ABSTRACT

Title of dissertation:	THEORIES OF BALDWIN-SHI HYPERGRAPHS: THEIR ATOMIC MODELS AND REGULAR TYPES
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In [1], Baldwin and Shi studied the properties of generic structures built from certain Fraïssé classes of weighted hypergraphs equipped with a notion of strong substructure. Here we focus on a particularly important class of such structures, where much stronger results are possible.

We begin by fixing a finite relational language and a set of weights $\overline{\alpha}$. After constructing certain weighted hypergraphs with carefully chosen properties, we use these constructions to obtain an $\forall\exists$-axiomatization for the theory of the generic, denoted by $S_{\overline{\alpha}}$, and a quantifier elimination result for $S_{\overline{\alpha}}$. These results, which extend those of Laskowski in [2] and Ikeda, Kikyo and Tsuboi in [3] are then used to study atomic and existentially closed models of $S_{\overline{\alpha}}$, resulting in a necessary and sufficient condition on the weights that yields the existence of atomic models of the corresponding theory.

We then proceed to obtain the stability of $S_{\overline{\alpha}}$ and a characertization of non-forking, simplifying the proofs of some of these well known results (see [1], [4]) in the

process. We identify conditions on $\overline{\alpha}$ that guarantee that $S_{\overline{\alpha}}$ is non-trivial and prove that $S_{\overline{\alpha}}$ has the dimensional order property, a result that has only been established under certain additional hypothesis (see [5], [2]).

Restricting ourselves to the case where the weights are all rational (excluding, what is essentially a single exception), we characterize the countable models up to isomorphism and show that they form an elementary chain of order type $\omega + 1$. We also characterize the regular types of $S_{\overline{\alpha}}$ and explore the corresponding pregeometries. We answer a question of Pillay in [6] by providing examples of pseudofinite stable theories with non-locally modular regular types.

We conclude by studying the aforementioned exception (characterized by having trivial forking) and extending some of the results to countably infinite languages.

THEORIES OF BALDWIN-SHI HYPERGRAPHS: THEIR ATOMIC MODELS AND REGULAR TYPES

by

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Dedication

For my parents, brother and wife

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Chapter 1: Introduction

The central result that provided the impetus for much of modern model theory is Morley's Categoricity Theorem (see [7]). A key ingredient found in modern treatments of Morley's Theorem (such as [8], [9]) are strongly minimal sets which allow us to define and use notions such as independence, bases and dimension via the pregeometry induced by taking (model theoretic) algebraic closures.

In a series of influential papers [10], [11], [12], Zilber explored the behavior of strongly minimal sets in the context of totally categorical theories and the behavior of the associated dimension function. He showed that strongly minimal sets in totally categorical theories were either trivial or did in fact interpret a group. He further conjectured any strongly minimal set that was non-locally modular would interpret a field.

This conjecture was famously refuted by Hrushovski in [13] who used a variant of Fraïssé's construction of a highly homogeneous countable structure to create a counterexample to Zilber's. A technically simpler variant of the construction was studied in depth by Baldwin and Shi in [1]. It is a generalization of these variants, that we term Baldwin-Shi hypergraphs, that we focus in here.

Baldwin-Shi hypergraphs and their theories, while not strongly minimal, are nevertheless of great interest: They are stable and non-forking has alternate, useful descriptions as found in [1], [4]. Similar to Hrushovski's original example, they do not interpret groups and have other interesting "geometric" properties such as CM-triviality (see [1], [4]). By the work of Baldwin and Shelah in [14], they are related to Zero-One laws studied by Shelah and Spencer in [15] (see also [16]). By work such as that of Evans and Ferreira in [17] and [18], there are strong connections between the pregeometries of Baldwin-Shi hypergraphs and variants of Hrushovski's original construction.

Our approach towards analyzing Baldwin-Shi hypergraphs and their theories builds on the work of Laskowski in [2]. As in Laskowski's work, at the center of many results is the construction of certain finite hypergraphs, a $\forall\exists$-axiomatization of the theory of the generic and a certain quantifier elimination result. In this chapter we briefly describe the setting, some definitions and the main results found herein. More in-depth and formal discussions of the featured results can be found at the beginning of each chapter.

Many of the results here in appear in the author's work [19], [20].

1.1 The Setting

With the exception of Chapter 9, we work with a fixed finite relational language L where each relation is at least binary. All structures we consider will be hypergraphs, i.e. each relation symbol of L will be interpreted irreflexively and symmetrically. Fix a function $\overline{\alpha} : L \to (0, 1]$ and let δ be a *rank* function on the class of hypergraphs $\delta(\mathfrak{A}) = |A| - \sum_{E \in L} \overline{\alpha}(E)|E^{\mathfrak{A}}|$. We take $K_{\overline{\alpha}}$ to be the class of finite hypegraphs with hereditarily non-negative rank, i.e. $K_{\overline{\alpha}} = \{\mathfrak{A} : \delta(\mathfrak{A}') \geq 0, \mathfrak{A}' \subseteq \mathfrak{A}\}$.

For any two hypergraphs $\mathfrak{A}, \mathfrak{B}$ with $\mathfrak{A} \subseteq \mathfrak{B}$ we say that $\mathfrak{A} \leq \mathfrak{B}$ if $\delta(\mathfrak{B}') \leq \delta(\mathfrak{B})$ for all $\mathfrak{A} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$. It is easily seen that $(K_{\overline{\alpha}}, \leq)$ is a Fraïssé class (see Definition 2.3.5) and as such there is (up to isomorphism) a unique countable structure with a high level of homogeneity (see Fact 2.3.7). It is this structure that we call the Baldwin-Shi hypergraph (for $\overline{\alpha}$) and it is the theory of this structure that we study throughout.

1.2 Key Results

In this section we highlight some key results from each chapter.

1.2.1 Some Key Results from Chapter 2

Chapter 2 describes the notation, delves into the setting, explores the properties of the rank δ in more detail, introduces the notion of *intrinsically closed sets* and contains a review of material related to pregeometries.

1.2.2 Key Results from Chapter 3

Chapter 3 is devoted to constructing certain finite hypergraphs. These result form the core of what is to follow. We begin this chapter by identifying *essential minimal pairs*: hypergraphs $\mathfrak{A}, \mathfrak{B} \in K_{\overline{\alpha}}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$, $\delta(\mathfrak{B}) < \delta(\mathfrak{A})$ but for any $\mathfrak{B}' \subsetneq \mathfrak{B}$, $\delta(\mathfrak{A} \cap \mathfrak{B}') \leq \delta(\mathfrak{B}')$. The existence of essential minimal pairs depends on $\overline{\alpha}$

not being *graph like with weight one* (i.e. each $E \in L$ is binary and $\overline{\alpha}(E) = 1$ for each in L).

One of the key results in this section is Theorem 3.2.15. In Theorem 3.2.15, we establish that given any $\mathfrak{A} \in K_{\overline{\alpha}}$, with $\delta(\mathfrak{A}) > 0$, we may construct infinitely many $\mathfrak{B} \in K_{\overline{\alpha}}$ such that $(\mathfrak{A}, \mathfrak{B})$ is an essential minimal pair and $|\delta(\mathfrak{A}) - \delta(\mathfrak{B})|$ is, in context, as small as desired. This theorem may be viewed as the appropriate generalization of Lemma 4.1 of [2] to the broader context here in. It plays a key part of the quantifier elimination result of Theorem 4.3.5 and features in establishing non-triviality of the theory of the Baldwin-Shi hypergraph in Theorem 6.3.3. It is also used through out Chapter 7, both to construct finite structures related to back and forth arguments and to construct those that witness various properties related to regular types.

The other key result is in this chapter is Theorem 3.3.6. In it we show that if $\overline{\alpha}$ is *coherent* (i.e. there exists positive integers $\langle m_E \rangle_{E \in L}$ such that $\sum_{E \in L} m_E \overline{\alpha}(E) \in \mathbb{Q}$), then for any $\mathfrak{A} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{A}) = 0$, there is some $\mathfrak{B} \supseteq \mathfrak{A}$ such that $\mathfrak{B} \in K_{\overline{\alpha}}$ and $\delta(\mathfrak{B}) = 0$. This result is used heavily throughout Chapter 5. Unlike the results regarding essential minimal pairs this result also holds in case $\overline{\alpha}$ is graph-like with weight one. We present a proof of this result in the case that $\overline{\alpha}$ is graph-like with weight one in Chapter 8.

1.2.3 Key Results from Chapter 4

In [2], Laskowski showed that if we assume that values $\overline{\alpha}(E)$ are irrational and that they are linearly independent over the rationals, then the theory of the associated Baldwin-Shi hypergraph has a $\forall\exists$-axiomatization and that theory admits quantifier elimination down to a boolean combination of formulas that is readily understood. These results improved the work of Baldwin and Shi in [1], who used an $\forall\exists\forall$-axiomatization of the Baldwin-Shi hypergraph and Baldwin and Shelah in [14] where quantifier elimination was studied in the context of near model completeness. In [3], Ikeda, Kikyo and Tsuboi showed that the restrictions on $\overline{\alpha}$ were not necessary to obtaining a $\forall\exists$-axiomatization of the theory of the generic. However their approach did not yield the quantifier elimination results of Laskowski in the more generalized context.

In this chapter we show that the results from [2], including the quantifier elimination result, can be generalized. To this end, throughout most of Chapter 4, we follow the same approach taken by Laskowski in [2], sometimes with minor modifications. We begin by defining $S_{\overline{\alpha}}$ as the smallest set of sentences insuring that, if $\mathfrak{M} \models S_{\overline{\alpha}}$, then

1. Every finite substructure of $\mathfrak{M}$ is in $K_{\overline{\alpha}}$
2. For all $\mathfrak{A} \leq \mathfrak{B}$ from $K_{\overline{\alpha}}$, every (isomorphic) embedding $f : \mathfrak{A} \to \mathfrak{M}$ extends to an embedding $g : \mathfrak{B} \to \mathfrak{M}$

The key result is Theorem 4.3.5. It states that $S_{\overline{\alpha}}$ admits quantifier elimination down to boolean combination of chain minimal formulas (see Definition 4.0.4). The proof makes use of many technical results, of which Proposition 4.1.1, Theorem 4.2.1 and Theorem 4.3.4 bears special mention. Theorem 4.2.1 is specially interesting. Its proof in the case that $\overline{\alpha}$ is not graph-like with weight makes heavy use of the existence of essential minimal pairs with extra properties. Nevertheless, in case the $\overline{\alpha}$ is graph-like with weight one, we can still establish Theorem 4.2.1 without appealing to the existence of essential minimal pairs. As Proposition 4.1.1 and Theorem 4.3.4 does not take into account the nature of $\overline{\alpha}$ in their proofs, it emerges that Theorem 4.2.1 may be viewed as providing a key technical property that enables the quantifier elimination result. This distinction between the structures provided for by Theorem 3.2.15 and those provided by Theorem 4.2.1 is not observable in [2] as the case that $\overline{\alpha}$ is graph-like with weight one is not studied there in.

Other important results include the various consequences of quantifier elimination result gathered in Chapter 4.4. They are used throughout the rest of the work in crucial ways.

1.2.4 Key Results from Chapter 5

In this section we study the atomic and existentially closed models of $S_{\overline{\alpha}}$ (see Definition 5.0.1). A key idea that runs throughout this chapter is the use of unions of chains of the universal sentences of $S_{\overline{\alpha}}$ to build new models of $S_{\overline{\alpha}}$, a technique applicable in the current setting because of the nature of $S_{\overline{\alpha}}$.

We begin by defining the function $d_{\mathfrak{N}}$ on the finite substructures of $\mathfrak{N} \models S_{\overline{\alpha}}$.

Let $d_{\mathfrak{N}} = \inf\{\delta(\mathfrak{B}) : \mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{N}, B \text{ is finite}\}$. We begin with Theorem 5.0.4. This Theorem, essentially due to Laskowski in [2], classifies the existentially closed models as those models $\mathfrak{N}$ of $S_{\overline{\alpha}}$ for which $d_{\mathfrak{N}}(\mathfrak{A}) = 0$ for all finite $\mathfrak{A} \subseteq \mathfrak{N}$.

The next key result is Theorem 5.1.7, which characterizes the atomic models of $S_{\overline{\alpha}}$ in several equivalent ways. One of the equivalences, that $\mathfrak{M}$ is atomic if and only if for every finite $\mathfrak{A} \subseteq \mathfrak{M}$, there is some finite $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{N}$, $\delta(\mathfrak{B}) = 0$ is particularly useful. We use this result to derive theorem 5.2.9, which identifies the coherence of $\overline{\alpha}$ as a necessary and sufficient condition for $S_{\overline{\alpha}}$ to have atomic models.

We conclude with Theorem 5.2.19, in which we show that $\overline{\alpha}$ being rational (i.e. if $\overline{\alpha}(E)$ is rational for all $E \in L$, in some sense the most natural form of coherence) is equivalent to each model of $S_{\overline{\alpha}}$ being embeddable in an atomic model.

1.2.5 Key Results from Chapter 6

This chapter is devoted to the exploring some stability theoretic related to $S_{\overline{\alpha}}$. Most results in this chapter are well known.

The first key result is of this chapter is Theorem 6.1.16, that states that $S_{\overline{\alpha}}$ is stable and ω-stable if $S_{\overline{\alpha}}$ is rational. This result is originally due to Baldwin and Shi in [1] (see also [4]). The main novelty is bypassing the technical conditions on amalgamation found therein via the use of Lemma 4.4.3.

The second key result is Theorem 6.2.25, which characterizes non-forking in a manner that is more intrinsic to $S_{\overline{\alpha}}$. This characterization will be particularly useful for obtaining the results in Chapter 7. Another key result is Lemma 6.2.27

which shows that $S_{\overline{\alpha}}$ has a form of weak elimination of imaginaries (i.e. types over algebraically closed sets are stationary). While a stronger form of weak elimination of imaginaries is possible (see [1] or [4]), the form found here turns out to be particularly useful for analyzing regular types in Chapter 7. In Theorem 6.3.3 we show that if $\overline{\alpha}$ is not graph-like with weight one, then $S_{\overline{\alpha}}$ is non-trivial, to which we will establish the converse in Theorem 8.3.1.

In Chapter 6.4, following Laskowski, we show that if $\overline{\alpha}$ is not rational then $S_{\overline{\alpha}}$ is strictly stable and make some observations about the spectrum (i.e. the number of non-isomorphic models) of $S_{\overline{\alpha}}$. We conclude by showing that $S_{\overline{\alpha}}$ has the dimensional order property, a result known for only special cases by the work of Baldwin and Shelah in [5] and Laskowski in [2].

1.2.6 Key Results from Chapter 7

In this chapter we focus our attention on the behavior of $S_{\overline{\alpha}}$ in the case $\overline{\alpha}$ is rational but not graph-like with weight one. Two key results of this chapter, Theorem 7.1.5 and Theorem 7.1.8. state that each countable model of $S_{\overline{\alpha}}$ is highly homogeneous and that the countable models of $S_{\overline{\alpha}}$ (up to isomorphism) form an elementary chain $\mathfrak{M}_0 \preccurlyeq \mathfrak{M}_1 \preccurlyeq \ldots \preccurlyeq \mathfrak{M}_\infty$ of order type $\omega + 1$ with $\mathfrak{M}_0$ being atomic and $\mathfrak{M}_\infty$ being the Baldwin-Shi hypergraph.

We then turn our attention towards to the study of regular types. We begin by arguing that in order to understand regular types it suffices to understand types over finite algebraically closed sets. We fix a model $\mathbb{M}$ and a finite algebraically closed set $A \subseteq \mathbb{M}$ and let c denote the least common multiple of the denominators of $\overline{\alpha}(E)$ (expressed in lowest terms). Given a type $p \in S(A)$, we define $d(p/A)$ by $d_{\mathbb{M}}(A\overline{b}) - d_{\mathbb{M}}(A)$ for any $\overline{b} \models p$ and introduce the notion of nugget-like types (see Definition 7.2.7).

In Theorem 7.2.10 we begin by showing that if p is nugget-like with $d(p/A) = 0, 1/c$, then p is regular (see Definition 7.2.2). We follow with Theorem 7.2.11, that any two $1/c$-nugget like types are non-orthogonal. We finish Chapter 7.2.1 with Theorem 7.2.13 that shows that if $d(p/A) \geq 2/c$, then p is not regular.

Chapter 7.2.2 is devoted to the geometric properties of the regular types. We begin with Theorem 7.2.16 that shows that the pregeometries associated with 0 nugget like types are trivial. We follow this up with Theorem 7.2.19 that shows that the pregeometries assumed with a $1/c$ nugget like types are not locally modular (i.e. behaves like transcendental dimension over algebraically closed fields).

We conclude the Chapter with Theorem 7.2.23. In this theorem we use results from [21] and Theorem 7.2.19 to exhibit a stable pseudofinite theory with a non-locally modular type. This answers a question of Pillay in [6] who noted that the statement fails if we replace “stable” with “strongly minimal”.

1.2.7 Key Results from Chapter 8

This section is devoted to a discussion of the case $\overline{\alpha}$ is graph-like with weight one. After setting up some terminology and initial lemmas, we provide ad hoc arguments that have been promised throughout the rest of the chapters. We conclude with Theorem 8.3.1 that provides a converse for Theorem 6.3.3.

1.2.8 Key Results from Chapter 9

In this chapter we relax the condition that L be finite and instead allow for the possibility that L be countable. After defining a corresponding version of $K_{\overline{\alpha}}$, we establish a strong connection between the reducts of the $(K_{\overline{\alpha}}, \leq)$ generic $\mathfrak{M}_{\overline{\alpha}}$ to finite sub-language $L_0 \subseteq L$ and the Baldwin-Shi hypergraphs that we have studied thus far in Theorem 9.2.1. We then use this connection to obtain the stability of $\mathfrak{M}_{\overline{\alpha}}$.

Chapter 2: Preliminaries

This section is devoted to introducing notation, definitions and some facts about the rank function δ (see Definition 2.1.5) that will be useful throughout. The results in this section are well known or follow from routine calculations involving δ. We work (barring in Chapter 9) with a finite relational language L where *each relation symbol $E \in L$ is at least binary*. Let $ar : L \to \{n : n \in \omega \text{ and } n \geq 2\}$ be a function that takes each relation symbol to its arity.

2.1 Some general notions

We begin with some notation.

Notation 2.1.1. Fraktur letters will denote L-structures. Their Latin counterparts will, as we shall see, denote either the structure or the underlying set. Let $\mathfrak{Z}$ be an L-structure and let $X, Y \subseteq Z$. We will adapt the practice of writing XY for $X \cup Y$. Since we are in a finite relational language X, Y, XY will have a natural L-structures associated with them, i.e. the L-structures with universe X, Y, XY that are substructures of $\mathfrak{Z}$, respectively. By a slight abuse of notation we write X, Y, XY for these L-structures. It will be clear by context what the notation refers to. We write $X \subseteq_{\mathrm{Fin}} Z$, $\mathfrak{X} \subseteq_{\mathrm{Fin}} \mathfrak{Z}$ to indicate that $|X|$ is finite.

Notation 2.1.2. We will use $\emptyset$ to denote the unique L-structure with no elements. Further given L-structures $\mathfrak{X}, \mathfrak{Y}$, there is a uniquely determined L-structure whose universe is $X \cap Y$. We denote this structure by $\mathfrak{X} \cap \mathfrak{Y}$.

Notation 2.1.3. We let K_L denote the class of all finite L structures $\mathfrak{A}$ (including the empty structure), where each $E \in L$ is interpreted symmetrically and irrelexively in A: i.e. $\mathfrak{A} \in K_L$ if and only if for every $E \in L$, if $\mathfrak{A} \models E(\overline{a})$, then $\overline{a}$ has no repetitions and $\mathfrak{A} \models E(\pi(\overline{a}))$ for every permutation π of $\{0, \ldots, n-1\}$. We let $\overline{K_L}$ denote the class of L-structures whose finite substructures lie in elements of K_L, i.e. $\overline{K_L} = \{\mathfrak{M} : \mathfrak{M} \text{ an } L-\text{structure and if } \mathfrak{A} \subseteq_{\text{Fin}} \mathfrak{M}, \text{ then } \mathfrak{A} \in K_L\}$

Notation 2.1.4. Fix any $E \in L$. Given $\mathfrak{A} \in K_L$, $N_E(\mathfrak{A})$ will denote the *number of distinct subsets* of A on which E holds positively inside of $\mathfrak{A}$. The set of such subsets will be denoted by $E^{\mathfrak{A}}$. Consider an L-structure whose finite substructures are all in K_L and let $A, B, C \subseteq Z$ be finite. Now $N_E(A, B)$ will denote the *number of distinct subsets* of AB on which E holds with *at least one element from* A and *at least one element from* B inside of AB. We further let $N_E(A, B, C)$ denote the number of *distinct subsets* of $A \cup B \cup C$ on which E holds with *at least one element* from A and *at least one element* from C.

We now introduce the class $K_{\overline{\alpha}}$ as a subclass of K_L.

Definition 2.1.5. Fix a function $\overline{\alpha} : L \to (0, 1]$. Define a function $\delta : K_L \to \mathbb{R}$ by $\delta(\mathfrak{A}) = |A| - \sum_{E \in L} \overline{\alpha}(E) N_E(\mathfrak{A})$ for each $\mathfrak{A} \in K_L$. We let $K_{\overline{\alpha}} = \{\mathfrak{A} | \delta(\mathfrak{A}') \geq 0 \text{ for all } \mathfrak{A}' \subseteq \mathfrak{A}\}$.

We adopt the convention $\emptyset \in K_L$ and hence $\emptyset \in K_{\overline{\alpha}}$ as $\delta(\emptyset) = 0$. It is easily observed that $K_{\overline{\alpha}}$ is closed under substructure. Further the rank function δ allows us to view both K_L and $K_{\overline{\alpha}}$ as collections of weighted hypergraphs. We proceed to use the rank function to define a notion of strong substructure $\leq$.

Definition 2.1.6. Given $\mathfrak{A}, \mathfrak{B} \in K_L$ with $\mathfrak{A} \subseteq \mathfrak{B}$, we say that $\mathfrak{A}$ *is strong in* $\mathfrak{B}$ (or alternatively $\mathfrak{A}$ is a strong substructure of $\mathfrak{B}$) if and only if $\delta(\mathfrak{A}) \leq \delta(\mathfrak{A}')$ for all $\mathfrak{A} \subseteq \mathfrak{A}' \subseteq \mathfrak{B}$. We denote this by $\mathfrak{A} \leq \mathfrak{B}$

Remark 2.1.7. Given some fixed $K \subseteq K_L$, K inherits a notion of strong substructure from K_L as follows: Let $\mathfrak{A}, \mathfrak{B} \in K$ with $\mathfrak{A} \subseteq \mathfrak{B}$. Now $\mathfrak{A}$ is strong in $\mathfrak{B}$ if and only if $\mathfrak{A} \leq \mathfrak{B}$ when $\mathfrak{A}, \mathfrak{B}$ are viewed as elements of K_L. We denote K with this inherited notion of strong substructure relation by $(K, \leq)$.

Typically the notion of $\leq$ is defined on $K_{\overline{\alpha}} \times K_{\overline{\alpha}}$ by letting $\mathfrak{A} \leq \mathfrak{B}$ if and only if $\delta(\mathfrak{A}) \leq \delta(\mathfrak{A}')$ for all $\mathfrak{A} \subseteq \mathfrak{A}' \subseteq \mathfrak{B}$ for $\mathfrak{A}, \mathfrak{B} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \subseteq \mathfrak{B}$ (see for example [1]). However, we define the concept on the broader class $K_L \times K_L$. This will allow us to make the exposition significantly simpler via Remark 2.3.2. It is easily seen that the notion of strong substructure inherited by $K_{\overline{\alpha}}$ in this setting is the same as the notion of strong substructure studied in existing literature such as [1].

2.2 Joins and some basic properties of the rank function

We introduce the notion of *joins and free joins* and explore the rank function δ in more detail. The properties of the rank function introduced here will be useful throughout.

Definition 2.2.1. Let n be a positive integer. A set $\{\mathfrak{B}_i : i < n\}$ of elements of $K_{\overline{\alpha}}$ is disjoint over $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}_i$ for each $i < n$ and $B_i \cap B_j = A$ for $i < j < n$. If $\{\mathfrak{B}_i : i < n\}$ is disjoint over $\mathfrak{A}$, then $\mathfrak{D}$ is a *join* of $\{\mathfrak{B}_i : i < n\}$ if the universe $D = \bigcup\{B_i : i < n\}$ and $\mathfrak{B}_i \subseteq \mathfrak{D}$ for all i. A join is called the *free join*, which we denote by $\oplus_{i<n}\mathfrak{B}_i$ if there are no additional relations, i.e. $E^{\mathfrak{D}} = \bigcup\{E^{\mathfrak{B}_i} : i < n\}$ for all $E \in L$. In the case $n = 2$ we will use the notation $\mathfrak{B}_0 \oplus_{\mathfrak{A}} \mathfrak{B}_1$ for $\oplus_{i<2}\mathfrak{B}_i$. We note that there are obvious extension of these notions to $K_L, \overline{K_L}, \overline{K_{\overline{\alpha}}}$ and to infinitely many structures $\{\mathfrak{X}_i : i < \kappa\}$ being disjoint/joined/freely joined over some fixed $\mathfrak{Y} \subseteq \mathfrak{X}_i$ for each $i < \kappa$.

Definition 2.2.2. Let $\mathfrak{Z} \in \overline{K_L}$ and let $A, B \subseteq_{\text{Fin}} Z$. Now $\delta(B/A) = \delta(BA) - \delta(A)$. We will call $\delta(B/A)$, the *relative rank* of B over A. When B and A are understood in context we will just say relative rank.

Remark 2.2.3. Let $\mathfrak{A}, \mathfrak{B} \in K_L$ with. Note that $\mathfrak{A} \leq \mathfrak{B}$ if and only if for all $\mathfrak{A} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$, $\delta(\mathfrak{B}'/\mathfrak{A}) \geq 0$.

We introduce some notation:

Notation 2.2.4. For readability, we will often write $\overline{\alpha}_E$ in place of $\overline{\alpha}(E)$. Given $\mathfrak{Z} \in \overline{K_L}$ and $A, B, C \subseteq_{\text{Fin}} Z$, we write $e(A)$ for $\sum_{E\in L} \overline{\alpha}_E N_E(A)$, $e(A, B)$ for $\sum_{E\in L} \overline{\alpha}_E N_E(A, B)$ and $e(A, B, C)$ for $\sum_{E\in L} \overline{\alpha}_E N_E(A, B, C)$ where N_E is defined as in 2.1.4.

The following collects some useful facts about the behavior of the rank function δ.

Fact 2.2.5. *Let $\mathfrak{Z} \in \overline{K_L}$ and let $A, B, C, B_i \subseteq_{Fin} Z$.*

1. $\delta(B/A) = \delta(B) - \delta(A \cap B) - e(A - B, A \cap B, B - A)$ *and hence if either A or B is in $K_{\overline{\alpha}}$, $\delta(B/A) \leq \delta(B) - e(A - B, A \cap B, B - A)$. Further if A, B are disjoint then $\delta(B/A) = \delta(B) - e(A, B)$.*

2. *Let $A' = A \cap B$. Now $\delta(B/A') \geq \delta(B/A) = \delta(AB/A)$, while $\delta(AB/A) + \overline{\alpha}_E = \delta(B/A) + \overline{\alpha}_E \leq \delta(B/A')$ whenever $E^{AB} \neq E^A \cup E^B$. Further if B, C are disjoint and freely joined over A, then $\delta(B/AC) = \delta(B/A)$*

3. *Assume that $BC \cap A = \emptyset$, $A \leq AB$ and $A \leq AC$. Then $\delta(BC/A) \leq \delta(B/A) + \delta(C/A)$.*

4. *If $\{B_i : i < n\}$ is disjoint over A and $Z = \oplus_{i<n} B_i$ is their free join over A, then $\delta(Z/A) = \sum_{i<n} \delta(B_i/A)$. In particular, if $A \leq B_i$ for each $i < n$, then $A \leq \oplus_{i<n} B_i$.*

5. $\delta(B_1 B_2 \ldots B_k/A) \;=\; \delta(B_1/A) + \sum_{i=2}^{k} \delta(B_i/AB_1 \ldots B_{i-1})$

6. *Assume that $A \leq B$ and $\delta(B/A) > 0$. Then there exists $b \in B - A$ such that for all B' with $bA \subseteq B'$, $\delta(B'/A) > 0$.*

Proof. (1):

$$\begin{aligned}
\delta(B/A) &= \delta(AB) - \delta(A) \\
&= |AB| - e(AB) - (|A| - e(A)) \\
&= |A| + |B| - |A \cap B| - (e(B) + e(A) - e(A, B) + e(A - B, A \cap B, \\
&\quad B - A) - |A| + e(A) \\
&= \delta(B) - \delta(A \cap B) - e(A - B, A \cap B, B - A)
\end{aligned}$$

If either A or B is in $K_{\overline{\alpha}}$, we obtain that $\delta(A \cap B) \geq 0$ as $A \cap B \in K_{\overline{\alpha}}$ and hence $\delta(B/A) \leq \delta(B) - e(A-B, A\cap B, B-A)$. As $\delta(A \cap B) = 0$ and $e(A-B, A\cap B, B-A) = e(A, B)$ if A, B are disjoint the rest of the claim follows.

(2): First note that $\delta(B - A'/A) = \delta(B/A)$ and that $B - A'$ and A are disjoint. Now using (1) we obtain that $\delta(B/A) = \delta(B - A') - e(B - A', A)$. However as $e(B - A', A') \leq e(B - A', A)$ we obtain that $\delta(B - A') - e(B - A', A) \leq \delta(B - A') - e(B - A', A')$. Since A', $B - A'$ are also disjoint $\delta(B/A) \leq \delta(B/A')$ now follows. Now note that if $E^{AB} \neq E^A \cup E^B$, then we have that $A' \neq A, B$. Further under the given conditions we have that $e(B - A', A') + \overline{\alpha}_E \leq e(B - A', A)$ and the result now follows similarly.

For the last part of the claim, note that $\delta(B/AC) = \delta(B - AC) - e(B - AC, AC)$ by (1). But under the given conditions $B - AC = B - A$ and $e(B - AC, AC) = e(B - A, A)$. Hence we obtain that $\delta(B/AC) = \delta(B - A) - e(B - A, A) = \delta(B/A)$.

(3): First note that $\delta(BC/A) = \delta(BC) - e(BC, A)$. But $\delta(BC) = |BC| - e(BC) \leq |B| + |C| - e(B) - e(C) = \delta(B) + \delta(C)$. Further $e(BC, A) \geq e(B, A) + e(C, A)$. Thus we obtain that $\delta(BC/A) \leq \delta(B) + \delta(C) - e(B, A) - e(C, A)$. An application of (1) now yields that $\delta(BC/A) \leq \delta(B/A) + \delta(C/A)$

(4): First consider the case of $\{B_1, B_2\}$. Given $A \subseteq B_1' \subseteq B_1$ and $A \subseteq B_2' \subseteq B_2$, we obtain that $\delta(B_1'B_2'/A) = \delta(B_1'B_2' - A/A) = \delta(B_1'B_2' - A) - e(B_1'B_2' - A, A)$. Note that $B_1'B_2' - A = (B_1' - A)(B_2' - A)$. As $B_1' - A, B_2' - A$ are freely joined over

A, it follows that $e(B_1'B_2' - A, A) = e(B_1' - A, A) + e(B_2' - A, A)$. Further an easy calculation shows that $\delta(B_1'B_2' - A) = \delta(B_1' - A) + \delta(B_2' - A)$. Thus we obtain that $\delta(B_1'B_2'/A) = \delta(B_1' - A) + \delta(B_2' - A) - e(B_1' - A, A) - e(B_2' - A, A)$. An application of (1) now yields that $\delta(B_1'B_2'/A) = \delta(B_1'/A) + \delta(B_2'/A)$. The rest of the statement now follows easily. For $\{B_1, B_2, \ldots, B_n\}$ the result can be obtained by an easy induction argument.

(5): Note that $\delta(B_1B_2/A) = \delta(B_1B_2A) - \delta(B_1A) + \delta(B_1A) - \delta(A)$. Thus $\delta(B_1B_2/A) = \delta(B_1/A) + \delta(B_2/B_1A)$. The required result now follows by induction.

(6): Assume not. Then for each $b \in B - A$, there is some B_b' with $Ab \subseteq B_b' \subseteq B$ and $\delta(B_b'/A) \leq 0$. Now $\delta(B/A) \leq \sum_{b \in B-A} \delta(B_b/A) \leq 0$, a contradiction that yields the required result. $\square$

2.3 Towards building the Baldwin-Shi hypergraph

We now work towards defining Baldwin-Shi hypergraphs. Along the way we observe several useful properties of K_L and $K_{\overline{\alpha}}$ that will be useful throughout.

Remark 2.3.1. The relation $\leq$ on $K_L \times K_L$ is reflexive, transitive and has the property that given $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K_L$, if $\mathfrak{A} \leq \mathfrak{C}$, $\mathfrak{B} \subseteq \mathfrak{C}$ then $\mathfrak{A} \cap \mathfrak{B} \leq \mathfrak{B}$: Suppose not and let $\mathfrak{A} \cap \mathfrak{B} \subseteq \mathfrak{D} \subseteq \mathfrak{B}$ be a $\subseteq$ minimal witness for $\mathfrak{A} \cap \mathfrak{B} \not\leq \mathfrak{B}$. An application of (2) of Fact 2.2.5 yields that $\delta(\mathfrak{D}/\mathfrak{A}) \leq \delta(\mathfrak{D}/\mathfrak{A} \cap \mathfrak{B}) < 0$ which contradicts $\mathfrak{A} \leq \mathfrak{C}$. The same statement holds true if we replace K_L by $K_{\overline{\alpha}}$ in the above. Further for any given $\mathfrak{A} \in K_{\overline{\alpha}}$, $\emptyset \leq \mathfrak{A}$.

Remark 2.3.2. Let $\mathfrak{A} \in K_{\overline{\alpha}}$, $\mathfrak{B} \in K_L$ with $\mathfrak{A} \subseteq \mathfrak{B}$. We claim that if $\mathfrak{A} \leq \mathfrak{B}$, then $\mathfrak{B} \in K_{\overline{\alpha}}$: Let $\mathfrak{B}' \subseteq \mathfrak{B}$. Then $\delta(\mathfrak{B}'/\mathfrak{A} \cap \mathfrak{B}') \geq \delta(\mathfrak{B}'/\mathfrak{A})$ by (2) of 2.2.5. But $\delta(\mathfrak{B}'/\mathfrak{A}) = \delta(AB'/A) \geq 0$ which yields our claim. Thus if we have some $\mathfrak{B} \in K_L$ and we show that there is some $\mathfrak{A} \subseteq \mathfrak{B}$ with $\mathfrak{A} \in K_{\overline{\alpha}}$ we can immediately conclude that $\mathfrak{B} \in K_{\overline{\alpha}}$

The following definition extends the notion of strong substructure to structures in $\overline{K_L}$:

Definition 2.3.3. Let $\mathfrak{X} \in \overline{K_L}$. For $\mathfrak{A} \subseteq_{\text{Fin}} \mathfrak{X}$, $\mathfrak{A}$ *is strong in* $\mathfrak{X}$, denoted by $\mathfrak{A} \leq \mathfrak{X}$, if $\mathfrak{A} \leq \mathfrak{B}$ for all $\mathfrak{A} \subseteq \mathfrak{B} \subseteq_{\text{Fin}} \mathfrak{Z}$. Given $\mathfrak{A}' \in K_L$ an embedding $f : \mathfrak{A}' \to \mathfrak{X}$ is called a *strong embedding* if $f(\mathfrak{A}')$ is strong in $\mathfrak{X}$.

Fact 2.3.4. *If* $\mathfrak{B}, \mathfrak{C} \in K_{\overline{\alpha}}$*,* $\mathfrak{A} = \mathfrak{B} \cap \mathfrak{C}$*, and* $\mathfrak{A} \leq \mathfrak{B}$*, then* $\mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C} \in K_{\overline{\alpha}}$ *and* $\mathfrak{C} \leq \mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C}$.

Proof. Let $\mathfrak{D} = \mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C}$. Due to Remark 2.3.2, it suffices to establish $\mathfrak{C} \leq \mathfrak{D}$. Let $\mathfrak{C} \subseteq \mathfrak{D}' \subseteq \mathfrak{D}$. Take $B' = B \cap D'$. Now $\delta(\mathfrak{D}'/\mathfrak{C}) = \delta(D' - C/C) = \delta(D' - C) - e(D' - C, C)$. Note that $D' - C = B' - A$ and as B, C are freely joined over A, $e(D' - C, C) = e(B' - A, A)$. Thus $\delta(D'/C) = \delta(B'/A)$. As $\mathfrak{A} \leq \mathfrak{B}$, it follows that $\mathfrak{C} \leq \mathfrak{D}$. $\square$

We now turn our attention towards *generic structures*. As all generic structures of interest will be built from subclasses of $K_{\overline{\alpha}}$, our definitions will be tailored to this context.

Definition 2.3.5. Let $K \subseteq K_{\overline{\alpha}}$ be closed under isomorphism and consider $(K, \leq)$. We say that $(K, \leq)$ is a *Fraïssé class* if

1. $(K, \leq)$ has the *amalgamation property*: For any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K$ and given strong embeddings $f_1 : \mathfrak{A} \to \mathfrak{B}$, $g_1 : \mathfrak{A} \to \mathfrak{C}$, there exists $\mathfrak{D} \in K$ and strong embeddings $f_2 : \mathfrak{B} \to \mathfrak{D}$, $g_2 : \mathfrak{C} \to \mathfrak{D}$ such that $f_2 \circ f_1(\mathfrak{A}) = g_2 \circ g_1(\mathfrak{A})$.

2. $(K, \leq)$ has the *joint embedding property*: For any given $\mathfrak{B}, \mathfrak{C} \in K$ there exists $\mathfrak{D} \in K$ and strong embeddings $f : \mathfrak{B} \to \mathfrak{D}$, $g : \mathfrak{C} \to \mathfrak{D}$.

Note that we do not require that K be closed under substructure. This is reflected in the fact that we require $\mathfrak{M} \in \overline{K_{\overline{\alpha}}}$ (as opposed $\mathfrak{M} \in \overline{K}$ which does not make sense as $\overline{K}$ is not well defined unless K is closed under substructure).

Definition 2.3.6. Let $K \subseteq K_{\overline{\alpha}}$. A countable structure $\mathfrak{M} \in \overline{K_{\overline{\alpha}}}$ is said to be a *generic* for $(K, \leq)$ if

1. $\mathfrak{M}$ is the union of an ω-chain $\mathfrak{A}_0 \leq \mathfrak{A}_1 \leq \dots$ with each $\mathfrak{A}_i \in K$.

2. If $\mathfrak{A}, \mathfrak{B} \in K$ with $\mathfrak{A} \leq \mathfrak{B}$ and $\mathfrak{A} \leq \mathfrak{M}$, then there is $\mathfrak{B}' \leq \mathfrak{M}$ such that $\mathfrak{B} \cong_{\mathfrak{A}} \mathfrak{B}'$.

3. If $\mathfrak{A} \in K$, then there is some embedding $f : \mathfrak{A} \to \mathfrak{M}$ such that $f(\mathfrak{A}) \leq \mathfrak{M}$.

Fact 2.3.7. *Let $K \subseteq K_{\overline{\alpha}}$ be such that $(K, \leq)$ be a Fraïssé class. Then a $(K, \leq)$ generic exists and is unique up to isomorphism. In particular $(K_{\overline{\alpha}}, \leq)$ is a Fraïssé class. Thus a generic structure for $(K_{\overline{\alpha}}, \leq)$ exists and is unique up to isomorphism.*

Proof. The fact that a generic for $(K, \leq)$ exists and is unique up to isomorphism is essentially the same as Fraïssé's original proof in [22] (see also Chapter 7.1 of [23] for more details).

Clearly $K_{\overline{\alpha}}$ is closed under isomorphisms. The fact that $(K_{\overline{\alpha}}, \leq)$ has amalgamation follows from Fact 2.3.4. Joint embedding is immediate for $(K_{\overline{\alpha}}, \leq)$ as $\emptyset \in K_{\overline{\alpha}}$. Thus the rest of the claim follows. $\square$

This justifies the following definition:

Definition 2.3.8. For a fixed $\overline{\alpha}$ we call the generic for $(K_{\overline{\alpha}}, \leq)$ the *Baldwin-Shi hypergraph for* $\overline{\alpha}$.

2.4 Closed sets

In this section we generalize the notion of strong substructure to substructures of arbitrary size by introducing the notion of a closed set. This will provide us with a useful tool for analyzing the various theories of Baldwin-Shi hypergraphs.

Definition 2.4.1. Let $\mathfrak{A}, \mathfrak{B} \in K_L$. Now $(\mathfrak{A}, \mathfrak{B})$ is a *minimal pair* if and only if $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \leq \mathfrak{C}$ for all $\mathfrak{A} \subseteq \mathfrak{C} \subset \mathfrak{B}$ but $\mathfrak{A} \not\leq \mathfrak{B}$.

Note that $(\mathfrak{A}, \mathfrak{B})$ is a minimal pair if and only if $\mathfrak{A} \subseteq \mathfrak{B}$, $\delta(\mathfrak{A}) \leq \delta(\mathfrak{C})$ for all $\mathfrak{A} \subseteq \mathfrak{C} \subset \mathfrak{B}$ but $\delta(\mathfrak{B}) < \delta(\mathfrak{A})$.

Definition 2.4.2. Let $\mathfrak{Z} \in \overline{K_L}$ and $X \subseteq Z$. We say X *is closed in* $\mathfrak{Z}$ if and only if for all $A \subseteq_{\text{Fin}} X$, if $(\mathfrak{A}, \mathfrak{B})$ is a minimal pair with $B \subseteq Z$, then $B \subseteq X$.

Remark 2.4.3. As any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K_L$ with $\mathfrak{A} \leq \mathfrak{C}$ and $\mathfrak{B} \subseteq \mathfrak{C}$ satisfies $\mathfrak{A} \cap \mathfrak{B} \leq \mathfrak{B}$ (see Remark 2.3.1) an easy argument yields that given $\mathfrak{Z} \in K_L$ and $\mathfrak{A} \subseteq_{\text{Fin}} \mathfrak{Z}$, $\mathfrak{A} \leq \mathfrak{Z}$ if and only if $\mathfrak{A}$ is closed in $\mathfrak{Z}$.

It is immediate from the above definition that any $\mathfrak{Z} \in \overline{K_L}$, Z is closed in $\mathfrak{Z}$ and that the intersection of a family of closed sets of $\mathfrak{Z}$ is again closed. These observations justify the following definition:

Definition 2.4.4. Let $\mathfrak{Z} \in \overline{K_L}$ and $X \subseteq Z$. The *intrinsic closure* of X in Z, denoted by $\mathrm{icl}_{\mathfrak{Z}}(X)$ is the smallest set X' such that $X \subseteq X' \subseteq Z$ and X' is closed in Z.

Remark 2.4.5. Fix $\mathfrak{Z}$ in K_L. We note that taking the map $\mathrm{icl}_{\mathfrak{Z}} : \mathbb{P}(Z) \to \mathbb{P}(Z)$ that takes a subset of $\mathfrak{Z}$ to its intrinsic closure is a *finitary closure operation*; i.e. for $X, Y \subseteq Z$, it satisfies $X \subseteq \mathrm{icl}(X)$, $\mathrm{icl}(\mathrm{icl}(X)) = \mathrm{icl}(X)$, if $X \subseteq Y$, then $\mathrm{icl}(X) \subset \mathrm{icl}(Y)$ and $\mathrm{icl}(X) = \bigcup_{X_0 \subseteq_{\mathrm{Fin}} X} \mathrm{icl}(X_0)$ (we have dropped the index $\mathfrak{Z}$ as it is the only structure with respect to which intrinsic closures are taken).

We show that $\mathrm{icl}(X) = \bigcup_{X_i \subseteq_{\mathrm{Fin}} X} \mathrm{icl}(X_i)$. The fact that $\bigcup_{X_i \subseteq_{\mathrm{Fin}} X} \mathrm{icl}(X_i) \subseteq \mathrm{icl}(X)$ is clear. Thus it suffices to show that $X \subseteq \bigcup_{X_i \subseteq_{\mathrm{Fin}} X} \mathrm{icl}(X_i)$ and that $\bigcup_{X_i \subseteq_{\mathrm{Fin}} X} \mathrm{icl}(X_i)$ is closed. The first requirement is clear. To see that the second requirement is satisfied, let $A \subseteq_{\mathrm{Fin}} \bigcup_{X_i \subseteq_{\mathrm{Fin}} X} \mathrm{icl}(X)_i$. Thus there exists $A_{i_1}, \ldots, A_{i_k}$ such that each A_{i_j} is finite and $A \subseteq \bigcup A_{i_j}$ with $A_{i_j} \subseteq \mathrm{icl}(X_{i_j})$ where $\{X_i\}_{i \in I}$ is a fixed indexing of the finite subsets of X. Let X' be the union of the X_{i_j} whose indexes appear above. Now X' is finite and it is clear that $\bigcup \mathrm{icl}(X_{i_j}) \subseteq \mathrm{icl}(X')$. Thus $A \subseteq \bigcup A_{i_j} \subseteq \bigcup \mathrm{icl}(X_{i_j}) \subseteq \mathrm{icl}(X')$. Thus if (A, B) is a minimal pair with $B \subseteq Z$, then $B \subseteq \mathrm{icl}(X') \subseteq \bigcup_{X_i \subseteq_{\mathrm{Fin}} X} \mathrm{icl}(X_i)$, i.e. $\bigcup_{X_i \subseteq_{\mathrm{Fin}} X} \mathrm{icl}(X_i)$ is closed from which our claim follows.

It should be noted that we may construct the closure of a finite $A \subseteq Z$ as follows. Let;

$$\mathbb{I}_0(A) = A \cup \bigcup \{B : A \subseteq B \subseteq Z, \text{ and } (A, B) \text{ is a minimal pair}\}$$

For each $n < \omega$, let $\mathbb{I}_{n+1}(A)$ be given by

$$A \cup \bigcup \{B : B \subseteq Z, \text{ and there is an } A' \subseteq \mathbb{I}_n(A) \text{ such that} (A', B) \text{ is a minimal pair}\}$$

Now by construction we have that $\bigcup_{n<\omega} \mathbb{I}_n(A)$ is closed and that it contains A. A routine argument shows that $\bigcup_{n<\omega} \mathbb{I}_n(A)$ is in fact the closure of A. From this it follows that the definition of a closed set in [1] and [4] and the notion here correspond.

2.5 Pregeometries

In this section, we briefly mention some details regarding *pregeometries*. As can be seen form the details below, pregeometries allows one to have a well defined notion of dimension. Detailed discussions regarding pregeometries can be found in Chapter 8 of [8] and Chapter 3 of [24].

Definition 2.5.1. A *pregeometry* (X, cl) is a set X with a closure operator $\mathrm{cl} : \mathbb{P}(X) \to \mathbb{P}(X)$, (where $\mathbb{P}(X)$ denotes the power set of X) such that for all $A, B \subseteq X$ and $a, b \in X$

1. (Reflexivity) $A \subseteq \mathrm{cl}(A)$.

2. (Monotonocity) If $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

3. (Finite character) $\mathrm{cl}(A)$ is the union of all $\mathrm{cl}(A^*)$, where the A^* range over all finite subsets of A.

4. (Transitivity) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

5. (Exchange) $a \in \mathrm{cl}(Ab) - \mathrm{cl}(A)$ implies that $b \in \mathrm{cl}(Aa)$.

   We say that A is *closed* if $A = \mathrm{cl}(A)$

   Pregeometries allow us to introduce a notion of independence as follows:

Definition 2.5.2. Let (X, cl) be a pregeometry and let $A, B, S \subseteq X$.

1. We say that A is *independent over* B if $a \notin \mathrm{cl}(B \cup (A - \{a\}))$ for all $a \in A$.

2. We say that A is a *basis for* S *over* B if A is a maximal subset of $\mathrm{cl}(S \cup B)$ that is independent over B.

In the case that $B = \emptyset$, we simply say that A is *independent* and A *is a basis for* S.

The following remark allows us to introduce a notion of dimension.

Remark 2.5.3. Let (X, cl) be a pregeometry and let $A, B, C, S \subseteq X$. If A, C are bases for S over B, then $|A| = |C|$.

Definition 2.5.4. Let (X, cl) be a pregeometry and let $A, B, S \subseteq X$.. If A is a basis for S over B, then we call $|A|$ the *dimension of* S over B and denote this by $\dim_{\mathrm{cl}}(S/B)$. If $B = \emptyset$ then we call the cardinality of a basis of S the dimension of S and denote this by $\dim_{\mathrm{cl}}(S)$

The following remark shows that $\dim_{\mathrm{cl}}$ satisfies a certain subadditivity property.

Remark 2.5.5. Let (X, cl) be a pregeometry and let $A, B \subseteq X$ be closed. Then $\dim_{\mathrm{cl}}(\mathrm{cl}(A \cup B)) + \dim_{\mathrm{cl}}(A \cap B) \leq \dim_{\mathrm{cl}}(A) + \dim_{\mathrm{cl}}(B)$.

Definition 2.5.6. Let (X, cl) be a pregeometry and let $A, B \subseteq X$, we can consider *the localization to* B given by $\mathrm{cl}_B(A) = \mathrm{cl}(A \cup B)$.

Remark 2.5.7. If (X, cl) is a pregeometry, then (X, cl_A) is a pregeometry.

Definition 2.5.8. Let (X, cl) be a pregeometry.

1. We say that (X, cl) is *trivial* if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{a\})$ for any $A \subseteq X$.

2. We say that (X, cl) is *modular* if for any finite-dimensional closed $A, B \subseteq X$,

   $\dim_{\mathrm{cl}}(\mathrm{cl}(A \cup B)) = \dim_{\mathrm{cl}}(A) + \dim_{\mathrm{cl}}(B) - \dim_{\mathrm{cl}}(A \cap B)$.

3. We say that (X, cl) is *locally modular* if (X, cl_a) is modular for some $a \in X$.

Chapter 3: Existence theorems

The results of this chapter, which consists of constructing certain finite weighted hypergraphs, forms the core of the results that follow. We use them throughout to construct various other finite structures satisfying carefully chosen properties. Such structures, in addition to giving us the ability to generalize the arguments of [2], allow us to explore the atomic models, non-forking, regular types, etc of the theory of the $(K_{\overline{\alpha}}, \leq)$ generic when combined with the results in Chapter 3. The results in this chapter are inspired by Lemma 4.1 of [2] by Laskowski. Laskowski himself was using a variant of a construction found in [25] by Ikeda.

We begin with the following:

Definition 3.0.1. We say that $\overline{\alpha}$ *is rational* if $\overline{\alpha}_E$ is rational for all $E \in L$.

Definition 3.0.2. If $\overline{\alpha}(E) = 1$ for all E in L and each $E \in L$ has arity 2, then we say that $\overline{\alpha}$ *is graph-like with weight one*.

Definition 3.0.3. Let $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{B}) > 0$. We call $\mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{B} \subseteq \mathfrak{D}$ an *essential minimal pair* if $(\mathfrak{B}, \mathfrak{D})$ is a minimal pair and for any $\mathfrak{D}' \subsetneq \mathfrak{D}$, $\delta(\mathfrak{D}'/\mathfrak{D}' \cap \mathfrak{B}) \geq 0$.

Definition 3.0.4. We use $ar(L)$ to denote $\max\{ar(E) : E \in L\}$.

One of the main results of Section 3 is Theorem 3.2.15. It states that given $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{B}) > 0$, there exists infinitely many non-isomorphic $\mathfrak{D} \in K_{\overline{\alpha}}$ where $(\mathfrak{B}, \mathfrak{D})$ is an essential minimal pair that satisfies $-\epsilon \leq \delta(\mathfrak{D}/\mathfrak{B}) < 0$ where ϵ is, *in context*, arbitrarily small. The overall proof of this theorem has the following structure:

1. We begin by introducing the notion of an L-*collection*. An L-collection r will be a multiset, i.e. a set with repeated elements, where each element is an element of L. For any E in L, we let $r(E)$ be the number of times E is repeated in r.

2. Next we introduce the notion of a *template*. A template, will be a triple $\langle n, \underline{r}, t\rangle$. Here n is a positive integer and $\underline{r} = \langle r_1 \ldots, r_n\rangle$ will index a collection L-collections. Further each r_i will have the property that for each $E \in L$, $r_i(E) < m_{pt}$, where m_{pt} is a fixed positive integer that we will introduce shortly. Finally $t = \{E_1, \ldots, E_{n-1}\}$ is an indexed L-collection. The idea is that the extension $\mathfrak{D} \supseteq \mathfrak{B}$ will have universe $D - B = \{d_1, \ldots, d_n\}$. Further, for each $E \in L$, it will have $\underline{r}(j)(E)$ many relations involving only subsets of B and d_j. Also there will be precisely one relation involving $t(j)$, $\{d_j, d_{j+1}\}$ and a subset of B and no other relations (besides the ones already in $\mathfrak{B}$) will hold.

3. A moments' reflection shows that under the above conditions above, not all $\mathfrak{B} \in K_{\overline{\alpha}}$ will have extensions by templates (for example L might contain only one relation symbol whose arity $ar(E)$ is much larger than $|B|$). We identify

crude bounds such as m_{pt} and on $|B|$ that will make the construction of an extension by a template feasible. Let $ar(L) = \max\{ar(E) : E \in L\}$ The bound on $|B|$ will be picked so that there are at least $m_{\mathrm{pt}} ar(L)$ disjoint subsets of B.

4. With these technical details aside, we isolate the notions of *acceptable* and *good* templates for a fixed $\mathfrak{B} \in K_{\overline{\alpha}}$ with positive rank. A good template Θ is set up in such a way that guarantees that an extension $\mathfrak{D}$ of $\mathfrak{B}$ using Θ will be an essential minimal pair. Thus we are left with generating good templates, which we carry out with the help of some number theoretic results (see Appendix A). The notion of acceptable, which is weaker than the notion of good, is isolated as it plays a part in the second main result of this section, i.e. Theorem 3.3.6.

5. We prove Lemma 3.2.13, which states: Given $\mathfrak{B} \in K_{\overline{\alpha}}$ with $|B|$ *sufficiently large* and $\delta(\mathfrak{B}) > 0$ that there are here exists infinitely many non-isomorphic $\mathfrak{D} \in K_{\overline{\alpha}}$ where $(\mathfrak{B}, \mathfrak{D})$ is an essential minimal pair that satisfies $-\epsilon \leq \delta(\mathfrak{D}/\mathfrak{B}) < 0$. Here again, ϵ is, in context, arbitrarily small. Finally in Theorem 3.2.15 we establish the desired result.

We now introduce some of the notions that we alluded to above:

Definition 3.0.5. We define m_{pt} be the least positive integer $m \in \omega$ such that $1 - m_{\mathrm{pt}}\overline{\alpha}_E < 0$ for all $E \in L$. We let m_{suff} be the product $m_{\mathrm{pt}} ar(L)$.

Remark 3.0.6. Note that if $\mathfrak{B} \in K_L$ and $\mathfrak{D} \in K_L$ is a one point extension of $\mathfrak{B}$ and $\delta(\mathfrak{D}/\mathfrak{B}) \geq 0$, then the number of relations that include the single point in $D - B$

and B is less than m_{pt}. It can be seen that given an essential minimal pair $(\mathfrak{A}, \mathfrak{C})$ and $c \in C - A$, then $N(c, A) < m_{\mathrm{pt}}$. Now m_{suff} gives a crude lower bound over the size of $\mathfrak{B} \in K_{\overline{\alpha}}$ over which we can construct essential minimal pairs. Here m_{suff} stands for sufficient.

The other main result in this section, Theorem 3.3.6, is concerned with building $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $\delta(\mathfrak{D}) = 0$ that extend $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{B}) > 0$. We will see that the existence of such structures can be characterized by the notion of *coherence*.

Definition 3.0.7. We say that $\overline{\alpha}$ is *coherent* if there exists $\langle m_E : E \in L, m_E \in \omega, m_E > 0\rangle$ such that $\sum_{E \in L} m_E \overline{\alpha}_E \in \mathbb{Q}$.

Remark 3.0.8. Clearly if $\overline{\alpha}$ is rational, then $\overline{\alpha}$ is coherent. We now give an example of a coherent $\overline{\alpha}$ that is not rational: Fix $0 < \beta < 1/2$ irrational. If $\overline{\alpha}(E_1) = \beta$ for some $E_1 \in L$ and $\overline{\alpha}(E_2) = 1 - \beta$ for some $E_2 \in L$ and $\overline{\alpha}(E) \in \{\beta, 1 - \beta\}$ for all $E \in L$, then $\overline{\alpha}$ is coherent but not rational.

In Section 5, we use these structures to classify the $\overline{\alpha}$ for which the corresponding theory of the Baldwin-Shi hypergraph has atomic models. The construction of the required $\mathfrak{D}$ will again be done with the help of templates and will reuse the ideas developed in the constructions of essential minimal pairs with some caveats.

3.1 Templates and Extensions

Throughout the rest of this section we work under the assumption that $\overline{\alpha}$ is not graph-like with weight one.

We begin by defining a template.

Definition 3.1.1. A multiset r where the elements of r are relation symbols from L will be called an *L-collection*. Given $E \in L$, $r(E)$ will denote the number of times that E is repeated in r. Further we let $|r| = \sum_{E \in L} r(E)$. Given a L-collections r and r', we say that r' is a *sub-collection* of L if $r' \subseteq r$.

Notation 3.1.2. Throughout the rest of Section 3, we will use the letters r, s (with or without various subscripts) to denote L-collections.

Definition 3.1.3. Let $n \geq 3$ be a fixed positive integer. Let $\underline{r} = \langle r_1, \ldots r_n \rangle$ where each r_i is an L-collection. Further let t be an *indexed* L-collection with $|t| = n - 1$, i.e. there is a fixed enumeration $E_1, \ldots, E_{n-1}$ of the elements of t. We call a triple $\Theta = \langle n, \underline{r}, t \rangle$ an *n-template* if for each $1 \leq i \leq n$, $E \in L$ we have that $r_i(E) < m_{\text{pt}}$.

