ABSTRACT

Title of dissertation: LOCAL DYNAMICS OF ESSENTIAL PROJECTIVE VECTOR FIELDS FOR LEVI-CIVITA CONNECTIONS

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We study metrizable projective structures near non-linearizable singularities of projective vector fields. We prove connected 3-dimensional Riemannian manifolds and closed connected pseudo-Riemannian manifolds admitting a projective vector field with a non-linearizable singularity are projectively flat. We also show that a 3-dimensional Lorentzian metric is projectively flat on a cone with its vertex at non-linearizable singularities of projective vector fields.

LOCAL DYNAMICS OF ESSENTIAL PROJECTIVE VECTOR FIELDS FOR LEVI-CIVITA CONNECTIONS

by

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1 Introduction

This thesis focuses on the dynamics of projective vector fields for Levi-Civita connections near their non-linearizable singularities, and implications on the global geodesic rigidity of semi-Riemannian manifolds. We start by studying the relationships between the dynamics of a projective vector field for a semi-Riemannian metric g near its singularity and the dynamics of the action of its flow on metrics projectively equivalent to g. In some situations, the metrizable connections in a given projective structure is unique. This property is sometimes referred as the geodesic rigidity of the projective class, since the projective class determines both the unparametrized curves and the specific parametrizations of the geodesics induced by the Levi-Civita connection in this projective structure.

To begin, we give a brief review of the following basic definitions in projective geometry. Let ∇ be a torsion-free affine connection on a manifold M^n. The projective class $[\nabla]$ of ∇ consists of the torsion-free affine connections on M having the same unparametrized geodesics as those defined by ∇. Two metrics on M are projectively equivalent if their Levi-Civita connections are in the same projective class. The class $[\nabla]$ is said to be metrizable if there is a Levi-Civita connection contained in $[\nabla]$. It is said to be flat if

$[\nabla]$ is induced by a flat affine connection. It is well known that:

$$\overline{\nabla} \in [\nabla] \quad \Longleftrightarrow \quad \overline{\nabla} = \nabla + \eta \otimes Id + Id \otimes \eta, \ \exists \eta \in \Gamma(T^*M).$$

Given (M, ∇), a smooth diffeomorphism $f : M \to M$ is a projective transformation of the projective class $[\nabla]$ if $f^*\nabla \in [\nabla]$. Let X be a vector field on M, and denote ϕ^t the flow generated by X. Then X is a projective vector field for ∇ if ϕ^t preserves the unparametrized geodesics defined by ∇. Denote by $\mathcal{L}_X\nabla$ the Lie derivative of ∇ with respect to X. This is equivalent to:

$$\mathcal{L}_X\nabla = \hat{\eta} \otimes Id + Id \otimes \hat{\eta}, \ \hat{\eta} \in \Gamma(T^*M).$$

The projective vector field X is affine for ∇ if $\mathcal{L}_X\nabla = 0$. It is essential if it is not affine for any connection in $[\nabla]$.

It is a classical topic to study projective structures induced by Levi-Civita connections. Some classical results have been obtained by mathematicians like Dini, Levi-Civita, Weyl, and Solodovnikov. One can refer to Theorems 7-10 from [4] for a summary of their results. The local description of projectively equivalent metrics is well understood by Bolsinov and Matveev in [10] and [8] in terms of BM structures (See Definition 2.4 in Section 2.3). Kobayashi and Nagano give a concrete description of projective structures in terms of Cartan geometries in [1].

One of the main motivations for my thesis is to understand on closed manifolds how a metrizable projective class and its projective transformation group determine each other. For example, I tried to study what additional assumptions on the projective transformation group or algebra are sufficient to deduce that the projective structure is flat on the manifold or some special subsets. Sometimes it turns out $[\nabla]$ is determined by assumptions less than expected, and we obtain rigidity results. One of the most important topics in the global theory of projective geometry about geodesic rigidity is the following projective Lichnerowicz-Obata conjecture.

Conjecture 1. *Let G be a connected Lie group acting on a complete connected or closed connected semi-Riemannian manifold (M^n, g) by projective transformations. Then either G acts on M by affine transformations, or (M^n, g) is Riemannian with positive constant sectional curvature.*

In addition to the Riemannian cases [4], this conjuncture has been proved for closed connected Lorentzian manifolds [21], and the case g has the degree of mobility of at least three [3]. (See Definition 2.4 in Section 2.3.) This dissertation focuses on the case that the degree of mobility of the metric is precisely two. In such cases, the applicable techniques are different from the cases where the degree of the mobility of the metric is at least three. Let $\mathrm{Isom}(M, g)$, $\mathrm{Proj}(M, g)$ and $\mathrm{Aff}(M, g)$ be the groups of isometric, projective and affine transformations of (M, g) respectively. One of the most useful approaches comes from [5] by Zeghib, where he proves the following important result for discrete groups of projective transformations on closed semi-Riemannian manifolds.

Theorem 1.1 (Zeghib [5])**.** *Let (M, g) be a closed semi-Riemannian manifold with* $\mathrm{Proj}(M, g)/\mathrm{Aff}(M, g)$ *infinite. Then the following holds:*

1. $|\mathrm{Aff}(M, g)/\mathrm{Isom}(M, g)|$ *is finite, and* $\mathrm{Aff}(M, g)$ *is a normal subgroup of* $\mathrm{Proj}(M, g)$.

2. *There is a representation* $\rho : \mathrm{Proj}(M, g) \to SL_2(\mathbb{R})$ *such that* $\mathrm{Ker}(\rho)$ *is a finite index subgroup of* $\mathrm{Aff}(M, g)$*, and* $\mathrm{Im}(\rho)$ *has a subgroup of finite index contained in a 1-parameter hyperbolic subgroup of* $SL_2(\mathbb{R})$.

Though the paper [5] focuses on global analysis on closed manifolds, the methods can be adapted in some cases to study the local properties of projective geometries near a singularity of a projective vector field. The key assumption in [5] is that the degree of mobility for the metric is precisely two, which will be explained in detail in this thesis.

Another motivation for this thesis is to find the maximal possible generalizations of the global and local results presented by Nagano and Ochiai in [2]. A vector field X vanishes to order 2 at o if $X_o = 0$ and the flow ϕ^t of X satisfies $(D\phi^t)_o \equiv Id$. Their result for projective vector fields on closed Riemannian manifolds is as follows.

Theorem 1.2 (Nagano, Ochiai [2])**.** *Let (M^n, g) with $n > 1$ be a closed connected Riemannian manifold. Suppose that X is a projective vector field* *for (M, g) such that it has a vanishing point of order 2 at some $o \in M$. Then (M, g) is either $\mathbb{S}^n$ or $\mathbb{RP}^n$.*

In my doctoral research, I studied what would be a good generalization for the assumption that a projective vector field vanishes up to order two at some point. It turns out when the singularity o of a projective vector field X is non-linearizable, as shown in Section 2.2, at least on some special sets containing o the flow generated by X will have dynamics similar to the case presented in Theorem 1.2. Also, for a projective vector field, if it has a non-linearizable singularity, its flow cannot preserve any connections in the projective class (See Section 2.2 for details), so we may use the terms "essential singularity" and "non-linearizable singularity" interchangeably. By analyzing the properties of the projective structures on these special sets together with the global techniques used by Zeghib and Matveev, I obtain the following result for closed semi-Riemannian manifolds.

Theorem 1.3. *Let (M^n, g) be a closed connected semi-Riemannian manifold with $n > 1$. Suppose X is a projective vector field for (M, g) which admits an essential singularity $o \in M$. Then g is Riemannian, and (M^n, g) is a quotient of the standard sphere S^n.*

For non-closed connected manifolds, how a projective vector field with an essential singularity could determine the global metrizable projective structure is still open. However, for the special cases of 3-dimensional Riemannian manifolds, the restriction on the upper bound of degree of mobility gives the following result analogous to Theorem 1.3.

Theorem 1.4. *Let (M^n, g) with $n \geq 3$ be a connected Riemannian manifold. Suppose it admits a projective vector field with an essential singularity $o \in M$. Then (M^n, g) has degree of mobility at least three. When $n = 3$, then (M^3, g) has constant sectional curvature.*

The local theory of projective structures near a singularity of a projective vector field is fundamental to the proofs of Theorem 1.3 and Theorem 1.4 in this thesis and Theorem 1.2 presented in [2]. For example, the key lemma in [2] is the following.

Lemma 1.1 (Nagano, Ochiai [2])**.** *Let ∇ be a symmetric affine connection on some open set $U \subset \mathbb{R}^n$ with $n \geq 3$. Suppose X is a projective vector field for ∇ vanishing to order 2 at o, then there exists an open subset $V \ni o$ of U where $[\nabla]$ is flat.*

This lemma is proved by analyzing the dynamics of the flow ϕ^t generated by X near o with the fact that the Weyl curvature of $[\nabla]$ is ϕ^t-invariant. Though a projective vector field may admit dynamics similar to the case in Lemma 1.1 on some subset of the manifold of smaller dimension containing its essential singularity o, if it is assumed that o is not a higher order zero of X, we may not be able to get an open set containing o on which $[\nabla]$ is flat. In fact for non-metrizable projective structures, we can construct examples of a projective class $[\nabla]$ which is not flat on any neighborhood of o while admitting a projective vector field X with an essential singularity at o (See

Section 5.1 for details). For metrizable projective structures, whether such examples exist leads to the following question for my doctoral research.

Problem 1. *Let g be a metric defined on some open set $U \subset \mathbb{R}^n$ with $n \geq 2$. Suppose X is a projective vector field for g with an essential singularity $o \in U$. Does there always exist an open $V \subset U$ containing o such that g is projectively flat on V?*

The answer to the problem still remains open, though I am able to give answers in some special cases. For example, for 3-dimensional Riemannian metrics, the metric g has to be projectively flat by Theorem 1.4 on the entire connected component containing o. The dynamics of the flow at a general essential singularity are much more complicated compared to the case in Lemma 1.1, especially for metrics with indefinite signatures. Determining the maximal possible open set containing o on which g is projectively flat leads to the following result for the 3-dimensional metrics.

Theorem 1.5. *Let g be a smooth metric defined on some open set $U \subset \mathbb{R}^3$ with $o \in U$. Let X be a projective vector field for g admitting an essential singularity at o. Then there is some open set V with $o \in \overline{V}$ such that g is projectively flat on V.*

Another motivation for Problem 1 comes from the observations in conformal geometries. In Cartan geometries, both projective and conformal geometries are $|1|$-graded parabolic geometries. In conformal geometries we have the following result from [13].

Theorem 1.6 (C. Frances, K. Melnick [13]). *Let X be a conformal vector field for a semi-Riemannian manifold (M^n, g) with $n \geq 3$ with a singularity o. If the 1-parameter group $\{(D\phi_X^t)_o : t \in \mathbb{R}\}$ is bounded, one of the following is true:*

- *There exists a neighborhood V of o on which X is complete and generates a bounded flow. In this case, it is linearizable.*
- *There is an open set $U_0 \subset M$, with $o \in \overline{U_0}$ such that g is conformally flat on U_0.*

There are several variations for the theorem above, see [13] and [12]. All of them assert the existence of some open set containing the non-linearizable singularity o of X in its closure on which the metric g is conformally flat. On the other hand, it is shown in Section 6 of [16] this estimate is sharp for Lorentzian metrics, so there are examples in which g is not conformally flat on any neighborhood of a non-linearizable singularity of a conformal vector field. Our construction of the example in Section 5.1 is analogous to the method used to obtain the examples in Section 6 of [16]. Since conformal and projective geometries have a lot of similarities in terms of Cartan geometries, it is natural to expect statements analogous to the results above in projective geometries.

In this thesis, the main methods used are the geometrical PDE methods for projectively equivalent metrics applied by Matveev and the dynamical methods by Zeghib in [5]. The local results for general projective structures near the essential singularities of projective vector fields use concepts from projective Cartan geometries established by people like Nagano, Kobayashi and Ochiai in [2], [1].

The general structure of this thesis is as follows. In Chapter 2, I show that the non-linearizable singularities are actually essential. Chapter 3 gives the adaptation of the dynamical method of Zeghib for closed manifolds to our settings which can be used to study the local theory of projective structures. The main theorems on the global analysis of projective geometries are proved in Chapter 4. In Chapter 5, the cases of 3-dimensional Lorentzian metrics are analyzed which leads to the conclusion of Theorem 1.5.

2 Background

The content my work in this chapter is essentially from the preprint [17]. We adopt the basic definitions of projective Cartan geometries in [1] as our focus in this chapter is entirely on projective structures induced by torsion-free affine connections.

2.1 Cartan model for projective geometries

We begin this section by reviewing the basic concepts of Cartan geometries used in this thesis. Let G be a Lie group, and G' is a closed subgroup of G. Denote $\mathfrak{g}, \mathfrak{g}'$ their Lie algebras, respectively. The definition of a Cartan geometry is as follows.

Definition 2.1. *A Cartan geometry modelled on $(\mathfrak{g}, \mathfrak{g}')$ with the structure group G' is a triple (M, B, ω). Here B is a G' principal bundle over M, and the Cartan connection ω is a $\mathfrak{g}$ valued 1-form. In addition, it satisfies the following conditions:*

- *$\forall b \in B$, the map $\omega_b : T_bB \to \mathfrak{g}$ is an isomorphism.*
- *$\forall g \in G'$, $R_g^*\omega = Ad(g^{-1})\omega$, here R_g is the right translation by g of the principal G'-bundle.*
- $\omega\left(\frac{d}{dt}|_{t=0} b\exp(t\tilde{g})\right) = \tilde{g}$, $\forall b \in B,\ \forall \tilde{g} \in \mathfrak{g}'$,

Here the $\mathfrak{g}$-valued 1-form ω is the Cartan connection, and $\kappa = d\omega + \frac{1}{2}[\omega, \omega]$ is the curvature of this Cartan geometry. The Cartan geometry is flat if κ vanishes. A flat Cartan geometry modelled on $(\mathfrak{g}, \mathfrak{g}')$ is locally isomorphic to the flat model $(G/G', G, \omega_G)$, where ω_G is the Maurer-Cartan form on G(See Page 116 of [19]). In addition, we have the following definition of exponential maps in Cartan geometries.

Definition 2.2. *Suppose (M, B, ω) is a Cartan geometry modelled on $(\mathfrak{g}, \mathfrak{g}')$. Given any $v \in \mathfrak{g}$, we have a vector field $\omega^{-1}(v)$ on B. Denote by Φ_v the flow generated by $\omega^{-1}(v)$. The exponential map of ω at $b \in B$ is a map from $\mathfrak{g}$ to B given by $\exp_b(v) = \Phi_v(1, b)$, wherever it is well defined. Thus, $\exp_b$ gives a local diffeomorphism between a neighborhood of 0 of $\mathfrak{g}$ and a neighborhood of $b \in B$.*

Because flows of vector fields on principal bundles commute with right translation if and only if they are right translation invariant, we define the infinitesimal automorphisms of Cartan bundles as follows.

Definition 2.3. *An automorphism of the Cartan bundle (M, B, ω) is a principal bundle automorphism F with $F^*\omega = \omega$. An infinitesimal automorphism on (M, B, ω) is a G'-invariant vector field $\tilde{X}$ on B together with $\mathcal{L}_{\tilde{X}}\omega = 0$.*

The projective classes on M can be described in terms of Cartan geometries by the following. Choose $e_0 = [1, 0, \cdots, 0] \in \mathbb{RP}^n$, and let H be its stabilizer. Denote by $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G and H, respectively. We know $G =$

$PGL(n+1,\mathbb{R})$ acting on $\mathbb{RP}^n$ transitively. Then, we have the following identification (see Page 216 of [2]):

$$\mathfrak{sl}(n+1,\mathbb{R}) = \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathbb{R}^n \oplus GL(n,\mathbb{R}) \oplus (\mathbb{R}^n)^*, \ \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1. \quad (1)$$

Note that the standard Euclidean metric gives an identification $\mathbb{R}^n \simeq (\mathbb{R}^n)^*$. The identification is given by

$$u \oplus A \oplus v^* \mapsto \begin{bmatrix} -\frac{1}{n+1}Tr(A) & v^T \\ u & A - \frac{1}{n+1}Tr(A) \cdot Id \end{bmatrix} \in \mathfrak{sl}(n+1,\mathbb{R}). \quad (2)$$

The following is the standard chart of $\mathbb{RP}^n$ near e_0.

$$i_0: \ [x_0, \cdots, x_n] \mapsto (\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0})$$

In this chart i_0, any $h \in H$ is a local diffeomorphism at $0 \in \mathbb{R}^n$ with $h(0) = 0$. If f is a local diffeomorphism at $0 \in \mathbb{R}^n$ with $f(0) = 0$, let $J^k(f)(0)$ be its k-jet at the origin. Define $G^k(n)$ as the $k-$jet at 0 of all such functions. Clearly elements in $G^k(n)$ form a group. Since every $h \in H$ is such a diffeomorphism in the standard chart i_0, we have the following subgroup $H^2(n)$ of $G^2(n)$:

$$H^2(n) = \{J^2(h)(0) : h \in H\}.$$

This gives an identification $H^2(n) \cong H \cong GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$. Since $G^1(n)$ is induced by invertible linear maps on $\mathbb{R}^n$, we can identify $G^1(n)$ with the subgroup $GL(n,\mathbb{R})$ of $H^2(n)$. Let $F^2(M)$ be the 2-jet frame bundle of M, then it is a $G^2(n)$ principal bundle. We can take $F^2(M)$ as a sub-bundle of $F^1(F^1(M))$. Denote θ the canonical form on $F^1(F^1(M))$, then it is a $\mathfrak{gl}_n(\mathbb{R}) \bigoplus \mathbb{R}^n$-valued 1-form. Then $\theta|_{F^2(M)}$ has the following decomposition:

$$\theta = \theta^i + \theta^i_j,\ \theta^i \in \Gamma(\mathrm{Hom}(T(F^2M), \mathbb{R}^n)),\ \theta^i_j \in \Gamma(\mathrm{Hom}(T(F^2M), \mathfrak{gl}_n(\mathbb{R}))).$$

Here $\theta = \theta_i + \theta^i_j$ is the canonical form on $F^2(M)$. One can refer to Page 224 of [1] for a more precise definition.

A projective Cartan geometry on M is a Cartan geometry (M, B, ω) modelled on the pair $(\mathfrak{g}, \mathfrak{h})$. It is normal if the components of its curvature κ satisfies Equation (2) and (3) from [1]. Under the identification given by Equations (1) and (2), we have by Proposition 3 of [1], on any $H^2(n)$ sub-bundle P of $F^2(M)$, there is a unique normal projective Cartan connection $\omega = \omega_i + \omega^i_j + \omega^i$ with $\omega_i = \theta_i$, and $\omega^i_j = \theta^i_j$. We call this connection the normal projective Cartan connection associated to P.

We give the following way of identifying torsion-free affine connections on M^n with GL_n sub-bundles of $F^2(M)$.

Given a torsion-free affine connection ∇, $\forall x \in M$, the exponential map of ∇ at x, denoted as $\exp_x^\nabla$, is a map:

$$\exp_x^\nabla : U \subset T_xM \to M,\ 0 \mapsto x$$

Here U is an open set of T_xM containing the origin.

