ABSTRACT

Title of dissertation:	SOME APPLICATIONS OF SET THEORY TO MODEL THEORY
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We investigate set-theoretic dividing lines in model theory. In particular, we are interested in Keisler's order and Borel complexity.

Keisler's order is a pre-order on complete countable theories T, measuring the saturation of ultrapowers of models of T. In Chapter 3, we present a self-contained survey on Keisler's order. In Chapter 4, we uniformize and sharpen several ultrafilter constructions of Malliaris and Shelah. We also investigate the model-theoretic properties detected by Keisler's order among the simple unstable theories.

Borel complexity is a pre-order on sentences of $\mathcal{L}_{\omega_1\omega}$ measuring the complexity of countable models. In Chapter 5, we describe joint work with Richard Rast and Chris Laskowski on this order. In particular, we connect the Borel complexity of $\Phi \in \mathcal{L}_{\omega_1\omega}$ with the number of potential canonical Scott sentences of Φ. In Chapter 6, we introduce the notion of thickness; when Φ has class-many potential canonical Scott sentences, thickness is a measure of how quickly this class grows in size. In Chapter 7, we describe joint work with Saharon Shelah on the Borel complexity of torsion-free abelian groups.

SOME APPLICATIONS OF SET THEORY TO MODEL THEORY

by

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Chapter 1: Introduction

Modern model theory began with Morley's categoricity theorem [65]:

Theorem 1.0.1. Suppose T is a complete countable theory. If T is κ-categorical for some uncountable cardinal κ, then T is κ-categorical for every uncountable cardinal κ.

Soon after, Morley's proof was refined by Baldwin and Lachlan [2] to give the modern formulation:

Theorem 1.0.2. Suppose T is a complete countable theory. Then the following are equivalent:

(A) T is κ-categorical for some uncountable κ.

(B) T is ω-stable and has no Vaughtian pairs.

(C) T is κ-categorical for every uncountable κ.

This theorem follows a template that has repeated often since. Namely, we begin with some semantic property of theories, frequently involving uncountable set theory. Then the hope is to find a syntactic characterization of this property. In this case, we view the property as being particularly significant, and likely to have applications in applied model theory.

In the case of Morley's categoricity theorem, the relevant semantic properties of T are κ-categoricity, for uncountable cardinals κ. The equivalent syntactic property is being

ω-stable and having no Vaughtian pairs. We note that the notion of being ω-stable (or totally transcendental, in Morley's terminology) was discovered by Morley in the course of proving the categoricity theorem; this has become a key hypothesis in applications of model theory to algebraic geometry and other fields.

We view classification theory as the program of hunting down or dreaming up semantic properties to play the role of κ-categoricity, and then isolating their syntactic equivalents; a large part of Shelah's career has been devoted to this. Most famously, in [75] Shelah took the spectrum of T as the semantic property of interest: namely, given T and an uncountable cardinal λ, let $I(T, \lambda)$ denote the number of models of T of size λ up to isomorphism. The function $I(T, \cdot)$ is called the spectrum of T; so always $I(T, \lambda) \leq 2^\lambda$. When sometimes $I(T, \lambda) < 2^\lambda$, this indicates that T is nice. In the transformational work [75], Shelah determined the asymptotic behavior of $I(T, \lambda)$ in terms of purely syntactic properties of T, and used this to solve a slew of open problems in model theory, including Morley's conjecture that $(I(T, \lambda) : \lambda \geq \aleph_1)$ is always nondecreasing.

There are many additional sources of semantic properties to investigate. In this thesis, we are concerned with two of them: Keisler's order and Borel complexity. In the remainder of the introduction, we give a summary of our main results.

1.1 Keisler's Order

Suppose M is a structure in a countable language, and $\mathcal{U}$ is an ultrafilter on $\mathcal{P}(\lambda)$. Then we can form the ultrapower $M^\lambda/\mathcal{U}$; by Łoś's theorem, this is an elementary extension of M, and in particular elementarily equivalent to M. We wish to understand what $M^\lambda/\mathcal{U}$ looks like. As a test question: is $M^\lambda/\mathcal{U}$ λ^+-saturated?

In [34], Keisler made the following definitions:

Definition 1.1.1. Suppose $\mathcal{U}$ is a λ-regular ultrafilter on $\mathcal{P}(\lambda)$. Then say that $\mathcal{U}$ λ^+-saturates T if for some or every $M \models T$, $M^\lambda/\mathcal{U}$ is λ^+-saturated.

Given complete countable theories T_0, T_1, say that $T_0 \trianglelefteq_\lambda T_1$ if whenever $\mathcal{U}$ is a λ-regular ultrafilter on $\mathcal{P}(\lambda)$, if $\mathcal{U}$ λ^+-saturates T_1 then $\mathcal{U}$ λ^+-saturates T_0. Say that $T_0 \trianglelefteq T_1$ if $T_0 \trianglelefteq_\lambda T_1$ for all λ.

$\trianglelefteq$ is called Keisler's order; we view $T_0 \trianglelefteq T_1$ as meaning that it is easier to saturate models of T_0 than those of T_1. We are interested in the dividing lines induced by Keisler's order. More precisely:

Definition 1.1.2. Temporarily let $\mathbf{T}$ denote the set of all complete countable theories; so $(\mathbf{T}, \trianglelefteq)$ is a preorder. Say that $\mathbf{D}$ is a dividing line in $\trianglelefteq$ if $\mathbf{D} \subseteq \mathbf{T}$ is downward-closed under $\trianglelefteq$ (this much actually makes sense for any preorder $\leq$ on $\mathbf{T}$). $\mathbf{D}$ is a principal dividing line if there is a single λ-regular ultrafilter $\mathcal{U}$ on some $\mathcal{P}(\lambda)$, such that $\mathbf{D}$ is the set of all $T \in \mathbf{T}$ which are λ^+-saturated by $\mathcal{U}$.

We wish to understand what the principal dividing lines in $\trianglelefteq$ are; these, in turn, would determine $\trianglelefteq$. This question has attracted a lot of attention, but progress has been, until recently, rather slow.

We will give a more detailed history of Keisler's order in Section 3.1. For now, we skip ahead to a fairly recent development of Malliaris and Shelah.

Namely, given an ultrafilter $\mathcal{U}$ on the complete Boolean algebra $\mathcal{B}$ and a complete countable theory T, Malliaris and Shelah define in [56] what it means for $\mathcal{U}$ to be $(\lambda, \mathcal{B}, T)$ moral, and they prove the following:

Theorem 1.1.3. Suppose $\mathcal{B}$ is a complete Boolean algebra with the λ^+-c.c. and with $|\mathcal{B}| \leq 2^\lambda$. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then there is a λ-regular ultrafilter $\mathcal{U}_*$ on

$\mathcal{P}(\lambda)$, such that for every complete countable theory T, $\mathcal{U}_*$ λ^+-saturates T if and only if $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$ moral.

This yields a new strategy for constructing dividing lines in Keisler's order: find a Boolean algebra $\mathcal{B}$ which is small in some sense (in [56], $\mathcal{B}$ has the $\aleph_1$-c.c.) and construct an ultrafilter $\mathcal{U}$ on $\mathcal{B}$ which is as generic as possible. Then the smallness of $\mathcal{B}$ will prevent $\mathcal{U}$ from being $(\lambda, \mathcal{B}, T)$ moral for some T, and its genericity will ensure that $\mathcal{U}$ is $(\lambda, \mathcal{B}, T')$-moral for other T'. This strategy has been used to great success by Malliaris and Shelah in [56], [57] and [58].

1.1.1 Chapter 3: A Survey of Keisler's Order

In Chapter 3, we present a self-contained and systematic treatment of Keisler's order, in what we believe to be the most logical fashion. Ultimately this means moving away from thinking about regular ultrafilters on $\mathcal{P}(\lambda)$, and instead focusing on arbitrary ultrafilters on complete Boolean algebras. Moreover, instead of considering ultrapowers of models, we instead look at specializations of Boolean valued models. Using this framework, we give many equivalences of what it means for $\mathcal{U}$ to be $(\lambda, \mathcal{B}, T)$-moral, for instance:

Theorem 1.1.4. Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$, and T is a complete countable theory. Then $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-moral if and only if for some or every λ^+-saturated $\mathcal{B}$-valued model $\mathbf{M}$ of T, the specialization $\mathbf{M}/\mathcal{U}$ is λ^+-saturated.

These equivalences justify the following generalization of Definition 1.1.1:

Definition 1.1.5. Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$, and suppose T is a complete countable theory. Then say that $\mathcal{U}$ λ^+-saturates T if $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-moral.

We take this as our basic notion to investigate. In view of Theorem 1.1.3, it would be equivalent to change the definition of $\trianglelefteq$ to allow the ultrafilter $\mathcal{U}$ to be on any complete Boolean algebra $\mathcal{B}$ with the λ^+-c.c. and with $|\mathcal{B}| \leq 2^\lambda$. Actually, in Corollary 3.16.20, we strengthen this to show that the hypothesis $|\mathcal{B}| \leq 2^\lambda$ can be weakened. In particular:

Theorem 1.1.6. Suppose T_0, T_1 are complete countable theories, and λ is a cardinal. Then $T_0 \trianglelefteq_\lambda T_1$ if and only if for every complete Boolean algebra $\mathcal{B}$ with the λ^+-c.c., and for every ultrafilter $\mathcal{U}$ on $\mathcal{B}$, if $\mathcal{U}$ λ^+-saturates T_1, then $\mathcal{U}$ λ^+-saturates T_0.

We remark that the bulk of the proof of this theorem consists of Malliaris and Shelah's Separation of Variables and Existence Theorems from [56], and involves complicated ultrafilter constructions on $\mathcal{P}(\lambda)$. We find it convenient to take Theorem 3.2.4 as our operating definition of $\trianglelefteq$ for most of the survey, deferring its proof until the end.

There are several new results in the survey; we describe some of them now.

In Section 3.15, we prove the following. Special cases are proved by Malliaris and Shelah in [57] and [56].

Theorem 1.1.7. Suppose $\mathcal{B}$ is a complete Boolean algebra with the λ-c.c. and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ does not λ^+-saturate any unsimple theory. If $\mathcal{U}$ is additionally $\aleph_1$-incomplete, then $\mathcal{U}$ does not λ^+-saturate any nonlow theory.

This makes clear the role the chain condition is playing: namely, to constrict the possible theories that we can saturate. We note this theorem is sharp: if $\mathcal{B}$ has an antichain of size λ, then we show in Theorem 3.16.5 that there is an ultrafilter $\mathcal{U}$ on $\mathcal{B}$ which λ^+-saturates every complete countable theory.

In Section 3.8, we discuss the interpretability orders $\trianglelefteq^*_\kappa$, for $\kappa \in \{\aleph_1, 1\}$; these are approximation to Keisler's order. $\trianglelefteq^*_{\aleph_1}$ was originally introduced in [78] by Shelah as a general context for proving positive reductions in Keisler's order. The point is that $\trianglelefteq^*_{\aleph_1} \subseteq \trianglelefteq$, and so to prove $T_0 \trianglelefteq T_1$ it is enough to prove $T_0 \trianglelefteq^*_{\aleph_1} T_1$; further, the arguments in some ways become cleaner. However, dealing with $\trianglelefteq^*_\kappa$ does introduce certain complications versus $\trianglelefteq$. We introduce our own interpretability orders $\trianglelefteq^\times_\kappa$, which eliminate these complications. We still have $\trianglelefteq^\times_1 \subseteq \trianglelefteq^\times_{\aleph_1} \subseteq \trianglelefteq$, and further, $\trianglelefteq^\times_\kappa$ allows an elegant general theory of combinatorial characteristics of models of ZFC^-.