Given a template and $\mathfrak{B} \in K_L$, we use the template to create an extension $\mathfrak{D}$ of $\mathfrak{B}$. As noted previously *The constructions of interest* are the ones where given $\mathfrak{B} \in K_{\overline{\alpha}}$ and we can create $\mathfrak{D}$ extending $\mathfrak{B}$ such that $\mathfrak{D} \in K_{\overline{\alpha}}$ and $\mathfrak{D}$ satisfies other desirable properties. We now make precise the notion of an extension by a template that was somewhat loosely described at the beginning of Section 3.

Definition 3.1.4. Let $\mathfrak{B} \in K_L$ such that $|B| \geq m_{\text{suff}}$. Let Θ be an n-template. An *extension of $\mathfrak{B}$ by Θ* is some $\mathfrak{D}$ in K_L that satisfies

1. $\mathfrak{B} \subseteq \mathfrak{D}$

2. The universe of $\mathfrak{D} - \mathfrak{B}$ is $\{d_1, \ldots, d_n\}$, i.e. it consists of n-points.

3. For each $1 \leq i \leq n-1$, there is a subset $Q \subseteq B$ of size $ar(E_i) - 2$ such that $\{d_i, d_{i+1}\} \cup Q \in E_i^D$ (where Q is possibly empty).

4. If $r_i(E) > 0$ for some $E \in L$, there are precisely $r_i(E)$ distinct subsets $Q_1, \dots, Q_{r_i(E)}$ of B of size $ar(E)-1$ such that $\{d_i\} \cup Q_j \in E^D$ for $1 \leq j \leq r_i(E)$.

5. There are no further relations in $\mathfrak{D}$ than the ones that were originally in $\mathfrak{B}$ and the ones that are described above.

In the case for any $b \in B$, there exists some d_j, $Q' \subseteq D$, $E \in L$ such that $\{b, d_j\} \cup Q' \in E^{\mathfrak{D}}$, we say that $\mathfrak{D}$ *covers* $\mathfrak{B}$.

Lemma 3.1.5. *Let $\mathfrak{B} \in K_L$ such that $|B| \geq m_{suff}$. Let Θ be an n-template. There is an an extension $\mathfrak{D} \supseteq \mathfrak{B}$ of $\mathfrak{B}$ by Θ. Moreover if $\sum_{i=1}^{n} |r_i| \geq |B|$ or if $\sum_{ar(E) \geq 3}(t(E) + \sum_{i=1}^{n} r_i(E)) \geq |B|$ there exists $\mathfrak{D}$ that covers $\mathfrak{B}$.*

Proof. Take $D_0 = \{d_1, \dots, d_n\}$ and consider the L structure $\mathfrak{D}_0$ with universe D_0 and no relations in $\mathfrak{D}_0$. Now $\mathfrak{D}$ will be a structure with universe $B \cup D_0$.

First note that since $|B| \geq m_{\text{suff}}$, B has at least m_{pt} distinct subsets of size $ar(E) - 1$ for each $E \in L$. For each $1 \leq i \leq n-1$ we may fix some subset $Q \subseteq B$ and add a relation so that $\{d_i, d_{i+1}\} \cup Q \in E_i^D$. Here Q is possibly empty: in fact Q is empty if and only if E_i is a binary relation symbol.

Now fix $1 \leq i \leq n$. For each $E \in L$ we have $r_i(E) < m_{\text{pt}}$. Thus for fixed $E \in L$, as $|B| \geq m_{\text{suff}}$, we may choose $r_i(E)$ distinct subsets Q_j as $1 \leq j \leq r_i(E)$, of B where each Q_j is of size $ar(E) - 1$. Add relations so that $\{d_i\} \cup Q_j \in E'^D$ for $1 \leq j \leq r_i(E)$. Do this for each relation symbol $E \in L$. Now assume that this process of adding relations has been carried out for each $1 \leq i \leq n$. Let the resulting structure be $\mathfrak{D}$. Note that the relations that hold on $\mathfrak{D}$ are precisely the ones that turn B to $\mathfrak{B}$ and the relations described so far. It is now clear that the resulting structure satisfies the properties required of $\mathfrak{D}$.

If $\sum_{i=1}^{n} |r_i| \geq |B|$ we may insist that the choice of Q_j, as E ranges through L, be made so that their union is B. If $\sum_{ar(E)\geq 3}(t(E) + \sum_{i=1}^{n} r_i(E)) \geq |B|$, then we may insist that the choice of the various Q and Q_j be made so that the union is B. In either case the statement that for any $b \in B$, there exists some d_j, $Q' \subseteq D$ such that $\{b, d_j\} \cup Q' \in E^{\mathfrak{D}}$ for some $E \in L$ holds. $\square$

Remark 3.1.6. Note that an extension by Θ need not be unique up to isomorphism over $\mathfrak{B}$. However given two non-isomorphic extensions $\mathfrak{D}, \mathfrak{D}'$ of $\mathfrak{B}$ by Θ their relative ranks are identical: $\delta(\mathfrak{D}/\mathfrak{B}) = \delta(\mathfrak{D}'/\mathfrak{B})$. Hence $\delta(\mathfrak{D}) = \delta(\mathfrak{D}')$.

Notation 3.1.7. Let $\Theta = \langle n, \underline{r}, t\rangle$ be an n-template. Fix $1 \leq j \leq n$. Let $\mathfrak{B} \in K_L$ such that $|B| \geq m_{\text{suff}}$ and let $\mathfrak{D}$ be an extension by Θ of $\mathfrak{B}$. Under the natural enumeration of $D - B = \{d_1, \ldots, d_n\}$ used to construct the extension; we let $\mathfrak{D}^j$ denote the substructure of $\mathfrak{D}$ with universe $B \cup \{d_1, \ldots, d_j\}$ for $1 \leq j \leq n$ and we let $\mathfrak{D}^{j,k}$ denote the substructure of $\mathfrak{D}$ with $= B \cup \{d_j, \ldots d_k\}$ for any $1 \leq j \leq k \leq n$.

We now define the *acceptable* and *good* templates. As noted previously, good templates are defined with the construction of essential minimal pairs in mind. Acceptable templates capture a weaker notion that is common to both the essential minimal pairs and the rank zero extensions that are dealt with in Section 3.3.

When dealing with templates it will often be convenient to focus on the sub-language of the symbols that occur in Θ. We make the following somewhat broader definition.

Definition 3.1.8. Given a *triple* $\Theta = \langle n, \underline{r}, t \rangle$, the *localization of* L *to* Θ, denoted by L^Θ is the subset of L such that $E \in L^\Theta$ if and only if E occurs positively in Θ, i.e. $r_j(E) > 0$ for some $1 \le j \le n$ or $E = E_j$ for some $1 \le j \le n-1$. Further we let $Gr_\Theta(2)$ denote the least positive value of $\sum_{E \in L^\Theta} \overline{\alpha}(E) n_E - 1$ for non-negative integers n_E.

Remark 3.1.9. The reason behind using the notation $Gr_\Theta(2)$ will become clear in Section 3.2.

Definition 3.1.10. Let $\mathfrak{B} \in K_{\overline{\alpha}}$ be such that $|B| \ge m_{\text{suff}}$ and $\delta(\mathfrak{B}) > 0$. Let Θ be a n-template and let $\mathfrak{D}$ be an extension of $\mathfrak{B}$ by Θ. We say that Θ is *acceptable* for $\mathfrak{B}$ if and only if

1. $0 < -\delta(\mathfrak{D}/\mathfrak{B}) \le \min\{\delta(\mathfrak{B}), Gr_\Theta(2)\}$.

2. $\delta(\mathfrak{D}^j/\mathfrak{B}) \ge 0$ for $1 \le j \le n-1$.

3. $\overline{\alpha}(E_j) - \delta(\mathfrak{D}^j/\mathfrak{B}) > 0$ for $1 \le j \le n-1$.

We say that Θ is *good* for $\mathfrak{B}$ if

1. Θ is acceptable for $\mathfrak{B}$.

2. $\overline{\alpha}(E_j) - \delta(\mathfrak{D}^j/\mathfrak{B}) + \delta(\mathfrak{D}/\mathfrak{B}) \ge 0$ for $1 \le j \le n-1$.

3. We may in addition assume that $\mathfrak{D}$ can be chosen so that it covers $\mathfrak{B}$.

The following lemma captures the key properties of extensions by acceptable and good templates.

Lemma 3.1.11. *Let $\mathfrak{B} \in K_{\overline{\alpha}}$ be such that $|B| \geq m_{suff}$ and $\delta(\mathfrak{B}) > 0$. Let Θ be an n-template and let $w = n - (\sum_{i=1}^{n-1} \overline{\alpha}_{E_i} + \sum_{i=1}^{n} \sum_{E \in L} \overline{\alpha}_E r_i(E))$. Let $\mathfrak{D}$ be an extension by Θ of $\mathfrak{B}$*

1. *If Θ is acceptable, then*

   1.a For any $\mathfrak{B} \subseteq \mathfrak{D}' \subsetneq \mathfrak{D}$ such that $d_n \notin \mathfrak{D}'$, $\delta(\mathfrak{D}'/\mathfrak{B}) \geq 0$

   1.b For any $\mathfrak{D}' \subsetneq \mathfrak{D}$ such that $d_n \notin \mathfrak{D}'$, $\delta(\mathfrak{D}'/\mathfrak{D}' \cap \mathfrak{B}) \geq 0$

   1.c For any $\mathfrak{B} \subseteq \mathfrak{D}' \subseteq \mathfrak{D}$, $\delta(\mathfrak{D}'/\mathfrak{B}) \geq w$

2. *If Θ is good for $\mathfrak{B}$, we may choose $\mathfrak{D}$ so that $\mathfrak{D}$ covers $\mathfrak{B}$ and then*

   2.a $\mathfrak{D} \in K_{\overline{\alpha}}$

   2.b For any proper $\mathfrak{B} \subseteq \mathfrak{D}' \subsetneq \mathfrak{D}$, $\delta(\mathfrak{D}'/\mathfrak{B}) \geq 0$

   2.c For any $\mathfrak{D}' \subsetneq \mathfrak{D}$, $\delta(\mathfrak{D}'/\mathfrak{B} \cap \mathfrak{D}') \geq 0$

   i.e. $(\mathfrak{B}, \mathfrak{D})$ is an essential minimal pair with $\delta(\mathfrak{D}/\mathfrak{B}) = w$.

Proof. We begin with (1): For (1.a), the case $\mathfrak{D}' = \mathfrak{D}^j$ for some $1 \leq j \leq n-1$ is immediate. Consider the case that $\mathfrak{D}' = \mathfrak{D}^{k+1,j}$ where $1 \leq k < j \leq n-1$. As there is only a single relation, namely E_k, that contains the points d_k, d_{k+1}, it follows that $\delta(\mathfrak{D}^{k+1,j}/\mathfrak{B}) = \delta(\mathfrak{D}^j/\mathfrak{D}^k) + \overline{\alpha}(E_k)$. Further $\delta(\mathfrak{D}^{k+1,j}/\mathfrak{B}) = \delta(\mathfrak{D}^j/\mathfrak{B}) - \delta(\mathfrak{D}^k/\mathfrak{B}) + \overline{\alpha}(E_k)$. But $\overline{\alpha}(E_k) - \delta(\mathfrak{D}^k/\mathfrak{B}) + \delta(\mathfrak{D}^j/\mathfrak{B}) \geq 0$ by using conditions 2 and 3 of Θ being acceptable. Since an arbitrary $\mathfrak{B} \subseteq \mathfrak{D}' \subsetneq \mathfrak{D}$ with $d_n \notin \mathfrak{D}$ can be written as the free join different $\mathfrak{D}^{k,j}$ over $\mathfrak{B}$, it follows that *for* $\mathfrak{B} \subseteq \mathfrak{D}' \subsetneq \mathfrak{D}'$, $\delta(\mathfrak{D}'/\mathfrak{B}) \geq 0$. Now consider an arbitrary $\mathfrak{D}' \subseteq \mathfrak{D}$ such that $d_n \notin \mathfrak{D}'$ and $\mathfrak{B} \not\subseteq \mathfrak{D}'$. By (2) of Fact

2.2.5 $\delta(\mathfrak{D}'/\mathfrak{B}\cap\mathfrak{D}') \geq \delta(BD'/B)$. But the above shows that $\delta(BD'/B) \geq 0$ and thus $(1.b)$ follows.

For $(1.c)$, note that for $1 \leq j \leq n$, $\delta(\mathfrak{D}^j/\mathfrak{B}) < 0$ if and only if $j = n$ if and only if $\delta(\mathfrak{D}^j/\mathfrak{B}) = w$. As $\delta(\mathfrak{D}^{k+1,j}/\mathfrak{B}) = \delta(\mathfrak{D}^j/\mathfrak{B}) + \overline{\alpha}(E_k) - \delta(\mathfrak{D}^k/\mathfrak{B})$ for $1 \leq k < j \leq n$ and since $\mathfrak{D}'$ can be written as the free join of several $\mathfrak{D}^{k,j}$ and over $\mathfrak{B}$ and at most one of the $\mathfrak{D}^{k,j}$ satisfies $0 > \delta(\mathfrak{D}^{k,j}/\mathfrak{B}) \geq w$, it follows that $\delta(\mathfrak{D}'/\mathfrak{B}) \geq w$.

Now consider (2): We are assuming $\mathfrak{D}$ covers $\mathfrak{B}$. As $\delta(\mathfrak{D}/\mathfrak{B}) = w$ by construction both $(2.a)$ and the statement regarding $(\mathfrak{B},\mathfrak{D})$ being an essential minimal pair follows from $(2.b)$ and $(2.c)$. For the proof of $2.b$, first consider $\mathfrak{D}' = \mathfrak{D}^{j+1,n}$ for $1 \leq j \leq n-1$. By arguing as above we obtain that $\delta(\mathfrak{D}^{j+1,n}/\mathfrak{B}) = \overline{\alpha}(E_j) - \delta(\mathfrak{D}^j/\mathfrak{B}) + \delta(\mathfrak{D}/\mathfrak{B})$. By using condition (2) of good, it follows that $\overline{\alpha}(E_j) - \delta(\mathfrak{D}^j/\mathfrak{B}) + \delta(\mathfrak{D}/\mathfrak{B}) \geq 0$. As Θ is good, it is also acceptable and thus $\delta(\mathfrak{D}^{k,j}/\mathfrak{B}) \geq 0$ for $1 \leq k \leq j \leq n-1$. Since an arbitrary $\mathfrak{B} \subseteq \mathfrak{D}' \subsetneq \mathfrak{D}$ can be written as the free join different $\mathfrak{D}^{k,j}$ over $\mathfrak{B}$ it follows that *for* $\mathfrak{B} \subseteq \mathfrak{D}' \subsetneq \mathfrak{D}'$, $\delta(\mathfrak{D}'/\mathfrak{B}) \geq 0$. $\delta(\mathfrak{D}'/\mathfrak{B}) \geq 0$.

It remains to show that for a general substructure $\mathfrak{D}' \subsetneq \mathfrak{D}$, we have that $\delta(\mathfrak{D}'/\mathfrak{B}\cap\mathfrak{D}') \geq 0$. If $D' - B \neq D - B$, then this follows easily by $(2.b)$ and (2) of Fact 2.2.5. So assume that $D' - B = D - B$. Since $\mathfrak{D}' \subsetneq \mathfrak{D}$, it follows that $\mathfrak{D}' \cap \mathfrak{B} \neq \mathfrak{B}$. Fix a relation $E \in L$ such that it holds with a point from $D' - B$ and at least one point from $B - B'$. By using (2) of Fact 2.2.5 we see that $\delta(\mathfrak{D}'/\mathfrak{D}'\cap\mathfrak{B}) \geq \delta(\mathfrak{D}/\mathfrak{B}) + \overline{\alpha}(E)$. Since $-Gr_\Theta(2) \leq \delta(\mathfrak{D}/\mathfrak{B})$, it follows that $0 \leq Gr_\Theta(2) + \overline{\alpha}_E \leq \delta(\mathfrak{D}/\mathfrak{B}) + \overline{\alpha}_E$. Thus $(2.c)$ follows. $\square$

3.2 Generating Templates

In this section we introduce the notions of *acceptable pairs* and *good pairs*. We will show how to construct a good/acceptable template by using a good/acceptable pair. The acceptable and good pairs are easily obtained by the well known number theoretic results that can be found in the Appendix. This allows us to establish that the constructions in Section 3.1 can indeed be carried out. We finish this section with Lemma 3.2.8 and Theorem 3.2.15 which generalize results in [2]. We begin by introducing the notion of granularity.

Definition 3.2.1. Given a positive integer $m \in \omega$ and $L_0 \subseteq L$, we define $Gr_{L_0}(m)$, *the granularity* m *relative to* L_0, to be the smallest positive value $\sum_{E\in L_0} \overline{\alpha}_E n_E - k$ where k is an integer satisfying $0 < k < m$ and each $n_E \in \omega$. In case $L = L_0$ we call $Gr_L(m)$ the *granularity of* m and denote it by $Gr(m)$.

Remark 3.2.2. Note that given a triple $\Theta = \langle n, \underline{r}, t\rangle$, $Gr_\Theta(2) = Gr_{L^\Theta}(2)$. Further if $Gr(2) = \sum_{E\in L} n_E\overline{\alpha}_E - 1$, then $\sum_{E\in L} n_E < m_{pt}$

The following are immediate from the definition of granularity.

Lemma 3.2.3. *For all* $E \in L$*,* $Gr(2) \leq \overline{\alpha}_E$*.*

Lemma 3.2.4. *Suppose* $\mathfrak{D} \in \overline{K_{\overline{\alpha}}}$ *and* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *are finite substructures of* $\mathfrak{D}$*, satisfying* $(\mathfrak{A}, \mathfrak{B})$ *is a minimal pair,* $|B - A| < m$*,* $\mathfrak{A} \subseteq \mathfrak{C}$*, but* $\mathfrak{B} \nsubseteq \mathfrak{C}$*. Then* $\delta(\mathfrak{D}'/\mathfrak{C}) \leq -Gr(m)$*, where* $\mathfrak{D}'$ *is the substructure of* $\mathfrak{D}$ *with universe* $B \cup C$*.*

Proof. Let $\mathfrak{B}^*$ be the substructure of $\mathfrak{D}$ with universe $\mathfrak{B} \cap \mathfrak{C}$. Then $\mathfrak{A} \leq \mathfrak{B}^* \subseteq \mathfrak{B}$ and $\mathfrak{B}, \mathfrak{C}$ are disjoint over B^*, so $\mathfrak{D}'$, the substructure of $\mathfrak{D}$ with universe $\mathfrak{B} \cup \mathfrak{C}$, is a join of $\mathfrak{B}, \mathfrak{C}$. Then $\delta(\mathfrak{D}'/\mathfrak{C}) \leq \delta(\mathfrak{B}/\mathfrak{B}^*) \leq -Gr(m)$. where the first inequality follows from (1) of Fact 2.2.5 and the second follows from $(\mathfrak{A}, \mathfrak{B})$ being a minimal pair and granularity. $\square$

We now turn our attention to good pairs and acceptable pairs. The goal will be to use good/acceptable pairs to generate good/acceptable templates, which we proceed to do in Lemma 3.2.7.

Definition 3.2.5. Given a non-negative integer n and an L-collection r, we let the weighted sum $n - \sum_{E \in L} \overline{\alpha}_E r(E)$ be denoted by $w(n, r)$.

Definition 3.2.6. Let $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{B}) > 0$. Let $n \in \omega$ and let s be an L-collection. Let $L_0 \subseteq L$ be such that $E \in L_0$ if and only if $s(E) > 0$. We say that $\langle n, s \rangle$ is an *acceptable pair* for $\mathfrak{B}$, if

1. $\min\{\delta(\mathfrak{B}), Gr_{L_0}(2)\} \geq -w(n, s) > 0$

2. $|s| \geq n$

We say that $\langle n, s \rangle$ is a *good pair* for $\mathfrak{B}$

1. $\langle n, s \rangle$ is acceptable

2. $|s| \geq |B| + (n-1)$ or $\sum_{ar(E) \geq 3}(t(E) + \sum_{i=1}^{n} r_i(E)) \geq |B|$

3. For all $m \leq n$ and sub-collections s' of s, $w(m, s')$ not in the interval $(w(n, s), 0)$.

Often we will not mention $\mathfrak{B}$ as it will be clear from context.

Lemma 3.2.7. *Let $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{B}) > 0$, $|B| \geq m_{suff}$. If $\langle n, s\rangle$ is an acceptable pair for $\mathfrak{B}$, then there exists an acceptable n-template $\Theta = \langle n, \underline{r}, t\rangle$. If $\langle n, s\rangle$ is good, then Θ will be good for $\mathfrak{B}$.*

Proof. We begin with the observation that if u is a sub-collection of s, then $s - u$ is the residual multiset with $(s - u)(E) = s(E) - u(E)$. Our first goal is to define the *triple* $\Theta = \langle n, \underline{r}, t\rangle$. We do this in Step 1. We do this using a "greedy algorithm". In Step 2, we establish that the triple Θ we have constructed is indeed a template and it is acceptable/good based on the corresponding properties of (n, s).

Step 1: We first define t. For $1 \leq j \leq n - 1$ inductively define E_j so that E_j is in the residual multiset $s - \{E_1, \ldots, E_{j-1}\}$ and $\alpha(E_j) = \max\{\alpha(E) : E \in s - \{E_1, \ldots, E_{j-1}\}\}$. If there is $E \in L$ with arity at least 3 such that $s(E) \geq n-1 \geq |B|$ and $\overline{\alpha}(E) \geq \overline{\alpha}(E^*)$ for all $E^* \in L$, then we insist that the above E_j satisfy $E_j = E$. Let t be the *ordered* L-collection $\langle E_1 \ldots, E_{n-1}\rangle$. Let s_1 be the residual multiset $s - \{E_1, \ldots, E_{n-1}\}$. For $1 \leq j \leq n$ define the *potential relative rank* $Rel(j) = \sum_{i=1}^{j} w(1, r_i) - \sum_{i=1}^{j-1} \overline{\alpha}(E_i)$.

First let $r_1 \subseteq s_1$ be an L-collection such that $Rel(1) = w(1, r_1)$ achieves the least possible non-negative value. Assume that for $1 \leq j \leq n - 1$ that r_j, s_j have been defined and take s_{j+1} to be the residual multiset $s_j - r_j$. For $1 \leq j < n - 1$ pick $r_{j+1} \subseteq s_{j+1}$ such that $Rel(j+1) = Rel(j) + w(1, r_{j+1}) - \alpha(E_j)$ attains the least possible non-negative value and let $r_n = s_n$. Let $\underline{r} = \langle r_1, \ldots, r_n\rangle$ and let Θ be the triple $\langle n, \underline{r}, t\rangle$.

Step 2: We first show that Θ is indeed an n-template. We begin with the following claims.

Claim 1: *For $1 \leq j < n$, s_{j+1} is non-empty*: We begin by noting that as $|s| \geq n$, s_1 is non-empty. Now assume to the contrary that s_{j+1} is empty for some $1 \leq j < n$ and let j_0 be the least positive integer for which s_{j_0+1} is empty. Then for all $j' \geq j_0 + 1$, s'_j, $w(1, r_{j'}) = 1$. Now it follows that $0 > w(n, s) = Rel(n) = Rel(j_0) + (n - j_0) - \sum_{i=j_0}^{n-1} \overline{\alpha}(E_i)$. By construction $Rel(j_0) \geq 0$. Further as for each $E \in L$, $\overline{\alpha}(E) \leq 1$ implies that $(n - j_0) - \sum_{i=j_0}^{n-1} \overline{\alpha}(E_i) \geq 0$. But this yields a contradiction that proves the claim.

Claim 2: *For $1 \leq j < n$, $Rel(j) < \overline{\alpha}(E_j)$*: If not, $Rel(j) \geq \overline{\alpha}(E_j)$ for some $1 \leq j < n$. From Claim 1 it follows that there is some $E \in L^{\Theta}$ such that $s_{j+1}(E) > 0$. By our choice of the E_i, it follows that $\overline{\alpha}(E_j) \geq \overline{\alpha}(E)$. However this shows that $Rel(j) - \overline{\alpha}(E) \geq \overline{\alpha}(E_j) - \overline{\alpha}(E) \geq 0$ which contradicts our choice of r_j.

Note that to show that Θ is an n-template it suffices to show that for $1 \leq j \leq n$, $w(1, r_j) \geq 0$. Now for all $1 \leq j < n-1$, $Rel(j+1) \geq 0$ and $Rel(j) < \overline{\alpha}(E_j)$ yields that $w(1, r_{j+1}) = Rel(j+1) + \overline{\alpha}(E_j) - Rel(j) \geq 0$. Now assume that $w(1, r_n) < 0$. Then $w(1, r_n) \leq -Gr_{\Theta}(2)$. Now $Rel(n) = w(1, r_n) + Rel(n-1) - \overline{\alpha}(E_{n-1}) < -Gr_{\Theta}(2)$ which contradicts $-Rel(n) \geq Gr_{\Theta}(2)$. Thus it follows that $w(1, r_n) \geq 0$. Hence Θ is indeed a n-template.

Let $\mathfrak{D}$ be an extension of $\mathfrak{B}$ by Θ as given by Lemma 3.1.5. Observe that $\delta(\mathfrak{D}^j/\mathfrak{B}) = Rel(j)$ for $1 \leq j \leq n$. It immediately follows that if $\langle n, s\rangle$ is acceptable, then Θ is also acceptable. Now assume that $\langle n, s\rangle$ is good. We claim that Θ is good. By construction $|s| = |t| + \sum_{i=1}^{n} |r_i|$. Recall condition (2) of good. If $|s| \geq |B| + (n-1)$, then $\sum_{i=1}^{n} |r_i| \geq |B|$. Else we have that $\sum_{ar(E)\geq 3}(t(E) + \sum_{i=1}^{n} r_i(E)) \geq |B|$. Now Lemma 3.1.5 shows that $\mathfrak{D}$ can be constructed in a manner covers $\mathfrak{B}$. Thus in order to establish that Θ is good it suffices to show $\overline{\alpha}(E_j) - \delta(\mathfrak{D}^j/\mathfrak{B}) + \delta(\mathfrak{D}/\mathfrak{B}) \geq 0$ for $1 \leq j \leq n-1$. Suppose to the contrary that $a = \overline{\alpha}(E_j) - \delta(\mathfrak{D}^j/\mathfrak{B}) + \delta(\mathfrak{D}/\mathfrak{B}) < 0$ for some $1 \leq j \leq n-1$. Thus we may write $a = w(m, s')$ for some $m \leq n$ and some sub-collection s' of s. Now by clause (3) of goodness and the fact that $\langle n, s\rangle$ is good, it follows that $a \leq w(n, s)$. But $w(n, s) = \delta(\mathfrak{D}/\mathfrak{B})$ and hence $\overline{\alpha}(E_j) - \delta(\mathfrak{D}^j/\mathfrak{B}) \leq 0$, a contradiction to Claim 2. Thus Θ is good. $\square$

Corollary 3.2.8. *Let $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{B}) > 0$, $|B| \geq m_{suff}$ and $\langle n, s\rangle$ a good pair with $n \geq 3$. Then there is an $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $(\mathfrak{B}, \mathfrak{D})$ is an essential minimal pair with $w(n, s) = \delta(\mathfrak{D}/\mathfrak{B}) < 0$.*

Proof. This follows directly from Lemma 3.1.11 and 3.2.7. $\square$

As it turns out, granularity offers us a very convenient way of establishing a connection between acceptable/good pairs and the number theoretic facts in the Appendix (See Lemma 3.2.11 and Theorem 3.2.15 below). Thus granularity takes on two *separate* roles: it's original role in [2], (given by Lemma 3.2.4) and the one just mentioned (replacing the role played by *local optimality*, in Section 4 of [2]).

We now turn our attention towards using the number theoretic results in the

Appendix to construct good pairs.

Lemma 3.2.9. *The sequence given by $\langle Gr(m) : m \in \omega \rangle$ is a monotonic decreasing sequence. If $\overline{\alpha}$ is not rational, then $\langle Gr(m) : m \in \omega \rangle$ converges to 0. If $\overline{\alpha}$ is rational, then $Gr(m)$ is eventually constant with $Gr(m) = 1/c$ for sufficiently large m.*

Proof. If $\overline{\alpha}$ is not rational then there is some $E \in L$ such that $\overline{\alpha}_E$ is irrational. Now the required result follows from Remark A.0.2. If $\overline{\alpha}$ is rational, then the required result follows from Remark A.0.1. $\square$

Notation 3.2.10. We fix some notation: Whenever the assumption that $\overline{\alpha}$ is rational is in effect, we assume that $\overline{\alpha}_E = \frac{p_E}{q_E}$ in reduced form and that $c = \operatorname{lcm}(q_E)$.

Lemma 3.2.11. *Let $n \in \omega$ with $n \geq 3$ and s be an L-collection. For $1 \leq m \leq n$ and any sub-collection s' of s, $w(m, s')$ is not in the interval $(-Gr(n+1), 0)$.*

Proof. Let n, s, m, s' be as above. As granularity is monotonically decreasing, $Gr(n+1) \geq Gr(m+1)$. Assume to the contrary that $w(m, s') \in (-Gr(n+1), 0)$. This yields that $Gr(n+1) > w(m, s') > 0$. But $w(m, s') \geq Gr(m+1) > 0$, a contradiction which established the claim. $\square$

Lemma 3.2.12. *Let $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{B}) > 0$ and $|B| \geq m_{suff}$.*

1. *Let $\epsilon > 0$ and assume that $\overline{\alpha}$ is not rational. Then for any $E \in L$ such that $\overline{\alpha}_E$ is irrational, there are infinitely many good pairs (n, s) for $\mathfrak{B}$ such that $0 < -w(n, s) < \epsilon$ and s is such that $s(E^*) > 0$ if and only if $E^* = E$ for all $E \in L$.*

2. *If $\overline{\alpha}$ is rational, then we may obtain infinitely many good pairs (n, s) for $\mathfrak{B}$ such that $-w(n, s) = 1/c$.*

Proof. (1): Let $E \in L$ be such that $\overline{\alpha}(E)$ is irrational. Let $L' = \{E\}$ and let $\alpha = \overline{\alpha}(E)$. Note that we may as well assume that $\epsilon \leq \min\{\delta(\mathfrak{B}), Gr_{L'}(2)\}$. As $\lim_n Gr_{L'}(n) = 0$, there is an infinite set A of positive integers such that $Gr_{L'}(n+1) < Gr_{L'}(k)$ for all $2 \leq k \leq n$. For each $n \in A$, let l_n be such that $Gr_{L'}(n+1) = l_n\alpha - n$. Since ϵ, $|B|$ are fixed *and* $\alpha < 1$, all but finitely many $n \in A$ satisfy $0 < l_n\alpha - n < \epsilon$ and $l_n \geq |B| + (n-1)$. Given such n, let s be the L-collection that contains l_n many E relation symbols and no other relation symbols. It is immediate that by our choice of n and s that (n, s) is a good pair with $0 < -w(n, s) < \epsilon$ and that s satisfies the other properties given in (1).

(2) : Assume that $\overline{\alpha}$ is rational. The proof now splits off into two cases depending on the value of c.

First consider the case $c > 1$: Then $Gr(n') = 1/c < 1$ for all sufficiently large n'. Note that $\delta(\mathfrak{B}) = k/c$ for some $k \in \omega, k \neq 0$ and thus $\delta(\mathfrak{B}) \geq 1/c$. Let $L' = \{L \in E : \overline{\alpha}_E < 1\}$. Using Remark A.0.1 of the Appendix, there is an infinite set A of positive integers n such that $Gr_{L'}(n+1) = 1/c$. For each $n \in A$, let $l_n : L' \to \omega$ be a function such that $Gr_{L'}(n+1) = \sum_{E \in L'} l_n(E)\alpha_E - n$. Since $|B|$ is fixed *and* $\overline{\alpha}_E < 1$ for each $E \in L'$, all but finitely many $n \in A$ satisfy $\sum_{E \in L'} \overline{\alpha}_E l_n(E) - n = 1/c$ and $\sum_{E \in L'} l_n(E) \geq |B| + (n-1)$. Given such n, let s be the L-collection that contains exactly $l_n(E)$ many E relation symbols for $E \in L'$

and no other relation symbols. Now by our choice of n, s it is immediate that (n, s) is a good pair with $-w(n, s) = 1/c$.

Now consider the case $c = 1$: Now for each $E \in L$, $\overline{\alpha}(E) = 1$, $Gr(m) = 1$ for all $m \geq 2$ and all finite structures have integer rank. Note that there is some $E \in L$ that has arity at least 3 as $\overline{\alpha}(E) = 1$ for each $E \in L$ implies that arity of each relation symbol cannot be 2. Fix such an $E \in L$ and let $L' = \{E\}$. Then for any $n \geq |B| + 1$ take s to be the L-collection with n many E relations and no other relations. A routine verification shows that $\langle n, s \rangle$ is a good pair. $\square$

We now put the previous results together to establish:

Lemma 3.2.13. *Let $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{B}) > 0$ and $|B| \geq m_{suff}$.*

1. *Let $\epsilon > 0$ and assume that $\overline{\alpha}$ is not rational. Now given any $E \in L$ such that $\overline{\alpha}_E$ is irrational, we can construct infinitely many non-isomorphic $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $(\mathfrak{B}, \mathfrak{D})$ is an essential minimal pair that satisfies $-\min\{\epsilon, \delta(\mathfrak{B})\} < \delta(\mathfrak{D}/\mathfrak{B}) < 0$ where the new relations that appear in $\mathfrak{D}$ that were not in $\mathfrak{B}$ are E relations.*

2. *If $\overline{\alpha}$ is rational, then we can construct infinitely many non-isomorphic $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $(\mathfrak{B}, \mathfrak{D})$ is an essential minimal pair that satisfies $\delta(\mathfrak{D}/\mathfrak{B}) = -1/c$.*

Proof. Use Lemma 3.2.12 to obtain a good pair (n, s) for $\mathfrak{B}$ that satisfies $0 < -w(n, s) \leq Gr(m)$. Now use Corollary 3.2.8 to construct an essential minimal pair $(\mathfrak{B}, \mathfrak{D})$ with $w(n, s) = \delta(\mathfrak{D}/\mathfrak{B}) < 0$. As (n, s) is a good pair, $\mathfrak{D} \in K_{\overline{\alpha}}$. We can obtain infinitely many $\mathfrak{D}$ as required by varying our choice of good pairs. Further

$(1),(2)$ can be obtained by choosing suitable good pairs using $(1),(2)$ (respectively) of Lemma 3.2.12. $\square$

The two clauses of the following lemma illustrate some routine argument patterns that can be used in constructing new structures by taking free joins. It will also yield a substantial part of Theorem 3.2.15 and Lemma 4.2.1.

Lemma 3.2.14. *Let $\mathfrak{A}, \mathfrak{B} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \leq \mathfrak{B}$. Assume that $(\mathfrak{B}, \mathfrak{C})$ is an essential minimal pair and let $\gamma = -\delta(\mathfrak{C}/\mathfrak{B})$. Then*

1. *We can construct $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $\mathfrak{B} \subseteq \mathfrak{D}$, $\mathfrak{A} \leq \mathfrak{D}$ and $0 \leq \delta(D/A) < \gamma$. Further if $(\mathfrak{B}, \mathfrak{G})$ is a minimal pair with $|G| < |C|$, then $\mathfrak{G}$ does not embed into $\mathfrak{D}$ over $\mathfrak{B}$.*

2. *Assume that $\delta(\mathfrak{A}) \geq \gamma$. Then we can construct $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $\mathfrak{B} \subseteq \mathfrak{D}$, $(\mathfrak{A}, \mathfrak{D})$ is an essential minimal pair that satisfies $0 > \delta(\mathfrak{D}/\mathfrak{A}) \geq -\gamma$*

Proof. Note that there is some *non-negative* integer k such that $k\gamma \leq \delta(\mathfrak{B}/\mathfrak{A}) < (k+1)\gamma$. Let $\mathfrak{D}$ be the free join of k-copies of $\mathfrak{C}$ over $\mathfrak{B}$ and enumerate the copies of $\mathfrak{C}$ in $\mathfrak{B}$ by $\{\mathfrak{C}_i : 1 \leq i \leq k\}$ (with $\mathfrak{B} = \mathfrak{D}$ if $k = 0$). We now show that $\mathfrak{D}$ has the required properties. We begin by establishing some notation: Let $\mathfrak{D}' \subseteq \mathfrak{D}$ be a nonempty substructure of $\mathfrak{D}$ and let $\mathfrak{C}'_i = \mathfrak{C}_i \cap \mathfrak{D}'$ and $\mathfrak{B}' = \mathfrak{D}' \cap \mathfrak{B}$.

Clearly $\mathfrak{B} \subseteq \mathfrak{D}$ and $\mathfrak{D} \in K_L$. By Remark 2.3.2, $\mathfrak{D} \in K_{\overline{\alpha}}$ follows if you show that $\mathfrak{A} \leq \mathfrak{D}$. This is equivalent to establishing $\delta(\mathfrak{D}'/\mathfrak{A}) \geq 0$ in the case that $\mathfrak{A} \subseteq \mathfrak{D}'$. So we will assume that $\mathfrak{A} \subseteq \mathfrak{D}'$. Since $\mathfrak{A} \leq \mathfrak{B}$, if $\mathfrak{D}' \subseteq \mathfrak{B}$, we have the required result. So consider $\mathfrak{D}' \nsubseteq \mathfrak{B}$. We may view $\mathfrak{D}'$ as the free join of $\mathfrak{D}'_i$ over $\mathfrak{B}'$. Note that $\delta(\mathfrak{D}'/\mathfrak{B}') = \sum_{i=1}^{k} \delta(\mathfrak{C}_i'/\mathfrak{B}')$ by (4) of Fact 2.2.5. Since $(\mathfrak{B}, \mathfrak{C})$ are essential minimal pairs, it follows that if $\mathfrak{B}' \neq \mathfrak{B}$, then $\delta(\mathfrak{C}_i'/\mathfrak{B}') \geq 0$. Further if $\mathfrak{B}' = \mathfrak{B}$, then $\delta(\mathfrak{D}'/\mathfrak{B}) \geq -k\gamma$ with equality holding if and only if $\mathfrak{D}' = \mathfrak{D}$.

Assume that $\mathfrak{A} \subseteq \mathfrak{D}' \subseteq \mathfrak{D}$. We need to establish that $\delta(\mathfrak{D}'/\mathfrak{A}) \geq 0$. First consider the case where $\mathfrak{B}' \neq \mathfrak{B}$. Now $\delta(\mathfrak{D}'/\mathfrak{B}') \geq 0$. Further $\delta(\mathfrak{D}'/\mathfrak{A}) = \delta(\mathfrak{D}'/\mathfrak{B}') + \delta(\mathfrak{B}'/\mathfrak{A})$. Since $\mathfrak{A} \leq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$, we have that $\delta(\mathfrak{B}'/\mathfrak{A}) \geq 0$. Thus $\delta(\mathfrak{D}'/\mathfrak{A}) \geq 0$. Now consider the case $\mathfrak{B}' = \mathfrak{B}$. In this case we have that $\delta(\mathfrak{D}'/\mathfrak{A}) = \delta(\mathfrak{D}'/\mathfrak{B}) + \delta(\mathfrak{B}/\mathfrak{A}) \geq -k\gamma + \delta(\mathfrak{B}/\mathfrak{A}) \geq 0$. Hence $\mathfrak{A} \leq \mathfrak{D}$.

A simple calculation yields $\delta(\mathfrak{D}/\mathfrak{A}) = -k\gamma + \delta(\mathfrak{B}/\mathfrak{A}) < \gamma$. We now show that *no* $\mathfrak{G}$ such that $(\mathfrak{B}, \mathfrak{G})$ is a minimal pair with $|G| < |C|$ embeds into $\mathfrak{D}$ over $\mathfrak{B}$. Assume such a minimal pair did embed into $\mathfrak{D}$ over $\mathfrak{B}$ and let its image be $\mathfrak{D}'$. Now $\delta(\mathfrak{D}'/\mathfrak{B}) = \sum_{i=1}^{k} \delta(\mathfrak{C}_i'/\mathfrak{B})$. But *each* $\delta(\mathfrak{C}_i'/\mathfrak{B}) \geq 0$ unless $\mathfrak{C}_i' = \mathfrak{C}$. Thus $|D'| \geq |C|$, a contradiction.

(2) Note that there is some *non-negative* integer k such that $k\gamma \leq \delta(\mathfrak{B}/\mathfrak{A}) < (k+1)\gamma$. Consider the structure $\mathfrak{D}$ which is the free join of $k+1$-copies of $\mathfrak{C}$ over $\mathfrak{B}$. Enumerate these copies of $\mathfrak{C}$ as $\mathfrak{C}_1 \ldots \mathfrak{C}_{k+1}$. Let $\mathfrak{D}' \subseteq \mathfrak{D}$ be non-empty, $\mathfrak{B}' = \mathfrak{B} \cap \mathfrak{D}'$ and $\mathfrak{C}_i' = \mathfrak{C} \cap D'$

We begin by showing that $\mathfrak{D} \in K_{\overline{\alpha}}$. We need to show that $\delta(\mathfrak{D}') \geq 0$. As this is immediate when $\mathfrak{D}' \subseteq \mathfrak{B}$, we may as well assume that this is not the case. Now as in (1), $\delta(\mathfrak{D}'/\mathfrak{B}') = \sum_{i=1}^{k+1} \delta(\mathfrak{C}_i'/\mathfrak{B}')$. As $(\mathfrak{B}, \mathfrak{C})$ is an essential minimal pair we need only consider $\mathfrak{B}' = \mathfrak{B}$ (the other case follows easily as in (1)). Then $\delta(\mathfrak{D}'/\mathfrak{B}) \geq -(k+1)\gamma$. But by our choice of k and using the assumption $\delta(\mathfrak{A}) \geq \gamma$, we see that

$\delta(\mathfrak{B}) \geq (k+1)\gamma$ and hence $\delta(\mathfrak{D}')$. Thus $\mathfrak{D}' \in K_{\overline{\alpha}}$.

Now we show that $(\mathfrak{A}, \mathfrak{D})$ is an essential minimal pair with $0 > \delta(\mathfrak{D}/\mathfrak{A}) \geq -\gamma$. So assume that $\mathfrak{A} \subseteq \mathfrak{D}' \subsetneq \mathfrak{D}^*$. If $\mathfrak{B}' \neq \mathfrak{B}$, then $\delta(\mathfrak{D}'/\mathfrak{A}) = \delta(\mathfrak{D}'/\mathfrak{B}') + \delta(\mathfrak{B}'/\mathfrak{A}) \geq 0$. So assume that $\mathfrak{B}' = \mathfrak{B}$. Thus $\delta(\mathfrak{D}'/\mathfrak{A}) \geq \delta(\mathfrak{D}'/\mathfrak{B}) + k\gamma$. Since each $(\mathfrak{B}, \mathfrak{C}_i)$ is an essential minimal pair, it follows that $\delta(\mathfrak{D}'/\mathfrak{B}) \geq -k\gamma$ *unless* $\mathfrak{D}' = \mathfrak{D}$ and $\delta(\mathfrak{D}'/\mathfrak{B}) = -(k+1)\gamma$ if and only if $\mathfrak{D}' = \mathfrak{D}$. Thus $(\mathfrak{A}, \mathfrak{D})$ forms an essential minimal pair with the required properties. □

Finally we are in a position to prove one of the key result of this section:

Theorem 3.2.15. *Let $\mathfrak{A} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{A}) > 0$.*

1. *If $\overline{\alpha}$ is not rational, then for any $\epsilon > 0$, we can construct infinitely many non-isomorphic $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $(\mathfrak{A}, \mathfrak{D})$ is an essential minimal pair that satisfies $-\epsilon < \delta(\mathfrak{D}/\mathfrak{A}) < 0$. Further if $\alpha(E_0)$ is irrational for some fixed E_0 in L, then we may assume that the only relations that hold in $\mathfrak{D}$ that did not hold in $\mathfrak{A}$ are E_0 relations.*

2. *If $\overline{\alpha}$ is rational, then we can construct infinitely many non-isomorphic $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $(\mathfrak{A}, \mathfrak{D})$ is an essential minimal pair that satisfies $\delta(\mathfrak{D}/\mathfrak{A}) = -1/c$. (Recall that c denotes the least common multiple of the denominators of the $\overline{\alpha}_E$).*

Proof. For $|A| \geq m_{\text{suff}}$, the required results are immediate from Lemma 3.2.13. So assume that $|A| < m_{\text{suff}}$. Let $\mathfrak{A}_0$ be an L-structure with m_{suff} many points such that no relations hold on A_0 and take $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{A}_0$. Clearly $\mathfrak{A} \leq \mathfrak{B}$. Using Theorem

3.2.13 fix a $\mathfrak{C}$ such that $(\mathfrak{B}, \mathfrak{C})$ is an essential minimal pair $\mathfrak{C} \in K_{\overline{\alpha}}$. Note that if $\overline{\alpha}_{E_0}$ is irrational for some $E_0 \in L$ and $\epsilon > 0$, then we may assume that $-\min\{\epsilon, \delta(\mathfrak{A})\} < \delta(\mathfrak{C}/\mathfrak{B}) < 0$ and if $\overline{\alpha}$ is rational, then we may assume $\delta(\mathfrak{C}/\mathfrak{B}) = -1/c$. By using (2) of Lemma 3.2.14, we obtain a required structure $\mathfrak{D}$. We observe that the non-isomorphic $\mathfrak{D}$ may be obtained by varying our choice of $\mathfrak{C}$ and leave it to the reader to verify that in the case $\overline{\alpha}$ is rational, we have $\delta(\mathfrak{D}/\mathfrak{A}) = -1/c$ as claimed. □

3.3 Coherence and rank 0 structures

This section is dedicated to building finite extensions of rank 0. Our goal is to show that if $\overline{\alpha}$ is coherent, then for any $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{B}) > 0$, there is some $\mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{B} \subseteq \mathfrak{D}$ such that $\delta(\mathfrak{D}) = 0$. If $\overline{\alpha}$ is rational, this is easily achieved by repeated use of (2) of Theorem 3.2.15. Thus we focus on the case that $\overline{\alpha}$ is coherent but not rational.

Definition 3.3.1. Let $\overline{\alpha}$ be coherent but not rational. Let $\beta(\overline{\alpha}) = \min\{\delta(\mathfrak{A}), Gr(2) : \mathfrak{A} \in K_{\overline{\alpha}}, \delta(\mathfrak{A}) > 0 \text{ and } |A| < m_{\text{suff}}\}$.

Remark 3.3.2. Note that $\beta(\overline{\alpha}) > 0$. Further if $\mathfrak{B} \in K_{\overline{\alpha}}$ is such that $0 < \delta(\mathfrak{B}) < \beta(\overline{\alpha})$, then $|B| \geq m_{\text{suff}}$.

Proposition 3.3.3. *Let $\mathfrak{B} \in K_{\overline{\alpha}}$. Then there is some $\mathfrak{Z} \subseteq B$ such that $\delta(\mathfrak{Z}) = 0$ and if $\mathfrak{C} \subseteq \mathfrak{B}$ is such that $\delta(\mathfrak{C}) = 0$, then $\mathfrak{C} \subseteq \mathfrak{Z}$.*

Proof. Let $\mathfrak{B} \in K_{\overline{\alpha}}$ and let $\mathfrak{A}, \mathfrak{C} \subseteq B$ with $\delta(\mathfrak{A}) = \delta(\mathfrak{C}) = 0$. Let $\mathfrak{D}$ be the join of $\mathfrak{A}, \mathfrak{C}$ in $\mathfrak{B}$. Now $0 \leq \delta(\mathfrak{D}) \leq \delta(\mathfrak{A}) + \delta(\mathfrak{C}) = 0$ by (3) of Fact 2.2.5. Thus there is a unique maximal (with respect to $\subseteq$) $\mathfrak{Z} \subseteq \mathfrak{B}$ such that $\delta(\mathfrak{Z}) = 0$. □

Definition 3.3.4. Let $\mathfrak{B} \in K_{\overline{\alpha}}$. The unique maximal (with respect to $\subseteq$) $\mathfrak{Z} \subseteq \mathfrak{B}$ such that $\delta(\mathfrak{Z}) = 0$ will be called the *zero set of* $\mathfrak{B}$ and we denote $\mathfrak{Z}$ by $\mathfrak{Z}_{\mathfrak{B}}$. We will let Z_B denote the universe of $\mathfrak{Z}_{\mathfrak{B}}$.

Lemma 3.3.5. *Let $\overline{\alpha}$ be coherent and assume that $\overline{\alpha}$ is not rational. Let $\mathfrak{A} \in K_{\overline{\alpha}}$ with $\beta(\overline{\alpha}) > \delta(\mathfrak{A}) > 0$. Then there exists $\mathfrak{A}^* \in K_{\overline{\alpha}}$ such that $\mathfrak{A}^* \supseteq \mathfrak{A}$, $0 \leq \delta(\mathfrak{A}^*) < \beta(\overline{\alpha})$ and $|A^* - Z_{A^*}| < |A - Z_A|$.*

Proof. Choose $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{Z}_{\mathfrak{A}} \subsetneq \mathfrak{B} \subseteq \mathfrak{A}$ and $\gamma := \delta(\mathfrak{B})$ is least possible. Clearly $\gamma > 0$ as $\mathfrak{Z}_{\mathfrak{A}} \subsetneq \mathfrak{B}$, $\mathfrak{B} \leq \mathfrak{A}$ as the rank of $\mathfrak{B}$ is minimal and $|B| \geq m_{\text{suff}}$ as $\gamma \leq \delta(\mathfrak{A}) < \beta(\overline{\alpha})$. Further using (2) of Fact 2.2.5, it follows that for any $\mathfrak{B}' \subseteq \mathfrak{B}$, either $\mathfrak{B}' \subseteq \mathfrak{Z}_{\mathfrak{A}}$ or $\delta(\mathfrak{B}') \geq \gamma$. We construct $\mathfrak{A}^*$ by taking a free join of $\mathfrak{A}$ over $\mathfrak{B}$ with a suitably constructed structure $\mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{B} \subseteq \mathfrak{D}$.

Now as $\overline{\alpha}$ is coherent there are infinitely many positive integers $\langle n', m'_E \rangle_{E \in L}$ such that $n' - \sum_{E \in L} m'_E \overline{\alpha}_E = 0$. Using the fact that $\gamma = \delta(\mathfrak{C})$, we obtain that $\delta(\mathfrak{C}) = n_0 - \sum_{E \in L} m_0(E) \overline{\alpha}_E$ for some non-negative integers $\langle n_0, m_0(E) \rangle_{E \in L}$. Hence we now obtain that there are infinitely many positive integers $\langle n'', m''_E \rangle_{E \in L}$ such that $n'' - \sum_{E \in L} m''_E \overline{\alpha}_E = -\gamma$. Thus we can construct acceptable $\langle n, s \rangle$ such that $w(n, s) = -\gamma$. Use Lemma 3.2.7 to construct an n-template Θ that corresponds to $\langle n, s \rangle$.

Fix any $b^* \in B - Z_A$. Let $\mathfrak{D}$ be an extension of $\mathfrak{B}$ by Θ with the additional property that there is some relation E and $Q \in E^{\mathfrak{D}}$ with $\{b^*, d_n\} \subseteq Q$ where d_n is as described in Notation 3.1.7. As $\delta(\mathfrak{D}/\mathfrak{B}) = -\gamma$ we have that $\delta(\mathfrak{D}) = 0$. We claim that $\mathfrak{D} \in K_{\overline{\alpha}}$.

First note that if $\mathfrak{B} \subseteq \mathfrak{D}' \subseteq \mathfrak{D}$, then $\delta(\mathfrak{D}'/\mathfrak{B}) \geq -\gamma$ by $(1.c)$ of Lemma 3.1.11. Hence we obtain that $\delta(\mathfrak{D}') \geq 0$. Now choose $\mathfrak{D}' \subseteq \mathfrak{D}$ arbitrary and and let $\mathfrak{B}' = \mathfrak{B} \cap \mathfrak{D}'$. There are now three possibilities. First consider the case $d_n \notin \mathfrak{D}'$. By $(1.b)$ of Lemma 3.1.11 we obtain that $\delta(\mathfrak{D}'/\mathfrak{B}') \geq 0$ and hence we obtain that $\delta(\mathfrak{D}') \geq 0$ as $\mathfrak{B}' \in K_{\overline{\alpha}}$. Now consider the case $b^* \in \mathfrak{D}'$. Then we have that $b^* \in B'$ and hence $\delta(\mathfrak{B}') \geq \gamma$. As $\delta(\mathfrak{D}'/\mathfrak{B}') \geq \delta(BD'/B)$ by (2) of Fact 2.2.5 and $\delta(BD'/B) \geq -\gamma$, we conclude that $\delta(\mathfrak{D}') \geq 0$. Finally consider the case $d_n \in D'$ but $b^* \notin D'$. Then we have that $Q \notin E^{\mathfrak{D}'}$. So $\delta(\mathfrak{D}'/\mathfrak{B}') \geq \delta(BD'/B) + \overline{\alpha}(E) \geq 0$. As $\delta(B') \geq 0$, $\delta(\mathfrak{D}') \geq 0$.

Let $\mathfrak{A}^*$ be the free join $\mathfrak{D} \oplus_{\mathfrak{B}} \mathfrak{A}$. As $\mathfrak{B} \leq \mathfrak{A}$ and $\mathfrak{D} \in K_{\overline{\alpha}}$, by Fact 2.3.4, we obtain that $\mathfrak{A}^* \in K_{\overline{\alpha}}$. Now $\delta(\mathfrak{A}^*/\mathfrak{B}) = \delta(\mathfrak{A}/\mathfrak{B}) + \delta(\mathfrak{D}/\mathfrak{B}) = \delta(\mathfrak{A}/\mathfrak{B}) - \gamma$ and hence $0 \leq \delta(\mathfrak{A}^*) < \beta(\overline{\alpha})$.

Finally note that the universe of $\mathfrak{A}^*$ is $A \cup D$. As $\delta(\mathfrak{D}) = 0$, we have that $\mathfrak{B} \subseteq \mathfrak{D} \subseteq \mathfrak{Z}_{\mathfrak{A}^*}$. As $b^* \in B - Z_A$, we conclude that $|A^* - Z_{A^*}| < |A - Z_A|$. □

Theorem 3.3.6. *Let $\overline{\alpha}$ be coherent. Then given any $\mathfrak{A} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{A}) > 0$ there is $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $\mathfrak{D} \supseteq \mathfrak{A}$ and $\delta(\mathfrak{D}) = 0$.*

Proof. Case 1: Assume that $\overline{\alpha}$ is not rational. Now there is some $E \in L$ such that $\overline{\alpha}_E$ is irrational. If $0 \leq \delta(\mathfrak{A}) < \beta(\overline{\alpha})$, then we are done. So assume that $\delta(\mathfrak{A}) \geq \beta(\overline{\alpha})$. Since $\overline{\alpha}_E$ is irrational, we can find a minimal pair $(\mathfrak{A}, \mathfrak{B})$ with $\delta(\mathfrak{B}/\mathfrak{A})$ as small as we like using Theorem 3.2.15. Now fixing a minimal pair such that $\delta(\mathfrak{B}/\mathfrak{A}) < \beta(\overline{\alpha})$ and taking sufficiently many isomorphic copies of $\mathfrak{B}$ freely joined over $\mathfrak{A}$, we can find a $\mathfrak{A}^* \supseteq \mathfrak{A}$ such that $\mathfrak{A}^* \in K_{\overline{\alpha}}$ and $0 < \delta(\mathfrak{A}^*) < \beta(\overline{\alpha})$. Let $l = |A^* - Z_{A^*}|$. By iterating Lemma 3.3.5 at most l times, we may construct $\mathfrak{D} \supseteq \mathfrak{A}^*$ with $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $|D - Z_D| = 0$, i.e. $\delta(\mathfrak{D}) = 0$.

Case 2: Assume that $\overline{\alpha}$ is rational. Then $\delta(\mathfrak{A}) = k/c$ for some positive integer k, where c is the least common multiple the q_E where $\overline{\alpha}_E = p_E/q_E$ (in reduced form). If $\overline{\alpha}$ is *not* graph-like with weight one, as noted in Theorem 3.2.15 we may create a minimal pair $\mathfrak{B}$ over $\mathfrak{A}$ such that $\delta(\mathfrak{B}/\mathfrak{A}) = -1/c$ and for all $\mathfrak{B}' \subsetneq B$, $\delta(\mathfrak{B}'/\mathfrak{A} \cap \mathfrak{B}') \geq 0$. Let $\mathfrak{D} = \oplus_{1 \leq i \leq k} \mathfrak{B}_i/\mathfrak{A}$, the free join of k isomorphic copies of $\mathfrak{B}$ over $\mathfrak{A}$. A routine argument now shows that $\delta(\mathfrak{D}) = 0$ and that $\mathfrak{D} \in K_{\overline{\alpha}}$.

In the case that $\overline{\alpha}$ *is* graph-like with weight one, we may producing the required $\mathfrak{D}$ by constructing a chain of minimal pairs (using Theorem 8.2.4) $\mathfrak{A} = \mathfrak{D}_0 = \subseteq \mathfrak{D}_1 \ldots \subseteq \mathfrak{D}_k = \mathfrak{D}$ with $\delta(\mathfrak{D}_{i+1}/\mathfrak{D}_i) = -1$.

We note that the approach in the case $\overline{\alpha}$ is graph-like with weight one can be used in the case that $\overline{\alpha}$ is rational eliminating the need to consider whether $\overline{\alpha}$ is graph-like with weight one or not separately. $\square$

Remark 3.3.7. We note that we may construct infinitely many such non-isomorphic $\mathfrak{D}$ by varying our choice of $\mathfrak{A}^*$ or $\mathfrak{B}$ accordingly.