We define a bundle inclusion $i_\nabla : F^1(M) \to F^2(M)$ as follows. Any $p \in F^1(M)$ in the fibre of x can be uniquely identified with a linear map $\tilde{p} : \mathbb{R}^n \to T_xM$. Then we define

$$i_\nabla(p) = J^2(\exp_x^\nabla \circ \tilde{p})(0),\ \forall p \in F^1(M).$$

Let $F_1^2(M) = F^2(M)/GL_n(\mathbb{R})$, and $\pi_1^2 : F^2(M) \to F^1(M)$ be the canonical projection. Notice that every section Γ of $F_1^2(M)$ induces a unique natural bundle inclusion:

$$\gamma_\Gamma : F^1(M) \to F^2(M), \quad \pi_1^2 \circ \gamma_\Gamma = id.$$

The identification $\nabla \mapsto i_\nabla$ in fact gives a 1-1 correspondence between torsion-free affine connections on M and GL_n reductions of $F^2(M)$ by the following summary of Proposition 10 and 11 of [1].

Theorem 2.1 (Nagano,Kobayashi[1])**.** *Let $\theta = \theta_i + \theta_j^i$ be the canonical form on $F^2(M)$ as usual. There is a 1-1 correspondence between sections of $F_1^2(M)$ and symmetric affine connections on M. For a symmetric connection ∇,* *denote Γ the corresponding section of $F_1^2(M)$, then the following holds:*

- *The natural bundle inclusion γ_Γ is exactly i_∇.*
- *$(i_\nabla)^*\theta^i$ is the canonical form on $F^1(M)$.*
- *$(i_\nabla)^*\theta^i_j$ is the connection form for ∇.*

For every torsion-free connection ∇ on M, the map i_∇ gives a GL_n reduction of the $G^2(n)$-principal bundle $F^2(M)$. Since $GL_n(\mathbb{R}) \ltimes \mathbb{R}^n \cong H^2(n) \leq G^2(n)$, it induces a $H^2(n)$ sub-bundle $P(\nabla)$ of $F^2(M)$. From Proposition 12 of [1], we have $P(\nabla) = P(\overline{\nabla})$ if and only if ∇ and $\overline{\nabla}$ are projectively equivalent. This gives a 1-1 correspondence between the projective structures on M and $H^2(n)$ reductions of $F^2(M)$. Here $P(\nabla)$, along with its associated normal projective Cartan connection, is called the projective Cartan geometry associated to $[\nabla]$.

2.2 Infinitesimal automorphisms of projective Cartan bundles

In this section we study the local theory of infinitesimal automorphisms of projective Cartan bundles induced by projective vector fields with singularities. Every projective vector field X on M for ∇ can be uniquely lifted to an infinitesimal automorphism $\tilde{X}$ on $P = P(\nabla)$. For the flat model $(\mathbb{RP}^n, G, \omega_G)$, the infinitesimal automorphisms are just right invariant vector fields on G.

Given any torsion-free connection ∇ on M^n, set $P = P(\nabla)$, and let ω be the normal projective Cartan connection associated to P. Denote by $\pi : P \to M$ the standard projection. If X vanishes at $o \in M$, then $\forall p \in \pi^{-1}(o)$, $\omega(\tilde{X})(p) \in \mathfrak{h}$. We can prove the following local result.

Proposition 2.1. *Let X be a projective vector field for (M, ∇). Assume $X_o = 0$ for some $o \in M$. Then the following are equivalent:*

- *X is linearizable at o.*
- *There exist a neighborhood U of o and a torsion-free affine connection $\nabla' \in [\nabla|_U]$ such that X is an affine vector field for ∇'.*

Before proving the proposition above, we need to derive the canonical forms of a projective vector field near its singularity. Denote by ω the normal projective Cartan connection associated to $P = P(\nabla)$ as before. Fix any p in the fiber of o, and let $\exp_p$ be the exponential map of ω at p. Then there is a small neighborhood U of $0 \in \mathfrak{g}_{-1} \simeq \mathbb{R}^n$ such that $\sigma_p = \pi \circ \exp_p : U \to M$ gives a local coordinate system of M at o. Such coordinates are the normal coordinates for $P(\nabla)$ at o. The GL_n sub-bundle given by local section $\exp_p(\mathfrak{g}_{-1})$ over U induces an affine connection $\nabla_U \in [\nabla|_U]$ near o. By Theorem 2.1, σ_p is also a normal coordinate for the affine connection $\nabla_U \in [\nabla|_U]$ at o.

Lemma 2.1. *Suppose X is a projective vector field for ∇ such that $X_o = 0$. Let $P = P(\nabla)$, and set ω to be the corresponding normal projective Cartan* *connection on P induced by ∇ as before. Choose any $p \in \pi^{-1}(o)$, then in the normal coordinate chart σ_p for P at p, the form of ϕ^t in the coordinate chart σ_p is uniquely determined by the value of $\omega(\tilde{X})(p)$, regardless of the choice of the projective Cartan connection ω induced by the projective structure $[\nabla]$.*

Proof. Let $\tilde{X}$ be the lift of X to P such that $\mathcal{L}_{\tilde{X}}\omega = 0$. Because $X_o = 0$, we have $\omega(\tilde{X})(p) = v_h \in \mathfrak{h}$. Define the following identification along fibers over o:

$$\Delta : H \to pH, \quad h \mapsto ph.$$

It follows that $\Delta^*\omega|_{\pi^{-1}(o)}$ is the Maurer–Cartan form ω_H on H. Let X_h be a right-invariant vector field on G with $\omega_G(X_h)(1) = v_h$. Note that $\omega_G(X_h)|_H \in \mathfrak{h}$, and $\mathcal{L}_{X_h}\omega_G = 0$. It follows that $\Delta_*(X_h) = \tilde{X}|_{\pi^{-1}(o)}$.

Denote Φ the flow generated by $\tilde{X}$ on P, so Φ projects to a flow ϕ^t on M fixing o. We have $\Phi(t,p) = ph(t)$, where the function $h(t) = \exp(tv_h)$ evidently depends only on v_h. Fix any $t_0 \in \mathbb{R}$ and $v \in \mathfrak{g}_{-1} = \mathbb{R}^n$, and define the curve $l(s) = \exp_p(sv)$. Note that $\pi \circ l(s)$ is a geodesic of $[\nabla]$. Because $\mathcal{L}_{\tilde{X}}\omega = 0$, the following equality holds:

$$l_{t_0}(s) := \Phi(t_0, l(s)) = \exp_{ph(t_0)}(sv).$$

We also obtain

$$\phi^{t_0} \circ \pi \circ l = \pi \circ l_{t_0} = \pi \circ R_{h(t_0)^{-1}} \circ l_{t_0}.$$

By the axioms of the Cartan connections, we have

$$R_{h(t_0)^{-1}} \circ l_{t_0}(s) = \exp_p(s(Ad(h(t_0)(v)))).$$

Define $v' = Ad(h(t_0)(v))$, then v' is totally determined by the values of v and $h(t_0)$. We define the curve $f(s)$ by

$$f(s) := R_{h(t_0)^{-1}} \circ l_{t_0}.$$

Because $\pi \circ l(s)$ is a geodesic of $[\nabla]$, the projected curve $\pi \circ f(s)$ is also a geodesic of $[\nabla]$. Denote v'_{-1} the $\mathfrak{g}_{-1}$ component of v'. We have that $\pi \circ f(s)$ and $\pi \circ \exp_p(sv'_{-1})$ are geodesics for $[\nabla]$ with the same initial condition. Then on a small interval I containing 0, we can write $f(s) : I \to P$ in the following form:

$$f(s) = \exp_p(r(s)v'_{-1})g(s), \quad r(s) : I \to \mathbb{R},\ g(s) : I \to H.$$

$$r(0) = 0,\ g(0) = 1.$$

Differentiating the equation above, we get

$$v' = \omega(\frac{df}{ds}) = Ad(g(s)^{-1})(r'(s)v'_{-1}) + \omega_H(g'(s)).$$

Given a pair of functions $\{r(s), g(s)\}$, whether this pair is a solution to this equation depends only on v', independent of the connection ω. On the other hand, the definition of the exponential map implies that the solution $\{r(s), g(s)\}$ satisfying the condition $g(0) = 1$ and $r(0) = 0$ is unique. Note that v' and v'_1 only depend on v and $h(t_0)$. It follows from the uniqueness that $\{r(s), g(s)\}$ depends only on v and $h(t_0)$. In particular, the function $r(t)$ and $v' \in \mathbb{R}^n$ depend only on the parameters v, v_h, t_0, regardless of the connection ω. Given any two projective connections ω and ω' on the $H^2(n)$ bundle P, as long as the parameters v, v_h, t_0 are the same, we get the same the function $r(t)$ and $v' \in \mathbb{R}^n$. It follows that the form of ϕ^{t_0} in the normal coordinates of P at p depends only on $h(t_0)$. This completes the proof. $\square$

Suppose X is a projective vector field for (M, ∇) vanishing at o, and fix any $p \in \pi^{-1}(o)$ as before. Because the algebra of the projective vector fields has the maximum dimension for the flat bundle, we can choose some right invariant vector field $\tilde{Y}$ on G such that $\omega_G(\tilde{Y})(1_G) = \omega(\tilde{X})(p) \in \mathfrak{h}$. Let Y be the projection of $\tilde{Y}$ on $\mathbb{RP}^n$. Then X in the normal coordinates of P at p has the same form of Y in the normal coordinates of the flat model at $1 \in G$. Thus, by computations on the flat model, we obtain all possible forms of projective vector fields with a singularity at o in the normal coordinates for P at p.

Lemma 2.2. *Let X be a projective vector field for (M, ∇) with $X_o = 0$. For any $p \in \pi^{-1}(o)$, the vector field X has the form $X_x = Ax + \langle w, x \rangle x$ in the normal coordinates of $P(\nabla)$ at p, where $A \in M_n(\mathbb{R})$, $w \in \mathbb{R}^n$. In addition, X is linearizable if and only if $w \in \mathrm{Im} A^T$.*

Proof. Let X be a projective vector field for (M, ∇) such that $X_o = 0$, and choose any $p \in \pi^{-1}(o)$. First we show X has the form: $X_x = Ax + \langle w, x \rangle x$ in the normal coordinates of $P(\nabla)$ at p. By Lemma 2.1 and the argument in the previous paragraph, we only need to prove for the flat bundle $P = (\mathbb{RP}^n, G, \omega_G)$, any projective vector field X vanishing at $[e_0]$ is in this form in the normal coordinates at $p = 1 \in G$. In this case, the exponential map $\exp_p$ gives the canonical coordinate i_0^{-1} of $\mathbb{RP}^n$ near e_0, where the chart i_0 is given by

$$i_0 : [x_0, x_1, \cdots, x_n] \mapsto (\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}).$$

The projective vector fields fixing $o = [e_0] \in \mathbb{RP}^n$ are induced by linear vector fields in $\mathbb{R}^{n+1}$ fixing the line $[e_0]$. Projecting these vector fields to $\mathbb{RP}^n$, we get X has the form $X_x = Ax + \langle w, x \rangle x$ in the normal coordinates at p.

Next we show X in this form is linearizable if and only if $w \in \mathrm{ImA}^T$. If $w \notin \mathrm{ImA}^T$, we write $w = w_k + w'$ with $w_k \neq 0$, where $w_k \in \mathrm{Ker}A$ and $w' \in \mathrm{Im}A^T$. Denote ϕ^t the flow generated by X as usual. In the normal coordinates for $P(\nabla)$ at p, for some small interval I containing 0, we have

$$\phi^t(sw_k) = \frac{s}{1 + tas} w_k, \quad s \in I, \ a \neq 0.$$

Note that $D\phi^t(w_k) = w_k \neq 0$. Without loss of generality, we can assume $a > 0$. For $s > 0$, we have $\dfrac{s}{1 + tas} \to 0$ as $t \to +\infty$. Then X is not linearizable by Lemma 4.6 of [12]. Conversely, if $w \in \mathrm{Im}A^T$, the following calculation in

Remark 1 shows we can find some $p' \in \pi^{-1}(o)$ such that $X_x = (A_{p'})x$ in the normal coordinates at p'. Hence it is linearizable. $\square$

Remark 1. *To simply the calculations later, Suppose X vanishes at o. Note that for any $A \in M_n(\mathbb{R})$, we have $\mathbb{R}^n = Im(A^T) \bigoplus \mathrm{Ker}A$. Then for any $p \in \pi^{-1}(o)$, this decomposition of $\mathbb{R}^n$ gives*

$$S_p = \omega(\tilde{X})(p) = \begin{bmatrix} -b & w_i^T A + w_k \\ 0 & B \end{bmatrix} \in \mathfrak{sl}_{n+1}(\mathbb{R}).$$

$$A = B + b \cdot Id,\ w_k \in \mathrm{Ker}A.$$

Define $C = \begin{bmatrix} 1 & -w_i^T \\ 0 & Id \end{bmatrix}$, *we have* $CS_pC^{-1} = \begin{bmatrix} -b & w_k \\ 0 & B \end{bmatrix}$. *In other words, given any local coordinate system $\tilde{\sigma} : U \subset \mathbb{R}^n \to M$, with $\tilde{\sigma}(0) = o$, we can choose some $\tilde{p} \in \pi^{-1}(o)$ such that the normal coordinate system $\sigma_{\tilde{p}}$ at $\tilde{p}$ for P satisfies:*

$$J^1(\tilde{\sigma})(0) = J^1(\sigma_{\tilde{p}})(0), \quad ((\sigma_{\tilde{p}}^{-1})_* X)_x = Ax + \langle w, x\rangle x,\ w \in \mathrm{Ker}A.$$

With the results above, we can prove Proposition 2.1.

Proof of Proposition 2.1. By Remark 1, we can always choose some $p \in \pi^{-1}(o)$ such that in the normal coordinate system σ_p of $P(\nabla)$ at p, X has the following form:

$$X_x = Ax + \langle w, x\rangle x,\ w \in \mathrm{Ker}A.$$

If X is linearizable at o, we have $w \in \mathrm{Im} A^T$ by Lemma 2.2. It follows that $w = 0$, then X is linear in σ_p. According to Theorem 2.1 by Nagano, the local section of $F_1^2(M)$ induced by the local section $\exp_p(\mathfrak{g}_{-1})$ corresponds to a connection ∇' projectively equivalent to ∇ locally defined near o. From the last statement of Theorem 2.1, it is clear that σ_p is a normal coordinate of ∇' at o. Thus X is an affine vector field for ∇'. The converse is trivial as affine vector fields of ∇' vanishing at o are clearly linear in the normal coordinates of ∇' at o. $\square$

Suppose that X is a non-linearizable projective vector field for (M, ∇) vanishing at $o \in M$. For each $a > 0$, we can choose a neighborhood U_a of o such that ϕ^t is well defined on U_a for $t \in I = [-a, a]$. Then on U_a, the connection $\nabla_t = \phi^t_* \nabla$ is projectively equivalent to ∇ for $t \in I$. If $\gamma(s)$ is a geodesic segment for ∇ contained in $\phi^{t_0}(U_a)$ with $t_0 \in I$, we have $\phi^{-t_0} \circ \gamma(s)$ is a geodesic segment on U_a for ∇_{t_0}. This leads to the following:

Corollary 2.1.1. *Let X be a projective vector field for (M, ∇) admitting a non-linearizable singularity $o \in M$. Then for each $t \neq 0$, we have*

$$\nabla_t = \nabla + \eta_t \otimes Id + Id \otimes \eta_t, \quad (\eta_t)_o \neq 0.$$

Proof. Suppose that $\eta_{t_0}(o) = 0$ for some $t_0 \neq 0$. The connection ∇ induces a GL_n sub-bundle P_1 of $P(\nabla)$. Choose $p \in \pi^{-1}(o) \cap P_1$. Let ∇_p be the connection induced by the local section $\exp_p(\mathfrak{g}_{-1})$ at p. Then the type (2,1)-tensor $(\nabla_p - \nabla)$ vanishes at o. Thus, we can assume ∇ is ∇_p in this proof.

In the normal coordinates of ∇ at o, denote by $\overline{\Gamma^k_{i,j}}$ and $\Gamma^k_{i,j}$ the Christoffel symbols of ∇ and ∇_{t_0}, respectively. It follows that $\overline{\Gamma^k_{i,j}}(o) = \Gamma^k_{i,j}(o) = 0$, because of $(\eta_{t_0})_o = 0$. Following the calculations of the proof of Theorem 2.1 of Nagano in [1], we can conclude the exponential maps of ∇ and ∇_{t_0} at o have the same 2-jets. Denote $\exp_o^{\nabla}$ and $\exp_o^{\nabla_{t_0}}$ the exponential maps of ∇ and ∇_{t_0} at o, respectively. Note σ_p is a normal coordinate of ∇ at o. Since X is non-linearizable at o, in the coordinate chart σ_p, we may write

$$X_x = Ax + \langle w, x\rangle x,\ 0 \neq w \notin \mathrm{Im}A^T.$$

In the coordinate chart σ_p, choose $w_k \in KerA$ with $\langle w, w_k\rangle \neq 0$. Then in the coordinate chart σ_p, the curve $\gamma(s) = sw_k$ is a non-trivial parametrized geodesic of ∇. In the coordinate chart σ_p, there exists some $s_0 > 0$ such that $\gamma^{t_0}(s) = \phi^{-t_0} \circ \gamma(s)$ is well defined for $|s| < s_0$. Note that $w_k \in \mathrm{Ker}A$ implies the flow ϕ^t preserves the unparametrized geodesic γ. Because $\langle w, w_k\rangle \neq 0$, we have

$$\gamma^{t_0}(s) = \phi^{-t_0} \circ \gamma(s) = \frac{s}{1+as} w_k,\ a \neq 0.$$

Then near $s = 0$, define the function

$$f(s) := (\gamma^{-1} \circ \gamma^{t_0})(s) = \frac{s}{1+as}.$$

It is a local diffeomorphism fixing $0 \in \mathbb{R}$. The map ϕ^{-t_0} takes geodesics of ∇

to geodesics of ∇_{t_0}, so $\gamma^{t_0}(s)$ is a geodesic for ∇_{t_0} such that

$$(\gamma^{t_0})'(0) = \gamma'(0) = w_k.$$

Near $s = 0$, we have

$$\gamma(s) = \exp_o^{\nabla}(sw_k),\ \gamma^{t_0}(s) = \exp_o^{\nabla_{t_0}}(sw_k).$$

The exponential maps ∇ and ∇_{t_0} have the same 2-jets at o, so $\gamma(s)$ and $\gamma^{t_0}(s)$ have the same 2-jets at $s = 0$. This implies the function $f(s)$ has a trivial 2-jet at $s = 0$. But we have

$$\left.\frac{d^2}{ds^2}\right|_{s=0} f(s) = -2a.$$

Thus, we have a contradiction. $\square$

2.3 Metrizable projective structures

This section provides a short review of the tools to study metrizable projective structures. Most of these are from papers [3] by Matveev. Fix a general symmetric affine connection ∇ on M^n, then there is a 1-1 correspondence between elements in the projective class $[\nabla]$ and 1-forms on M^n. The latter is an infinite dimensional vector space. However, if the connection is a Levi-Civita connection induced by g, the metrics projectively equivalent to g form a finite dimensional manifold. The following gives a way to identify those metrics.