We prove in Section 3.18 that $\trianglelefteq^\times_{\aleph_1} \subseteq \trianglelefteq^*_{\aleph_1}$, and also $\trianglelefteq^\times_1 \subseteq \trianglelefteq^*_1$ except perhaps on pairs of stable theories. Thus, all of the positive reductions we prove in $\trianglelefteq^\times_{\aleph_1}$ carry over to $\trianglelefteq^*_{\aleph_1}$. Moreover, we use this machinery to deduce the following in Corollaries 3.18.9 and 3.18.10:

Theorem 1.1.8. $\trianglelefteq^*_1$ and $\trianglelefteq^*_{\aleph_1}$ coincide on pairs of theories which are not both stable. Hence, if suitable instances of GCH hold, then $NSOP_2$ theories are nonmaximal in $\trianglelefteq^*_{\aleph_1}$, and in ZFC, simplicity is a dividing line in $\trianglelefteq^*_{\aleph_1}$.

The hence portion follows from corresponding results for $\trianglelefteq^*_1$, proved in [8], [7] and [61].

1.1.2 Chapter 4: Amalgamation Properties and Keisler's Order

In Chapter 4, we give a streamlined treatment of many existing ultrafilter constructions in Keisler's order.

The following theorem is due to Malliaris and Shelah [57]:

Theorem 1.1.9. Suppose there is a supercompact cardinal σ; set $\lambda = \sigma^+$. Then there is a complete Boolean algebra $\mathcal{B}$ with the λ-c.c., and an ultrafilter $\mathcal{U}$ on $\mathcal{B}$, which λ^+-saturates exactly the simple theories. Hence, simplicity is a principal dividing line in Keisler's order.

Using similar arguments, I proved the following in [87]:

Theorem 1.1.10. Set $\lambda = (2^{\aleph_0})^+$. Then there is a complete Boolean algebra $\mathcal{B}$ with the λ-c.c., and an $\aleph_1$-incomplete ultrafilter $\mathcal{U}$ on $\mathcal{B}$, which λ^+-saturates exactly the low theories. Hence, lowness is a principal dividing line in Keisler's order.

Finally, we have the following improvement of a theorem of Malliaris and Shelah [58]; here, $T_{k+1,k}$ is the theory of the random k-ary $k+1$-clique free hypergraph. Malliaris and Shelah proved the special case where $k < k' - 1$, which is already enough to deduce that Keisler's order has infinitely many classes.

Theorem 1.1.11. For each $k < k'$, $T_{k+1,k} \not\trianglelefteq T_{k'+1,k'}$.

We present all of these theorems as a single instance of a general ultrafilter construction, parametrized by a suitable sequence of cardinals. Moreover, we are able to obtain purely model-theoretic upper and lower bounds for the relevant principal dividing lines in Theorem 1.1.11, which detect weaker and stronger notions of k-dimensional amalgamation. We do not have any examples of theories witnessing a gap between our bounds; we introduce the notion of a "well-behaved" simple theory, and show that for these theories, all of our notions of k-dimensional amalgamation coincide.

1.2 Borel Complexity

We now move on from Keisler's order and consider Borel complexity. The motivation here is to find interesting dividing lines for *countable* model theory. The naïve method of just counting the number of countable models of isomorphism does not give enough information, since the maximum number of $2^{\aleph_0}$ is achieved even in relatively simple cases.

Given a sentence $\Phi \in \mathcal{L}_{\omega_1\omega}$, we can form $\mathrm{Mod}(\Phi)$, the set of models of Φ with universe ω. This is naturally a standard Borel space, where the Borel sets are taken to be solution sets to formulas of $\mathcal{L}_{\omega_1\omega}$. In [12], Friedman and Stanley make the following definition.

Definition 1.2.1. Suppose Φ, Ψ are sentences of $\mathcal{L}_{\omega_1\omega}$. Then $f : \Phi \leq_B \Psi$ is a Borel reduction if $f : \mathrm{Mod}(\Phi) \to \mathrm{Mod}(\Psi)$ is a Borel map, such that for all $M, N \in \mathrm{Mod}(\Phi)$, $M \cong N$ if and only if $f(M) \cong f(N)$. We say that $\Phi \leq_B \Psi$ if there is some $f : \Phi \leq_B \Psi$. Say that $\Phi \sim_B \Psi$ (Φ and Ψ are Borel bireducible) if $\Phi \leq_B \Psi \leq_B \Phi$.

We describe some of the initial results on $\leq_B$ obtained by Friedman and Stanley in [12].

First, Friedman and Stanley showed that there is a maximal class of sentences under $\leq_B$, namely the *Borel complete sentences.* For example, graphs are Borel complete, as are groups, rings, linear orders, and trees.

Also, Friedman and Stanley introduced the Friedman-Stanley tower. There are many equivalent formulations of this; we will find a certain family $(\Phi_\alpha : \alpha < \omega_1)$ of sentences of $\mathcal{L}_{\omega_1\omega}$ to be the most convenient to work with. The countable models of Φ_α up to isomorphism are, in a precise sense, identifiable with $\mathrm{HC}_{\omega+\alpha}$, the hereditarily countable sets of foundation rank less than $\omega + \alpha$. It is easy to see that $\Phi_\alpha \leq_B \Phi_\beta$ for $\alpha \leq \beta$; using sophisticated methods of descriptive set theory, Friedman and Stanley show that when $\alpha < \beta$, then $\Phi_\alpha <_B \Phi_\beta$ (i.e. $\Phi_\alpha \leq_B \Phi_\beta$ but $\Phi_\beta \not\leq_B \Phi_\alpha$).

1.2.1 Chapter 5: Potential Canonical Scott Sentences and Borel Complexity

In Chapter 5, we include the results of [89], joint work with Richard Rast and Chris Laskowski, although we make some small changes to notation.

One of the central ideas of [89] is the following. Given a structure M, let $\mathrm{css}(M)$ denote its canonical Scott sentence; this is a canonical sentence of $\mathcal{L}_{|M|^+\omega}$ characterizing

M up to back-and-forth equivalence; in particular, if M is countable, then $\mathrm{css}(M) \in \mathcal{L}_{\omega_1\omega}$ characterizes M up to isomorphism. Given $\Phi \in \mathcal{L}_{\omega_1,\omega}$, let $\mathrm{CSS}(\Phi)$ denote the set $\{\mathrm{css}(M) : M \in \mathrm{Mod}(\Phi)\}$. Then any Borel map $f : \mathrm{Mod}(\Phi) \to \mathrm{Mod}(\Psi)$ induces an injection $f^* : \mathrm{CSS}(\Phi) \to \mathrm{CSS}(\Psi)$.

Next, given a sentence $\Phi \in \mathcal{L}_{\omega_1\omega}$, let $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ be the class of all $\varphi \in \mathcal{L}_{\infty\omega}$, such that there is some forcing extension $\mathbb{V}[G]$ of the universe with $\varphi \in \mathrm{CSS}(\Phi)^{\mathbb{V}[G]}$. These are the potential canonical Scott sentences of Φ.

We define Φ to be *short* if $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ is a set as opposed to a proper class and define the *potential cardinality* of Φ, denoted $\|\Phi\|$, to be the (usual) cardinality of $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ if Φ is short, or ∞ otherwise.

Now, if $f : \Phi \leq_B \Psi$, then this induces some injection $f_* : \mathrm{CSS}(\Phi) \leq_{\mathrm{HC}} \mathrm{CSS}(\Psi)$, which in turn induces an injection $(f_*)_{\mathrm{ptl}} : \mathrm{CSS}(\Phi)_{\mathrm{ptl}} \to \mathrm{CSS}(\Psi)_{\mathrm{ptl}}$. This is the content of Theorem 5.3.11:

> If $\|\Psi\| < \|\Phi\|$, then $\Phi \not\leq_B \Psi$.

The advantage of this is that the potential cardinality $\|\Phi\|$ is, in applications, something we can calculate; thus, this gives an important new method for proving nonreducibilities.

As a particular example: we define (a version of) the Friedman-Stanley tower $(\Phi_\alpha : \alpha < \omega_1)$ in Section 5.3.4, and show that each $\|\Phi_\alpha\| = \beth_\alpha$. This gives a much simpler proof that $\Phi_\alpha <_B \Phi_\beta$ for $\alpha < \beta$ than the original proof of Friedman and Stanley.

We use the machinery of potential canonical Scott sentences to analyze several specific first order theories. In particular, in [42], Koerwien defined a certain first order theory K, proved K does not have Borel isomorphism relation, and asked whether or not K is Borel complete. We resolve this negatively by showing that $\|\mathrm{K}\| = \beth_2$. This gives the first example of a first order theory which is not Borel complete, and which does not have Borel isomorphism relation.

1.2.2 Chapter 6: Borel Complexity, Thickness, and the Schröder-Bernstein property

One limitation of potential cardinality is that there exist sentences Φ which are not short (i.e. $\|\Phi\| = \infty$) and yet Φ is not Borel complete. For example, let $\mathrm{TAG}_1 \in \mathcal{L}_{\omega_1\omega}$ axiomatize torsion abelian groups. By Ulm's classification of abelian p-groups, we can identify $\mathrm{CSS}(\mathrm{TAG}_1)$ with $[\omega_1]^{\aleph_0}$ (countable sets of ordinals), and so we can identify $\mathrm{CSS}(\mathrm{TAG}_1)_{\mathrm{ptl}}$ with $\mathcal{P}(\mathrm{ON})$ (the class of all sets of ordinals). In particular, TAG_1 is not short. But Friedman and Stanley showed in [12] that TAG_1 is not Borel complete, and in fact $\Phi_2 \not\leq_B \mathrm{TAG}_1$.

In Chapter 6, we introduce the notion of thickness. Namely, for each sentence $\Phi \in \mathcal{L}_{\omega_1\omega}$, we get the thickness spectrum $\tau(\Phi, \cdot)$ of Φ, a function from cardinals to cardinals; $\tau(\Phi, \cdot)$ is closely related to $|\mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap \mathbb{V}_{\lambda^+}|$. It follows immediately from the definition that for every λ, $\tau(\Phi, \lambda) \leq |\mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap \mathbb{V}_{\lambda^+}| \leq \beth_{\lambda^+}$, and $\tau(\Phi, \cdot)$ is monotonically increasing, and $\lim_{\lambda \to \infty} \tau(\Phi, \lambda) = \|\Phi\|$.

The definition of thickness is arranged so that it is a Borel-reducibility invariant:

Theorem 1.2.2. Suppose $\Phi \leq_B \Psi$. Then for every cardinal λ, $\tau(\Phi, \lambda) \leq \tau(\Psi, \lambda)$.

As a first application of thickness, we show that $\tau(\Phi_\alpha, \aleph_0) = \tau(\mathrm{TAG}_\alpha, \aleph_0) = \beth_\alpha$ for all $1 \leq \alpha < \omega_1$, and thus, $\Phi_{\alpha+1} \not\leq_B \mathrm{TAG}_\alpha$ for all $\alpha < \omega_1$, generalizing the theorem of Friedman and Stanley in [12] that $\Phi_2 \not\leq_B \mathrm{TAG}_1$.

We then present another application of the thickness machinery, namely to the Schröder-Bernstein property. We make the following definitions, which are slightly modified from previous contexts:

Definition 1.2.3. Suppose $\mathcal{L}$ is a language, and M, N are $\mathcal{L}$-structures. Then say that $f : M \leq N$ is an embedding if it is a homomorphism; that is, f commutes with the function symbols, and if R is an n-ary relation, then $f[R^M] \subseteq R^N$.

Definition 1.2.4. Say that Φ has the Schröder-Bernstein property if for all $M, N \models \Psi$ countable, if $M \sim N$ then $M \cong N$.

Then we are able to prove the following. $\kappa(\omega)$, the *ω'th Erdös cardinal*, is the least cardinal satisfying $\kappa \to (\omega)_2^{<\omega}$. $\kappa(\omega)$ is a large cardinal, i.e. it cannot be proven to exist in ZFC; nonetheless, it is relatively low in the hierarchy of large cardinal axioms.

Theorem 1.2.5. Assume $\kappa(\omega)$ exists, and suppose Φ has the Schröder-Bernstein property. Then for every cardinal λ, $\tau(\Phi, \lambda) \leq \lambda^{<\kappa(\omega)}$, so in particular $\mathrm{TAG}_1 \not\leq_B \Phi$.