Chapter 4: Quantifier elimination and the completeness of $S_{\overline{\alpha}}$

Quantifier Elimination for the theory of the $(K_{\overline{\alpha}}, \leq)$ generic is briefly explored by Baldwin and Shelah in Section 1 of [14]. They establish that the theory of the $(K_{\overline{\alpha}}, \leq)$ generic is near model complete (i.e. each formula is equivalent to a boolean combination of existentials), using a $\forall\exists\forall$-axiomatization of the theory of the generic. In [2], under the additional assumption that the values of $\overline{\alpha}(E)$ are irrational and linearly independent over the rationals obtained the quantifier elimination result and a $\forall\exists$-axiomatization of the theory of the generic. He utilized the quantifier elimination result to explore the existentially closed models of the theory of the generic. Further work by Ikeda, Kikyo and Tsuboi extended the $\forall\exists$-axiomatization to arbitratry $\overline{\alpha} : L \to (0, 1]$. However the consequences of the quantifier elimination in this setting were not explored.

In this section we begin by introducing a collection of $\forall\exists$-axioms that we denote by $S_{\overline{\alpha}}$ (see Definition 4.0.1). In Theorem 4.3.5 we observe that $S_{\overline{\alpha}}$ admits quantifier elimination down to the level of chain minimal extension formulas (see Definition 4.0.4) by generalizing the arguments of Laskowski in [2]. In Theorem 4.4.1 we collect useful results about $S_{\overline{\alpha}}$ including the fact that $S_{\overline{\alpha}}$ is the theory for the Baldwin-Shi hypergraph for $\overline{\alpha}$. Lemma 4.4.2 gathers useful consequences of the quantifier elimination.

Definition 4.0.1. The theory $S_{\overline{\alpha}}$ is the smallest set of sentences insuring that if $\mathfrak{M} \models S_{\overline{\alpha}}$, then

1. $\mathfrak{M} \in \overline{K_{\overline{\alpha}}}$, i.e. every finite substructure of $\mathfrak{M}$ is in $K_{\overline{\alpha}}$

2. For all $\mathfrak{A} \leq \mathfrak{B}$ from $K_{\overline{\alpha}}$, every (isomorphic) embedding $f : \mathfrak{A} \to \mathfrak{M}$ extends to an embedding $g : \mathfrak{B} \to \mathfrak{M}$

Remark 4.0.2. We note that $S_{\overline{\alpha}}$ is a collection of $\forall\exists$-sentences. Further since $\emptyset \leq \mathfrak{A}$ for each $\mathfrak{A} \in K_{\overline{\alpha}}$ it follows that $\mathfrak{M} \models \exists \overline{y} \Delta_A(\overline{x})$

Notation 4.0.3. Let $\mathfrak{N} \in \overline{K_L}$. Given $\mathfrak{A} \in K_L$ with a fixed enumeration $\overline{a}$ of A, we write $\Delta_{\mathfrak{A}}(\overline{x})$ for the atomic diagram of $\mathfrak{A}$. Also for $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K_L$ with $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$ and fixed enumerations $\overline{a}, \overline{b}, \overline{c}$ respectively with $\overline{a}$ an initial segment of $\overline{b}$ and $\overline{b}$ an initial segment of $\overline{c}$; we let $\Delta_{\mathfrak{A},\mathfrak{B}}(\overline{x}, \overline{y})$ the atomic diagram of $\mathfrak{B}$ with the universe of $\mathfrak{A}$ enumerated first according to the enumeration $\overline{a}$. Similarly $\Delta_{\mathfrak{A},\mathfrak{B},\mathfrak{C}}(\overline{x}, \overline{y}, \overline{z})$ will denote the atomic diagram of $\mathfrak{C}$ with the universe of $\mathfrak{A}$ enumerated first by $\overline{x}$, the remainder $B - A$ by $\overline{y}$ and then $C - B$ by $\overline{z}$ according to the enumerations $\overline{a}, \overline{b}, \overline{c}$.

Definition 4.0.4. Let $\mathfrak{A}, \mathfrak{B} \in K$ and assume $\mathfrak{A} \subseteq \mathfrak{B}$. Let $\Psi_{\mathfrak{A},\mathfrak{B}}(\overline{x}) = \Delta_{\mathfrak{A}}(\overline{x}) \wedge \exists \overline{y} \Delta_{(\mathfrak{A},\mathfrak{B})}(\overline{x}, \overline{y})$. Such formulas are collectively called *extension formulas* (over $\mathfrak{A}$). A *chain minimal extension formula* is an extension formula $\Psi_{\mathfrak{A},\mathfrak{B}}$ where $\mathfrak{B}$ us the union of a minimal chain over $\mathfrak{A}$.

4.1 Some Finiteness Results

This results in this section are due to Laskowski and can be found in Section 3 of [2]. We follow his development exactly. This section is devoted to setting notation and obtaining two finiteness results which will be used throughout the paper. Both of these are achieved by combining the notion of granularity with the definition of $K_{\overline{\alpha}}$. The first, Proposition 3.1, asserts that any sufficiently large collection of substructures of an element of $K_{\overline{\alpha}}$ contains an arbitrarily large free join. We freely use the Δ-System Lemma (see, for example, Lemma III.2.6 of [26]), König's Lemma (see Lemma III.5.6 of [26]) and Ramsey's Theorem (see Theorem 5.1.1 of [8]) in its proof.

Proposition 4.1.1. *Fix $m \in \omega$ and $\mathfrak{D} \in \overline{K_{\overline{\alpha}}}$. For any infinite set $\{\mathfrak{B}_i : i \in \omega\}$ of m-element substructures of $\mathfrak{D}$ there is an infinite subset $Y \subseteq \omega$ and a finite $\mathfrak{A} \subseteq \mathfrak{D}$ such that*

1. *$\{\mathfrak{B}_i : i \in \omega\}$ is a free join over $\mathfrak{A}$ and are pairwise isomorphic over $\mathfrak{A}$; and*

2. *$\mathfrak{A} \leq \mathfrak{B}_i$ for every $i \in Y$.*

Moreover, for any $m, s \in \omega$ there is an integer $N(m, s)$ large enough such that for any set $\{\mathfrak{B}_i : i < N(m, s)\}$ of substructures, each of size at most m, of any $\mathfrak{D} \in K_{\overline{\alpha}}$, there is a subset $Y \subseteq N(m, s)$ and an $\mathfrak{A}$ such that $\{\mathfrak{B}_i : i \in Y\}$ is a free join over $\mathfrak{A}$ and $\mathfrak{A} \leq \mathfrak{B}_i$ for all $i \in Y$.

Proof. Fix a set $\{\mathfrak{B}_i : i \in \omega\}$ of m-element substructures of a fixed $\mathfrak{D} \in \overline{K_{\overline{\alpha}}}$. By replacing ω by an infinite subset of itself, it follows from the finite Δ-system lemma that we may assume that there is a fixed $\mathfrak{A}$ such that $B_i \cap B_j = A$ for all $i < j < \omega$. Fix an enumeration $\overline{a}$ of $\mathfrak{A}$ and enumerations $\overline{b}_i$ of each $\overline{b}_i$ extending $\overline{a}$. Recall that $ar(L)$ denote the maximum arity of the relations $E \in L$. Since L is finite and $|B_i| = m$ for all i, there are only finitely many possibilities for the quantifier-free type $qftp(\overline{b}_{i_i}, \ldots, \overline{b}_{i_{ar(L)}}/A)$ over A among all possible sequences $i_1 < \ldots < i_{ar(L)} < \omega$. Thus, by Ramsey's theorem there is an infinite $Y \subseteq \omega$ so that the quantifier-free type $qftp(\overline{b}_{i_i}, \ldots, \overline{b}_{i_{ar(L)}}/A)$ over A is constant among all sequences $i_1 < \ldots < i_{ar(L)} < \omega$ from Y. Since $B_i \cap B_j = A$ for all distinct i, j from Y , $\{\mathfrak{B}_i : i \in Y\}$ is clearly a join over A. That they are pairwise isomorphic over A is immediate since $qftp(\overline{b}_i/A)$ is constant. Assume by way of contradiction that it is not a free join. Then there are $E \in L$, $2 \le t \le ar(L)$, and $X_{(i_1,\ldots,i_t)} \subseteq E^{\mathfrak{B}_{i_1} \cup \ldots \cup \mathfrak{B}_{i_t}} - \bigcup\{E^{\mathfrak{B}_{i_l}} : 1 \le l \le t\}$ for every increasing sequence $i_1 < \ldots < i_t$ from Y. For every integer N , let Y_N be the first N elements of Y and let $\mathfrak{C}_N$ be the finite substructure of $\mathfrak{D}$ with universe $\bigcup\{B_i : i \in Y_N\}$. Now $|\mathfrak{C}_N|$ grows linearly in N, while (since $t \ge 2$) the number of subsets of $\mathfrak{C}_N$ satisfying E grows at least quadratically. So, if N is large enough, $\delta(\mathfrak{C}_N)$ would be negative, contradicting $\mathfrak{D} \in K_{\overline{\alpha}}$. Thus $\{\mathfrak{B}_i : i \in Y\}$ is a free join over A. Arguing similarly, if $\mathfrak{A} \not\leq \mathfrak{B}_i$ for some (equivalently for every) $i \in \omega$, then choose $\mathfrak{A}_i$ such that $\mathfrak{A} \subseteq \mathfrak{A}_i \subseteq \mathfrak{B}_i$ and $\delta(\mathfrak{A}_i/\mathfrak{A}) < 0$. Since $|A_i - A| < m$, it follows from granularity that $\delta(\mathfrak{A}_i/\mathfrak{A}) \le -Gr(m)$. So, for any integer N if we let $\mathfrak{C}_N$ be the substructure of $\mathfrak{D}$ with universe $\{A_j : j \in Y_N\}$ (where $\mathfrak{A}_j$ is the substructure of $\mathfrak{B}_j$ corresponding to A_j) then by (5) of Fact 2.2.5, $\delta(\mathfrak{C}_N/\mathfrak{A}) \le NGr(m)$. Thus,

$\delta(\mathfrak{C}_N) < 0$ whenever N is sufficiently large, which again contradicts $\mathfrak{D} \in K_{\overline{\alpha}}$.

The "Moreover" clause follows from the infinitary version by the standard König's Lemma argument. $\square$

Definition 4.1.2. Fix $m \in \omega$ and $\mathfrak{A} \in K_{\overline{\alpha}}$. An *m-minimal chain over* $\mathfrak{A}$ is a sequence $\{\mathfrak{A}_i : i \leq j\}$ of structures from $K_{\overline{\alpha}}$ such that $\mathfrak{A}_0 = \mathfrak{A}$, $|A_{i+1} - A_i| < m$, and $(\mathfrak{A}_i, \mathfrak{A}_{i+1})$ is a minimal pair for all $i < j$.

The following is almost immediate:

Lemma 4.1.3. *Fix $m \in \omega$ and $\mathfrak{A} \in K_{\overline{\alpha}}$. Every m-minimal chain $\{\mathfrak{A}_i : i \leq j\}$ over $\mathfrak{A}$ has length $j \leq \delta(\mathfrak{A})/Gr(m)$.*

Proof. Since $|A_{i+1} - A_i| < m$ and $\delta(\mathfrak{A}_{i+1}/\mathfrak{A}_i) < 0$, it follows immediately from the definition of $Gr(m)$ that $\delta(\mathfrak{A}_{i+1}/\mathfrak{A}_i) \leq -Gr(m)$. Thus, for each $i \leq j, 0 \leq \delta(\mathfrak{A}_i) \leq \delta(\mathfrak{A}) - iGr(m)$, so $j \leq \delta(\mathfrak{A})/Gr(m)$. $\square$

Lemma 4.1.4. *Let $\mathfrak{D} \in \overline{K_{\overline{\alpha}}}$ and let $\{\mathfrak{A}_i : i \leq j\}$ be an m-minimal chain over $\mathfrak{A}$ of substructures of $\mathfrak{D}$, and suppose that $\mathfrak{B} \subseteq \mathfrak{D}$ is finite, $\mathfrak{A} \subseteq \mathfrak{B}$, but $\mathfrak{A}_j \nsubseteq \mathfrak{B}$. Then $\delta(\mathfrak{D}_j/\mathfrak{B}) \leq -Gr(m)$, where $\mathfrak{D}_j$ is the substructure of $\mathfrak{A}_j$ with universe $A_j \cup B$.*

Proof. For each $i \leq j$, let $\mathfrak{D}_i$ denote the substructure of $\mathfrak{D}$ with universe $A_i \cup B$. Note that $D_0 = B$. By iterating Lemma 3.2.4 $\delta(\mathfrak{D}_{i+1}/\mathfrak{D}_i) \leq 0$ for all $i < j$, with equality holding when $\mathfrak{D}_{i+1} = \mathfrak{D}_i$ and $\delta(\mathfrak{D}_{i+1}/\mathfrak{D}_i) \leq Gr(m)$ otherwise. Since $\mathfrak{A}_j \nsubseteq \mathfrak{B}$, $\mathfrak{D}_{i+1} \neq \mathfrak{D}_i$ for at least one i, so $\delta(\mathfrak{D}_j/\mathfrak{B}) = \delta(\mathfrak{D}_j/\mathfrak{D}_0) = \sum_{i=0}^{j-1} \delta(\mathfrak{D}_{i+1}/\mathfrak{D}_i) \leq -Gr(m)$. $\square$

Definition 4.1.5. Fix $m \in \omega$ and $\mathfrak{D} \in \overline{K_{\overline{\alpha}}}$. A finite $\mathfrak{B} \subseteq \mathfrak{D}$ is m-strong in $\mathfrak{D}$ if $\mathfrak{B} \leq \mathfrak{C}$ for all $\mathfrak{C}$ satisfying $|C - B| < m$ and $\mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{D}$.

Lemma 4.1.6. *Fix $m \in \omega$, $\mathfrak{D} \in \overline{K_{\overline{\alpha}}}$, and a finite $\mathfrak{A} \subseteq \mathfrak{D}$. Let $\{\mathfrak{A}_i : i \leq j\}$ be a maximal m-chain over $\mathfrak{A}$ in $\mathfrak{D}$. Then $\mathfrak{A}_j$ is m-strong and $\mathfrak{A}_j \subseteq \mathfrak{B}$ for any m-strong $\mathfrak{B}$ satisfying $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{D}$. In particular, $\mathfrak{A}_j = \mathfrak{A}'_k$ whenever $\{\mathfrak{A}'_i : i \leq k\}$ is any maximal m-chain over $\mathfrak{A}$ in $\mathfrak{D}$.*

Proof. We first argue that $\mathfrak{A}_j$ is m-strong in $\mathfrak{D}$. By way of contradiction, assume there were $\mathfrak{B}$ satisfying $\mathfrak{A}_j \subseteq \mathfrak{B} \subseteq \mathfrak{D}$, $|B - A_j| < m$, and $\delta(\mathfrak{B}/\mathfrak{A}_j) < 0$. Let $\mathfrak{C}$ be $\subseteq$-minimal such that $\mathfrak{A}_j \subseteq \mathfrak{C} \subseteq \mathfrak{B}$ and $\delta(\mathfrak{C}/\mathfrak{A}_j) < 0$. Then $(\mathfrak{A}_j, \mathfrak{C})$ is a minimal pair, contradicting the maximality of the m-chain. So $\mathfrak{A}_j$ is m-strong in $\mathfrak{D}$.

Now suppose that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{D}$ and that $\mathfrak{B}$ is m-strong in $\mathfrak{D}$. We argue that $\mathfrak{A}_j \subseteq B$. If this were not the case, then choose the largest $i < j$ such that $\mathfrak{A}_i \subseteq \mathfrak{B}$. Let $\mathfrak{C}$ be the substructure of $\mathfrak{D}$ with universe $\mathfrak{A}_{i+1} \cup \mathfrak{B}$. Then $\delta(\mathfrak{C}/\mathfrak{B}) < 0$ by Lemma 4.1.4, contradicting $\mathfrak{B}$ being m-strong in $\mathfrak{D}$. ∎

Remark 4.1.7. As a special case of Lemma 4.1.6, suppose that $\mathfrak{A} \subseteq \mathfrak{B}$ are from $K_{\overline{\alpha}}$. Let $m = |B|$ and let $\{\mathfrak{A}_i : i \leq j\}$ be a maximal m-chain over $\mathfrak{A}$ of substructures of $\mathfrak{B}$. Then $\mathfrak{A}_j \leq \mathfrak{B}$. As well, it is easily checked that $\delta(\mathfrak{A}_j)$ is minimal among all $\mathfrak{C}$ satisfying $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$.

4.2 Towards Quantifier Elimination: The existence of even more particular finite structures

This Section contains several Lemmas that will be needed in the proof of the quantifier elimination result of 4.3.5. We begin by generalizing Proposition 4.2 of [2]. Recall that if $\overline{\alpha}$ is not rational, then $\lim_n Gr(n) = 0$. Thus in the case $\overline{\alpha}$ is not rational we may replace clause (1) of the following lemma with $0 \leq \delta(\mathfrak{D}^*/\mathfrak{A}) < \mu$ where $\mu > 0$. The new statement thus obtained is precisely Proposition 4.2 of [2].

Throughout the rest of this section we work under the assumption that $\overline{\alpha}$ is not graph-like with weight one.

Lemma 4.2.1. *Suppose that $\mathfrak{A} \leq \mathfrak{B} \in K_{\overline{\alpha}}$ and $\Phi \subseteq_{Fin} K_{\overline{\alpha}}$ are given such that $\mathfrak{B} \subseteq \mathfrak{C}$ with $\mathfrak{B} \not\leq \mathfrak{C}$ for all $\mathfrak{C} \in \Phi$. Let $m \in \omega$. Then there is a $\mathfrak{D}^* \supseteq \mathfrak{B}$, $\mathfrak{D}^* \in K_{\overline{\alpha}}$ such that*

1. $0 \leq \delta(\mathfrak{D}^*/\mathfrak{A}) < Gr(m)$
2. $\mathfrak{A} \leq \mathfrak{D}^*$
3. *No $\mathfrak{C} \in \Phi$ isomorphically embeds into $\mathfrak{D}^*$ over $\mathfrak{B}$*

If $\overline{\alpha}$ is rational then we can always find $\mathfrak{D}^*$ such that $\delta(\mathfrak{D}^*/\mathfrak{A}) = 0$.

Proof. Fix $\mathfrak{A}, \mathfrak{B}$ and Φ as above. Note that we may replace each $\mathfrak{C} \in \Phi$ by $\mathfrak{B} \subseteq \mathfrak{C}' \subseteq \mathfrak{C}$ that is minimal and thus we may as well assume that $(\mathfrak{B}, \mathfrak{C})$ is a minimal pair for any given $\mathfrak{C} \in \Phi$. Now if $\delta(\mathfrak{A}) = \delta(\mathfrak{B})$, then take $\mathfrak{D}^* = \mathfrak{B}$. So we may assume that $\delta(\mathfrak{A}) < \delta(\mathfrak{B})$. There are now two possibilities.

Case 1: $\overline{\alpha}$ is *not* graph-like with weight one. Let u be a positive integer such that $u > |\mathfrak{C}|$ for each $\mathfrak{C} \in \Phi$. Now using Theorem 3.2.15, fix a $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $|D - B| > u$ and $(\mathfrak{B}, \mathfrak{D})$ is an essential minimal pair that satisfies $-\min\{Gr(m), \delta(\mathfrak{B}/\mathfrak{A})\} \leq \delta(\mathfrak{D}/\mathfrak{B}) < 0$. Using (1) of Lemma 3.2.14, we may obtain $\mathfrak{D}^*$ with the required properties.

Case 2: $\overline{\alpha}$ *is* graph-like with weight one. We refer the reader to Lemma 8.2.6 □

Definition 4.2.2. Let $\mathfrak{B} \in K_{\overline{\alpha}}$ and let $\Phi \subseteq_{\text{Fin}} K_{\overline{\alpha}}$ such that each $\mathfrak{C} \in \Phi$ extends $\mathfrak{B}$. For any $\mathfrak{M} \models S_{\alpha}$, an embedding $g : \mathfrak{B} \to \mathfrak{M}$ *omits* Φ if there is no embedding $h : \mathfrak{C} \to \mathfrak{M}$ extending g for any $\mathfrak{C} \in \Phi$.

The following is a Proposition 4.4 of [2]. It's proof follows along the same lines there in with obvious modifications made to allow for the existence of structures $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $\delta(\mathfrak{D}) = 0$ in the case that $\overline{\alpha}$ is coherent.

Theorem 4.2.3. *Suppose that $\mathfrak{A} \leq \mathfrak{B}$ are from $K_{\overline{\alpha}}$ and Φ is a finite subset of $K_{\overline{\alpha}}$ such that for each $\mathfrak{C} \in \Phi$, $\mathfrak{A} \leq \mathfrak{C}$, $\mathfrak{B} \subseteq \mathfrak{C}$ but $\mathfrak{B} \not\leq \mathfrak{C}$. Then for any $\mathfrak{M} \models S_{\overline{\alpha}}$, for any embedding $f : \mathfrak{A} \to \mathfrak{M}$ there are infinitely many embeddings $g_i : \mathfrak{B} \to \mathfrak{M}$ extending f such that each g_i omits Φ and $\{g_i(\mathfrak{B}) : i \in \omega\}$ is disjoint over $f(\mathfrak{A})$.*

Proof. If $\delta(\mathfrak{B}) = 0$ (which is possible only if $\overline{\alpha}$ is coherent), then there are no $\mathfrak{C} \in K_{\overline{\alpha}}$ such that $\mathfrak{B} \not\leq \mathfrak{C}$ and thus the statement follows vacuously. So we may assume that $\delta(\mathfrak{B}) > 0$. To ease notation we may assume $f = id$, i.e., $\mathfrak{A} \subseteq \mathfrak{M}$. By replacing each $\mathfrak{C} \in \Phi$ by a $\subseteq$-minimal $\mathfrak{C}'$ satisfying $\mathfrak{B} \subseteq \mathfrak{C}' \subseteq \mathfrak{C}$ and $\delta(\mathfrak{C}'/\mathfrak{B}) < 0$, we may assume that $(\mathfrak{B}, \mathfrak{C})$ is a minimal pair for all $\mathfrak{C} \in \Phi$. Choose an integer m so that $|C - A| < m$

for all $\mathfrak{C} \in \Phi$. Using Lemma 4.2.1, choose $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $\mathfrak{A} \leq \mathfrak{D}$, $\mathfrak{B} \subseteq \mathfrak{D}$, but $\delta(D/A) < Gr(m)$. Choose a disjoint family $\{\mathfrak{D}_i : i \in \omega\}$ over $\mathfrak{A}$ and isomorphisms $k_i : \mathfrak{D} \to \mathfrak{D}_i$ over $\mathfrak{A}$ for each i. Since $i < n$ $\oplus_{i<n}\mathfrak{D}_i \leq \oplus_{i\leq n}\mathfrak{D}_i$ for each n and since $\mathfrak{M} \models S_{\overline{\alpha}}$, one can inductively construct an embedding $j : \oplus_{i\in\omega}\mathfrak{D}_i \to \mathfrak{M}$ extending f . As notation, for each $i \in \omega$ let $g_i = j \circ k_i$, let $\mathfrak{B}'_i = g_i(\mathfrak{B})$, and let $\mathfrak{D}'_i = g_i(\mathfrak{D})$. So $\mathfrak{A} \subseteq \mathfrak{B}'_i \subseteq \mathfrak{D}'_i \subseteq \mathfrak{M}$ for each i and $\{\mathfrak{D}'_i : i \in \omega\}$ is disjoint over $\mathfrak{A}$.

We complete the proof by showing that the set $Z = \{i \in \omega : g_i \text{ does not omit } \Phi\}$ is finite. Assume by way of contradiction that Z were infinite. For each $i \in Z$, choose $\mathfrak{C}_i \in \Phi$ and an embedding $h_i : \mathfrak{C}_i \to \mathfrak{M}$ extending $g_i|\mathfrak{B}$. For each such i, let $\mathfrak{H}_i$ be the substructure of $\mathfrak{M}$ with universe $D'_i \cup h_i(C_i)$. Note that $|H_i| < |D| + m$ for each $i \in Z$. By Proposition 4.1.1 there is an $\mathfrak{F}$ and an infinite $Y \subseteq Z$ such that $\{\mathfrak{H}_i : i \in Y\}$ is disjoint over $\mathfrak{F}$ and $\mathfrak{F} \leq \mathfrak{H}_i$ for each $i \in Y$. Fix any $i(*) \in Y$. Since $\{\mathfrak{D}'_i : i \in Y\}$ are disjoint over $\mathfrak{A}$, $\mathfrak{A} \subseteq \mathfrak{F} \subseteq h_i(C_{i(*)})$. Since $\mathfrak{A} \leq \mathfrak{C}_{i(*)}$ by hypothesis, this implies $\mathfrak{A} \leq \mathfrak{F}$, hence $\mathfrak{A} \leq \mathfrak{H}_{i(*)}$ by transitivity. But this is impossible, since $\delta(\mathfrak{H}_{i(*)}/\mathfrak{D}'_{i(*)}) < 0$ (hence $\leq -Gr(m)$), while $\delta(\mathfrak{D}'_{i(*)}/\mathfrak{A}) < Gr(m)$ and $\delta(\mathfrak{H}_{i(*)}/\mathfrak{A}) = \delta(\mathfrak{H}_{i(*)}/\mathfrak{D}'_{i(*)}) + \delta(\mathfrak{D}'{}_{i(*)}/\mathfrak{A})$. □

Corollary 4.2.4. *Suppose that $\mathfrak{A}, \mathfrak{B} \in K_{\overline{\alpha}}$ and $\mathfrak{A} \leq \mathfrak{B}$ and $f : \mathfrak{A} \to \mathfrak{M}^*$ is strong where $\mathfrak{M}^* \models S_{\overline{\alpha}}$ is $\aleph_0$-saturated. Then there is a strong embedding $g : \mathfrak{B} \to \mathfrak{M}^*$ extending f. In particular, every $\mathfrak{B} \in K_\alpha$ embeds strongly into $\mathfrak{M}^*$.*

Proof. First, note that if $\mathfrak{C} \in K_{\overline{\alpha}}$ extends $\mathfrak{B}$, but $\mathfrak{A} \not\leq \mathfrak{C}$, then since f is strong, any embedding $g : \mathfrak{B} \to \mathfrak{M}^*$ omits $\mathfrak{C}$. So let Φ be the (infinite) set of all isomorphism types (over $\mathfrak{B}$) of $\mathfrak{C} \in K_{\overline{\alpha}}$ such that $\mathfrak{B} \subseteq \mathfrak{C}, \mathfrak{A} \leq \mathfrak{C}$, but $\mathfrak{B} \not\leq \mathfrak{C}$. By Proposition

4.2.3, for every finite $\Phi_0 \subseteq \Phi$ there is an embedding $g : \mathfrak{B} \to \mathfrak{M}^*$ extending f. Since $\mathfrak{M}^*$ $\aleph_0$-saturated there is $g : \mathfrak{B} \to \mathfrak{M}^*$ extending f that omits all of Φ. Combining this with the note above, g omits every extension $\mathfrak{C} \supseteq \mathfrak{B}$ such that $\mathfrak{B} \not\leq \mathfrak{C}$. Thus g is a strong embedding. The final sentence follows immediately since $\emptyset \leq \mathfrak{B}$ for any $\mathfrak{B} \in K_{\overline{\alpha}}$. $\square$

4.3 Quantifier elimination for $S_{\overline{\alpha}}$

In this section we give a description of how to genaralize the results of [2] mentioned at the beginning of this section. The arguments are originally due to Laskowski and we follow the development closely. They are included here for completeness.

Definition 4.3.1. For each $\mathfrak{A} \in K_{\overline{\alpha}}$ and $m \in \omega$, we say $\mathfrak{B} \in K_{\overline{\alpha}}$ is constructed by an m-chain over $\mathfrak{A}$ if there is an m-chain $\langle \mathfrak{A}_i : i \leq j \rangle$ over $\mathfrak{A}$, $\mathfrak{B} = \mathfrak{A}_j$. Let $X_m(\mathfrak{A})$ be a set of representatives of isomorphism types of $K_{\overline{\alpha}}$ that are constructed by m-chains over $\mathfrak{A}$.

Clearly $\mathfrak{A} \in X_m(\mathfrak{A})$, every $\mathfrak{A}' \in X_m(\mathfrak{A})$ extends $\mathfrak{A}$ and by 4.1.3, $X_m(\mathfrak{A})$ is finite.

Definition 4.3.2. For $\mathfrak{A}', \mathfrak{A}'' \in X_m(\mathfrak{A})$, write $\mathfrak{A}' \sqsubset \mathfrak{A}''$ if there is an embedding $g : \mathfrak{A} \to \mathfrak{A}''$ over $\mathfrak{A}$ such that $g(A') \neq A''$. If $\mathfrak{M} \models S_{\overline{\alpha}}$ and $\mathfrak{A} \to \mathfrak{M}$ is an embedding, a structure $\mathfrak{A}^* \in X_m(\mathfrak{A})$ is *maximally embeddable in* $\mathfrak{M}$ *over* f if there is an embedding $f' : \mathfrak{A}^* \to \mathfrak{M}$ extending f, but for any $\mathfrak{A}'$ such that $\mathfrak{A}^* \sqsubset \mathfrak{A}'$, there is no embedding $g : \mathfrak{A}' \to \mathfrak{M}$ that extends f.

Remark 4.3.3. Fix $\mathfrak{A} \in K_{\overline{\alpha}}$, $m \in \omega$, $\mathfrak{M} \models S_{\overline{\alpha}}$ and an embedding $f : \mathfrak{A} \to \mathfrak{M}$. Since $\mathfrak{A} \in X_m(\mathfrak{A})$ and $X_m(\mathfrak{A})$ is finite, a maximally embeddable $\mathfrak{A}^* \in X_m(\mathfrak{A})$ in $\mathfrak{M}$ over f exists. For any such $\mathfrak{A}^*$, if $f : \mathfrak{A}^* \to \mathfrak{M}$ is an embedding extending f, then $f'(\mathfrak{A}^*)$ is m-strong. Conversely, if $\langle \mathfrak{A}_i : i \leq j \rangle$ is a maximal m-chain in $\mathfrak{M}$ over $f(\mathfrak{A})$, then by Lemma 4.1.6 $\mathfrak{A}_j$ is isomorphic (over f) to some $\mathfrak{A}^*$ that is maximally embeddable in $\mathfrak{M}$ over f.

Fix $\mathfrak{A}, \mathfrak{B} \in K_{\overline{\alpha}}$, Φ a finite subset of $K_{\overline{\alpha}}$ and $m \in \omega$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and for each $\mathfrak{C} \in \Phi$, $\mathfrak{C} \supseteq \mathfrak{B}$ and $|C - A| < m$. For each such quadruple, let $Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$ denote the (finite) set of all $\mathfrak{A}^* \in X_m(\mathfrak{A})$ such that there is $\mathfrak{D} \in K_{\overline{\alpha}}$ and an embedding $g : \mathfrak{B} \to \mathfrak{D}$ over $\mathfrak{A}$ such that $\mathfrak{A}^* \leq \mathfrak{D}$, $D = A^* \cup g(B)$, and it is NOT the case that there are $\mathfrak{H} \in K_{\overline{\alpha}}$, $\mathfrak{C} \in \Phi$, and $h : \mathfrak{C} \to \mathfrak{H}$ extending g such that $\mathfrak{D} \leq \mathfrak{H}$.

The following Theorem forms the crux of our quantifier elimination. The significance is that the existence of an extension g omitting Φ is described in terms of extensions (and nonextensions) of f itself.

Theorem 4.3.4. *Fix any $\mathfrak{A}, \mathfrak{B} \in K_{\overline{\alpha}}$, Φ a finite subset of $K_{\overline{\alpha}}$, and $m \in \omega$ such that $\mathfrak{A} \subseteq \mathfrak{B}, \mathfrak{B} \subseteq \mathfrak{C}$, and $|C - A| < m$ for all $\mathfrak{C} \in \Phi$. As well, fix $\mathfrak{M} \models S_{\overline{\alpha}}$ and an embedding $f : \mathfrak{A} \to \mathfrak{M}$. There is an embedding $g : \mathfrak{B} \to \mathfrak{M}$ extending f and omitting Φ if and only if there is $\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$ that is maximally embeddable in $\mathfrak{M}$ over f .*

Proof. First suppose that there is $g : \mathfrak{B} \to \mathfrak{M}$ extending f and omitting Φ. Let $\langle \mathfrak{A}'_i : i \leq j \rangle$ be a maximal m-chain of minimal pairs in $\mathfrak{M}$ over $f(\mathfrak{A})$. By Remark 4.3.3 there is $\mathfrak{A}^* \in X_m(\mathfrak{A})$ that is maximally embeddable in $\mathfrak{M}$ over f via an isomorphism $f' : \mathfrak{A}^* \to \mathfrak{A}'_j$ extending f. Also, by Lemma 4.1.6, $\mathfrak{A}'_j$ is m-strong in $\mathfrak{M}$.

It suffices to show that $\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$. Let $\mathfrak{D}'$ be the substructure of $\mathfrak{M}$ with universe $A'_j \cup g(B)$. Let $\mathfrak{D} \supseteq \mathfrak{A}^*$ be isomorphic to $\mathfrak{D}'$ via an isomorphism $j : \mathfrak{D} \to \mathfrak{D}'$ that extends f' . Since $\mathfrak{A}'_j$ is m-strong in $\mathfrak{M}$, $\mathfrak{A}'_j \leq \mathfrak{D}'$, hence $\mathfrak{A}^* \leq \mathfrak{D}$. Put $g^* := j^{-1} \circ g$. Then $g : \mathfrak{B} \to \mathfrak{D}$ and $D = A^* \cup g^*(B)$. To finish this direction, assume by way of contradiction that there is $\mathfrak{H} \geq \mathfrak{D}$, $\mathfrak{C} \in \Phi$ and $h : \mathfrak{C} \to \mathfrak{H}$ extending g^* . Since $\mathfrak{M} \models S_{\overline{\alpha}}$ and $\mathfrak{D} \leq \mathfrak{H}$, the embedding $j : \mathfrak{D} \to \mathfrak{M}$ extends to an embedding $j^* : \mathfrak{H} \to \mathfrak{M}$. But then $j^* \circ h : \mathfrak{C} \to \mathfrak{M}$ extends g, contradicting the fact that g omitted Φ.

Conversely, suppose that $\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$ and that $\mathfrak{A}^*$ is maximally embeddable in $\mathfrak{M}$ over f. Choose an embedding $f' : \mathfrak{A} \to \mathfrak{M}$ extending f. By Remark 4.3.3, $f'(\mathfrak{A}^*)$ is m-strong in $\mathfrak{M}$.

Choose $\mathfrak{D} \in K_{\overline{\alpha}}$ and $g : \mathfrak{B} \to \mathfrak{D}$ over $\mathfrak{A}$ witnessing $\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$. Fix Φ^*, a (finite) set of representatives of all isomorphism types over $\mathfrak{D}$ of all $\mathfrak{H} \in K_{\overline{\alpha}}$ that satisfy $\mathfrak{A}^* \leq \mathfrak{H}$, $|H - A^*| < m$, $\mathfrak{D} \subseteq \mathfrak{H}$, but $\mathfrak{D} \not\leq \mathfrak{H}$. By Proposition 4.2.3 there is an embedding $j : \mathfrak{D} \to \mathfrak{M}$ extending f' that omits every $\mathfrak{H} \in \Phi^*$. We argue that $g' : \mathfrak{B} \to \mathfrak{M}$ omits every $\mathfrak{C} \in \Phi$, where $g' := j \circ g$.

By way of contradiction, suppose that there were $\mathfrak{C} \in \Phi$ and $h : \mathfrak{C} \to \mathfrak{M}$ extending g'. Let $\mathfrak{H}'$ be the substructure of $\mathfrak{M}$ with universe $j(D) \cup h(C)$. There are two cases. On one hand, if $j(\mathfrak{D}) \not\leq \mathfrak{H}'$ then we would contradict j omitting Φ^*. On the other hand, if $j(\mathfrak{D}) \leq \mathfrak{H}$ then we would contradict $\mathfrak{D}$ being a witness to $\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$. $\square$

Suppose that $\mathfrak{A} \subseteq \mathfrak{B}$ are from $K_{\overline{\alpha}}$. Let $\mathfrak{C}$ be the union of a maximal minimal chain of minimal pairs over $\mathfrak{A}$ in $\mathfrak{B}$. Then clearly $\mathfrak{C} \leq \mathfrak{B}$. Since the sentence $\forall \overline{x}[\Delta_{\mathfrak{C}}(\overline{x}) \to \exists \overline{y} \Delta_{(\mathfrak{C},\mathfrak{B})}(\overline{x}, \overline{y})]$ is an axiom $S_{\overline{\alpha}}$, the extension formula $\Psi_{\mathfrak{A},\mathfrak{B}}$ is $S_{\overline{\alpha}}$ equivalent to the chain-minimal extension formula $\Psi_{\mathfrak{A},\mathfrak{C}}$, i.e. every extension formula is $S_{\overline{\alpha}}$ equivalent to a chain minimal extension formula.

Theorem 4.3.5. *Every L-formula is $S_{\overline{\alpha}}$-equivalent to a boolean combination of chain-minimal extension formulas.*

Proof. It suffices to show that every L-formula is $S_{\overline{\alpha}}$-equivalent to a boolean combination of extension formulas. By taking $\mathfrak{A} = \mathfrak{B}$, every Δ-formula describing the isomorphism type of any $\mathfrak{A}$ is equivalent to an extension formula. It is easily seen that every atomic formula $\varphi(\overline{x})$ is equivalent to a disjunction of $\Delta_{\mathfrak{A}}$ -formulas for which φ holds. Thus, every quantifier-free formula is equivalent to a boolean combination of extension formulas.

It suffices to show that if $\theta(\overline{x}, \overline{y})$ is a boolean combination of extension formulas, then $\exists \overline{y} \theta(\overline{x}, \overline{y})$ is $S_{\overline{\alpha}}$-equivalent to a boolean combination of extension formulas. Since existential quantification commutes with disjunction we may assume that $\theta(\overline{x}, \overline{y}) \vdash \Delta_{\mathfrak{A}}(\overline{x}) \wedge \Delta_{\mathfrak{A},\mathfrak{B}}(\overline{x}, \overline{y})$ for some $\mathfrak{A} \subseteq \mathfrak{B}$ and that θ is a conjunction of extension formulas and negations of extension formulas over $\mathfrak{B}$. We must show that $\exists \overline{y} \theta(\overline{x}, \overline{y})$ is $S_{\overline{\alpha}}$-equivalent to a boolean combination of extension formulas over $\mathfrak{A}$.

Fix such a θ, let Γ be the set of $\mathfrak{C}$ such that $\Psi_{\mathfrak{B},\mathfrak{C}}$ occurs positively in θ, and Φ be the set of $\mathfrak{C}$ for which $\neg \Psi_{\mathfrak{B},\mathfrak{C}}$ occurs as a conjunct of θ. Let $m = \sum_{\mathfrak{C} \in \Gamma \cup \Phi} |\mathfrak{C}|$. Call a $\mathfrak{D} \in K_{\overline{\alpha}}$ a *candidate* if $\mathfrak{B} \subseteq \mathfrak{D}$, $|D| < m$, for every $\mathfrak{C} \in \Gamma$ there is $h : \mathfrak{C} \to \mathfrak{D}$, while for each $\mathfrak{C} \in \Phi$, there is NO $h : \mathfrak{C} \to \mathfrak{D}$. For each candidate $\mathfrak{D}$, let $\Phi^*_{\mathfrak{D}}$ consist of representatives of all isomorphism types of $\mathfrak{F} \in K_{\overline{\alpha}}$ such that $\mathfrak{D} \subseteq \mathfrak{F}$, $|F - D| < \max\{|\mathfrak{C}| : \mathfrak{C} \in \Phi\}$, and there is an embedding $h : \mathfrak{C} \to \mathfrak{F}$ over $\mathfrak{B}$. Let Z consist of a representative of every isomorphism type over $\mathfrak{B}$ of candidates. We claim that $\exists \overline{y}\theta(\overline{x}, \overline{y})$ is $S_{\overline{\alpha}}$-equivalent to:

$$\chi(\overline{x}) := \bigvee_{\mathfrak{D} \in Z} \bigvee_{\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{D}, \Phi_{\mathfrak{D}^*, m})} \left[\Psi_{\mathfrak{A}, \mathfrak{A}^*}(\overline{x}) \wedge \bigwedge_{\mathfrak{A}' \in X_m(\mathfrak{A}), \mathfrak{A}' \sqsupseteq \mathfrak{A}^*} \neg \Psi_{\mathfrak{A}, \mathfrak{A}'}(\overline{x}) \right]$$

To see this, fix $\mathfrak{M} \models S_{\overline{\alpha}}$ and $\overline{a}$ from M. Let $\mathfrak{A}$ be the substructure of $\mathfrak{M}$ with universe $\overline{a}$. First assume that $\mathfrak{M} \models \exists \overline{y}\theta(\overline{x}, \overline{y})$. Fix a tuple $\overline{b}$ from $\mathfrak{M}$ realizing $\theta(\overline{a}, \overline{y})$ and let $\mathfrak{B}$ be the substructure of $\mathfrak{M}$ with universe $\overline{a} \cup \overline{b}$ For each $\mathfrak{C} \in \Gamma$ choose an embedding $g_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{M}$ over $\mathfrak{B}$. Let $\mathfrak{D} = \bigcup\{g_{\mathfrak{C}}(\mathfrak{C}) : \mathfrak{C} \in \Gamma\} \subseteq \mathfrak{M}$. Since each $\mathfrak{C} \in \Phi$ is omitted over $\mathfrak{B}$, $\mathfrak{D}$ is a candidate. Moreover, the identity map $id : \mathfrak{D} \to \mathfrak{M}$ omits Φ^*_D , so $\mathfrak{M} \models \chi(\overline{x})$ by Theorem 4.3.4.

Conversely, suppose that $\mathfrak{M} \models \chi(\overline{x})$. Choose a candidate $\mathfrak{D}$ witnessing this. By Theorem 4.3.4 again, there is an embedding $g : \mathfrak{D} \to \mathfrak{M}$ over $\mathfrak{A}$ omitting $\Psi^*_{\mathfrak{D}}$. Let $\overline{b}$ enumerate the image of the restriction $g|_B$. It is easily checked that $\mathfrak{M} \models \theta(\overline{a}, \overline{b})$. $\square$

4.4 Some immediate consequences of the quantifier elimination

Of the following results, (1) and (2) of Theorem 4.4.1 was first proved (in near full generality) in [3] by Ikeda, Kikyo and Tsuboi. However their proof does not yield the quantifier elimination result of Theorem 4.3.5. The proofs we present here are due to Laskowski [2].

Theorem 4.4.1. *1. The theory $S_{\overline{\alpha}}$ is complete.*

2. $S_{\overline{\alpha}}$ is the theory of the $(K_{\overline{\alpha}}, \leq)$-generic $\mathfrak{M}_{\overline{\alpha}}$.

3. Fix $\mathfrak{M} \models S_{\overline{\alpha}}$ and $\mathfrak{X} \subseteq \mathfrak{M}$. The following are equivalent:

(a) X is algebraically closed

(b) For any minimal pair $(\mathfrak{B}, \mathfrak{C})$ with $\mathfrak{C} \subseteq \mathfrak{M}$, if $B \subseteq X$, then $C \subseteq X$.

(c) For any finite $B \subseteq M$, $B \cap X \leq B$

Proof. Claim 1: Since the empty structure is an element of $K_{\overline{\alpha}}$ and since $\emptyset \leq \mathfrak{A}$ for all $\mathfrak{A} \in K_{\overline{\alpha}}$, $S_{\overline{\alpha}}$ decides every extension sentence (i.e., extension formula with no free variables). Thus, $S_{\overline{\alpha}}$ decides every L-sentence by Theorem 4.3.5.

Claim 2: Since $S_{\overline{\alpha}}$ is complete, it suffices to show that $\mathfrak{M} \models S_{\overline{\alpha}}$ where $\mathfrak{M}$ is the $(K_{\overline{\alpha}}, \leq)$ generic. Say $\mathfrak{M} = \bigcup\{\mathfrak{A}_n : n \in \omega\}$, where each $\mathfrak{A}_n \in K_{\overline{\alpha}}$, $\mathfrak{A}_n \leq \mathfrak{A}_{n+1}$, and as a result $\mathfrak{A}_n$ is a strong substructure of $\mathfrak{M}$. First, let $\mathfrak{B}$ be any finite substructure of $\mathfrak{M}$. Choose n such that $\mathfrak{B} \subseteq \mathfrak{A}_n$. Since membership in $K_{\overline{\alpha}}$ is hereditary, it follows that $\mathfrak{B} \in K_{\overline{\alpha}}$.

Second, suppose that $\mathfrak{B} \leq \mathfrak{C}$ and $f : \mathfrak{B} \to \mathfrak{M}$ is given. Choose n such that $f(\mathfrak{B}) \subseteq \mathfrak{A}_n$. Let $f : \mathfrak{C} \to \mathfrak{C}'$ be any isomorphism extending f such that $\{\mathfrak{A}_n, \mathfrak{C}'\}$ are disjoint over $f(\mathfrak{B})$. (We do NOT require that $\mathfrak{C}' \subseteq \mathfrak{M}$.) Let $\mathfrak{D}$ be the free join of $\{\mathfrak{A}_n, \mathfrak{C}'\}$ over $f(\mathfrak{B})$. Since $f(\mathfrak{B}) \leq \mathfrak{C}'$, Fact 2.3.4 implies that $\mathfrak{A}_n \leq \mathfrak{D}$. Since $\mathfrak{M}$ is $(K_{\overline{\alpha}}, \leq)$-generic, choose an embedding $g : \mathfrak{D}' \to \mathfrak{M}$ over $\mathfrak{A}_n$. Then $h = g \circ f'$ is an embedding of $\mathfrak{C}$ into $\mathfrak{M}$ extending f.

Claim 3: $a) \implies b)$ Assume $\mathfrak{X}$ is algebraically closed and fix $\mathfrak{B} \subseteq \mathfrak{X}$ and a minimal pair $(\mathfrak{B}, \mathfrak{C})$ with $\mathfrak{C} \subseteq \mathfrak{M}$. Then, letting $\bar{b}$ be an enumeration of $\mathfrak{B}$, $\Delta_{\mathfrak{C}}(\overline{x}, \bar{b})$ is an algebraic formula in $\mathfrak{M}$, hence $\mathfrak{C} \supseteq \mathfrak{X}$.

$b) \implies c)$ Choose any finite $\mathfrak{B} \subseteq \mathfrak{M}$. If $\mathfrak{B} \cap \mathfrak{X} \not\leq \mathfrak{B}$ then let $\mathfrak{C}$ be minimal such that $\mathfrak{B} \cap \mathfrak{X} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$ and $\mathfrak{B} \cap \mathfrak{X} \not\leq \mathfrak{C}$. Then $C \subseteq X$, so $\mathfrak{B} \cap X = \mathfrak{C}$, contradiction . $c) \implies a)$ Assume that (c) holds. Let $b \in M - X$ and let $\varphi(x, \overline{a})$ be any $L(X)$-formula such that $\mathfrak{M} \models \varphi(\bar{b}, \overline{a})$. We argue that $\varphi(x, \overline{a})$ is not algebraic. Let $\mathfrak{B}$ denote the substructure of $\mathfrak{M}$ with universe $\overline{a}b$. By Theorem 4.3.5, we may assume that φ is a boolean combination of chain-minimal extension formulas. By writing φ in Disjunctive Normal Form it suffices to assume that $\varphi(x, \overline{a})$ has the form

$$\bigwedge_{\mathfrak{C} \in \Gamma} \exists \overline{z} \Delta_{\mathfrak{C}}(x, \overline{a}, \overline{z}) \wedge \bigwedge_{\mathfrak{C} \in \Phi} \neg \exists \overline{z} \Delta_{\mathfrak{C}}(x, \overline{a}, \overline{z})$$

for finite sets Γ, Φ of chain-minimal extensions of $\mathfrak{B}$. Choose m large (at least $|B| + \sum_{\mathfrak{C} \in \Gamma \cup \Phi} |\mathfrak{C}|$). Recall that by Lemma 4.1.6, for every finite $\mathfrak{A} \subseteq \mathfrak{M}$ and every $n \in \omega$, there is a unique smallest m-strong $\mathfrak{B}$ satisfying $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}$. Denote this $\mathfrak{B}$ by $\mathrm{cl}_m(\mathfrak{B})$ and set $\mathfrak{B}^* = \mathrm{cl}_m(\mathfrak{B})$. Let $\mathfrak{A}_0 = \mathfrak{B} \cap \mathfrak{A}$, and let Φ^* be a (finite) set of isomorphism types of all $\mathfrak{D} \supseteq \mathfrak{B}^*$ with $|D - B| < m$. Clearly $\Phi \subseteq \Phi^*$ By (3) $\mathfrak{A}_0 \leq \mathfrak{B}$, so by Proposition 4.2.3 there are infinitely many embeddings $g_i : \mathfrak{B}^* \to \mathfrak{M}$, each omitting Φ, such that $\{g_i(\mathfrak{B}^*) : i \in \omega\}$ is disjoint over $\mathfrak{A}_0$. It is easily checked that $\mathfrak{M} \models \varphi(g_i(b), \overline{a})$ for each $i \in \omega$. □

The following lemma, will be useful in both Section 5. It is an immediate consequence of the quantifier elimination:

Lemma 4.4.2. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$ and A be a finite closed set of $\mathfrak{M}$. Suppose that π is a consistent partial type over A any realization of π has the same quantifier free type over A. Then*

1. *If $\mathfrak{M}$ is $\aleph_0$-saturated and any realization $\overline{b}$ of π in $\mathfrak{M}$ has the property that $\overline{b}A$ is closed in $\mathfrak{M}$, then π has a unique completion to a complete type p over A.*

2. *If any realization $\overline{b}$ of the quantifier free type of π (over A) has the property $\delta(\overline{b}/A) = 0$, then π has a unique completion p over A and further p is isolated by the formula $\Delta_{A,A\bar{b}}(\overline{a}, \overline{x})$.*

Proof. (1): Note that by Theorem 4.3.5 it suffices to show that all chain minimal formulas over A are determined by the given conditions. Let $\bar{b} \models \pi$. Fix $\overline{b}A \subseteq D \in K_{\overline{\alpha}}$ and let $\phi_D(\overline{x}) = \Delta_{\overline{a},\overline{a}\bar{b}}(\overline{a}, \overline{x}) \wedge \exists \overline{y} \Delta_{\overline{a},\overline{a}\bar{b},D-\overline{a}\bar{b}}(\overline{a}, \overline{x}, \overline{y})$ be the corresponding extension formula. Suppose that $\overline{b}A \leq D$. Now as $\overline{b}A \leq M$ and $\mathfrak{M} \models S_{\overline{\alpha}}$, we obtain that $\mathfrak{M} \models \phi_D(\overline{b})$. Thus it follows that $p \vdash \phi_D$. Now suppose that $\overline{b}A \not\leq D$. If $\pi^* = \pi \cup \neg\phi_D(\overline{x})$ is consistent, then there is some realization of π^* *in* $\mathfrak{M}$ by $\aleph_0$-saturation. Clearly no realization of π^* can be strong in $\mathfrak{M}$, and hence $\pi \vdash \neg\phi_D(\overline{x})$. Thus π determines all *extension formulas* including the chain minimal formulas over A and thus is complete. So simply take $p = \pi$ to obtain the required complete type.

(2): Consider a partial type given as above. We may as well assume that $\Delta_{A,A\bar{b}}(\overline{a}, \overline{x}) \in \pi$. Arguing as in part (1), we see that if $\overline{b}A \leq D$, then $\phi_D(\overline{x}) \in \pi$. So assume that $\overline{b}A \not\leq D$ and that $\neg\phi_D(\overline{x})$ is consistent with π. As $\mathfrak{M}$ is a model, there is some $\overline{b'}$ realizing $\phi_D(\overline{x})$. But then, there is some $C \subseteq M$ such that $(\overline{b}A, \overline{b}AC)$ is a minimal pair. Now $\delta(\bar{b}AC/A) = \delta(\bar{b}AC/\bar{b}A) + \delta(\bar{b}A/A) < 0$. But this contradicts $A \leq M$. Thus the required result follows. $\square$

The following lemma shows that isomorphisms between closed sets are in fact elementary.

Lemma 4.4.3. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$ and assume that $X, X' \subseteq M$ are intrinsically (or equivalently algebraically), closed. If there is an isomorphism from X to X', i.e. the quantifier free types of X, X' are the same, then $tp_{\mathfrak{M}}(X) = tp_{\mathfrak{M}}(X')$.*

Proof. Let $\mathfrak{M}$, X, X' be above. Note that we are not assuming that X, X' are finite. Assume that there is an isomorphism from X to X'. Denote this isomorphism f. We claim that $\mathrm{tp}_{\mathfrak{M}}(X) = \mathrm{tp}_{\mathfrak{M}}(X')$. Thus we need to establish that given an L formula $\phi(\overline{x})$ and a finite tuple $\overline{a}$ of elements from X with the corresponding length, $\mathfrak{M} \models \varphi(\overline{a})$ if and only if $\mathfrak{M} \models \varphi(f(\overline{a}))$.

Since any formula is equivalent to a boolean combination of chain minimal formulas, the result follows if we establish the above result for chain minimal formulas. Let $\Delta_B(\overline{x})$ be the atomic diagram of $\overline{a}$. Assume that $\varphi(\overline{x}) = \Delta_B(\overline{x}) \wedge \exists \overline{y} \Delta_{B,C}(\overline{x}, \overline{y})$ is a chain minimal extension formula and that $B = B_0 \subsetneq ... \subsetneq B_n = C$ with (B_i, B_{i+1}) a minimal pair. Assume that $\mathfrak{M} \models \phi(\overline{a})$. Since X is intrinsically closed, it follows that if $\mathfrak{M} \models \Delta_{B,C}(\overline{a}, \overline{c})$, then $\overline{c} \subseteq X$. By using the fact that f preserves the quantifier free formulas, it follows that $\mathfrak{M} \models \Delta_{B,C}(f(\overline{a}))$ and $\mathfrak{M} \models \Delta_{B,C}(f(\overline{a}), f(\overline{c}))$ and thus $\mathfrak{M} \models \varphi(f(\overline{a}))$. The reverse direction is immediate by a similar argument. Hence we obtain that $\mathrm{tp}_{\mathfrak{M}}(X) = \mathrm{tp}_{\mathfrak{M}}(X')$. $\square$

Theorem 4.4.4. *Let $\mathfrak{M}, \mathfrak{N} \models S_{\overline{\alpha}}$. If $M \leq N$, then $\mathfrak{M} \preccurlyeq \mathfrak{N}$.*

Proof. Let $\psi(\overline{x},\overline{y})$ be an L formula. Let $\overline{a} \in M^{lg(\overline{x})}$. Assume that $\mathfrak{N} \models \exists\overline{y}\psi(\overline{a},\overline{y})$. But $\psi(\overline{x},\overline{y})$ is equivalent to the boolean combination of chain minimal formulas, say $S_{\overline{\alpha}} \vdash \forall(\overline{x})(\exists\psi(\overline{x},\overline{y}) \leftrightarrow \bigwedge_{i<n}\varphi_i(\overline{x},\overline{y}))$ where each $\varphi(\overline{x},\overline{y})$ is either a chain minimal formula or the negation of a chain minimal formula. Suppose that $\overline{b} \in N^{lg(\overline{y})}$ is such that $\mathfrak{N} \models \psi(\overline{a},\overline{b})$. If φ_i is a chain minimal formula then it follows that $\overline{b} \in M^{lg(\overline{y})}$ as M is a closed set. So assume that each φ_i is the negation of a chain minimal formula. Note that we may split $\overline{b} = \overline{b}_1\overline{b}_2$ where $\overline{b}_1$ is formed via a minimal chain and $A\overline{b}_1 \leq N$. As above, it follows that $\overline{b}_1 \subseteq\in M^{lg(\overline{y})-lg(\overline{b}_1)}$. But as $\mathfrak{M} \models S_{\overline{\alpha}}$, it follows that there exists a $\overline{b'}_2 \in M^{lg(\overline{y})-lg(\overline{b}_1)}$ that is isomorphic to $\overline{b}_2$ over $A\overline{b}_1$. It is now easily seen that the $\overline{b}_1\overline{b'}_2 \in M^{lg(\overline{y})}$ and $\mathfrak{N} \models \varphi_i(\overline{a},\overline{b}_1\overline{b'}_2)$ for each i. Thus $\mathfrak{N}$ is an elementary extension of $\mathfrak{M}$. $\square$

Chapter 5: Atomic Models of $S_{\overline{\alpha}}$

In this section we study the atomic models of the theories of Baldwin-Shi hypergraphs. Our main results begin with Theorem 5.1.7, in which we characterize the atomic models as the existentially closed models of $S_{\overline{\alpha}}$ with *finite closures* (see Definition 5.1.1) or equivalently those with finite closures where the closed finite substructures are those with rank 0. This immediately yields coherence of $\overline{\alpha}$ as a necessary condition for the existence of atomic models for $S_{\overline{\alpha}}$. We then proceed to combine the results in Section 3.3 and chain arguments to obtain Theorem 5.2.9 which establishes coherence of $\overline{\alpha}$ is also sufficient for the existence of atomic models. We also explore the effect that rationality of $\overline{\alpha}$, arguably the most natural form of coherence, has on atomic models of $S_{\overline{\alpha}}$. Our exploration leads to Theorem 5.2.19 which allows us to categorize rational $\overline{\alpha}$ as precisely the coherent $\overline{\alpha}$ with theories of Baldwin-Shi hypergraphs whose models isomorphically embed into an atomic model of the same cardinality. We begin with the following definitions.

Definition 5.0.1. Let $\mathfrak{M} \models S_{\overline{\alpha}}$. We say that $\mathfrak{M}$

1. is *atomic* if any finite tuple $\overline{a} \subseteq M^n$ of length n, $\mathrm{tp}(\overline{a})$ is isolated.

2. is existentially closed if for all $\mathfrak{N}$ such that $M \subseteq N$ any quantifier free formula $\varphi(\overline{x}, \overline{y})$ any $\overline{b} \in M^{\lg(\overline{y})}$, if $\mathfrak{N} \models \exists \overline{x} \varphi(\overline{x}, \overline{b})$, then $\mathfrak{M} \models \exists \overline{x} \varphi(\overline{x}, \overline{b})$.

Remark 5.0.2. We note that since $S_{\overline{\alpha}}$ is a collection of $\forall\exists$-sentences any model of $S_{\overline{\alpha}}$ embeds into a existentially closed model of $S_{\overline{\alpha}}$ (see for example Theorem 8.2.1 of [23]).