Fix a metric g on M. Then for any metric $\overline{g}$ on M, the g-strength of $\overline{g}$ is defined to be the (1,1)-tensor $K_{\overline{g}}$ such that

$$\overline{g}(u,v) = g\left(\frac{K_{\overline{g}}^{-1}}{|\det(K_{\overline{g}})|}\cdot u, v\right),\ \forall u,v \in T_xM,\ \forall x \in M. \tag{3}$$

To proceed, we need the following definition from Section 2 of [4]:

Definition 2.4. *Suppose g is a metric on M^n, the space of BM-structures on M for g, denoted as $B(M,g)$, is the space of g-adjoint (1,1)-tensors on M satisfying the following linear PDE, $\forall u,v,w \in T_xM,\ \forall x \in M$:*

$$g((\nabla_w K)u, v) = \frac{1}{2}(d(trK)(u)g(v,w) + d(trK)(v)g(u,w)). \tag{4}$$

The degree of mobility of g on M^n, denoted as $D(M^n, g)$, is the dimension of the vector space $B(M^n, g)$.

According to Equation (7)-(9) of [3], the non-degenerate elements of $B(M,g)$ are exactly the g-strengths of the metrics projectively equivalent to g on M. Equation (4) is finite-type by Remark 5 of [3], so the solutions on each connected component are uniquely determined by the k-th jet at a single point for some $k \in \mathbb{N}$. Thus we always have $D(M^n, g) < \infty$. In fact, according to Section 3 of [7] , $[\nabla]$ defines a linear connection on some vector bundle $VM \simeq \bigodot^2 TM \oplus TM \oplus C^\infty(M)$. By Theorem 3.1 of [7], solutions to

Equation (4) are in 1-1 correspondence with parallel sections on VM. From Introduction of [6], if M^n is connected, then $D(M^n, g)$ is at most equal to the rank of VM:

$$D(M^n, g) \leq \frac{(n+1)(n+2)}{2}$$

For any $K \in B(M^n, g)$, the eigenfunctions of K, counting multiplicity, can always be chosen to be continuous. Suppose λ_i with $1 \leq i \leq n$ is such a choice. Fix any $x \in M$. We say the eigenfunctions of K admit a partition on some neighborhood U_x of x if there are non-empty sets $\mathcal{S}_1, \mathcal{S}_2$ with $\mathcal{S}_1 \cup \mathcal{S}_2 = \{\lambda_i\}_{1 \leq i \leq n}$ so that the following holds:

$$\lambda_i(y_1) \neq \lambda_j(y_2), \quad \forall \lambda_i \in \mathcal{S}_1, \ \forall \lambda_j \in \mathcal{S}_2, \ \forall y_1, y_2 \in U_x. \tag{5}$$

Suppose the eigenfunctions of K admit such a partition on U_x. Denote the $\chi(K)$ the characteristic polynomial of K in z. Then $\chi(K)$ admits a factorization according to the partition above, namely:

$$\chi(K) = \chi(K_1)\chi(K_2), \quad \chi(K_i) = \prod_{\lambda_j \in \mathcal{S}_i} (z - \lambda_j).$$

With the notations above, the Splitting Lemma in Section 2.1 of [8] gives coordinates to write g and K in block-diagonal forms as follows.

Theorem 2.2 (Bolsinov, Matveev [8]). *Suppose $K \in B(M^n, g)$ admits a partition on some neighborhood of x as above. Then there is a local coordinate system $(x_1, \cdots, x_r, y_1, \cdots, y_{n-r})$ at x so that the pair (g, K) can be written* *in the following block diagonal form.*

$$g = \begin{bmatrix} h_1\chi_2(K_1) & 0 \\ 0 & h_2\chi_2(K_1) \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \tag{6}$$

Here the pair (h_1, K_1) and (h_2, K_2) depend only on the x_i and y_j coordinates, respectively. In addition, K_i is a BM-structure for the metric h_i on each corresponding sub-manifold.

If the metric g is Riemannian, any $K \in B(M, g)$ is clearly real-diagonalizable. For closed connected semi-Riemannian manifolds, the non-constant eigenfunctions of BM-structures are always real-valued by Theorem 6 of [10]. In addition, by the following theorem from Section 2.2 of [4], the eigenfunctions of K are globally ordered on connected convex sets for Riemannian metrics.

Theorem 2.3 (Matveev [4])**.** *Let (M^n, g) be a Riemannian manifold such that every two points can be connected by a geodesic. Suppose $K \in B(M^n, g)$, and let $(\lambda_i)_{1\leq i\leq n}$ with $\lambda_i \leq \lambda_{i+1}$ be the eigenfunctions of K. The following statements hold:*

- $\lambda_i(x) \leq \lambda_{i+1}(y), \quad \forall x, y \in M, \forall 1 \leq i \leq n-1,$
- *If $\lambda_i(x) < \lambda_{i+1}(x)$ for some $x \in M$, then $\lambda_i < \lambda_{i+1}$ almost everywhere on M.*

3 Local dynamics of projective vector fields for metric connections

In this chapter we adapt the dynamical method by Zeghib in [5] to our setting to study the local behavior of metrizable projective structures. For a metrizable projective structure $[\nabla]$ induced by a metric g on M^n, the available methods used in studying the projective structure of (M^n, g) depend on $D(M^n, g)$. We cannot use the methods from [11] when $D(M, g) = 2$, instead the adapted dynamical method from [5] by analyzing the action of $\mathrm{Proj}(M, g)$ on $B(M, g)$ for closed manifolds can be applied our problems after making proper adaptations.

3.1 Dynamics of a projective vector field near its singularity

We start with a brief review of the main approach in [5]. Suppose (M^n, g) is a closed semi-Riemannian manifold. According to Section 2 of [5], the natural action of the group $\mathrm{Proj}(M, g)$ on metrics projectively equivalent to g defines a representation $\rho : \mathrm{Proj}(M, g) \to GL(B(M, g))$ as follows. For any $f \in \mathrm{Proj}(M, g)$, let K_f be the g-strength of f^*g. We have

$$\rho(f)(L) = f_* L \circ K_f, \quad \forall L \in B(M, g). \tag{7}$$

Since M is closed, we can always choose a basis of $B(M, g)$ consisting of non-degenerate elements.

Now further assume that $D(M,g) = 2$ and $f \in \mathrm{Proj}(M,g)$ is non-affine for g. Then, $\{K_f, Id\}$ is a basis of $B(M,g)$. As in Section 4 of [5], there are some constants $\alpha, \beta \in \mathbb{R}$ such that

$$\rho(f)(Id) = K_f, \quad \rho(f)(K_f) = f_* K_f \circ K_f = \alpha K_f + \beta Id. \tag{8}$$

That is to say, for the basis $\{K_f, Id\}$, the linear map $\rho(K_f)$ has the matrix representation: $\begin{bmatrix} \alpha & 1 \\ \beta & 0 \end{bmatrix}$.

Then we define the Möbius map $T_f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ associated to f by

$$T_f(z) = \frac{\alpha z + \beta}{z}. \tag{9}$$

According to Equation (8), we have

$$T_f(\mathrm{Spec}(K_f)(x)) = \mathrm{Spec}(K_f)(f(x)), \quad \forall x \in M. \tag{10}$$

In addition, the map T_f preserves the Jordan types of each eigenvalue of K_f; see Section 4 of [5] for details.

Now we adapt the approach above so it can be applied to study local theory of incomplete projective vector fields near the singularities. Let $f : (M^n, g) \to (N^n, g')$ be a smooth projective embedding. Denote by K_f the g-strength of

f^*g'. Define the linear map $\rho^f(g,g') : T^{1,1}N \to T^{1,1}M$ by

$$\rho^f(g,g')(L) = f_*L \circ K_f. \tag{11}$$

We claim that the map above actually maps $B(N,g')$ into $B(M,g)$. For any $L \in B(N,g')$ and any $y \in N$, choose a neighborhood U_y of y so that $B(U_y,g')$ has a basis $\{K_i\}$ with each $\det(K_i)|_{U_y}$ non-vanishing. Because f^*g' is projectively equivalent to g, the map $\rho^f(g,g')$ takes the g'-strength of a metric projectively equivalent to g' to the g-strength of a metric projectively equivalent to g. Thus we have $\rho^f(g,g')(K_i) \in B(f^{-1}(U_y),g)$ for all i. Since $L|_{U_y}$ is a linear combinations of K_i, it follows that

$$\rho^f(g,g')(L)|_{f^{-1}(U_y)} \in B(f^{-1}(U_y),g).$$

Then $\rho^f(g,g')(L)$ is linear solution to Equation (4) on all of M^n. We have $\rho^f(g,g')(L) \in B(M,g)$.

In addition, the map defined by Equation (11) is multiplicative. Let $f_1 : (N_1,g_1) \to (N_2,g_2)$ and $f_2(N_2,g_2) \to (N_3,g_3)$ be smooth projective embeddings. We have

$$\rho^{f_1\circ f_2}(g_1,g_3) = \rho^{f_1}(g_1,g_2) \circ \rho^{f_2}(g_2,g_3). \tag{12}$$

From now on, assume M is connected. Let U be an open subset of M. We have $\forall K \in B(M,g),\ K|_U \in B(U,g)$. Since M is connected, the following restriction map is injective.

$$R_U : B(M,g) \to B(U,g), \quad K' \mapsto K'|_U.$$

We can view $B(M,g)$ as a linear subspace of $B(U,g)$. Suppose X is a projective vector field for (M^n,g), and denote ϕ^t the flow generated by X. Also suppose that $\exists a > 0$ such that $\phi^t(x)$ is defined for $\forall x \in U$, and $\forall t \in I = [-a,a]$. Then the flow ϕ^t induces a 1-parameter family of maps $L_t : B(M,g) \to B(U,g)$ for $t \in I$ simply by

$$L_t(K) = \rho^{\phi^t}(g,g)(K'), \quad \forall t \in I,\ \forall K' \in B(M,g). \tag{13}$$

If we further assume that $D(U,g) = D(M,g)$, every $K' \in B(U,g)$ can be uniquely extended to an element in $B(M,g)$. Then we can take L_t as a map $B(M,g) \to B(M,g)$ for each $t \in I$. To simplify the notation, set $B = B(M,g)$ from now on. A natural question to ask is whether L_t can be extended to a 1-parameter subgroup of $GL(B)$. This leads to the following lemma.

Lemma 3.1. *Let (M^n,g) be connected with a projective vector field X. Suppose X vanishes at $o \in M$. Assume that U with $D(U,g) = D(M,g)$ is a connected open set containing o such that ϕ^t is defined on U for $t \in I = [-a,a]$*

for some $a > 0$. Then the map $L_t : B \to B$ defined in the previous paragraph satisfies the following:

- *$L_{t+s} = L_t \circ L_s$ for $t, s, t+s \in I$.*
- *The representation $L : I \to GL(B)$ is continuous in t.*

In other words, we can extend L_t to a 1-parameter subgroup of $GL(B)$.

Proof. Fix any $K' \in B = B(M,g)$. For any $t \in I$, $L_t(K')$ is the unique element in $B(M,g)$ such that

$$L_t(K')|_U = \phi^t_*(K') \circ K_t \in B(U,g).$$

Note that given the embedding $\phi^t : U \to M$, we have on U:

$$L_t(K')|_U = \rho^{\phi^t}(g,g)(K').$$

The embedding $\phi^s : U \to M$ gives

$$L_s(L_t(K'))|_U = \rho^{\phi^s}(g,g)(L_t(K')).$$

Because X vanishes at o, there is some neighborhood U_o of o such that $\phi^s(U_o) \subset U$. Then we have a sequence of embeddings:

$$U_o \xrightarrow{\phi^s} U \xrightarrow{\phi^t} M.$$

Because $t, s, t+s \in I$, by Equation (12) we have on U_o:

$$L_s(L_t(K'))|_{U_o} = \left(\rho^{\phi^s}(g,g) \circ \rho^{\phi^t}(g,g)\right)(K') \tag{14}$$

$$= \rho^{\phi^{t+s}}(g,g)(K') \tag{15}$$

$$= L_{t+s}(K')|_{U_o} \tag{16}$$

Since U is connected, any BM-structure on U is uniquely determined by its k-th jet at o for some $k \geq 0$. Then $L_{t+s}(K') = L_s \circ L_t(K')$ on U_o implies $L_{t+s}(K') = L_s \circ L_t(K')$ in B.

Next we show the representation $L_t : I \to GL(B)$ is continuous in t. Because L_t is linear for each t, and B is a finite dimensional vector space, it is sufficient to show for any fixed $K' \in B$, $L_t(K')$ is continuous in t. Fix a compact neighborhood $V_o \subset U$ of o and a basis $\{K^i\}$ for B. Then we can write $L_t(K') = \sum c_i(t)K^i$, where $c_i : I \to \mathbb{R}$. Equation (4) is of finite type, implying $\{K^i\}$ are linearly independent over V_o. On $U \supset V_o$, we have $L_t(K') = \phi^t_*(K') \circ K_t$. Then for any fixed $t_0 \in I$, as $t \to t_0$, we have $L_t(K') \to L_{t_0}(K')$ uniformly on V_0. It follows that $c_i(t) \to c_i(t_0)$ for each i as $t \to t_0$. This proves the continuity of $L_t : I \to GL(B)$. $\square$

The following shows the neighborhood U in Lemma 3.1 always exists.

Lemma 3.2. *Let (M^n, g) be a connected manifold. Suppose X is a projective vector field for g vanishing at $o \in M$. Then there exists a connected open set*

U containing o such that $D(U, g) = D(M^n, g)$, and $\exists a > 0$ such that ϕ^t is well defined on U for $t \in I = [-a, a]$.

Proof. Define the following sets:

$$S_i = \{x \in M : \phi^t(x) \text{ is well defined for t} \in [-\frac{1}{\mathrm{i}}, \frac{1}{\mathrm{i}}]\}.$$

Without loss of generality, we can assume $o \in Int(S_i)$ for all i. Let U_i be the component of $Int(S_i)$ containing o. Since each U_i is open and connected, it is also path connected. Given any $x \in U_i$, let γ_x be a curve in U_i joining o and x. Then we have $\gamma_x \subset Int(S_{i+1})$. It follows that $U_i \subset U_{i+1}$. Similarly, given any $x \in M$, we can choose a curve γ'_x in M joining o and x. Then there exists $\epsilon > 0$ and a neighborhood U_ϵ of γ'_x such that ϕ^t is well defined on U_ϵ for $t \in [-\epsilon, \epsilon]$. It follows that $x \in U_i$ for some i, hence $\bigcup_{i=1}^{\infty} U_i = M$. We obtain an increasing sequence of open sets containing o:

$$o \in U_1 \subset U_2 \subset \cdots, \quad \bigcup_{i=1}^{\infty} U_i = M.$$

Because each U_i is connected, the restriction map gives a sequence of injective linear maps:

$$B(U_1, g) \xleftarrow{r_1} B(U_2, g) \xleftarrow{r_2} \cdots$$

We have $D(U_i, g) \geq D(M, g)$, and $D(U_1, g) < \infty$. It follows that there exists some i_0 such that $r_j : B(U_{j+1}, g) \to B(U_j, g)$ are linear isomorphisms for all $j \geq i_0$. Then any $\tilde{K} \in B(U_{i_0}, g)$ can be uniquely extended to an element in $B(U_j, g)$ for all $j \geq i_0$. Because a BM-structure on a connected manifold is uniquely determined by its finite jet at some point, we have $\tilde{K}$ can be extended to an element in $B(M, g)$. Thus $D(U_{i_0}, g) = D(M, g)$. This completes the proof. □

Let X be a projective vector field of (M, g) vanishing at o. We have shown although the projective vector field X may be incomplete, it is possible to obtain a 1-parameter group L_t of $GL(B)$ from ϕ^t. Next thing we need to check is whether the action of L_t on B agrees with the one induced by metric pull-back by the flow ϕ^t near o. This is clearly true by the following.

Corollary 3.0.1. *Let X be a projective vector field for (M, g) vanishing at o. Suppose M is connected. Let U,I, and L_t be constructed as above. Given any $t_0 \in \mathbb{R}$, there exists some neighborhood V_{t_0} of o such that ϕ^t is well defined for $|t| \leq |t_0|$, and $L_{t_0}(K')|_{V_{t_0}} = \phi_*^{t_0}(K') \circ K_{t_0}$ on V_{t_0}.*

Proof. Without loss of generality, assume $t_0 > 0$. Let U, I be the same as in Lemma 3.2, and $t_0 = nt_1$ with $t_1 \in I$. Given any $K' \in B \simeq B(U, g)$ and $t \in I$, there is some neighborhood V_t of o such that $\phi^t(V_t) \subset U$. In particular, we have$L_{t_1}(K')|_{V_{t_1}} = \phi_*^{t_1}(K') \circ K_{t_1}$. Assume there is some neighborhood $V_{mt_1} \subset U$ of o such that $\phi^s(V_{mt_1})$ is defined for $s \in [-mt_1, mt_1]$, and

$$L_{mt_1}(K')|_{V_{mt_1}} = \phi_*^{mt_1}(K') \circ K_{mt_1}.$$

We can choose some $V_{(m+1)t_1}$ such that

$$o \in V_{(m+1)t_1} \subset V_{mt_1} \subset U, \quad \phi^{t'}(V_{(m+1)t_1}) \subset V_{mt_1} \ for \ t' \in I.$$

Then ϕ^s is well defined on $V_{(m+1)t_1}$ for $s \in [-(m+1)t_1, (m+1)t_1]$. This implies on $V_{(m+1)t_1}$, we have

$$\begin{aligned} L_{(m+1)t_1}(K')|V_{(m+1)t_1} &= L_{t_1}(L_{mt_1}(K'))|V_{(m+1)t_1} && (17)\\ &= \phi_*^{t_1}(L_{mt_1}(K')) \circ K_{t_1} && (18)\\ &= \phi_*^{t_1}(\phi_*^{mt_1}(K') \circ K_{mt_1}) \circ K_{t_1} && (19)\\ &= \phi_*^{(m+1)t_1}(K') \circ K_{(m+1)t_1} && (20) \end{aligned}$$

By induction, on $V_{t_0} = V_{nt_1}$, we have $L_{t_0}(K')|_{V_{t_0}} = \phi_*^{t_0}(K') \circ K_{t_0}$. □

3.2 The case that the degree of mobility is exactly 2

In this section we study the local dynamics of projective vector fields on (M^n, g) with $D(M^n, g) = 2$. The case $D(M^n, g) \geq 3$ is well understood from works like [4], [11], [3]. For $D(M, g) \geq 3$, we always have the so-called Gallot-Tanno Equation, see [11] for details. As in [11], we can study the parallel structures on the cone-manifold to obtain the results for $B(M, g)$. In addition, for $D(M, g) \geq 3$, the parametrizations of geodesics for a metric projectively equivalent to g are restricted by Equation (68) from [3]. The case $D(M, g) = 2$ is more difficult to analyze as generally there are not enough symmetries of the projective structure.