The following is essentially a special case.

Theorem 1.2.6. Assume $\kappa(\omega)$ exists. Then there is no Borel reduction from graphs to colored trees, which takes nonisomorphic graphs to nonbiembeddable trees.

We also introduce a hierarchy of α-ary Schröder Bernstein properties for each ordinal α, and prove analogous statements for them.

1.2.3 Chapter 7: Borel Complexity of Torsion-Free Abelian Groups

In Chapter 7, we describe the results of [82], joint with Shelah, where we investigate the complexity of countable torsion free abelian groups.

In [12], Friedman and Stanley leverage the Ulm analysis [86] to show that torsion abelian groups are far from Borel complete. They then pose the following question:

Question. Let TFAG be the theory of torsion-free abelian groups. Is TFAG Borel complete?

This has attracted considerable attention, but has nonetheless remained open. The following theorem of Hjorth [23] is the best known so far:

Theorem 1.2.7. $\Phi_\alpha \leq_B$ TFAG for every $\alpha < \omega_1$.

This means that if TFAG is not Borel complete, then it represents a very new phenomenon. In fact, in [12], Friedman and Stanley separately described the following question as one of the basic open problems of the general theory: if Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ and if $\Phi_\alpha \leq_B \Phi$ for each $\alpha < \omega_1$, must Φ be Borel complete?

Using the basic idea of Theorem 1.2.7, we are able to prove the following. Here, $\leq_{a\Delta^1_2}$ is the absolute Δ^1_2-reducibility notion, defined like $\leq_B$, except we allow the reduction to be absolutely Δ^1_2, rather than Borel.

Theorem 1.2.8. Suppose there is no transitive model of $ZFC^- + \kappa(\omega)$ exists. Then Graphs $\leq_{a\Delta^1_2}$ TFAG.

Corollary 1.2.9. It is consistent with ZFC that Graphs $\leq_{ZFC^-}$ TFAG, and hence that TFAG is $\leq_{ZFC^-}$-complete.

The key set-theoretic contribution is the following partial converse to Theorem 1.2.6:

Theorem 1.2.10. Suppose there is no transitive model of $ZFC^- + \kappa(\omega)$ exists. Then there is an absolutely Δ^1_2-reduction from graphs to colored trees, which takes nonisomorphic graphs to nonbiembeddable trees.

This suggests that perhaps TFAG has some α-ary Schröder-Bernstein property. Then $\alpha = 0$ case has already been investigated: the Schröder-Bernstein property for

TFAG fails, as first proved by Goodrick [17] (in fact, the failure was with elementary embedding). Recently, Calderoni and Thomas have shown in [85] that the relation of biembeddability on models of TFAG is Σ^1_1-complete, which is as bad as possible.

For $\alpha > 0$, we are able to prove the following (where we use injective group homomorphisms as our notion of embedding):

Theorem 1.2.11. For every $\alpha < \kappa(\omega)$, TFAG fails the α-ary Schröder-Bernstein property.

The construction breaks down at $\kappa(\omega)$, so the following remains open:

Question. Does TFAG have the $\kappa(\omega)$-ary Schröder-Bernstein property?

If the answer is yes, then this would imply that TFAG is not Borel complete.

Chapter 2: Preliminaries

In this chapter, we collect together some background results that will be needed later.

We work in ZFC; $\mathbb{V}$ denotes the universe of all sets. We formalize all of model theory within ZFC. For instance, countable languages $\mathcal{L}$ are construed as being elements of HC (the set of hereditarily countable sets).

If X is a set and κ is a cardinal, then $[X]^\kappa$ denotes the set of all subsets of X of cardinality less than κ. $\mathcal{P}(X)$ denotes the powerset of X. Also, if X is a class, then $\mathcal{P}(X)$ denotes the class of all subsets of X.

2.1 Forcing Notions and Boolean Algebras

For background, see Jech [27] or Kunen [44].

A forcing notion is a pre-order $(P, \leq)$ with a maximal element 1; in other words, $\leq$ is a transitive relation. We will always identify P with its separative quotient, defined by putting $p \sim q$ if for all p', p' is compatible with p if and only if p' is compatible with q. When we say that P is a forcing notion, we always mean P is a set (rather than a proper class), unless explicitly stated otherwise.

A complete boolean algebra $\mathcal{B}$ is a structure $(\mathcal{B}, \leq, 0, 1, \bigwedge, \bigvee, \neg)$ satisfying the axioms for a Boolean algebra, with the greatest lower bound property (equivalently, the least upper bound property). When we view $\mathcal{B}$ as a forcing notion, we always mean $\mathcal{B}_+$, the set of positive elements of $\mathcal{B}$.

Suppose $\mathcal{B}_0, \mathcal{B}_1$ are complete Boolean algebras. Then say that $\mathcal{B}_0$ is a complete subalgebra of $\mathcal{B}_1$ if $\mathcal{B}_0$ is a subalgebra of $\mathcal{B}_1$, and for every $X \subseteq \mathcal{B}_0$, the join of X as computed in $\mathcal{B}_0$ is the same as computed in $\mathcal{B}_1$. (This implies the corresponding statements for meets.)

Given a forcing notion P, let $\mathcal{B}(P)$ be its unique Boolean algebra completion; that is the unique Boolean algebra, up to isomorphism, such that P densely embeds into $\mathcal{B}(P)$. ($\mathcal{B}(P)$ exists and is unique by Theorem 14.10 of [27].) We always view P as a dense subset of $\mathcal{B}(P)$ even though when P is not separative, the canonical map from P to $\mathcal{B}(P)$ is not injective; this is not a real problem, since the map is injective on the separative quotient of P, and as mentioned above we always identify P with its separative quotient. Every element of $\mathcal{B}(P)$ can be written as $\bigvee X$ for some $X \subseteq P$, which in fact can be chosen to be an antichain. Further, $\bigvee X \leq \bigvee Y$ if and only if for every $x \in X$, there is $x' \leq x$ and $y \in Y$ such that $x' \leq y$.

Definition 2.1.1. For sets X, Y and a regular cardinal θ, let $P_{XY\theta}$ be the forcing notion of all functions partial functions from X to Y of cardinality less than θ, ordered by reverse inclusion. Let $\mathcal{B}_{XY\theta}$ be its Boolean algebra completion.

Definition 2.1.2. Suppose $\mathcal{B}$ is a complete Boolean algebra. An antichain on $\mathcal{B}$ is a subset $\mathbf{C}$ of $\mathcal{B}$, such that for all distinct $\mathbf{c}, \mathbf{c}' \in \mathbf{C}$, $\mathbf{c} \wedge \mathbf{c}' = 0$. If $\mathbf{C}, \mathbf{C}'$ are maximal antichains, then $\mathbf{C}$ refines $\mathbf{C}'$ if for every $\mathbf{c} \in \mathbf{C}$ there is some $\mathbf{c}' \in \mathbf{C}'$ such that $\mathbf{c} \leq \mathbf{c}'$; in this case, $\mathbf{c}'$ is unique. Easily, every finite set of maximal antichains has a common refinement.

Here are three important properties of forcing notions.

- Say that P is θ-closed if every descending chain from P of length less than θ has a lower bound in P.

- Say that P is $< \theta$-distributive if the intersection of every family of $< \theta$-many dense, downward closed subsets of P is dense. Say that P is θ-distributive if it is $< \theta^+$-distributive.

- Say that P is κ-c.c. if every antichain from P has size less than κ. (Historically, this is also called κ-saturated, although this term is no longer used.)

Clearly θ-closed implies $< \theta$-distributive, and every forcing notion P is $< \aleph_0$-distributive. Also, P is $< \theta$-distributive if and only if $\mathcal{B}(P)_+$ is, and similarly P is κ-c.c. if and only if $\mathcal{B}(P)_+$ is.

Example 2.1.3. c.c.$(\mathcal{P}(\lambda)) = \lambda^+$, and $\mathcal{P}(\lambda)$ is θ-distributive for all θ.

$P_{XY\theta}$ is θ-closed and $(|Y|^{<\theta})^+$-c.c., and so $\mathcal{B}_{XY\theta}$ is $< \theta$-distributive and $(|Y|^{<\theta})^+$-c.c. The chain condition is proved using the Δ-system lemma, e.g. Theorem II.1.2 of [44].

We want the following lemma on distributivity; see [27] Lemma 7.16 for (A) if and only if (B) if and only if (C), and see [13] for (A) if and only if (D).

Lemma 2.1.4. Suppose $\mathcal{B}$ is a complete Boolean algebra. Then the following are equivalent:

(A) For all $\alpha_* < \theta$, for all $(\mathbf{b}_{\alpha,i} : \alpha < \alpha_*, i \in I_\alpha)$, we have

$$\bigwedge_{\alpha<\alpha_*} \bigvee_{i \in I_\alpha} \mathbf{b}_{\alpha,i} = \bigvee_{f \in \prod_{\alpha<\alpha_*} I_\alpha} \bigwedge_{\alpha<\alpha_*} \mathbf{b}_{\alpha,f(\alpha)}.$$

(B) $\mathcal{B}$ is $< \theta$-distributive.

(C) Every family of $< \theta$-many maximal antichains of $\mathcal{B}$ has a common refinement.

(D) For each $\alpha_* < \theta$, Player I has no winning strategy in the following transfinite game: Players I and II alternate picking a descending chain $(\mathbf{a}_\alpha : \alpha \leq \alpha_*)$ from $\mathcal{B}_+$ (the chain need not be strictly descending) where Player I gets to pick $\mathbf{a}_0$ and $\mathbf{a}_\delta$ for δ limit. Player I wins if at some point play cannot continue (i.e. at some limit stage $\delta \leq \alpha_*$, $\bigwedge_{\alpha<\delta} \mathbf{a}_\alpha = 0$).

We deduce the following consequences: first, if $\mathcal{B}$ is $< \theta$ distributive for θ singular, then $\mathcal{B}$ is θ-distributive. So the least θ such that $\mathcal{B}$ is not $< \theta$-distributive, is regular.

Second, suppose P is a forcing notion and $p, q \in P$. Say that p decides q if either $p \leq q$ or else p and q are incomparable. Trivially, then, if P is σ-distributive and $X \in [P]^\sigma$, then the set of all $p \in P$ which decide each $q \in X$ is dense.

Now we discuss the κ-c.c. First of all, the following is Theorem 7.15 from [27]:

Theorem 2.1.5. If κ is least such that P has the κ-c.c., then $\kappa + \aleph_0$ is regular.

This κ will come up enough that we make the following definition, following [27]. Note that κ-c.c. is sometimes also referred to as being κ-saturated.

Definition 2.1.6. If P is a forcing notion, then let c.c.(P) be the least κ such that P has the κ-c.c.

So always c.c.$(P) \leq |P|^+$, and as long as c.c.$(P) \geq \aleph_0$, it is regular. The latter fails exactly when P is finite, as the following facts indicate:

Proposition 2.1.7. Let $\mathcal{B}$ be a complete Boolean algebra; write $\lambda = $ c.c.$(\mathcal{B})$. If $\mathcal{B}$ is λ-distributive, then $\lambda = \mu^+$ is a successor cardinal, and $\mathcal{B} \cong \mathcal{P}(\mu)$. Hence $\mathcal{B}$ is θ-distributive for all θ.

Proof. I claim that the atoms of $\mathcal{B}$ (i.e. minimal nonzero elements of $\mathcal{B}$) are dense in $\mathcal{B}$.

This suffices, since then $\mathcal{B} \cong \mathcal{P}(\mu)$, where μ is the cardinality of the set of atoms of $\mathcal{B}$, and then necessarily $\lambda = \mu^+$.