Definition 5.0.3. Let $\mathfrak{M} \models S_{\overline{\alpha}}^{\forall}$. By $d_{\mathfrak{M}}$ we denote the function $d_{\mathfrak{M}} : \{\mathfrak{A} : \mathfrak{A} \subseteq_{\text{Fin}} \mathfrak{M}\} :\rightarrow \mathbb{R}$ such $d_{\mathfrak{M}}(\mathfrak{A}) = \inf\{\delta(\mathfrak{B}) : \mathfrak{A} \subseteq \mathfrak{B}, \mathfrak{B}$ finite and $\mathfrak{B} \subseteq \mathfrak{M}\}$.

Our starting point is the following theorem due to Laskowski (Theorem 6.5 of [2]). Its proof only uses the quantifier elimination result of Theorem 4.3.5 and thus holds in our generalized context.

Theorem 5.0.4. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$. Now $d_{\mathfrak{M}}(\mathfrak{A}) = 0$ for all finite $\mathfrak{A} \subseteq \mathfrak{M}$ if and only if $\mathfrak{M}$ is an existentially closed model.*

Proof. Fix $\mathfrak{M} \models S_{\overline{\alpha}}$. By virtue of Theorem 4.3.5 $\mathfrak{M}$ is an existentially closed model of $S_{\overline{\alpha}}$ if and only if for every extension formula $\Psi_{\mathfrak{A}.\mathfrak{B}}(\overline{x})$ and every $\overline{a}$ from $\mathfrak{M}$, *if* $\mathfrak{N} \models \Psi_{\mathfrak{A}.\mathfrak{B}}(\overline{a})$ for some $\mathfrak{N} \supseteq \mathfrak{M}$ modelling $S_{\overline{\alpha}}$, *then* $\mathfrak{M} \models \Psi_{\mathfrak{A}.\mathfrak{B}}(\overline{a})$.

Now assume that every finite $\mathfrak{A} \subseteq \mathfrak{M}$ satisfies $d_{\mathfrak{M}}(\mathfrak{A}) = 0$. By way of contradiction, assume that $\mathfrak{M}$ is not an existentially closed model of $S_{\overline{\alpha}}$. Then there are triples $(\mathfrak{A}, \mathfrak{B}, \mathfrak{N})$ such that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{N}$, $\mathfrak{N} \supseteq \mathfrak{M}$ is a model of $S_{\overline{\alpha}}$, $\mathfrak{A} \subseteq \mathfrak{M}$, but there is no embedding of $\mathfrak{B}$ into $\mathfrak{M}$ over $\mathfrak{A}$. Among all such triples, choose $(\mathfrak{A}, \mathfrak{B}, \mathfrak{N})$ such that $|B - A|$ is as small as possible. Note that this minimality implies that $B \cap M = A$.

We claim that $\mathfrak{A} \leq \mathfrak{B}$. To see this, assume by way of contradiction that $\delta(\mathfrak{B}'/\mathfrak{A}) < 0$ for some $\mathfrak{B}'$ satisfying $\mathfrak{A} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$. Since $d_{\mathfrak{M}}(A) = 0$ there is a substructure $\mathfrak{C}$ such that $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{M}$ with $\delta(\mathfrak{C}) < -\delta(\mathfrak{B}'/\mathfrak{A})$. It follows from our minimality condition that $\mathfrak{B}' \cap \mathfrak{C} = \mathfrak{A}$. Thus, taking $\mathfrak{D}$ to be the substructure of $\mathfrak{N}$ with universe $B' \cup C$, $\mathfrak{D}$ is a join of $\{\mathfrak{B}', \mathfrak{C}\}$ over $\mathfrak{A}$. Applying Lemma 2.2.5 yields $\delta(\mathfrak{D}/\mathfrak{C}) \leq \delta(\mathfrak{B}'/\mathfrak{A})$. But then $\delta(\mathfrak{D}) = \delta(\mathfrak{C}) + \delta(\mathfrak{D}/\mathfrak{C}) \leq \delta(\mathfrak{C}) + \delta(\mathfrak{B}'/\mathfrak{A}) < 0$ which contradicts $\mathfrak{N} \models S_{\overline{\alpha}}$. But now, since $\mathfrak{A} \leq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{M}$, there is an embedding of $\mathfrak{B}$ into $\mathfrak{M}$ over $\mathfrak{A}$ since $\mathfrak{M} \models S_{\overline{\alpha}}$.

For the converse, suppose that $\mathfrak{M}$ is an existentially closed model of $S_{\overline{\alpha}}$, $\mathfrak{A}$ is a finite substructure of $\mathfrak{M}$, and $\epsilon > 0$ (in case $\overline{\alpha}$ is rational, we assume that $\epsilon = \frac{1}{c}$). In order to show that $d_{\mathfrak{M}}(\mathfrak{A}) = 0$ it suffices to find a finite substructure $\mathfrak{D}'$ such that δ and $\delta(\mathfrak{D}') < \epsilon$. Since $\emptyset \leq \mathfrak{A}$ we can apply Proposition 4.2.1 to get $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $\mathfrak{A} \subseteq \mathfrak{D}$ and $\delta(\mathfrak{D}) \leq \epsilon$. By replacing $\mathfrak{D}$ by an isomorphic copy we may assume that $D \cap M = A$.

The free join $\mathfrak{X} = \mathfrak{M} \oplus_{\mathfrak{A}} \mathfrak{D}$ is a model of $S_{\overline{\alpha}}^{\forall}$, so there is a model $\mathfrak{N}$ of $S_{\overline{\alpha}}$ containing $\mathfrak{X}$. Without loss, we may assume that $\mathfrak{N} \supseteq \mathfrak{M}$. Now $\mathfrak{A} \subseteq \mathfrak{D} \subseteq \mathfrak{N}$, $\mathfrak{A} \subseteq \mathfrak{M}$, and $\mathfrak{M}$ is an existentially closed model of $S_{\overline{\alpha}}$, so there is an embedding $g : \mathfrak{D} \to \mathfrak{M}$ over $\mathfrak{A}$. Then $g(\mathfrak{D})$ is as desired. $\square$

5.1 Atomic Models

Our goal in this section is to prove Theorem 5.1.7. We begin with the following:

Definition 5.1.1. Given $\mathfrak{M} \models S_{\overline{\alpha}}^{\forall}$, we say that $\mathfrak{M}$ has *finite closures* if for any finite $\mathfrak{A} \subseteq \mathfrak{M}$, there is some finite $\mathfrak{B} \supseteq \mathfrak{A}$ with $\mathfrak{B} \leq \mathfrak{M}$. We say an L theory $T \supseteq S_{\overline{\alpha}}^{\forall}$ has finite closures if every model of T has finite closures.

Remark 5.1.2. Given a countable model $\mathfrak{M} \models S_{\overline{\alpha}}$, $\mathfrak{M}$ has finite closures if and only if $\mathfrak{M}$ is the union of a strong chain $\langle \mathfrak{A}_i : i \in \omega \rangle$ of elements of $K_{\overline{\alpha}}$.

Remark 5.1.3. Let $\mathfrak{A} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{A}) > 0$. We note that there are infinitely many non-isomorphic minimal pairs $(\mathfrak{A}, \mathfrak{C})$ over $\mathfrak{A}$. Indeed if $\overline{\alpha}$ is *not* graph-like with weight one, then this is immediate from Theorem 3.2.15. In the case that $\overline{\alpha}$ *is* graph-like with weight one, then this is an immediate consequence of Theorem 8.2.4.

Lemma 5.1.4. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$ and $A \subseteq_{Fin} M$ with $\delta(A) = 0$. Let $\overline{a}$ be a fixed enumeration of A. Then $A \leq \mathfrak{M}$ and $\Delta_A(\overline{x})$ isolates the tp$(\overline{a})$ in $\mathfrak{M}$.*

Proof. This follows from an application of Lemma 4.4.2, by noting that $\emptyset \leq \mathfrak{M}$ and $\delta(A/\emptyset) = 0$. □

Lemma 5.1.5. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$ be atomic.*

1. *Let $A \subseteq_{Fin} M$. Now $A \leq \mathfrak{M}$ if and only if $\delta(A) = 0$.*

2. *$\mathfrak{M}$ has finite closures.*

Proof. Claim 1: Let $A, \mathfrak{M}$ be as stated above. Clearly if $\delta(A) = 0$, then $A \leq \mathfrak{M}$. For the converse assume that $A \leq \mathfrak{M}$. Assume by way of contradiction that $\delta(A) > 0$ but there is a formula $\varphi(\overline{x})$ that isolates $\mathrm{tp}(A/\emptyset)$. We may as well assume that φ is a boolean combination of chain minimal formulas over A (using Theorem 4.3.5). Note that as $A \leq \mathfrak{M}$, it follows that there are *no* minimal pairs over A realized in M. Thus φ contains entirely of negations of chain minimal formulas. Fix an integer m larger than $\sum_{\mathfrak{C} \in \Phi} |\mathfrak{C}|$ where Φ is a finite set that contains the isomorphism types of finite structures that appear in the formula $\varphi(\overline{x})$. By Remark 5.1.3 there are infinitely many non-isomorphic $\mathfrak{C} \in K_{\overline{\alpha}}$ with $(A, \mathfrak{C})$ a minimal pair. Thus there is some $\mathfrak{C} \in K_{\overline{\alpha}}$ with $|\mathfrak{C}| > m$ and take $\psi(\overline{x}) := \Delta_A(\overline{x}) \wedge \exists \overline{y} \Delta_{A,\mathfrak{C}}(\overline{x}, \overline{y})$. As $\varphi(\overline{x})$ isolates $\mathrm{tp}(A/\emptyset)$, we obtain that $S_{\overline{\alpha}} \vdash \forall \overline{x}(\varphi(\overline{x}) \to \neg\psi(\overline{x}))$.

There exists an isomorphic copy of $\mathfrak{C}$ inside of the $(K_{\overline{\alpha}}, \leq)$ generic $\mathfrak{M}^*$ (which by an abuse of notation we denote by C) such that $\mathfrak{C} \leq \mathfrak{M}^*$. Now since $(A, \mathfrak{C})$ is a minimal pair, there is a copy of A inside of $\mathfrak{C}$ ((which by an abuse of notation we denote by C) and an enumeration of $\overline{a}$ of the isomorphic copy such that $\mathfrak{C} \models \varphi(\overline{a})$. We claim that $\mathfrak{M}^* \models \varphi(\overline{a}) \wedge \psi(\overline{a})$. It is clear that $\mathfrak{M}^* \models \psi(\overline{a})$. So we show that $\mathfrak{M}^* \models \varphi(\overline{a})$. Note that as $\mathfrak{C} \leq \mathfrak{M}^*$, i.e. C is *closed*. Thus any minimal pair over A that lies inside $\mathfrak{M}^*$ lies inside of $\mathfrak{C}$ (see Definition 2.4.2 and Remark 2.4.3). By our choice of $\mathfrak{C}$, it is immediate that $\mathfrak{M}^* \models \varphi(\overline{a})$. Hence we have that $\mathfrak{M}^* \models \varphi(\overline{a}) \wedge \psi(\overline{a})$. But this contradicts $S_{\overline{\alpha}} \vdash \forall \overline{x}(\varphi(\overline{x}) \to \neg\psi(\overline{x}))$, which establishes our claim.

Claim 2: We claim that $\mathfrak{M}$ has finite closures. Assume to the contrary that $\mathfrak{M}$ does not have finite closures. Let $\mathfrak{A} \subseteq_{\mathrm{Fin}} \mathfrak{M}$ be such that there is no finite $\mathfrak{C} \leq \mathfrak{M}$ such that $\mathfrak{A} \subseteq \mathfrak{M}$. It now follows that there is a $\subseteq$ increasing sequence $\{\mathfrak{A}_i : i \in \omega, \mathfrak{A}_i \subseteq \mathfrak{M}$ such that $\mathfrak{A}_0 = \mathfrak{A}$ and each $(\mathfrak{A}_i, \mathfrak{A}_{i+1})$ is a minimal pair$\}$. Using the downward Lowenhiem Skolem Theorem, we may construct a countable $\mathfrak{M}' \preccurlyeq \mathfrak{M}$ such that $\bigcup_{i<\omega} A_i \subseteq M'$. Note that M' is a countable, atomic and hence prime model of $S_{\overline{\alpha}}$. We may as well assume that $\mathfrak{M}' \preccurlyeq \mathfrak{M}^*$ for notational convenience where $\mathfrak{M}^*$ is the $(K_{\overline{\alpha}}, \leq)$ generic. Recall that $\mathfrak{M}^*$ has finite closure and let $\mathfrak{A} \subseteq \mathfrak{C} \leq \mathfrak{M}^*$ where $|C|$ is finite. Let i be the least integer such that $\mathfrak{A}_i \nsubseteq \mathfrak{C}$. Clearly $i \geq 1$ and $C \neq A_{i-1}$ (for if $A_{i-1} = C$, then A_i is a minimal pair over C, which contradicts $C \leq M^*$).

Now $C \leq CA_i$ as $C \leq M^*$ and $A_i \subseteq CA_i$. By using Fact 2.3.1 we obtain that $C \cap A_i \leq A_i$. Further $A_{i-1} \subseteq C \cap A_i \subsetneq A_i$ as $A_i \not\subseteq C$. But then $A_{i-1} \leq C \cap A_i$ as (A_i, A_{i+1}) is a minimal pair. By the transitivity of $\leq$ we then obtain $A_{i-1} \leq A_i$, a contradiction that shows $\mathfrak{M}$ has finite closures. $\square$

Lemma 5.1.6. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$. Assume that $d_{\mathfrak{M}}(\mathfrak{A}) = 0$ for all finite $\mathfrak{A} \subseteq \mathfrak{M}$ and that $\mathfrak{M}$ has finite closures. Then $\mathfrak{M}$ is atomic.*

Proof. Let $\mathfrak{A} \subseteq \mathfrak{M}$. We begin by fixing an enumeration $\overline{a}$ of $\mathfrak{A}$. Let $\mathrm{icl}_{\mathfrak{M}}(\mathfrak{A}) = \mathfrak{C}$. As $\mathfrak{M}$ has finite closures, it follows that $\mathfrak{C}$ is finite. It is clear that $d_{\mathfrak{M}}(\mathfrak{A}) = d_{\mathfrak{M}}(\mathfrak{C}) = \delta(\mathfrak{C}) = 0$. Note that if $\mathfrak{A} = \mathfrak{C}$ then we have already established the result by 5.1.4 and that if $\mathfrak{A} \neq \mathfrak{C}$, then there is no $\mathfrak{A} \subseteq \mathfrak{B} \subsetneq \mathfrak{C}$ such that $\delta(\mathfrak{B}) = 0$. We claim that the formula $\Psi_{A,C}(\overline{x}) = \Delta_A(\overline{x}) \wedge \exists \overline{y} \Delta_{A,C}(\overline{x}, \overline{y})$ isolates $\mathrm{tp}(\overline{a})$. Now it suffices to show that $\Psi_{A,C}(\overline{x})$ decides the chain minimal extension formulas.

Let $\mathfrak{M}' \models S_{\overline{\alpha}}$ and assume that $\mathfrak{A}' \subseteq \mathfrak{M}'$. Let $\overline{a'}$ be a fixed enumeration of $\mathfrak{A}'$ and assume that $\mathfrak{M}' \models \Psi_{A,C}(\overline{a'})$. Let $\mathfrak{A}' \subseteq \mathfrak{C}' \subseteq \mathfrak{M}'$ and $\overline{c'}$ be an enumeration of $C' - A'$ such that $\mathfrak{M}' \models \Delta_A(\overline{a'}) \wedge \Delta_{A,C}(\overline{a'}, \overline{c'})$. Note that $C' \leq M'$ as $\delta(\mathfrak{C}') = 0$. Now given a chain of minimal pairs $\mathfrak{A}' = \mathfrak{B}_0 \subseteq \ldots \subseteq \mathfrak{B}_n \subseteq \mathfrak{M}'$, we have that $\mathfrak{B}_n \subseteq \mathfrak{C}'$ as $\mathfrak{C}'$ is closed in $\mathfrak{M}'$. Thus $\Psi_{A,C}(\overline{x})$ decides all chain minimal extension formulas thus isolates the type of $\mathfrak{A}$. $\square$

We now obtain the following theorem:

Theorem 5.1.7. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$. The following are equivalent*

1. *$\mathfrak{M}$ is atomic*

2. $d_{\mathfrak{M}}(\mathfrak{A}) = 0$ *for all finite* $\mathfrak{A} \subseteq \mathfrak{M}$ *and* $\mathfrak{M}$ *has finite closures.*

3. $\mathfrak{M}$ *is existentially closed and has finite closures.*

4. *For any* $\mathfrak{A} \subseteq \mathfrak{M}$ *finite, there is* $\mathfrak{B} \supseteq \mathfrak{A}$ *such that* $\mathfrak{B} \subseteq \mathfrak{M}$, $\mathfrak{B}$ *is finite and* $\delta(\mathfrak{B}) = 0$

Proof. The equivalence of (1) and (2) is immediate from Lemma 5.1.5 and Lemma 5.1.6. The equivalence of (2) and (3) is immediate from Theorem 5.0.4. We now show the equivalence of (2) and (4):

Assume (2). Then take $\mathrm{icl}(\mathfrak{A}) = \mathfrak{B}$. Since $\mathfrak{M}$ has finite closures, it follows that $\mathfrak{B}$ is finite. Since $d_{\mathfrak{M}}(\mathfrak{A}) = 0$ it follows that $d_{\mathfrak{M}}(\mathfrak{A}) = \delta(\mathfrak{B}) = 0$ and thus (4) follows. Now assume (4) holds. Since any $\mathfrak{B}$ with $\delta(\mathfrak{B}) = 0$ is strong in $\mathfrak{M}$. Now pick a $\mathfrak{B}'$ such that $\mathfrak{A} \subseteq \mathfrak{B}' \subseteq \mathfrak{M}$ and $\mathfrak{B}'$ is finite, $\subseteq$ minimal and $\delta(\mathfrak{B}') = 0$. $\square$

5.2 Existence of atomic models

We begin this section by developing tools to prove Theorem 5.2.9 which establishes that coherence is necessary and sufficient for the existence of atomic models. The proof of sufficiency will involve several steps. The idea is to use the $\forall\exists$-axiomatization of $S_{\overline{\alpha}}$ to construct atomic models as the union of a chain under $\subseteq$. However, as dictated by Theorem 5.1.7, atomic models of $S_{\overline{\alpha}}$ must have finite closures. This introduces the need to carefully keep track of how closures change as you go up along the chain.

We then proceed to prove Theorem 5.2.19 which establishes that for coherent $\overline{\alpha}$, the rationality of $\overline{\alpha}$ is equivalent to every model of $S_{\overline{\alpha}}$ being isomorphically embeddable in an atomic model of $S_{\overline{\alpha}}$. A key step in the proof is Lemma 5.2.18, which constructs a model that does not embed into *any* atomic model by exploiting the fact that there is no decreasing sequence of real numbers of order type ω_1.

Definition 5.2.1. We use $S_{\overline{\alpha}}^{\forall}$ to denote the set of universal sentences of $S_{\overline{\alpha}}$. Note that an L-structure $\mathfrak{M}$ models $S_{\overline{\alpha}}^{\forall}$ if and only if $\mathfrak{M} \in \overline{K_{\overline{\alpha}}}$, i.e. for any finite $\mathfrak{A} \subseteq \mathfrak{M}$, $\mathfrak{A} \in K_{\overline{\alpha}}$.

Definition 5.2.2. Let $\mathfrak{M}, \mathfrak{N} \models S_{\overline{\alpha}}^{\forall}$ with $\mathfrak{M} \subseteq \mathfrak{N}$. We say that $\mathfrak{N}$ *preserves closures for* $\mathfrak{M}$ if $X \subseteq M$ is closed in M, then X is closed in N.

Lemma 5.2.3. *Let $\mathfrak{M} \models S_{\overline{\alpha}}^{\forall}$ and $\mathfrak{A}, \mathfrak{B} \in K_{\overline{\alpha}}$. Assume that $\mathfrak{B} \cap \mathfrak{M} = \mathfrak{A}$ and let $\mathfrak{N} = \mathfrak{M} \oplus_{\mathfrak{A}} \mathfrak{B}$.*

1. *If $\mathfrak{A} \leq \mathfrak{B}$ or $\mathfrak{A} \leq \mathfrak{M}$, then $\mathfrak{N} \models S_{\overline{\alpha}}^{\forall}$.*

2. *If $\mathfrak{A} \leq \mathfrak{B}$, then $\mathfrak{N}$ preserves closures for $\mathfrak{M}$*

3. *If $\mathfrak{A} \leq \mathfrak{M}$, then $\mathfrak{B} \leq \mathfrak{N}$*

4. *If $\mathfrak{A} \leq \mathfrak{B}$ or $\mathfrak{A} \leq \mathfrak{M}$ and $\mathfrak{M}$ has finite closures, then so does $\mathfrak{N}$.*

Proof. (1): Assume that $\mathfrak{A} \leq \mathfrak{B}$ or $\mathfrak{A} \leq \mathfrak{M}$. We show that $\mathfrak{N} \models S_{\overline{\alpha}}^{\forall}$. Note that if not, there is some $\mathfrak{A} \subseteq \mathfrak{C} \subseteq_{\mathrm{Fin}} \mathfrak{M}$ such that for some $\mathfrak{B}' \subseteq \mathfrak{B}$, $\mathfrak{A}' \subseteq \mathfrak{A}$ and $\mathfrak{C}' \subseteq \mathfrak{C}$, $\mathfrak{B}' \oplus_{\mathfrak{A}'} \mathfrak{C}' \notin K_{\overline{\alpha}}$. But if this were the case then $\mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C} \notin K_{\overline{\alpha}}$. However we have that $\mathfrak{A} \leq \mathfrak{C}$ or $\mathfrak{A} \leq \mathfrak{B}$ by our assumption and hence $\mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C} \in K_{\overline{\alpha}}$ by Fact 2.3.4. A contradiction that establishes the our claim.

(2): Assume that $\mathfrak{A} \leq \mathfrak{B}$. Let $X \subseteq M$ be closed in $\mathfrak{M}$. By way of contradiction assume that X is not closed in $\mathfrak{N}$. Thus there is some $D \subseteq_{\mathrm{Fin}} X$, $E \subseteq_{\mathrm{Fin}} N$, (D, E) is a minimal pair but $E \not\subseteq X$. Let $A' = E \cap A$, $B' = E \cap (B - A)$ and $D' = E \cap (D - A)$. Now note that $0 > \delta(E/D) = \delta(B'/D'A') = \delta(B') - e(B', D'A') \geq \delta(B') - e(B', D'A) = \delta(B') - e(B', A) \geq 0$ using (1) of Fact 2.2.5. Thus it follows that $\mathfrak{N}$ preserves closures for $\mathfrak{M}$.

For the proof of $(3), (4)$, first note that if $B \subseteq F \subseteq_{\mathrm{Fin}} N$, then we may write $F = B \oplus_A F'$ with $F' \subseteq M$. Further if $F \subseteq G \subseteq N$ with $G = B \oplus_A G'$, then $\delta(G/F) = \delta(G'/F')$. Also to show that $F \subseteq_{\mathrm{Fin}} N$ is strong in N, it suffices to show that $\delta(G/F) \geq 0$ for all finite $F \subseteq G \subseteq_{\mathrm{Fin}} N$.

(3): Assume that $\mathfrak{A} \leq \mathfrak{M}$. Given $B \subseteq G \subseteq_{\mathrm{Fin}} N$. Take $F = B \oplus_A A = B$ and $G = B \oplus_A G'$ where $G' = G \cap M$. Now it follows that $\delta(G/F) = \delta(G'/A)$. Since $\mathfrak{A} \leq \mathfrak{M}$, it follows that $\delta(G'/A) \geq 0$. Thus $\mathfrak{B} \leq \mathfrak{N}$.

(4): Assume that $\mathfrak{M}$ has finite closures. We wish to show that $\mathfrak{N}$ has finite closures. Let $X \subseteq_{\mathrm{Fin}} N$. Since intrinsic closures are monotonic with respect to $\subseteq$, we may as well assume that $B \subseteq X$. Let $F = \mathrm{icl}_M(X \cap M)$. Note that F' is finite because $\mathfrak{M}$ has finite closures. Take $F = B \oplus_A F'$ and note that $X \subseteq F$. Fix $F \subseteq G \subseteq_{\mathrm{Fin}} N$ with $G = B \oplus_A G'$ where $G' = G \cap N$. Now $\delta(G/F) = \delta(G'/F')$ from which the result follows as $\delta(G'/F') \geq 0$ as $F' \leq M$. $\square$

Lemma 5.2.4. *Let $\langle \mathfrak{M}_\beta \rangle_{\beta<\kappa}$ be a $\subseteq$-chain of models of $S^{\forall}_{\overline{\alpha}}$ with $\mathfrak{M}_\gamma = \bigcup_{\beta<\gamma} \mathfrak{M}_\beta$ for limit γ. Assume that $\mathfrak{M}_{\beta+1}$ preserves closures for $\mathfrak{M}_\beta$ for each $\beta < \kappa$. Then $\mathfrak{M} = \bigcup_{\beta<\kappa} \mathfrak{M}_\beta$ preserves closures for each $\mathfrak{M}_\beta$, $\beta < \kappa$. Further if $\mathfrak{M}_\beta$ has finite closures for each $\beta < \kappa$, then so does $\mathfrak{M}$*

Proof. Let $\mathfrak{M}$ be as above and let $\mathfrak{X} \subseteq \mathfrak{M}_\beta$ be closed. We claim that if $\mathfrak{X}$ is closed in $\mathfrak{M}$, then it is closed in $\mathfrak{N}$. By way of contradiction, suppose not. Then there is some minimal pair $(\mathfrak{A}, \mathfrak{B})$ with $\mathfrak{B} \subseteq \mathfrak{M}, \mathfrak{A} \subseteq \mathfrak{X}$ and $\mathfrak{B} \subsetneq \mathfrak{X}$ that witnesses this. Let $\gamma > \beta$ be the least ordinal such that $\mathfrak{B} \subseteq \mathfrak{M}_\gamma$. As closures are preserved for successor ordinals, it follows that γ is not a successor ordinal. Thus γ must be a limit ordinal. But $\mathfrak{M}_\gamma = \bigcup_{\beta<\gamma} \mathfrak{M}_\beta$ which implies $\mathfrak{B} \subseteq \mathfrak{M}_{\gamma'}$ for some $\gamma' < \gamma$. But then $\mathfrak{X}$ is not closed in $\mathfrak{M}_{\gamma'}$, which contradicts the minimality of γ. Thus the first claim is true. The second claim follows by a similar argument. $\square$

We now illustrate how to extend a model of the universal sentences of $S_{\overline{\alpha}}$ to a model of $S_{\overline{\alpha}}$, while preserving closures, a key step towards building atomic models.

Lemma 5.2.5. *Let $\mathfrak{M} \models S^{\forall}_{\overline{\alpha}}$ be infinite. There exists $\mathfrak{N} \models S_{\overline{\alpha}}$ such that $\mathfrak{M} \subseteq \mathfrak{N}$, $|M| = |N|$, $\mathfrak{N}$ preserves closures for $\mathfrak{M}$. Further if $\mathfrak{M}$ has finite closures, then $\mathfrak{N}$ has finite closures too.*

Proof. Let $\mathfrak{M} \models S^{\forall}_{\overline{\alpha}}$. Fix a finite $\mathfrak{A} \subseteq \mathfrak{M}$. A routine chain argument using Lemma 5.2.3 allows us to create $\mathfrak{M}'$ with the following properties:

1. $\mathfrak{M}'$ preserves closures for $\mathfrak{M}$ and $|M'| = |M|$

2. If $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \leq \mathfrak{B}$, there is some g that embeds $\mathfrak{B}$ into $\mathfrak{N}$ over $\mathfrak{A}$.

3. If $\mathfrak{B}_1, \mathfrak{B}_2 \in K_{\overline{\alpha}}$ with $\mathfrak{A} \leq \mathfrak{B}_1, \mathfrak{B}_2$ and $\mathfrak{B}_1, \mathfrak{B}_2$ are not isomorphic over $\mathfrak{A}$, then there are embeddings g_1, g_2 of $\mathfrak{B}_1, \mathfrak{B}_2$ over $\mathfrak{A}$ such that $g_1(\mathfrak{B}_1), g_2(\mathfrak{B}_2)$ are freely joined over $\mathfrak{A}$.

Note that $\mathfrak{A}$, when considered as a substructure of $\mathfrak{M}'$, satisfies the extension formulas required by $S_{\overline{\alpha}}$. Further, by an application of Lemma 5.2.4, it follows that if $\mathfrak{M}$ has finite closures, then so does $\mathfrak{M}'$. Iterating this process and using a routine chain argument, we can construct $\mathfrak{N}$ as required. The fact that $\mathfrak{N}$ has finite closures if $\mathfrak{M}$ does follows from an application of Lemma 5.2.4. $\square$

We now introduce the class K_0. It contains *all* the finite structures of $K_{\overline{\alpha}}$ that *may sit* strongly inside an *atomic model* of $S_{\overline{\alpha}}$.

Definition 5.2.6. We let $K_0 = \{\mathfrak{A} : \mathfrak{A} \in K_{\overline{\alpha}} \text{ and } \delta(\mathfrak{A}) = 0\}$. Further we let $\overline{K_0} = \{\mathfrak{X} : \mathfrak{X} \models S^{\forall}_{\overline{\alpha}}$ and for any $\mathfrak{A} \subseteq_{\text{Fin}} \mathfrak{Y}$ there exists $\mathfrak{B} \subseteq_{\text{Fin}} \mathfrak{X}$ with $\mathfrak{A} \subseteq \mathfrak{B}$ and $\delta(\mathfrak{B}) = 0\}$.

Remark 5.2.7. Let $\mathfrak{D} \in K_{\overline{\alpha}}$, and $\mathfrak{X} \models S^{\forall}_{\overline{\alpha}}$ with $\mathfrak{D} \subseteq \mathfrak{X}$. Note that if $\delta(\mathfrak{D}) = 0$, then $\mathfrak{D} \leq \mathfrak{X}$. Thus it follows that if $\mathfrak{X} \in \overline{K_0}$, then $\mathfrak{X}$ has finite closures.

We are now in a position to show that coherence of $\overline{\alpha}$ is a sufficient condition for the existence of atomic models.

Lemma 5.2.8. *Let $\overline{\alpha}$ be coherent. Suppose $\mathfrak{M} \in \overline{K_0}$ with $|M| = \kappa$. Then we can construct $\mathfrak{N} \models S_{\overline{\alpha}}$ such that $\mathfrak{N} \supseteq \mathfrak{M}$, $\mathfrak{N}$ is atomic and $|M| = |N|$. Thus for any κ there is an atomic model of $S_{\overline{\alpha}}$ of size κ.*

Proof. Assume that $|M| = \kappa$. Enumerate the finite substructures of $\mathfrak{M}_0 = \mathfrak{M}$ by $\{\mathfrak{B}_0, \ldots\}$. Let $\{\mathfrak{B}_0^n : n < \omega\}$ enumerate, up to isomorphism $\mathfrak{F} \in K_{\overline{\alpha}}$ such that $\mathfrak{B}_0 \leq \mathfrak{F}$. Now consider $\mathfrak{C}_0 = \mathrm{icl}_{\mathfrak{M}_0}(\mathfrak{B}_0)$ which is finite and has rank 0 as $\mathfrak{M} \in \overline{K_0}$. Let $\mathfrak{C}_1' = \mathfrak{C}_0 \oplus_{\mathfrak{B}_0} \mathfrak{B}_0^0$. Since $\mathfrak{B}_0 \leq \mathfrak{B}_0^0$ we have that $\mathfrak{C}_1' \in K_{\overline{\alpha}}$. As $\overline{\alpha}$ is coherent, we can fix $\mathfrak{D}_0 \in K_{\overline{\alpha}}$ such that $\mathfrak{C}_1' \subseteq \mathfrak{D}_0$ and $\delta(\mathfrak{D}_0) = 0$. Now consider $\mathfrak{M}_1 = \mathfrak{M}_0 \oplus_{\mathfrak{C}_0} \mathfrak{D}_0$. Note that as $\delta(\mathfrak{C}_0) = 0$, $\mathfrak{C}_0 \leq \mathfrak{D}_0$. By (1) of Lemma 5.2.3, $\mathfrak{M}_1 \models S_{\overline{\alpha}}^{\forall}$ and by (2) of $\mathfrak{M}_1$ preserves closures for $\mathfrak{M}$.

We claim that $\mathfrak{M}_1 \in \overline{K_0}$. From (4) of Lemma 5.2.3 we obtain that $\mathfrak{M}_1$ has finite closures. Let $H = G_1F_1$ be a finite substructure of $\mathfrak{M}_1$ with $G_1 \subseteq \mathfrak{M}_0$ and $F_1 \subseteq \mathfrak{D}_1$. Now let $G' = \mathrm{icl}_{\mathfrak{M}_0}(G_1)$. Since $\mathfrak{M} \in \overline{K_0}$, G' is finite and $\delta(G') = \delta(\mathrm{icl}_{\mathfrak{M}}(G_1)) = 0$. Thus it follows that $\mathrm{icl}_{\mathfrak{M}_1}(G_1) = G'$ as well. Now $\delta(G'D_1) \leq \delta(G') + \delta(D_1) - e(G' - D_1, D_1 - G') = -e(G' - D_1, D_1 - G') \leq 0$ by using (1) of Fact 2.2.5. But as we have already established that $\mathfrak{M}_1 \models S_{\overline{\alpha}}^{\forall}$, it follows that $\delta(GD_1) = 0$. Thus any finite substructure of $\mathfrak{M}_1$ is contained in a finite substructure with rank 0. Hence $\mathfrak{M}_1 \in \overline{K_0}$.

Now as noted above $\mathrm{icl}_{\mathfrak{M}_1}(\mathfrak{B}_0) = \mathfrak{C}_0$. Thus we may recursively form a chain $\langle \mathfrak{M}_i \rangle_{i<\omega}$ such that $\mathfrak{M}_{n+1} = \mathfrak{M}_n \oplus_{\mathfrak{C}_n} \mathfrak{D}_n$ so that $\delta(\mathfrak{D}_n) = 0$, $\mathfrak{B}_0^n \subseteq \mathfrak{D}_n$, $\mathfrak{M}_{n+1} \in \overline{K_0}$ and $\mathrm{icl}_{\mathfrak{M}_{n+1}}(\mathfrak{B}_0) = \mathfrak{C}_{n+1} = \mathfrak{C}_0$. Now consider $\mathfrak{M}^1 = \bigcup_{i<\omega} \mathfrak{M}_n$. Now since $\mathfrak{M}_n \in \overline{K_0}$ for each n, it follows immediately that $\mathfrak{M}^1 \in \overline{K_0}$. Note that $\mathfrak{M}^1$ satisfies all the extension formulas demanded by $S_{\overline{\alpha}}$ for $\mathfrak{B}_0$. It is clear that, by using the ideas behind the above construction of $\mathfrak{M}^1$ and taking unions at limit ordinals, we can build a chain $\mathfrak{M}^\beta \in \overline{K_0}$, $\beta < \kappa$ such that each $\mathfrak{M}^\beta \in \overline{K_0}$ and for all $\gamma < \beta$, $\mathfrak{M}^\beta$ contains all finite extensions of $\mathfrak{B}_\gamma$ needed to satisfy the extensions dictated by $S_{\overline{\alpha}}$.

Now clearly $\mathfrak{M}^\kappa \in \overline{K_0}$ and all finite substructures of $\mathfrak{M}$ have the extensions needed to satisfy the extensions dictated by $S_{\overline{\alpha}}$ *in* $\mathfrak{M}^\kappa = \mathfrak{N}_0$. Now repeating this procedure we may form a $\subseteq$-chain $\langle \mathfrak{N}_\beta \rangle$ (taking unions at limit stages) where $\mathfrak{N} = \bigcup_{\beta<\kappa} \mathfrak{N}_\beta$ satisfies $\mathfrak{N} \in \overline{K_0}$ and $\mathfrak{N} \models S_{\overline{\alpha}}$.

Since there are $\mathfrak{M} \in \overline{K_0}$ with $|M| = \kappa_0$ for all infinite cardinals κ (for example, the free join over $\emptyset$ of all the elements of K_0 up to isomorphism, each repeated κ many times in the free join) there are atomic models of size κ. □

We now obtain the following:

Theorem 5.2.9. *There exists atomic models of the theory $S_{\overline{\alpha}}$ if and only if $\overline{\alpha}$ is coherent.*

Proof. We begin by showing that if $S_{\overline{\alpha}}$ has atomic models, then $\overline{\alpha}$ is coherent. To see this for each $E \in L$, fix a finite L structure $\mathfrak{A}_E$ such that at E holds on at least one subset of $\mathfrak{A}_E$ and no other relation holds on $\mathfrak{A}_E$. Let $\mathfrak{A} = \oplus_{E\in L}\mathfrak{A}_E$ be the free join of the $\mathfrak{A}_E$ over $\emptyset$. Let $\mathfrak{M} \models S_{\overline{\alpha}}$ be atomic with $A \subseteq M$. Thus there is some $\mathfrak{B} \supseteq \mathfrak{A}$ with $B \subseteq_{\text{Fin}} M$ and $\delta(\mathfrak{B}) = 0$. It follows that $\delta(\mathfrak{B}) = 0 = n - \sum_{E\in L} m_E \overline{\alpha}_E$. Thus $\overline{\alpha}$ is coherent.

The converse is immediate by Lemma 5.2.8. □

Remark 5.2.10. The Shelah-Spencer almost sure theories do not have atomic models.

In the case that $\overline{\alpha}$ is rational, an even stronger result than Theorem 5.2.9 is possible. In this case the models of $S_{\overline{\alpha}}$ displays similar behavior to that of classical

Fraïssé limits (i.e. theories of generics built from Fraïssé classes where $\leq$ corresponds to $\subseteq$).

Lemma 5.2.11. *Assume that $\overline{\alpha}$ is rational. Let $\mathfrak{M} \models S_{\overline{\alpha}}$. Now $\mathfrak{M}$ is atomic if and only if $\mathfrak{M}$ is an existentially closed model. Hence every model of $S_{\overline{\alpha}}$ embeds isomorphically into an atomic model of $S_{\overline{\alpha}}$.*

Proof. Assume that $\overline{\alpha}$ is rational and as a result $S_{\overline{\alpha}}$ has finite closures. Let $\mathfrak{M} \models S_{\overline{\alpha}}$. By Theorem 5.1.7 we immediately obtain that $\mathfrak{M}$ is atomic if and only if $\mathfrak{M}$ is an existentially closed model. By Remark 5.0.2, there is some $\mathfrak{N} \models S_{\overline{\alpha}}$ such that $\mathfrak{N}$ is atomic and $\mathfrak{M} \subseteq \mathfrak{N}$. $\square$

Remark 5.2.12. Assume that $\overline{\alpha}$ is rational. It is easily seen that any $\mathfrak{X} \models S_{\overline{\alpha}}^{\forall}$ has finite closures. Thus it follows from Lemma 5.2.5 that any $\mathfrak{X} \models S_{\overline{\alpha}}^{\forall}$ embeds isomorphically into some $\mathfrak{N} \models S_{\overline{\alpha}}$ (taking the free join of $\aleph_0$ many non-isomorphic copies of $\mathfrak{X}$ over $\emptyset$ if $\mathfrak{X}$ is finite). Thus from Lemma 5.2.11, it follows that $\mathfrak{X}$ embeds into an atomic $\mathfrak{N}' \models S_{\overline{\alpha}}$.

We will now explore the behavior of atomic models when $\overline{\alpha}$ is coherent but $\overline{\alpha}$ is not rational. We begin by showing that *any* countable $\mathfrak{X} \models S_{\overline{\alpha}}^{\forall}$ with finite closures embeds isomorphically into the countable atomic model of $S_{\overline{\alpha}}$ mimicking the behavior of Remark 5.2.12. Recall that if $\mathfrak{X} \in \overline{K_0}$, then $\mathfrak{X}$ has finite closures.

Lemma 5.2.13. *Let $\overline{\alpha}$ be coherent and let $\mathfrak{M} \models S_{\overline{\alpha}}^{\forall}$ be countable with finite closures. Then*

1. *There exists a countable $\mathfrak{M}^* \in \overline{K_0}$ with $\mathfrak{M}^* \supseteq \mathfrak{M}$.*

2. There exists a countable atomic $\mathfrak{N} \models S_{\overline{\alpha}}$ such that $\mathfrak{M} \subseteq \mathfrak{N}$.

Proof. (1): Since $\mathfrak{M}$ has finite closures, we may write $\mathfrak{M} = \bigcup_{i<\omega} \mathfrak{A}_i$ where $\mathfrak{A}_i \leq \mathfrak{A}_{i+1}$ for each $i < \omega$. We will now construct $\mathfrak{M}^*$ as the union of a countable $\subseteq$-chain $\mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq ...$ with $\mathfrak{M} = \mathfrak{M}_0$ and $|M_n - M_0|$ finite for all $n < \omega$ as follows: Let $\mathfrak{M}_0 = \mathfrak{M}$ and given $\mathfrak{M}_n$, let $\mathfrak{A}_n^* = \mathrm{icl}_{\mathfrak{M}_n}(A_n \cup (M_n - M_0))$. Using Theorem 3.3.6 choose $\mathfrak{B}_n \in K_{\overline{\alpha}}$ with $\mathfrak{A}_n^* \subseteq \mathfrak{B}_n^*$ and $\delta(\mathfrak{B}_n) = 0$. Let $\mathfrak{M}_{n+1} = \mathfrak{M}_n \oplus_{\mathfrak{A}_n^*} \mathfrak{B}_n$. As $\mathfrak{A}_n^* \leq \mathfrak{M}^*$, it follows from Lemma 5.2.3 that each $\mathfrak{M}_n \models S_{\overline{\alpha}}^{\forall}$. Clearly $|M_n - M_0|$ is finite as claimed. As each $\mathfrak{M}_n \models S_{\overline{\alpha}}^{\forall}$, $\mathfrak{M}^* \models S_{\overline{\alpha}}^{\forall}$ where $\mathfrak{M}^* = \bigcup_{i<\omega} \mathfrak{M}_n$. Note that given *any* finite set of $\mathfrak{A} \subseteq \mathfrak{M}^*$, there is some $n < \omega$ such that $\mathfrak{A} \subseteq \mathfrak{M}_n$. By construction, it follows that there is some $k < \omega$ such that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}_{n+k}$ with $\mathfrak{B}$ finite and $\delta(\mathfrak{B}) = 0$. Thus it follows that $\mathfrak{M}^* \in \overline{K_0}$.

(2): We now do an alternating chain argument: We let $\mathfrak{M}_0^* = \mathfrak{M}$. Thus $\mathfrak{M}_0^*$ has finite closures. We build $\mathfrak{M}_{2n+1}^* \models S_{\overline{\alpha}}$ with $\mathfrak{M}_{2n}^* \subseteq \mathfrak{M}_{2n+1}^*$ such that $\mathfrak{M}_{2n+1}^*$ has finite closures, preserves closures for $\mathfrak{M}_{2n}^*$ and is countable by use of Lemma 5.2.5. We let $\mathfrak{M}_{2n+2}^*$ be such that $\mathfrak{M}_{2n+1}^* \subseteq \mathfrak{M}_{2n+2}^*$ and $\mathfrak{M}_{2n+2}^* \in \overline{K_0}$ which exists by use of (1). We let $\mathfrak{N} = \bigcup_{n<\omega} \mathfrak{M}_n^*$. Let $\mathfrak{B} \subseteq_{\mathrm{Fin}} \mathfrak{N}$. Now as $\mathfrak{B} \subseteq \mathfrak{M}_{2n_0+1}^*$ for some n_0, a routine argument shows that $\mathfrak{N} \models S_{\overline{\alpha}}$. As $\mathfrak{B} \subseteq \mathfrak{M}_{2n_0+1}^* \subseteq \mathfrak{M}_{2n_0+2}^*$ it follows that $\mathfrak{D} = icl_{\mathfrak{M}_{2n_0+2}}(\mathfrak{B})$ is finite and $\delta(\mathfrak{D}) = 0$. Thus it follows that $icl_{\mathfrak{M}_{2n_0+2}}(\mathfrak{B}) = icl_{\mathfrak{N}}(\mathfrak{B})$ and hence $\mathfrak{N} \in \overline{K_0}$. Thus $\mathfrak{N}$ is (up to isomorphism), the unique countable atomic model of $S_{\overline{\alpha}}$ by Theorem 5.1.7. $\square$

We now proceed to show that this behavior may fail for uncountable $\mathfrak{X} \models S_{\overline{\alpha}}^{\forall}$.

Definition 5.2.14. Call a structure $\mathfrak{N} \models S_{\alpha}^{\forall}$ *tent-like* over $\mathfrak{M}$ if

1. M is a set of points with no relations between them

2. For all pairs $\{a, b\}$ of distinct elements from M, there is a unique minimal pair $(\{a, b\}, \mathfrak{F}_{a,b})$ in $\mathfrak{N}$.

3. $N = \bigcup_{a,b \in A, a \neq b} F_{(a,b)}$

   (a) For distinct $a, b, b' \in M$, $\mathfrak{F}_{a,b}, \mathfrak{F}_{a,b'}$ are freely joined over a

   (b) For distinct $a, a', b, b' \in M$, $\mathfrak{F}_{a,b}, \mathfrak{F}_{a,b'}$ are freely joined over $\emptyset$

4. $\mathrm{icl}_{\mathfrak{N}}(\{a\}) = \{a\}$ for each $a \in M$

We will refer to M as the *base* of the tent $\mathfrak{N}$ over $\mathfrak{M}$.

Remark 5.2.15. Note that given a finite subset $A_0 = \{a_{n_1}, \ldots, a_{n_k}\}$ of M we have that $A' = \bigcup \mathfrak{F}_{a,b} \subseteq icl_{\mathfrak{N}}(A_0)$ where (a, b) ranges through distinct pairs from A_0. We claim that this set is closed. Assume to the contrary that there is a minimal pair (D, DG) where $D \subseteq A'$ and G is disjoint from A'. Note $\delta(G/D) \geq \delta(G/A')$ using (2) of Fact 2.2.5. Since $\mathfrak{N}$ is tent-like over $\mathfrak{M}$, $\delta(G/A') = \delta(G/A_0)$. From the tent-likeness of $\mathfrak{N}$ over $\mathfrak{M}$ and our choice of A' and G, it follows that $\delta(G/A_0) = \sum_{(a,b) \notin A_0 \times A_0, a \neq b} \delta(G \cap F_{a,b}/A_0)$. Thus $\delta(G \cap F_{a,b}/A_0)$ for $(a, b) \notin A_0 \times A_0, a \neq b$ reduces to either $\delta(G \cap F_{a,b})$ or $\delta(G \cap F_{a,b}/c)$ where $c = a$ or $c = b$. But since each $a' \in A$ is its own closure in $\mathfrak{N}$ it follows that the $\delta(G \cap F_{a,b}/c) \geq 0$. Thus it follows that A' is closed. Now by noting that each finite subset lies in finitely many of the $F_{a,b}$ it follows that $\mathfrak{N}$ has finite closures.

Remark 5.2.16. Note that if $\mathfrak{N} \models S^{\forall}_{\overline{\alpha}}$ is tent like over $\mathfrak{M}$, then $\mathfrak{N} \notin \overline{K_0}$ as $\delta(\mathrm{icl}(a)) = 1$ for each $a \in M$.

Lemma 5.2.17. *Let $\overline{\alpha}$ be coherent but not rational. Suppose $\mathfrak{N} \models S^{\forall}_{\overline{\alpha}}$ tent-like over $\mathfrak{M}$ where M is countable. Then there is an extension $\mathfrak{N}^*$ of $\mathfrak{N}$ over $\mathfrak{M}^*$ where $\mathfrak{M} \subseteq \mathfrak{M}^*$ and M^* has universe $M\{a^*\}$, where a^* is a single new point such that $\mathfrak{N}^*$ is tent-like over $\mathfrak{M}^*$. Thus there is some $\mathfrak{N}'$ where the corresponding base $\mathfrak{M}'$ has $|M'| = \aleph_1$.*

Proof. Enumerate $M = \{a_n : n \in \omega\}$. Fix $E \in L$ such that $\overline{\alpha}_E$ is irrational. Now for each $n \in \omega$ we may choose an essential minimal pair $\mathfrak{F}_{(a_n,a^*)}$ over $\{a_n, a^*\}$ such that $-1/2^{n+1} < \delta(\mathfrak{F}_{(a_n,a^*)}/\{a_n, a^*\}) < 0$ using Theorem 3.2.15. Let $D' \subseteq F_{a_n,a^*}$. Now if $D' \cap \{a_n, a^*\}$ contains exactly one element, then $\delta(D'/D \cap \{a_n, a^*\}) \geq 0$. So suppose that $D' \cap \{a_n, a^*\} = \{a_n, a^*\}$. Since $\delta(\{a_n, a^*\}/\{a^*\}) = \delta(\{a_n, a^*\}/\{a_n\}) = 1$ and $\delta(D'/\{c\}) = \delta(D'/\{a_n, a^*\}) + \delta(\{a_n, a^*\}/c) \geq -1/2^{n+1} + 1 \geq 0$ where $c = a_n$ or $c = a^*$ it follows that $\{a_n\}, \{a^*\} \leq \mathfrak{F}_{a_n,a^*}$. Now consider the structure $\mathfrak{N}^*$ with universe $N \cup \{a^*\} \cup \bigcup_{a_n \in A} F_{a^*,a_n}$ with

1. For distinct $a, b, b' \in Ma^*$, $\mathfrak{F}_{a,b}, \mathfrak{F}_{a,b'}$ are freely joined over a

2. For distinct $a, a', b, b' \in Ma^*$, $\mathfrak{F}_{a,b}, \mathfrak{F}_{a,b'}$ are freely joined over $\emptyset$

Clearly $M\{a^*\}$ is a set of points with no relations between them. Note that we have shown that $\{a^*\}, \{a_n\} \leq \mathfrak{F}_{a_n,a^*}$. Let $\mathfrak{G} \subseteq_{\mathrm{Fin}} \mathfrak{N}^*$. Suppose that the $G \cap M\{a^*\} = \emptyset$. Then because of the conditions regarding free joins we see that $\delta(\mathfrak{G}) = \sum \delta(\mathfrak{F}_{a,b} \cap \mathfrak{G}) \geq 0$. Now consider the case $G \cap M\{a^*\} \neq \emptyset$. Put $G' = G \cap A\{a^*\}$.

Now $\delta(\mathfrak{G}/\mathfrak{G}') = \delta((\mathfrak{N} \cap \mathfrak{G})/\mathfrak{G}') + \delta((\mathfrak{N}^* - \mathfrak{N}) \cap \mathfrak{G}/\mathfrak{G}') = \delta((\mathfrak{N} \cap \mathfrak{G})/\mathfrak{N} \cap \mathfrak{G}') + \delta((\mathfrak{N}^* - \mathfrak{N}) \cap \mathfrak{G}/\mathfrak{G}')$ where $\delta((\mathfrak{N} \cap \mathfrak{G})/\mathfrak{G}') = \delta((\mathfrak{N} \cap \mathfrak{G})/\mathfrak{N} \cap \mathfrak{G}')$ follows by considering the fact that the underlying finite structures are freely joined. Now $\delta(\mathfrak{G}) = \delta(\mathfrak{G}') + \delta((\mathfrak{N} \cap \mathfrak{G})/\mathfrak{N} \cap \mathfrak{G}') + \delta((\mathfrak{N}^* - \mathfrak{N}) \cap \mathfrak{G}/\mathfrak{G}')$. Suppose that $a^* \notin G'$. Then $\delta(\mathfrak{G}') + \delta((\mathfrak{N} \cap \mathfrak{G})/\mathfrak{N} \cap \mathfrak{G}') = \delta((\mathfrak{N} \cap \mathfrak{G}))$ and $\delta((\mathfrak{N}^* - \mathfrak{N}) \cap \mathfrak{G}/\mathfrak{G}') \geq 0$ by using an argument similar to that in Remark 5.2.15. So assume that $a^* \in G'$. Now $\delta(\mathfrak{G}) = \delta(\mathfrak{G}' \cap \mathfrak{N}) + \delta((\mathfrak{N} \cap \mathfrak{G})/\mathfrak{N} \cap \mathfrak{G}') + \delta(a^*) + \delta((\mathfrak{N}^* - \mathfrak{N}) \cap \mathfrak{G}/\mathfrak{G}')$. It follows that $\delta(\mathfrak{G}' \cap \mathfrak{N}) + \delta((\mathfrak{N} \cap \mathfrak{G})/\mathfrak{N} \cap \mathfrak{G}') \geq 0$ by an argument similar to the above. But by construction of the new minimal pairs $\delta(a^*) + \delta((\mathfrak{N}^* - \mathfrak{N}) \cap \mathfrak{G}/\mathfrak{G}') \geq 1 - \sum 1/2^{n+1} \geq 0$. Thus $\mathfrak{N}^* \models S_{\overline{\alpha}}^{\forall}$.

Now each pair of points $\{a, b\}$ from $M\{a^*\}$ has a minimal pair over it; i.e. $(ab, \mathfrak{F}_{a,b})$ is a minimal pair. Now consider $\mathfrak{F}_{a,b}$. Note that since $a \leq \mathfrak{F}_{a,b'}$ and $b \leq \mathfrak{F}_{b,b'}$ and using the various properties regarding how the $\mathfrak{F}_{c,d}$ are freely joined and arguing in a similar manner to Remark 5.2.15 yields that $\mathfrak{F}_{a,b}$ is closed in $\mathfrak{N}^*$ which establishes that there is a unique minimal pair over ab. Now it also follows that for any $a \in M\{a^*\}$ the closure of a is itself. Thus $\mathfrak{N}^*$ is also tent-like.

By iterating this ω_1 many times we obtain a tent-like structure where the corresponding $\mathfrak{N}'$ over $\mathfrak{M}'$ where $|M'| = \aleph_1$. □

Lemma 5.2.18. *Let $\overline{\alpha}$ be coherent but not rational. Then there is $\mathfrak{X} \models S_{\overline{\alpha}}^{\forall}$ of size $\aleph_1$ such that $\mathfrak{X}$ has finite closures but there is no atomic model $\mathfrak{N}$ of $S_{\overline{\alpha}}$ such that $\mathfrak{N} \supseteq \mathfrak{X}$. Thus there is $\mathfrak{M} \models S_{\overline{\alpha}}$ such that $\mathfrak{M}$ does not embed isomorphically into any atomic model of $S_{\overline{\alpha}}$.*

Proof. Let $\mathfrak{X} \models S^{\forall}_{\overline{\alpha}}$ be tent-like over $\mathfrak{Y}$ where $Y = \{a_i : i < \omega_1\}$. We claim that there is no $\mathfrak{N} \supseteq \mathfrak{X}$ such that $\mathfrak{N}$ is an atomic model of $S_{\overline{\alpha}}$.

Assume to the contrary that there is such a $\mathfrak{N}$. Now for any a_β, $\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\})$ would be finite and $\delta(\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\})) = 0$ by use of Theorem 5.1.7. Note that for β, γ distinct, $\mathfrak{F}_{\beta,\gamma} \subseteq \mathrm{icl}_{\mathfrak{N}}(\{a_\beta, a_\gamma\})$. Now either $(\mathfrak{F}_{\beta,\gamma} - \{a_\beta, a_\gamma\}) \cap \mathrm{icl}_{\mathfrak{N}}(\{a_\beta\}) \neq \emptyset$ or $(\mathfrak{F}_{\beta,\gamma} - \{a_\beta a_\gamma\}) \cap \mathrm{icl}_{\mathfrak{N}}(\{a_\beta\}) \neq \emptyset$. For if not

$$\begin{aligned}\delta(\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\})\mathrm{icl}_{\mathfrak{N}}(\{a_\gamma\})) &= \delta(\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\})) + \delta(\mathrm{icl}_{\mathfrak{N}}(\{a_\gamma\})) - \delta(\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\}) \cap \mathrm{icl}_{\mathfrak{N}}(\{a_\gamma\})) \\ &\quad - e(\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\}) - \mathrm{icl}_{\mathfrak{N}}(\{a_\gamma\}), \mathrm{icl}_{\mathfrak{N}}(\{a_\gamma\}) - \mathrm{icl}_{\mathfrak{N}}(\{a_\beta\}))\end{aligned}$$

by use of (1) of Fact 2.2.5). This implies that $\delta(\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\})\mathrm{icl}_{\mathfrak{N}}(\{a_\gamma\})) = 0$. But then $\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\})\mathrm{icl}_{\mathfrak{N}}(\{a_\gamma\})$ is closed. Thus we obtain that, $(\mathfrak{F}_{\{a_\beta,a_\gamma\}} - \{a_\beta a_\gamma\}) \subsetneq \mathrm{icl}_{\mathfrak{N}}(\{a_\beta a_\gamma\}) \subseteq \mathrm{icl}_{\mathfrak{N}}(\{a_\beta\})\mathrm{icl}_{\mathfrak{N}}(\{a_\gamma\})$, a contradiction.

Now for each β, $\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\})$ is finite. Thus there is some $\beta^* > \beta$ such that

$$\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\}) \cap (\mathfrak{F}_{\{a_\beta,a_\gamma\}} - \{a_\beta, a_\gamma\}) = \emptyset$$

for all $\gamma > \beta^*$. But now by doing a standard catch your tail argument, we can find $\beta' < \omega_1$ such that for all $\beta < \beta'$, if $\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\}) \cap (\mathfrak{F}_{\{a_\beta,a_\gamma\}} - \{a_\beta, a_\gamma\}) \neq \emptyset$, then $\gamma < \beta'$. Choose $\gamma > \beta'$. For all $\beta < \beta'$, $\mathrm{icl}_{\mathfrak{N}}(\{a_\beta\}) \cap (\mathfrak{F}_{\{a_\beta,a_\gamma\}} - \{a_\beta, a_\gamma\}) = \emptyset$. Hence $\mathrm{icl}_{\mathfrak{N}}(\{a_\gamma\}) \cap (\mathfrak{F}_{\{a_\beta,a_\gamma\}} - \{a_\beta, a_\gamma\}) \neq \emptyset$. But this is contradictory as $\mathrm{icl}_{\mathfrak{N}}(\{a_\gamma\})$ is finite and the $\mathfrak{F}_{\{a_\beta,a_\gamma\}} - \{a_\beta a_\gamma\}$ are distinct non-empty sets.