Let (M^n, g) be a connected manifold with $D(M, g) = 2$. Let X be a projective vector field for g with a singularity o. Denote ϕ^t the flow generated by X as before. Suppose X is not linearizable at o. Then L_t is a 1-parameter subgroup of $GL(B) \simeq GL_2(\mathbb{R})$. By Corollary 3.0.1, for any fixed $t \in \mathbb{R}$, on some neighborhood V_t of o, we have

$$L_t(K') = \phi^t_*(K') \circ K_t.$$

Then on V_t, we have $L_t(Id) = K_t$. By Corollary 2.1.1, for any $t \neq 0$, the metrics g_t and g are not affine equivalent on any neighborhood of o. This implies the eigenfunctions of K_t are not all constant on any neighborhood of o. Otherwise Using Equation (4), we get $\nabla K_t = 0$ near o, implying g_t and g are affine equivalent near o. If L_t is elliptic, we have $\exists t_0 \neq 0$ such that $K_{t_0} = L_{t_0}(Id) = r_{t_0} Id$ for some $r_{t_0} \neq 0$. It follows that L_t cannot be an elliptic 1-parameter subgroup of $GL(B)$. The following theorem shows L_t is indeed parabolic.

Theorem 3.1. *Let (M^n, g) be a connected semi-Riemannian manifold with $D(M, g) = 2$. Let X be a projective vector field for g vanishing at o. Suppose X is not linearizable at $o \in M$, then L_t is a 1-parameter parabolic subgroup of $GL(B)$.*

The idea of the proof of Theorem 3.1 follows from [5] by Zeghib. Before proving the theorem, we make the following observations: Let U, I, L_t be as before. Fix any $t_0 \neq 0$, then $\{L_{t_0}(Id), Id\}$ is a basis for B. Write $\overline{K}$ for $L_{t_0}(Id)$ for simplicity. Let T be the Möbius map associated to ϕ^{t_0} by (9). Then Equation (10) becomes:

$$T(\mathrm{Spec}(\overline{K}_x)) = \mathrm{Spec}\left(\overline{K}_{\phi^{t_0}(x)}\right), \quad \forall x \in U. \tag{21}$$

To prove Theorem 3.1, first we need the following lemma.

Lemma 3.3. *Suppose L_t is induced by a projective vector field admitting a non-linearizable vanishing point $o \in M$. Fix any $t_0 \neq 0$, and define $\overline{K}$ and T as before. Note that L_t defines a non-trivial 1-parameter parabolic or hyperbolic subgroup of $PGL(B)$ acting on $\mathbb{P}(B)$. Its fixed set on $\mathbb{P}(B)$ is exactly the following:*

$$D_o = \{[\overline{K} - rId] : r \in \mathrm{Spec}((\overline{K})_o) \cap \mathbb{R}\}.$$

Moreover, the fixed set of the Möbius map T on $\widehat{\mathbb{C}}$ is exactly $\mathrm{Spec}(\overline{K}_o)$.

Proof. We know L_t is either hyperbolic or parabolic. Then for any $t_0 \neq 0$, the fixed set of L_{t_0} on $\mathbb{P}(B)$ is the fixed set of L_t on $\mathbb{P}(B)$. It is clearly non-empty. For any fixed $t_0 \neq 0$, by Corollary 3.0.1, there is a neighborhood V of o such that

$$L_{t_0}(K')|_V = \phi_*^{t_0}(K') \circ K_{t_0}, \ \forall K' \in B.$$

Then $(L_{t_0}(K'))_o$ is degenerate if and only if $(K')_o$ is degenerate. Note D_o is the set of elements in B degenerate at o. This implies L_{t_0} takes $D_o \subset \mathbb{P}(B)$ to itself. Because D_o is a finite discrete subset of $\mathbb{P}(B)$, we have L_t fixes all elements in D_o.

Suppose there is some $[\overline{K} - r_0 Id] \notin D_o$ fixed by L_t. We seek to derive a contradiction. Let $K^1 = \overline{K} - r_0 Id$, then $L_t(K^1) = e^{ct}K^1$ for some $c \in \mathbb{R}$. Because K^1 is non-degenerate near o, we have K^1 defines a metric g_{K^1} projectively equivalent to g on some neighborhood $V_o \subset U$ of o. Because $L_t(K^1)|_U = \phi_*^t(K^1) \circ K_t$ for $t \in I$, then X is a homothetic vector field for g_{K^1}. This is impossible. Also note that L_t does not fix the line $[Id]$, otherwise it is a homothetic vector field for g. This proves the fixed set of L_t on $\mathbb{P}(B)$ is exactly D_o

For any fixed $t_0 \neq 0$, the associated Möbius map is of the form $T(z) = \dfrac{\alpha z + \beta}{z}$. Under the basis $\{\overline{K}, Id\}$, L_{t_0} has the following matrix representation:

$$\begin{bmatrix} \alpha & 1 \\ \beta & 0 \end{bmatrix}$$

Denote $F(T)$ the fixed set of T on $\widehat{\mathbb{C}}$. The fixed set of L_{t_0} is exactly D_o. This implies $F(T) \cap \mathbb{R}$ is exactly $\mathrm{Spec}(\overline{K}_o) \cap \mathbb{R}$. Note $F(T) \cap \mathbb{R}$ is non-empty, because L_t is not elliptic. Then the equation $z^2 = \alpha z + \beta$ has 1 or 2 distinct real root. In both cases $F(T)$ has to be a subset of $\mathbb{R}$, so we get

$F(T) = \mathrm{Spec}(\overline{K}_o) \cap \mathbb{R}$. In addition, the finite subsets of $\widehat{\mathbb{C}}$ preserved by T are subsets of $F(T)$. According to Equation (21), we have $\mathrm{Spec}((\overline{K})_o)$ is a finite set fixed by T. It follows that $F(T) = \mathrm{Spec}((\overline{K})_o)$. This completes the proof. $\square$

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. The general scheme of the proof is as follows. First we fix some $t_0 \neq 0$, and use the normal forms of projective vector fields to obtain the dynamics of ϕ^t on some special geodesic curve γ for g_{t_0} and g. For the hyperbolic case, the Splitting Lemma allows us to write g_{t_0} and K_{t_0} in block diagonal forms. The dynamics of ϕ^{t_0} on γ and the dynamics of the associated Möbius map T are related by (21). Using this and the properties of the map T, we derive a contradiction.

By Lemma 3.3, L_t is either hyperbolic or parabolic. Suppose L_t is hyperbolic. Choose $0 \neq t_0 \in I$, then K_{t_0} is the g-strength of g_{t_0} on U. Denote ∇ the Levi-Civita connection for g. Let $P = P(\nabla)$ be the projective Cartan bundle for ∇. Then ∇ induces a GL_n sub-bundle Γ of P. Choose $p \in \Gamma \cap \pi^{-1}(o)$. The section given by $\exp_p(\mathfrak{g}_{-1})$ locally defines a symmetric affine connection $\overline{\nabla} \in [\nabla|_V]$ on some neighborhood V of o. Let σ_p be a normal coordinate of P at p. Clearly by Theorem 2.1, σ_p is a normal coordinate of $\overline{\nabla}$ at o. Because

X is not linearizable at o, by Lemma 2.2, $(\sigma_p)_*^{-1}X$ has the following form:

$$X_x = Ax + \langle w, x\rangle x, \quad w \notin Im(A^T).$$

Choose $v \in \mathrm{Ker}A$ such that $\langle w, v\rangle \neq 0$. In the coordinate chart σ_p, there exists $a \neq 0$ and $\epsilon > 0$ such that

$$\phi^t(yv) = \left(\frac{y}{1+tay}\right) v,\ y \in (-\epsilon, \epsilon),\ t \in I. \tag{22}$$

Let $\gamma(s)$ and $\gamma(s(y))$ be geodesics with initial vector $(\sigma_p)_*v$ for ∇ and $\overline{\nabla}$, respectively. Denote $E : T_oM \to M$ and $\overline{E} : T_oM \to M$ the exponential maps for ∇ and $\overline{\nabla}$ at o, respectively. From Theorem 2.1 by Nagano, we have $J^2(E)(0) = J^2(\overline{E})(0)$, because $p \in \Gamma \cap \pi^{-1}(o)$, we get

$$\frac{ds}{dy}(0) = 1, \quad \frac{d^2s}{dy^2}(0) = 0. \tag{23}$$

Note that ϕ^t preserves the unparametrized geodesic given by γ. Then for small s, we can define a parametrized family of functions τ_t with $\tau_t(0) = 0$ for $t \in I$ by

$$\phi^t \circ \gamma(s) = \gamma(\tau_t(s)).$$

Let $\tau = \tau_{t_0}$ for simplicity. From Equation (22), we have $\dfrac{d\tau}{ds}(0) = 1$. As in Equation (5) of [3], define the function:

$$\psi(s) = -\frac{1}{2}\log(\det(K_{t_0}))(\gamma(s)).$$

Then for small s, by Equation (2) and (3) of [3], we obtain

$$\frac{d\psi}{ds} = \frac{1}{2}\frac{d}{ds}(\log(\frac{d\tau}{ds})).$$

It follows that $\dfrac{d\psi}{ds}(0) = \dfrac{1}{2}\dfrac{d^2\tau}{ds^2}(0)$. According to Lemma 3.3, $\mathrm{Spec}((K_{t_0})_o) = \{\lambda_u, \lambda_b\} \subset \mathbb{R}$. Here λ_u, λ_b are the unstable and stable fixed point of the associated Möbius map $T(z) = \dfrac{\alpha z + \beta}{z}$, respectively. We can apply the Splitting Lemma by Matveev and Bolsinov stated in Theorem 2.2. On some neighborhood $V' \subset V$ of o, there is a smooth local coordinate system in which K_{t_0} can be written in the following block-diagonal form:

$$K_{t_0} = \begin{bmatrix} K_u & 0 \\ 0 & K_b \end{bmatrix}, \quad \mathrm{Spec}((K_u)_o) = \{\lambda_u\},\ \mathrm{Spec}((K_b)_o) = \{\lambda_b\}.$$

We may choose V' small enough so that $\mathrm{Spec}(K_u)|_{V'} \subset D_u$, and $\mathrm{Spec}(K_b)|_{V'} \subset D_b$. Here D_u, D_b are pairwise disjoint disks in $\mathbb{C}$ centered at λ_u, λ_b, respectively. It follows that

$$\psi(s) = -\frac{1}{2}[\log(\det(K_u))(\gamma(s)) + \log(\det(K_b))(\gamma(s))]. \tag{24}$$

Define $f_u(s) = \det(K_u)(\gamma(s))$, and $f_b(s) = \det(K_b)(\gamma(s))$. Without loss of generality, let us assume $t_0 a > 0$. From Equation (22), for small $s > 0$, we have $\tau(s) < s$, and $\phi^{mt_0}(\gamma(s)) \to o$ as $m \to +\infty$.

Choosing the eigenfunctions of K_u and K_b to be continuous on V', we use Equation (24) and $\frac{d\psi}{ds}(0) = \frac{1}{2}\frac{d^2\tau}{ds^2}(0)$ to derive a contradiction. First we show the eigenfunctions of K_u have to be constant on $\gamma(s)$ for small $s > 0$. Suppose this is not the case. Let $\tilde{k_u}$ be an eigenfunction of K_u, and write $k_u(s) = \tilde{k_u}(\gamma(s))$. Then there is some $s_0 > 0$ such that $\gamma([0, s_0]) \subset V'$ and $k_u(s_0) \neq \lambda_u$. Then we have

$$\gamma([0, s_0]) \subset V' \subset V \Longrightarrow \phi^{t_0} \circ \gamma([0, s_0]) \subset \gamma([0, s_0]).$$

Because T is a continuous map on $\widehat{\mathbb{C}}$, we have $T^m \circ k_u : [0, s_0] \to \widehat{\mathbb{C}}$ is a continuous map for each m. For large m, we get $T^m(k_u(s_0)) \in D_b$. On the other hand, for any $s' \in [0, s_0]$, we have

$$T^m(k_u(s')) \in Spec\left((K_{t_0})(\phi^{mt_0} \circ \gamma(s'))\right) \subset D_u \cup D_b.$$

Because $T^m(k_u(0)) = \lambda_u$ for all m, we have $T^m \circ k_u([0, s_0])$ is not connected for large m. This contradicts the continuity.

The above implies $f_u(s)$ is constant for small $s \geq 0$. Similarly, we can prove $f_b(s)$ is constant for small $s \leq 0$. From Equation (24), we have $\frac{d\psi}{ds}(0) = 0$. It follows that

$$\frac{d^2\tau}{ds^2}(0) = 0.$$

Define the Möbius map $\widehat{T}(y) = \frac{y}{1 + t_0 a y}$. From Equation (22), we have near

0 that

$$\tau \circ s(y) = s \circ \widehat{T}(y).$$

By Equation (23), we get $J^2(\tau)(0) = J^2(\widehat{T})(0)$. This gives $\dfrac{d^2}{dy^2}(\widehat{T})(0) = 0$, which is clear impossible because $t_0 a \neq 0$. This gives a contradiction. Hence L_t can only be a 1-parameter parabolic subgroup of $GL(B)$. $\square$

4 Application to global results for metrizable projective structures

4.1 Proof of the theorem for 3-dimensional Riemannian manifolds

In this section, we give the proof of Theorem 1.4 stated in the introduction.

Before proving the theorem, we make the following observations. First suppose $(\hat{M}^3, g)$ is a simply-connected and connected manifold admitting a projective vector field X with an essential singularity. By Theorem 1 of [6], the possible values of $D(\hat{M}^3, g)$ are either 1,2 or 10. According to Section 1.2 of [9], the degree of mobility of an n-dimensional connected manifold with $n > 1$ achieves the upper bound $\dfrac{(n+1)(n+2)}{2}$ only when the manifold is projectively flat. It follows that $D(\hat{M}^3, g) = 2$ if $(\hat{M}^3, g)$ is not projectively flat. For a connected 3-dimensional manifold (M^3, g), after lifting everything to its universal cover, we see that $D(M^3, g) \leq 2$ unless (M^3, g) is projectively flat.

Let (M^n, g) with $n \geq 3$ be a connected Riemannian manifold with $D(M^n, g) = 2$. Then $\forall K' \in B(M, g)$, the BM-structure K' is real diagonalizable, because it is a self-adjoint operator for the Riemannian metric g. Let U, I, L_t be as before. Fix any $0 \neq t_0 \in I$, by Lemma 3.3, $(K_{t_0})_o$ has only 1 real eigenvalue

$\lambda > 0$. Thus $(K_{t_0})_o = \lambda Id$. Because X is not linearizable at o, by Lemma 2.2, we have $(D\phi^t)_o$ fixes some non-zero $v \in T_oM$. It follows that

$$g(v, v) = g_{t_0}(v, v) = \frac{1}{\det((K_{t_0})_o)} g((K_{t_0})_o^{-1} v, v).$$

This gives $\lambda = 1$, and $(K_{t_0})_o = Id$. By Lemma 3.3, the associated Möbius map for L_{t_0} is $T(z) = \dfrac{2z-1}{z}$.

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. First we show $D(M^n, g) \geq 3$. Suppose that $D(M, g) = 2$, and we try to obtain a contradiction.

Let U, I, L_t be constructed as before. Fix some $0 < t_0 \in I$. We have

$$(\phi^t)^* g(o) = g(o), \ \forall t \in I.$$

This implies $(D\phi^t)_o$ is a 1-parameter subgroup of $SO(g)$ at o. By Remark 1, we can choose $p \in \pi^{-1}(o)$ such that in the normal coordinate system σ_p for $P = P(\nabla)$ at p, the projective vector field X has the following form:

$$X_x = Ax + \langle w, x \rangle x, \quad A \in \mathfrak{so}(n), \ w = -e_1 \in \mathrm{Ker} A.$$

Then in the local coordinate system σ_p, the flow ϕ^t of X has the following form:

$$\phi^t(x) = \frac{1}{1+tx_1}\left(e^{tA}x\right), \quad x = (x_1, \cdots, x_n). \tag{25}$$

Choose a convex neighborhood C of o which lies in the image of the coordinate chart σ_p. According to Theorem 2.3 by Matveev, for all $i \in \{1, \cdots, n-1\}$, the eigenfunctions λ_i of K_{t_0} are globally ordered on C in the following sense:

- $\lambda_i(x) \leq \lambda_{i+1}(y)$ for all $x, y \in C$.
- If $\exists x \in C$ such that $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(y) < \lambda_{i+1}(y)$ for almost all $y \in C$.

At o, we have $\lambda_i(o) = 1$ for all i. For $n \geq 3$, this implies $\lambda_2 = \cdots = \lambda_{n-1} \equiv 1$ on C. It follows that for $n \geq 3$, $\lambda_1(x) \leq \lambda_2(x) = 1$, and $\lambda_n(x) \geq \lambda_{n-1}(x) = 1$ for all $x \in C$.

We show all eigenfunctions λ_i have to be constant on C. In the coordinate chart σ_p, define the following subsets of C:

$$C^+ = \{x \in C : x_1 > 0\},\ C^- = \{x \in C : x_1 < 0\}.$$

If $\exists x_1 \in C$ such that $\lambda_1(x_1) < 1$, we can find $x_0 \in C^+$ such that $\lambda_1(x_0) < 1$, and $\phi^t(x_0) \in C^+$ for all $t \geq 0$. Denote by $\mathcal{D}$ the closure of the integral curve of $\phi^t(x_0)$ for $t \geq 0$, then clearly $\mathcal{D} \subset C$. From Equation (25), we can see that $\mathcal{D}$

is compact and connected. Hence $\lambda_1(\mathcal{D})$ is an interval $I_1 = [d, 1]$ with $d < 1$. The eigenfunctions of K_{t_0} are all positive on U, so we have $0 < d < 1$ and $0 < \lambda_1(x) \leq 1$ $\forall x \in \mathcal{D}$. Because $T(z) = \dfrac{2z-1}{z}$ is monotonically increasing on $\mathbb{R}^+$, we have $T(\lambda_1(x)) = \lambda_1(\phi^{t_0}(x))$ for all $x \in \mathcal{D}$. It follows that

$$T([d,1]) = T(\lambda_1(\mathcal{D})) = \lambda_1(\phi^{t_0}(\mathcal{D})) \subset \lambda_1(\mathcal{D}) = [d,1],\ 0 < d < 1.$$

This is clearly impossible for the Möbius map $T(z) = \dfrac{2z-1}{z}$ as $T(d) < d$ for $0 < d < 1$. Hence $\lambda_1 \equiv 1$ on C. Replacing C^+ with C^-, and T with T^{-1}, respectively, we can show $\lambda_n \equiv 1$ on C. It follows that all eigenfunctions of K_{t_0} are constant on C.

If all eigenfunctions of K_{t_0} are constant on C, then $\phi^{t_0}g$ and g are affine equivalent on C. This is clearly impossible by Corollary 2.1.1. It follows that $D(M,g) \neq 2$.

Since X is a projective vector field for (M^n, g), according to Section 2.1 of [4], we have

$$K' = g^{-1}\mathcal{L}_X g - \frac{1}{n+1} Tr(g^{-1}\mathcal{L}_X g) \cdot Id \in B(M,g).$$

Then $D(M,g) = 1$ implies that X is a homothetic vector field for g, which is impossible. Hence we have $D(M,g) \geq 3$.

When $n = 3$, it follows from the discussion earlier in this section that (M^3, g) has constant sectional curvature. $\square$

4.2 Proof of theorem for closed connected semi-Riemannian manifolds

In this section, we give the proof of Theorem 1.3 stated in the introduction.