So suppose towards a contradiction that the atoms are not dense in $\mathcal{B}$. Say there are no atoms below $\mathbf{c}$. Inductively choose a sequence $(\mathbf{C}_\alpha : \alpha \leq \lambda)$ of maximal antichains of $\mathcal{B}$ below $\mathbf{c}$ (i.e., each $\mathbf{C}_\alpha \cup \{\neg\mathbf{c}\}$ is a maximal antichain of $\mathcal{B}$), such that $\mathbf{C}_\alpha$ refines $\mathbf{C}_\beta$ for all $\alpha > \beta$, and such that for every $\alpha < \lambda$ and for every $\mathbf{d} \in \mathbf{C}_\alpha$, there are at least two elements $\mathbf{d}_0, \mathbf{d}_1 \in \mathbf{C}_{\alpha+1}$ such that $\mathbf{d}_i \leq \mathbf{d}$. At the successor stages, we are using that there are no atoms below $\mathbf{c}$; at the limit stages, we are using λ-distributivity.

Now let $\mathbf{c}_\lambda \in \mathbf{C}_\lambda$; for all $\alpha \leq \lambda$, let $\mathbf{c}_\alpha$ be the unique element of $\mathbf{C}_\alpha$ with $\mathbf{c}_\lambda \leq \mathbf{c}_\alpha$. Then $(\mathbf{c}_\alpha : \alpha < \lambda)$ is a strictly descending chain in $\mathcal{B}$. Thus $(\mathbf{c}_\alpha \wedge \neg\mathbf{c}_{\alpha+1} : \alpha < \lambda)$ is an antichain from $\mathcal{B}$, contradicting that $\mathcal{B}$ has the λ-c.c. $\square$

In particular:

Corollary 2.1.8. If $\mathcal{B}$ is a complete Boolean algebra and c.c.$(\mathcal{B}) = n < \aleph_0$, then $\mathcal{B} \cong \mathcal{P}(n)$. If $\mathcal{B}$ is infinite, then c.c.$(\mathcal{B}) \geq \aleph_1$.

Proof. The first claim follows since $\mathcal{B}$ is $< \aleph_0$-distributive. For the second, if c.c.$(\mathcal{B}) \leq \aleph_0$, then the atoms must be dense in $\mathcal{B}$, since otherwise we could find an infinite descending chain from $\mathcal{B}$ and thus get an infinite antichain as above. Thus $\mathcal{B} \cong \mathcal{P}(n)$ for some $n < \omega$. $\square$

2.2 σ-complete ultrafilters

The ultrafilter $\mathcal{U}$ on the complete Boolean algebra $\mathcal{B}$ is σ-complete if for all $\kappa < \sigma$ and for all $(\mathbf{a}_\alpha : \alpha < \kappa)$ from $\mathcal{U}$, $\bigwedge_{\alpha<\kappa} \mathbf{a}_\alpha \in \mathcal{U}$. This is the same as asking that for every $\kappa < \sigma$ and for every descending sequence $(\mathbf{a}_\alpha : \alpha < \kappa)$ from $\mathcal{U}$, $\bigwedge_{\alpha<\kappa} \mathbf{a}_\alpha \neq 0$. We define the completeness of $\mathcal{U}$ to be the least σ such that $\mathcal{U}$ is σ^+-incomplete, or ∞ if $\mathcal{U}$ is σ-complete for all σ. Also, note that $\mathcal{U}$ is ∞-complete if $\mathcal{U}$ is principal, i.e. $\bigwedge \mathcal{U} \neq 0$.

We need a couple of straightforward lemmas on σ-complete ultrafilters. They are both easy generalizations of the $\mathcal{B} = \mathcal{P}(\lambda)$ case, but I don't know if they have been recorded anywhere in generality.

Lemma 2.2.1. Suppose $\mathcal{U}$ is a nonprincipal ultrafilter on the complete Boolean algebra $\mathcal{B}$, with completeness σ. Then $\sigma < \text{c.c.}(\mathcal{B})$, and there is a maximal antichain $\mathbf{C}$ of $\mathcal{B}$ of size σ such that for each $\mathbf{c} \in \mathbf{C}$, $\mathbf{c} \notin \mathcal{U}$. Thus $\mathcal{U}$ induces a σ-complete ultrafilter on $\mathbf{C}$, and so σ is measurable.

Proof. Choose $(\mathbf{a}_\alpha : \alpha < \sigma)$ a descending sequence from $\mathcal{U}$ such that $\bigwedge_\alpha \mathbf{a}_\alpha = 0$. We can suppose $\mathbf{a}_0 = 1$ and for limit $\delta < \sigma$, $\mathbf{a}_\delta = \bigwedge_{\alpha<\delta} \mathbf{a}_\alpha$. Let $\mathbf{c}_\alpha = \mathbf{a}_\alpha \backslash \mathbf{a}_{\alpha+1}$ for each $\alpha < \sigma$; then I claim $\mathbf{C} = \{\mathbf{c}_\alpha : \alpha < \lambda\}$ is a maximal antichain of $\mathcal{B}$ and each $\mathbf{c}_\alpha \notin \mathcal{U}$.

The only part that is not clear is maximality; suppose $\mathbf{c} \wedge \mathbf{c}_\alpha = 0$ for each $\alpha < \sigma$. Then it is easy to prove by induction on α that $\mathbf{c} \leq \mathbf{a}_\alpha$ for each $\alpha < \sigma$. □

The special case of the following lemma when $\mathcal{B} = \mathcal{P}(\lambda)$ is Proposition 4.1 of [28]. A cardinal σ is strongly compact if, whenever Γ is a set of $\mathcal{L}_{\sigma\sigma}$-formulas, if every subset of Γ of size less than σ is satisfiable, then Γ is satisfiable.

Lemma 2.2.2. Suppose σ is strongly compact, and $\mathcal{B}$ is a complete Boolean algebra, and $\mathcal{D}$ is a σ-complete filter on $\mathcal{B}$. Suppose $\mathcal{B}$ is $< \sigma$-distributive. Then $\mathcal{D}$ extends to a σ-complete ultrafilter on $\mathcal{B}$.

Proof. Let $\mathcal{L}$ be the language with a constant symbol for each element $\mathbf{a} \in \mathcal{B}$ (also denoted $\mathbf{a}$), and with a unary relation symbol U. Let Γ assert the following:

- $\{\mathbf{a} \in \mathcal{B} : U(\mathbf{a})\}$ is an ultrafilter (this is first-order);

- For every descending chain $(\mathbf{a}_\gamma : \gamma < \gamma_*)$ from $\mathcal{B}$ of length less then σ, if $U(\mathbf{a}_\gamma)$ holds for each γ then $U(\bigcap_\gamma \mathbf{a}_\gamma)$ holds;
- $U(\mathbf{a})$ holds for all $\mathbf{a} \in \mathcal{D}$.

To see that this is σ-satisfiable: let $\Gamma_0 \subseteq \Gamma$ have size less than σ. Choose $X \in [\mathcal{B}]^{<\sigma}$ containing all the constants appearing in Γ_0. Let $\mathbf{a}_0 = \bigcap\{\mathbf{a} : \mathbf{a} \in X \cap \mathcal{D}\}$, so $\mathbf{a} \in \mathcal{D}$ is nonzero (as $\mathcal{D}$ is σ-complete). Choose $\mathbf{a}_1 \leq \mathbf{a}_0$ nonzero such that $\mathbf{a}_1$ decides every element of X (this is possible since $\mathcal{B}$ is $< \sigma$-distributive). For all $\mathbf{b} \in \mathcal{B}$, define $U(\mathbf{b})$ to hold if and only if $\mathbf{a} \leq \mathbf{b}$. Then this clearly defines a model of Σ_0. $\square$

2.3 Forcing

We briefly review forcing, following Jech [27].

If P is a forcing notion, then define the class $\mathbb{N}_P$ of P-names inductively, as follows: $\dot\sigma$ is a P-name if every $x \in \dot\sigma$ is of the form $(p, \dot\tau)$ for some $p \in P$ and some P-name $\dot\tau$. If $X \subseteq P$ and $\dot\sigma$ is a P-name, then define $\mathrm{val}_P(\dot\sigma, X)$ inductively, by $\mathrm{val}_P(\dot\sigma, X) = \{\mathrm{val}_P(\dot\tau, X) :$ there is some $p \in X$ and some $q \leq p$ with $(q, \dot\tau) \in \dot\sigma\}$.

If P is a forcing notion, then say that G is P-generic over $\mathbb{V}$ if G is a filter on P living in some larger model of set theory $\mathbb{W}$, and for every dense subset D of P with $D \in \mathbb{V}$, $G \cap D$ is nonempty. (It is equivalent that $G \cap A$ is nonempty, whenever $A \in \mathbb{V}$ is a a maximal antichain from P.)

Define $\mathbb{V}[G] \subseteq \mathbb{W}$ to be $\{\mathrm{val}_P(\dot\sigma, G) : \dot\sigma \in \mathbb{N}_P\}$. I claim $\mathbb{V} \subseteq \mathbb{V}[G]$. Indeed, given $X \in \mathbb{V}$, define $\check X \in \mathbb{N}_P$, the canonical name for X, inductively via $\check X = \{(\check x, 1) : x \in X\}$. Then obviously $\mathrm{val}_P(\check X, G) = X$.

We call $\mathbb{V}[G]$ a P-generic forcing extension of $\mathbb{V}$, and say that we have forced over P.

In the special case where $P = \mathcal{B}_+$ where $\mathcal{B}$ is a complete Boolean algebra, the definition of names can be simplified.

Definition 2.3.1. Suppose $\mathcal{B}$ is a complete Boolean algebra. Then define $\mathbb{N}_{\mathcal{B}}$, the set of nice $\mathcal{B}$-names, as follows: $\dot{\sigma} \in \mathbb{N}_{\mathcal{B}}$ if $\dot{\sigma}$ is a function from $\mathbb{V}$ into $\mathbb{N}_{\mathcal{B}}$. We have $\mathbb{N}_{\mathcal{B}} \subseteq \mathbb{N}_{\mathcal{B}_+}$, and further every $\mathcal{B}_+$-name is equivalent to a nice $\mathcal{B}$-name.

Generally, if P is a forcing notion, we say that $\dot{a}$ is a nice P-name if $\dot{a} \in \mathbb{N}_{\mathcal{B}}$; so we can always restrict to nice P-names without loss of generality.

Suppose $\mathcal{B}$ is a complete Boolean algebra, $\varphi(x_i : i < n)$ is a formula of set theory, and $\dot{\sigma}_i : i < n$ is a sequence of $\mathcal{B}$-names. Then define $\|\varphi(\dot{\sigma}_i : i < n)\|_{\mathcal{B}} \in \mathcal{B}$ inductively via:

- $\|\dot{\sigma}_0 \in \dot{\sigma}_1\|_{\mathcal{B}} = \bigvee_{\dot{\tau}_1 \in \mathrm{dom}(\dot{\sigma}_1)}(\dot{\sigma}_1(\dot{\tau}_1) \wedge \|\dot{\sigma}_0 = \dot{\tau}_1\|_{\mathcal{B}})$;
- $\|\dot{\sigma}_0 = \dot{\sigma}_1\|_{\mathcal{B}} = \bigwedge_{i<2}\left(\bigwedge_{\dot{\tau}_i \in \mathrm{dom}(\dot{\sigma}_i)}(\neg\dot{\tau}_i \vee \|\dot{\tau}_i \in \dot{\sigma}_{1-i}\|_{\mathcal{B}})\right)$;
- $\|\varphi(\dot{\sigma}_i : i < n) \wedge \psi(\dot{\sigma}_i : i < n)\|_{\mathcal{B}} = \|\varphi(\dot{\sigma}_i : i < n)\|_{\mathcal{B}} \wedge \|\psi(\dot{\sigma}_i : i < n)\|_{\mathcal{B}}$;
- $\|\neg\varphi(\dot{\sigma}_i : i < n)\|_{\mathcal{B}} = \neg\|\varphi(\dot{\sigma}_i : i < n)\|_{\mathcal{B}}$;
- $\|\exists x \varphi(x, \dot{\sigma}_i : i < n)\|_{\mathcal{B}} = \bigvee_{\dot{\sigma} \in \mathbb{V}^{\mathcal{B}}} \|\varphi(\dot{\sigma}, \dot{\sigma}_i : i < n)\|_{\mathcal{B}}$.