We can do an easy chain argument argument to show that there is some $\mathfrak{X} \subseteq \mathfrak{M}$ and $\mathfrak{M} \models S_{\overline{\alpha}}$. Clearly no such $\mathfrak{M}$ embeds into an atomic model as otherwise, $\mathfrak{X}$ would too. This finishes the proof. □

We finally finish with Theorem 5.2.19, which shows that when $\overline{\alpha}$ is coherent, $\overline{\alpha}_E$ being rational for all $E \in L$ can be characterized in terms of isomorphic embeddability into atomic models.

Theorem 5.2.19. *Let $\overline{\alpha}$ be coherent. The following are equivalent*

1. *$\overline{\alpha}$ is rational*

2. *Every $\mathfrak{M} \models S_{\overline{\alpha}}$ embeds isomorphically into an atomic model of $S_{\overline{\alpha}}$*

Proof. The proof of this statement is immediate from Lemma 5.2.11 and Lemma 5.2.18. $\square$

Chapter 6: Stability and related matters

We begin this chapter with a proof that $S_{\overline{\alpha}}$ is stable. This well known result appears in several places, including Baldwin and Shi's original work in [1] (see also [27] for a treatment of the case $\overline{\alpha}$ is rational, an easier case to study as $S_{\overline{\alpha}}$ has finite closures). However as noted in [4], one of the key lemmas in [1] (Lemma 3.26) is incorrect. An alternate proof, in the spirit of Baldwin and Shi's original arguments was given by Verbovskiy and Yoneda in [4]. We offer a proof, combining ideas in [4] and making use of the quantifier elimination. It is possible to obtain the stability of $S_{\overline{\alpha}}$ based solely on the quantifier elimination result using the exact same arguments of Laskowski in [2]. We also offer a proof of the well known characterization of non-forking by bringing together the work and ideas found in [4] and [2].

We then show that $S_{\overline{\alpha}}$ is non-trivial if $\overline{\alpha}$ is not graph-like with weight one $S_{\overline{\alpha}}$. We show that the converse to this statement holds in Chapter 8. The converse seems to be a known result, though the author is unable to find any written account. The final result of this chapter, that $S_{\overline{\alpha}}$ has the dimensional order property (without placing any additional constraints on L or $\overline{\alpha}$, see [5] and [2]) is new. However the arguments used are not, as it is essentially the same argument given by Laskowski in [2]. We will also make some observations about the spectrum of $S_{\overline{\alpha}}$.

We work with in a monster model $\mathbb{M}$ of $S_{\overline{\alpha}}$. For *this chapter only*, we adopt the practice of writing $\overline{B}$ for the intrinsic closure (equivalently algebraic closure) of a set $B \subseteq \mathbb{M}$ except in places that would cause confusion (i.e. in cases where we have both (algebraic) closures sets and parameters involved in the discussion). We do this to improve redability. A line over lowercase letters $\overline{a}, \overline{b}$ etc. will denote parameters as is customary. We assume that the reader is familiar with the basic stability theory (such as definitions and basic facts related to non-forking) as found in [28] (or alternatively [29] and [30]).

6.1 Stability of $S_{\overline{\alpha}}$

In this section we provide a proof of the fact that $S_{\overline{\alpha}}$ is stable. The arguments from existing literature establishes that the generic is *full* (see Definition 4.4 of [1]) and uses this to establish that $S_{\overline{\alpha}}$ satisfies *amalgamation over closed sets* (see Definition 2.20 of [1]) which is at the heart of the stability argument. We replace the use of fullness by (1) of Lemma 4.4.3 which shows that the $S_{\overline{\alpha}}$ has amalgamation over closed sets directly using the quantifier elimination results. The rest of the argument basically follows that by Verbovskiy and Yoneda in [4].

We begin by extending the notion of relative rank. Recall our definition of $e(A, B)$ from 2.2.4.

Definition 6.1.1. Let $A, X \subseteq \mathbb{M}$ with A finite. Let $P = \{e(A, X_1) : X_1 \subseteq_{\text{Fin}} X\}$. We define $e(A, X) = \sup P$ where we allow for the possibility that the supremum may be ∞.

Remark 6.1.2. Note that given $X_0 \subseteq X_1 \subseteq_{\text{Fin}} X$ we have that $e(A, X_0) \leq e(A, X_1)$ Thus it follows that the notion agrees for finite A, X. Further there is some countable $X_0 \subseteq X$ such that $e(A, X_0) = e(A, X)$.

We now extend the relative rank as follows:

Definition 6.1.3. Let $A, X \subseteq \mathbb{M}$ with A finite. We extend the definition of $\delta(A/X)$ by letting $\delta(A/X) = \delta(A - X) - e(A - X, X)$. Further we say that (X, AX) is an intrinsic minimal pair if $-\infty < \delta(A/X) < 0$ but for all $A' \subsetneq A$, $\delta(A'/X) \geq 0$.

Now we extend the notion of closed sets to that of almost closed sets:

Definition 6.1.4. Let $X \subseteq \mathbb{M}$. We say that X is *almost closed* if there is a positive real γ such that $e(B, X) \leq \delta(B) + \gamma$ for every finite $B \subseteq \mathbb{M}$ disjoint from X, or alternatively $-\gamma \leq \delta(B) - e(B, X) = \delta(B/X)$. The infimum of such γ will be denoted by t_X.

Remark 6.1.5. Let $A, B \subseteq \mathbb{M}$ be disjoint and finite. Now

$$0 \leq \delta(AB) = |A| + |B| - e(A) - e(B) - e(B, A) = \delta(A) + \delta(B) - (e(B, A))$$

Hence $e(B, A) \leq \delta(A) + \delta(B)$. Thus every finite set is almost closed.

Remark 6.1.6. Let $X \subseteq \mathbb{M}$. Note that X is closed if and only if for any finite $A \subseteq \mathbb{M}$, $\delta(A/X) \geq 0$.

Remark 6.1.7. Suppose $X, Y \subseteq \mathbb{M}$ with Y finite. Let $Y' \subseteq Y$ Now $\delta(Y'/X \cap Y') \geq \delta(Y'/X)$. This property will be referred to as the *monotonicity of* δ.

Definition 6.1.8. We call a chain $\langle X_n \rangle_{n<\omega}$ an intrinsic chain if $X_i \subseteq X_{i+1}$ and if either

1. There is some k such that for all $i < k$, $(X_i, (X_{i+1} - X_i)X_i)$ is an intrinsic minimal pair and for all $i \geq k$, $X_i = X_k$

   or

2. For all i, $(X_i, (X_{i+1} - X_i)X_i)$ is an intrinsic minimal pair.

We say that $\langle B_n \rangle_{n<\omega}$ is a minimal intrinsic chain, if for all i, $(X_i, (X_{i+1} - X_i)X_i)$ is an intrinsic minimal pair.

Next we show that certain properties of $e(A, B)$ that hold for finite A, B, C can be extended.

Lemma 6.1.9. *Let $X, B, C \subseteq \mathbb{M}$, B, C finite and assume that X, B, C are pairwise disjoint. Now $e(BC, X) = e(B, XC) + e(C, X) - e(B, C)$.*

Proof. Let $X_0 \subseteq_{\mathrm{Fin}} X$ be arbitrary. Now $e(BC, X_0) = e(B, X_0C) + e(C, X_0) - e(B, C)$. Thus we obtain that $e(BC, X) \geq e(B, X_0C) + e(C, X_0) - e(B, C)$. Further for any $X_0 \subseteq X_1 \subseteq_{\mathrm{Fin}} X$ we have $e(B, X_0C) \leq e(B, X_1C)$. Thus we obtain $e(BC, X) \geq e(B, XC) + e(C, X_0) - e(B, C)$ and hence we obtain that $e(BC, X) \geq e(B, XC) + e(C, X) - e(B, C)$. Further from, $e(BC, X_0) = e(B, X_0C) + e(C, X_0) - e(B, C)$ it follows that $e(BC, X_0) \leq e(B, XC) + e(C, X) - e(B, C)$ as $e(B, X_0C) \leq e(B, XC)$ and $e(C, X_0) \leq e(C, X)$. Thus we obtain that

$$e(BC, X) \leq e(B, XC) + e(C, X) - e(B, C)$$

which establishes the result. $\square$

Lemma 6.1.10. *Let $X, B, C \subseteq \mathbb{M}$ be pairwise disjoint. Suppose that $\delta(BC/X) > -\infty$. Then we have that $\delta(BC/X) = \delta(B/XC) + \delta(C/X)$. Further if $\delta(B/XC)$, $\delta(C/X) > -\infty$ then $\delta(BC/X) > -\infty$.*

Proof. Note that $\delta(BC/X) = \delta(BC) - e(BC, X)$. Since $\delta(BC/X) > -\infty$ it follows that $e(BC, X)$ must be finite. Thus, using Lemma 6.1.9 we see that $e(B, XC)$ and $e(C, X)$ must be finite. Now

$$\begin{aligned} \delta(BC/X) &= \delta(BC) - e(BC, X) \\ &= \delta(B) + \delta(C) - e(B, C) - (e(B, XC) + e(C, X) - e(B, C)) \\ &= \delta(B) - e(B, XC) + \delta(C) - e(C, X) \\ &= \delta(B/XC) + \delta(C/X) \end{aligned}$$

For the second claim note that the finiteness of $\delta(B/XC)$ and $\delta(C/X)$ implies that $e(B, XC)$ and $e(C, X)$ must be finite. Thus the above calculation may be repeated to obtain the result. □

The following is based on Lemma 3.4 of [4].

Lemma 6.1.11. *Suppose that $X \subseteq \mathbb{M}$, $C \subseteq_{Fin} \mathbb{M}$ with X, C disjoint and X is almost closed. Then $-t_X + e(X, C) \leq \delta(C)$ and XC is almost closed in $\mathbb{M}$. If $X \leq \mathbb{M}$. then $e(X, C) \leq \delta(C)$.*

Proof. For any $X_0 \subseteq_{\text{Fin}} X$ we have that $-t_X \leq \delta(C/X_0) = \delta(C) - e(C, X_0)$. So it follows that $-t_X + e(C, X_0) \leq \delta(C)$ and hence $-t_X + e(X, C) \leq \delta(C)$. Now in order to show that XC is almost closed consider $B \subseteq \mathbb{M}$ disjoint from XC. Now $e(B, C) = \delta(B) + \delta(C) - \delta(BC)$. Further arguing as above we have that

$-t_X + e(BC, X) \leq \delta(BC)$. Now since $-t_X + e(BC, X) \leq \delta(BC)$ and $e(B, C)$ is finite (if not $0 \leq \delta(BC) = \delta(B) + \delta(C) - e(B, C)$ leads to a contradiction), we have that

$$\begin{aligned} e(B, XC) &= e(BC, X) + e(B, C) - e(C, X) \\ &\leq \delta(BC) + t_X + \delta(B) + \delta(C) - \delta(BC) \\ &\leq \delta(B) + (\delta(C) + t_X) \end{aligned}$$

Since $\delta(C) + t_X$ is fixed it follows that XC is almost closed. The other half of the claim follows by noting that $t_X = 0$ if X is closed. $\square$

The following terminology is borrowed form [4].

Definition 6.1.12. Let $X, Y \subseteq \mathbb{M}$. Assume that X is almost closed. Y is said to be *calculable* over X if $\langle X_n \rangle_{n<\omega}$ is an intrinsic chain such that $X \subseteq X_0$, $|X_0 - X|$ is finite and $Y = \bigcup_n X_n$. We define $\delta(Y/X) = \lim_n \delta(Y_n/X)$.

Remark 6.1.13. Lemma 6.2.3 tells us that $\delta(Y/X)$ is finite and the value doesn't depend on the intrinsic chain used. We have postponed its proof to the next section as it is somewhat lengthy and distracts from the key ingredients of the proof.

Remark 6.1.14. Note that if $X \subseteq M$ be closed and Y calculable over X, then $\delta(Y/X) \geq 0$. For if not we can find a finite $B \subseteq Y$, $B \nsubseteq X$ with $\delta(B/X) < 0$, which in turn implies that there is some finite $A \subseteq X$ with $\delta(B/A) < 0$ and hence that X is not closed.

The following is Lemma 3.9 of [4]. The proof proceeds similarly. Recall that $\delta(X/Y) = \delta(X - Y/Y)$ when $X - Y$ is finite and if (X, Y) is an intrinsic minimal pair, then $|Y - X| < \aleph_0$.

Lemma 6.1.15. *Let $X \subseteq \mathbb{M}$ be possibly infinite and almost closed. Then $acl(X)$ is calculable over X. In particular $|acl(X) - X| \leq \aleph_0$. Further if $\overline{\alpha}$ is rational, then $|acl(X) - X| < \aleph_0$.*

Proof. First note that as X is almost closed, given C, D such that they are finite, disjoint from X, (X, XC) an intrinsic minimal pair and $C \nsubseteq D$, then

$$\delta(C/XD) \leq \delta(C/X(D \cap C)) = \delta(C/X) - \delta(D \cap C/X) \leq \delta(C/X)$$

Next, suppose that there is a sequence of finite $\langle C_i \rangle$ such that (X, XC_i) is a minimal intrinsic extension of X with $\delta(C_i/X) \leq -1/n$ and $C_i \nsubseteq X \cup \bigcup_{j<i} C_j$. Choose $k > nt_X$. Then

$$\delta(\bigcup_{i<k} C_i/X) = \sum_{i<k} \delta(C_i/(X \cup \bigcup_{j<i} C_j)) \leq \sum_{i<k} \delta(C_i/X) \leq -k/n < -t_X$$

a contradiction. Hence there are only finitely many minimal intrinsic extensions C with $\delta(C/X) \leq -1/n$.

It follows that we can recursively construct an intrinsic chain $X = X_0 \subseteq X_1 \subseteq \ldots$ such that $\delta(X_{n+1}/X_n)$ is minimum possible given X_n. First note that in such a chain X_n is almost closed being a finite extension of X_{n-1} which is also almost closed. Now given X_n we pick X_{n+1} as follows: if X_n is not closed look at all possible intrinsic minimal extensions B of X_n and choose one such that $\delta(B/X_n)$ has the *least* possible value, i.e. $-\delta(B/X_n)$ is as large as possible. This value is finite by the fact that X_n is almost closed and as there are only finitely many minimal intrinsic extensions B with $\delta(B/X_n) \leq -1/k$ for some fixed k and the existence of such a k is guaranteed by the fact that X_n is not closed for all $n < \omega$. If at some stage X_i is closed put $X_i = X_j$ for all $j > i$. Take $X' = \bigcup_{n<\omega} X_n$. Then clearly X' is calculable over X. Now $\delta(X_{n+1}/X_0) = \sum_{i=0}^{n} \delta(X_{i+1}/X_i)$. Note that as X_0 is almost closed, it follows that for any n, $-t_{X_0} \leq \delta(X_{n+1}/X_0) = \sum_{i=0}^{n} \delta(X_{i+1}/X_i)$. Further it is easily seen that $\langle \sum_{i=0}^{n} \delta(X_{i+1}/X_i) \rangle_{n<\omega}$ is monotonic decreasing. Hence $\sum_{i=0}^{\infty} \delta(X_{i+1}/X_i)$ is convergent and as a result $\lim_n \delta(X_{n+1}/X_n) = 0$.

Suppose that $C \subseteq_{\mathrm{Fin}} \mathbb{M}$ is finite and disjoint from X. Now if $\delta(C/X) < 0$ there is some finite $X_0 \subseteq X$ such that $\delta(C/X_0) < 0$. Thus using $\delta(X_{n+1}/X_0) = \sum_{i=0}^{n} \delta(X_{i+1}/X_i)$ it follows that $X \subseteq X' \subseteq \mathrm{icl}X = \mathrm{acl}(X)$. Now suppose that X' is not closed. Then there is some C disjoint from X' with $(X', X'C)$ an intrinsic minimal pair. Since $0 > \delta(C/X') = \delta(C) - e(C, X') = \delta(C) - \lim e(C, X_n) = \lim \delta(C/X_n)$ there is $k < \omega$ with $\delta(C/X_k) < 0$. Thus from monotonicity, it follows that $\delta(C/X_n) \leq \delta(C/X_k)$ for all $n \geq k$. Since $(X', X'C)$ is an intrinsic minimal pair and $X_n \subseteq X'$, it follows that (X_n, X_nC) is a minimal pair: For if there is some non empty $C' \subsetneq C$ such that (X_n, X_nC') forms an intrinsic minimal pair, it follows that $(X', X'C')$ forms a minimal pair. Now as $\delta(X_{n+1}/X_n) \leq \delta(C/X_n) \leq \delta(C/X_k)$ by our choice of the X_i we obtain that $\lim \delta(X_{n+1}/X_n) \neq 0$, a contradiction.

For the second half of the claim, assume that $\overline{\alpha}$ is rational. Recall our notation of c for the least common multiple of the denominators of the $\overline{\alpha}_E$. Note that for any intrinsic minimal pair (X, XC), $\delta(C/X) \leq -1/c$. Thus there is some X_i such that $X_i = X'$ (as $\lim \delta(X_{n+1}/X_n) = 0$ fails otherwise). Thus the required statement follows. $\square$

We are finally in a position to give a proof of the stability of $S_{\overline{\alpha}}$.

Theorem 6.1.16. *$S_{\overline{\alpha}}$ is stable. Further if $\overline{\alpha}$ is rational, then $S_{\overline{\alpha}}$ is ω-stable.*

Proof. Recall that we are working within a monster model $\mathbb{M}$ of $S_{\overline{\alpha}}$. Fix an infinite cardinal $\kappa \geq \aleph_0$. Fix a positive integer n. We prove the stability of $S_{\overline{\alpha}}$ by counting n-types over a fixed algebraically closed set X where $|X| = \kappa$. Let $\overline{a}, \overline{b} \in \mathbb{M}^n$ with $\overline{a}, \overline{b} \notin X^n$.

We claim that $\mathrm{tp}(\overline{a}/X) = \mathrm{tp}(\overline{b}/X)$ if and only if there is a partial *isomorphism* $f : \mathrm{acl}(X\overline{a}) \to \mathrm{acl}(X\overline{b})$ where $f(\overline{a}) = \overline{b}$ and f is the identity on X. In order to establish the claim, first note that if $\mathrm{tp}(\overline{a}/X) = \mathrm{tp}(\overline{b}/X)$, then there exists an *elementary* map f with the required properties, So assume that there is a partial isomorphism $f : \mathrm{acl}(X\overline{a}) \to \mathrm{acl}(X\overline{b})$ where $f(\overline{a}) = \overline{b}$ and f is the identity on X. By Lemma 4.4.3, $tp(\mathrm{acl}(X\overline{a})) = tp(\mathrm{acl}(X\overline{b}))$. Thus we obtain that $\mathrm{tp}(\overline{a}/X) = \mathrm{tp}(\overline{b}/X)$ establishing the claim.

Note that $X\overline{a}$ is almost closed as X is closed. Now by Lemma 6.1.15, we have that $|\mathrm{acl}(X\overline{a}) - X\overline{a}| \leq \aleph_0$. Define an equivalence relation $\sim$ on the one point extensions of X by setting $\overline{a} \sim \overline{b}$ if and only if there is a partial isomorphism f : $\mathrm{acl}(X\overline{a}) \to \mathrm{acl}(X\overline{b})$ where $f(\overline{a}) = \overline{b}$ and f is the identity on X. Now combining this with $|\mathrm{acl}(X\overline{a}) - X\overline{a}| \leq \aleph_0$ we see that $|\{[a] : a \sim b\}| \leq \kappa^{\aleph_0}$. As $\mathrm{tp}(\overline{a}/X) = \mathrm{tp}(\overline{b}/X)$ if and only if there is a partial isomorphism $f : \mathrm{acl}(X\overline{a}) \to \mathrm{acl}(X\overline{b})$ where $f(\overline{a}) = \overline{b}$ and f is the identity on X, it follows that $|S_n(X)| \leq \kappa^{\aleph_0}$. For κ that satisfy $\kappa = \kappa^{\aleph_0}$, we have that $|S_n(X)| = \kappa$, Thus it follows that $S(X) = \kappa$ for all κ such that $\kappa^{\aleph_0} = \kappa$ and hence $S_{\overline{\alpha}}$ is stable.

Now assume that $\overline{\alpha}$ is rational. By Lemma 6.1.15, we have that $|\mathrm{acl}(X\overline{a}) -$

$X\overline{a}| < \aleph_0$. An argument similar to the above yields that $|S_n(X)| \leq \kappa^{<\aleph_0} = \kappa$ for any infinite cardinal κ. Thus we obtain that $|S(X)| = \kappa$ for any infinite κ and hence $S_{\overline{\alpha}}$ is ω-stable. □

6.2 Characterizing non-forking

We now work towards characterizing non-forking. The characterizations we present are well known, going back to the original work of Baldwin and Shi in [1]. We offer a proof that incorporates proofs from [4] and [2] (some of the ideas used by Laskowski in [2] also appear in [5]). We would like to note that the results in [4] and [1] attempts to incorporate a broader context and are crouched in technical details related to amalgamation properties.

6.2.1 Further properties of d

This section is devoted to obtaining various properties of d that we will use throughout. Our first goal is to show that $d(A)$ for finite $A \subseteq \mathbb{M}$ captures what the rank δ of $\overline{A}$ ought to be (see Lemma 6.2.7). This generalizes a similar result that is much more easily obtained in the $\overline{\alpha}$ is rational. We closely follow the development of Verbovskiy and Yoneda in [4]. Recall that we are working $\mathbb{M}$, a monster model of $S_{\overline{\alpha}}$. All sets, not otherwise mentioned, will be assumed to be small subsets of $\mathbb{M}$.

Lemma 6.2.1. *Let $\langle B_n\rangle_{n<\omega}$ be an intrinsic chain. Then $\delta(B_n/B_0)$ is finite for all n.*

Proof. We show this by induction on n. The statement is clearly true for $n = 0$ by the definition of a minimal intrinsic extension. Assume that the statement holds for $n < \omega$. Now we look at $\delta(B_{n+1}/B_0)$. Now $|B_{i+1} - B_i|$ is finite for each i. Further $(B_{n+1} - B_n)(B_n - B_0) = B_{n+1} - B_0$. Now $e((B_{n+1} - B_n)(B_n - B_0), B_0) = e((B_{n+1} - B_n), B_n) + e(B_n - B_0, B_0) - e(B_{n+1} - B_n, B_n - B_0)$ by Lemma 6.1.9. Since $\delta(B_n/B_0)$ is finite by the induction hypothesis, we have that $e(B_n - B_0, B_0)$ is finite. Further (B_n, B_{n+1}) is an either a intrinsic minimal pair or $B_{n+1} = B_n$ and it follows that $e(B_{n+1} - B_n, B_0)$ is finite. Then by Lemma 6.1.10 we have that $\delta(B_{n+1}/B_0) = \delta(B_{n+1}/B_n) + \delta(B_n/B_0)$ and the result follows. □

The following is a modified form of Lemma 3.5 of [4].

Lemma 6.2.2. *Let $\langle B_n \rangle_{n<\omega}$ be an intrinsic chain. Let $F \subseteq_{Fin} \bigcup B_n$. Then there exists an n_0 such that $\delta(F/B_0) \geq \delta(B_{n_0}/B_0)$.*

Proof. We show by induction on m that $\delta(F \cap B_m/B_0) \geq \delta(B_m/B_0)$. As there is an n_0 such that $F \subseteq B_{n_0}$ this establishes the claim. For $m = 0$, the assertion is trivial. Suppose that the induction hypothesis holds for $m < \omega$. By Lemma 6.2.1, $\delta(B_{m+1}/B_0) = -\infty$ is impossible. Thus $\delta(B_{m+1}/B_0) > -\infty$ and hence

$$\begin{aligned}
\delta(B_{m+1}/B_0) &= \delta(B_{m+1}/B_m) + \delta(B_m/B_0) \\
&\leq \delta(B_{m+1} \cap F/B_m) + \delta(B_m \cap F/B_0) \\
&\leq \delta(B_{m+1} \cap F/B_0(B_m \cap F)) + \delta(B_m \cap F/B_0) \\
&= \delta(B_{m+1} \cap F/B_0)
\end{aligned}$$

where the first inequality holds because (B_m, B_{m+1}) is an intrinsic minimal pair or $B_{m+1} = B_m$ and by the inductive hypothesis while the second inequality holds by monotonicity. □

The following is a modified form of Lemma 3.6 of [4].

Lemma 6.2.3. *Let A be an almost closed subset of $\mathbb{M}$. Let $\langle B_n \rangle_{n<\omega}$ be an intrinsic chain such that $A \subseteq B_0$ and $|B_0 - A|$ is finite. Let $B := \bigcup_{n<\omega} B_n$. Then*

1. $\lim_n \delta(B_n/A)$ *exists and is finite.*

2. *B is almost closed.*

3. $\lim_n \delta(B_n/A) = \delta(B_k/A) + \lim_n \delta(B_n/B_k) = \delta(B_k/A) + \sum_{n=k}^{\infty} \delta(B_{n+1}/B_n)$ *for any $k < \omega$; in particular,* $\lim_n \delta(B_{n+1}/B_n) = 0$.

4. *If $\langle B'_n \rangle_{n<\omega}$ is another intrinsic chain such that $B = \bigcup_{n<\omega} B'_n$ then* $\lim_n \delta(B_n/A) = \lim_n \delta(B'_n/A)$.

Proof. We note that clause (1) can not hold simultaneously with clause (2) in the definition of an intrinsic chain (see definition 6.1.8). Thus if $\langle B_n \rangle_{n<\omega}$ is eventually constant (i.e. satisfies clause (1)), then $B - A$ is finite and any other intrinsic chain that yields $B = \bigcup B'_n$ is also eventually constant.

First we prove (1). Since $\langle B_n \rangle_{n<\omega}$ is an intrinsic chain it follows that $-\infty < \delta(B_{n+1}/B_n) \leq 0$. Since A is almost closed we have that $-t_A \leq \delta(B_{n+1}/A) = \delta(B_{n+1}/B_n) + \delta(B_n/A) \leq \delta(B_n/A)$. Thus the sequence $\langle \delta(B_n/A) \rangle_{n<\omega}$ is decreasing and bounded and hence the limit exists and is finite.

Next we prove (2). Consider a finite C disjoint from B. Now $\delta(C/B_{n+1}) \leq \delta(C/B_n)$ by monotonicity. So $\lim_n \delta(C/B_n)$ exists. Further as A is almost closed $\delta(B_nC/A) > -\infty$. Using Lemma 6.1.10 we obtain that $\delta(B_nC/A) = \delta(C/B_n) + \delta(B_n/A) \geq -t_A$. Now $e(C, B_n) \leq e(C, B_{n+1}) \leq e(C, B)$ is clear as $B_n \subseteq B_{n+1} \subseteq B$

and thus $\lim_n e(C, B_n) \leq e(B, C)$. Also any finite subset B' of B is a finite subset of some B_{n_0}, it follows that $e(C, B) \leq \lim_n e(C, B_n)$ Hence it now follows that

$$\begin{aligned} \delta(C/B) &= \delta(C) - e(C, B) \\ &= \delta(C) - \lim_n e(C, B_n) \\ &= \lim_n \delta(C/B_n) \\ &\geq -t_A - \lim_n \delta(B_n/A) \end{aligned}$$

Thus B is almost closed.

Now we prove (3). Fix $k \in \mathbb{N}$. Note that Lemma 6.1.10 and A being closed tells us that for $n > k$, $\delta(B_n/A) = \delta(B_n/B_k) + \delta(B_k/A)$. Letting n tend to infinity we obtain that $\lim_n \delta(B_n/A) = \delta(B_k/A) + \lim_n \delta(B_n/B_k)$ Further an induction argument and the use of Lemma 6.1.10 shows us that $\delta(B_n/B_k) = \sum_{i=k}^{n-1} \delta(B_{i+1}/B_i)$. This shows us that $\delta(B_n/A) - \delta(B_k/A) = \sum_{i=k}^{n-1} \delta(B_{i+1}/B_i)$. Since $\lim_n \delta(B_n/A)$ exists, it follows that $\lim_n \delta(B_n/A) - \delta(B_k/A) = \lim_n \sum_{i=k}^{n-1} \delta(B_{i+1}/B_i)$ and hence $\lim_n \delta(B_n/A) = \delta(B_k/A) + \lim_n \sum_{i=k}^{n-1} \delta(B_{i+1}/B_i) = \delta(B_k/A) + \sum_{n=k}^{\infty} \delta(B_{n+1}/B_n)$.

Finally we prove (4). As in (1), $\lim_n \delta(B'_n/A)$ exists. Now using Lemma 6.2.2 for any k there is some n_0 such that $\delta(B_k/A) \geq \delta(B'_{n_0}/A) \geq \lim_n \delta(B'_n/A)$. The reverse inequality follows by symmetry. □

Remark 6.2.4. Note that any finite extension B of an almost closed set A

1. is calculable over A since we can take $B_n = B$ for each n to be the intrinsic chain witnessing calculability of B over A.

2. is almost closed by a (2) of Lemma 6.2.3.

The following is a modified version of Lemma 3.8 of [4].

Lemma 6.2.5. *Let A be almost closed and B calculable over A. Let C be calculable over B, which is almost closed by Lemma 6.2.3. Then C is calculable over A and $\delta(C/A) = \delta(C/B) + \delta(B/A)$. In particular if B is calculable over A and A is calculable over $\emptyset$, then $\delta(B/A) = \delta(BA) - \delta(A)$.*

Proof. Let $\langle B_n \rangle_{n<\omega}$ and $\langle C_n \rangle_{n<\omega}$ be corresponding intrinsic chains over A and B respectively that witness the calculability of B over A and C over B. Note that if $C - A$ is finite then C is calculable over A by remark 6.2.4. Further if $B - A$ is finite then since $C - B$ we have that $C - A$ is finite and thus we see that $\langle C_n \rangle_{n<\omega}$ witnesses the calculability of C over A. Thus we may assume that $B - A$ is infinite. Hence each (B_i, B_{i+1}) forms an intrinsic minimal pair. Put $C'_n = C_n - B$. Now

$$\begin{aligned} 0 &\geq \delta(C_{n+1}/C_n) \\ &= \delta(C_{n+1} - C_n) - e(C_{n+1} - C_n, C_n) \\ &= \delta(C'_{n+1} - C'_n) - \lim_k e(C'_{n+1} - C'_n, B_k C'_n) \\ &= \lim_k \delta(C'_{n+1}/B_k C'_n) \end{aligned}$$

Thus there is $\tau(n) < \omega$ such that $\delta(C'_n/B_{\tau(n)}C'_n) \leq 0$. We may assume that this τ is an increasing function.

Claim: We may choose $\tau(n) < \omega$ such that for all $n < \omega$

$$0 \leq \delta(C'_n/B_{\tau(n)}) - \delta(C'_n/B) < 1/n$$

Proof of Claim: The first inequality follows from monotonicity regardless of the value of $\tau(n)$. For the second inequality note that

$$\delta(C'_n/B_k) - \delta(C'_n/B) \quad = \quad e(C'_n, B) - e(C'_n, B_k)$$

Note that $e(C'_n, B)$ is finite by the fact that B is almost closed. Further $0 \leq e(C'_n, B_k) \leq e(C'_n, B)$. Thus the difference above is finite. As $\lim_k e(C'_n, B_k) = e(C_n, B)$ we may choose $\tau(n)$ such that the required inequality holds. Further $e(C'_n, B_k) \leq e(C'_n, B_{k+1})$. So we may assume that $\tau(n+1) - \tau(n) > 3$.

There are now two possibilities:

Case 1: Suppose that C is calculable over B and the sequence $\langle C_n \rangle_{n<\omega}$ is such that for each i, (C_i, C_{i+1}) is an intrinsic minimal pair. Now

$$\begin{aligned}
0 &> \delta(C_{n+1}/C_n) \\
&= \delta(C_{n+1} - C_n) - e(C_{n+1} - C_n, C_n) \\
&= \delta(C'_{n+1} - C'_n) - \lim_k e(C'_{n+1} - C'_n, B_k C'_n) \\
&= \lim_k \delta(C'_{n+1}/B_k C'_n)
\end{aligned}$$

Thus it also follows that $\delta(C'_n/B_{\tau(n)}C'_n) < 0$. Now we consider the increasing (with respect to $\subseteq$) sequence given by $D_0 = B_{\tau(1)}C'_1$. $D_1 = B_{\tau(1)+1}C'_1$. For $n \geq 2$, suppose that $D_n = B_{\tau(i)+l}C'_k$ is given. Now

$$D_{n+1} = \begin{cases}
B_{\tau(i)+l+1}C'_k & \text{if } \tau(i)+l+1 = \tau(i+1) \\
B_{\tau(i)+l}C'_{k+1} & \text{if } l = 0 \text{ and } D_{n-1} = B_{\tau(i)-1}C'_k \\
B_{\tau(i)+l+1}C'_k & \text{if } l = 0 \text{ and } D_{n-1} \neq B_{\tau(i)-1}C'_k \\
B_{\tau(i)+l+1}C'_k & \text{if } l > 0 \text{ and } \tau(i)+l+1 < \tau(i+1)
\end{cases}$$

Since each of the B_i, C'_i are distinct and $\tau(n+1) - \tau(n) > 3$ this is well defined. Clearly $B_{\tau(0)}C'_1 - A$ is finite, as is the difference between two successive sets in the sequence. Note that the sequence $\langle D_n \rangle$ is increasing and the union of this sequence is C.

Claim: The sequence $\langle D_n \rangle$ can be refined to an increasing minimal intrinsic chain that shows C is calculable over A.

Proof of Claim: First consider $B_{\tau(i)+l}C'_k$ and $B_{\tau(i)+l+1}C'_k$. Note that

$$B_{\tau(i)+l+1}C'_k - B_{\tau(i)+l}C'_k = B_{\tau(i)+l+1} - B_{\tau(i)+l} = B_{\tau(i)+l+1} - B_{\tau(i)+l}C'_k$$

. Hence it follows that:

$$\begin{aligned}
\delta\left(\frac{B_{\tau(i)+l+1}C'_k}{B_{\tau(i)+l}C'_k}\right) &= \delta(B_{\tau(i)+l+1}C'_k - B_{\tau(i)+l}C'_k) - e(B_{\tau(i)+l+1}C'_k - B_{\tau(i)+l}C'_k, B_{\tau(i)+l}C'_k) \\
&= \delta(B_{\tau(i)+l+1} - B_{\tau(i)+l}C'_k) - e(B_{\tau(i)+l+1} - B_{\tau(i)+l}C'_k, B_{\tau(i)+l}C'_k)
\end{aligned}$$

By monotonicity $\delta(B_{\tau(i)+l+1}/B_{\tau(i)+l}C'_k) \leq \delta(B_{\tau(i)+l+1}/B_{\tau(i)+l}) < 0$. This indicates the existence of a minimal pair. If we consider a set D such that $B_{\tau(i)+l}C'_k \subseteq D \subsetneq B_{\tau(i)+l+1}C'_k$, we may write $D = B'C'_k$ with $B_{\tau(i)+l} \subseteq B' \subsetneq B_{\tau(i)+l+1}$. Arguing as above we see that $\delta(B_{\tau(i)+l}C'_k/B'C'_k) = \delta(B_{\tau(i)+l}/B'C'_k) \leq \delta(B_{n+1}/B') < 0$. Thus we may refine $B_{\tau(i)+l}C'_k$, $B_{\tau(i)+l+1}C'_k$ into a finite sequence $D'_1, \ldots D'_k$ such that $D'_0 = B_{\tau(i)+l}C'_k$ and $D'_k = B_{\tau(i)+l}C'_k$ and (D_i, D_{i+1}) is an intrinsic minimal pair.

Now consider $B_{\tau(i)}C'_k$ and $B_{\tau(i)}C'_{k+1}$. Now given $B_{\tau(i)}C'_k \subseteq D' \subseteq B_{\tau(i)}C'_{k+1}$. If $D' \neq B_{\tau(i)}C'_{k+1}$, then we may write $D' = C'B_{\tau(i)}$ where $C'_k \subseteq C' \subsetneq C'_{k+1}$. Now by the monotonicity of δ it follows that $\delta(C'/B_{\tau(n)}C'_k) \geq \delta(C'/BC'_k) = \delta(C'/C_k) \geq 0$ as $C_k = BC'_k$. Further note that all sets in the sequence are almost closed by the second clause of remark 6.2.4. Thus $\delta(C'_{n+1}/B_{\tau(n)}C') = \delta(C'_{n+1}/B_{\tau(n)}C'_n) - \delta(C'/B_{\tau(n)}C'_n)$ follows by the use of Lemma 6.1.10. But then $\delta(C'_{n+1}/B_{\tau(n)}C') < 0$

as $\delta(C'_{n+1}/B_{\tau(n)}C'_n) < 0$ and $\delta(C'/B_{\tau(n)}C'_n) \geq 0$. Setting $C' = C'_k$ we see that $(B_{\tau(i)}C'_k, B_{\tau(i)}C'_{k+1})$ is an intrinsic minimal pair. This establishes the claim

Case 2: $|C - B|$ is finite. We may as well assume that $C_i = C_0$ for each i. Consider the increasing sequence $\langle D_n \rangle_{n<\omega}$ with $D_0 = B_{\tau(1)}C'_0$ and given $D_n = B_{\tau(1)+n}C'_0 = B_{\tau(i)+n}C'_l$ where l is any positive integer. The proof now proceeds as the initial part of case 1 above since (B_i, B_{i+1}) is an intrinsic minimal pair.

Thus we have established that C is calculable over A. Now it remains to show that the additivity properties of δ extends to calculable sets as found in the statement of the theorem. To this end note that if $C - A$ is finite this is just the content of Lemma 6.1.10. Now if $B - A$ is finite, and $C - B$ is infinite then under our hypothesis we may write $\delta(C/A) = \lim_n \delta(C_n/A) = \lim_n \delta(C_n/B) + \delta(B/A) = \delta(C/B) + \delta(B/A)$. Thus we may assume that $B - A$ is infinite. Note now that for $j < k$, $\delta(B_k) < \delta(B_j)$. Now for any subsequence $\langle B_{n_k} \rangle_{n_k<\omega}$ it follows that $\lim_n \delta(B_n/A) = \lim_k \delta(B_{n_k}/A)$. Similar comments hold about $\lim_n \delta(C_n/B)$ and $\lim_n \delta(D_n/A)$.

Now an induction argument shows that $\langle B_{\tau(n)C'_n} \rangle$ is a subsequence of $\langle D_n \rangle$ in case 1 above. It is clear for case 2 since the C'_i are constant. Thus

$$\begin{aligned} \delta(BC/A) &= \lim_n \delta(B_{\tau(n)}C'_n/A) \\ &= \lim_n \delta(C'_n/B_{\tau(n)}) + \delta(B_{\tau(n)}/A) \end{aligned}$$

Note that $\lim_n \delta(B_{\tau(n)}/A) = \delta(B/A)$. Thus if we show that $\lim_n \delta(C'_n/B_{\tau(n)}) = \delta(C/B)$, then the result follows. But by our choice of $\tau(n)$ is such that $0 \leq$

$\delta(C'_n/B_{\tau(n)}) - \delta(C'_n/B) < 1/n$. Since $\lim_n \delta(C'_n/B) = \delta(C/B)$, an application of the squeeze lemma yields that $\lim_n \delta(C'_n/B_{\tau(n)}) = \delta(C/B)$. This shows that

$$\delta(BC/A) = \delta(C/B) + \delta(B/A)$$

□

It now follows that we can define $\delta(\overline{B})$ for finite $B \subseteq \mathbb{M}$ as follows (recall that $\overline{B}$ denotes the intrinsic, equivalently algebraic, closure of B):

Definition 6.2.6. Let $B \subseteq \mathbb{M}$. We define $\delta(\overline{B}) = \lim_n(\delta(B_n/B)) + \delta(B)$ where $\langle B_n \rangle_{n<\omega}$ is some (equivalently any) intrinsic chain with $B_0 = B$ and union $\overline{B}$.

The following is Lemma 3.10 of [4].

Lemma 6.2.7. *Let B be a finite subset of $\mathbb{M}$. Then $d(B) = \delta(\overline{B})$*

Proof. Let $\langle B_n \rangle$ be an intrinsic chain such that $B_0 = B$ and $\overline{B} = \bigcup_{n<\omega} \delta(B_n)$. Then $\delta(B_n) \geq d(B)$ for each $n < \omega$. Thus $\delta(\overline{B}) = \lim_n \delta(B_n) \geq d(B)$.

Take an arbitrary finite $F \supseteq B$. Since $\overline{B}$ is closed, it follows that $\delta(F \cap \overline{B}) \leq \delta(F)$. By Lemma 6.2.2, there exists an n such that $\delta(B_n) \leq \delta(F \cap B_n)$. Now $\delta(\overline{B}) \leq \delta(B_n) < d(B) + 1/m$. from which the above follows. □

Definition 6.2.8. Let $A, B \subseteq \mathbb{M}$ be finite. We let $d(A/B) = d(AB) - d(B)$. For infinite X we take $d(A/X) = \inf\{d(A/X_0) : X_0 \subseteq_{\mathrm{Fin}} X\}$.

Remark 6.2.9. In light of Lemma 6.2.11, we see that for $A, X \subseteq \mathbb{M}$ with A finite $d(A/X) = \inf\{d(A/X_0) : X_0 \subseteq_{\mathrm{Fin}} X\}$. This establishes the compatibility of the definitions.

We prove a couple of useful technical lemmas before we begin characterizing non-forking. The following is Remark 3.11 of [4].

Remark 6.2.10. Let $A, B \subseteq \mathbb{M}$ be finite. By Lemma 6.2.7 and the definition of $d(A/B)$, we obtain that for $d(A/B) = d(BA) - d(B) = \delta(\overline{BA}) - \delta(\overline{B})$. Further, as $\overline{\overline{B}A} = \overline{AB}$ and A, B is finite, we obtain that $\overline{B}$ is calculable over $\emptyset$ and $\overline{AB}$ is calculable over $\overline{B}$ (by using intrinsic chains $\langle B_n \rangle_{n<\omega}, \langle A_n \rangle_{n<\omega}$ with $A_0 = \overline{B}A$ and $B_0 = B$ respectively). Thus by Lemma 6.2.5 we obtain that $\delta(\overline{AB}/\overline{B}) = \delta(\overline{AB}) - \delta(\overline{B})$. Thus for finite $A, B \subseteq \mathbb{M}$ we have that $d(A/B) = \delta(\overline{AB}/\overline{B})$.

The following is Lemma 3.12 of [4].

Lemma 6.2.11. *Let $A, B, C \subseteq \mathbb{M}$ with A finite and $B \subseteq C$. Then $d(A/B) \geq d(A/C)$.*

Proof. First assume that C (and hence B) are finite. Let $(B_n)_{n<\omega}$ be an intrinsic chain with union $\overline{AB}$, $B_0 = A\overline{B}$ and let $(C_n)_{n<\omega}$ be a chain with union $\overline{AC}$, $C_0 = A\overline{C}$. Now

$$
\begin{aligned}
\delta(B_n/\overline{B}) &= \delta(B_n/\overline{B}(\overline{C} \cap B_n)) + \delta((\overline{C} \cap B_n)/\overline{B}) \text{ by Lemma 6.2.5} \\
&\geq \delta(B_n/\overline{B}(\overline{C} \cap B_n) \text{ as } \overline{B} \text{ is closed} \\
&\geq \delta(B_n/\overline{C}) \text{ by monotonocity of } \delta \\
&= \delta(B_n - \overline{C}/\overline{C}) \text{ by definition} \\
&= \delta(B_n - \overline{C}/C_0) + \delta(C_0/\overline{C}) \text{ by Lemma 6.2.5} \\
&\geq \delta(C_k/C_0) + \delta(C_0/\overline{C}) \text{ by Lemma 6.2.2 for sufficiently large } k < \omega \\
&= \delta(C_k/\overline{C}) \text{ by Lemma 6.2.5} \\
&\geq \lim_k \delta(C_k/\overline{C}) \text{ as Lemma 6.2.3 tells us the sequence is decreasing} \\
&\geq \delta(\overline{AC}/(\overline{C})) \text{ by definition} \\
&= d(A/C) \text{ by Remark 6.2.10}
\end{aligned}
$$

Therefore $d(A/B) = d(AB) - d(B) = \delta(\overline{AB}) - \delta(\overline{B})$. Further $\delta(\overline{AB}/\overline{B}) = \lim_n(\overline{B_n}/\overline{B}) \geq d(A/C)$ as claimed for finite A, B. Now let C be arbitrary. Then $d(A/B) = \inf_{B_0 \subseteq B} d(A/B_0)$ where B_0 is finite. But as $d(A/C) = \inf_{C_0 \subseteq B} d(A/C_0)$ where C_0 is finite, it easily follows that $d(A/B) \geq d(A/C)$. □

6.2.2 d-independence and Free Joins of Algebraically Closed Sets

We now begin characterizing non-forking. Our goal is to describe non-forking in two unique ways. One involves the notion of $\mathop{\smile\!\!\!\!\!\!|}^d$, a notion that has been highly useful in studying theories if structures constructed with a rank (or alternatively pre-dimension) function (see [1], [4], [27] and [13] for example). The second notion in terms of closed sets and free joins. We begin by introducing the notion of $\mathop{\smile\!\!\!\!\!\!|}^d$.

Definition 6.2.12. Let A, B be finite. We say that A, B *are d-independent over Z* and write $A \mathop{\smile\!\!\!\!\!\!|}^d_Z B$ if

1. $d(A/Z) = d(A/ZB)$
2. $\overline{AZ} \cap \overline{BZ} \subseteq \overline{Z}$

For arbitrary X, Y, Z, we say that X and Y *are d-independent over Z* if for any $X_0 \subseteq_{\mathrm{Fin}} X$, $Y_0 \subseteq_{\mathrm{Fin}} Y$, $X_0 \mathop{\smile\!\!\!\!\!\!|}^d_Z Y_0$. We denote this by $X \mathop{\smile\!\!\!\!\!\!|}^d_Z Y$.

We note some easy consequences of this definition in the remarks and the lemma below.

Remark 6.2.13. Let $A, XY \subseteq \mathbb{M}$ with A finite. Note that as $d(A/X) \geq d(A/XY)$ and the union o two finite sets is again finite, we may calculate $d(A/X)$ by calculating

$\inf\{d(A/X'X_0) : X_0 \subseteq_{\text{Fin}} X\}$ where

Remark 6.2.14. Fix $X_0 \subseteq_{\text{Fin}} X$ and assume that $X \underset{Z}{\overset{d}{\downarrow}} Y$. Given any $X_1 \subseteq_{\text{Fin}} X_0$, $Y_0 \subseteq_{\text{Fin}} Y$, we have that $X_1 \underset{Z}{\overset{d}{\downarrow}} Y_0$. Thus it follows that $X_0 \downarrow^d_Z Y$.

The following lemma appears is Lemma 3.13 of [4]

Lemma 6.2.15. *Let $A, B \subseteq \mathbb{M}$ with B finite. Then $d(B/A) = \delta(\overline{AB}/\overline{A})$. In particular $d(B/A) = d(B - \overline{A}/A) = d(B/\overline{A})$.*

Proof. We begin by showing $d(B/A) \leq \delta(\overline{AB}/\overline{A})$. Fix an intrinsic chain $\langle B_n \rangle_{n<\omega}$ with $B_0 = \overline{A}B$ and $\bigcup_{n<\omega} B_n = \overline{AB}$. Put $B'_n = B_n - \overline{A}$, a finite set. Note that $\delta(B_n/\overline{A}) = \delta(B'_n) - e(B'_n - \overline{A}, \overline{A})$. Given an $n \in \omega$, we may pick a $A_n \subseteq_{\text{Fin}} A$ sufficiently large so that $e(B'_n, \overline{A'}) > e(B'_n, \overline{A}) - 1/n$. As $B'_n - \overline{A_n} = B'_n - \overline{A}$, it follows that $\delta(B'_n/\overline{A_n}) < \delta(B'_n/\overline{A}) + 1/n$.

Now $d(B/A) \leq d(B/A_n) = \delta(\overline{BA_n}/\overline{A_n})$ by Remark 6.2.10. Further note that $\overline{A_n}B \subseteq \overline{A_n}B'_n$ and hence $\overline{\overline{A_n}B} = \overline{A_nB} \subseteq \overline{\overline{A_n}B'_n} = \overline{A_nB'_n}$. As $\overline{A_nB'_n}$ is calculable over $\overline{A_n}B'_n$ and $\overline{A'_n}B'_n$ is calculable over $\overline{A_n}B$, we obtain that $\overline{A_nB'_n}$ is calculable over $\overline{A_nB}$. Since $\overline{A_nB}$ is closed, we obtain that $\delta(\overline{A_nB'_n}/\overline{A_nB}) \geq 0$. Now using Lemma 6.2.5 we obtain that $\delta(\overline{A'_nB'_n}/\overline{A_n}) = \delta(\overline{A_nB'_n}/\overline{A_nB}) + \delta(\overline{A_nB}/\overline{A_n})$. As $\overline{A_nB}$ is closed, we obtain that $\delta(\overline{A_nB'_n}/\overline{A_nB}) \geq 0$. Hence we have that $\delta(\overline{A'_nB'_n}/\overline{A_n}) \geq \delta(\overline{A_nB}/\overline{A_n})$. Note that there is an intrinsic chain $\langle D_k \rangle_{k<\omega}$ such that $D_0 = \overline{A_n}B'_n$ and $\bigcup_{k<\omega} D_k = \overline{A'_nB_n}$. Hence it follows that $\delta(\overline{A_nB'_n}/\overline{A_n}) \leq \delta(D_0/\overline{A_n}) = \delta(\overline{A_n}B'_n/\overline{A_n}) = \delta(B'_n/\overline{A_n})$ and thus $\delta(d(B/A)) \leq \delta(B'_n/\overline{A_n})$. By our choice of A_n it now follows that $d(B/A) < \delta(B_n/\overline{A}) + 1/n$.Taking limits we see that $d(B/A) \leq \delta(\overline{AB}/\overline{A})$.

We now show that $\delta(\overline{AB}/\overline{A}) \leq d(B/A)$. The crux of the argument is similar to that of 6.2.11 and we will be terser with the details here. Consider a finite $A' \subseteq A$ and let $\langle C_n \rangle$ be an intrinsic chain with $C_0 = \overline{A'}B$ and $\bigcup_{n<\omega} C_n = \overline{\overline{A'}B} = \overline{A'B}$. Then

$$\begin{aligned}
\delta(C-n/\overline{A'}) &= \delta(C_n/\overline{A'}(\overline{A} \cap C_n)) + \delta(\overline{A'} \cap C_n/\overline{A}) \text{ by Lemma 6.2.5} \\
&\geq \delta(C_n/\overline{A'}(\overline{A} \cap C_n)) \text{ since } \overline{A'} \text{ is closed} \\
&\geq \delta(C_n/\overline{A}) \text{ by monotonocity of } \delta \\
&\geq \delta(B_k/\overline{A}) \text{ for sufficiently large } k \text{ by Lemma 6.2.2, Lemma 6.2.3} \\
&\qquad \text{and an argument similar to that of Lemma 6.2.11} \\
&\geq \delta(\overline{AB}/\overline{A})
\end{aligned}$$

Since $d(B/A) = \inf\{d(B/A') : A' \subseteq_{\text{Fin}} A\} = \inf\{\delta(\overline{A'B/A'}) : A' \subseteq_{\text{Fin}} A\}$, we obtain that $d(B/A) \geq \delta(\overline{AB}/A)$.

The last assertion follows as $\overline{BA} = \overline{(B - \overline{A})A}$. $\square$

We now show that $\mathop{\smile\!\!\!\!\!\!\,|}^{d}$ of algebraically closed sets is equivalent to a statement regarding algebraic closedness of their free join. We begin by extending the notion of $e(A, B, C)$

Definition 6.2.16. Let $X, Y, Z \subseteq \mathbb{M}$. We let $e(X, Y, Z) = \sup\{e(X, Y, Z) : X_0 \subseteq_{\text{Fin}} X, Y_0 \subseteq_{\text{Fin}} Y, Z_0 \subseteq_{\text{Fin}} Z\}$ allowing for the possibility that $e(X, Y, Z) = \infty$. Since for $X_0 \subseteq X_1 \subseteq_{\text{Fin}} X$, $Y_0 \subseteq Y_1 \subseteq_{\text{Fin}} Y, Z_0 \subseteq Z_1 \subseteq_{\text{Fin}} Z$ we have that $e(X_0, Y_0, Z_0) \leq e(A, X_1, Y_1, Z_0)$, the definition agrees for the finite case.

Remark 6.2.17. We may also extend the definition $e(X, Y) = \sup\{e(X_0, Y_0) : X_0 \subseteq_{\text{Fin}} X, Y_0 \subseteq_{\text{Fin}} Y\}$.

The following is a modified form of Theorem 3.14 of [4].

Theorem 6.2.18. *Let $A, B, C \subseteq \mathbb{M}$ with A, B where A, B are finite. The following statements are equivalent*

1. $A \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^{d}_{C} B$
2. $\overline{AC}$, $\overline{BC}$ *are freely joined over* $\overline{C}$ *and* $\overline{AB} \cup \overline{BC}$ *is closed in* $\mathbb{M}$

Proof. Note that both statements imply $\overline{AC} \cap \overline{BC} = \overline{C}$ and hence we can assume this equality. Further if $\overline{AC} \subseteq \overline{BC}$, then either $\overline{AC} \subseteq \overline{C}$ or $\overline{AC} \nsubseteq \overline{C}$. In the case $\overline{AC} \subseteq \overline{C}$, we obtain that $\overline{AC} = \overline{A}$ and the two statements hold vacuously: By definition, $\overline{C}$ is freely joined with $\overline{BC}$ over $\overline{C}$ and $0 = d(A/C) \geq d(A/BC) \geq 0$. In the case $\overline{AC} \nsubseteq \overline{C}$, we obtain that $\overline{C} \subsetneq \overline{AC} \cap \overline{BC}$, a contradiction. Thus we may assume that $\overline{AC} \nsubseteq \overline{BC}$.

Claim: $\overline{AC}$ is calculable over $\overline{BC}$ and $\delta(\overline{AC}/\overline{BC}) = \delta(\overline{AC}/\overline{C}) - e(\overline{ACC}, \overline{C}, \overline{BC} - \overline{C})$.

Proof of Claim: Since $\overline{AC}$ is calculable over $\overline{C}$, there is some intrinsic chain $\langle A_n \rangle_{n\omega}$ with $A_0 = \overline{A}C$ and $\bigcup_{n<\omega} A_n = \overline{AC}$. Note that (A_n, A_{n+1}) is a minimal pair. Put $A'_n = A_n \cup \overline{BC}$. For any A'_{n+1}, A'_n and A' such that $A'_n \subseteq A' \subsetneq A'_{n+1}$ we have that $A'_{n+1} - A' = A_{n+1} - A'$. Now $\delta(A'_{n+1}/A') = \delta(A_n/A')$. By the monotonocity of δ and the fact that $A_{n+1} - A_n$ is finite, we obtain that $\delta(A_n/A') \leq \delta(A_{n+1}/A_n(A_{n+1} \cap A'))$. But we may write $\delta(A_{n+1}/A_n) = \delta(A_{n+1}/A_n(A_{n+1} \cap A')) + \delta(A_n(A_{n+1} \cap A')/A_n)$ using Theorem 6.2.5 and thus $\delta(A_{n+1}/A_n) - \delta(A_n(A_{n+1} \cap A')/A_n) = \delta(A_{n+1}/A_n(A_{n+1} \cap A'))$. Note that $\delta(A_{n+1}/A_n) < 0$ and $\delta(A_n(A_{n+1} \cap A')/A_n) \geq 0$. Thus $\delta(A'_{n+1}/A') < 0$. In particular $\delta(A'_{n+1}/A'_n) < 0$.