Proof of Theorem 1.3. Since X is not linearizable at o, we have $D(M, g) \geq 2$. First suppose $D(M, g) = 2$, then L_t is a 1-parameter parabolic subgroup by Theorem 3.1. This is in fact impossible by the following argument. This argument is analogous to the proof of the parabolic case of Theorem 1.7 of [5], see page 51 of [5] for details.

Because L_t is parabolic, there exists $K \in B = B(M, g)$ such that

$$L_t(Id) = e^{tb}(tK + Id),\ b \in \mathbb{R}.$$

We know X is complete because M is compact. Just fix $t = 1$, then $L_1(Id) = e^b(K + Id)$ is the g-strength of $(\phi^1)^*g$ on M. Because M is closed and connected, all non-real eigenfunctions of $L_1(Id)$ are constant by Theorem 6 of [10]. It follows that all non-real eigenfunctions of K are constant on M. On the other hand, all real eigenfunctions of K are identically zero. Otherwise, $\exists t_0 \in \mathbb{R}$ such that $L_{t_0}(Id) = K_{t_0}$ is degenerate. Then all eigenfunctions of K are constant. This implies g_t and g are affine equivalent for all $t \in \mathbb{R}$, which is impossible.

From above we have $D(M, g) \geq 3$. The projective Lichnerowicz conjecture is proved for an arbitrary closed connected manifold (M^n, g) with $n > 1$ and $D(M^n, g) \geq 3$, see Corollary 5.2 of [11] for details. Thus, g is Riemannian with positive constant sectional curvature. $\square$

5 Local dynamics for 3-dimensional Lorentzian metrics

5.1 Examples of non-metric connections

In this section, we give an example of a torsion-free affine connection defined on a neighborhood of $o \in \mathbb{R}^n$ admitting a projective vector field X with a non-linearizable singularity at o while not projectively flat on any neighborhood of o.

First we start with the case $n = 2$. Let ∇ be the canonical flat connection on (x, y)-plane, i.e. with all vanishing Christoffel symbols. Note that $X_{(x,y)} = (y - x^2)\partial_x - xy\partial_y$ is a projective vector field for ∇. Clearly X admits a non-linearizable singularity at the origin. Denote ϕ_X^t the flow generated by X. We have

$$\phi_X^t(x, y) = \left(\frac{x + ty}{1 + tx + t^2 y/2}, \frac{y}{1 + tx + t^2 y/2} \right).$$

Denote by H the lower half plane.

$$H = \{z = (x, y) \in \mathbb{R}^2 : y < 0\}.$$

It is straightforward to check the half line $\{z = (x, y) : x = 0, y < 0\}$ is a cross section for H. Then on H, we have the following change of coordinates:

$$\psi(t, r) = \phi_X^t(0, r)$$

Note that $r(x, y) = \dfrac{2y^2}{2y - x^2}$, so we have $|r(x, y)| < |y|$ on H.

Clearly on H, the 1-forms dt, dr are well defined. Also ∂_t is just X on H. Define the following connection:

$$\tilde{\nabla} = \nabla + \omega, \quad \omega(x, y) = \begin{cases} \exp(1/r(x, y))dt \otimes dt \otimes \partial_t, \ y < 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

Clearly X is a projective vector field for $\tilde{\nabla}$. We claim that $\tilde{\nabla}$ is a well-defined smooth connection near $0 \in \mathbb{R}^2$ while not projectively flat on any neighborhood of 0.

First we show that ω is smooth near 0. We have on H that:

$$dt = d\left(\frac{x}{y}\right) = \frac{1}{y^2}(ydx - xdy).$$

We define the following function:

$$h(x,y) = \begin{cases} \dfrac{e^{1/r(x,y)}}{y^4}, \ y < 0, \\ 0, \ y \geq 0. \end{cases}$$

Define the tensor ω_1 by $\omega = h\omega_1$. We can see that ω_1 always has bounded partial derivatives of all orders. Also note ω_1 is smooth except possibly on the x-axis. According to the formulas above, if we can show that h is smooth near 0, then all partials of all orders of h vanish on the x-axis. Then we can deduce that ω is smooth near 0. Since $|r(x,y)| < |y|$ on H, we have h is continuous on the x-axis. By induction, assume that h is C^k. Let g_k be one of the k-th partials of h. Note that g_k vanishes on the x-axis by continuity. On H, any partial of any order of h is a linear combination of products of rational functions and $e^{1/r}$. In addition, the denominator of every term is a polynomial in y and $(2y - x^2)$. Note we have the following on H:

$$|r(x,y)| < |y| < |2y - x^2|.$$

We need to show $\partial_x g_k$ and $\partial_y g_k$ exist and are continuous on the x-axis. Fix any $d > 0$, and let $B = B_d(0)$ be the ball centered at 0. Because of the inequality above, we have $\partial_x g_k$ and $\partial_y g_k$ go to zero as $y \to 0^-$ on $B \cap H$. Pick any $p = (x_0, 0)$. Note $\partial_x g_k(x_0, 0) = 0$ as g_k vanishes on the x-axis by continuity. For any fixed x_0, we have

$$\lim_{y\to 0^-}\left|\frac{g_k(x_0,y)}{y}\right| \leq \lim_{y\to 0^-}\left|\frac{g_k(x_0,y)}{r(x_0,y)}\right| = 0.$$

Hence partials of g_k are continuous. By induction, we can see h is smooth near 0. It follows that $\tilde{\nabla}$ is a smoothly defined connection on $\mathbb{R}^2$.

Next we show that $\tilde{\nabla}$ is not projectively flat on any neighborhood of 0. It is straightforward to compute the components of the Ricci curvature tensor Ric of $\tilde{\nabla}$ on H, which yields the following on H:

$$(Ric)_{tt} = -e^{1/r}\left(\frac{tr}{1+t^2r/2}\right),\ (Ric)_{rr} = (Ric)_{tr} = 0,\ (Ric)_{rt} = \frac{1}{r^2}e^{1/r}.$$

For $n = 2$, the projective Schouten tensor P is given by the following:

$$P(Z_1, Z_2) = Ric(Z_1, Z_2) + \frac{1}{3}(Ric(Z_2, Z_1) - Ric(Z_1, Z_2)).$$

Let $C_{abc} = \tilde{\nabla}_a P_{bc} - \tilde{\nabla}_b P_{ac}$ be the projective Cotton tensor (See Sec 2.1 of [18]). Using the formulas above, we can conclude that in the (t, r) coordinate of H:

$$C_{trr} = -\frac{2}{3r^3}e^{1/r}.$$

It is clear the term C_{trr} does not vanish identically on any neighborhood of 0. It follows that the class $[\tilde{\nabla}]$ is not flat on any neighborhood of 0. This completes the proof.

This example can be generalized to arbitrary dimension as follows. Define the vector field $X_x = Ax + \langle w, x\rangle x$, and denote ϕ_X^t its flow. Here we set

$$w = -(1, 0, \cdots, 0), \; A_{ij} = \begin{cases} 1, & i = 1, j = 2, \\ 0, & otherwise. \end{cases}$$

The flow is given by

$$\phi_X^t = \frac{1}{1 + tx_1 + t^2 x_2/2}(x_1 + tx_2, x_2, \cdots, x_n).$$

Then the open set $H = \{x \in \mathbb{R}^n : x_2 < 0\}$ has a cross section where $x_1 = 0$. Then we have a change of coordinate analogous to the case $n = 2$:

$$\Phi(t, r) = \phi_X^t(0, r), \quad r = (r_2, \cdots, r_n) \in \mathbb{R}^{n-1}.$$

On H, let $r_2(x)$ be the x_2-component of p_x, which is the intersection of the curve $\phi_X^t(x)$ with hyperplane $\{x_1 = 0\}$. Note that for $x \in H$, we have $0 < |r_2(x)| < |x_2|$. Denote ∇ the canonical flat connection on $\mathbb{R}^n$ as before. Now define the connection $\tilde{\nabla}$ on $\mathbb{R}^n$ analogously:

$$\tilde{\nabla} = \nabla + \omega, \quad \omega = \begin{cases} e^{1/r_2(x)} dt \otimes dt \otimes \partial_t, \; if \; x \in H, \\ 0, \; otherwise. \end{cases}$$

Then similar to the case $n = 2$, we can prove $\tilde{\nabla}$ is smoothly defined on $\mathbb{R}^n$. On the negative x_2-axis, it is straightforward compute the component $C_{tr_2r_2}$ to check it is not identically zero on any neighborhood of 0. It follows that $[\nabla]$ is not flat on any neighborhood of 0.

5.2 Normal forms near the essential singularity of the projective vector field

In this section, we will find a criterion to divide the proof of Theorem 1.5 into several cases and write the projective vector fields in normal forms case by case to reduce the calculations.

5.2.1 List of metrics and projective vector fields in normal forms

Denote ϕ^t the flow generated by X and ∇ the Levi-Civita connection of the Lorentzian metric g as usual. Let $P = P(\nabla)$ be the projective Cartan bundle and $\pi : P \to U$ be the projection. By the Remark 1, for any local coordinate system σ' at o with $\sigma'(0) = o$, we can choose some $p \in \pi^{-1}(o) \cap P$ such that X has the following form in the normal coordinate system σ_p at p with $J^1(\sigma_p)(0) = J^1(\sigma')(0)$:

$$X_x = Ax + \langle w, x \rangle x, \quad w \in \mathrm{Ker}A, . \tag{26}$$

By shrinking U if necessary, we can assume that σ_p is a normal coordinate at o of $\overline{\nabla} \in [\nabla|_U]$.

We only need to prove the theorem when g is not projectively flat on any neighborhood of o. For any connected open set U' with $o \in U' \subset U$, we have $D(U', g) = 2$ according to Section 4.1. In addition, the flow ϕ^t defines a non-trivial 1-parameter parabolic subgroup L_t by Theorem 3.1. We can choose some basis $\{K, Id\}$ of $B(U, g)$ with K_o nilpotent so that L_t has the following matrix representation:

$$e^{tD}, \quad D = \begin{bmatrix} \hat{\alpha} & 1 \\ 0 & \hat{\alpha} \end{bmatrix}. \tag{27}$$

Thus, we have the following:

$$K_t = L_t(Id) = e^{t\hat{\alpha}}(tK + Id). \tag{28}$$

Denote $g_t = \phi^t_* g$, which is well defined on some open set containing o. It follows the definition of BM-structures that for any $v_1, v_2 \in T_xU,\ x \in U$:

$$g_t(v_1, v_2) = g(\hat{K}_t v_1, v_2),\ \hat{K}_t = \frac{K_t^{-1}}{|K_t|} = \frac{1}{e^{4t\hat{\alpha}}|tK + Id|}(tK + Id)^{-1}. \tag{29}$$

We split the problem into cases by the values of $\dim(\mathrm{Ker}A)$ and $\hat{\alpha}$. For each case we can choose some special normal coordinate of σ_p at o so that the forms X and g are relatively easy to analyze by computation. The following is a complete list of all such cases. The detailed calculations on how to obtain them are given in the next section.

For the case in which Ker$A$ is 2-dimensional, we have the constant $\hat{\alpha} = 0$. The flow ϕ^t can be written in one of the following forms in some normal coordinate σ_p.

$$\phi^t(x) = \frac{1}{1 + tx_2 + t^2 x_3/2}(x_1, x_2 + tx_3, x_3), \tag{\textbf{I}}$$

$$\phi^t(x) = \frac{1}{1 + tx_1}(x_1, x_2 + tx_3, x_3). \tag{\textbf{II}}$$

When Ker$A$ is 1-dimensional and $\hat{\alpha} \neq 0$, we can choose the normal coordinate system σ_p in which the flow has the following form:

$$\phi^t(x) = \frac{1}{1 + tx_1}(x_1, e^{-2t}x_2, e^{-t}x_3). \tag{\textbf{III}}$$

In addition, the metric g has the following matrix form at o under the canonical basis $\{\partial_i\}$ in σ_p:

$$(M_g)_o = \begin{bmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \epsilon = \pm 1.$$

When Ker$A$ is 1-dimensional and $\hat{\alpha} = 0$, we find that the flow has one of the following forms in the coordinate chart σ_p.

$$\phi^t(x) = \frac{1}{p(t,x)} \begin{pmatrix} x_1 + tx_2 + \dfrac{1}{2}t^2 x_3 \\ x_2 + tx_3 \\ x_3 \end{pmatrix}, \ p(t,x) = 1 + tx_1 + \frac{t^2 x_2}{2} + \frac{t^3 x_3}{6}. \tag{IV}$$

$$\phi^t(x) = \frac{1}{1 + tx_1} \begin{pmatrix} x_1 \\ x_2 \cos t - x_3 \sin t \\ x_2 \sin t + x_3 \cos t \end{pmatrix}. \tag{V}$$

$$\phi^t(x) = \frac{1}{1+tx_1}\begin{pmatrix} x_1 \\ e^t x_2 \\ e^{-t}x_3 \end{pmatrix}. \qquad \textbf{(VI)}$$

5.2.2 The case in which Ker$A$ is 2-dimensional

In this case, we first show the constant $\hat{\alpha}$ is zero. Suppose that $\hat{\alpha} \neq 0$. Without loss of generality, we can assume $\hat{\alpha} > 0$. Because K_o is nilpotent, we have the following at o by Equation (29):

$$\lim_{t\to+\infty}(g_t)_o = \lim_{t\to+\infty} e^{-4t\hat{\alpha}}(g(tK+Id)^{-1})_o = 0. \tag{30}$$

On the other hand, we have $g_t|_{\mathrm{Ker}A} = g|_{\mathrm{Ker}A}$ at o. Because g is non-zero on any 2-dimensional subspace of T_oU, this gives a contradiction. Hence in this case, we have $\hat{\alpha} = 0$ and $K_t = tK + Id$.

Next we deduce all the possible forms K_o. For a given basis of T_oU, denote M_g and M_K the matrix representation of g and K at o, respectively. Then we have

$$M_g(\hat{K}_t)_o = (e^{tA})^T M_g e^{tA}.$$

Differentiating with respect to t at $t = 0$, we obtains

$$-M_g M_K = A^T M_g + M_g A. \tag{31}$$

Define $B = (B_{ij}) = M_g A$, we obtain

$$-M_g M_K = B^T + B.$$

If M_K is the zero matrix, we have $A \in \mathfrak{so}(g)$. This contradicts the assumption Ker$A$ has dimension 2. Using the canonical forms of self-adjoint operators for Minkowski metrics given in Case 2 of Appendix A, we can choose some basis $\{e_i\}$ of T_oU so that g and K has one of the following matrix representations:

$$M_g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad M_K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \ \textbf{(a)}$$

$$M_g = \begin{bmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \epsilon = \pm 1, \ M_K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \ \textbf{(b)}$$

For case **(a)**, Under the basis $\{e_i\}$ we have

$$B^T + B = -M_g M_K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It follows that $(B_{ii}) = 0$ and $B_{12} + B_{21} = B_{13} + B_{31} = 0$. Using $B = M_g A$ has a 2-dimensional kernel, we have $B_{12} = 0$. It follows that either $B_{13} = B_{23} = 0$

or $B_{13} = B_{32} = 0$. Then under the basis $\{e_i\}$, we have A is in one of the following forms:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

For case **(b)**,similarly, we can choose under some basis $\{e_i\}$ so that

$$M_g = \begin{bmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \epsilon = \pm 1, \ A = \begin{bmatrix} 0 & 0 & -\frac{\epsilon}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that for both **(a)** and **(b)**, the matrix A has the Jordan form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

It follows from Equation (26) we can choose coordinate σ_p so that X has the following from in σ_p:

$$X_x = A'x + \langle w', x\rangle x, \quad A' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ w' = (w_1, w_2, 0) \neq 0.$$

Now we make the following change of coordinate to simplify computations in later sections. First suppose that $w_1 \neq 0$. Under the basis $\{-w', \partial_2, \partial_3\}$, we have

$$X_x = Ax + \langle w, x\rangle x, \quad A' = A, \ w = (-1, 0, 0). \tag{32}$$

Note that the above is a linear change of coordinates. Then the flow ϕ^t is in the following form:

$$\phi^t(x) = \frac{1}{1 + tx_2 + t^2x_3/2}(x_1, x_2 + tx_3, x_3). \tag{33}$$

For the case $w_1 = 0$, similarly, under the basis $\{\partial_1, -w', -w_2\partial_3\}$ we have

$$X_x = Ax + \langle w, x\rangle x, \quad A = A', \ w = (0, -1, 0). \tag{34}$$

In this case, the flow is given by

$$\phi^t(x) = \frac{1}{1 + tx_1}(x_1, x_2 + tx_3, x_3). \tag{35}$$

5.2.3 The case in which Ker$A$ is 1-dimensional and $\hat{\alpha} \neq 0$

By changing X to $-X$ if necessary, we assume $\hat{\alpha} > 0$. In this case, we have $(g_t)_o \to 0$ as $t \to +\infty$ by Equation (30). Because g is non-zero on any 2-

dimensional subspace of T_oU, the characteristic space for 0 of A is at most 1-dimensional. Furthermore, the matrix A has no eigenvalue with positive real-part because Ker$A \subset T_oU$ is light-like. Then A has one of the following real Jordan form:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \textbf{(1)}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & -d \\ 0 & d & b \end{bmatrix} \textbf{(2)}, \quad b \leq c < 0,\ d \neq 0.$$

Denote $\{f_i\}$ the basis of T_oU under which A has the Jordan form. First note that $f_1 \in$ Ker$A$ is null, so we have $g(f_1, f_1)(o) = 0$. Because K_o is nilpotent with $(K_o)^3 = 0$, according to Equation (29), for any $u, v \in T_oU$ we have

$$g_t(u,v) = g(\hat{K}_t u, v) = e^{-4t\hat{\alpha}} g((Id - tK + t^2K^2)u, v) \tag{36}$$

$$= e^{-4t\hat{\alpha}}(g(u,v) - tg(Ku,v) + t^2 g(K^2u,v)). \tag{37}$$

For fixed u, v, the expression in Equation (37) is a product of an exponential term with a polynomial. The exponential term $e^{-4t\hat{\alpha}}$ is universal at o. Then regardless of $u, v \in T_oU$ chosen, the term $g_t(u,v)$ decays exponentially with the same exponential rate $e^{-4t\hat{\alpha}}$

For Case **(1)**, we have $g(f_1, f_2)$ and $g(f_1, f_3)$ cannot both vanish. In addition, g is non-zero on space spanned by $\{f_2, f_3\}$. For any $1 \leq i, j \leq 3$, we can write $g_t(f_i, f_j)$ in terms of $g(f_i, f_j), b, c$. For example:

$$g_t(f_2, f_3) = e^{(b+c)t} g(f_2, f_3).$$

On the other hand, the exponential term $e^{-4t\hat{\alpha}}$ is universal, so the terms $g_t(f_i, f_j)$ shall have same exponential decay rate for all $1 \leq i, j \leq 3$. Under the basis $\{f_i\}$, the only possibility at o is the following:

$$b = 2c < 0, \ (g_{11})_o = (g_{22})_o = (g_{13})_o = (g_{23})_o = 0.$$

Similarly, we can show Case **(2)** is actually impossible. Hence by a scaling of the vector field X, we assume A has the Jordan form:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

By a linear change coordinate if necessary, in σ_p we have

$$X_x = Ax + \langle w, x \rangle x, \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ w = (-1, 0, 0). \tag{38}$$

$$(M_g)_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{39}$$

In this case, the flow is given by

$$\phi^t(x) = \frac{1}{1+tx_1}(x_1, e^{-2t}x_2, e^{-t}x_3). \tag{40}$$

5.2.4 When Ker$A$ is 1-dimensional and $\hat{\alpha} = 0$

In this case, we can also find some basis $\{e_i\}$ of T_oU so that g and K have one of the following matrix representations by Case 2 and Case 3 of Appendix A.

$$M_g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ M_K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{41}$$

$$M_g = \begin{bmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ M_K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ or \ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \epsilon = \pm 1. \tag{42}$$

In all the cases above, we have $M_g^2 = Id$. It follows that

$$-M_K = M_g A^T M_g + A.$$

By taking trace of both sides of the equation above, we get $tr(A) = 0$. Scaling X if necessary, the matrix A is in one of the following Jordan forms:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{43}$$

By taking a linear change of coordinate if necessary, we can assume in σ_p that X has the form $X_x = Ax + \langle w, x\rangle x$, with A in one of the forms in Equation (43) and $w = (-1, 0, 0)$. It follows that the flow ϕ^t is in one of the following forms in coordinate σ_p, depending on A.

$$\phi^t(x) = \frac{1}{p(t,x)} \begin{pmatrix} x_1 + tx_2 + \frac{1}{2}t^2x_3 \\ x_2 + tx_3 \\ x_3 \end{pmatrix}, \quad p(t,x) = 1 + tx_1 + \frac{t^2x_2}{2} + \frac{t^3x_3}{6}. \tag{44a}$$

$$\phi^t(x) = \frac{1}{1+tx_1} \begin{pmatrix} x_1 \\ x_2 \cos t - x_3 \sin t \\ x_2 \sin t + x_3 \cos t \end{pmatrix}. \tag{44b}$$

$$\phi^t(x) = \frac{1}{1+tx_1} \begin{pmatrix} x_1 \\ e^t x_2 \\ e^{-t} x_3 \end{pmatrix}. \tag{44c}$$

5.3 Finding the open set V where g is projectively flat

In this section, we obtain the open set V on which g is projectively flat for each case (**I-VI**) listed in Section 5.2.1.