If P is a forcing notion and $p \in P$, say that p forces $\varphi(\dot{\sigma}_i : i < n)$, and write $p \Vdash_P \varphi(\dot{\sigma}_i) : i < n$, if $p \leq \|\varphi(\dot{\sigma}_i : i < n)\|_{\mathcal{B}(P)}$. We say that P forces $\varphi(\dot{\sigma}_i : i < n)$, and write $P \Vdash \varphi(\dot{\sigma}_i : i < n)$, if $1_P \Vdash_P \varphi(\dot{\sigma}_i : i < n)$.

The following key theorem on forcing is a straightforward induction, see Theorem 14.29 of [27]:

Theorem 2.3.2. Suppose P is a forcing notion, and $\mathbb{V}[G]$ is a P-generic forcing extension, and $\varphi(x_i : i < n)$ is a formula of set theory with a parameter for $\check{V}$, and $(\dot{\sigma}_i : i < n)$ is a sequence from $\mathbb{V}^P$, then $(\mathbb{V}[G], \mathbb{V}) \models \varphi(\mathrm{val}_P(\dot{\sigma}_i, G) : i < n)$ if and only if there is some $p \in G$ such that p forces $\varphi(\dot{\sigma}_i : i < n)$.

Also, the following is Theorem 14.24 of [27]:

Theorem 2.3.3. Suppose P is a forcing notion and φ is an axiom of ZFC. Then $P \Vdash \varphi$. In particular, if $\mathbb{V}[G]$ is a P-generic forcing extension, then $\mathbb{V}[G] \models ZFC$. Also, $\mathbb{V}[G]$ has the same ordinals as $\mathbb{V}$.

Finally, the following is one of many ways of justifying taking forcing extensions of $\mathbb{V}$ (see the next section for a discussion of ZFC^-):

Theorem 2.3.4. Suppose T is any extension of ZFC. Then T is equiconsistent with T_*, the theory in the language (V, W, ϵ), which asserts:

- $V \subseteq W$, and $\epsilon \subseteq W \times W$;
- $(V, \epsilon) \models T^*$;
- $(W, \epsilon) \models ZFC^- +$ every set is countable;
- V, W have the same ordinals.

Proof. Technically, we view this as a theorem of ZFC; i.e., we are working in some $\mathbb{V} \models ZFC$.

Clearly if T_* is consistent, then so is T. So suppose T is consistent. Let ZFC_0^- be a finite fragment of ZFC^-, and let ZFC_0 be a large enough finite fragment of ZFC; in particular we need ZFC_0 to prove that ZFC_0^- is forced by the Levy collapse of the ordinals.

Choose $\varphi \in T$. Let $(ZFC_0 + \varphi)_*$ be defined from $ZFC_0 + \varphi$ in the same way T_* is defined from T, except replacing ZFC^- with ZFC_0^-. We show that $(ZFC_0 + \varphi)_*$ is consistent, which suffices.

Choose $V_* \models T$. V_* need not be well-founded, but nonetheless it suffices to show that V_* believes $(ZFC_0 + \varphi)_*$ is consistent. We work within V_* and forget that V_* is possibly ill-founded; so we are assuming that $\mathbb{V} \models T$, and we are trying to show that $(ZFC_0 + \varphi)_*$ is consistent. Choose a countable transitive model V of $ZFC_0 + \varphi$, let P^V be the Levy collapse of the ordinals (a class forcing notion in V), and let $V[G]$ be a forcing extension by P^V. Then $(V[G], V, \in) \models (ZFC_0 + \varphi)_*$. $\square$

Note that if $(\mathbb{V}, \mathbb{W}, \in) \models T^*$, then for every forcing notion $P \in \mathbb{V}$, $(\mathcal{P}(P))^{\mathbb{V}}$ is countable in $\mathbb{W}$, so it is easy to construct a P-generic filter over $\mathbb{V}$ in $\mathbb{W}$. Henceforward, we will make use of this to freely suppose that P-generic forcing extensions of $\mathbb{V}$ exist.

A frequent theme in forcing is to understand how combinatorial properties of the forcing notion P correspond to properties of P-generic forcing extensions. For example, we have the following consequences of distributivity and the chain condition:

The following is Theorem 15.6 of [27]:

Theorem 2.3.5. Suppose P is θ-distributive, P forces that $\dot{f} : \theta \to \check{\mathbb{V}}$. Then P forces that $\dot{f} \in \check{\mathbb{V}}$. So forcing by P does not add θ-sequences.

Thus, if P is $< \theta$-distributive, then whenever $\kappa \leq \theta$ is a cardinal, P forces that κ remains a cardinal.

And the following is Lemma VII.6.8 of [44]:

Theorem 2.3.6. Suppose P is κ-c.c. and $\dot{f}$ is a P-name such that for some set X, P forces $\mathrm{dom}(\dot{f}) = \check{X}$, then we can find some map F with domain X, such that for all $x \in X$,

$|F(x)| < \kappa$, and such that P forces that for all $\check{x} \in \check{X}$, $\dot{f}(\check{x}) \in \check{F}(\check{x})$.

In particular, if $\lambda \geq \kappa$ is a cardinal then P forces that λ remains a cardinal.

2.4 ZFC^-

Let $\mathcal{L}_\in = \{\in\}$ be the language of set theory. Our metatheory will always be ZFC. Frequently we will need to work in transitive models of set theory; but there are not guaranteed to be set models of ZFC. ZFC^- is a convenient fragment of ZFC for this purpose.

There are some pathologies to be avoided in the definition of ZFC^-. In particular, the naïve formulation of ZFC^- where one just removes power set is ill-behaved, see [15]. The problem is that various formulations of our axioms are no longer equivalent: we really want collection instead of replacement, and we want the well-ordering principle instead of the axiom of choice.

Definition 2.4.1. Let ZFC^- be ZFC but: remove power set, and strengthen choice to the well-ordering principle, and strengthen replacement to the collection principle (this is as in [15]).

Example 2.4.2. If χ is a regular cardinal, then $H(\chi) \models ZFC^-$, where $H(\chi)$ is the set of sets of hereditary cardinality less than χ. Thus, if A is any transitive set, then there is some transitive $V \models ZFC^-$ with $|V| = |\mathrm{tcl}(A)| + \aleph_0$.

Note that we usually denote $H(\aleph_1)$ as HC.

Most arguments that do not appeal explicitly appeal to powerset go through in ZFC^-. For instance, successor cardinals are regular. Transfinite induction works fine. Every set X is in bijection with an ordinal α; thus it makes sense to define the cardinality of X to be the least such ordinal α.

The following lemma must be reproven for every fragment of ZFC one works with. For ZFC^- it is standard; for example, it is a (very) special case of theorems proved in [24].

Lemma 2.4.3. Suppose $\mathbb{V} \models ZFC^-$, and suppose P is a forcing notion. Then the forcing theorem holds for P, in other words: we have a definable forcing relation $\Vdash_P$ in $\mathbb{V}$, and if G is P-generic over $\mathbb{V}$, then $\mathbb{V}[G] \models \varphi(\dot{a}_1, \ldots, \dot{a}_n)$ if and only if there is some $p \in G$ which forces $\varphi(\dot{a}_1, \ldots \dot{a}_n)$. Also, if G is P-generic over $\mathbb{V}$, $\mathbb{V}[G] \models ZFC^-$.

2.5 Density and Independence

The following is a slight generalization of the classical Hewitt-Marczewski-Pondiczery theorem of topology; see [10] for a reference.

Theorem 2.5.1. Suppose $\theta \leq \mu \leq \lambda$ are infinite cardinals such that θ is regular, $\mu = \mu^{<\theta}$ and $\lambda \leq 2^\mu$. Suppose $(X_\alpha : \alpha < \lambda)$ are topological spaces such that for each $\alpha < \lambda$, X_α has a dense subset of size at most μ. Let X be the $< \theta$-support product of $(X_\alpha : \alpha < \lambda)$. Then X has a dense subset of size at most μ.

In [10], only the classical case where $\theta = \aleph_0$ is explicitly considered, but as remarked there, the generalization is easy.

We will be interested in the following special case. Actually, it can be used to prove Theorem 2.5.1, as is done in [10] (although historically, Theorem 2.5.1 was proved first).

Corollary 2.5.2. Suppose $\theta \leq \mu \leq \lambda$ are infinite cardinals such that θ is regular, $\mu = \mu^{<\theta}$, and $\lambda \leq 2^\mu$. Then there is a sequence $(\mathbf{f}_\gamma : \gamma < \mu)$ from ${}^\lambda\mu$ such that for all partial functions f from λ to μ of cardinality θ, there is some $\gamma < \mu$ such that $\mathbf{f}_\gamma$ extends f.

We now discuss the alternative viewpoint of independent antichains, and use them to prove Corollary 2.5.2, as in [10].

Definition 2.5.3. Suppose $\mathcal{B}$ is a Boolean algebra. Suppose $\mathbb{C}$ is a family of maximal antichains of $\mathcal{B}$. Then say that $\mathbb{C}$ is a θ-independent family of maximal antichains if for every $s \in [\mathbb{C}]^{<\theta}$, and for each choice function f on s (i.e. $f(\mathbf{C}) \in \mathbf{C}$ for all $\mathbf{C} \in s$), $\bigwedge_{\mathbf{C} \in s} f(\mathbf{C})$ is nonzero. If we omit θ then we mean $\theta = \aleph_0$.

The key observation of [10] is that Theorem 2.5.1 and Corollary 2.5.2 are actually both reformulations of the following combinatorial statement. In [10], just the case $\mathcal{B} = \mathcal{P}(\lambda)$ is considered, but the general case is the same.

Theorem 2.5.4. Suppose $\mathcal{B}$ is a complete Boolean algebra with an antichain of size λ. Suppose $\theta \leq \lambda$ and $\lambda = \lambda^{<\theta}$. Then $\mathcal{B}$ admits a θ-independent family $\mathbb{C}$ of maximal antichains with $|\mathbb{C}| = 2^\lambda$; moreover we can arrange each $\mathbf{C} \in \mathbb{C}$ has cardinality λ.

Proof. Let I be the set of all pairs (s, r) where $s \in [\lambda]^{<\theta}$ and $r : \mathcal{P}(s) \to \lambda$. Then $|I| = \lambda$.

We can find a maximal antichain $(\mathbf{a}_{s,r} : (s, r) \in I)$ of $\mathcal{B}$. (Suppose $(\mathbf{a}_\gamma : 1 \leq \gamma < \lambda)$ be an antichain which is not maximal, and define $\mathbf{a}_0 = \neg \left(\bigvee_{\gamma \geq 1} \mathbf{a}_\gamma\right)$; this is a maximal antichain, and then we can reindex.)

For each $A \subset \lambda$ and for each $\alpha < \lambda$, let $\mathbf{c}_{A,\alpha} = \bigvee\{\mathbf{a}_{(s,r)} : (s, r) \in I, r(A \cap s) = \alpha\}$ and let $\mathbf{C}_A = \{\mathbf{c}_{A,\alpha} : \alpha < \lambda\}$, and finally let $\mathbb{C} = \{\mathbf{C}_A : A \subseteq \lambda\}$. I claim this works. Clearly each $\mathbf{C}_A$ is an antichain of size λ, and clearly $|\mathbb{C}| = 2^\lambda$, so it suffices to show that $\mathbb{C}$ is θ-independent. So suppose $(A_\gamma : \gamma < \gamma_*)$ is a sequence of distinct subsets of λ of length $\gamma_* < \theta$, and suppose $\mathbf{c}_{A_\gamma, \alpha_\gamma} \in \mathbf{C}_{A_\gamma}$ for each $\gamma < \gamma_*$. Choose $s \in [\lambda]^{<\theta}$ large enough so that each $A_\gamma \cap s \neq A_{\gamma'} \cap s$, for $\gamma \neq \gamma'$. Choose $r : \mathcal{P}(s) \to \lambda$ such that for all $\gamma < \gamma_*$, $r(A_\gamma \cap s) = \alpha_\gamma$. Then $\mathbf{a}_{(s,r)} \leq \mathbf{c}_{A_\gamma,\alpha_\gamma}$ for all $\gamma < \gamma_*$, witnessing that $\bigwedge_{\gamma < \gamma_*} \mathbf{c}_{A_\gamma,\alpha_\gamma} \neq \emptyset$. $\square$

We now give a proof of Corollary 2.5.2, following [10].