On the other hand we may write $\delta(A'/A'_n) = \delta(A'' \cup \overline{BC}/A_n \cup \overline{BC})$ for some $A_n \subseteq A'' \subsetneq A_{n+1}$. As $\overline{AC} \nsubseteq \overline{BC}$, for sufficiently large n, $A_n \nsubseteq \overline{BC}$. Note that

$A'' \cup \overline{BC} - (A_n \cup \overline{BC}) = A'' - (A_n \cup \overline{BC})$. Now

$$\begin{aligned}
\delta(A'/A'_n) &= \delta(A'' \cup \overline{BC} - (A_n \cup \overline{BC})) - e(A'' \cup \overline{BC} - (A'_n \cup \overline{BC}), A_n \cup \overline{BC}) \\
&\geq \delta(A'' - (A_n \cup \overline{BC})) - e(A'' - (A_n \cup \overline{BC}), A_n) \\
&= \delta(A'' - (A_n \cup \overline{BC})/A_n) \text{ as } A'' \cup \overline{BC} - (A_n \cup \overline{BC}) \text{ is finite,} \\
&\quad \text{this is defined} \\
&= \delta(A'' - (A_n \cup \overline{BC})A_n/A_n) \\
&\geq 0 \text{ as } A_n \subseteq A'' - (A_n \cup \overline{BC})A_n \subsetneq A_{n+1}
\end{aligned}$$

Thus for any A'_n, A'_{n+1} with $A'_n \subsetneq A'_{n+1}$, (A'_n, A'_{n+1}) is a minimal pair. Further there is some N such that for all $n \geq N$, $A_{n'} - \overline{BC}$ is nonempty. Thus after suppressing the elements of $A'_{n+1} = A'_n$ we may extract an intrinsic chain $\langle A'_{n_k} \rangle_{k<\omega}$, $A''_0 = A\overline{BC}$ and $\bigcup_{n<\omega} A''_n = \overline{AC} \cup \overline{BC}$. Thus $\overline{AC}$ is calculable over $\overline{BC}$. Moreover, as $\overline{AC} \cap BC$, it follows that $A_n - \overline{BC} = A_n - \overline{C}$. Then

$$\begin{aligned}
\delta(\overline{AC}/\overline{BC}) &= \lim_n \delta(A'_n/\overline{BC}) \\
&= \lim_n \delta(A_n/\overline{BC}) \\
&= \lim_n \delta(A_n - \overline{BC}) - e(A_n - \overline{BC}, \overline{BC}) \\
&= \lim_n \delta(A_n - \overline{C}) - e(A_n - \overline{C}, \overline{BC}) \\
&= \lim_n \delta(A_n - \overline{C}) - [e(A_n - \overline{C}, \overline{C}) + e(A_n - \overline{C}, \overline{C}, \overline{BC} - \overline{C})] \\
&= \lim_n \delta(A_n/\overline{C}) - e(A_n - \overline{C}, \overline{C}, \overline{BC} - \overline{C}) \\
&= \lim_n \delta(A_n/\overline{C}) - \lim_n e(A_n - \overline{C}, \overline{C}, \overline{BC} - \overline{C}) \text{ the limit can be split as} \\
&\quad \delta(\overline{AC}/\overline{BC}) \text{ is finite as } \overline{BC} \text{ is calculable over } \overline{AC} \\
&= \delta(\overline{AC}/\overline{C})
\end{aligned}$$

Now as $\overline{AC} \cup \overline{BC}$ is almost closed by (2) of Lemma 6.2.3 since $\overline{AC}$ is calculable over $\overline{BC}$. As $\overline{ABC} = \overline{\overline{AC} \cup \overline{BC}}$, $\overline{ABC}$ is calculable by Lemma 6.1.15. Now by

Lemma 6.2.5 and Lemma 6.2.15.

$$\begin{aligned}
d(A/BC) &= \delta(\overline{ABC}/\overline{BC}) \\
&= \delta(\overline{AC}/\overline{BC}) + \delta(\overline{ABC}/\overline{AC} \cup \overline{BC}) \\
&= \delta(\overline{AC}/\overline{C}) - e(\overline{AC} - \overline{C}, \overline{C}, \overline{BC} - \overline{C}) + \delta(\overline{ABC}/\overline{AC} \cup \overline{BC}) \\
&= d(\overline{AC}/\overline{C}) - e(\overline{AC} - \overline{C}, \overline{C}, \overline{BC} - \overline{C}) + \delta(\overline{ABC}/\overline{AC} \cup \overline{BC})
\end{aligned}$$

Note that $e(\overline{AC}-\overline{C}, \overline{C}, \overline{BC}-\overline{C}) \geq 0$ and that $\overline{AC} \cup \overline{BC} \subseteq \overline{\overline{AC} \cup \overline{BC}} = \overline{ABC}$. As we can calculate $\delta(\overline{\overline{AC} \cup \overline{BC}}/\overline{AC} \cup \overline{BC})$ using an intrinsic chain D_n with $D_0 = \overline{AC} \cup \overline{BC}$, it follows that $\delta(\overline{ABC}/\overline{AC} \cup \overline{BC}) \leq 0$.

Thus $d(A/BC) = d(A/C)$ if and only if $e(\overline{AC} - \overline{C}, \overline{C}, \overline{BC} - \overline{C}) = 0 = \delta(\overline{ABC}/\overline{AC} \cup \overline{BC})$. But $e(\overline{AC} - \overline{C}, \overline{C}, \overline{BC} - \overline{C}) = 0 = \delta(\overline{ABC}/\overline{AC} \cup \overline{BC})$ if and only if $\overline{AC}, \overline{BC}$ is freely joined over $\overline{AC} \cup \overline{BC} = \overline{ABC}$ is strong in $\mathbb{M}$ (Note that we may construct $\overline{ABC}$ with a intrinsic chain $\langle D_n \rangle_{n<\omega}$ with $D_0 = \overline{AC} \cup \overline{BC}$ as $\overline{AC} \cup \overline{BC} \subseteq \overline{\overline{AC} \cup \overline{BC}} = \overline{ABC}$ and hence $\delta(\overline{ABC}/\overline{AC} \cup \overline{BC}) = 0$ implies that $\overline{AC} \cup \overline{BC} = \overline{ABC}$). $\square$

The following is a pared down version Corollary 3.15 of [4].

Theorem 6.2.19. *Let $X, Y \subseteq \mathbb{M}$ be closed and let $Z = X \cap Y$. The following are equivalent*

1. $X \mathop{\smile\!\!\!\!\!\!|}^{d}_{Z} Y$
2. *XY is closed and X, Y are freely joined over Z*

Proof. First assume that $X \mathop{\smile\!\!\!\!\!\!|}^{d}_{Z} Y$. Then for any finite $X_0 \subseteq X, Y_0 \subseteq Y$ we have that $X_0 \mathop{\smile\!\!\!\!\!\!|}^{d}_{Z} Y_0$ and hence X_0, Y_0 are freely joined over Z and $\overline{X_0 Z} \cup Y_0 Z$ is closed in $\mathbb{M}$. The required result now follows as (algebraic) closure is finitary.

For the converse assume that XY is closed and X, Y are freely joined over Z. Given any finite $X_0 \subseteq X$, $Y_0 \subseteq Y$, we immediately obtain that $\overline{X_0 Z}, \overline{Y_0 Z}$ are freely joined over Z. Thus we need to show that $\overline{X_0 Z} \cap \overline{Y_0 Z}$ is closed. Assume to the contrary that it is not. Then there is some finite $F \subseteq \overline{X_0 Y_0 Z}$ with $\delta(F/\delta(\overline{X_0 Z} \cup \overline{Y_0 Z})) < 0$. Since XY is closed, we obtain that $F \subseteq XY$. Take $F_X = F \cap XZ$ and $F_Y = F \cap YZ$. Since X, Y are freely joined over Z, we have that

$$\begin{aligned}
\delta(F/\delta(\overline{X_0 Z} \cup \overline{Y_0 Z})) &= \delta(F_X/\delta(\overline{X_0 Z} \cup \overline{Y_0 Z})) + \delta(FY/\delta(\overline{X_0 Z} \cup \overline{Y_0 Z})) \\
&= \delta(F_X/\delta(\overline{X_0 Z}) + \delta(FY/\delta(\overline{X_0 Z}) \\
&\geq 0
\end{aligned}$$

which yields a contradiction. $\square$

6.2.3 Non-forking and Free joins of Algebraically Closed Sets

We begin by exploring some consequences of non-forking. Recall that algebraically closed sets and intrinsically closed sets correspond. The following is Lemma 7.2 of [2]

Lemma 6.2.20. *For any cardinal κ, every algebraically closed set B, and every $\overline{c}$, there is a set $\{C_i : i \in \kappa\}$ of algebraically closed extensions of B that are pairwise isomorphic over B, $C_0 = acl(B\overline{c})$, and satisfy $\{C_i : i \in \kappa\}$ freely joined over B and $\{C_i : i \in \kappa\}$ is algebraically closed. In particular, $\{C_i : i \in \kappa\}$ is fully indiscernible over B. Moreover, if A is any set disjoint from C_0, then we may additionally assume that $A \cap C_i = \emptyset$ for each i.*

Proof. By compactness it suffices to show that for any finite $B_0 \subseteq B$ and any

$n, m \in \omega$ there are finite sets B_0' and $C^* = \bigcup\{C_i : i < n\}$ such that $B_0 \subseteq B_0' \subseteq B$, $C_0 = cl_m(B_0'\overline{c})$, C^* is the free join of n copies of C_0 over B_0 and C^* is m-closed.

So fix B_0, n, m as above. Let $C_0 = cl_m(B_0\overline{c})$ and let $B_0' = C_0 \cap B$. Since B is algebraically closed, $\mathfrak{B}'_0 \leq \mathfrak{C}_0$ (considered as finite structures in $K_{\overline{\alpha}}$). Let $\mathfrak{C}^* = \oplus_{i<n}\mathfrak{C}_i$ be the free join of n copies of $\mathfrak{C}_0$ over $\mathfrak{B}_0$. By 4 of Fact 2.2.5 we have $\mathfrak{B}_0' \leq \mathfrak{C}^*$, so Corollary 4.2.4 gives a strong embedding of $\mathfrak{C}^*$ over $\mathbb{M}$. It follows from (3) of Theorem 4.2.4 that the image of $\mathfrak{C}^*$ is algebraically closed, hence m-closed in $\mathbb{M}$. □

The following describes properties of non-forking in a manner that is more intrinsic to $S_{\overline{\alpha}}$. The proof we give here is due to Laskowski in [2] (see Proposition 7.3 of [2])

Lemma 6.2.21. *Let $X, Y, Z \subseteq \mathbb{M}$ be closed sets with $Z = X \cap Y$ and X, Y freely joined over Z. If $X \mathop{\smile\!\!\!\!\!\!\perp}_Z Y$, then X, Y are freely joined over Z and XY is closed.*

Proof. We begin by showing that if $\{X, Y\}$ is not a free join over Z, then $tp(X/Y)$ contains a formula that divides over Z. Suppose that there are $\overline{a} \subseteq X - Z$, $\overline{b} \subseteq Z$, $\overline{c} \subseteq Y - Z$, and $E \in L$ such that $E(\overline{a}, \overline{b}, \overline{c})$ with $\overline{a}, \overline{c}$ non-empty. By Lemma 6.2.20 choose $\{C_i : i \in \omega\}$ freely joined and indiscernible over Z with $C_0 = \text{acl}(B\overline{a})$ and $\{C_i : i \in \omega\}$ algebraically closed. For each $i \in \omega$ fix an isomorphism $f_i : C_0 \to C_i$ over B and let $\overline{c}_i = f(\overline{c})$. Fix $m > |\overline{a}|$ and let C_i^m denote the m-closure of $\overline{b}\overline{c}_i$ in C_i. We wish to show that $\varphi(\overline{x}, \overline{b}, \overline{c}) := E(\overline{x}, \overline{b}, \overline{c}) \wedge_j x_j \notin C_0^m$ divides over Z. Clearly the $\overline{c}_i$ all realize $\text{tp}(\overline{c}/Z)$. If $\{\varphi(\overline{x}, \overline{b}, \overline{c}_i) : i \in \omega\}$ is not k-inconsistent for all $k \in \omega$, then $\{\varphi(\overline{x}, \overline{b}, \overline{c}_i) : i \in \omega\}$ (where $\varphi(\overline{x}, \overline{b}, \overline{c}_i) = E(\overline{x}, \overline{b}, \overline{c_i}) \wedge_j x_j \notin C_i^m$) would be consistent.

Assume that some $\overline{a'}$ from realizes $\{\varphi(\overline{x}, \overline{b}, \overline{c}_i) : i \in \omega\}$. Choose n large enough such that $n\alpha_E > |\overline{a'}|$ and let $D_n = \bigcup\{C_i^m : i < n\}$. Since $E(\overline{a'}, \overline{b}, \overline{c_i})$ holds for all $i < n$, $\delta(D_n\overline{c_i}/D_n)|\overline{a'}| - n\alpha < 0$, which contradicts D_n being m-closed.

Finally, assume that X, Y are freely joined over Z, but XY is not algebraically closed. Choose $\overline{a} \subseteq X - Z$, $\overline{b} \subseteq Z$, $\overline{c} \subseteq Y - Z$ and $\overline{d}$ disjoint from XY, so that letting D be the substructure with universe $\overline{a}\overline{b}\overline{c}\overline{d}$, $(\overline{a}\overline{b}\overline{c}, D)$ is a minimal pair. Since X is closed, we may assume that for at least one $E \in L$, at least one element of E^D contains at least one element of $\overline{c}$. Let $m > |D|$, choose n such that $nGr(m) > |\overline{a}|$, and let $m > n|D|$. By Lemma 6.2.20 choose $\{X_i : i \in \omega\}$ to be fully indiscernible and freely joined over Z with $\{C_i : i \in \omega\}$ algebraically closed with $C_0 = \mathrm{acl}(X\overline{c})$. Let $C_0^{m^*} = cl_{m^*}(\overline{b}\overline{c})$ and let $C_i^{m^*} = f_i(C_0^{m^*})$, where $f_i : C_0 \to C_i$ is an isomorphism over Z. Let $D_n = \bigcup\{C_i^{m^*} : i < n\}$. We argue that $\varphi(\overline{x}, \overline{b}, \overline{c}) := \exists\overline{z}[\Delta_{\mathfrak{G}}(\overline{x}, \overline{b}, \overline{c}, \overline{z}) \wedge \bigwedge_j x_j \notin C_0^{m^*}]$ divides over Z. Clearly the $\overline{c_i}$ realize $\mathrm{tp}(\overline{c}/A)$. If $\{\varphi(\overline{x}, \overline{b}, \overline{c}_i) : i \in \omega\}$ is not k-inconsistent for all $k \in \omega$, then $\{\varphi(\overline{x}, \overline{b}, \overline{c}_i) : i \in \omega\}$ is consistent. Assume that $\{\varphi(\overline{x}, \overline{b}, \overline{c}_i) : i \in \omega\}$ is not k-inconsistent for all $k \in \omega$. There is some $\overline{a'}$ such that $\varphi(\overline{a'}, \overline{b}, \overline{c_i})$ holds for all $i \in \omega$. Now for each $i \in \omega$ we may choose d_i such that $\Delta_G(\overline{a'}, \overline{b}, \overline{c_i}, \overline{d_i})$. Apply the -system lemma to $\{di : i \in \omega\}$. There are now two cases:

Case 1: For infinitely many i, $\overline{d_i} = \overline{d}^*$for some fixed d^*. In this case, arguing as above we can show that $\delta(\overline{a'}\overline{d^*}D_n/D_n) < 0$, contradicting the fact that D_n is m-strong in $\mathbb{M}$.

Case 2: For infinitely many i, $\overline{d_i} = \overline{d^*}^\frown\overline{e_i}$ for some $\overline{d^*}$ and some pairwise disjoint $\{\overline{e_i} : i \in \omega\}$. For each $l \leq n$, let F_l denote the substructure with universe $D_n\overline{a'} \cup \{\overline{d_i} : i < l\}$. Since $(\overline{a}\overline{b}\overline{c}, D)$ is a minimal pair and the $\overline{e_i}$ are pairwise disjoint, it follows from Lemma 4.1.4 that $\delta(D_{l+1}/\delta(l)) \leq Gr(m)$ for each $l < n$, so $\delta(F_n/D_n) \leq |\overline{a'}|nGr(m) < 0$, contradicting the fact that D_n was m-strong in $\mathbb{M}$. $\square$

We now work towards establishing a converse for the above lemma.

Lemma 6.2.22. *Let $X, Y, Z \subseteq \mathbb{M}$ be closed sets with $Z = X \cap Y$ and X, Y freely joined over Z. For any automorphism of $\mathbb{M}$ that fixes Z such that $f(X)Y$ is closed and $f(X), Y$ are freely joined over Z, tp$(XY/Y) =$ tp$(f(X)Y/Y)$.*

Proof. Note that under the given conditions XY and $f(X)Y$ have the same quantifier free type. From Lemma 4.4.3, we obtain that $\mathrm{tp}(f(X)Y) = \mathrm{tp}(XY)$. The result now follows. $\square$

Lemma 6.2.23. *Let $X, Y, Z \subseteq \mathbb{M}$ be closed sets with $Z = X \cap Y$ and X, Y freely joined over Z. Then $X \mathop{\smile\!\!\!\!\!\!|}_Z Y$*

Proof. By general properties of nonforking (see for example [28]), there is some auotmorphism of f fixing Z such that $f(X) \mathop{\smile\!\!\!\!\!\!|}_Z Y$. By Lemma 6.2.21 we obtain that $f(X), Y$ is freely joined over Z and $f(X)Y$ is closed. Further by Lemma 6.2.22, we have that $\mathrm{tp}(XY/Y) = \mathrm{tp}(f(X)Y/Y)$. Since nonforking is automorphism invariant it follows that $X \mathop{\smile\!\!\!\!\!\!|}_Z Y$. $\square$

We collect the results to obtain:

Theorem 6.2.24. *Let $X, Y \subseteq \mathbb{M}$ be closed and let $Z = X \cap Y$. The following are equivalent*

1. $X \mathop{\smile\!\!\!\!\!\!|}_Z Y$

2. *XY is closed and X, Y are freely joined over Z*

Proof. The proof is immediate from the lemmas 6.2.21, 6.2.23. $\square$

6.2.4 Characterization of non-forking and weak elimination of imaginaries

We now come to one of the main results for this section. It characterizes non-forking in terms of d-independence and free joins of algebraically closed sets. It should be noted that different a proof of the following, bypassing the quantifier elimination result and using various amalgamation properties is possible (it may be viewed as an amalgamation of Corollary 3.22, Fact 5.1 of [4] and Lemma 4.4 of [1]).

Theorem 6.2.25. *Let X, Y and hence $Z = X \cap Y$ be closed. Then the following are equivalent:*

1. *$X \mathop{\smile\!\!\!\!\!\!|}_Z Y$*

2. *$X \mathop{\smile\!\!\!\!\!\!|}^d_Z Y$*

3. *X, Y are freely joined over Z and XY is closed.*

Proof. This follows by Theorem 6.2.19 and Theorem 6.2.24. $\square$

The following result appears in [2]. A stronger form of this result is possible (see Proposition 4.2 of [4]).

Definition 6.2.26. Recall that a type $p \in S(X)$ is *stationary* if it has a unique non-forking extension (to an ideal type, i.e. a type over $\mathbb{M}$). Given a stationary type

p and $Y \subseteq \mathbb{M}$, we let $p|_Y$ denotes the unique nonforking extension of p restricted to Y.

Lemma 6.2.27. *The theory $S_{\overline{\alpha}}$ has weak elimination of imaginaries, i.e. every complete type over an algebraically closed set in the home sort is stationary.*

Proof. Fix an algebraically closed X and a type $p \in S(X)$. To prove that p is stationary, it suffices to show that it has a unique nonforking extension to any algebraically closed $Y \supseteq X$. Fix such a Y and choose a, a' realizing p such that neither $\text{tp}(a/Y)$ and $\text{tp}(a'/Y)$ fork over X. Let $X' = \text{acl}(Xa)$ and $X'' = \text{acl}(Xa')$. Since $\text{tp}(a/X) = \text{tp}(a'/X)$, Lemma 4.4.3 gives an isomorphism $f : X' \to X''$ over Z such that $f(a) = a'$. Since neither $tp(X'/Y)$ and $tp(X''/Y)$ fork over X, it follows from Lemma 6.2.21 that $X'Y$,$X''Y$ are algebraically closed over and $\{X', Y\}$, $\{X'', Y\}$. Now Lemma 6.2.22 yields that $\text{tp}(X'Y) = \text{tp}(X''Y)$ and hence $\text{tp}(X'Y) = \text{tp}(X''Y)$. $\square$

We make the following useful observation:

Remark 6.2.28. Let $A \leq \mathbb{M}$ and let $p \in S(A)$. Suppose that for some $k \in \omega$, $d(p/A) = k/c$. Let $A \subseteq X \leq \mathbb{M}$. Suppose that $q \in S(X)$ extends p. If $d(q/X) < d(p/A)$, then q is a forking extension of p follows easily by theorem 6.2.25. Further p, q are stationary.

6.3 Non-triviality

In this section we establish that in the case $\overline{\alpha}$ is not graph-like with weight one, $S_{\overline{\alpha}}$ is trivial. We refer the reader to [31] for a discussion of triviality.

Definition 6.3.1. A stable theory T is said to be *trivial* if for any given $A, B, C, X \subseteq \mathbb{M}'$ (where $\mathbb{M}'$ is a monster model of T), with $A \downarrow_X B$, $A \downarrow_X C$ and $B \downarrow_X C$ we have that $A \downarrow_X BC$.

We begin with the following lemma:

Lemma 6.3.2. *Assume the $\overline{\alpha}$ is not graph-like with weight one. Let $\mathfrak{A}, \mathfrak{C}_1 \ldots \mathfrak{C}_n \in K_{\overline{\alpha}}$ be such that $\mathfrak{A} \leq \mathfrak{C}_i$ and $\delta(\mathfrak{C}_i/\mathfrak{A}) > 0$ for each $1 \leq i \leq n$. Let $\gamma = \min\{\delta(\mathfrak{C}_i/A) : 1 \leq i \leq n\}$ and let $\mathfrak{C} = \oplus \mathfrak{C}_i$ be the free join of the $\mathfrak{C}_i$ over $\mathfrak{A}$. Then there is an essential minimal pair $\mathfrak{C}, \mathfrak{D}$ such that $\mathfrak{A}, \mathfrak{C}_1 \ldots \mathfrak{C}_n \leq \mathfrak{D}$ and $-\gamma \leq \delta(\mathfrak{D}/\mathfrak{C}) < 0$.*

Proof. Note that under the given conditions, we can apply Theorem 3.2.15 to obtain a $\mathfrak{D} \in K_{\overline{\alpha}}$ with $(\mathfrak{C}, \mathfrak{D})$ an essential minimal pair satisfying $-\gamma \leq \delta(\mathfrak{D}/\mathfrak{C}) < 0$.

We claim that for any $\Phi_0 \subsetneq \{1, \ldots, n\}$, $\mathfrak{C}_{\Phi_0} = \oplus_{i \in \Phi_0} \mathfrak{C}_i \leq \mathfrak{D}$. If $\Phi_0 = \emptyset$, then $\mathfrak{C}_\emptyset = \mathfrak{A}$ and the claim will follow if we establish this result for even one $\Phi_0 \neq \emptyset$ as $\leq$ is transitive. So assume that $\Phi_0 \neq \emptyset$. Consider $\mathfrak{C}_{\Phi_0} \subseteq \mathfrak{D}' \subseteq \mathfrak{D}$. Now $\delta(\mathfrak{D}'/\mathfrak{C}_{\Phi_0}) = \delta(\mathfrak{D}'/\mathfrak{D}' \cap \mathfrak{C}) + \delta(\mathfrak{D}' \cap \mathfrak{C}/\mathfrak{C}_{\Phi_0})$. But $\delta(\mathfrak{D}' \cap \mathfrak{C}/\mathfrak{C}_{\Phi_0}) \geq 0$ as the $\mathfrak{C}_i$ are freely joined over $\mathfrak{A}$. Thus we need to consider $\delta(\mathfrak{D}'/\mathfrak{D}' \cap \mathfrak{C}) < 0$. Now $\delta(\mathfrak{D}'/\mathfrak{D}' \cap \mathfrak{C}) < 0$ if and only if $\mathfrak{D}' = \mathfrak{D}$ as $(\mathfrak{C}, \mathfrak{D})$ is an essential minimal pair with $-\gamma \leq \delta(\mathfrak{D}/\mathfrak{C}) < 0$. But then $\delta(\mathfrak{D}/\mathfrak{C}_{\Phi_0}) = \delta(\mathfrak{D}/\mathfrak{C}) + \delta(\mathfrak{C}/\mathfrak{C}_{\Phi_0})$. But $\delta(\mathfrak{C}/\mathfrak{C}_{\Phi_0}) \geq \gamma$ and $\delta(\mathfrak{D}/\mathfrak{C}) \geq -\gamma$ and hence it follows that $\delta(\mathfrak{D}/\mathfrak{C}_{\Phi_0}) \geq 0$ from which our claim follows. $\square$

Theorem 6.3.3. *Assume the $\overline{\alpha}$ is not graph-like with weight one. Then $S_{\overline{\alpha}}$ is not trivial.*

Proof. Take $\mathfrak{A} = \emptyset$ and $\mathfrak{C}_i \in K_{\overline{\alpha}}$ such that $\delta(\mathfrak{C}_i) > 0$ for $i = 1, 2, 3$. Note that $\emptyset \leq \mathfrak{C}_i$. Let $\mathfrak{D}$ be an essential minimal pair obtained by the use of Lemma 6.3.2.

Let $f : \mathfrak{D} \to \mathbb{M}$ be a strong embedding. Now as the $\mathfrak{C}_i$ are freely joined over $\emptyset$ and for a fixed $\Phi_0 \subsetneq \{1, 2, 3\}$, $f(\oplus_{i \in \Phi_0} \mathfrak{C}_i) \leq \mathbb{M}$ (using the transitivity of $\leq$ and $\oplus_{i \in \Phi_0} \mathfrak{C}_i \leq \mathfrak{D}$) it follows from the characterization of forking in Theorem 6.2.25 that $f(\mathfrak{C}_i) \perp_{\emptyset} f(\mathfrak{C}_j)$. However $f(\mathfrak{C}_1\mathfrak{C}_2) \not\perp_{\emptyset} f(\mathfrak{C}_3)$ as $\delta(f(\mathfrak{D})/f(\oplus_{i \in \Phi} \mathfrak{C}_i)) < 0$ which establishes that $S_{\overline{\alpha}}$ is not trivial. □

6.4 Strict stability for non-rational $\overline{\alpha}$ and the Dimensional Order Property

In this section we give a proof of the fact that $S_{\overline{\alpha}}$ is superstable for non-rational α. We also take this opportunity to give a proof of the fact that $S_{\overline{\alpha}}$ has *Dimensional Order Property* (DOP, see [30], or [29] for a definition) for $\overline{\alpha}$ *not* graph-like with weight one. In Chapter 8 we will extend the result to the case that $\overline{\alpha}$ is graph-like with weight one, thus establishing the (long expected) result for all $\overline{\alpha}$. The result has been known for special cases. In [5], Baldwin and Shelah gave a proof that $S_{\overline{\alpha}}$ has DOP *assuming* that L has a binary relation. In Corollary 7.10 of [2], Laskowski gave a proof of DOP by explicitly constructing a type that witnesses the DOP. He did not assume that L contained a binary symbol, however he did assume $\overline{\alpha}$ satisfied certain properties. Finally we show that for non-rational $\overline{\alpha}$, $|S(\emptyset)| = 2^{\aleph_0}$.

Before we prove the results mentioned above, we observe the following consequence of the stated results as it relates to the *spectrum* of $S_{\overline{\alpha}}$.

Definition 6.4.1. Let $I(\kappa)$ denote the number of non-isomorphic models of $S_{\overline{\alpha}}$ of size κ. The function I is called the *spectrum of* $S_{\overline{\alpha}}$,

Remark 6.4.2. Note that DOP implies that $I(\kappa) = 2^\kappa$ for any $\kappa \geq \aleph_1$ (see [30] or [29]). In the case that $\overline{\alpha}$ is not rational, the fact that $S(\emptyset) = 2^{\aleph_0}$ will yield that $I(\kappa) = 2^\kappa$ for all infnite κ. The case where $\overline{\alpha}$ is rational is discussed in Chapter 7 (for the case $\overline{\alpha}$ is rational but not graph-like with weight one) and Chapter 8 (for the case $\overline{\alpha}$ is graph-like with weight one).

6.4.1 Strict Stability of $S_{\overline{\alpha}}$ for non-rational $\overline{\alpha}$

We now start proving the results mentioned at the beginning of the section. The following argument is essentially due to Laskowski (see Proposition 7.6 of [2], which has the same result). He states that the proof is based on ideas from Ikeda in [25].

Theorem 6.4.3. *If $\overline{\alpha}$ is not rational, then $S_{\overline{\alpha}}$ is strictly stable, i.e. $S_{\overline{\alpha}}$ is stable but not superstable*

Proof. Choose $a \in \mathbb{M}$ such that $\operatorname{acl}(\{a\}) = \{a\}$, let $B_0 = \emptyset$ and let $D_0 = \{a\}$. We will produce a nested sequence $\langle B_n : n \in \omega \rangle$ of finite substructures of $\mathbb{M}$ such that $a \not\downarrow_{B_n} B_{n+1}$ for all $n \in \omega$ using the fact that if $\overline{\alpha}$ is not rational, then there is some $E \in L$ for which $\overline{\alpha}(E)$ is irrational.

To accomplish this, we also construct an ancillary sequence $\langle D_n \rangle$ of finite substructures of $\mathbb{M}$ such that:

1. $\{a\} \cup \bigcup\{B_n : n \in \omega\}$ is discrete (i.e., no E'-relations hold among any subset for any $E' \in L$).

2. $\delta(B_n) < \delta(D_n)$

3. $B_n \leq B_{n+1}$, $D_n \leq D_{n+1}$ and $B_n \leq D_n$.

4. $B_n = \mathrm{acl}(B_n), D_n = \mathrm{acl}(\{a\}B_n)$, but $D_{n+1} \neq D_n \cup B_{n+1}$

It follows from the characterization of forking given in Theorem 6.2.25 that these conditions imply $a \not\downarrow_{B_n} B_{n+1}$ for each $n \in \omega$, so it suffices to perform the construction.

Assume that B_n and D_n have been defined and satisfy the conditions. Choose $\mathfrak{B}_{n+1} \in K_{\overline{\alpha}}$ (not necessarily in $\mathbb{M}$ but is isomorphic to B_n) such that $B_{n+1} = B_n \cup \{b_n\}$, B_{n+1} is discrete, and $\mathfrak{B}_{n+1}, \mathfrak{D}_n$ are disjoint over $\mathfrak{B}_n$ (here $\mathfrak{D}_n$ is isomorphic to D_n). Let $\mathfrak{F}$ denote the free join of and $\mathfrak{B}_{n+1}, \mathfrak{D}_n$ over $\mathfrak{B}_n$. We apply (1) of Theorem 3.2.15 (with the relation symbol E) to obtain $\mathfrak{D}_{n+1}$ for $\mathfrak{F}$ and $\epsilon = \delta(\mathfrak{D}_n/\mathfrak{B}_n)$, i.e. $(\mathfrak{F}, \mathfrak{D}_{n+1})$ is an essential minimal pair with $-\epsilon < \delta(\mathfrak{D}_{n+1}/\mathfrak{F}) < 0$. Clearly $\mathfrak{D}_n \leq \mathfrak{D}_{n+1}$ and $\mathfrak{B}_n \leq \mathfrak{B}_{n+1}$. Further as $\{a\} \leq \mathfrak{D}_1$, it follows that $\{a\} \leq \mathfrak{D}_{n+1}$ by the transitivity of $\leq$.

We now have to show that $\mathfrak{B}_{n+1} \leq \mathfrak{D}_{n+1}$ and that $\delta(\mathfrak{D}_{n+1}/\mathfrak{B}_{n+1}) > 0$. Let $\mathfrak{B}_{n+1} \subseteq \mathfrak{B}' \subseteq \mathfrak{D}_{n+1}$. Note that as $\mathfrak{B}_{n+1}, \mathfrak{D}_n$ are freely joined over $\mathfrak{B}_n$, $\mathfrak{B}_{n+1} \leq \mathfrak{F}$. Now $\delta(\mathfrak{B}'/\mathfrak{B}_{n+1}) = \delta(\mathfrak{B}'/\mathfrak{F} \cap \mathfrak{B}') + \delta(\mathfrak{F} \cap \mathfrak{B}'/\mathfrak{B}_{n+1})$. As $(\mathfrak{F}, \mathfrak{D}_{n+1})$ is an essential minimal pair and $\mathfrak{B}_{n+1} \leq \mathfrak{F}$, $\delta(\mathfrak{B}'/\mathfrak{F} \cap \mathfrak{B}'), \delta(\mathfrak{F} \cap \mathfrak{B}'/\mathfrak{B}_{n+1}) > 0$ for $\mathfrak{B}' \neq \mathfrak{D}_{n+1}$. Further if $\mathfrak{B}' = \mathfrak{D}_{n+1}$, then $\delta(\mathfrak{B}'/\mathfrak{F}) + \delta(\mathfrak{F}/\mathfrak{B}_{n+1}) > 0$ by our choice of $\mathfrak{D}_{n+1}$. Note that the inequalities are strict as the only relation symbol that appears in $\mathfrak{D}_{n+1}$ is E (and hence $\overline{\alpha}(E)$ is irrational).

Now apply Corollary 4.2.4 to get a strong embedding $g : \mathfrak{D}_{n+1} \to \mathbb{M}$ over D_n. The rest now follows. $\square$

6.4.2 The Dimensional Order Property

We now turn our attention towards showing that $S_{\overline{\alpha}}$ has the *Dimensional Order Property* (DOP). Our approach will be to combine a sufficient condition for $S_{\overline{\alpha}}$ to have the DOP given by Baldwin and Shelah in [5] with a generalization of an argument given by Laskowski in [2] to obtain the required result. We refer the reader to Chapter XVI of [30] for the definition of DOP and related facts. We begin by extending the function d to the space of types.

Definition 6.4.4. Let $X \subseteq \mathbb{M}$ and let $p \in S(X)$. We let $d(p/X) = d(\overline{b}/X)$ for some (equivalently any) realization $\overline{b}$ of p.

We begin by showing that partial types whose realizations have a d-value of 0 are complete.

Lemma 6.4.5. *Let $A \leq \mathbb{M}$ and π be a partial type over A. Suppose that any realization of π has the same quantifier free type over A. If for any $\overline{b} \models \pi$, $d(\overline{b}/A) = 0$, then π is complete.*

Proof. Let $\overline{b} \models \pi$. Since $d(\overline{b}/A) = 0$, π implies that there is an intrinsic minimal chain $\langle D_n : n < \omega \rangle$ with $D_0 = A\overline{b}$ such that $\lim_n \delta(D_n/A) = 0$. Let $X = \bigcup_{n<\omega} D_n$ (note that X may not be finite). We claim that X is algebraically closed. Suppose to the contrary that it is not. Then there is some finite C, disjoint from X, such that $\delta(C/X) < 0$. Now consider $D'_n = CD_n$. Note that for sufficiently large D'_n, $\delta(D'_n/A) = \delta(D'_n/D_n) + \delta(D_n/A) < 0$ which contradicts that A is closed. As $\delta(D_n) < \delta(D_{n+1})$, it follows that X is the smallest algebraically closed set that contains $A\bar{b}$. Thus π fully determines the algebraic closure of any of its realizations and thus π is complete. $\square$

In [5] Baldwin and Shelah gave the sufficient condition below to show that $S_{\overline{\alpha}}$ has the DOP (see Theorem 2.8 of [5] for details).

Lemma 6.4.6. *If there are independent points a, b in $\mathbb{M}$ and some $p \in S(\{a, b\})$ such that $d(p/\{a, b\}) = 0$ but $d(p/\{a\}) > 0$ and $d(p/\{b\}) > 0$, then $S_{\overline{\alpha}}$ has the DOP.*

Following Laskowski's approach in [2], we now construct a type that will witnesses the DOP. The following lemma (Proposition 7.8 of [2]) shows that types of dimension 0 occur in abundance.

Lemma 6.4.7. *For any $\mathfrak{A} \leq \mathfrak{B}$ from $K_{\overline{\alpha}}$ with $\delta(\mathfrak{A}) \neq \delta(\mathfrak{B})$ there is an isomorphic embedding f of $\mathfrak{B}$ into $\mathbb{M}$ such that, taking $A' = f(A)$ and an enumeration $\overline{e}$ of $f(B) - f(A)$, $d(p/A') = 0$, where $p = tp(\overline{e}/A')$.*

Proof. There are two cases to consider. First assume that $\overline{\alpha}$ is *not* graph-like with weight one. Under the given conditions we can repeatedly apply Lemma 4.2.1 to obtain a sequence $\langle \mathfrak{D}_n : n \in \omega \rangle$ of elements of $K_{\overline{\alpha}}$ such that

1. $\mathfrak{D}_0 = \mathfrak{B}$
2. $\mathfrak{D}_n \subseteq \mathfrak{D}_{n+1}$ for all $n \in \omega$

3. $\mathfrak{A} \leq \mathfrak{D}_n$ for all $n \in \omega$

4. $\delta(\mathfrak{D}_n/\mathfrak{A}) < 1/n$ for all $n \geq 1$

Note that if $\overline{\alpha}$ is not rational, we may strengthen condition (4) to $0 < \delta(\mathfrak{D}_n/\mathfrak{A}) < 1/n$ for all $n \geq 1$. Now, given such a sequence, let $X = \bigcup\{\mathfrak{D}_n : n \in \omega\}$, there is an embedding $f : X \to \mathbb{M}$ such that $\mathrm{acl}(X) = X$.

Let $A' = f(A)$, $\overline{e}$ enumerate $f(B) - f(A)$, and let $p = \mathrm{tp}(\overline{e}/A')$. Note that by definition it is immediate that for any $A^* \subseteq_{\mathrm{Fin}} A'$, $d(\overline{e}/A^*) \geq 0$. Thus it follows that $d(\overline{e}/A') \geq 0$ and hence $d(p/A') \geq 0$. Since $\delta(f(D_n)/A') < 1/n$ for each n, it follows that $d(p/A) < 1/n$ for all integers $n \geq 1$. It follows that $d(p/A) = 0$.

The proof in case that $\overline{\alpha}$ is graph-like with weight one is similar. However we need to use Lemma 8.2.6 to obtain the sequence $\langle \mathfrak{D}_m : n < \omega \rangle$.

Note that in the case that $\overline{\alpha}$ is rational, there is some $N \in \omega$ such that $\delta(\mathfrak{D}_n) = \delta(\mathfrak{D}_m)$ for all $m, k \geq N$ and as such implies that the associated type is isolated by (2) of Lemma 4.4.2. □

We now establish that $S_{\overline{\alpha}}$ has the DOP. It follows the argument given by Laskowski in [2]

Theorem 6.4.8. *$S_{\overline{\alpha}}$ has the DOP.*

Proof. Since the notion of DOP is invariant under the addition of finitely many constants to the language, we may do so and reduce to the case where we have some distinguished relation symbol E of arity 2. To simplify notation let $\alpha = \alpha(E)$. Let $\mathfrak{B}$ be the L-structure whose universe has four points $\{a, b, x, y\}$, with the sets

$\{a, x\}, \{b, y\} \in E^{\mathfrak{B}}$ and no other relations, and let $\mathfrak{A}$ be the substructure of $\mathfrak{B}$ with universe $A = \{a, b\}$. It is easily seen that $\mathfrak{A} \leq \mathfrak{B}$. Apply Lemma 6.4.7 and get an embedding f of $\mathfrak{A}$ into $\mathbb{M}$ (as notation let $A' = \{a', b'\} = \{f(a), f(b)\}$) and a type $p(x, y) \in S(A')$ such that $\{e_1, e_2\} \cup A' \cong \mathfrak{B}$ over A' and $d(\overline{e}/A') = 0$ for any $\overline{e} = (e_1, e_2)$ realizing p. Since extensions of nonnegative dimension occur in abundance, p is nonalgebraic (this follows from Lemma 6.2.20). Now fix any $\overline{e} = (e_1, e_2)$ realizing p. We will finish the proof by showing that $d(\overline{e}/a') \geq \alpha$ (the argument for showing that $d(\overline{e}/b') > 0$ is symmetric). Choose any finite $F \subseteq \mathbb{M}$ such that $\overline{e}a' \subseteq F$. Since $\delta(\{a'\}) = 1$, it suffices to show that $\delta(F) \geq 1 + \alpha$. To accomplish this, consider the substructure with universe Fb' . On one hand, since $\{e_2, b'\} \in E$, $\delta(Fb'/F) \leq 1-\alpha$. On the other hand, since $Fb' \supseteq \overline{e}A'$ and $d(p/A') \geq 0$, $(Fb/A') \geq 0$. As $\delta(A) = 2$, this implies $\delta(Fb) \geq 2$. Since $\delta(Fb') = \delta(Fb/F) + \delta(F)$ we obtain that $\delta(F) = (Fb') - \delta(Fb'/F) \geq 2 - (1 - \alpha) = 1 + \alpha$ and we finish. $\square$

The following is Remark 7.9 of [2]. It is a consequence of Lemma 6.4.7 and it shows that $|S(\emptyset)| = 2^{\aleph_0}$ when $\overline{\alpha}$ is not rational.

Remark 6.4.9. Assume that $\overline{\alpha}$ is not rational. The reader should note that by choosing appropriate finite sets Φ in our application of Lemma 4.2.1 in the proof of Lemma 6.4.7, we can inductively construct a perfect tree of types of dimension zero. More precisely, for any $\mathfrak{A} \in K_{\overline{\alpha}}$ there is a family $\{f_\eta : \eta \in {}^{\omega}2\}$ of isomorphisms taking $\mathfrak{A}$ into $\mathbb{M}$ and a family $\{p_\eta \in S(f_\eta(A)) : \eta \in {}^{\omega}2\}$ of complete types of dimension zero over their base, both indexed by a perfect tree, that are not conjugate, i.e., $f_\eta(f_\mu^{-1}(p_\mu)) \neq p_\eta$ for $\eta \neq \mu$. Note that the condition $\overline{\alpha}$ is not rational is required:

If $\overline{\alpha}$ is rational then for sufficiently large n, $\delta(\mathfrak{A}) = \delta(\mathfrak{D}_n)$ and as such implies that the associated type is isolated by (2) of Lemma 4.4.2.

Chapter 7: Rational $\overline{\alpha}$ and the corresponding $S_{\overline{\alpha}}$

In this chapter we study $S_{\overline{\alpha}}$ when $\overline{\alpha}$ *is rational* focusing on the case that $\overline{\alpha}$ is *not* graph-like with weight one. We start by studying the countable models of $S_{\overline{\alpha}}$, the spectrum already being determined for uncountable κ by the DOP and the countable models in the case that α is not rational by the work of Laskowski (see Remark 6.4.2). We begin by defining a notion of dimension for (countable) models. We then show that this notion of dimension is able to categorize countable models up to both isomorphism and elementary embeddability. We then focus our attention on the regular types that occur in relation to these theories and classify large classes of types as either regular or non-regular. We end this chapter by answering a question of Pillay's (see [6]) in the negative by providing examples of a pseudofinite ω-stable theories with non-locally modular regular types.

Recall that c is the least common multiple of the denominators of the $\overline{\alpha}_E$ (in reduced form) and note that $S_{\overline{\alpha}}$ has finite closures.

Lemma 7.0.1. *Let $k \in \omega$. Given any $\mathfrak{B} \in K_{\overline{\alpha}}$, there is some $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $\mathfrak{D} \supseteq \mathfrak{B}$, $\delta(\mathfrak{D}) = k/c$ and for any $\mathfrak{A} \leq \mathfrak{B}$ with $\delta(\mathfrak{A}) \leq k/c$, $\mathfrak{A} \leq \mathfrak{D}$.*

Proof. Given $\mathfrak{B}$ take $\mathfrak{D}_0$ to be the free join of $\mathfrak{B}$ with a structure with $k+1$ many points with no relations among them over $\emptyset$. Note that $\mathfrak{B} \leq \mathfrak{D}_0$. Let $l = c\delta(\mathfrak{D}_0) - k$.

Consider a sequence $\mathfrak{D}_0 \subseteq \ldots \subseteq \mathfrak{D}_l$ where each $(\mathfrak{D}_i, \mathfrak{D}_{i+1})$ is an essential minimal pair with $\delta(\mathfrak{D}_{i+1}/\mathfrak{D}_1) = -1/c$. We claim that $\mathfrak{D} = \mathfrak{D}_l$ is as required. Fix any $\mathfrak{A} \leq \mathfrak{B}$ with $\delta(\mathfrak{A}) \leq k/c$. We show by induction on $i < l$ that if $\mathfrak{A} \leq \mathfrak{D}_i$, then $\mathfrak{A} \leq \mathfrak{D}_{i+1}$. Clearly $\mathfrak{A} \leq \mathfrak{D}_0$ as $\mathfrak{A} \leq \mathfrak{B} \leq \mathfrak{D}_0$. Fix $i < l$ and consider any $\mathfrak{F}$ such that $\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{D}_{i+1}$. If $\mathfrak{F} = \mathfrak{D}_{i+1}$ then $\delta(\mathfrak{F}) \geq k/c \geq \delta(\mathfrak{A})$ and so $\delta(\mathfrak{F}/\mathfrak{A}) \geq 0$. On the other hand, if $\mathfrak{F} \neq \mathfrak{D}_{i+1}$, then, $\delta(\mathfrak{F}/\mathfrak{D}_{i+1} \cap \mathfrak{F})$ since $(\mathfrak{D}_i, \mathfrak{D}_{i+1})$ is an essential minimal pair and $\delta(\mathfrak{D}_i \cap \mathfrak{F}/\mathfrak{A}) \geq 0$ as $\mathfrak{A} \leq \mathfrak{D}_i$. Thus $\delta(\mathfrak{F}/\mathfrak{A}) = \delta(\mathfrak{F}/\mathfrak{D}_i \cap \mathfrak{F}) + \delta(\mathfrak{F} \cap \mathfrak{D}_i/\mathfrak{A}) \geq 0$ as required. □

7.1 The Number of Countable Models

Definition 7.1.1. Let $\mathfrak{M} \models S_{\overline{\alpha}}$. Let $\mathfrak{A} \leq \mathfrak{M}$. We let $\dim(\mathfrak{M}/\mathfrak{A}) = \max\{\delta(\mathfrak{B}/\mathfrak{A}) : \mathfrak{A} \leq \mathfrak{B} \leq \mathfrak{M}\}$. If there is no maximum, i.e. given any $z > 0$, there will be some $\mathfrak{B} \leq \mathfrak{M}$ with $\delta(\mathfrak{B}/\mathfrak{A}) > z$, we let $\dim(\mathfrak{M}/\mathfrak{A}) = \infty$. We write $\dim(\mathfrak{M})$ for $\dim(\mathfrak{M}/\emptyset)$.

Definition 7.1.2. Fix an integer $k \geq 0$ and let $K_{k/c} = \{\mathfrak{A} : \mathfrak{A} \in K_{\overline{\alpha}}$ and $\delta(\mathfrak{A}) = k/c\}$. Let $(K_{k/c}, \leq)$ be such that $\leq$ is *inherited by* $K_{\overline{\alpha}}$ i.e. $\mathfrak{A} \leq \mathfrak{B}$ for $\mathfrak{A}, \mathfrak{B} \in K_{k/c}$ if and only if for all $\mathfrak{A} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$ with $\mathfrak{B}' \in K_{\overline{\alpha}}$, $\mathfrak{A} \leq \mathfrak{B}'$

We begin with the following technical lemma:

Lemma 7.1.3. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \leq \mathfrak{B}, \mathfrak{C}$; $\delta(\mathfrak{C}/\mathfrak{A}) \geq \delta(\mathfrak{B}/\mathfrak{A})$ and $\mathfrak{D} = \mathfrak{B} \oplus \mathfrak{C}$ the free join of $\mathfrak{B}, \mathfrak{C}$ over $\mathfrak{A}$. We can construct $\mathfrak{H} \in K_{\overline{\alpha}}$ such that $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \leq \mathfrak{H}$, $\mathfrak{D} \subseteq \mathfrak{H}$ and $\delta(\mathfrak{H}/\mathfrak{C}) = 0$. Further if $\delta(\mathfrak{B}/\mathfrak{A}) = \delta(\mathfrak{C}/\mathfrak{A})$, the $\mathfrak{H}$ that was constructed has the property $\delta(\mathfrak{H}/\mathfrak{B}) = 0$.*

Proof. This follows from an easy application of Lemma 7.0.1 on $\mathfrak{D}$. □

We now work towards showing that certain countable models of $S_{\overline{\alpha}}$ can be built as Fraïssé limits $(K_{k/c}, \leq)$. In Theorem 7.1.7 we show that these are in fact, all of the countable models up to isomorphism.

Lemma 7.1.4. *For any fixed integer $k \geq 0$, $(K_{k/c}, \leq)$, where $\leq$ is inherited from $K_{\overline{\alpha}}$ is a Fraïssé class.*

Proof. Fix an integer $k \geq 0$ and consider $K_{k/c}$. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K_{k/c}$. Note that for the purposes of proving amalgamation, we may as well assume $\mathfrak{B}, \mathfrak{C}$ are freely joined over $\mathfrak{A}$ and that $\mathfrak{A} \leq \mathfrak{B}, \mathfrak{C}$. Note that $\delta(\mathfrak{B}/\mathfrak{A}) = \delta(\mathfrak{C}/\mathfrak{A}) = 0$. The required statement follows by a simple application of Lemma 7.1.3 on $\mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C}$. For joint embedding consider $\emptyset \leq \mathfrak{B}, \mathfrak{C}$. Note that $\delta(\mathfrak{B}/\emptyset) = \delta(\mathfrak{C}/\emptyset) = k/c$. Apply Lemma 7.1.3 on $\mathfrak{B} \oplus_{\emptyset} \mathfrak{C}$, the free join of $\mathfrak{B}, \mathfrak{C}$ *over* $\emptyset$. □

We now prove the following theorem. Note that unlike the class $K_{\overline{\alpha}}$, the class $K_{k/c}$ is not closed under substructure, however as $(K_{k/c}, \leq)$ generic still exists as it satisfies amalgamation and joint embedding.

Theorem 7.1.5. *Let k be a fixed integer with $k \geq 0$. Let $\mathfrak{M}_{k/c}$ be the generic for the Fraïssé class $(K_{k/c}, \leq)$ where $\leq$ is inherited from $K_{\overline{\alpha}}$. Now $\mathfrak{M}_{k/c} \models S_{\overline{\alpha}}$ and $\dim(\mathfrak{M}_{k/c}) = k/c$.*

Proof. Fix an integer $k \geq 0$. From Lemma 7.1.4, it follows that $(K_{k/c}, \leq)$ where $\leq$ is inherited from $K_{\overline{\alpha}}$ is a Fraïssé class. Let $\mathfrak{M}_{k/c}$ be the $(K_{k/c}, \leq)$ generic. Note that given $\mathfrak{B} \in K_{\overline{\alpha}}$, there is some $\mathfrak{D} \in K_{k/c}$ such that $\mathfrak{D} \supseteq \mathfrak{B}$ by Lemma 7.0.1. Thus it suffices to show that $\mathfrak{M}_{k/c}$ satisfies the extension formulas in $S_{\overline{\alpha}}$.

Let $\mathfrak{A}, \mathfrak{B} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \leq \mathfrak{B}$ and assume that $\mathfrak{A} \subseteq_{\text{Fin}} \mathfrak{M}_{k/c}$. As $\mathfrak{M}_{k/c}$ is the $(K_{k/c}, \leq)$ generic, there is some $\mathfrak{C} \leq \mathfrak{M}_{k/c}$ with $\mathfrak{A} \subseteq \mathfrak{C}$ and $\delta(\mathfrak{C}) = k/c$. By Fact 2.3.4, we have that $\mathfrak{D} = \mathfrak{B} \oplus \mathfrak{C}$, the free join of $\mathfrak{B}, \mathfrak{C}$ over $\mathfrak{A}$ is in $K_{\overline{\alpha}}$ and that $\mathfrak{C} \leq \mathfrak{D}$. Now using Lemma 7.0.1, we can find $\mathfrak{G} \in K_{k/c}$ such that $\mathfrak{D} \subseteq \mathfrak{G}$ and $\mathfrak{C} \leq \mathfrak{G}$. But as $\mathfrak{M}_{k/c}$ is the $(K_{k/c}, \leq)$ generic we can find a strong embedding of $\mathfrak{G}$ into $\mathfrak{M}_{k/c}$ over $\mathfrak{C}$. Thus it follows that $\mathfrak{M}_{k/c} \models \forall \overline{x} \exists \overline{y} (\Delta_A(\overline{x}) \implies \Delta_{A,B}(\overline{x}, \overline{y}))$. Hence it follows that $\mathfrak{M}_{k/c} \models S_{\overline{\alpha}}$. Further as noted above, given any $\mathfrak{A} \subseteq_{\text{Fin}} \mathfrak{M}_{k/c}$, there is some $\mathfrak{C} \leq \mathfrak{M}_{k/c}$ with $\mathfrak{A} \subseteq \mathfrak{C}$ and $\delta(\mathfrak{C}) = k/c$. Hence $\dim(\mathfrak{M}_{k/c}) = k/c$. $\square$

We now work towards classifying the countable models of $S_{\overline{\alpha}}$ up to isomorphism using our notion of dimension.

Lemma 7.1.6. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$ and $\mathfrak{A} \leq \mathfrak{M}$ be finite. Let $\mathfrak{D} \in K_{\overline{\alpha}}$ be such that $\mathfrak{A} \leq \mathfrak{D}$. Then $\dim(\mathfrak{M}/\mathfrak{A}) \geq \delta(\mathfrak{D}/\mathfrak{A})$ if and only if there is some g such that g strongly embeds $\mathfrak{D}$ into $\mathfrak{M}$ over $\mathfrak{A}$.*

Proof. The statement that if there is some g such that g strongly embeds $\mathfrak{D}$ into $\mathfrak{M}$ over $\mathfrak{A}$, then $\dim(\mathfrak{M}/\mathfrak{A}) \geq \delta(\mathfrak{D}/\mathfrak{A})$ is immediate from the definition. Thus we prove the converse. Let $\mathfrak{A} \leq \mathfrak{M}$ be finite. Let $\mathfrak{D} \in K_{\overline{\alpha}}$ be such that $\mathfrak{A} \leq \mathfrak{D}$.

First assume that $\delta(\mathfrak{D}/\mathfrak{A}) = 0$. Now as $S_{\overline{\alpha}} \models \forall \overline{x} \exists \overline{y} (\Delta_{\mathfrak{A}}(\overline{x}) \implies \Delta_{\mathfrak{A},\mathfrak{D}}(\overline{x}, \overline{y}))$. Thus there is some $\mathfrak{A} \subseteq \mathfrak{D}' \subseteq \mathfrak{M}$ such that $\mathfrak{D} \cong_{\mathfrak{A}} \mathfrak{D}'$. Further as $\delta(\mathfrak{D}'/\mathfrak{A}) = 0$, from (2) of Lemma 4.4.2, $\mathfrak{D}' \leq \mathfrak{M}$. Thus regardless of the value of $\dim(\mathfrak{M}/\mathfrak{A})$, if $\delta(\mathfrak{D}/\mathfrak{A}) = 0$ then there is some g such that g strongly embeds $\mathfrak{D}$ into $\mathfrak{M}$ over $\mathfrak{A}$.

Now assume that $m/c = \delta(\mathfrak{D}/\mathfrak{A}) \leq \dim(\mathfrak{M}/\mathfrak{A})$ with $m \geq 1$ and further assume that $\dim(\mathfrak{M}/\mathfrak{A}) \geq k/c$ with $k \geq m$. Let $\mathfrak{A} \leq \mathfrak{F} \leq \mathfrak{M}$ be such that $\delta(\mathfrak{F}/\mathfrak{A}) = k/c$.

Let $\mathfrak{G} = \mathfrak{D} \oplus \mathfrak{F}$, the free join of $\mathfrak{D}, \mathfrak{F}$ over $\mathfrak{A}$. By Lemma 7.1.3, there exists $\mathfrak{H} \in K_{\overline{\alpha}}$ with $\mathfrak{G} \subseteq \mathfrak{H}$ and $\mathfrak{A}, \mathfrak{D}, \mathfrak{F} \leq \mathfrak{H}$ and $\delta(\mathfrak{H}/\mathfrak{F}) = 0$. Since $\mathfrak{F} \leq \mathfrak{M}$ and $\delta(\mathfrak{H}/\mathfrak{F}) = 0$ we are in the setting above. So take a strong embedding g of $\mathfrak{H}$ into $\mathfrak{M}$ over $\mathfrak{F}$. Clearly g fixes $\mathfrak{A}$ and $\mathfrak{D}$ has the property that $g(\mathfrak{D}) \leq \mathfrak{F} \leq \mathfrak{M}$ and thus $g(\mathfrak{D}) \leq \mathfrak{M}$. □

We now obtain:

Theorem 7.1.7. *Let $\mathfrak{M}, \mathfrak{N} \models S_{\overline{\alpha}}$ be countable. Now $\mathfrak{M} \cong \mathfrak{N}$ if and only if $\dim(\mathfrak{M}) = \dim(\mathfrak{N})$ and $\dim(\mathfrak{M}) = \infty$ if and only if $\mathfrak{M}$ is the generic for $K_{\overline{\alpha}}$. Thus there are precisely $\aleph_0$ many non-isomorphic models of $S_{\overline{\alpha}}$ of size $\aleph_0$. Further each countable model of $S_{\overline{\alpha}}$ can be built up from a subclass of $(K_{\overline{\alpha}}, \leq)$.*

Proof. Since δ is invariant under isomorphism, it immediately follows that if $\mathfrak{M} \cong \mathfrak{N}$, then $\dim(\mathfrak{M}) = \dim(\mathfrak{N})$. Now from Theorem 7.1.5, it follows that the number of non-isomorphic countable models is at least $\aleph_0$.

Case 1: $\dim(\mathfrak{M}) = \dim(\mathfrak{N}) = k/c$ for some $k \in \omega$. Fix enumerations for M, N. Let $\mathfrak{A} \leq \mathfrak{M}$ with $\dim(\mathfrak{M}/\mathfrak{A}) = 0$. Thus $\delta(\mathfrak{A}) = \dim(\mathfrak{M}) = \dim(\mathfrak{N})$. Assume that we have constructed a *strong embedding* $g : \mathfrak{A} \to \mathfrak{N}$. Pick $b \in \mathfrak{N} - g(\mathfrak{A})$, where b in the enumeration corresponds to the element of N with least index not in $g(A)$. Consider $icl_{\mathfrak{N}}(\{b\} \cup g(\mathfrak{A})) = \mathfrak{B} \leq \mathfrak{N}$. Now $\mathfrak{B}$ is finite. Since $g(\mathfrak{A}) \leq \mathfrak{N}$ and $g(\mathfrak{A}) = \dim(\mathfrak{N})$, it follows that $\delta(\mathfrak{B}/g(\mathfrak{A})) = 0$ and $g(\mathfrak{A}) \leq \mathfrak{B}$. Now as $\mathfrak{A} \cong g(\mathfrak{A})$ by Lemma 7.1.6, there exists a *strong embedding* $g' : \mathfrak{B} \to \mathfrak{M}$ and $g'|_{g(\mathfrak{A})} = g^{-1}$. Clearly this allows us to form a back and forth system between $\mathfrak{M}, \mathfrak{N}$.

Thus all that remains to be shown is that we can find a strong embedding of $\mathfrak{A} \leq \mathfrak{M}$ where $\delta(\mathfrak{A}) = \dim(\mathfrak{M})$. To see this first note that $\emptyset \leq \mathfrak{N}$. Further $\dim(\mathfrak{N}/\emptyset) = \delta(\mathfrak{A}/\emptyset)$. Thus there exists some strong embedding of $\mathfrak{A}$ over $\emptyset$ into $\mathfrak{N}$ by an application of Lemma 7.1.6 as required.