5.3.1 The cases in which Ker$A$ is 2-dimensional

Case (**I**):

In the coordinate chart σ_p, the flow is in the following form given by Equation (**I**):

$$\phi^t(x) = \frac{1}{1 + tx_2 + t^2x_3/2}(x_1, x_2 + tx_3, x_3).$$

Define the function $p_1(t,x)$ by

$$p_1(t,x) = 1 + tx_2 + t^2x_3/2. \tag{45}$$

Define the subset $D' = \{x \in \mathbb{R}^3 : x_3 > 0\}$. Clearly for $x \in D'$, the polynomial $p_1(t,x)$ in t cannot have real roots of opposite signs. Then for $x \in D'$, Equation (33) is well defined for $t \geq 0$ or $t \leq 0$. We have $\phi^t(x) \to 0$ as $t \to +\infty$ or $-\infty$, provided it is well defined. Suppose that the vector field X is defined on the Euclidean ball $B_\delta(0)$ in coordinate σ_p for some $\delta > 0$. Define the subset

$$D = \{x \in B_\delta(0) : x_3 > 0, x_3 < \frac{1}{2}x_2^2\}.$$

Then for $x \in D$, the flow $\phi^t(x)$ is defined for either $t \geq 0$ or $t \leq 0$.

We want to show that on D, the projective Weyl curvature W vanishes. Firstly, we prove under the standard basis $\{\partial_i\}$, the tensor $W^l_{ijk} = 0$ if at least one of the covariant indices is 1. The differential of the flow is given by

$$D\phi^t(x) = \frac{1}{p_1^2(t,x)} \begin{bmatrix} 1 + tx_2 + x_3t^2/2 & -x_1t & -x_1t^2/2 \\ 0 & 1 - \frac{1}{2}t^2x_3 & t + x_2t^2/2 \\ 0 & -tx_3 & 1 + tx_2 \end{bmatrix}. \tag{46}$$

Now fix an arbitrary point $x \in D$, and define $p_1(t) = p(t, x)$. Suppose for some fixed j, k, we have $W(\partial_1, \partial_j, \partial_k)(x) = v = v_i(\partial_i)_x \neq 0$. This gives

$$D\phi^t(x)(v) = \frac{1}{p_1^2(t)}(q_1(t), q_2(t), q_3(t)).$$

Here each $q_i(t)$ is a polynomial in t. Then it is either constant zero or has finitely many zeros. Now suppose that $q_{i'}(t)$ is a non-zero polynomial for some $1 \leq i' \leq 3$. In this case, we have $q_{i'}(t)$ is a polynomial in t with degree $0 \leq d \leq 2$. On the other hand, we have

$$W_{\phi^t(x)}(D\phi^t_*\partial_1, D\phi^t_*\partial_j, D\phi^t_*\partial_k) = \frac{1}{p_1(t)} W_{\phi^t(x)}(\partial_1, D\phi^t_*\partial_j, D\phi^t_*\partial_k). \tag{47}$$

$$= \frac{1}{p_1^2(t)}(q_1(t), q_2(t), q_3(t)). \tag{48}$$

Now denote $A(t) := (D\phi^t)_x$. Using that $q_{i'}(t)$ is a non-zero polynomial, we obtain

$$\frac{q_{i'}(t)}{p_1(t)} = A_j^r(t) A_k^s(t) (W_{1rs}^{i'})_{\phi^t(x)}. \tag{49}$$

The functions $|(W_{1rs}^{i'})_{\phi^t(x)}|$ are uniformly bounded by some constant $C_2 > 0$ along the part of the integral curve $\phi^t(x)$ approaching the origin. Moreover, all coefficients in the matrix $A(t)$ are rational functions with the absolute values bounded above by C_1/t^2 for some $C_1 > 0$ for t large enough. Then, the right hand side of the Equation (49) has norm bounded above by C/t^4 for some $C > 0$ for large t. On the other hand, for t large enough, there exists $C' > 0$ so that $|\frac{q_{i'}(t)}{p_1(t)}| > \frac{C'}{t^2}$. This gives a contradiction. Note this argument above does not depend on the position of the lower index $i_0 = 1$.

Next, we show all components of the Weyl curvature vanish on D. By the argument above, we only need to show that W^l_{232} and W^l_{233} are zero on D. Since W is totally trace-free, the following sums vanish.

$$W^l_{jkl} = W^l_{jlk} = W^l_{ljk} = 0. \tag{50}$$

This gives the equations:

$$W^3_{232} + W^2_{222} + W^1_{212} = 0,\ W^1_{132} + W^2_{232} + W^3_{332} = 0.$$

Then we have $W^3_{232} = W^2_{232} = 0$. Similarly, using the equations:

$$W^l_{23l} = W^l_{l33} = 0.$$

We get $W^2_{233} = W^3_{233} = 0$. Then at a particular point $x \in D$ we have

$$D\phi^t_* W_x(\partial_2, \partial_3, \partial_2) = v_1 D\phi^t_*(\partial_1)_x.$$

It follows that

$$v_1/p_1(t) = (W^1_{ijk})_{\phi^t(x)} A^i_2(t) A^j_3(t) A^k_2(t).$$

Analogous to the argument after Equation (49), if the equation holds for all large t, we need to have $v_1 = 0$. This gives $W^1_{232} = 0$. Similarly, we can get $W^1_{233} = 0$. Then it is proved that $W = 0$ on D. If we set $V = D$, it follows that $W = 0$ on V and $0 \in \overline{V}$.

Case (**II**):

The flow ϕ^t has the following formula as in Equation (**II**):

$$\phi^t(x) = \frac{1}{1 + tx_1}(x_1, x_2 + tx_3, x_3).$$

The differential is given by

$$D\phi^t(x) = \frac{1}{(1 + tx_1)^2} \begin{bmatrix} 1 & 0 & 0 \\ -t(x_2 + tx_3) & 1 + tx_1 & t(1 + tx_1) \\ -tx_3 & 0 & 1 + tx_1 \end{bmatrix}.$$

Suppose the vector field X is defined on some open set U containing the origin. Define $C_r = \{x \in \mathbb{R}^3 : |x_3| < r|x_1|\}$ for $r \geq 0$. It follows that $\exists r_0 > 0$ such that on $U \cap C_{r_0}$, the flow is defined for either $t \geq 0$ or $t \leq 0$, depending on the sign of x_1, and stays in $U \cap C_{r_0}$. Note the origin is in the closure of $U \cap C_{r_0}$. Define the following vector fields on $U \cap C_{r_0}$:

$$u_1 = (1, \frac{x_1 x_2 - x_3}{x_1^2}, \frac{x_3}{x_1}), u_2 = \partial_2, u_3 = \partial_3.$$

Then under the basis $\{u_i\}$, the differential $D\phi^t_x$ for any $x \in U \cap C_{r_0}$ can be written in the following matrix form:

$$\begin{bmatrix} \frac{1}{(1+tx_1)^2} & 0 & 0 \\ 0 & \frac{1}{1+tx_1} & \frac{t}{1+tx_1} \\ 0 & 0 & \frac{1}{1+tx_1} \end{bmatrix}.$$

Note the matrix above has 2 distinct eigenfunctions: $\lambda_1 = \frac{1}{(1+tx_1)^2}$, and $\lambda_2 = \frac{1}{1+tx_1}$. Since W is ϕ^t-invariant, we have

$$D\phi^t_* W_x(u_i, u_j, u_k) = W_{\phi^t(x)}(D\phi^t_* u_i, D\phi^t_* u_j, D\phi^t_* u_k). \tag{51}$$

Denote W^l_{ijk} the components of W under the basis $\{u_i\}$. If $(W^l_{ijk})_x \neq 0$ for $x \in U \cap C_{r_0}$, at least one of the lower indices has to be 3 by an argument analogous to Case (**I**). In addition, let $(W^3_{233})_x = v_3$. It follows that

$$\frac{v_3}{1+tx_1} = \frac{t}{(1+tx_1)^3}(W^3_{232})_{\phi^t(x)} + \frac{1}{(1+tx_1)^3}(W^3_{233})_{\phi^t(x)}.$$

Then we have $v_3 = 0$. We obtain the following equation:

$$(W^2_{233})_x = \frac{t}{(1+tx_1)^2}(W^2_{232})_{\phi^t(x)} + \frac{1}{(1+tx_1)^2}(W^2_{233})_{\phi^t(x)}.$$

This gives $W^2_{233} = 0$ at x. Using the same method, we can show that $(W^l_{232})_x = 0$, for any $1 \leq l \leq 3$. It follows from the symmetries of W

that $W^1_{133} = 0$ at x, because $W^1_{133} = -W^2_{233} = 0$. By comparing the growth of both sides of Equation (51), we have for $x \in U \cap C_{r_0}$, if one of the lower indexes is 1, then $(W^l_{ijk})_x = 0$. Assume that at x, $W^1_{233} = v_1$. Then we have

$$\frac{v_1}{(1+tx_1)^2} = \frac{t}{(1+tx_1)^3}(W^1_{232})_{\phi^t(x)} + \frac{1}{(1+tx_1)^3}(W^1_{233})_{\phi^t(x)}.$$

Since $(W^1_{232})_{\phi^t(x)} = 0$ by above, it follows that $v_1 = 0$. Hence all components of the Weyl curvature shall vanish on $U \cap C_{r_0}$. Then ∇ is projectively flat on $V = U \cap C_{r_0}$.

5.3.2 The cases in which Ker$A$ is 1-dimensional and $\hat{\alpha} \neq 0$

Case (**III**):

Without loss of generality, we assume $\hat{\alpha} > 0$. In the coordinate chart σ_p, the flow and the metric have the following forms as in Section 5.2.1:

$$\phi^t(x) = \frac{1}{1+tx_1}\begin{pmatrix} x_1 \\ e^{-2t}x_2 \\ e^{-t}x_3 \end{pmatrix}, \quad (M_g)_0 = \begin{bmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \epsilon = \pm 1.$$

The differential of the flow is given by

$$D\phi^t(x) = \frac{1}{(1+tx_1)^2}\begin{bmatrix} 1 & 0 & 0 \\ -te^{-2t}x_2 & (1+tx_1)e^{-2t} & 0 \\ -te^{-t}x_3 & 0 & (1+tx_1)e^{-t} \end{bmatrix}.$$

Choose some $\delta > 0$ so that $(M_g)_{12} \neq 0$ on $B_\delta(0)$. We can assume if $x \in B_\delta(0)$ with $x_1 > 0$, the flow $\phi^t(x)$ is well defined for $t \geq 0$. Moreover, $\phi^t(x) \to 0$ as $t \to +\infty$.

First we prove $W^l_{ijk} = 0$ if $x \in B_\delta(0)$ with $x_1 > 0$, and one of the lower indices is 2. Set $i = 2$. Fix such an x, and suppose $v = W_x(\partial_2, \partial_j, \partial_k) \neq 0$. Define $\tilde{p}(t) = (1 + tx_1)^2$, and $A(t) = D\phi^t(x)$. Now define the functions $q_{i'}(t)$ for each $1 \leq i' \leq 3$ using the following equation:

$$D\phi^t_* v = \frac{e^{-2t}}{\tilde{p}(t)}(q_1(t), q_2(t), q_3(t)).$$

One of the $q_{i'}(t)$ for some $1 \leq i' \leq 3$ is not identically zero. In addition, we have $|q_{i'}(t)| > C > 0$ for large $t > 0$. Similar to the Case (**I**), we have

$$q_{i'}(t) = (1 + tx_1)(W^{i'}_{2sr})_{\phi^t(x)} A^r_j(t) A^s_k(t).$$

The right hand side of the equation above approaches 0 as $t \to +\infty$. This gives a contradiction. Similarly, we can prove that if $j = 2$ or $k = 2$, we have $W^l_{ijk} = 0$ at x.

Next, by the argument above and (50), we can show on $B^+ = \{x \in B_\delta(0) : x_1 > 0\}$ the components of W of the form W^i_{jkl} are zero, where $i, j, k, l \in \{1, 3\}$. Then on V, the non-zero components of W can only be: $W^2_{133}, W^2_{313}, W^2_{131}, W^2_{311}$. Suppose that $(W^2_{133})_x = v_2 \neq 0$. By using that $\phi^t_* W = W$ and $W^1_{133} = 0$, we obtain

$$\frac{(1+tx_1)v_2e^{-2t}}{\tilde{p}(t)} = \frac{e^{-2t}}{\tilde{p}^2(t)}(W^2_{133})_{\phi^t(x)}. \tag{52}$$

This gives $v_2 = \dfrac{1}{(1+tx_1)^3}(W^2_{133})_{\phi^t(x)} \neq 0$. However, the right hand side of the equation tends to 0 as $t \to +\infty$. This leads to a contradiction. Similarly, we have $W^2_{313} = 0$. Hence the only possible non-zero components of W are W^2_{131} and W^2_{311}, with $W^2_{131} + W^2_{311} = 0$. In addition, at the origin o, we can show $(W^2_{131})_o = (W^2_{311})_o = 0$, using a calculation similar to Equation (52). Hence all components of W vanishes at o.

For the 3-dimensional case, denote $P_{ij} = \dfrac{1}{2}(Ric)_{ij} = R^k_{ikj}$ and $W^2_{131} = f$. Note that $P_{ij} = P_{ji}$. We show the Weyl tensor is not metrizable near the origin unless $f = 0$. By a classical result of Weyl (See Page 101 of [22]), we have the following decomposition of the curvature tensors of Levi-Civita connections:

$$R^l_{ijk} = W^l_{ijk} + \delta^l_i P_{kj} - \delta^l_j P_{ki}. \tag{53}$$

We may write the following curvature endomorphisms in the matrix forms:

$$R(\partial_1, \partial_2) = \begin{bmatrix} P_{12} & P_{22} & P_{32} \\ -P_{11} & -P_{21} & -P_{31} \\ 0 & 0 & 0 \end{bmatrix},$$

$$R(\partial_2, \partial_3) = \begin{bmatrix} 0 & 0 & 0 \\ P_{13} & P_{23} & P_{33} \\ -P_{12} & -P_{22} & -P_{32} \end{bmatrix},$$

$$R(\partial_1, \partial_3) = \begin{bmatrix} P_{13} & P_{23} & P_{33} \\ f & 0 & 0 \\ -P_{11} & -P_{21} & -P_{31} \end{bmatrix}.$$

Since $\forall T \in \mathfrak{so}(g)$, the matrix $M_g T$ is skew symmetric. This leads to the following equation:

$$(M_g)_{1i} R^i_{121} = (M_g)_{1i} R^i_{231} = (M_g)_{1i} R^i_{131} = 0.$$

It follows the following matrix consisting of the first columns from the curvature endomorphisms $R(\partial_1, \partial_2), R(\partial_2, \partial_3), R(\partial_1, \partial_3)$ has zero determinant.

$$\begin{bmatrix} P_{12} & 0 & P_{13} \\ -P_{11} & P_{13} & f \\ 0 & -P_{12} & -P_{11} \end{bmatrix}.$$

This gives $(P_{12})^2 f = 0$. Suppose at some point $x \in B^+$, we have $f(x) \neq 0$. It follows that $P_{12}(x) = 0$. Similarly, we also have the following equation:

$$(M_g)_{2i} R^i_{122} = (M_g)_{2i} R^i_{232} = (M_g)_{2i} R^i_{132} = 0.$$

This equation gives

$$(M_g)_{21}(x)P_{22}(x) = (M_g)_{22}(x)P_{23}(x)+(M_g)_{23}(x)P_{22}(x) = (M_g)_{21}(x)P_{23}(x) = 0.$$

Since $(M_g)_{12}(x) \neq 0$ by assumption, it follows that $P_{22} = P_{23} = 0$. Because $R(e_i, e_j) \in \mathfrak{so}(g)$, we have $|Ker(R(\partial_i, \partial_j))| = 1$ or $R(\partial_i, \partial_j) = 0$. Note that the matrix $R(\partial_1, \partial_2)$ has a kernel of dimension at least 2. This implies $P_{11} = P_{31} = 0$. Similarly, we obtain $P_{33} = 0$ by using the matrix $R(\partial_2, \partial_3)$. It follows that at x, we have $P_{ij}(x) = 0$. This implies $f = 0$, otherwise the matrix $R(\partial_1, \partial_3)$ has a precisely 2-dimensional kernel which leads to a contradiction.

From the above, we conclude that $W = 0$ on $V = B^+$.