It suffices to consider the case $\lambda = 2^\mu$. So suppose θ is regular and $\mu = \mu^{<\theta}$. By

Theorem 2.5.4 applied to $\mathcal{B} = \mathcal{P}(\mu)$, we can find a θ-independent family $\mathbb{C}$ of maximal antichains of $\mathcal{P}(\mu)$ with $|\mathbb{C}| = 2^\mu$, such that each for each $\mathbf{C} \in \mathbb{C}$, $|\mathbf{C}| = \mu$. Enumerate $\mathbb{C} = (\mathbf{C}_\alpha : \alpha < 2^\mu)$, and for each α let $F_\alpha : \mu \to \mu$ be surjective, so that $\{F_\alpha^{-1}(\gamma) : \gamma < \mu\}$ enumerates $\mathbf{C}_\alpha$. For each $\gamma < \mu$, define $\mathbf{f}_\gamma : 2^\mu \to \mu$ via $\mathbf{f}_\gamma(\alpha) = F_\alpha(\gamma)$. I claim this works. Indeed, suppose f is a function from 2^μ to μ of cardinality less than θ. The θ-independence of $\mathbb{F}$ just means that $\bigcap_{\alpha \in \mathrm{dom}(f)} F_\alpha^{-1}(f(\alpha))$ is nonempty. Choose γ in the intersection; so for all $\alpha \in \mathrm{dom}(f)$, $F_\alpha(\gamma) = f(\alpha)$. This just means that $\mathbf{f}_\gamma$ extends f.

2.6 Model-Theoretic Notation and Terminology

Our model-theoretic notation is standard; see, for instance, [37]. We will typically be dealing with a complete first order theory in a countable language; in this case, $\mathfrak{C}$ denotes its monster model. $\overline{x}, \overline{y}$ denote finite tuples of variables. Sometimes, to reduce visual clutter, x, y are also used to denote finite tuples of variables, but this is always explicitly stated; by default, x, y are single variables.

For the reader's convenient reference, we give a list of some of the model theoretic terminology we will be using, and the implications between them.

- OP is the order property; a theory is stable if it does not have the order property, so usually we write unstable instead of OP. See [37].

- IP is the independence property; SOP is the strict order property. NIP and $NSOP$ are their negations, respectively. A theory is stable if and only if it is NIP and $NSOP$. The canonical example of a theory with IP is T_{rg}, the theory of the random graph. The canonical example of a theory with SOP is $\mathrm{Th}(\mathbb{Q}, <)$, the theory of dense linear orders. See [37].

- FCP is the finite cover property; $NFCP$ is its negation. We use the definition of finite cover property from [75]; thus $NFCP$ implies stability, but typically we will write stable without the finite cover property, for emphasis. See [75].

- $SOP_n : n \geq 1$ is a family of strict order properties weakening SOP, with negations $NSOP_n$. We have $SOP \to SOP_{n+1} \to SOP_n$ for all n. For $n \geq 3$ the implications are strict, but all of the implications $SOP_1 \to SOP_2 \to SOP_3$ are open. See [38].

- TP is the tree property. A theory is simple if it does not have the tree property, so we usually write unsimple instead of TP. Simplicity is a weaking of stability; for example, T_{rg} is a simple, unstable theory. See [37].

- TP_1, TP_2 are the tree properties of the first and second kind, respectively, with negations NTP_1, NTP_2. TP_1 is equivalent to SOP_2, and henceforward we use SOP_2 exclusively. A theory is simple if and only if it is $NSOP_2$ and NTP_2, in particular, simple theories are all $NSOP$. See [38].

- Lowness is a property intermediate between stability and simplicity. Most natural examples of simple theories are low, for instance T_{rg} is low. See [37].

Chapter 3: A Survey on Keisler's Order

In this chapter, we present a self-contained and systematic treatment of Keisler's order, in what we believe to be the most logical fashion. Ultimately this means moving away from thinking about regular ultrafilters on $\mathcal{P}(\lambda)$, and instead focusing on arbitrary ultrafilters on complete Boolean algebras. Our treatment revolves around the notion of full $\mathcal{B}$-valued structures.

3.1 A History of Keisler's Order

We give a brief history of Keisler's order.

First of all, we review the ultrapower construction. Suppose M is an $\mathcal{L}$-structure, λ is a cardinal (or generally any index set), and $\mathcal{U}$ is an ultrafilter on $\mathcal{P}(\lambda)$. Then we can form the ultrapower $M^\lambda/\mathcal{U}$ of M by $\mathcal{U}$ as follows. $M^\lambda/\mathcal{U}$ will be an $\mathcal{L}$-structure with universe M^λ/E, where M^λ is the set of all functions $f : \lambda \to M$, and where E is the equivalence relation defined by: fEg if and only if $\{\alpha < \lambda : f(\alpha) = g(\alpha)\} \in \mathcal{U}$. Then given an n-ary relation $R \in \mathcal{L}$, we put $([f_i]_E : i < n) \in R^{M^\lambda/\mathcal{U}}$ if and only if $\{\alpha < \lambda : R^M(f_i(\alpha) : i < n)\} \in \mathcal{U}$, and similarly for function symbols (we will spell out the details in greater generality later). We have a natural embedding $\mathbf{j} : M \to M^\lambda/\mathcal{U}$ sending $a \in M$ to the constant map $f_a : \lambda \to \{a\}$. Then $\mathbf{j}$ is in fact an elementary embedding; this is Łoś's theorem, and $\mathbf{j}$ is called the Łoś embedding [48].

We wish to understand what $M^\lambda/\mathcal{U}$ looks like. As a test question: is $M^\lambda/\mathcal{U}$ λ^+-

saturated?

In order to obtain a satisfactory answer for an arbitrary structure M, we need some regularity hypothesis:

Definition 3.1.1. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{P}(\lambda)$. Then say that $\mathcal{U}$ is λ-regular if there is a sequence $(X_\alpha : \alpha < \lambda)$ from $\mathcal{U}$, such that for every infinite $I \subseteq \lambda$, $\bigcap_{\alpha \in I} X_\alpha = \emptyset$.

Keisler proved the following fundamental theorem in [34]:

Theorem 3.1.2. Suppose T is a complete countable theory, and $\mathcal{U}$ is an ultrafilter on $\mathcal{P}(\lambda)$, and $M_0, M_1 \models T$. Then the following both hold:

(A) If $\mathcal{U}$ is λ-regular, then $M_0^\lambda/\mathcal{U}$ is λ^+-saturated if and only if $M_1^\lambda/\mathcal{U}$ is.

(B) If M_0, M_1 are both λ^+-saturated, then $M_0^\lambda/\mathcal{U}$ is λ^+-saturated if and only if $M_1^\lambda/\mathcal{U}$ is.

This gives many new semantic properties to investigate. In particular, given a λ-regular ultrafilter $\mathcal{U}$ on $\mathcal{P}(\lambda)$, one can ask: for which complete countable theories T is it true that for some or any $M \models T$, $M^\lambda/\mathcal{U}$ is λ^+-saturated? We view this as detecting how difficult it is to saturate models of T.

Motivated by these questions, Keisler made the following definitions:

Definition 3.1.3. Suppose $\mathcal{U}$ is a λ-regular ultrafilter on $\mathcal{P}(\lambda)$. Then say that $\mathcal{U}$ λ^+-saturates T if for some or every $M \models T$, $M^\lambda/\mathcal{U}$ is λ^+-saturated.

Given complete countable theories T_0, T_1, say that $T_0 \trianglelefteq_\lambda T_1$ if whenever $\mathcal{U}$ is a λ-regular ultrafilter on $\mathcal{P}(\lambda)$, if $\mathcal{U}$ λ^+-saturates T_1 then $\mathcal{U}$ λ^+-saturates T_0. Say that $T_0 \trianglelefteq T_1$ if $T_0 \trianglelefteq_\lambda T_1$ for all λ.

$\trianglelefteq$ is called Keisler's order; we view $T_0 \trianglelefteq T_1$ as meaning that it is easier to saturate models of T_0 than those of T_1. We are interested in the dividing lines induced by Keisler's order. More precisely:

Definition 3.1.4. Temporarily let $\mathbf{T}$ denote the set of all complete countable theories; so $(\mathbf{T}, \trianglelefteq)$ is a preorder. Say that $\mathbf{D}$ is a dividing line in $\trianglelefteq$ if $\mathbf{D} \subseteq \mathbf{T}$ is downward-closed under $\trianglelefteq$ (this much actually makes sense for any preorder). $\mathbf{D}$ is a principal dividing line if there is a single λ-regular ultrafilter $\mathcal{U}$ on some $\mathcal{P}(\lambda)$, such that $\mathbf{D}$ is the set of all $T \in \mathbf{T}$ which are λ^+-saturated by $\mathcal{U}$.

We wish to understand what the principal dividing lines in $\trianglelefteq$ are; these, in turn, would determine $\trianglelefteq$. This question has attracted a lot of attention, but progress has been, until recently, rather slow.

We summarize most of what is known on Keisler's order. In [34], Keisler proved that there is a maximal class in $\trianglelefteq$, and it includes $\mathrm{Th}(\mathcal{P}(\omega))$. Further, he proved that there is a minimal class in $\trianglelefteq$; in fact, it is not hard to see that any uncountably categorical theory is minimal. Finally, Keisler showed that any theory with the finite cover property is not minimal:

Matters stood there for a while; so it was conceivable at this point that Keisler's order had only two classes. Then, Shelah illuminated the situation considerably in [75] with the following two theorems:

Theorem 3.1.5. The $\trianglelefteq$-minimal class of theories is the class of stable theories without the finite cover property. The next-least $\trianglelefteq$-class of theories (exists and) is the class of stable theories with the finite cover property. In other words:

(A) If T_0 is stable without the finite cover property, then for all T_1, $T_0 \trianglelefteq T_1$.

(B) If T_1 is stable with the finite cover property, and T_2 is either unstable or is stable with the finite cover property, then $T_1 \trianglelefteq T_2$.

(C) If T_0 is stable without the finite cover property, and T_1 is stable with the finite cover property, and T_2 is unstable, then $T_1 \ntrianglelefteq T_0$ and $T_2 \ntrianglelefteq T_1$. In fact, the finite cover property is a principal dividing line in $\trianglelefteq$, as is stability.

Theorem 3.1.6. If T is a complete countable theory with SOP, then T is maximal in $\trianglelefteq$.

Thus, it follows that $\trianglelefteq$ has at least three classes (where two theories T_0, T_1 are Keisler-equivalent if $T_0 \trianglelefteq T_1 \trianglelefteq T_0$). Shelah also showed in [75] that consistently, T_{rg}, the theory of the random graph, is not $\trianglelefteq$-maximal; thus, consistently there are at least four classes. Shelah asked if these were all of them. Apparently [58] this was viewed as likely.

There were two key technical innovations in Shelah's work in [75], although both were in nascent forms. Note that Theorem 3.1.5(A), (B) and Theorem 3.1.6 both concern positive reductions in $\trianglelefteq$, while Theorem 3.1.5(C) involves a negative reduction. It turns out these involve almost completely different toolsets. Work on Keisler's order has since bifurcated along these two directions. We first describe the positive reduction aspect, and trace its development to the present; then we discuss the negative reduction aspect.