Case 2: $\mathfrak{M} \models S_{\overline{\alpha}}$ and $\dim(\mathfrak{M}) = \infty$. We claim that in this case $\mathfrak{M}$ is isomorphic to the generic. Clearly $\mathfrak{M}$ has finite closures and hence condition (1) of the generic is satisfied. Note that if we show that $\dim(\mathfrak{M}) = \infty$ implies that for any $\mathfrak{A} \leq \mathfrak{M}$, $\dim(\mathfrak{M}/\mathfrak{A}) = \infty$, then condition (2) follows immediately from Lemma 7.1.6. We claim that this is indeed the case. By way of contradiction, assume that there is some $\mathfrak{A} \leq \mathfrak{M}$ such that $\dim(\mathfrak{M}/\mathfrak{A})$ is finite. Now there is some $\mathfrak{A} \leq \mathfrak{D} \leq \mathfrak{M}$ such that $\dim(\mathfrak{M}/\mathfrak{A}) = \delta(\mathfrak{D}/\mathfrak{A})$. It is immediate from the definition that $\dim(\mathfrak{M}/\mathfrak{D}) = 0$. As $\dim(\mathfrak{M}) = \infty$, fix a $\mathfrak{B} \leq \mathfrak{M}$ with $\delta(\mathfrak{B}) > \delta(\mathfrak{D})$. Consider G, the closure of BD in M. Now G is finite and since $B, D \leq M$, $B, D \leq G$. Further $\delta(G/D) = 0$ as $\dim(M/D) = 0$. So $\delta(G) = \delta(D)$. But $B \leq M$, so $\delta(G/B) \geq 0$ and hence $\delta(G) \geq \delta(B)$. Thus $\delta(B) \leq \delta(D)$, a contradiction to our choice of B that establishes the claim. Hence it follows that the number of non-isomorphic countable models of $S_{\overline{\alpha}}$ is $\aleph_0$.

From Theorem 7.1.5, it follows that we can construct a countable model of a fixed dimension (the $\dim(\mathfrak{M}) = \infty$ case being the generic as seen above) as the generic of a subclass of $(K_{\overline{\alpha}}, \leq)$. But as the dimension determines the countable model up to isomorphism, we obtain the result. $\square$

We now use our notion of dimension to characterize elementary embedability.

Theorem 7.1.8. *Let $\mathfrak{M}, \mathfrak{N}$ be countable models of $S_{\overline{\alpha}}$. If $\dim(\mathfrak{M}) \leq \dim \mathfrak{N}$, then there is some elementary embedding $f : \mathfrak{M} \to \mathfrak{N}$. Thus there is an elementary chain $\mathfrak{M}_0 \preccurlyeq \ldots \preccurlyeq \mathfrak{M}_{\mathfrak{n}} \ldots \preccurlyeq \mathfrak{M}_\omega$ of countable models of $S_{\overline{\alpha}}$ with each countable model isomorphic to some element of the chain.*

Proof. Let $\mathfrak{M}, \mathfrak{N}$ be countable models of $S_{\overline{\alpha}}$ with $\dim(\mathfrak{M}) \leq \dim(\mathfrak{N})$. Note that if $\dim(\mathfrak{M}) = \dim(\mathfrak{N})$, then by Theorem 7.1.7, $\mathfrak{M} \cong \mathfrak{N}$. So assume that $\dim(\mathfrak{M}) < \dim(\mathfrak{N})$ and fix an enumeration $\{m_i : i \in \omega\}$. Now we have that $\dim(\mathfrak{M}) < \infty$. Let $\mathfrak{A} \leq \mathfrak{M}$ be such that $\delta(\mathfrak{A}) = \dim(\mathfrak{M})$. Now by Lemma 7.1.6, there exists a strong embedding f_1 of $\mathfrak{A}$ into $\mathfrak{N}$. Let $\mathfrak{B} \leq \mathfrak{M}$ be such that $A\{m_i\} \subseteq B$ where i is the least index such that $m_i \notin A$. Note that as $\delta(\mathfrak{A}) = \dim(\mathfrak{M})$, $\delta(\mathfrak{B}) = \delta(\mathfrak{A})$. Again using Lemma 7.1.6, we can extend f_1 to f_2 so that f_2 is a strong embedding of $\mathfrak{B}$ into $\mathfrak{N}$ *over* $\mathfrak{A}$.

Proceeding iteratively we can find a $\leq$ chain $\{\mathfrak{A}_i : i \in \omega\}$ such that $\mathfrak{M} = \bigcup_{i<\omega} \mathfrak{A}_i$ and $f : \mathfrak{M} \to \mathfrak{N}$ such that $f(\mathfrak{A}_i) \leq \mathfrak{N}$ for each $i \in \omega$. It is easily seen that f is an isomorphic embedding. We claim that f is actually an elementary embedding of $\mathfrak{M}$ into $\mathfrak{N}$. Note that given $\mathfrak{C} \leq \mathfrak{M}$ with C finite, there is some $\mathfrak{A}_{\mathfrak{i}}$ with $\mathfrak{C} \leq \mathfrak{A}_i \leq \mathfrak{M}$. Using the transitivity of $\leq$, it easily follows that $f(\mathfrak{C}) \leq \mathfrak{N}$. In particular $f(\mathfrak{M})$ is (algebraically) closed in $\mathfrak{N}$. Now Theorem 4.4.4 yields that $f(\mathfrak{M}) \preccurlyeq \mathfrak{N}$.

Now given an elementary chain $\mathfrak{M}_0 \preccurlyeq \ldots \preccurlyeq \mathfrak{M}_n$ with $\dim(\mathfrak{M}_k) = k/c$ for all $k \leq n$ of models of $S_{\overline{\alpha}}$ we may construct $\mathfrak{M}_{n+1}$ such that $\mathfrak{M}_1 \preccurlyeq \ldots \preccurlyeq \mathfrak{M}_n \preccurlyeq \mathfrak{M}_{n+1}$ and $\dim(\mathfrak{M}_{n+1}) = (n+1)/c$. Given an elementary chain $\mathfrak{M}_0 \preccurlyeq \ldots \preccurlyeq \mathfrak{M}_{\mathfrak{n}} \ldots \preccurlyeq$ set

$\mathfrak{M}_\omega = \bigcup_{n<\omega} \mathfrak{M}_n$. As elementary embeddings preserve closed sets it is easily seen that $\dim(\mathfrak{M}_\omega) = \infty$. The rest of the claim now follows from Theorem 7.1.7. $\square$

7.2 Regular Types

In Section 7.2 we turn our attention towards the study of regular types. We fix a monster model $\mathbb{M}$ of $S_{\overline{\alpha}}$. Now, due to ω-stability and weak elimination of imaginaries (see Theorem 6.1.16 and Lemma 6.2.27), it suffices to restrict our attention to non-algebraic types over finite algebraically closed sets in the home sort for the study of regular types. So fix some finite $A \leq \mathbb{M}$ (recall that algebraically closed sets are precisely the intrinsically closed ones). In what follows we assume that the user is familiar with notions such as regular types, orthogonality, modular types etc. and facts about them. We will give the salient definitions and facts regarding the a fore mentioned, but we will be brief. The reader may find an in depth discussion of the relevant definitions and results in [32] (for non-geometric matters the reader may also see [30], [29] or [28]).

Remark 7.2.1. Let $A \leq \mathbb{M}$ be finite and $\overline{b}$ be finite such that $\overline{b} \cap A = \emptyset$. Now let $A \subseteq C$ also be finite. Note that $\overline{b} \perp_A C$ if and only if $\operatorname{acl}(\overline{b}A) \perp_{\operatorname{acl}(A)} \operatorname{acl}(C)$. Since $S_{\overline{\alpha}}$ has finite closures it follows that $\operatorname{acl}(bA), \operatorname{acl}(C)$ are both finite. Thus in order to understand non-forking, it suffices to look at types $p \in S(A)$ such that $x \neq a \in p$ for all $a \in A$ such that for any $\overline{b} \models p$, $\overline{b}A \leq \mathbb{M}$. Note that this information, along with the atomic diagram of some (of any) realization of p is sufficient to determine p uniquely as noted in (1) of Lemma 4.4.2. Also such a type p is non-algebraic and stationary as A is algebraically closed.

Definition 7.2.2. Let $X \subseteq \mathbb{M}$, $p, q \in S(X)$

1. p, q are *orthogonal* to each other if for any $Y \supseteq X$, and non-forking extensions p', q' of p, q respectively, any $\overline{a} \models p'$, $\overline{b} \models q'$, we have that $\overline{a} \downarrow_Y \overline{b}$. We denote this with $p \perp q$.

2. p is *regular* if it is non-algebraic, stationary and orthogonal to any forking extension of it self.

3. If p, q are stationary, then we say that they are *parallel* if they have the same nonforking type (i.e. same non-forking extension as a type over $\mathbb{M}$).

It is easily seen that parallelism is an equivalence relation on the space of types. Combining this with our comments at the beginning of Section 7.2 and Remark 7.2.1, it suffices to study *basic types over finite sets* in order to understand regular types (i.e. we can choose a basic type to represent the required parallelism class).

Definition 7.2.3. Let $A \leq \mathbb{M}$ be finite and $p \in S(A)$, we say that p is a *basic type* if $x \neq a \in p$ for all $a \in A$ and for some (equivalently any) $\overline{b} \models p$, $\overline{b}A \leq \mathbb{M}$.

Remark 7.2.4. Note that if $\mathfrak{A}, \mathfrak{B} \in K_L$ with $\mathfrak{A} \in K_{\overline{\alpha}}$ and $\mathfrak{A} \leq \mathfrak{B}$, then $\mathfrak{B} \in K_{\overline{\alpha}}$.

Lemma 7.2.5. *Let $\mathfrak{A} \in K_{\overline{\alpha}}$. Then there exists $\mathfrak{B} \in K_{\overline{\alpha}}$ such that $\mathfrak{A} \leq \mathfrak{B}$ and $\delta(\mathfrak{B}/\mathfrak{A}) = 1/c$.*

Proof. Consider the structure given by $\mathfrak{A}^* = \mathfrak{A} \oplus \mathfrak{A}_0$ where $\mathfrak{A}_0 \in K_{\overline{\alpha}}$ consists of a single point. Now an application of Lemma 7.0.1 to $\mathfrak{A}^*$ yields the required result. $\square$

We begin by studying basic types such that $d(p/A) = 0, 1/c$ where $A \leq \mathbb{M}$ is finite. The choice to restrict our attention to such types will be justified by Theorem 7.2.13, where we show any type p with $d(p/A) \geq 2/c$ *cannot* be regular. We begin our analysis of types that can be regular types by defining nuggets and nugget-like types.

Definition 7.2.6. Let $\mathfrak{A}, \mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \subsetneq \mathfrak{D}$ with $D = AB$. Let $k \in \omega$. We say that B is a *k/c-nugget over* $\mathfrak{A}$ if $A \cap B = \emptyset$, $\delta(B/A) = k/c$ and $\delta(B'/A) > k/c$ for all $A \subsetneq AB' \subsetneq AB$.

Definition 7.2.7. Let $A \leq \mathbb{M}$ be finite. We say that a basic type $p \in S(A)$ is *nugget-like* over A, if given B where B realizes the quantifier free type of p over $\mathfrak{A}$, then B is a k/c-nugget over A for some $k \in \omega$.

Lemma 7.2.8. *Let $A \leq \mathbb{M}$ be finite and let $p \in S(A)$ be nugget-like. Let $A \subseteq X$ with X closed. For any $\overline{b} \models p$, either $\overline{b} \cap X = \emptyset$ or $\overline{b} \subseteq X$.*

Proof. Assume that $\overline{b} \cap X \neq \emptyset$. Let $\overline{b}' = \overline{b} \cap X$ assume that $\overline{b}' \neq \overline{b}$. Then as $\delta(\overline{b}'/A) > \delta(\overline{b}/A)$, it follows that there is some minimal pair $(A\overline{b}', D)$ with $D \subseteq A\overline{b}$ but $D \nsubseteq X$. But this contradicts that X is closed. Hence $\overline{b} \subseteq X$. $\square$

We now explore how the behavior of the d function interacts with nugget-like types.

Lemma 7.2.9. *Let $A \leq \mathbb{M}$ be finite and let $p \in S(A)$ is nugget-like. Let $A \subseteq Y \subseteq \mathbb{M}$ with Y closed. Let q be an extension of p to Y. Now q is a forking extension of p if and only if $d(q/Y) < d(p/A)$ or given $\overline{b} \models q$, $\overline{b} \subseteq Y$.*

Proof. If $d(q/Y) < d(p/A)$, then Remark 6.2.28 tells us that q is a forking extension of p. Further Y is algebraically closed. So if for any $\bar{b} \models q$, $\bar{b} \subseteq Y$, it follows that b is an algebraic type over Y. Since p is not an algebraic type over A, it follows that q is a forking extension of p.

For the converse assume that q is a forking extension of p and that $d(q/Y) = d(p/A)$. As q is a forking extension of p, it follows from Theorem 6.2.25 that $\mathrm{icl}(\bar{b}A) \cap \mathrm{icl}(Y) \supsetneq \mathrm{icl}(A)$. But $\mathrm{icl}(A) = A$, $\mathrm{icl}(Y) = Y$ and as $\bar{b}$ realizes p over A, $\mathrm{icl}(\bar{b}A) = \bar{b}A$. Thus $\bar{b} \cap Y \neq \emptyset$. Now by Lemma 7.2.8, $\bar{b} \subseteq Y$. $\square$

7.2.1 Identifying Regular and Non-regular types

We now present some results towards identifying regular and non-regular types. The following theorem allows us to identify certain regular types. Further it establishes that 0-nuggets are, in some sense, orthogonal to almost all other types.

Theorem 7.2.10. *Let $A \leq \mathbb{M}$ be finite and let $p \in S(A)$ be nugget-like. Now if $d(p/A) = 0$ or $d(p/A) = 1/c$, then p is regular. Further if $d(p/A) = 0$, then p is orthogonal to any other nugget-like type over A.*

Proof. Under the given conditions p is clearly non-algebraic and stationary. We directly establish that it will be orthogonal to any forking extension of itself. Let $A \subseteq X \subseteq \mathbb{M}$ with X closed. Since $S_{\overline{\alpha}}$ is ω-stable and has finite closures we may as well assume that X is finite, i.e. if $q \in S(X)$ with $q \supseteq p$ a forking extension, there is some finite closed $X_0 \subseteq X$ such that $q \restriction_{X_0}$ is a forking extension. Let $\bar{b} \models p$. We have that $\bar{b} \mathop{\smile\!\!\!\!\!\!\downarrow}_A X$. As $A\bar{b}$, X are closed and $A\bar{b} \cap X = A$, from an application of

Theorem 6.2.25, we obtain that $X\bar{b}$ is closed.

First assume that $d(p/A) = 0$. Let p' be a forking extension of p to X and let$\overline{f} \models p'$. It follows easily from Lemma 6.2.28, that $d(\overline{f}/A) \geq d(\overline{f}/X)$. As $d(\overline{f}/A) = 0$ and $d(\overline{f}/X) \geq 0$, it now follows that $d(\overline{f}/X) = 0$. Thus by Lemma 7.2.9, we have that $\overline{f} \subseteq X$ and hence $\bar{b} \downarrow_X \overline{f}$ as $\bar{b} \downarrow_A X$.

So assume that $d(p/A) = 1/c$. Let p', $\overline{f}$ be as above. By Lemma 7.2.9, $d(p'/X) = 0$ or $\overline{f} \subseteq X$. As above $\overline{f} \subseteq X$ yields that $\bar{b} \downarrow_X \overline{f}$. So assume that $\overline{f} \nsubseteq X$ and note that by Lemma 7.2.8 we have that $\overline{f} \cap X = \emptyset$. Now by Theorem 6.2.25 it suffice to show that $X\bar{b} \cap \mathrm{acl}(X\overline{f}) = X$ to establish that $\bar{b} \downarrow_X \overline{f}$. Consider $d(\mathrm{acl}(X\overline{f})\,\bar{b}/X)$. On the one hand we have that $d(\mathrm{acl}(X\overline{f})\,\bar{b}/X) \geq d(\bar{b}/X) = 1/c$ (see Lemma 6.2.15). On the other hand $d(\mathrm{acl}(X\overline{f})\,\bar{b}/X) = d(\bar{b}/acl(X\overline{f})) + d(\mathrm{acl}(X\overline{f})/X)$. As $d(\mathrm{acl}(X\overline{f})/X) = d(\overline{f}/X) = 0$, we obtain that $d(\bar{b}/\mathrm{acl}(X\overline{f})) \geq 1/c$. In particular $\bar{b} \nsubseteq \mathrm{acl}(X\overline{f})$. But then by Lemma 7.2.8, $\bar{b} \cap \mathrm{acl}(X\overline{f}) = \emptyset$ and thus $X\bar{b} \cap \mathrm{acl}(X\overline{f}) = \emptyset$ as required.

For the second half of the claim, assume that $d(p/A) = 0$. Let $q \in S(A)$ be nugget-like and distinct from p. Now $d(p/A) = d(p|_X/X)$ and $d(q/A) = d(q|_X/X)$. Let $\overline{f} \models q|_X$. Note that $\overline{f} \downarrow_A X$ implies that $X\overline{f}$ is closed. Now using Lemma 7.2.8, we can easily show that $\bar{b}X \cap \overline{f}X \neq X$, then $\bar{b} = \overline{f}$. But this contradicts $p \neq q$. Thus it follows that $\bar{b}X \cap \overline{f}X = X$. Further $0 = d(\bar{b}/X) \geq d(\bar{b}/X\overline{f}) \geq 0$. Again by Theorem 6.2.25, we obtain that $\bar{b} \downarrow_X \overline{f}$ and thus p, q are orthogonal. $\square$

The following theorem shows that while there are many regular types with $d(p/A) = 1/c$, all such types are non-orthogonal. Thus up to non-orthogonality, there is only one regular type with $d(p/A) = 1/c$. This is in contrast to distinct 0-nuggets, any two of which are orthogonal to each other. We also show that the number of independent realizations of a $1/c$ nugget determines the dimension of a model.

Theorem 7.2.11. *Let A be closed and finite and let $p, q \in S(A)$ be distinct basic types and satisfy $d(p/A) = d(q/A) = 1/c$. Then p, q are non-orthogonal. Hence any two regular types over $p', q' \in S(X)$ where X is closed and $d(p'/X) = d(q'/X) = 1/c$ are non-orthogonal. Further if we take $A = \emptyset$ and let $\mathfrak{M} \preccurlyeq \mathbb{M}$. The dimension of $\mathfrak{M}$ is determined by the number of independent realizations of p in $\mathfrak{M}$. Thus a single regular type determines the dimension of $\mathfrak{M}$.*

Proof. Let A be as given. Consider A as a finite structure that lives in $K_{\overline{\alpha}}$. Now consider the finite structures AB, AC where B, C realize the quantifier free types of p, q respectively. Consider D, the free join of AB, AC over A. Apply Lemma 7.0.1 to obtain a finite G with $\delta(G/D) = -1/c$ and $A, AB, AC \leq G$. Let f be a strong embedding of G into $\mathbb{M}$ where f is the identity on A. From (1) of Lemma 4.4.2 and the transitivity of $\leq$ it follows that $f(B) \models p$ and $f(C) \models q$. Now from Theorem 6.2.25, it follows that $f(B) \not\perp_A f(C)$ and thus $p \not\perp q$. Now given $p', q' \in S(X)$, there exists a finite closed set, which by an abuse of notation we call A, such that p', q' are based and stationary over A. Since regularity is parallelism invariant both $p|_A$ and $q|_A$ are regular. Arguing as above we see that $p'|_A \not\perp q'|_A$ and thus they are non-orthogonal.

Let $\mathfrak{M} \preccurlyeq \mathbb{M}$ and assume that $A = \emptyset$. Given $n \in \omega$, consider the finite structure C_n that is the free join of n-copies of the quantifier free type of p over $\emptyset$. If $\dim(\mathfrak{M}) \geq n/c$, by Lemma 7.1.6, there is a strong embedding of C_n into $\mathfrak{M}$. It is easily checked that the strong embedding witnesses n-independent realizations of p. The rest follows easily. $\square$

The following result shows that a broad class of types cannot be regular types and justifies the choice to study types $p \in S(A)$ with $d(p/A) = 0, 1/c$ in our study of regular types. We begin with the following fact regarding regular types. It is an immediate consequence of the well known fact that regular types have *weight* one (here weight is in the sense of stability theory, see for example Definition D.1 of [28]).

Fact 7.2.12. *Let $A \subseteq \mathbb{M}$ and $p \in S(A)$ be regular. If $\bar{b} \models p$, then there is no $C_1, C_2 \subseteq \mathbb{M}$ such that $C_1 \mathop{\smile\!\!\!\!\!\!\mid}_A C_2$ but $\bar{b} \not\mathop{\smile\!\!\!\!\!\!\mid}_A C_i$ for $i = 1, 2$.*

Theorem 7.2.13. *Let A be finite and closed in $\mathbb{M}$. Let $p \in S(A)$ be a basic type such that $d(p/A) \geq 2/c$. Then p is not regular.*

Proof. Our strategy is similar to the one used in Theorem 7.2.11: we consider A as living inside of $K_{\overline{\alpha}}$. We then construct a finite structure G over the finite structure A that we then embed strongly into $\mathbb{M}$ over A using saturation. Finally we argue that the strong embedding witnesses that there are C_1, C_2 such that $C_1 \mathop{\smile\!\!\!\!\!\!\mid}_A C_2$ but $\bar{b} \not\mathop{\smile\!\!\!\!\!\!\mid}_A C_i$ for $i = 1, 2$, where $\bar{b} \models p$.

Consider A as a finite structure that lives in $K_{\overline{\alpha}}$. By Lemma 7.2.5 we may construct $D \in K_{\overline{\alpha}}$ such that the $D = AC$, $A \cap C = \emptyset$ (as sets) and $A \leq D$ with $\delta(D/A) = \delta(C/A) = 1/c$. Let AB be such that B realizes the quantifier free type of p

over A. Consider the finite structures F_i, $i = 1, 2$ where each F_i is the free join of AB and an isomorphic copy of D over A and $F_1 \cap F_2 = AB$. We label the isomorphic copies of D as AC_1, AC_2 and thus $F_i = ABC_i$, the free join of AB, AC_i over A. Apply Theorem 3.2.15 to obtain G_i for $i = 1, 2$ such that (F_i, G_i) is an essential minimal pair and $\delta(G_i/F_i) = -1/c$. It is easily verified that $A, AB, AC_i \leq G_i$. Let G be the free join of G_1, G_2 over AB. Note that $G \in K_L$ and that we may now regard the finite structures A, AB, AC_1 etc. as substructures of G.

We claim that $G \in K_{\overline{\alpha}}$, $A, AB, AC_1, AC_2, AC_1C_2 \leq G$ but F_1, F_2, is *not* strong in G. Using Remark 2.3.2 and the transitivity of $\leq$, we obtain that it suffices to show that $AB, AC_1C_2 \leq G$ along with $F_1, F_2 \nleq G$ to obtain the claim.

First, as $AB \leq G_i$ and G is the free join of G_1, G_2 over AB, we obtain $AB \leq G$ by an application of (4) of Fact 2.2.5. We now show that $AC_1C_2 \leq G$. Let $AC_1C_2 \subseteq G' \subseteq G$ and let $B' = B \cap G'$, $G'_i = G_i - AC_i$. Now $\delta(G'/AC_1C_2) = \delta((G'_1 - B')(G'_2 - B')/AC_1C_2B') + \delta(B'/AC_1C_2)$ using (5) of Fact 2.2.5. Further, since AB, AC_1C_2 is freely joined over A $\delta(B'/AC_1C_2) = \delta(B'/A)$ follows from (2) of Fact 2.2.5. Arguing similarly we obtain that $\delta(G'_i - B'/AC_1C_2B') = \delta(G'_i/AB'C_i)$. Thus it follows that $\delta(G'/AC_1C_2) = \delta(G'_1/AC_1B') + \delta(G'_2/AC_2B') + \delta(B'/A)$. Now as $A \leq AB$, it follows that $\delta(B'/A) \geq 0$. The claim now follows by considering the cases $B' \neq B$ and $B' = B$ using that fact that (ABC_i, G_i) forms an essential minimal pair. Finally, and easy calculation shows that $\delta(G/F_1F_2) = -2/c$. Now $\delta(G/F_i) = \delta(G/F_1F_2) + \delta(F_1F_2/F_i) = -2/c + 1/c = -1/c$ for $i = 1, 2$.

Fix a strong embedding of f of G into $\mathbb{M}$ over A, which we assume to be the identity on A to simplify notation. Arguing as we did in Theorem 7.2.11, we obtain that $f(B) \models p$. Further using Theorem 6.2.25, $f(C_1) \mathop{\smile\!\!\!\!\!\!\downarrow}_A f(C_2)$ as $AC_1, AC_2 \leq G$ but $f(B) \not\mathop{\smile\!\!\!\!\!\!\downarrow}_A f(C_i)$ as $F_i \not\leq G$ for $i = 1, 2$. Assume that p is regular. Now the above argument contradicts Fact 7.2.12 and thus p is not regular. $\square$

Remark 7.2.14. In the above proof, we have shown that the pre-weight of p is at least two and hence the weight of p is at least two. This yields a contradiction with the fact that p is regular as regular types have weight one.

7.2.2 Some Geometric Matters

In this section we study geometric properties of the regular types. We establish that 0-nuggets have a trivial pregeometry. We also show that the pregeometry associated to a $1/c$-nugget is not locally modular. Finally, we draw on some known results to prove that there are pseudofinite ω-stable theories with non-locally modular regular types. This answers a question of Pillay's in [6] regarding whether pseudofinite stable theories always have locally modular regular types. We assume that the reader is familiar with basic facts about pseudofinite theories.

The following fact is well known. For example, see the Comment before Theorem D.8 of [28].

Fact 7.2.15. *Let $A \subseteq \mathbb{M}$ and $p \in S(A)$ be regular. Let $p^{\mathbb{M}}$ be the set of realizations of p in $\mathbb{M}$. For $B \subseteq p^{\mathbb{M}}$, let $cl^p(B) = \{\bar{c} \in p^{\mathbb{M}} : \bar{c} \not\mathop{\smile\!\!\!\!\!\!\downarrow}_A B\}$. Then $(p^{\mathbb{M}}, cl)$ is a pregeometry.*

We begin by studying the pregeometry associated with 0-nugget like types.

Theorem 7.2.16. *The pregeometry induced by forking closure on the realizations of a* 0*-nugget-like type* p *(over some finite* $A \leq \mathbb{M}$*), is trivial*

Proof. Assume to the contrary that it is not. Then there exists $\overline{a}, \overline{b}, \overline{c} \models p$ that is pairwise independent over A but dependent over A, say $\overline{a} \not\perp_A \overline{b}\overline{c}$. Note that repeated applications of (3) of Fact 2.2.5 yields that $\delta(\overline{a}\overline{b}\overline{c}/A) \leq \delta(\overline{a}/A) + \delta(\overline{b}/A) + \delta(\overline{a}/A) = 0$. Since A is strong in $\mathbb{M}$, we have $\delta(\overline{a}\overline{b}\overline{c}/A) \geq 0$ and thus $\delta(\overline{a}\overline{b}\overline{c}/A) = 0$. Assume that $\overline{a}\overline{b}\overline{c}A$ is not closed. Then there is some D such that $\delta(D/\overline{a}\overline{b}\overline{c}A) < 0$. But as $\delta(D/A) = \delta(D/\overline{a}\overline{b}\overline{c}A) + \delta(\overline{a}\overline{b}\overline{c}A/A)$, this yields a contradiction (recall that $\delta(\overline{a}\overline{b}\overline{c}A/A) = \delta(\overline{a}\overline{b}\overline{c}/A)$). Thus $\overline{a}\overline{b}\overline{c}A$ is closed.

Thus it now suffices to show that $\overline{a}, \overline{b}\overline{c}$ is not freely joined over A. Arguing as above, we easily obtain that $\delta(\overline{b}\overline{c}/A) = 0$. Assume towards a contradiction that there is some relation E that holds on $\overline{a}\overline{b}\overline{c}A$. Clearly this relation cannot be binary, as pairwise freely joined structures would be freely joined in this case. So we may assume that E is ternary.

We now give an argument similar to that used to establish (3) of Fact 2.2.5. There are two possibilities: First assume that E holds with at least one element each from $\overline{a}, \overline{b}, \overline{c}, A$. Now note that $\delta(\overline{a}\overline{b}\overline{c}/A) = \delta(\overline{a}\overline{b}\overline{c}) - e(\overline{a}\overline{b}\overline{c}, A)$. But $\delta(\overline{a}\overline{b}\overline{c}) \leq |\overline{a}| + |\overline{b}\,\overline{c}| - e(\overline{a}) - e(\overline{b}\overline{c}) = \delta(\overline{a}) + \delta(\overline{b}\overline{c})$ and $e(\overline{a}\overline{b}\overline{c}, A) \geq e(\overline{a}, A) + e(\overline{b}\overline{c}, A) + \overline{\alpha}(E)$. Thus it follows that $\delta(\overline{a}\overline{b}\overline{c}/A) \leq \delta(\overline{a}/A) + \delta(\overline{b}\overline{c}/A) - \overline{\alpha}(E) < 0$, a contradiction that yields that $\overline{a}, \overline{b}\overline{c}$ is freely joined over A. So assume that the relation E holds with at least one point each $\overline{a}$, $\overline{b}$, $\overline{c}$ but no points from A. Now note that $\delta(\overline{a}\overline{b}\overline{c}/A) = \delta(\overline{a}\overline{b}\overline{c}) - e(\overline{a}\overline{b}\overline{c}, A)$. But $\delta(\overline{a}\overline{b}\overline{c}) \leq |\overline{a}| + |\overline{b}\overline{c}| - e(\overline{a}) - e(\overline{b}\overline{c}) - \overline{\alpha}(E) = \delta(\overline{a}) + \delta(\overline{b}\overline{c}) - \overline{\alpha}(E)$.

Using the same inequalities involving $e(\overline{a}\overline{b}\overline{c}, A)$, we again obtain that $\delta(\overline{a}\overline{b}\overline{c}/A) \leq \delta(\overline{a}/A) + \delta(\overline{b}\overline{c}/A) - \overline{\alpha}(E) < 0$, a contradiction that yields that $\overline{a}, \overline{b}\overline{c}$ is freely joined over A.

Thus we obtain that $\overline{a} \downarrow_A \overline{b}\overline{c}$ and hence p is a trivial type. $\square$

We now turn our attention towards showing that $1/c$ nugget-like types are not locally modular. We begin by discussing *flatness*.

Definition 7.2.17. We say that an L-structure N is *flat* if for any finite number of finite closed sets $\{D_i\}_{i \in I}$ in N, we have that $\sum_{S \subseteq I} (-1)^{|S|} d(D_S) \leq 0$ where $D_\emptyset = \bigcup_{i \in I} D_i$ and $D_S = \bigcap_{i \in S} D_i$.

Lemma 7.2.18. *Any model of $S_{\overline{\alpha}}$ is flat.*

Proof. Fix $N \models S_{\overline{\alpha}}$. Let $\{D_i\}_{i \in I}$ be a finite collection of closed finite sets in N. We begin by estimating $\delta(D_\emptyset)$. By using the inclusion-exclusion principle to count the points in $D_\emptyset$ we obtain that

$$\begin{aligned} |D_\emptyset| &= \textstyle\sum_{\{i\} \subseteq I} |D_i| - \sum_{\{i,j\} \subseteq I} |D_i \cap D_j| + \sum_{\{i,j,k\} \subseteq I} |D_i \cap D_j \cap D_k| - \ldots \\ &\quad + (-1)^{|I|-1} |\textstyle\bigcap_{i \in I} D_i| \end{aligned}$$

Similarly for any relation $E \in L$ we obtain that

$$\begin{aligned} N_E(D_\emptyset) &= \textstyle\sum_{\{i\} \subseteq I} N_E(D_i) - \sum_{\{i,j\}} N_E(D_i \cap D_j) + \sum_{\{i,j,k\} \subseteq I} N_E(D_i \cap D_j \cap D_k) \\ &\quad - \ldots + (-1)^{|I|-1} N_E(\textstyle\bigcap_{i \in I} D_i) \end{aligned}$$

Hence we obtain that

$$\begin{aligned} \delta(D_\emptyset) &= \textstyle\sum_{\{i\} \subseteq I} \delta(D_i) - \sum_{\{i,j\}} \delta(D_i \cap D_j) + \sum_{\{i,j,k\} \subseteq I} \delta(D_i \cap D_j \cap D_k) \\ &\quad - \ldots + (-1)^{|I|-1} \delta(\textstyle\bigcap_{i \in I} D_i) \end{aligned}$$

Note that for $S \neq \emptyset$, we have that $d(D_S) = \delta(D_S)$. Hence it follows that $\sum_{S \subseteq I} (-1)^{|S|} d(D_S) = d(D_\emptyset) + \sum_{S \subseteq I, S \neq \emptyset} (-1)^{|S|} \delta(D_S) = d(D_\emptyset) - \delta(D_\emptyset)$. But $d(D_\emptyset) \leq \delta(D_\emptyset)$. Thus we obtain that $\sum_{S \subseteq I} (-1)^{|S|} d(D_S) \leq 0$. □

The following shows that $1/c$ nugget-like types are not locally modular.

Theorem 7.2.19. *Let $A \leq \mathbb{M}$ be finite and let $p \in S(A)$ be a nugget-like with $d(p/A) = 1/c$. Then p is not locally modular, in particular it is non-trivial.*

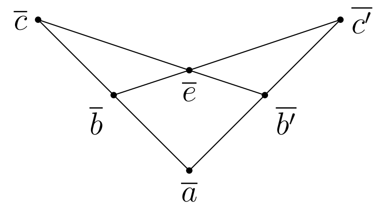
Proof. Recall that given a regular type p, the realizations of p form a pregeometry with respect to forking closure. In order to simplify the presentation, we will let $A = \emptyset$. We let $p^{\mathbb{M}}$ denote the realizations of p in $\mathbb{M}$, cl^p denote the forking closure (or p-closure) of $p^{\mathbb{M}}$ and $\dim^p$ (p-dimension) denote the associated dimension.

We begin with a proof that p is non-trivial. Let B_1, B_2, B_3 be three finite structures that has the same quantifier free type as p and are disjoint over $\emptyset$. Consider $C = \oplus B_i$, the free join of the B_i over $\emptyset$. Using Lemma 7.0.1 we obtain a finite structure $D \in K_{\overline{\alpha}}$ with $\delta(D) = 2/c$, $B_i \leq C$ and $B_i \oplus B_j \leq C$ for any $i \neq j$. Note that $C \not\leq D$ as $\delta(C) > \delta(D)$. Let g be a strong embedding of C into $\mathbb{M}$. An argument similar to that found in Theorem 7.2.11 shows that $g(B_1), g(B_2), g(B_3)$ are pairwise independent but dependent realizations of p and thus p is non-trivial.

We will now establish that p is not modular. To show that p is not locally modular, we can simply choose a realization $\overline{h}$ of p independent from the configuration used in the following argument and relativize the argument over $\overline{h}$. Fix realizations $\overline{a}, \overline{b}, \overline{c} \models p$ such that they are pairwise independent but are dependent. As $S_{\overline{\alpha}}$ is stable we can find $\overline{b'}, \overline{c'} \models p$ such that $\overline{b'}\,\overline{c'} \equiv_{\overline{a}} \overline{b}\,\overline{c}$ and $\overline{b'}\,\overline{c'} \mathop{\smile\!\!\!\!\!\mid}_{\overline{a}} \overline{b}\,\overline{c}$.

Let $X = \mathrm{cl}^p(\{\overline{c'}, \overline{b}\})$, $Y = \mathrm{cl}^p(\{\overline{c}, \overline{b'}\})$. Let $Z = \mathrm{cl}^p(X \cup Y)$. We will show that $\dim^p(\mathrm{cl}^p(Z)) + \dim^p(X \cap Y) < \dim^p(X) + \dim^p(Y)$. As $\dim^p(X) = 2 = \dim^p(Y)$ and $\dim^p(Z) = 3$, it suffices to show that $X \cap Y \cap p^{\mathbb{M}}$ is empty.

Suppose to the contrary that there is some $\overline{e} \in X \cap Y \cap p^{\mathbb{M}}$. We obtain the following configuration:



with every single (labeled) point having p-dimension 1, any three (labeled) colinear points (as found in the configuration) having p-dimension 2 and any three (labeled) non-colinear points having p-dimension 3 (a discussion on calculating the dimensions can be found in Appendix B).

Let $C^* = \mathrm{acl}(\overline{a}\,\overline{b}\,\overline{c}\overline{b'}\,\overline{c'}\,\overline{e})$. We will obtain a contradiction by estimating $d(C^*)$ in two different ways. On the one hand, as $\{\overline{a}, \overline{b}, \overline{b'}\}$ is independent and $\mathrm{acl}(\overline{a}\,\overline{b}\,\overline{b'}) \subseteq C^*$, it follows that $d(C^*) \geq 3/c$ (recall that $d(C^*/\mathrm{acl}(\overline{a}\,\overline{b}\,\overline{b'})) \geq 0$). On the other hand $\mathrm{acl}(C_1C_2C_3C_4) = C^*$ where $C_1 = \mathrm{acl}(\overline{c}\,\overline{e}\,\overline{b'})$, $C_2 = \mathrm{acl}(\overline{b}\,\overline{e}\,\overline{c'})$, $C_3 = \mathrm{acl}(\overline{a}\,\overline{b}\,\overline{c})$ and $C_4 = \mathrm{acl}(\overline{a}\,\overline{b'}\,\overline{c'})$. We estimate $d(C_1C_2C_3C_4)$ using *flatness*.

We begin by showing $d(C_1) = 2/c$. Note that as $\overline{b} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}} \overline{c'}$, by Theorem 6.2.25 we obtain that $\overline{b}\,\overline{c'}$ is closed. As $\overline{e} \not\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}} \overline{c'}\,\overline{b}$ another application of Theorem 6.2.25 tells us that $\overline{e} \cap \overline{b}\,\overline{c'}$ is non-empty or that $d(\overline{e}/\emptyset) > d(\overline{e}/\overline{b}\,\overline{c'})$. An application of Lemma 7.2.8 easily shows that $\overline{e} \cap \overline{b}\,\overline{c'}$ is empty and hence $d(\overline{e}/\emptyset) > d(\overline{e}/\overline{b}\,\overline{c'})$. But as $d(\overline{e}/\emptyset) = d(\overline{e}) = 1/c$ we obtain that $d(\overline{e}/\overline{b}\,\overline{c'}) = 0$. But then $d(C_1) = d(\overline{e}\,\overline{b}\,\overline{c'}) = d(\overline{b}\,\overline{c'}) = 2/c$

as $\overline{b}\,\overline{c'}$ is closed. Similar arguments show that $d(C_i) = 2/c$ for $i = 2, 3, 4$.

We now claim that $d(C_i \cap C_j) = 1/c$ for each $1 \leq i < j \leq 4$. Fix $1 \leq i < j \leq 4$. Note that as $C_i \cap C_j$ contains a realization of p, and hence $d(C_i \cap C_j) \geq 1/c$. Further $C_i \cup C_j$ contains three independent realizations of p and hence $d(C_iC_j) \geq 3/c$. Using flatness on $I' = \{i, j\}$, we obtain that $d(C_iC_j) \leq d(C_i) + d(C_j) - d(C_i \cap C_j)$ and the claim follows. A similar argument shows that $d(C_i \cap C_j \cap C_k) = 0$ for $1 \leq i < j < k \leq 4$. Now as $C_1 \cap C_2 \cap C_3 \cap C_4 \subseteq C_1 \cap C_2 \cap C_3$ and $d(C_1 \cap C_2 \cap C_3) = 0$, it follows that $d(C_1 \cap C_2 \cap C_3 \cap C_4) = 0$.

Hence we obtain that $3/c \leq d(C_1C_2C_3C_4) \leq 4(2/c) - 6(1/c) - 4(0) + 0 = 2/c$, a contradiction which shows that $X \cap Y \cap p^{\mathbb{M}}$ is empty. Thus p is not modular. Hence p is not locally modular. □

Remark 7.2.20. We sketch an alternate proof of Theorem 7.2.19 that uses only the non-triviality of p: Well known results of Hrushovski in [33] state that any stable theory with a non-trivial locally modular regular type interprets a group. As these structures do not interpret groups (see [27] by Wagner) the result now follows.

The above proof can be viewed as an explicit manifestation of ideas. We assume that a group configuration exists (which by the work of Hrushovski in [33] implies that the theory interprets a group) and derive a contradiction. One should also note that flatness is a key element in the proof that $S_{\overline{\alpha}}$ does not interpret groups.

In [6], Pillay asked the question whether every regular type in a stable pseudofinite theory is locally modular, a statement that holds true if we replace stable with strongly minimal. In the following we show that $S_{\overline{\alpha}}$ is a pseudofinite ω-stable theory with a non-locally modular regular type, answering Pillay's question in the negative. At the time of writing these are the only known examples with this property.

Definition 7.2.21. A complete L theory T in a countable language is *pseudofinite* if for any $\theta \in T$, there is some finite L structure A such that $A \models \theta$. We call an L-structure $\mathfrak{M}$ pseudofinite if $Th(\mathfrak{M})$ is pseudofinite.

Fact 7.2.22. *Let $\langle \mathfrak{M}_i \rangle_{i \in \omega}$ be a sequence of L structures that are pseudofinite. Let $\mathcal{U}$ be an ultrafilter on ω. Then $\Pi_{\mathcal{U}} \mathfrak{M}_i$, the ultraproduct of $\langle \mathfrak{M}_i \rangle_{i \in \omega}$ (with respect to $\mathcal{U}$), is also pseudofinite.*

Theorem 7.2.23. *There is a pseudofinite ω-stable theory with a non-locally modular regular type.*

Proof. Consider the case where $L = \{E\}$ contains only one relation symbol (of arity at least 2) and let $\overline{\alpha} \in (0, 1)$ be rational. We claim that $S_{\overline{\alpha}}$ has the required properties.

Let $\{\alpha_n\}$ be an increasing sequence of irrationals in $(0, 1)$ that converge to $\overline{\alpha}$. By the results of [14], it follows that $Th(\mathfrak{M}_{\alpha_n})$ can be obtained as a almost sure theory with respect to a certain probability measure. Thus, in particular, each theory $Th(\mathfrak{M}_{\alpha_n})$ is pseudofinite. Now by Theorem 4.2 of [21], it follows that $S_{\overline{\alpha}} = Th(\Pi_{\mathcal{U}} \mathfrak{M}_{\alpha_n})$ where $\mathcal{U}$ is a non-principal ultrafilter on ω. Since taking the ultraproduct of structures with pseudofinite theories results in a structure with a pseudofinite theory, it follows that $S_{\overline{\alpha}}$ is pseudofinite. Further as we have shown in Theorem 7.2.19 that $1/c$-nuggets are non-locally modular and the result follows. $\square$

Remark 7.2.24. Assume that L has just one reation symbol whose arity we denote by r. We observe here that the pseudofiniteness of $S_{\overline{\alpha}}$ has a curious property, namely that given $\theta \in S_{\overline{\alpha}}$, it is *not necessary* that any finite $\mathfrak{B}$ such that $\mathfrak{B} \models \theta$ is in $K_{\overline{\alpha}}$.

In order to see this we first describe a *k-fan* over some fixed point a. Fix a point a and some positive integer k. Let $\mathfrak{A}$ be an L structure with exactly r points, $a \in A$ and E hods on $\mathfrak{A}$. A k-fan over a, is simply the free join of k-copies of $\mathfrak{A}$ over $\{a\}$.Clearly any k fan lies in $K_{\overline{\alpha}}$.

Fix $k > r/\alpha$. Now take θ_k to be the sentence saying, that there is at least two elements, and every element has a k-fan over it. Clearly $S_{\overline{\alpha}} \models \theta_k$. Now assume that there is some $\mathfrak{B} \in K_{\overline{\alpha}}$ such that $\mathfrak{B} \models \theta_k$. Now $\delta(\mathfrak{B}) = |B| - \alpha|E^B|$. But $|E^B| \geq |B|k/r$. Thus we obtain that $\delta(\mathfrak{B}) < 0$, a contradiction which establishes our claim.

Chapter 8: Wrapping things up: Graph-like with weight one

In this chapter we study the case in which each relation $E \in L$ is binary and $\overline{\alpha}(E) = 1$ for each $E \in L$. This case, which we denote by $\overline{\alpha}$ *is graph-like with weight one*, shares many of the results of their counterparts such as the quantifier elimination, number of countable models, regularity of types etc. However Theorem 3.2.15 does not hold in this context and as such we must use ad hoc arguments to establish the various results. It should be noted that the faliure of Theorem 3.2.15 is reflected in the forking properties of the corresponding $S_{\overline{\alpha}}$ (see Chapter 8.3).

8.1 Some Prelimanaries

This section is devoted to setting up terminology and results that will allow us to study the case that $\overline{\alpha}$ is graph-like with weigh one in depth.

Definition 8.1.1. Let $\mathfrak{F} \in K_{\overline{\alpha}}$ and $a, b \in F$. We say that *there is a path between a and b* if there is $k \geq 1$, distinct $a_0, \ldots, a_k \in F$ with $\langle E_i : 1 \leq i \leq k-1 \rangle$ such that $(a_i, a_{i+1}) \in E_i^{\mathfrak{F}}$. We say that a, b are *connected* if there is a path between a and b. By definition $\{a\}$ is always connected, i.e. a is connected to a.

Definition 8.1.2. Let $\mathfrak{A} \in K_{\overline{\alpha}}$ and let $a \in A$. The *connected component of a* (in $\mathfrak{A}$) is the substructure of $\mathfrak{A}$ which contains all the points of A that is path connected to

a. We say that $\mathfrak{A}$ is connected if for some (alternatively for all) $a \in A$, the connected component of a is $\mathfrak{A}$.

Definition 8.1.3. Let $\mathfrak{A} \in K_{\overline{\alpha}}$ and let $a, b \in A$. The *distance* from a to b in $\mathfrak{A}$; denoted by $\mathrm{dis}(a, b)$ is $\min\{n : \langle E_n : E_n \in L\rangle$ is a path from a to $b\}$ when $a \neq b$. $\mathrm{dis}(a, a) = 0$ and if there is no path between a, b when $a \neq b$ we let $\mathrm{dis}(a, b) = \infty$.

Remark 8.1.4. Recall that by definition a point is always connected to itself. So if $a \in A$ such that there is no path to a $b \in A$ distinct from a, then the connected component of a in $\mathfrak{A}$ is $\{a\}$. It is also clear the the different path components of $\mathfrak{A}$ will be freely joined over the $\emptyset$.

Proposition 8.1.5. *Let $\mathfrak{A} \in K_{\overline{\alpha}}$ be connected. Then $\delta(\mathfrak{A}) \leq 1$. Further if $\delta(\mathfrak{A}) = 1$, then for all $\emptyset \neq \mathfrak{A}' \subseteq \mathfrak{A}$, $\delta(\mathfrak{A}') \geq 1$.*

Proof. We prove the first half of the claim by induction on the size of the structure. If $|A| = 1$, then the result follows. Assume that the statement holds true for all connected structures of size up to and including n. Let $\mathfrak{A}$ be a connected structure of size $n+1$. Fix a point $a \in A$. Consider $A' = A - \{a\}$. There are two possibilities: If A' is connected, then $\delta(A') \leq 1$. Since A is connected this means there has to be some $E \in L$ such that $(a, b) \in E^{\mathfrak{A}}$ for some $b \in A'$. Now $\delta(a/A') \leq 1 - 1$ and hence it follows that $\delta(\mathfrak{A}) \leq \delta(A')$. So suppose that A' is disconnected. Thus it splits up into at most $k \leq n$ connected components $A'_1, \ldots, A'_k$, each of size $< n$. Since A is connected it follows that for each $1 \leq i \leq k$, there must be at least one relation symbol $E_k \in L$, $a_k \in A'_k$ such that $(a, a'_k) \in E_k^{\mathfrak{A}}$. Now $\delta(a/A') = 1 - e(a, A')$. So

$\delta(aA') = 1 - e(a, A') + \delta(A') = 1 - e(a, A') + \sum_{i=1}^{k} \delta(A_i)$. Now $e(a, A') \geq k$ and $\sum_{i=1}^{k} \delta(A_i) \leq k$. Thus $\delta(A) \leq 1$ and the result holds.

For the second half of the claim, let $\mathfrak{A} \in K_{\overline{\alpha}}$ be connected and assume that $\emptyset \neq A' \subseteq A$. By the above, the statement holds if $A' = A$. So assume that $A' \subsetneq A$. We can view A' as the free join of connected components $A'_1, \ldots, A'_k$ for some $k \in \mathbb{N}$. Now $\delta(A') = \sum_{i=1}^{k} \delta(A'_i)$. Assume that $\delta(A'_i) = 0$ for each $1 \leq i \leq k$ and further assume that $k \geq 2$. Now since $\mathfrak{A}$ is connected, there is some path (in $\mathfrak{A}$) between some point $a_1 \in A'_1$ and $a_2 \in A'_2$. Picking a a_1, a_2 such that they will be a minimal distance apart and calculating $\delta(A_1 A_2 P)$ where P contains the rest of the points in the path from a_1 to a_2, we see that $\delta(A_1 A_2 P) \leq \delta(A_1) + \delta(A_2) + |P| - (|P| + 1) < 0$, a contradiction. So assume that $k = 1$ and thus $A' = A'_1$ and A' is connected.

We claim that $\delta(A'_1) = 1$. Suppose to the contrary that $\delta(A'_1) = 0$. Note that $A - A'_1$ also decomposes into connected components, say $A''_1, \ldots, A''_{k'}$. If $\delta(A''_{i_0}) = 0$ for some $1 \leq i_0 \leq k'$, then a path of minimal length connecting a point $a \in A'_1$ and $b \in A''_{i_0}$ yields a contradiction as above. Thus $\delta(A''_i) \geq 1$ for each $1 \leq i \leq k'$. But as the A''_i are connected, this implies that $\delta(A''_i) = 1$ for each i. Pick points $a_0 \in A'_1, b_0 \in A - A'_1$ such that it witnesses a path between A'_1 and $A - A'_1$ of least distance. It is clear that such a path satisfies $\mathrm{dis}(a_0, b_0) = 1$. Now this connects up with some A''_i. But $\delta(A'_1 A''_i) \leq \delta(A'_1) + \delta(A''_i) - 1 = 0$. Thus $\delta(A'_1 A''_i) = 0$. Now $\delta(A'_1 A''_1) = 0$. An easy induction argument using the existence of a minimal path yields that $\delta(A'_1 A''_1 \ldots A''_i) = 0$ for each $1 \leq i \leq k'$. But as $A = A'_1 A''_1 \ldots A''_{k'}$ we see that $\delta(A) = 0$ which contradicts $\delta(\mathfrak{A}) = 1$ and establishes our claim. $\square$

Definition 8.1.6. Let $\mathfrak{A}, \mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \leq \mathfrak{D}$. Let k be a positive integer. We call a partition $D = AD_1D_2H_1...H_k$ (with A, D_1, D_2 possibly empty) a *decomposition of D over A of length k* if AD_1 is the connected component of A (in D), $H_1, \dots H_k$ are connected components with $\delta(H_i/A) = 1$, $\delta(D_i/A) = 0$ and no point in D_2 connects to any point in A

Remark 8.1.7. Given a decomposition of D over A of length k, it is immediate that $D_1, D_2, H_1 \dots H_k \subseteq D - A$ and that $D_1, D_2, H_1 \dots H_k$ are freely joined over A with no relations between AD_1 and any of $D_2, H_1 \dots H_k$.

The following allows us to decompose a given structure.

Proposition 8.1.8. *Suppose that $\overline{\alpha}$ is graph-like with weight one. Let $\mathfrak{A}, \mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \leq \mathfrak{D}$ and $\delta(\mathfrak{D}/\mathfrak{A}) = k > 0$. There exists $\{d_1, \dots d_k\} \subseteq D - A$ such that their connected components $H_{d_1}, \dots H_{d_k}$ along with A, D_1, D_2 forms a decomposition of D of length k.*

Proof. First assume that the statement "there is at least k points in $D - A \supseteq \{d_1, \dots d_k\}$ such that for any $1 \leq i \leq k$, $a \in A$, a is not connected to d_i" fails. Then the number of points in $D - A$ not connected to a point in A is less than k. Further, note that if $d \in D - A$ is connected to some point $a \in A$ by some path of length n, then $0 = n - 1 - (n - 1) \geq \delta(d/A)$. Let $D_1 \subseteq D - A$ be the (possibly empty) set of points in $D - A$ that is connected to some point in A. Let $C = (D - A) - D_1$, i.e. the set of points in $D - A$ that is not connected to any point in A.

Note that C, D_1 are freely joined over A. Now $\delta(D/A) = \delta(CD_1/A) = \delta(C/A) + \delta(D_1/A)$. But $\delta(D_1/A) \leq \sum_{d \in D_1} \delta(d/A) = 0$ by Fact 2.2.5. Further

$\delta(C/A) = \delta(C) - e(C, A) = |C| - e(C) - e(C, A) = |C| - e(C) = \delta(C)$. Thus it follows that $|C| \geq k$ which establishes the weaker claim that there is at least k points in $D - A \supseteq \{d_1, \ldots d_k\}$ such that for any $1 \leq i \leq k$, $a \in A$, a is not connected to d_i. This further shows that $\delta(D/A) = \delta(C/A)$ as $\mathfrak{A} \leq \mathfrak{D}$ implies that $\delta(D_1/A) = 0$.

Let $c \in C$ and let H_c be the set of points in $D - A$ that is connected to c. Note that $\delta(H_c/A) = \delta(H_c) \leq 1$ by Proposition 8.1.5. Let $D_2 \subseteq C$ be such that $d \in D_2$ if and only if $\delta(H_d) = 0$. Fix $C' \subseteq C$ such that for each $c \in C - D_2$, there is precisely one $c' \in C'$ such that $H_{c'} = H_c$. An easy argument shows that $|C'| = k$ and that an enumeration of C' provides the required points $d_1, \ldots, d_k$ with the required properties. That $d_1, \ldots, d_k, D_1, D_2$ are as required follows easily. $\square$

Definition 8.1.9. Let $E \in L$ be binary and let $\mathfrak{B} \in K_{\overline{\alpha}}$. We say that $\mathfrak{B}$ is an *n-cycle* (in E) if there exists an enumeration $\{b_1, \ldots, b_n\}$ of B such that $\{b_i, b_{i+1}\} \in E^{\mathfrak{B}}$ for each $1 \leq i \leq n-1$, $\{b_n, b_1\} \in E^{\mathfrak{B}}$ and for any other pair b_i, b_j; $\{b_i, b_j\} \notin E^{\mathfrak{B}}$. Abusing notation we say that $\mathfrak{B}$ is a *cycle* if there is a sequence $\langle E_i : 1 \leq i \leq n\rangle$ of binary relations from L such that $\{b_i, b_{i+1}\} \in E_i^{\mathfrak{B}}$ for each $1 \leq i \leq n-1$, $\{b_n, b_1\} \in E_n^{\mathfrak{B}}$

We end the section with the following:

Proposition 8.1.10. *Let $\mathfrak{A} \in K_{\overline{\alpha}}$ be connected. Now $\delta(\mathfrak{A}) = 0$ if and only if $\mathfrak{A}$ contains a cycle.*

Proof. Note that if $\mathfrak{B} \in K_{\overline{\alpha}}$ is a cycle, a simple calculation shows that $\delta(\mathfrak{B}) = 0$. Thus if $\mathfrak{A}$ contains a cycle $\mathfrak{B}$, then $\mathfrak{A}$ has a non-trivial substructure with rank zero and hence Proposition 8.1.5 yields the required result.

We prove the opposite direction by induction. Note that the statement holds true for $|A| \leq 3$ by an examination of the possibilities for $\mathfrak{A}$. Assume that the statement holds true for all structures with at most n elements. Let $\mathfrak{A}$ be such that $|A| = n+1$. Fix a point $a \in A$ and let $A' = A - a$. If A' contains a cycle then we are done. So assume that it does not. Then by the induction hypothesis and Proposition 8.1.5 we see that $\delta(A') \geq 1$. Now $\delta(\mathfrak{A}) = \delta(A'a) = \delta(A') + |1| - e(a, A')$ which in turn yields that $e(a, A') = \delta(A') + 1$. In particular $e(a, A') \geq 2$. We claim this implies that there is a cycle that contains a.

To see this, first assume that there is a path in A' between two distinct points which witness $e(a, A') \geq 2$. Then clearly there is a cycle. So assume this fails. Then A' splits into k connected components for some k. By the induction assumption this implies that for each component ${A'}_i$, $\delta({A'}_i) = 1$. Thus $e(a, \mathfrak{A}') = k + 1$. Using the fact that the A'_i are freely joined over $\emptyset$, we obtain that $e(a, A') = \sum_{i=1}^{k} e(a, A'_i)$. A simple application of the pigeonhole principle yields that there must be some connected component A'_{i_0} for which $e(a, A'_{i_0}) \geq 2$, a contradiction. □

8.2 Shared Results

In the previous chapters we have obtained a number of results regarding the case $\overline{\alpha}$ is graph-like with weight one, often postponing the proof of technical results. In this section, we prove such technical results. We also show how the proof of theorems regarding countable models and DOP can be extended to cover the case that $\overline{\alpha}$ is rational. Note that under the current condition that c, the least common multiple of the denominators of the weights is 1 and that $Gr(m) = 1$ for all positive integers $m \geq 2$.

8.2.1 Quantifier Elimination and Atomic Models

We begin this section by constructing various finite structures in the spirit of Chapter 3. Our first goal is to obtain the quantifier elimination result.

Definition 8.2.1. Let $\mathfrak{B} \in K_L$ be non-empty and let $n \geq 3$ be a positive integer. Given $b \in B$ we say that $\mathfrak{D} \in K_L$ *is obtained by attaching an n-cycle (of type $E \in L$) to b* if

1. $\mathfrak{B} \subseteq \mathfrak{D}$
2. The set $D - B = \{d_1, \ldots, d_{n-1}\}$
3. $\{b, d_1, \ldots, d_{n-1}\}$ is an n-cycle in E
4. The only new relations that hold in $\mathfrak{D}$ are the E relation symbols just described.