5.3.3 The case in which Ker$A$ is 1-dimensional and $\hat{\alpha} = 0$

Case (**IV**):

In this case, the flow is given by Equation (**IV**) as follows.

$$\phi^t(x) = \frac{1}{p(t,x)} \begin{pmatrix} x_1 + tx_2 + \frac{1}{2}t^2 x_3 \\ x_2 + tx_3 \\ x_3 \end{pmatrix}, \ p(t,x) = 1 + tx_1 + \frac{t^2 x_2}{2} + \frac{t^3 x_3}{6}.$$

Write p for $p(t,x)$ for simplicity. The differential of the flow is the following.

$$D\phi^t(x) = \frac{1}{p^2}\begin{bmatrix} p - tx_1 - t^2x_2 - \frac{1}{2}t^3x_3 & -\frac{1}{2}t^2x_1 + t(p - \frac{1}{2}t^2x_2) - \frac{1}{4}t^4x_3 & -\frac{1}{6}t^3(x_1 + tx_2) + \frac{1}{2}t^2(p - \frac{1}{6}t^3x_3) \\ -t(x_2 + tx_3) & p - \frac{1}{2}t^2x_2 - \frac{1}{2}t^3x_3 & -\frac{1}{6}t^3x_2 + t(p - \frac{1}{6}t^3x_3) \\ -tx_3 & -\frac{1}{2}t^2x_3 & p - \frac{1}{6}t^3x_3 \end{bmatrix}.$$

Let E be the following cone containing the x_3-axis:

$$E = \{x \in \mathbb{R}^3 : |x_1| \leq |x_3| \text{ and } |x_2| \leq |x_3|\}.$$

For $\delta > 0$, define the sets

$$B_\delta = \{x \in \mathbb{R}^3 : |x_i| < \delta,\ 1 \leq i \leq 3\}.$$

Then $\exists \delta' > 0$ such that $B = B_{\delta'}$ satisfies: $\forall x \in B \cap E$, $\phi^t(x)$ is defined and stays in U for $t \geq 0$ whenever $x_3 \geq 0$, and for $t \leq 0$ whenever $x_3 \leq 0$. Define $V = Int(E) \cap B$.

We show that on $V^+ = \{x \in V : x_3 > 0\}$, the Weyl curvature vanishes. Fix any $x \in V^+$ and set the matrix representation $A(t) = D\phi^t(x)$ under the canonical under $\{\partial_i\}$. We see that $p(t,x)$ is a polynomial in t with degree 3. Also, all the components in the matrix $\tilde{A}(t) = p^2A(t)$ are polynomials in t with degree at most 4. Suppose that $W_x(\partial_i, \partial_j, \partial_k) = v$ is non-zero. We have $dx_2(A(t)(v)) = \dfrac{1}{p^2}q(t,x)$, where $q(t,x)$ is a non-zero polynomial in t. We prove for $x \in V^+$ and $v \neq 0$, the function $q(t,x)$ is a polynomial in t with degree at least 1. By expanding the polynomial $q(t,x)$, we have for $v \neq 0$, the polynomial $q(t,x)$ has degree zero only if the following matrix has zero determinant.

$$\begin{bmatrix} -x_2 & x_1 & 1 \\ -x_3 & 0 & x_1 \\ 0 & -\dfrac{x_3}{3} & \dfrac{x_2}{3} \end{bmatrix}.$$

The matrix above has determinant $\dfrac{x_3^2}{3}$, so $q(t,x)$ is non-constant polynomial in t.

The fact that $\phi_*^t W = W$ gives the following equation:

$$\frac{q(t,x)}{p^2} = (W^2_{rsl})_{\phi^t(x)} A_i^r(t) A_j^s(t) A_k^l(t). \tag{54}$$

For any fixed $x \in V^+$, all terms $|W^m_{rsl}|$ are uniformly bounded by some constant C along $\phi^t(x)$ for $t \geq 0$. Each component in $A(t)$ is a rational function with degree at most -2. Then Equation (54) cannot hold by comparing the degrees of both sides, which is a contradiction. Similarly, we can show that on $V^- = \{x \in D : x_3 < 0\}$, the Weyl curvature also vanishes.

Case (**V**):

In this case, the flow ϕ^t has the following form as in Equation ($\mathbf{V}$).

$$\phi^t(x) = \frac{1}{1+tx_1}\begin{pmatrix} x_1 \\ x_2\cos t - x_3\sin t \\ x_2\sin t + x_3\cos t \end{pmatrix}.$$

One can choose a smaller open set $V \subset U$ such that $\forall x \in V$, $\phi^t(x)$ is defined for $t \geq 0$ or $t \leq 0$, for $x_1 \geq 0$ or $x_1 \leq 0$, respectively. Moreover, for $x \in V$ with $x_1 > 0$, we have $\phi^t(x) \to 0$ as $t \to +\infty$. Also, for $x \in V$ with $x_1 < 0$, we have $\phi^t(x) \to 0$ as $t \to -\infty$.

The differential of the flow is given by

$$D\phi^t(x) = \begin{bmatrix} \dfrac{1}{(1+tx_1)^2} & 0 & 0 \\ \dfrac{-t(x_2\cos t - x_3\sin t)}{(1+tx_1)^2} & \dfrac{\cos t}{1+tx_1} & \dfrac{-\sin t}{1+tx_1} \\ \dfrac{-t(x_2\sin t + x_3\cos t)}{(1+tx_1)^2} & \dfrac{\sin t}{1+tx_1} & \dfrac{\cos t}{1+tx_1} \end{bmatrix}. \tag{55}$$

Define the open set $V^+ = \{x \in V : x_1 > 0\}$. First we prove if $x \in V^+$, then $W^1_{ijk} = 0$. Suppose for some $x \in V^+$, we have $W_x(\partial_i, \partial_j, \partial_k) = v = (v_1, v_2, v_3)$ with $v_1 \neq 0$ for some i, j, k. As for previous cases, write $A(t) = (D\phi^t)_x$. It follows that

$$\frac{v_1}{(1+tx_1)^2} = (W^1_{rsl})_{\phi^t(x)} A^r_i(t) A^s_j(t) A^l_k(t). \tag{56}$$

By a simple observation, we have $\exists C > 0$ such that $|A_n^m(t)| < C/t$ for $t > 0$ large enough. Because all components of the Weyl tensor are bounded along the integral curve $\phi^t(x)$ for $t \geq 0$, we obtain a contradiction similar to the first case (**I**).

Next, we prove all other components of W vanish on V^+. Fix any $x \in V^+$, and let $W_x(\partial_i, \partial_j, \partial_k) = v \neq 0$ for some i, j, k. For $t \geq 0$, we have that $\|D\phi_*^t(v)\| = \dfrac{\|v\|}{1 + tx_1} > 0$, since $v_1 = 0$ by above. Here the norms is induced by the standard Euclidean metric defined on this geodesic normal coordinate. On the other hand, we have

$$W_{\phi^t(x)}(D\phi_*^t\partial_i, D\phi_*^t\partial_j, D\phi_*^t\partial_k) = A_i^r(t)A_j^s(t)A_k^l(t)(W_{rsl}^2\partial_2 + W_{rsl}^3\partial_3)_{\phi^t(x)}. \tag{57}$$

Then the norm of right hand side of Equation (57) is bounded by C'/t^3 for large t with a constant $C' > 0$. But the quantity give by Equation (57) shall have norm $\dfrac{\|v\|}{1 + tx_1}$, where $\|v\| > 0$. This gives a contradiction. Thus $W = 0$ on V^+. Define $V^- = \{x \in V : x_1 < 0\}$. Analogously, we can show that W also vanishes on V^-. It follows that $W = 0$ on V.

Case (**VI**):

For the last case, in the coordinate chart σ_p, the flow is in the following form as in Equation (**VI**).

$$\phi^t(x) = \frac{1}{1+tx_1}(x_1, e^t x_2, e^{-t} x_3).$$

The differential of the flow is the following:

$$D\phi^t(x) = \begin{bmatrix} \dfrac{1}{(1+tx_1)^2} & 0 & 0 \\ \dfrac{-te^t x_2}{(1+tx_1)^2} & \dfrac{e^t}{1+tx_1} & 0 \\ \dfrac{-te^{-t} x_3}{(1+tx_1)^2} & 0 & \dfrac{e^{-t}}{1+tx_1} \end{bmatrix}. \tag{58}$$

Denote g_{ij} the matrix representation of g under the basis $\{\partial_i\}$ in σ_p. Note that Equation $(36, 37)$ also hold when $\hat{\alpha} = 0$, since $(K)_o$ is nilpotent. Then for any $u, v \in T_o U$, the following equality holds:

$$g_t(u, v) = g(u, v) - tg(Ku, v) + t^2 g(K^2 u, v).$$

The right hand side of this equation is a polynomial in t. On the other hand, the differential of the flow gives

$$g_t(\partial_2, \partial_2)(o) = e^{2t}(g_{22})_o.$$

Then $g_t(\partial_2, \partial_2)(o)$ has exponential growth with respect to t. This implies $(g_{22})_o = 0$. Using the same method, we conclude that

$$(g_{22})_o = (g_{33})_o = (g_{12})_o = (g_{13})_o = 0, \ (g_{11})_o \neq 0. \tag{59}$$

Hence by a linear change of coordinate and a scaling on the metric g, we can assume in σ_p:

$$(g_{ij})_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{60}$$

Equation (58) and (60) imply that $(D\phi^t)_o$ is 1-parameter subgroup of $SO(g)_o$ in σ_p. Thus K_o is in fact the zero matrix

We first show that the Weyl curvature vanishes at the origin o of σ_p. We exam the values of W^l_{ijk} on the x_1-axis. On the x_1-axis, the differential $D\phi^t$ becomes a diagonal matrix under the basis $\{\partial_i\}$. Pick $x = (x_1, 0, 0)$ with $x_1 \neq 0$, and suppose that $W_x(\partial_i, \partial_j, \partial_k) = v = (v_1, v_2, v_3)$. By $\phi^t_* W = W$, we have

$$(W_{\phi^t(x)})^l_{ijk} = \frac{e^{mt} v_l}{(1 + tx_1)^n}, \quad m, n \in \mathbb{Z}. \tag{61}$$

Depending on the sign of x_1, we have $\phi^t(x) \to o$ as $t \to +\infty$ or $-\infty$. As $\phi^t(x) \to o$, the right hand side of Equation (61) tends to 0 or ∞, unless $m = n = 0$. Observe that it is impossible to have $n = 0$. If right hand side of Equation (61) approaches ∞, this implies W blows up at the origin, which is impossible. On the other hand, if this value approaches zero, this implies $(W_o)^l_{ijk} = 0$. Hence W vanishes at the origin.

Next we show that Ric is a multiple of g at the origin o. Because $W^l_{ijk}(o) = 0$, at the origin, Equation (53) gives

$$R^l_{ijk}(o) = \delta^l_j P_{ki}(o) - \delta^l_i P_{kj}(o).$$

Similar to Case (III), we can write the components of $R(e_i, e_j)$ at o in terms of the components of the Schouten tensor P, for example:

$$R(e_1, e_2)(o) = \begin{bmatrix} P_{12} & P_{22} & P_{32} \\ -P_{11} & -P_{21} & -P_{31} \\ 0 & 0 & 0 \end{bmatrix}.$$

As $R(e_i, e_j) \in \mathfrak{so}(g)$, we know the matrices $gR(e_i, e_j)$ shall all be skew symmetric. Then by using Equation (60), we have $P_{11} = P_{23} = P_{32}$, and all other P_{ij} are 0 at o. This shows $(Ric_{ij})_o$ is a multiple of $(g_{ij})_o$.

Now we study the behavior of the eigenfunctions of K along the x_1-axis of σ_p.

Because in this case $\hat{\alpha} = 0$, it follows from Equation (28) that $K_t = tK + Id$. Define the function $\psi^t = -\frac{1}{2}\log(\det K_t)$ as in Section 2.1 of [3]. Let ∇ be the Levi-Civita connection for g as before. Denote Ric^t and Ric the Ricci curvatures of g_t and g, respectively. We have the following from Equation (3)-(5) of [20]:

$$\nabla\nabla\psi^t - \nabla\psi^t \otimes \nabla\psi^t = \frac{1}{n-1}(Ric^t - Ric). \tag{62}$$

We have $(Ric_{ij})_o = c(g_{ij})_o$ for some constant c. Then $(D\phi^t_o) \subset SO(g)_o$ gives

$$(Ric^t)_o = (\phi^t_* Ric)_o = \phi^t_*(cg_o) = cg_o = (Ric)_o.$$

Let $\gamma(s)$ be a geodesic for g with $\gamma(0) = o$. We obtain the following equation from Equation (62):

$$\frac{d^2\psi^t}{ds^2}(0) - \left(\frac{d\psi^t}{ds}(0)\right)^2 = 0. \tag{63}$$

Denote λ_i for $1 \leq i \leq 3$ the eigenfunctions of K defined on some neighborhood of the origin. We have the following:

$$\psi^t = -\frac{1}{2}\log(\det(tK + Id)) = -\frac{1}{2}\log(\prod_i (t\lambda_i + 1)) = -\frac{1}{2}\sum_i \log(t\lambda_i + 1). \tag{64}$$

For x on the positive x_1-axis, the curve $\phi^t(x)$ is well defined for $t \geq 0$, and $\phi^t(x) \to 0$ as $t \to +\infty$. By Equation (27) and $\hat{\alpha} = 0$, the orbit of K under the action of the 1-parameter subgroup L_t of $GL(B(U, g))$ satisfies the following:

$$L_t(K) = K.$$

According to Corollary 3.0.1, for any $t_0 > 0$, there is some neighborhood $V_{t_0} \subset U$ of o and an interval $I = [-t_0, t_0]$ so that

$$L_t(K)|_{V_{t_0}} = \phi_*^t K \circ K_t = \phi_*^t K \circ (tK + Id),\ t \in I.$$

The positive x_1-axis is contracted by the flow ϕ^t as $t \to +\infty$. On the positive x_1-axis, this gives

$$\phi_*^t K = K(tK + Id)^{-1},\ t \geq 0. \tag{65}$$

Then for $t \geq 0$, the parametrized Möbius map $T^t(z) = \dfrac{z}{tz + 1}$ takes the eigenvalues of K_x to eigenvalues of $K_{\phi^t(x)}$ while preserving the forms of Jordan blocks. It follows that λ_i can be chosen smoothly on the positive x_1-axis with $\lambda_i(\phi^t(x)) = \dfrac{\lambda_i(x)}{t\lambda_i(x) + 1}$, for $t \geq 0$. Taking the derivative with respect to t at $t = 0$, this gives

$$L_X \lambda_i = -\lambda_i^2. \tag{66}$$

In the coordinate chart σ_p, the projective vector field X is in the following form:

$$X_x = -x_1^2\partial_1 + (x_2 - x_1x_2)\partial_2 - (x_3 + x_1x_3)\partial_3.$$

Then in σ_p, for $x = (x_1, 0, 0)$ with small $x_1 > 0$, we have

$$-x_1^2\partial_1\lambda_i(x) = -\lambda_i^2.$$

For $x = (x_1, 0, 0)$ with $x_1 > 0$, we have

$$\lambda_i(x) = \frac{x_1}{c_ix_1 + 1}, \text{ or } \lambda_\mathrm{i} \equiv 0. \tag{67}$$

We want to combine Equations (63) and (64) to get the information of λ_is on the x_1-axis. However, the eigenfunctions λ_is may not be ∂_1-differentiable at o. The eigenfunctions of K can be chosen smoothly on the negative x_1-axis, but the left and right derivatives of a given λ_i on the x_1-axis may not agree at o. To work around this difficulty, we extend the functions λ_is to $\hat{\lambda}_i$s smoothly defined on some interval on the x_1-axis containing o using (67). Note ψ^t is always a smooth function. Then we may define equations analogous to (63) and (64) to study the values of $\hat{\lambda}_i$s on the x_1-axis near o instead. This allows us to examine the values of λ_is on the positive x_1-axis near o.

The functions in the form of Equation (67) are actually smooth on an open interval $\hat{I}$ of the x_1-axis containing 0. We can define the functions $\hat{\lambda}_i$ on $\hat{I}$

for $1 \leq i \leq 3$ so that the following equations hold.

$$\hat{\lambda}_i(x) = \frac{x_1}{c_i x_1 + 1}, \text{ or } \hat{\lambda}_{\mathrm{i}}(\mathrm{x}) \equiv 0. \tag{68}$$

$$\hat{\lambda}_i(x) = \lambda_i(x), \ \textit{if}\ x \in \hat{I}, x_1 > 0. \tag{69}$$

In other words, the function $\hat{\lambda}_i$ is the unique extension of λ_i on $\hat{I}$ by formulas in (67). Then for any t, there is some interval $\hat{I}_t$ on the x_1-axis containing o so that the following function $\hat{\psi}^t$ is well defined.

$$\hat{\psi}^t(x) = -\frac{1}{2}\sum_{i=1}^{3} \log(t\hat{\lambda}_i(x) + 1). \tag{70}$$

We have $\psi^t(x) = \hat{\psi}^t(x)$ for x on the positive x_1-axis.

To simplify the calculation using (63), let σ be a normal coordinate of ∇ at o having the same 1-jet as σ_p at o. Denote by $\overline{\nabla}$ the connection induced by the local section $\exp_p(\mathfrak{g}_{-1})$ as on Page 16. Remember that σ_p is a normal coordinate of $\overline{\nabla}$ at o. By Equation (63), in the coordinate chart σ we have

$$\partial_1^2 \psi^t(o) - \left(\partial_1 \psi^t(o)\right)^2 = 0, \ t \geq 0. \tag{71}$$

Because ∇ and $\overline{\nabla}$ are projectively equivalent, then σ and σ_p have the same positive x_1-axis with possibly different parametrizations. It follows that in the coordinate chart σ:

$$\partial_1^m \psi^t(o) = \partial_1^m \hat{\psi}^t(o), \; m \in \mathbb{N}. \tag{72}$$

It follows from (71) and (72) that the following holds for $t \geq 0$ in the coordinate chart σ.

$$\partial_1^2 \hat{\psi}^t(o) - \Big(\partial_1 \hat{\psi}^t(o)\Big)^2 = 0.$$

For simplicity, denote by $\partial_1 \hat{\lambda}_i = \hat{\lambda}'_i$ and $\partial_1^2 \hat{\lambda}_i = \hat{\lambda}''_i$ in the coordinate chart σ. Because each $\hat{\lambda}_i$ is smooth on $\hat{I}$, substituting Equation (70) into the equation above, we get the following for $t \geq 0$.

$$-\frac{1}{2}\sum_{i=1}^{3} \frac{t\hat{\lambda}''_i(t\hat{\lambda}_i + 1) - (t\hat{\lambda}'_i)^2}{(t\hat{\lambda}_i + 1)^2}(o) - \frac{1}{4}(\sum_{i=1}^{3} \frac{t\hat{\lambda}'_i}{t\hat{\lambda}_i + 1})^2(o) = 0. \tag{73}$$

Suppose K has exactly k non-identically vanishing eigenfunctions on the positive x_1-axis, counting multiplicity. Because σ and σ_p have the same 1-jet at o, if $\hat{\lambda}|_{\hat{I}} = \dfrac{x_1}{c_i x_1 + 1}$ in σ_p, we have $\hat{\lambda}'_i(o) = \partial x_1(\dfrac{x_1}{c_i x_1 + 1})(o) = 1$ by (68, 69). Also, it is clear $\hat{\lambda}_i(o) = 0$ for all i. Substituting these into Equation (73), it is reduced to the following

$$(\frac{k}{2} - \frac{k^2}{4})t^2 - \frac{1}{2}\sum_{i=1}^{3} \hat{\lambda}''_i t = 0.$$

Because all coefficients of the left hand side of the equation above vanish, we have $k = 0$ or 2. Hence K has either exactly 0 or 2 non-zero eigenfunctions on the positive x_1-axis, counting multiplicity.