First, to prove the positive reductions above, Shelah made the conceptual shift to studying the ultrapower $M^\lambda/\mathcal{U}$ as situated within the class ultrapower $\mathbb{V}^\lambda/\mathcal{U}$. As notation, let $\mathbf{j} : \mathbb{V} \preceq \mathbb{V}^\lambda/\mathcal{U}$ be the Łoś-embedding. It is not quite true that $M^\lambda/\mathcal{U} = \mathbf{j}(M)$; this is because $\mathbf{j}(M)$ is, in $\mathbb{V}^\lambda/\mathcal{U}$, a $\mathbf{j}(\mathcal{L})$-structure. If $\mathcal{L}$ is infinite, then $\mathbf{j}(\mathcal{L})$ will contain extra symbols; if the arities of the symbols in $\mathcal{L}$ are unbounded, then some of the "symbols" of $\mathbf{j}(\mathcal{L})$ will have nonstandard arity. But if we let $\mathbf{j}_{\mathrm{std}}(M)$ denote the reduct of $\mathbf{j}(M)$ to $\mathcal{L}$, then $M^\lambda/\mathcal{U} \cong \mathbf{j}_{\mathrm{std}}(M)$.

Now, suppose $p(x)$ is a partial type over $M^\lambda/\mathcal{U}$ of cardinality at most λ. Then one can try to find some set $\check{p}(x) \in \mathbb{V}^\lambda/\mathcal{U}$, such that $\mathbb{V}^\lambda/\mathcal{U}$ believes $\check{p}(x)$ is a consistent, finite set of $\mathbf{j}(\mathcal{L})$-formulas over $\mathbf{j}(M)$, and such that $p(x) \subseteq \check{p}(x)$. Then necessarily, $\mathbb{V}^\lambda/\mathcal{U}$ must believe that $\check{p}(x)$ is realized in $\mathbf{j}(M)$. This implies $p(x)$ is realized in $M^\lambda/\mathcal{U}$. In [75], for example, Shelah constructs $\check{p}(x)$ as a union of a chain of approximations; at limit stages, we need some saturation hypothesis on $\mathbb{V}^\lambda/\mathcal{U}$, which is met if $\mathcal{U}$ λ^+-saturates some theory with SOP.

In [78], Shelah generalized this approach to studying Keisler's order by introducing the interpretability order $\trianglelefteq^*_\kappa$. Actually, in [78], only the case $\kappa = \aleph_1$ is considered. As a convention (following [61]), we define that every structure M is 1-saturated.

Definition 3.1.7. Suppose T, T_* are complete countable theories with monster models $\mathfrak{C}, \mathfrak{C}_*$ respectively. Then an interpretation of T in T_* is given by some definable subset X of $\mathbf{C}^n_*$, and for each m-ary relation symbol $R \in \mathcal{L}_T$, an m-ary definable subset $R_* \subseteq X^m$, and for each m-ary function symbol $f \in \mathcal{L}_T$, an m-ary definable function $f_* : X^m \to X$, such that $(X, \ldots) \models T$. Given $M_* \models T_*$ we always get an interpreted model $M \models T$.

Suppose T_0, T_1 are complete countable theories. Suppose κ is either an infinite cardinal or else 1. Then say that $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$ if there is some countable theory T_* interpreting both T_0 and T_1, such that for all κ-saturated $M_* \models T_*$, if we let M_i be the interpreted model of T_i, then if M_1 is λ^+-saturated, so is M_0. Say that $T_0 \trianglelefteq^*_\kappa T_1$ if $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$ for all λ.

So, for $\kappa < \kappa'$, $\trianglelefteq^*_\kappa \subseteq \trianglelefteq^*_{\kappa'}$.

We warn the reader that differing indexing systems for $\trianglelefteq^*_\kappa$ have been used in later papers (for instance, in [61]); for these other indexing systems, we can no longer prove that $\trianglelefteq^*_{\aleph_1} \subseteq \trianglelefteq$, and so we stick to Shelah's original formulation from [78] (which in any case seems more natural). The differences among these versions are minor, and the reader attached to the alternative versions will have no problem adapting our arguments.

The reason these orders were conceived was due to the following: if $T_0 \trianglelefteq^*_{\lambda\aleph_1} T_1$ then $T_0 \trianglelefteq_\lambda T_1$, i.e. $\trianglelefteq^*_{\lambda\aleph_1} \subseteq \trianglelefteq_\lambda$. Indeed, suppose T_* witnessed $T_0 \trianglelefteq^*_{\lambda\aleph_1} T_1$, and let $\mathcal{U}$ be a λ-regular ultrafilter on $\mathcal{P}(\lambda)$. Let $M_* \models T_*$. Let M_i be the interpreted model of T_i. Since $\mathcal{U}$ is λ-regular, it is $\aleph_1$-incomplete, and so as proved by Keisler [34], $M_*^\lambda/\mathcal{U}$ is $\aleph_1$-saturated. But $M_i^\lambda/\mathcal{U}$ is (isomorphic to) the interpreted model of T_i in $M_*^\lambda/\mathcal{U}$, and if $M_1^\lambda/\mathcal{U}$ is λ^+-saturated, so is M_0. Thus, if $\mathcal{U}$ λ^+-saturates T_1, then it also λ^+-saturates T_0.

It follows that $\trianglelefteq^*_{\aleph_1} \subseteq \trianglelefteq$. In all cases where we can prove $T_0 \trianglelefteq T_1$, we can actually prove $T_0 \trianglelefteq^*_{\aleph_1} T_1$ (although frequently with extra work); thus, $\trianglelefteq^*_{\aleph_1}$ captures all of the currently known techniques for proving positive reductions in Keisler's order.

In [78], Shelah notes that Theorem 3.1.5 holds for $\trianglelefteq^*_{\aleph_1}$, and strengthens Theorem 3.1.6 to show that any theory with SOP_3 is maximal in $\trianglelefteq^*_{\aleph_1}$ (and hence in $\trianglelefteq$).

The interpretability orders were further investigated in [8] (Džamonja and Shelah) and [80] (Shelah and Usvyatsov), which together imply that any theory T with $NSOP_2$ is nonmaximal in $\trianglelefteq^*_1$. Note that this is a negative reduction result, namely: if T has $NSOP_2$ then $\mathrm{Th}(\mathbb{Q}, <) \ntrianglelefteq^* T$. Thus the corresponding statement for $\trianglelefteq$ does not follow, and in fact it is a major open problem whether or not theories with $NSOP_2$ must be nonmaximal in Keisler's order.

Malliaris entered the scene with [53], where she proved the following:

Theorem 3.1.8. Suppose $\mathcal{U}$ is a λ-regular ultrafilter on $\mathcal{P}(\lambda)$ and M is a structure. Then $M^\lambda/\mathcal{U}$ is λ^+-saturated if and only if $M^\lambda/\mathcal{U}$ is locally λ^+-saturated; i.e. for every formula $\varphi(x, \overline{y})$, every positive φ-type over M of cardinality at most λ is λ^+-saturated.

One can summarize this as saying "Keisler's order is local." This is compelling evidence that Keisler's order detects model-theoretically significant dividing lines, in particular that it has a syntactic formulation.

In [49] [50] [51], Malliaris used similar techniques to obtain several further results, including: there is a $\trianglelefteq$-minimal unstable theory (for instance, the theory of the random graph), and there is a $\trianglelefteq$-minimal TP_2 theory. These results were later extended to $\trianglelefteq_1^*$ by Malliaris and Shelah in [61].

Recently, Malliaris and Shelah prove in [54] that every SOP_2 theory is $\trianglelefteq$-maximal. This proof is substantially more involved than Shelah's proof in [78] that SOP_3-theories are $\trianglelefteq$-maximal. Remarkably, Malliaris and Shelah were able to leverage the same techniques to solve a seemingly unrelated problem on cardinal invariants of the continuum, namely they showed that $\mathfrak{p} = \mathfrak{t}$. Soon after, in [59], Malliaris and Shelah show that theories with SOP_2 are $\trianglelefteq^*_{\aleph_1}$-maximal.

The second key innovation in Shelah's work on Keisler's order in [75] was for proving the negative reduction, Theorem 3.1.5(C). This amounts to the construction of special ultrafilters. Shelah's insight was to find a λ-regular filter $\mathcal{D}$ on $\mathcal{P}(\lambda)$, such that $\mathcal{P}(\lambda)/\mathcal{D}$, considered as a Boolean algebra, has the $\aleph_1$-c.c, but is still sufficiently rich.

In the papers [60], [55], and [56], Malliaris and Shelah clarified this construction technique: namely, they translated the problem of constructing λ-regular ultrafilters on $\mathcal{P}(\lambda)$ to constructing arbitrary ultrafilters on complete Boolean algebras $\mathcal{B}$.

Specifically, given an ultrafilter $\mathcal{U}$ on the complete Boolean algebra $\mathcal{B}$ and a complete countable theory T, Malliaris and Shelah define in [56] what it means for $\mathcal{U}$ to be $(\lambda, \mathcal{B}, T)$ moral; one should think of this as an abstract version of $\mathcal{U}$ λ^+-saturating T, although the definition is purely combinatorial and does not mention saturation. For ultrafilters on $\mathcal{P}(\lambda)$ at least, one can give a nice definition: $\mathcal{U}$ is $(\lambda, \mathcal{P}(\lambda), T)$ moral if and only if for some or any λ^+-saturated $M \models T$, $M^\lambda/\mathcal{U}$ is λ^+-saturated. (The choice of M does not matter, by Theorem 3.1.2(B).) In particular, if $\mathcal{U}$ is λ-regular, then this is the same as $\mathcal{U}$ λ^+-saturating T.

In [56], Malliaris and Shelah prove the following (it is the key consequence of their Existence Theorem and Separation of Variables Theorem):

Theorem 3.1.9. Suppose $\mathcal{B}$ is a complete Boolean algebra with the λ^+-c.c. and with $|\mathcal{B}| \leq 2^\lambda$. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then there is a λ-regular ultrafilter $\mathcal{U}_*$ on $\mathcal{P}(\lambda)$, such that for every complete countable theory T, $\mathcal{U}_*$ λ^+-saturates T if and only if $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$ moral.

The strategy set forth for ultrafilter constructions in [56] is as follows: find a Boolean algebra $\mathcal{B}$ which is small in some sense (in [56], $\mathcal{B}$ has the $\aleph_1$-c.c.) and construct an ultrafilter $\mathcal{U}$ on $\mathcal{B}$ which is as generic as possible. Then the smallness of $\mathcal{B}$ will prevent $\mathcal{U}$ from being $(\lambda, \mathcal{B}, T)$ moral for some T, and its genericity will ensure that $\mathcal{U}$ is $(\lambda, \mathcal{B}, T')$-moral for other T'.

In [56], Malliaris and Shelah use these ideas to show that $T \not\trianglelefteq T_{rg}$ whenever T is nonlow. Then, in [57], they refine the technique to show that if there is a supercompact cardinal, then simplicity is a principal dividing line in $\trianglelefteq$.

In [58], Malliaris and Shelah push these techniques further to show that Keisler's order has infinitely many classes. Namely, for each $n > k \geq 3$, let $T_{n,k}$ be the theory of the random k-ary n-clique free graph (Malliaris and Shelah subtract 1 from the indices). They showed that for $3 \leq k' < k - 1$, $T_{k'+1,k'} \not\trianglelefteq T_{k+1,k}$. The idea here is that $T_{k+1,k}$ fails k-dimensional amalgamation, which is worse when k is small. Note that there is a gap; Malliaris and Shelah left open whether or not $T_{k+1,k} \trianglelefteq T_{k+2,k+1}$ is possible.

In [88], I show that if there is a supercompact cardinal, then Keisler's order is not linear, using the techniques from results in [57] and [58]; this was obtained independently by Malliaris and Shelah [61]. Also, I show in [87] that lowness is a principal dividing line in Keisler's order (with no set-theoretic hypotheses).

3.2 The Approach Via Boolean-Valued Models

Suppose we have an ultrafilter $\mathcal{U}$ on the complete Boolean algebra $\mathcal{B}$. We would like to understand what it means for $\mathcal{U}$ to be $(\lambda, \mathcal{B}, T)$-moral. (The combinatorial definition of Malliaris and Shelah is rather technical.)

