these models, a discrete variable n is added and, for instance, the intervention operator M becomes

$$M\varphi(x,n) = \inf\big\{\varphi(\xi,m) + c(x,n,\xi,m) : (\xi,m) \in \Gamma(x,n)\big\}$$

for any x in E and n in $N \subset \mathbb{N}$. Essentially, the variable x contains several components and the value of n determines which component is highlighted and, clearly, there are several other interpretations. The reader is referred to the book Jasso-Fuentes et al. [10], to appear soon. Similar problems with constraint of this type are found in Liang and Wei [14], Wang [22], for diffusion processes.

Summing up, it is simple to review the proofs given earlier in the text to accommodate a multivalued function Γ under the assumption (5.1). Moreover, based on the current large bibliography on control by interventions, this technique can be adapted to other problems with constraint, like switching (or hybrid) models.

5.2. Unbounded data. Using the same tools as in our previous paper [17], we can extend the results of the present paper to the case $E = \mathbb{R}^d$ or, more generally, E locally compact and f with polynomial growth, using the Banach space $C_p(\mathbb{R}^d)$ of real uniformly continuous functions on any ball and with a growth bounded by the norm (in $\mathbb{R}^d$) to the p-power (i.e., continuous functions with a polynomial growth), or the most delicate case when E is a Hilbert or Banach space of infinite dimension; e.g., see Menaldi and Sritharan [18].

Beyond this, let us point out that for unbounded data, the quasi-VIs (QVIs) or impulse control problems present more challenges than the VIs or optimal stopping problems. The reader is referred to Menaldi [15], where some general arguments are used to close this gap, at least for diffusion with jumps.

5.3. Other type of signals. Modelling the signal with a sequence $\{T_1, T_2, \ldots\}$ of IID (conditionally to $\{x_t : t \geq 0\}$) as presented, it seems very efficient to include the cases described in our previous paper [17] as sources of the sequence of signal, namely, pure jump Markov processes, semi-Markov processes, piecewise-deterministic Markov processes, and diffusion processes with jumps. Essentially, it suffices to add a new variable to the initial process $\{x_t : t \geq 0\}$ to be reduced to the current situation.

As other assumptions on the process $\{y_t : t \geq 0\}$:

(a) Instead of assuming y_t in $\mathbb{R}^+$, we can consider the case y_t in $[0, y_\star[$, for some finite $y_\star > 0$, and therefore $\pi([0, y_\star[) = 1$. The results of section 4 can be extended to this situation, except for the result involving the infinitesimal generator in Theorem 4.3.

(b) In the case of IID variables independent of $\{x_t : t \geq 0\}$, we can consider a "general" distribution π_0 without a density. If $\pi_0(]0, t[)$ is continuous with respect to

t and $\pi_0(]t,\infty[) > 0$ for every t, then the results of section 4 are still valid, except that which concerns the infinitesimal generator.

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