We prove K has exactly two non-zero eigenfunctions on the positive x_1-axis. If K is nilpotent on the positive x_1-axis, then ψ^t is always constant on the positive x_1-axis. Thus in the coordinate chart σ, the curve $\gamma(s) = (s, 0, 0)$ with $s > 0$ is a parametrized geodesic segment for g_t for any $t \in \mathbb{R}$. On the other hand, $\phi^{-t} \circ \gamma(s)$ is a geodesic for g_t with the same initial vector. Then we have $\phi^{-t} \circ \gamma(s) = \gamma(s)$ for $s \geq 0$. This is clearly impossible by (**VI**). Hence K has exactly 2 non-zero eigenfunctions on the positive x_1-axis.

In the coordinate chart σ_p, fix a point $x = (x_1, 0, 0)$ with $x_1 > 0$. From now on, denote $u' = \partial_1(x), u = \partial_3(x)$ in σ_p. Then $K(x)$ has one of the following real Jordan forms:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix} \quad \textbf{(VI-a)}, \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & z & 1 \\ 0 & 0 & z \end{bmatrix} \quad \textbf{(VI-b)},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & d \end{bmatrix} \quad \textbf{(VI-c)}, \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix} \quad \textbf{(VI-d)}.$$

$$bdz \neq 0, \quad d - b \neq 0.$$

Next, we show that for all cases listed above, a contradiction can always be derived.

Case **(VI-a)**:

In this case, $K(x)$ has a pair of complex conjugate eigenvalues. Then we have $\lambda_1(x) = 0, \lambda_2(x) = a + bi, \lambda_3(x) = a - bi$ with $b \neq 0$. According to the normal forms of self-adjoint operators of 3-dimensional Minkowski metrics (all possible cases of Appendix A), we can choose a basis $\{\hat{e}_i\}$ of T_xU such that $K(x)$ and $g(x)$ have the following matrix representations:

$$M'_K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix}, \ M'_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{74}$$

Define the polynomial $\hat{p}(t) = (a^2 + b^2)t^2 + 2at + 1$, which is the determinant of $K_t(x) = (tK + Id)(x)$. Under the basis $\{\hat{e}_i\}$, we have the following matrix representation for $(tK + Id)^{-1}(x)$:

$$M'_{(tK+Id)^{-1}} = \frac{1}{\hat{p}(t)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & ta+1 & -bt \\ 0 & -bt & ta+1 \end{bmatrix}.$$

Remember that $(g_t)_{ij} = g_{ik} \left(\frac{K_t^{-1}}{\det(K_t)} \right)_j^k$. Under the basis $\{\hat{e}_i\}$, the metric $g_t(x)$ has the matrix representation:

$$M'_{g_t} = \frac{1}{\hat{p}^2(t)} \begin{bmatrix} \hat{p}(t) & 0 & 0 \\ 0 & ta+1 & bt \\ 0 & bt & -(ta+1) \end{bmatrix}. \tag{75}$$

In the coordinate chart σ_p, denote g_{ij} the components of g under the frame $\{\partial_i\}$. We have

$$g_t(u,u) = \phi_*^t g(u,u) = e^{-2t}(1+tx_1)^{-2} g_{33}(\phi^t(x)). \tag{76}$$

For $x = (x_1, 0, 0)$ and $x_1 > 0$, we have $\phi^t(x) \to o$ as $t \to +\infty$. Then $g_{33}(\phi^t(x))$ is uniformly bounded for $t \geq 0$. Then the right hand side of Equation (76) has exponential decay. On the other hand, under the basis $\{\hat{e}_i\}$, let $u = (r_1, r_2, r_3)$. We obtain the following using Equation (75):

$$g_t(u,u) = \frac{1}{\hat{p}^2(t)} \left(r_1^2 \hat{p}(t) + a(r_2^2 - r_3^2)t + 2r_2 r_3 bt + r_2^2 - r_3^2 \right). \tag{77}$$

Note that right hand side of Equation (77) is a rational function. This implies both (76) and (77) shall vanish identically for $t \geq 0$. Because (77) is identically zero, we have

$$r_1 = 0,\ 2r_2r_3b = 0,\ r_2^2 - r_3^2 = 0.$$

Then we have $r_i = 0$ for all i, since $b \neq 0$. We get $u = 0$, which is impossible.

Case **(VI-b)**:

In this case, we have $\lambda_1(x) = 0$, $\lambda_2(x) = \lambda_3(x) = z \neq 0$. Similar to Case (**VI-a**), we can find a basis $\{\hat{e}_i\}$ of $T_x M$ such that $K(x)$ and $g(x)$ have the following matrix representations:

$$M'_K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & z & 1 \\ 0 & 0 & z \end{bmatrix}, \; M'_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{bmatrix}, \; \epsilon = \pm 1. \tag{78}$$

This gives the following

$$\det(tK + Id)(x) = (tz + 1)^2.$$

Under the basis $\{\hat{e}_i\}$, we have the following matrix representation of $(tK + Id)^{-1}(x)$:

$$M'_{(tK+Id)^{-1}} = \frac{1}{(tz+1)^2} \begin{bmatrix} (tz+1)^2 & 0 & 0 \\ 0 & tz+1 & -t \\ 0 & 0 & tz+1 \end{bmatrix}.$$

Then $g_t(x)$ has the following matrix representation:

$$M'_{g_t} = \frac{1}{(1+tz)^4}\begin{bmatrix} (1+tz)^2 & 0 & 0 \\ 0 & 0 & \epsilon(1+tz) \\ 0 & \epsilon(1+tz) & -\epsilon t \end{bmatrix}. \tag{79}$$

Under the basis $\{\hat{e_i}\}$, let $u' = (r'_1, r'_2, r'_3)$ and $u = (r_1, r_2, r_3)$. Then Equation (79) gives

$$g_t(u,u) = \frac{1}{(1+tz)^4}(r_1^2(1+tz)^2 + 2\epsilon r_2 r_3(1+tz) - \epsilon r_3^2 t). \tag{80}$$

On the other hand, the following calculation analogous to (76) gives

$$g_t(u,u) = \phi_*^t g(u,u) = e^{-2t}(1+tx_1)^{-2} g_{33}(\phi^t(x)). \tag{81}$$

This equation has exponential decay. It follows that $g_t(u,u) = 0$ for $t \geq 0$. Similarly, we can show $g_t(u',u) \equiv 0$ for $t \geq 0$. Using Equation (80) and $g_t(u,u) = 0$ for $t \geq 0$, we have

$$r_1 = r_3 = 0, \;\; r_2 \neq 0.$$

Then $g_t(u',u) = 0$ gives $r'_3 = 0$. It follows that

$$g_t(u',u') = \frac{(r'_1)^2}{(1+tz)^2}. \tag{82}$$

On the other hand, denote g_{ij} the components of g for the canonical frame $\{\partial_i\}$ of σ_p. We have

$$g_t(u', u') = \frac{g_{11}(\phi^t(x))}{(1+tx_1)^4}. \tag{83}$$

Because $g_{11}(\phi^t(x))$ is uniformly bounded for $t \geq 0$, we get $r'_1 = 0$. This implies u' is a light-like vector for g. But the x_1-axis of σ_p is a space-like geodesic for g. This leads to a contradiction.

Case **(VI-c)**:

For this case, we can choose $\{\hat{e}_i\}$ so that $g(x)$ and $K(x)$ have the following matrix forms:

$$M'_K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & d \end{bmatrix}, \; b - d \neq 0. \tag{84}$$

$$M'_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \textbf{(i)}, \; or \; \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \textbf{(ii)} \tag{85}$$

Then under the basis $\{\hat{e}_i\}$, we have the following matrix representation of $(tK + Id)^{-1}(x)$:

$$M'_{(tK+Id)^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{tb+1} & 0 \\ 0 & 0 & \dfrac{1}{td+1} \end{bmatrix}.$$

We also have

$$\det(tK + Id)(x) = (tb+1)(td+1).$$

Define $\beta_1(t) = tb + 1, \beta_2(t) = td + 1$. Under the basis $\{\hat{e}_i\}$, denote $u = (r_1, r_2, r_3)$.

For Case **(i)**, under the basis $\{\hat{e}_i\}$, we have

$$M'_{g_t} = \frac{1}{\beta_1^2 \beta^2} \begin{bmatrix} \beta_1\beta_2 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & -\beta_1 \end{bmatrix}. \tag{86}$$

Then Equation (86) gives

$$g_t(u,u) = \frac{1}{(tb+1)^2(td+1)^2}\left(r_1^2(tb+1)(td+1) + r_2^2(td+1) - r_3^2(tb+1)\right). \tag{87}$$

Similar to Case (**VI-b**), we have $g_t(u,u) \equiv 0$ for $t \geq 0$. Using Equation (87), we obtain

$$r_1^2 = 0,\ r_2^2 - r_3^2 = 0,\ r_2^2 d = r_3^2 b. \tag{88}$$

If $r_2^2 = r_3^2 \neq 0$, we obtain $b = d$, contradicting the assumption. In addition, $r_2 = r_3 = r_1 = 0$ gives $u = 0$, which is also impossible.

For Case **(ii)**, we have under the basis $\{\hat{e}_i\}$:

$$M'_{g_t} = \frac{1}{\beta_1^2\beta_2^2}\begin{bmatrix} -\beta_1\beta_2 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_1 \end{bmatrix}.$$

Then we have

$$g_t(u,u) = \frac{1}{(tb+1)^2(td+1)^2}\left(-r_1^2(tb+1)(td+1) + r_2^2(td+1) + r_3^2(tb+1)\right).$$

We have $g_t(u,u) = 0$ for $t \geq 0$ analogous to **(i)**. This gives the following equalities:

$$r_1^2 = 0,\ r_2^2 + r_3^2 = 0.$$

Then again we have $u = 0$, which is impossible.

Case **(VI-d)**:

For this case, fix some x on the positive x_1-axis. We have $K(x)$ is real diagonalizable, and $\lambda_1(x) = 0,\ \lambda_2(x) = \lambda_3(x) = b \neq 0$. We can choose $\{\hat{e_i}\}$ so that $g(x), K(x)$ have the following matrix representations:

$$M'_g = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix},\ M'_K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix},\ \epsilon_i = \pm 1.$$

Then we have

$$M'_{(tK+Id)^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{tb+1} & 0 \\ 0 & 0 & \dfrac{1}{tb+1} \end{bmatrix},\ \det(tK + Id)(x) = (tb+1)^2.$$

Under the basis $\{\hat{e_i}\}$, we have the following matrix representation for $g_t(x)$:

$$M'_{g_t} = \frac{1}{(tb+1)^3} \begin{bmatrix} \epsilon_1(tb+1) & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix},\ \epsilon_i = \pm 1. \tag{89}$$

Under the basis $\{\hat{e_i}\}$, define $u' = (r'_1, r'_2, r'_3)$. Remember that

$$g_t(u', u') = \frac{g_{11}(\phi^t(x))}{(1 + tx_1)^4}. \tag{90}$$

We have $g_{11}(\phi^t(x))$ is uniformly bounded for $t \geq 0$. On the other hand, we have by Equation (89):

$$g_t(u', u') = \frac{1}{(tb+1)^3}(\epsilon_1 (r'_1)^2(tb+1) + \epsilon_2 (r'_2)^2 + \epsilon_3 (r'_3)^2). \tag{91}$$

By comparing (90) and (91), we get $g_t(u', u') = 0$ for $t \geq 0$. This implies $g(u', u') = 0$. This is impossible since x_1-axis is a space-like geodesic for g.

In summary all the cases listed above are impossible.

5.3.4 The case in which Ker$A$ is 3-dimensional

In this case, the projective vector field X vanishes at o with $O(X, o) = 2$. It follows from Lemma 5.6 of [2] that g is projectively flat on a neighborhood of o.

In conclusion, we have showed there is a always an open set V with $o \in \overline{V}$ on which g is projectively flat. This proves Theorem 1.5.

A Normal forms of 3-dimensional Minkowski Self-adjoint operators

We give the normal forms of self-adjoint operators of 3-dimensional Minkowski space-times, starting with the following well-known result of algebra (See Proposition 2 of [10]).

Proposition A.1. *Let g be a real non-degenerate quadratic form defined on $\mathbb{R}^n$. Suppose T is a self-adjoint operator for g. Then there exists an ordered basis $\{e_i\}$ such that g and T can be simultaneously reduced to the following block diagonal canonical forms.*

$$g_{can} = \begin{bmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_k \end{bmatrix}, \; T_{can} = \begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_k \end{bmatrix}. \tag{92}$$

For each $1 \leq i \leq k$, the matrices g_i and T_i are square matrices of the same size, where

$$g_i = \pm \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}. \tag{93}$$

Each T_i is a real Jordan block under this basis. In the case T_i has a real eigenvalue λ, we have

$$T_i = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & \\ & & & \lambda \end{bmatrix}.$$

In the case T_i has a pair of complex conjugate eigenvalues $a \pm ib$, let Λ and D_2 be the following matrices:

$$\Lambda = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \; D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have

$$T_i = \begin{bmatrix} \Lambda & D_2 & & \\ & \Lambda & \ddots & \\ & & \ddots & \\ & & & \Lambda \end{bmatrix}.$$

Note we have may have $\mathrm{Spec}(T_i) = \mathrm{Spec}(T_j)$ with $i \neq j$. Denote E_i the T-invariant subspace corresponding to T_i. It is clear from this proposition $E_i \perp E_j$, for any $i \neq j$. Hence $g|_{E_i}$ is non-degenerate for all $1 \leq i \leq k$.

As we are only dealing with 3-dimensional Lorentzian metrics in this thesis, we assume g is a Minkowski metric on $\mathbb{R}^3$ from now on. Then the normal forms of the pair (g, T) split into the following cases.

1. The case T has a pair of complex conjugate eigenvalues.

   In this case, the spectrum of T is the following:

   $$\{a + ib, a - ib, \lambda : a, b, \lambda \in \mathbb{R}, b \neq 0\}.$$

   Because a Jordan block T_i of T is at most 2-dimensional, it is clear that T is complex diagonalizable. Since T is not real diagonalizable on the characteristic space E_1 of $\{a + ib, a - ib\}$, we know $g|_{E_1}$ is not positive definite. It follows that the eigenspace of λ has to be space-like. By Proposition $A.1$, we have under the basis $\{e_i\}$:

   $$g_{can} = \begin{bmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; T_{can} = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \; \epsilon = \pm 1. \tag{94}$$

2. The case T is not complex diagonalizable.

   We know from Case 1 that all eigenvalues of T have to be real. If T has two Jordan blocks T_1 and T_2, we can assume T_1 is non-diagonalizable with the corresponding T-invariant subspace E_1. Then $g|_{E_1}$ is not positive definite. In this case we have

$$g_{can} = \begin{bmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; T_{can} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \; \epsilon = \pm 1. \tag{95}$$

If T has a single 3-dimensional Jordan block, it is immediate from Proposition A.1 that

$$g_{can} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; T_{can} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}. \tag{96}$$

3. The case T is real diagonalizable.

From Proposition A.1, we have the following normal forms.

$$g_{can} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; T_{can} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{97}$$

References

[1] S. Kobayashi, T. Nagano, *On projective connections*, J. Math. Mech. 13, 215-236 (1964)

[2] T. Nagano, T. Ochiai, *On compact Riemannian manifolds admitting essential projective transformations*, Journal of the Faculty of Science, the University of Tokyo. Sect. 1 A, Mathematics. 33, no. 2, 233-246 (1986).

[3] V. Kiosak, V. Matveev, *Proof of the projective Lichnerowicz conjecture for pseudo-Riemannian metrics with degree of mobility greater than two*, Comm. Mat. Phys. 297, 401-426 (2010).

[4] V. Matveev, *Proof of the projective Lichnerowicz-Obata conjecture*, J. Differential Geom. 75, no. 3, 459-502 (2007).

[5] A. Zeghib, *On discrete projective transformation groups of Riemannian manifolds*, Adv. Math. 297, 26-53 (2016).

[6] V. Matveev, V. Kiosak, *Degree of mobility for metrics of Lorentzian signature and parallel (0,2)-tensor fields on cone manifolds*, Proceedings of the LMS 108, 1277-1312 (2014).

[7] V. Matveev, M. Eastwood, *Metric Connections in Projective Differential Geometry*, Symmetries and Overdetermined Systems of Partial Differential Equations, Volume 144 of the series The IMA Volumes in Mathematics and its Applications, 339-350 (2008)

[8] A. Bolsinov, V. Matveev, *Splitting and gluing lemmas for geodesically equivalent pseudo-Riemannian metrics*, Trans. Amer. Math. Soc. 363 , 4081-4107 (2011).

[9] J. Mike, *Geodesic mappings of affine-connected and Riemannian spaces*, J. Math. Sci. 78, no. 3, 311-333 (1996).

[10] A. Bolsinov, V. Matveev, *Local normal forms for geodesically equivalent pseudo-Riemannian metrics*, Trans. Amer. Math. Soc. 367, 6719–6749 (2015).

[11] V.Matveev, P. Mounoud, *Gallot-Tanno Theorem for closed incomplete pseudo-Riemannian manifolds and applications*, Ann. Global Anal. Geom. 38, no. 3, 259-271 (2010).

[12] C. Frances, *Local dynamics of conformal vector fields*, Geom. Dedic. 158, 35-59 (2012).

[13] K. Melnick, C. Frances, *Formes normales pour les champs conformes pseudoriemanniens [Normal forms for pseudo-Riemannian conformal vector fields]*, Bulletin de la Société Mathématique de France 141, Issue 3, 377-421 (2013).

[14] R. Bryant, G. Manno, V. Matveev, *A solution of a problem of Sophus Lie: Normal forms of two-dimensional metrics admitting two projective vector fields*, Math. Ann. 340, no. 2, 437–463 (2008).

[15] V. Matveev, *Two-dimensional metrics admitting precisely one projective vector field*, Math. Ann. 352, no. 4, 865–909 (2012).

[16] C. Frances, *Causal conformal vector fields, and singularities of twistor spinors*, Ann. Global Anal. Geom. 32, no. 3, 277-295 (2007).

[17] Tianyu Ma, *Geodesic rigidity of Levi-Civita connections admitting essential projective vector fields*, arXiv:1705.00460.

[18] V. Matveev, A. Gover, *Projectively related metrics, Weyl nullity, and metric projectively invariant equations*, Proc. Lond. Math. Soc. 114 , no. 2, 242-292 (2017).

[19] R. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Springer-Verlag New York (1997).

[20] S. Kim, *Volume and Projective Equivalence Between Riemannian Manifolds*, Ann. Global Anal. Geom. 27, 47-52 (2005).

[21] V. Matveev, A. Bolsinov, S. Rosemann, *Local normal forms for c-projectively equivalent metrics and proof of the Yano-Obata conjecture in arbitrary signature. Proof of the projective Lichnerowicz conjecture for Lorentzian metrics*, arXiv:1510.00275.

[22] H. Weyl, *Zur Infinitisimalgeometrie: Einordnung der projektiven und der konformen Auffasung*, Gött. Nachr. (1921), 99-112