Remark 8.2.2. Given non-empty $\mathfrak{B} \in K_L$, $n \in \mathbb{N}$ be such that $n \geq 3$ and $b \in B$ we see that we can attach an n-cycle (of type $E \in L$) to b.

Lemma 8.2.3. *Let $\mathfrak{A} \in K_{\overline{\alpha}}$ be such that $\delta(\mathfrak{A}) > 0$. Then there exists an $a \in A$ such that if we attach any n-cycle to a, the resulting structure $\mathfrak{B}$ will be in $K_{\overline{\alpha}}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$, $\delta(\mathfrak{B}/\mathfrak{A}) = -1$.*

Proof. Using Fact 2.2.5 with $\emptyset \leq \mathfrak{A}$, fix some point $a \in A$ such that if $a \in A' \subseteq A$, then $\delta(\mathfrak{A}') > 0$. Let $n \geq 3$ be a positive integer. We claim that the structure $\mathfrak{B}$

obtained by attaching an n-cycle to a satisfies the required properties. it is clear by construction that $\delta(\mathfrak{B}/\mathfrak{A}) = -1$ so it remains to establish that $\mathfrak{B} \in K_{\overline{\alpha}}$. Let $\mathfrak{B}' \subseteq \mathfrak{B}$. Let $\mathfrak{B}^* = (\mathfrak{B} - \mathfrak{A}) \cap \mathfrak{B}'$ and let $\mathfrak{A}^* = \mathfrak{A} \cap \mathfrak{B}'$. There are two cases to consider: If $a \notin A^*$ then $\delta(\mathfrak{B}^*/\mathfrak{A}^*) \geq \delta(B^*/\mathfrak{A}) \geq k - (k-1)$ for some k If $a \in A^*$, then $\delta(\mathfrak{B}^*/\mathfrak{A}^*) \geq \delta(B^*/A) \geq -1$ and hence $\delta(B^*A^*) = \delta(\mathfrak{B}') = -1 + \delta(\mathfrak{A}^*) \geq 0$. $\square$

With this lemma in hand, we obtain the following weak version of Theorem 3.2.15 that encompasses the case that $\overline{\alpha}$ is graph-like with weight zero:

Theorem 8.2.4. *Let $\mathfrak{A} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{A}) > 0$. we can construct infinitely many non-isomorphic $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $(\mathfrak{A}, \mathfrak{D})$ is a minimal pair that satisfies $\delta(\mathfrak{D}/\mathfrak{A}) = -1$.*

Proof. The required structure can be obtained by attaching an n-cycle as in Lemma 8.2.3. Varying the value of n yields the non-isomorphic minimal pairs. $\square$

Remark 8.2.5. It should be noted that $(\mathfrak{A}, \mathfrak{D})$, in general, will not be an essential minimal pair. Further we are able to build rank 0 extension of finite structures with positive rank.

We now extend Lemma 4.2.1 in the following manner:

Lemma 8.2.6. *Suppose that $\mathfrak{A} \leq \mathfrak{B} \in K_{\overline{\alpha}}$ and $\Phi \subseteq_{Fin} K_{\overline{\alpha}}$ are given such that $\mathfrak{B} \subseteq \mathfrak{C}$ with $\mathfrak{B} \not\leq \mathfrak{C}$ for all $\mathfrak{C} \in \Phi$. Let $m \in \mathbb{N}$. Then there is a $\mathfrak{D}^* \supseteq \mathfrak{B}$, $\mathfrak{D}^* \in K_{\overline{\alpha}}$ such that*

1. $0 = \delta(\mathfrak{D}^*/\mathfrak{A})$
2. $\mathfrak{A} \leq \mathfrak{D}^*$

3. No $\mathfrak{C} \in \Phi$ isomorphically embeds into $\mathfrak{D}^*$ over $\mathfrak{B}$

Proof. Fix $\mathfrak{A}, \mathfrak{B}$ and Φ as above. Note that we may replace each $\mathfrak{C} \in \Phi$ by $\mathfrak{B} \subseteq \mathfrak{C}' \subseteq \mathfrak{C}$ that is minimal and thus we may as well assume that $(\mathfrak{B}, \mathfrak{C})$ is a minimal pair for any given $\mathfrak{C} \in \Phi$. Now if $\delta(\mathfrak{A}) = \delta(\mathfrak{B})$, then take $\mathfrak{D}^* = \mathfrak{B}$. So we may assume that $\delta(\mathfrak{A}) < \delta(\mathfrak{B})$. Let u be a positive integer such that $u > |\mathfrak{C}|$ for each $\mathfrak{C} \in \Phi$.

We construct $\mathfrak{D}^*$ in two steps. First we chow that certain extension of $\mathfrak{B}$ by n-cycles have certain desired properties. Then we iterate this process as required.

Using (6) of Fact 2.2.5, fix $b \in B - A$ such that for all $B' \subseteq B$ with $aA \subseteq B'$, $\delta(\mathfrak{B}'/\mathfrak{A}) > 0$. Also fix a positive integer $n > u+3$ and a relation symbol E. Consider the structure $\mathfrak{D}$ obtained by attaching an n-cycle to b. We claim that $\delta(\mathfrak{D}/\mathfrak{B}) = -1$, $\mathfrak{D} \in K_{\overline{\alpha}}$ and $\mathfrak{A} \leq \mathfrak{D}$.

It is clear by construction that $\delta(\mathfrak{D}/\mathfrak{B}) = -1$. In order to establish the remaining claims, we set up some notation: Let $\mathfrak{D}' \subseteq \mathfrak{D}$, $\mathfrak{B}' = \mathfrak{D}' \cap \mathfrak{B}$ and $\mathfrak{A}' = \mathfrak{D}' \cap \mathfrak{A}$. If we establish that $\delta(\mathfrak{D}'/\mathfrak{A}') \geq 0$ both our claims follow: $\delta(\mathfrak{D}') \geq \delta(\mathfrak{A}') \geq 0$ yields $\mathfrak{D} \in K_{\overline{\alpha}}$ and the case that $\mathfrak{A}' = \mathfrak{A}$ yields $\mathfrak{A} \leq \mathfrak{D}$. Note that $\delta(\mathfrak{D}'/\mathfrak{A}') = \delta(\mathfrak{D}'/\mathfrak{B}') + \delta(\mathfrak{B}'/\mathfrak{A}')$. Further $\delta(\mathfrak{B}'/\mathfrak{A}') \geq \delta(AB'/A) \geq 0$ and $\delta(\mathfrak{D}'/\mathfrak{B}') \geq \delta(BD'/B)$. Now there are two cases to consider. First assume that $b \notin B'$. Now $\delta(BD'/B) \geq 1$ by construction and $\delta(AB'/A) \geq 0$ as $\mathfrak{A} \leq \mathfrak{B}$. Thus we obtain the required result. So assume that $b \in B'$. Now $\delta(BD'/B) \geq -1$ by construction and $\delta(Ab'/A) \geq 1$ by our choice of b. Hence the required result again follows.

We now construct $\mathfrak{D}^*$. If $\delta(\mathfrak{D}/\mathfrak{A}) = 0$, then setting $\mathfrak{D}^* = \mathfrak{D}$ yields a $\mathfrak{D}^*$ with the required properties. So assume that $\delta(\mathfrak{D}/\mathfrak{A}) > 0$. Now note that b *does not* have the property that for all $bA \subseteq D' \subseteq D$, $\delta(\mathfrak{D}'/\mathfrak{A}) > 0$. However using Fact 2.2.5 we may fix $d \in D - A$ (necessarily $d \neq b$), such that for all $dA \subseteq D' \subseteq D'$ $\delta(\mathfrak{D}'/\mathfrak{A}) > 0$. Attaching an n-cycle to d yields a structure $\mathfrak{D}_1$ with $\mathfrak{A} \leq \mathfrak{D}_k$, $\delta(\mathfrak{D}_1/\mathfrak{B}) = -2$ and $\delta(\mathfrak{A}) \leq \delta(\mathfrak{D}_1) = \delta(\mathfrak{B}) - 2$. It is now clear that iterating this process sufficiently many times we can obtain a structure $\mathfrak{D}^* = \mathfrak{D}_k$ with $\mathfrak{A} \leq \mathfrak{D}^*$, $\delta(\mathfrak{D}^*/\mathfrak{B}) = -k$ and $\delta(\mathfrak{A}) = \delta(\mathfrak{D}^*) = \delta(\mathfrak{B}) - k$. Note that $\mathfrak{D}^*$ may be viewed as being obtained by attaching k many n-cycles to a specially selected set of k points in $B - A$. Now if $\mathfrak{B} \subseteq \mathfrak{D}'\mathfrak{D}^*$ is such that $\delta(\mathfrak{D}'/\mathfrak{B}) < 0$, then $\mathfrak{D}'$ must contain one of the newly attached n-cycles. Thus it follows that $|D' - B| \geq n > u + 3$ and hence *no* $\mathfrak{C} \in \Phi$ embeds into $\mathfrak{D}_k$ □

Remark 8.2.7. The above lemma is the key to establishing the quantifier elimination result Theorem 4.3.5 by allowing us to prove Lemma 4.2.1. As a result, the results of Section 4.4 hold when $\overline{\alpha}$ is graph-like with weight one.

Further the above lemma also allows us to prove Theorem 3.3.6. Note that the the results of Chapter 5 depends only the quantifier elimination result and Theorem 3.3.6. Thus the above Lemma is the key to obtaining these results in the case $\overline{\alpha}$ is graph-like with weight one.

8.2.2 Countable Models, DOP and Regular Types

In this section, we extend our results regarding countable models and the DOP to the case that $\overline{\alpha}$ is graph-like with weight one.

We now establish that Lemma 7.1.3 holds in the case that $\overline{\alpha}$ is graph-like with weight one.

Lemma 8.2.8. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \leq \mathfrak{B}, \mathfrak{C}$; $\delta(\mathfrak{C}/\mathfrak{A}) \geq \delta(\mathfrak{B}/\mathfrak{A})$ and $\mathfrak{D} = \mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C}$ the free join of $\mathfrak{B}, \mathfrak{C}$ over $\mathfrak{A}$. We can construct $\mathfrak{H} \in K_{\overline{\alpha}}$ such that $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \leq \mathfrak{H}$, $\mathfrak{D} \subseteq \mathfrak{H}$ and $\delta(\mathfrak{H}/\mathfrak{C}) = 0$. Further if $\delta(\mathfrak{B}/\mathfrak{A}) = \delta(\mathfrak{C}/\mathfrak{A})$, the $\mathfrak{H}$ that was constructed has the property $\delta(\mathfrak{H}/\mathfrak{B}) = 0$.*

Proof. Clearly $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \leq \mathfrak{D}$ and $\mathfrak{D} \in K_{\overline{\alpha}}$. Let $\delta(\mathfrak{B}/\mathfrak{A}) = m/c$ and $\delta(\mathfrak{C}/\mathfrak{A}) = k/c$. It is easily seen that $\delta(\mathfrak{D}/\mathfrak{A}) = (m+k)/c$; $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \leq \mathfrak{D}$; $\delta(\mathfrak{D}/\mathfrak{A}) = \delta(\mathfrak{D}/\mathfrak{C}) + \delta(\mathfrak{C}/\mathfrak{A})$ and that $\delta(\mathfrak{D}/\mathfrak{A}) = \delta(\mathfrak{D}/\mathfrak{B}) + \delta(\mathfrak{B}/\mathfrak{A})$. Assume that $\delta(\mathfrak{B}/\mathfrak{A}) = 0$. Take $\mathfrak{H} = \mathfrak{D}$. Now a routine verification using $\delta(D/A) = \delta(C/A) + \delta(B/A)$ yields $\delta(D/C) = 0$ and hence the required result. So suppose that $\delta(\mathfrak{B}/\mathfrak{A}) = m$ with $m \geq 1$.

Using Proposition 8.1.8, we may decompose $C - A$, $B - A$ as $H_{d_1}, \ldots, H_{d_k}, D_1, D_2$; $H_{d'_1}, \ldots, H_{d'_m}$ D'_1, D'_2 respectively. Fix some $E \in L$. Consider the structure $\mathfrak{H}$ with the underlying *set* D, $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \subseteq \mathfrak{H}$ and m new edges $(d_i, d'_i) \in E^{\mathfrak{H}}$ for $1 \leq i \leq m$. Let $\mathfrak{H}' \subseteq \mathfrak{H}$, $\mathfrak{A}' = \mathfrak{H}' \cap \mathfrak{A}$, $\mathfrak{B}' = \mathfrak{H}' \cap \mathfrak{B}$ and $\mathfrak{C}' = \mathfrak{H}' \cap \mathfrak{C}$.

We first show $\mathfrak{B} \leq \mathfrak{H}$. This establishes $\mathfrak{H} \in K_{\overline{\alpha}}$ and $\mathfrak{A} \leq \mathfrak{H}$ as $\mathfrak{A} \leq \mathfrak{B}$. So assume that $\mathfrak{B} \subseteq \mathfrak{H}'$ and hence $\mathfrak{B}' = \mathfrak{B}$. Now $\delta(\mathfrak{H}'/\mathfrak{B}) = \delta((H' \cap C)/B)$. Note that $H' \cap C = (H_{d_1} \cap C') \ldots (H_{d_k} \cap C')(D_2 \cap C')(D_1 \cap C')$ and the structures that appear here are *pairwise* freely joined over $\mathfrak{B}$. By Proposition 8.1.5, it follows that no non-empty substructure $\mathfrak{G}$ of $\mathfrak{H}_i$ (or $\mathfrak{H}'_i$) has rank zero. Thus it follows that $\delta((H_{d_j} \cap C')/B) \geq 1 - 1 = 0$. Hence $\delta(\mathfrak{H}'/\mathfrak{B}) = \sum_{i=1}^{k} \delta(H_{d_i} \cap C'/B) + \delta(D_1/B) + \delta(D_2/B)$. But $\delta(D_i/B) = \delta(D_i/A)$ and the claim follows.

The proof $\mathfrak{C} \leq \mathfrak{H}$ is similar. Further $\delta(\mathfrak{H}/\mathfrak{C}) = 0$ follows as we have added exactly m new edges. Note that unlike when $\overline{\alpha}$ was graph-like with weight one $\mathfrak{D} \not\subseteq \mathfrak{H}$. □

The following is Lemma 7.1.6 for the case that $\overline{\alpha}$ is graph-like with weight one.

Lemma 8.2.9. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$ and $\mathfrak{A} \leq \mathfrak{M}$ be finite. Let $\mathfrak{D} \in K_{\overline{\alpha}}$ be such that $\mathfrak{A} \leq \mathfrak{D}$. Then $\dim(\mathfrak{M}/\mathfrak{A}) \geq \delta(\mathfrak{D}/\mathfrak{A})$ if and only if there is some g such that g strongly embeds $\mathfrak{D}$ into $\mathfrak{M}$ over $\mathfrak{A}$.*

Proof. The statement that if there is some g such that g strongly embeds $\mathfrak{D}$ into $\mathfrak{M}$ over $\mathfrak{A}$, then $\dim(\mathfrak{M}/\mathfrak{A}) \geq \delta(\mathfrak{D}/\mathfrak{A})$ is immediate from the definition. Thus we prove the converse. Let $\mathfrak{A} \leq \mathfrak{M}$ be finite. Let $\mathfrak{D} \in K_{\overline{\alpha}}$ be such that $\mathfrak{A} \leq \mathfrak{D}$.

First assume that $\delta(\mathfrak{D}/\mathfrak{A}) = 0$. Now as $S_{\overline{\alpha}} \models \forall \overline{x} \exists \overline{y}(\Delta_{\mathfrak{A}}(\overline{x}) \implies \Delta_{\mathfrak{A},\mathfrak{D}}(\overline{x}, \overline{y}))$. Thus there is some $\mathfrak{A} \subseteq \mathfrak{D}' \subseteq \mathfrak{M}$ such that $\mathfrak{D} \cong_{\mathfrak{A}} \mathfrak{D}'$. Further as $\delta(\mathfrak{D}'/\mathfrak{A}) = 0$, from (2) of Lemma 4.4.2, $\mathfrak{D}' \leq \mathfrak{M}$. Thus regardless of the value of $\dim(\mathfrak{M}/\mathfrak{A})$, if $\delta(\mathfrak{D}/\mathfrak{A}) = 0$ then there is some g such that g strongly embeds $\mathfrak{D}$ into $\mathfrak{M}$ over $\mathfrak{A}$.

Assume that $\overline{\alpha}$ is graph-like with weight one. Now using Proposition 8.1.8 we can decompose $D - A$ into the free join of $k \leq i \leq k+2$ *non-empty* substructures over $\mathfrak{A}$: $H_{d_1}, \ldots, H_{d_k}, D_2, D_1$ (note that D_2 or D_1 might be empty). Now $\delta(\mathfrak{D}/\mathfrak{A}) = \sum_{j=1}^{k} \delta(H_{d_j}/A) + \delta(D_1 D_2/A) = \sum_{j=1}^{k} \delta(H_{d_j}/A)$. By (2) of Lemma 4.4.2 it suffices to show that $AH_{d_1} \ldots AH_{d_k}$ embeds strongly into $\mathfrak{M}$ over $\mathfrak{A}$. Note that $\delta(\mathfrak{M}/\mathfrak{A}) \geq k$ means there are at least k distinct connected components H_i' that do not contain cycles (by a simple argument using Proposition 8.1.10). An exploration of the axioms shows that any two such components are isomorphic. It also follows that the H_{b_i} are isomorphic to substructures of those components. Fix an isomorphism that fixes $\mathfrak{A}$ and each distinct H_{b_i} is mapped on to a distinct isomorphic substructure of H'_i. We claim that this embedding is strong. To see this simply note that the existence of a minimal pair implies the existence of some cycle in H'_i or that the H'_i are not freely joined over $\mathfrak{A}$, a contradiction. □

The following theorems are now immediate.

Theorem 8.2.10. *Let $\overline{\alpha}$ be graph-like with weight one. Let $\mathfrak{M}, \mathfrak{N} \models S_{\overline{\alpha}}$ be countable. Now $\mathfrak{M} \cong \mathfrak{N}$ if and only if $\dim(\mathfrak{M}) = \dim(\mathfrak{N})$. Thus there are precisely $\aleph_0$ many non-isomorphic models of $S_{\overline{\alpha}}$ of size $\aleph_0$. Further each countable model of $S_{\overline{\alpha}}$ can be built up from a subclass of $(K_{\overline{\alpha}}, \leq)$.*

Proof. Is the same as the proof of Theorem 7.1.7. □

We now give a proof that $S_{\overline{\alpha}}$ has the DOP.

Theorem 8.2.11. *$S_{\overline{\alpha}}$ has the DOP*

Proof. The proof is the same as the proof of Theorem 6.4.8 (as noted therein). However the proof would utilize Lemma 8.2.6 instead of Lemma 4.2.1 in the proof.

□

We will finish this section with some results on regular types. As in the case that $\overline{\alpha}$ is rational but not graph-like with weight one, 0 and 1 nugget-like types will regular.

Theorem 8.2.12. *Let $A \leq \mathbb{M}$ be finite and let $p \in S(A)$ be nugget-like. Now if $d(p/A) = 0$ or $d(p/A) = 1$, then p is regular. Further if $d(p/A) = 0$, then p is orthogonal to any other nugget-like type over A.*

Proof. Note that a single point consists of a 1-nugget over $\emptyset$ while any l-cycle will form a 0-nugget for $l \geq 3$ over $\emptyset$. It is easy to see that this extends to nuggets over some fixed $\mathfrak{B} \in K_{\overline{\alpha}}$. The rest of the proof is the same as that of 7.2.10. □

As in the case that $\overline{\alpha}$ is rational but not graph-like with weight one distinct types p, q with $d(p/A) = d(q/A) = 1$ are non-orthogonal.

Theorem 8.2.13. *Let A be closed and finite and let $p, q \in S(A)$ be distinct and satisfy $d(p/A) = d(q/A) = 1$. Then they are non-orthogonal. Hence any two regular types over $p', q' \in S(X)$ where X is closed and $d(p'/X) = d(q'/X) = 1$ are non-orthogonal. Further if we take $A = \emptyset$ and let $\mathfrak{M} \preccurlyeq \mathbb{M}$. The dimension of $\mathfrak{M}$ is determined by the number of independent realizations of p in $\mathfrak{M}$. Thus a single regular type determines the dimension of $\mathfrak{M}$.*

Proof. Let A be as given. As in Theorem 7.2.13, consider A as a finite structure that lives in $K_{\overline{\alpha}}$. Consider the finite structures AB, AC where B, C realize the quantifier free types of p, q respectively.

By Proposition 8.1.8 there exists $d_1 \in B$ such that d_1 is not connected to any $a \in A$ and $\delta(H_{d_1}/A) = 1$. Now similar comments hold with $d_1' \in C$. Fix $E \in L$. Consider the structure D with universe $ABCd$ where $d \notin ABC$. The structure D contains ABC as a substructure, $(d_1, d), (d_1', d) \in E^D$ and no other relations.

Routine arguments now yield that $D \in K_{\overline{\alpha}}$, $AB, AC \leq D$ but $ABC \not\leq D$. The rest of claim now follows as in Case 1 above.

For the second half of the claim note that given $p', q' \in S(X)$, there exists A' finite and closed such that p' is based and stationary over A' and B' such that q is based and stationary over B'. Let X' be the closure of $A'B'$ and consider $p|_{X'}, q|_{X'}$. Since regularity is parallelism invariant both $p|_{X'}$ and $q|_{X'}$ are regular. Arguing as above we see that $p'|_{X'} \not\perp q'|_{X'}$. Thus the first half of the result now follows.

Let $\mathfrak{M} \preccurlyeq \mathbb{M}$ and assume that $A = \emptyset$. Given $n \in \omega$, consider the finite structure C_n that is the free join of n-copies of the quantifier free type of p over $\emptyset$. If $\dim(\mathfrak{M}) \geq n$, by Lemma 8.2.9, there is a strong embedding of C_n into $\mathfrak{M}$. It is easily checked that the strong embedding witnesses n-independent realizations of p. The rest follows easily. $\square$

The result that any basic type p with $d(p) \geq 2$ cannot be be regular also holds.

Theorem 8.2.14. *Let A be finite and closed in $\mathbb{M}$. Let $p \in S(A)$ be a basic type such that $d(p/A) \geq 2$. Then p is not regular.*

Proof. Our strategy is the same as that in the proof of Theorem 7.2.13: we consider A as living inside of $K_{\overline{\alpha}}$, i.e. as a finite structure. We then construct a finite structure G over the finite structure A that we then embed strongly into $\mathbb{M}$ over $\mathfrak{A}$ using saturation. Finally we argue that the strong embedding witnesses the fact that p is not regular.

Let $p \in S(A)$ be a basic type such that $d(p/A) \geq 2$. Consider A as a finite structure that lives in $K_{\overline{\alpha}}$. Let $\mathfrak{D} \in K_{\overline{\alpha}}$ be such that $D = AB$ with $A \cap B = \emptyset$

and B realizes the quantifier free type of p over A. Now using Proposition 8.1.8, fix $d_1, \ldots, d_p$ such that d_i is not connected to any point in A, d_i, d_j are not connected for $i \neq j$ and $\delta(H_{b_i}/A) = \delta(H_{b_j}/A) = 1$ where H_{b_i} is the set of points connected to b_i. Fix $E \in L$. Consider the finite structure G where the underlying domain is $AB \cup \{c_1, c_2\}$ where $AB \subseteq G$, c_1, c_2 two points not in AB. Assume that G is endowed with the structure given by considering AB as a substructure of G, $(d_i, c_i) \in E^G$ and *no* other relations hold in G.

We will show that given $A \subseteq G' \subseteq G$ $\delta(G'/A) \geq 0$. Note that by Proposition 8.1.8, we can decompose $B^* = (G - A) \cap B$ into $H_{d_1}, \ldots H_{d_k}, D_1, D_2$ that are freely joined over A. Now $\delta(B \cap G'/A) = \sum_{j=1}^{k} \delta(H_{d_j} \cap G'/A) + \delta(D_2 \cap G'/A) + \delta(D_1 \cap G'/A)$. Thus it follows that $\delta(B \cap G'/A) = \sum_{j=1}^{k} \delta(H_{d_j} \cap G') + \delta(D_2 \cap G'/A) + \delta(D_1 \cap G'/A)$.

Now observe that $B^* c_1 c_2$ and A are freely joined over $\emptyset$. The argument reduces to showing $\delta(B^* c_1 c_2 \cap G') \geq 0$. Note that H_{d_1}, H_{d_2}, the connected components in AB to which d_1 and d_2 belongs (respectively) has the property that for any $\emptyset \neq H'_{d_i} \subseteq H'_{d_i}$, $\delta(H'_{d_i}) \geq 1$. This allows us to establish that $H_{d_i} c_i \in K_{\overline{\alpha}}$ for $i = 1, 2$. Now showing $\delta(B^* c_1 c_2 \cap G') \geq 0$ reduces to a simple argument that utilizes the connected components of $G \cap B'_1$ are $H_{d_1} c_1$, $H_{d_2} c_2$ and H_{d_i} for $3 \leq i \leq p$ where the H_{d_i} are the connected components in B'_1.

Now routine arguments will show that $AB \leq G$, $Ac_i \leq G$ and $Ac_1 c_2 \leq G$. Let f be a strong embedding G to $\mathbb{M}$ such that it is the identity on A. As in the previous argument this yields that $f(B)$ realizes p over A. Now $f(B) \not\downarrow_A f(c_i)$ as B and c_i is not freely joined over A. But $Ac_i \leq G$, $Ac_1 c_2 \leq G$ and c_1, c_2 is freely joined over

A which implies $f(c_1) \mathop{\smile\!\!\!\!\!\!\!\;\;|}_A f(c_2)$. As in Theorem 7.2.13, this configuration witnesses the fact that p cannot be regular. $\square$

8.3 Where Graph-like with weight one differs

In this section, we show that if $\overline{\alpha}$ is graph-like with weight one, then $S_{\overline{\alpha}}$ is trivial. Recall that in the case that $\overline{\alpha}$ is *not* graph-like with weight one, then $S_{\overline{\alpha}}$ is non-trivial by Theorem 6.3.3. This allows us to characterize the trivial $S_{\overline{\alpha}}$ as precisely those where $\overline{\alpha}$ is graph-like with weight one.

Theorem 8.3.1. *$S_{\overline{\alpha}}$ is trivial if and only if $\overline{\alpha}$ is graph-like with weight one.*

Proof. In Theorem 6.3.3 we have established that if $\overline{\alpha}$ is not graph-like with weight one, then $S_{\overline{\alpha}}$ is non-trivial. We now show that if $\overline{\alpha}$ is graph-like with weight one, then $S_{\overline{\alpha}}$ is trivial.

So assume that $\overline{\alpha}$ is graph-like with weight one but $S_{\overline{\alpha}}$ is not trivial. Note that $S_{\overline{\alpha}}$ has finite closures and that algebraic and intrinsic closures correspond. As $S_{\overline{\alpha}}$ is not trivial, there exists W, X, Y closed in $\mathbb{M}$ whose pairwise intersection is Z such that $W \mathop{\smile\!\!\!\!\!\!\!\;\;|}_Z X, W \mathop{\smile\!\!\!\!\!\!\!\;\;|}_Z Y, X \mathop{\smile\!\!\!\!\!\!\!\;\;|}_Z Y$ but $W \not\mathop{\smile\!\!\!\!\!\!\!\;\;|}_Z XY$. By the characterization of forking in Theorem 6.2.25, W, X, Y are freely joined over Z and WX, WY, WZ are closed in $\mathbb{M}$. Note that it follows that W, XY are freely joined over Z. Thus another application of Theorem 6.2.25, yields that WXY is not closed in $\mathbb{M}$. Thus there is $W_0 \subseteq_{\text{Fin}} W$, $X_0 \subseteq_{\text{Fin}} X$ and $Y_0 \subseteq_{\text{Fin}} Y$ such that the closure of $W_0X_0Y_0$ is not contained in WXY. Replace W_0 by its closure, which clearly lies in W and is finite as $S_{\overline{\alpha}}$ has finite closures. By an abuse of notation we will call this set W_0.

Further replace X_0 by $\mathrm{acl}(X_0Y_0) \cap X$ and Y_0 by $\mathrm{acl}(X_0Y_0) \cap Y$ and call them X_0, Y_0 by a similar abuse of notation. Note that X_0, Y_0 are also closed and finite. Let $F = \mathrm{acl}(W_0X_0Y_0)$. Now $F \not\subseteq WXY$. Let $(W_0X_0Y_0, D)$ be a minimal pair with D (where $D \subseteq M$).

We claim that there is some $a \in W_0$, $b \in X_0Y_0$ such that there is a path between a, b in $D - WXY$. Suppose not. For any $a \in W_0$, let H_a be the set of points in D that lie in a path that starts at a. Let $H = \bigcup_{a \in W_0} H_a$. Note that no point in H is connected to any point in X_0Y_0 for if it were, then there would be a path between a point in W_0 and a point in X_0Y_0. Note that this implies that $H, D-H$ are freely joined over Z. Now $\delta(D/W_0X_0Y_0) = \delta(H/W_0X_0Y_0) + \delta(D - H/W_0X_0Y_0) = \delta(H/W_0) + \delta(D - H/X_0Y_0) \geq 0$ as W_0, X_0Y_0 is closed in $\mathbb{M}$. But this contradicts $(W_0X_0Y_0, D)$ is not a minimal pair. Thus it follows (from picking a path with minimum distance) that there is a path between some $a \in W_0$ and $b \in X_0Y_0$ where except for a, b the other points all lie in $D - W_0X_0Y_0$. We may as well assume that $b \in X$. And let P denote the set of points in the path that lie in $D - W_0X_0Y_0$. Now, $0 > |P| - (|P+1|) \geq \delta(P/ab) \geq \delta(P/W_0X_0)$. However this contradicts WX is closed in $\mathbb{M}$ as there is some minimal pair (W_0X_0, W_0X_0P). Thus $S_{\overline{\alpha}}$ is trivial. $\square$

Remark 8.3.2. This result shows that every regular type of $S_{\overline{\alpha}}$ is trivial when $\overline{\alpha}$ is graph-like with weight one.

Chapter 9: Infinite relational languages

In this chapter we explore the analogues of Baldwin-Shi hypergraphs in the setting of countably infinite languages. Let L is a countably infinite relational language with no unary relation symbols. As we did earlier, we focus on K_L, the class of finite L-structures where the relation symbols are interpreted irreflexively and symmetrically and we let $\overline{\alpha} : L \to (0,1]$ and $\delta(\mathfrak{A}) = |A| - \sum_{E \in L} \overline{\alpha}(E)|A^E|$. By an abuse of notation, let $K_{\overline{\alpha}} = \{\mathfrak{A} : \delta(\mathfrak{A}') \geq 0$ for all $\mathfrak{A}' \subseteq \mathfrak{A}$ and the set of $E \in L$ such that $|A^E| \neq 0$ is finite$\}$. For any $\mathfrak{A}, \mathfrak{B} \in K_L$, we say that $\mathfrak{A} \leq \mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\delta(\mathfrak{A}') \geq \delta(\mathfrak{A})$ for all $\mathfrak{A} \subseteq \mathfrak{A}' \subseteq \mathfrak{B}$. The class $K_{\overline{\alpha}}$ inherits the notion of strong substructure from K_L as in Chapter 2. Unlike in Chapter 2 however, there may be finite structures $\mathfrak{A} \in K_L$ with infinitely many relation symbols holding on them (i.e. $\sum_{E \in L} |E^A| = \infty$) and even with $\delta(\mathfrak{A}) = -\infty$. We note that we may extend the notation $e(A), e(A, B), e(A, B, C)$ and the notion of closed sets, minimal pairs, etc to this setting.

Remark 9.0.1. Note that the contents of Remark 2.3.1 holds true in this setting as does Fact 2.3.4.

Using the fact that the set of isomorphism types of $K_{\overline{\alpha}}$ is countable along with Remark 9.0.1, we obtain that a $(K_{\overline{\alpha}}, \leq)$ generic exists. In this chapter we discuss some of the properties of this generic and its theory. It should be noted that if $\mathfrak{M}$ is a model of the theory of the $(K_{\overline{\alpha}}, \leq)$ generic and $\mathfrak{A} \subseteq \mathfrak{M}$ is finite, then it is not necessary for $\mathfrak{A}$ to be in $K_{\overline{\alpha}}$ though $\mathfrak{A} \restriction_{L'} \in K_{\overline{\alpha}}$ for any *finite* $L' \subseteq L$. Further given any such $\mathfrak{A}$ we obtain that $\delta(\mathfrak{A}) \geq 0$, for else we can find a finite L' such that $\delta(\mathfrak{A} \restriction_{L'}) < 0$ which is easily seen to be ruled out by the universal sentences of the theory of the generic. In the same manner, if $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}$ are finite and $(\mathfrak{A}, \mathfrak{B})$ is a minimal pair (as evaluated in K_L), then there is some finite $L' \subseteq L$ such that $(\mathfrak{A} \restriction_{L'}, \mathfrak{B} \restriction_{L'})$ is a minimal pair.

Our main result is Theorem 9.2.1, which establishes a link between the reducts of the $(K_{\overline{\alpha}}, \leq)$ generics and the Baldwin-Shi hypergraphs. We end with Theorem 9.2.2, which establishes the stability of the $(K_{\overline{\alpha}}, \leq)$ generic.

9.1 The reducts of $K_{\overline{\alpha}}$

Definition 9.1.1. Let $L' \subseteq L$. Then K' is the class of structures that contain the reducts of the structures in $K_{\overline{\alpha}}$ in the language L'; i.e. $K' = \{\mathfrak{A}\restriction_{L'} | \mathfrak{A} \in K_{\overline{\alpha}}\}$.

It is clear that K', like $K_{\overline{\alpha}}$, is closed under substructure and that $K' \subseteq K_{\overline{\alpha}}$. The class K' has a natural candidate for the notion of induced strong substructure $\leq'$ on $K' \times K'$ given by the following:

Definition 9.1.2. Fix $L' \subseteq L$. Given $\mathfrak{A}, \mathfrak{B} \in K'$, we have $\mathfrak{A} \leq' \mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\delta'(\mathfrak{B}') \geq \delta'(\mathfrak{A})$ for any $\mathfrak{B}'$ such that $\mathfrak{A} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$. Here $\delta' = \delta \restriction_{L'}$ i.e. $\delta'(\mathfrak{C}) = |C| - \sum_{E \in L'} \overline{\alpha}(E)|E^C|$ for any $C \in K_L$.

Lemma 9.1.3. *Let $\mathfrak{A}, \mathfrak{B} \in K_L$ and let $L' \subseteq L$. If $\mathfrak{A} \leq \mathfrak{B}$, then $\mathfrak{A}' \leq' \mathfrak{B}'$ where $\mathfrak{A}', \mathfrak{B}'$ are the L' reducts of $\mathfrak{A}, \mathfrak{B}$ respectively.*

Proof. Let $\mathfrak{C}$ be such that $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$. Now $0 \leq \delta(\mathfrak{C}/\mathfrak{A})$.

$$\delta(\mathfrak{C}/\mathfrak{A}) \quad = \quad |C| - |A| - \textstyle\sum_{E \in L} \overline{\alpha}(E)|(C^E| - |A^E|)$$

Now note that $0 \leq |C^E| - |A^E|$ for each $E \in L$. Thus we obtain;

$$\begin{aligned} \delta(\mathfrak{C}/\mathfrak{A}) \quad &\leq \quad |C| - |A| - \textstyle\sum_{E \in L'} \overline{\alpha}(E)|(C^E| - |A^E|) \\ &= \quad \delta'(\mathfrak{C}'/\mathfrak{A}') \end{aligned}$$

Thus $A' \leq' B'$. ☐

Remark 9.1.4. Fix a finite $L' \subseteq L$, and the corresponding $K' \subseteq K$. We easily see that $(K', \leq')$ may be used to construct a Baldwin-Shi hypergraph.

9.2 The theory of the generic for $(K_{\overline{\alpha}}, \leq)$

Here we explore the interplay between the generic for $(K_{\overline{\alpha}}, \leq)$ and the generic for $(K', \leq')$ for some $L' \subseteq L$. This approach towards establishing stability for the theory of the generic for $(K_{\overline{\alpha}}, \leq)$ appears to be new. *Given an L-structure $\mathfrak{X}$ and some $L' \subseteq L$, we follow the convention that $\mathfrak{X}'$ denotes $\mathfrak{X} \restriction_{L'}$.*

Theorem 9.2.1. *Let $L' \subseteq L$. If $\mathfrak{M}$ is the generic for $(K_{\overline{\alpha}}, \leq)$, then $\mathfrak{M}'$ is the generic for $(K', \leq')$. Further if L' is finite, then $\mathfrak{M}'$ is a Baldwin-Shi hypergraph.*

Proof. We know that $\mathfrak{M} = \bigcup \mathfrak{A}_n$ is the union of a strong chain of $(\mathfrak{A}_n)_{n \in \omega}$ of elements in $K_{\overline{\alpha}}$. Now from Lemma 9.1.3 it follows that $(\mathfrak{A}'_n)_{n \in \omega}$ is a strong chain of elements from K'. Since $\mathfrak{M}'$ is a reduct of $\mathfrak{M}$, we obtain that $\mathfrak{M}' = (\bigcup_{n<\omega} \mathfrak{A}_n)' = \bigcup_{n<\omega} \mathfrak{A}'_n$ and thus $\mathfrak{M}'$ is the union of a strong chain of elements from $(K', \leq')$.

Note that any element of K' is a reduct of a element in $K_{\overline{\alpha}}$. Thus in order to show that $\mathfrak{M}'$ is the $(K', \leq')$ generic, it suffices to show for $\mathfrak{A}', \mathfrak{B}' \in K'$ with $\mathfrak{A}' \leq' \mathfrak{B}'$, if $f : \mathfrak{A}' \to \mathfrak{M}'$ is a strong embedding in the sense of L' then there exists g extending f such that $g : \mathfrak{B}' \to \mathfrak{M}'$ is strong in the sense of L'. Now $f(\mathfrak{A}')$ generates a substructure of $\mathfrak{A}^*$ of $\mathfrak{M}$ with universe $f(A')$. Since $\mathfrak{M}$ is L-generic, there is a $\mathfrak{C} \in K_{\overline{\alpha}}$ such that $\mathfrak{C} \leq \mathfrak{M}$ and $\mathfrak{A}^* \subseteq \mathfrak{C}$. Note that as $\mathfrak{A}^*, \mathfrak{C} \subseteq \mathfrak{M}$ only finitely many relations hold on $\mathfrak{A}^*, \mathfrak{C}$.

We begin by showing that a suitably chosen copy of $\mathfrak{B}'$ embeds strongly into $\mathfrak{M}$ over $\mathfrak{A}^*$. Towards this consider the following $\mathfrak{B}^* \in K_L$ that satisfies

1. $\mathfrak{A}^* \subseteq \mathfrak{B}^*$.

2. There is a bijection $h : B' \to B^*$ such that $h \restriction_{A'} = f$.

3. For any $E \in L$, $G_1 \subseteq B' - A'$, $G_2 \subseteq A'$ with G_1 non-empty, $G_1 \cup G_2 \in E^{B'}$ if and only if $h(G_1 \cup G_2) \in E^{B^*}$

4. For any $E \in L - L'$, $G_1 \subseteq B^* - A^*$, $G_2 \subseteq A^*$ with G_1 non-empty $G_1 \cup G_2 \notin E^{B^*}$.

We claim that $\mathfrak{B}^* \in K_{\overline{\alpha}}$ and $\mathfrak{A}^* \leq \mathfrak{B}^*$. As noted above $\mathfrak{A}^*$ has only finitely many relations that hold positively on it and hence it follows that $\mathfrak{B}^*$ has only finitely many relations that hold positively on it. Let $L'' \subseteq L$ be the collection of relation symbols that appear positively on $\mathfrak{B}^*$ and note that L'' is finite. Let K''

be the collection of finite L'' structures with hereditarily non-negative rank δ'' (note that the notation here corresponds to that in Definition 9.1.2). Let $\mathfrak{A}'' = \mathfrak{A}^* \restriction_{L''}$ and $\mathfrak{B}'' = \mathfrak{B}^* \restriction_{L''}$. Note that as L'' contains all the relations that occur positively in $\mathfrak{B}^*$, establishing $\mathfrak{A}'' \leq'' \mathfrak{B}''$, is equivalent to establishing $\mathfrak{A}^* \leq \mathfrak{B}^*$. Further if we show that $\mathfrak{A}'' \leq'' \mathfrak{B}''$, we obtain that $\mathfrak{B}'' \in K''$ by Remark 2.3.2. Again using the fact that all of the relations that appear positively on $\mathfrak{B}^*$ lie in L'' we can conclude that $\mathfrak{B}^* \in K_{\overline{\alpha}}$. Thus let $\mathfrak{A}'' \subseteq \mathfrak{D} \subseteq \mathfrak{B}''$. Now it is easily seen that $\delta''(\mathfrak{D}/\mathfrak{A}'') = \delta''(D - A'') - e''(D - A'', A'')$ where the e'' denotes the fact that only the relations in L'' are taken into account when calculating the relevant weighted sum. But $D - A'' = D - A'$ and $A'' = A'$ (i.e. underlying universes are equal) and the only relations that hold on $D - A''$ and between $D - A''$ and A'' all lie in L'. Thus we obtain that $\delta''(\mathfrak{D}/\mathfrak{A}'') = \delta'(\mathfrak{D}'/\mathfrak{A}')$ where $\mathfrak{D}'$ is the reduct of $\mathfrak{D}$ to L'. But $\delta'(\mathfrak{D}'/\mathfrak{A}') \geq 0$ as $\mathfrak{A}' \leq' \mathfrak{B}'$. Hence our claim follows.

Thus we may form $\mathfrak{H} = \mathfrak{B}^* \otimes_{\mathfrak{A}^*} \mathfrak{C}$ (taking an isomorphic copy of $\mathfrak{B}^*$ if $B^* \cap C \supsetneq A$). It is easily verified that $\mathfrak{H} \in K_{\overline{\alpha}}$ and that $\mathfrak{C} \leq \mathfrak{H}$. By the genericity of $\mathfrak{M}$, there is a strong embedding $j : \mathfrak{H} \to \mathfrak{M}$ over $\mathfrak{C}$. Let $\mathfrak{H}'$ be the L'-reduct of $\mathfrak{H}$. Consider j as a map from the set H to the set M. Using Lemma 9.1.3 and the definition of a strong embedding we obtain that $j(\mathfrak{H}') \leq' \mathfrak{M}'$. Extend f to g by taking $g = j \restriction_{B'}$.

By construction $f(\mathfrak{A}') = g(\mathfrak{A}') \leq' g(\mathfrak{B}')$. Further $f(\mathfrak{A}') \leq' \mathfrak{C}'$ as f is a strong embedding of $\mathfrak{A}'$ into $\mathfrak{M}'$. Since $j(\mathfrak{H}') \leq' \mathfrak{M}'$, it suffices to show that $g(\mathfrak{B}') \leq' j(\mathfrak{H}')$. But $\mathfrak{D}' = \mathfrak{B}' \otimes_{f(\mathfrak{A}')} C'$ and $f(\mathfrak{A}') \leq' \mathfrak{C}'$ which implies that $\mathfrak{B}' \leq' \mathfrak{D}'$ from which our claim follows. Thus $\mathfrak{M}'$ is isomorphic to the generic for $(K', \leq')$.

If L' is finite, it is clear from the definition of $(K', \leq')$ that the $(K', \leq')$-generic is a Baldwin-Shi hypergraph. Thus the latter part of the claim follows. $\square$

With the above result at hand and with another abuse of notation, we let $S_{\overline{\alpha}}$ denote the theory of $(K_{\overline{\alpha}}, \leq)$ generic. The following argument shows that $S_{\overline{\alpha}}$ is stable.

Theorem 9.2.2. *$S_{\overline{\alpha}}$ is stable.*

Proof. Note that if $S_{\overline{\alpha}}$ is unstable, there exists some formula $\varphi(\overline{x}, \overline{y})$, that has the order property. Let L' be the language consisting of exactly the relation symbols that appear in φ. Let $\mathfrak{M}$ be the $(K_{\overline{\alpha}}, \leq)$ generic. For each $n \in \omega$ there are $(\overline{a_i}, \overline{b_i})$, $i < n$, such that $\mathfrak{M} \models \varphi(\overline{a_i}, \overline{b_j})$ for all $i < j \leq n$ and $\mathfrak{M} \models \neg\varphi(\overline{a_i}, \overline{b_j})$ for all $j \leq i \leq n$. Now $Th(\mathfrak{M}')$ is stable by Theorem 9.2.1 and Theorem 6.1.16. Since $\mathfrak{M}'$ is a reduct of $\mathfrak{M}$, it follows that $\mathfrak{M}' \models \varphi(\overline{a_i}, \overline{b_j})$ for all $i < j \leq n$ and hence $Th(\mathfrak{M}')$ is unstable. This contradiction establishes the claim. $\square$

Appendix A: Some Relevant Number Theoretic Facts

The number theoretic results concerning Diophantine equations can be found in Chapter 5 of [34] and the number theoretic results concerning continued fractions can be found in Chapter 7 therein. We use standard notation for the greatest common divisor, divides, etc in what follows.

Remark A.0.1. We will show that in the case all the $\overline{\alpha}_E$ are rational the equation $n - \sum_{E \in L} \overline{\alpha}_E m_E = -\frac{1}{c}$ has infinitely many positive integer solutions, i.e. solutions where n and all of the m_E are positive integers. Note that multiplying through by c, we obtain a linear Diophantine equation whose solutions will yield the required n, m_E. Our proof of the existence of infinitely many suitable solutions will use the following fact regarding linear Diophantine equations: Given a linear Diophantine equation $a_1x_1 + a_2x_2 + \dots a_nx_n = d$, the equation has a solution if and only if $\gcd(a_1, \dots, a_n)|d$.

Note that if $|L| = 1$, then the associated Diophantine equation becomes $q_E n - p_E m_E = -1$. As p_E, q_E are relatively prime, we obtain that $\gcd(p_E, q_E)| - 1$ and hence the required result follows. So we may as well assume that $|L| \geq 2$. Fix an enumeration $\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n}$ of the values of $\overline{\alpha}(E)$, where $|L| = n$. Note that $c = \mathrm{lcm}(q_1, \dots, q_n)$. Using the fact that $\mathrm{lcm}(a, b) = \frac{ab}{\gcd(a,b)}$ and $\mathrm{lcm}(a_1, \dots, a_n) =$

$\mathrm{lcm}(a_1, \mathrm{lcm}(a_2, \ldots, a_n))$ repeatedly, we obtain that

$$c = \frac{\prod_{1 \le i \le n} q_i}{\prod_{1 \le i < n} \gcd(q_i, \mathrm{lcm}(q_{i+1}, \ldots, q_n))}$$

Further note that given a different enumeration $\frac{p'_1}{q'_1}, \frac{p'_2}{q'_2}, \ldots, \frac{p'_n}{q'_n}$ of the values of $\overline{\alpha}(E)$, using the fact that $c = \mathrm{lcm}(q_1, \ldots, q_n) = \mathrm{lcm}(q'_1, \ldots q'_n)$, we obtain

$$\prod_{1 \le i < n} \gcd(q_i, \mathrm{lcm}(q_{i+1}, \ldots, q_n)) = \prod_{1 \le i < n} \gcd(q'_i, \mathrm{lcm}(q'_{i+1}, \ldots, q'_n))$$

Let $d = \prod_{1 \le i < n} \gcd(q_i, \mathrm{lcm}(q_{i+1}, \ldots, q_n))$

We now need to show that

$$n(\prod_{1 \le i \le n} q_i) - \sum_{1 \le i \le n} p_i m_i (\prod_{1 \le j \le n, i \ne j} q_j) = -\prod_{1 \le i < n} \gcd(q_i, \mathrm{lcm}(q_{i+1}, \ldots, q_n))$$

.

In order to show that the above equation has solutions, we need to show that $s|d$ where $s = \gcd(\prod_{1 \le i \le n} q_i, p_1 \prod_{2 \le j \le n,} q_j, \ldots, p_i \prod_{1 \le j \le n, j \ne i} q_j, \ldots, p_n \prod_{1 \le j \le n-1} q_j)$ using the following facts

1. $\gcd(ab, af) = a \gcd(b, f)$
2. $\gcd(a_1, \ldots, a_n) = \gcd(\gcd(a_1, a_2), \ldots a_n)$
3. For relatively prime b, f $\gcd(b, af) = \gcd(b, a)$.

Note that $s = \gcd(\gcd(\prod_{1 \le i \le n} q_i, p_1 \prod_{2 \le j \le n,} q_j), \ldots, p_n \prod_{1 \le j \le n-1} q_j)$. But as we know that $\gcd\left(\prod_{1 \le i \le n} q_i, p_1 \prod_{2 \le j \le n,} q_j\right) = \left(\prod_{2 \le i, \le n} q_i\right) \gcd(q_1, p_1)$. As p_1, q_1 are relatively prime, we obtain that $\gcd(q_1, p_1) = 1 = \gcd(q_1, 1)$ and hence we obtain that $\gcd\left(\prod_{1 \le i \le n} q_i, p_1 \prod_{2 \le j \le n,} q_j\right) = \gcd\left(\prod_{1 \le i \le n} q_i, \prod_{2 \le j \le n,} q_j\right)$. Thus $s =$

$\gcd(\prod_{1\leq i\leq n} q_i, \prod_{2\leq j\leq n,} q_j, \ldots, p_n \prod_{1\leq j\leq n-1} q_j)$. Proceeding in a similar manner we obtain that $s = \gcd(\prod_{1\leq i\leq n} q_i, \prod_{2\leq j\leq n,} q_j, \ldots, \prod_{1\leq j\leq n, i\neq j} q_j, \ldots, \prod_{1\leq j\leq n-1} q_j)$. But then it follows that $s = \gcd(\prod_{2\leq j\leq n,} q_j, \ldots, \prod_{1\leq j\leq n, i\neq j} q_j, \ldots, \prod_{1\leq j\leq n-1} q_j)$. We note that given a different enumeration $\frac{p'_1}{q'_1}, \frac{p'_2}{q'_2}, \ldots, \frac{p'_n}{q'_n}$ of the values of $\overline{\alpha}(E)$, we have that $s = \gcd(\prod_{2\leq j\leq n,} q'_j, \ldots, \prod_{1\leq j\leq n, i\neq j} q'_j, \ldots, \prod_{1\leq j\leq n-1} q'_j)$.

We have to show that $s|d$. If $s = 1$, then there is nothing to prove. So let p be some prime such that $p|d$. By choosing a different enumeration of $\overline{\alpha}$ if necessary, we may assume that $r_1 \leq r_2 \leq, \ldots \leq r_n$ where r_i is the largest integer for which $p^{r_i}|q_i$. It now follows easily that $p^{\sum_{1\leq i\leq n-1} r_i}|s$ but $p^{(\sum_{1\leq i\leq n-1} r_i)+1} \not| s$. We have to show that $p^{\sum_{1\leq i\leq n-1} r_i}|d$. Note that by our choice of enumeration $p^{r_i}|\gcd(q_i, \mathrm{lcm}(q_{i+1}, \ldots q_n))$ for all $1 \leq i < n$. But then it is immediate that $p^{\sum_{1\leq i\leq n-1} r_i}|d$. From this it easily follows that $s|d$.

Thus the equation $n - \sum_{E\in L} \overline{\alpha}_E m_E = -\frac{1}{c}$ has infinitely many positive integer solutions (using general facts about Diophantine equations).

Remark A.0.2. Let $0 < \beta < 1$ be irrational. We claim that for any $\epsilon > 0$, there are infinitely many positive m, n such that $-\epsilon < n - m\beta < 0$.

We begin by noting that β has a simple continued fraction form $[0 : a_1, a_2, \ldots] = 0 + 1\frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$ where $a_i \in \omega$ is positive for $i \geq 1$. Let $p_k/q_k = [0 : a_1, \ldots, a_k]$ be the simple continued fraction approximation restricted to k-terms. Now:

1. p_k, q_k are increasing sequences (and hence $p_k, q_k \to \infty$)

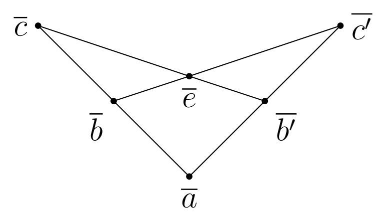
2. $\langle p_{2k}/q_{2k} : k \in \omega\rangle$ is a strictly increasing sequence that converges to β

3. For even k, $\frac{1}{q_k(q_k+q_{k+1})} < \beta - \frac{p_k}{q_k} < \frac{1}{q_k q_{k+1}}$

Now it follows that $-\frac{1}{q_{2k}} < p_{2k} - q_{2k}\beta < -\frac{1}{q_{2k}+q_{2k+1}}$. This easily yields that $\lim_k p_{2k} - q_{2k}\beta = 0$. Taking $n_i = p_{2k}, m_i = q_{2k}$ for sufficiently large values of k now yields the required result.

Appendix B: More on the Group Configuaration

The configuration found in Theorem 7.2.19. is the celebrated group configuration (originally due to Zilber). In this appendix we provide some details regarding the calculation of dimensions that was omitted from Theorem 7.2.19. Recall that we are working with, $p^{\mathbb{M}}$, the set of realizations of a regular type p in a monster model $\mathbb{M}$ of $S_{\overline{\alpha}}$ and that $(p^{\mathbb{M}}, \mathrm{cl}^p)$ is a pregeometry. The configuration in question can be represented visually by



where $\overline{a}, \overline{b}, \overline{c} \models p$ are such that they are pairwise independent but are dependent, $\overline{b'}, \overline{c'} \models p$ such that $\overline{b'}\,\overline{c'} \equiv_{\overline{a}} \overline{b}\,\overline{c}$, $\overline{b'}\,\overline{c'} \downarrow_{\overline{a}} \overline{b}\,\overline{c}$ and $\overline{e} \in X \cap Y \cap p^{\mathbb{M}}$ where $X = \mathrm{cl}^p(\overline{b}\,\overline{c'})$, $Y = \mathrm{cl}^p(\overline{c}\overline{b'})$. Let $Z = \mathrm{cl}^p(X \cup Y)$. Recall our claims regrading the dimensions from Theorem 7.2.19.

Clearly the dimension of every single point is 1. Note that by our choice of $\overline{a}, \overline{b}, \overline{b'}$, we obtain that $\overline{b} \downarrow \overline{a}$ and $\overline{b} \downarrow_{\overline{a}} \overline{a}\overline{b'}$. So by transitivity of non-forking we obtain that $\overline{b} \downarrow \overline{a}\overline{b'}$. Further as $\overline{b'} \downarrow_{\overline{a}} \overline{c}$ and $\overline{b'} \downarrow \overline{a}$ transitivity again yields $\overline{b'} \downarrow \overline{c}$. Thus it follows that $\dim^p(\{\overline{a}, \overline{b}, \overline{b}'\}) = 3$ and $\dim^p(X) = 2$. Similar arguments yield that (the closure of) any three non-colinear points that don't include $\overline{e}$ has dimension 3 and any two points that don't include $\overline{e}$ has dimension 2

Consider the set $\{\overline{a}, \overline{b}, \overline{b'}, \overline{c}, \overline{c'}, \overline{e}\}$. We claim that if $x, y \in \{\overline{a}, \overline{b}, \overline{b'}, \overline{c}, \overline{c'}, \overline{e}\}$ are distinct then $x \notin \mathrm{cl}^p(y)$. Note that if $x, y \in \{\overline{a}, \overline{b}, \overline{b'}, \overline{c}, \overline{c'}\}$ this follows by our choice of $\{\overline{a}, \overline{b}, \overline{b'}, \overline{c}, \overline{c'}\}$ using the properties of non-forking as above. So assume that either x or y is $\overline{e}$. Since $(p^{\mathbb{M}}, \mathrm{cl}^p)$ satisfies exchange, we may as well assume that $y = \overline{e}$ and $x \in \mathrm{cl}^p(y)$. By way of contradiction assume that $x \in \mathrm{cl}^p(\overline{e})$. We will consider the case $x = \overline{b'}$, the other cases will be handled similarly. Note that $\overline{b}, \overline{e}, \overline{c'} \in X$ and thus $\overline{b'}, \overline{c'} \in X$. But as $\overline{a} \in \mathrm{cl}^p(\{\overline{c'}, \overline{b'}\})$, it follows that $\overline{a}, \overline{b}, \overline{b'} \in X$. This now contradicts the fact that dimension of $\mathrm{cl}^p(\{\overline{a}, \overline{b}, \overline{b'}\})$. From this it follows that a set of with two points has dimension two.

We now show that three any colinear points has dimension two. By our choice of $\overline{e}$, if one of the three points is $\overline{e}$, the result is immediate. So consider three colinear point that does not contain $\overline{e}$, such as $\overline{a}, \overline{b}, \overline{c}$. As $\overline{a}, \overline{b}, \overline{c}$ are pairwise independent but are dependent it follows that $\dim^p(\mathrm{cl}^p\{\overline{a}, \overline{b}, \overline{c}\}) = 2$. The case of $\overline{a}, \overline{b'}, \overline{c'}$ similar using the fact that non-forking is automorphism invariant.

It remains to show that any three non-colinear points has dimension 3. We have established this result in the case that non of the points involved is $\overline{e}$. So assume that one of the points is $\overline{e}$. We will establish this result for $\overline{a}, \overline{c'}, \overline{e}$, the other cases being similar. By way of contradiction, assume that this is not the case. As $(p^{\mathbb{M}}, \mathrm{cl}^p)$ satisfies exchange we may as well assume that $\overline{c'} \in \mathrm{cl}^p(\{\overline{a}, \overline{e}\})$. As $\overline{e} \in \mathrm{cl}^p(\{\overline{b}, \overline{c'}\})$, using exchange we obtain that $\overline{b} \in \mathrm{cl}^p(\{\overline{e}, \overline{c'}\}) \subseteq \mathrm{cl}^p(\{\overline{e}, \overline{a}\})$. But then $\overline{a}, \overline{b}, \overline{c'} \in \mathrm{cl}^p(\{\overline{e}, \overline{a}\})$ as $\dim^p(\mathrm{cl}^p(\{\overline{e}, \overline{a}\}) = 2$ and $\dim^p(\{\overline{a}, \overline{b}, \overline{c'}\}) = 3$.

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