One can try to characterize this in terms of ultrapowers: suppose $M \models T$. Then one can form the Boolean-valued ultrapower $M^{\mathcal{B}}/\mathcal{U}$ similarly to the case when $\mathcal{B} = \mathcal{P}(\lambda)$ (in particular, if $\mathcal{B} = \mathcal{P}(\lambda)$ then $M^{\mathcal{P}(\lambda)}/\mathcal{U} \cong M^\lambda/\mathcal{U}$). These generalized ultrapowers have been around for a while, for instance they were investigated by Mansfield [62]. It is natural to ask:

Question. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$ and T is a complete countable theory. Are the following equivalent?

- $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-moral.
- For some or every λ^+-saturated $M \models T$, $M^{\mathcal{B}}/\mathcal{U}$ is λ^+-saturated.

As remarked above, this is true when $\mathcal{U} = \mathcal{P}(\lambda)$. When $\mathcal{B}$ is not λ^+-distributive, however, the proof breaks down, and as far as we know it is open if this holds in general.

Nonetheless, there is another way of making sense of $\mathcal{U}$ being $(\lambda, \mathcal{B}, T)$-moral, namely through full $\mathcal{B}$-valued structures.

To give the reader an idea of what these objects are: suppose first M is an ordinary $\mathcal{L}$-structure. Let $\mathcal{L}(M)$ be the set of formulas with parameters from M. View the elementary diagram of M as a map $\|\cdot\|_M$ from $\mathcal{L}(M)$ into the complete Boolean algebra $\{0,1\}$. To get a full $\mathcal{B}$-valued $\mathcal{L}$-structure, replace $\{0,1\}$ by $\mathcal{B}$. (The formal definition is given in Section 3.3.) So, a full $\mathcal{B}$-valued $\mathcal{L}$-structure $\mathbf{M}$ is in particular a pair $(\mathbf{M}, \|\cdot\|)$ where $\|\cdot\| : \mathcal{L}(\mathbf{M}) \to \mathcal{B}$. We say that $\mathbf{M}$ is a full $\mathcal{B}$-valued model of T, and write $\mathbf{M} \models^{\mathcal{B}} T$, if $\|\varphi\|_{\mathbf{M}} = 1$ for all $\varphi \in T$. For example, if M is an ordinary $\mathcal{L}$-structure, then M^λ is a full $\mathcal{P}(\lambda)$-valued structure, with $\|\varphi(f_i : i < n)\|_{M^\lambda} = \{\alpha < \lambda : M \models \varphi(f_i(\alpha) : i < n)\}$. More generally, if $M \models T$ and if $\mathcal{B}$ is a complete Boolean algebra, then $M^{\mathcal{B}} \models^{\mathcal{B}} T$.

In Section 3.3, we prove Corollary 3.3.8, a compactness theorem for full $\mathcal{B}$-valued models; this is the cornerstone of our development. The following is a simplified version:

Theorem 3.2.1. Suppose $\mathcal{B}$ is a complete Boolean algebra, X is a set, and $F : \mathcal{L}(X) \to \mathcal{B}$. Then the following are equivalent:

(A) There is some full $\mathcal{B}$-valued structure $\mathbf{M} \supseteq X$ such that $\|\cdot\|_{\mathbf{M}}$ extends F;

(B) For every finite $\Gamma \subseteq \mathcal{L}(X)$, there is some full $\mathcal{B}$-valued structure $\mathbf{M} \supseteq X$ such that $\|\cdot\|_{\mathbf{M}}$ extends $F \restriction_\Gamma$.

In Sections 3.5 and 3.6, we prove the following:

Theorem 3.2.2. Suppose T is a complete countable theory, $\mathcal{B}$ is a complete Boolean algebra, and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then the following are equivalent, for all λ:

(I) $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-moral;

(II) For some λ^+-saturated full $\mathcal{B}$-valued model $\mathbf{M}$ of T, the specialization $\mathbf{M}/\mathcal{U}$ is λ^+-saturated;

(III) For every λ^+-saturated full $\mathcal{B}$-valued model $\mathbf{M}$ of T, the specialization $\mathbf{M}/\mathcal{U}$ is λ^+-saturated.

We believe this justifies the following definition:

Definition 3.2.3. Suppose T is a complete countable theory, and $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$. Then $\mathcal{U}$ λ^+-saturates T if $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-moral.

So if $\mathcal{U}$ is an ultrafilter on $\mathcal{P}(\lambda)$, then $\mathcal{U}$ λ^+-saturates T if and only if for some or any λ^+-saturated $M \models T$, $M^\lambda/\mathcal{U}$ is λ^+-saturated; if $\mathcal{U}$ is λ-regular, then the saturation condition on M can be dropped.

In view of Theorem 1.1.3, it would be equivalent to change the definition of $\trianglelefteq$ to allow the ultrafilter $\mathcal{U}$ to be on any complete Boolean algebra $\mathcal{B}$ with the λ^+-c.c. and with $|\mathcal{B}| \leq 2^\lambda$. Actually, in Corollary 3.16.20, we strengthen this to show that the hypothesis $|\mathcal{B}| \leq 2^\lambda$ can be dropped. Thus:

Theorem 3.2.4. Suppose T_0, T_1 are complete countable theories, and λ is a cardinal. Then $T_0 \trianglelefteq_\lambda T_1$ if and only if for every complete Boolean algebra $\mathcal{B}$ with the λ^+-c.c., and for every ultrafilter $\mathcal{U}$ on $\mathcal{B}$, if $\mathcal{U}$ λ^+-saturates T_1, then $\mathcal{U}$ λ^+-saturates T_0.

We remark that the bulk of the proof of this theorem consists of Malliaris and Shelah's Separation of Variables and Existence Theorems from [56], and involves complicated ultrafilter constructions on $\mathcal{P}(\lambda)$. We view these constructions as essentially unimportant: most of of our arguments only deal with ultrafilters on some much nicer algebra $\mathcal{B}$, typically with the λ-c.c., and we only care about $\mathcal{P}(\lambda)$ because of the definition of $\trianglelefteq$. For this reason, we take Theorem 3.2.4 as our operating definition of $\trianglelefteq$ for most of the survey, and defer its proof until the end.

We draw the reader's attention to the new results we prove in Chapter 3.

In Section 3.14, we show that there is a $\trianglelefteq$-minimal nonlow theory T_{Cas}. I first proved this in [87]. We prove the main theorem of [87], namely that lowness is a dividing line in Keisler's order, in Chapter 4.

In Section 3.15, we prove the following. Special cases are proved by Malliaris and Shelah in [57] and [56].

Theorem 3.2.5. Suppose $\mathcal{B}$ is a complete Boolean algebra with the λ-c.c. and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ does not λ^+-saturate any unsimple theory. If $\mathcal{U}$ is additionally $\aleph_1$-incomplete, then $\mathcal{U}$ does not λ^+-saturate any nonlow theory.

This makes clear the role the chain condition is playing: namely, to constrict the possible theories that we can saturate. We note this theorem is sharp. First of all, if $\mathcal{B}$ has an antichain of size λ, then we show in Theorem 3.16.5 that there is an ultrafilter $\mathcal{U}$ on $\mathcal{B}$ which λ^+-saturates every complete countable theory. Second, Malliaris and Shelah show the following in [57]: suppose there is a supercompact cardinal σ; set $\lambda = \sigma^+$. Then there is a complete Boolean algebra $\mathcal{B}$ with the λ-c.c., and an ultrafilter $\mathcal{U}$ on $\mathcal{B}$ which λ^+-saturates exactly the simple theories. Further, in [87], I show in ZFC that for some λ and some complete Boolean algebra $\mathcal{B}$ with the λ-c.c., there is an $\aleph_1$-incomplete ultrafilter $\mathcal{U}$ on $\mathcal{B}$ which λ^+-saturates exactly the low theories.

In Section 3.8, we discuss the interpretability orders $\trianglelefteq^*_\kappa$. Dealing with $\trianglelefteq^*_\kappa$ does introduce certain complications versus $\trianglelefteq$. We introduce our own interpretability orders $\trianglelefteq^\times_\kappa$, which eliminate these complications. We still have $\trianglelefteq^\times_1 \subseteq \trianglelefteq^\times_{\aleph_1} \subseteq \trianglelefteq$, and further, $\trianglelefteq^\times_\kappa$ allows an elegant general theory of combinatorial characteristics of models of ZFC^-. We prove in Section 3.18 that $\trianglelefteq^\times_{\aleph_1} \subseteq \trianglelefteq^*_{\aleph_1}$, and also $\trianglelefteq^\times_1 \subseteq \trianglelefteq^*_1$ except perhaps on pairs of stable theories. Thus, all of the positive reductions we prove in $\trianglelefteq^\times_{\aleph_1}$ carry over to $\trianglelefteq^*_{\aleph_1}$. Moreover, we use this machinery to deduce Corollaries 3.18.9 and 3.18.10:

Theorem 3.2.6. $\trianglelefteq^*_1$ and $\trianglelefteq^*_{\aleph_1}$ coincide on pairs of theories which are not both stable. Hence, if suitable instances of GCH hold, then $NSOP_2$ theories are nonmaximal in $\trianglelefteq^*_{\aleph_1}$, and in ZFC, simplicity is a dividing line in $\trianglelefteq^*_{\aleph_1}$.

The hence portion follows from corresponding results for $\trianglelefteq^*_1$, proved in [8], [7] and [61].

3.3 A Compactness Theorem for Boolean-Valued Models

In view of Theorem 3.1.9, we are led to consider arbitrary ultrafilters on complete Boolean algebras $\mathcal{B}$ in our analysis of Keisler's order. Malliaris and Shelah always consider such ultrafilters accompanied by a pullback to a regular ultrafilter on $\mathcal{P}(\lambda)$, but we avoid this by working in the generality of $\mathcal{B}$-valued models. In this section, we define $\mathcal{B}$-valued models and prove a compactness theorem for them. This will be the cornerstone of our development.

The idea for $\mathcal{B}$-valued models appears to originate with Mostowski [66]; however, they have mainly been investigated as a tool for forcing, as in [27] Chapter 14. There does not seem to be a completely standard definition of $\mathcal{B}$-valued model; we remark on variants at the end of the section.

Definition 3.3.1. If $\mathcal{L}$ is a theory and X is a set, then let $\mathcal{L}(X)$ be the set of all $\mathcal{L}$-formulas with parameters taken from X. To be formal, we view the elements of X as new constant symbols, but it would work equally well to view them as variables.

Suppose $\mathcal{B}$ is a complete Boolean algebra and $\mathcal{L}$ is a language. A $\mathcal{B}$-valued $\mathcal{L}$-structure is a pair $(\mathbf{M}, \|\cdot\|_{\mathbf{M}})$ where:

1. $\mathbf{M}$ is a set;
2. $\varphi \mapsto \|\varphi\|_{\mathbf{M}}$ is a map from $\mathcal{L}(\mathbf{M})$ to $\mathcal{B}$;
3. If φ is a logically valid sentence then $\|\varphi\|_{\mathbf{M}} = 1$;

4. For every formula $\varphi \in \mathcal{L}(\mathbf{M})$, we have that $\|\neg\varphi\|_{\mathbf{M}} = \neg\|\varphi\|_{\mathbf{M}}$;

5. For all φ, ψ, we have that $\|\varphi \wedge \psi\|_{\mathbf{M}} = \|\varphi\|_{\mathbf{M}} \wedge \|\psi\|_{\mathbf{M}}$;

6. For every formula $\varphi(x)$ with parameters from $\mathbf{M}$, $\|\exists x\varphi(x)\|_{\mathbf{M}} = \bigvee_{a\in\mathbf{M}} \|\varphi(a)\|_{\mathbf{M}}$;

7. For all $a, b \in \mathbf{M}$ distinct, $\|a = b\|_{\mathbf{M}} < 1$.

$(\mathbf{M}, \|\cdot\|_{\mathbf{M}})$ is full if for every formula $\varphi(x)$ with parameters from $\mathbf{M}$, there is some $a \in \mathbf{M}$ such that $\|\exists x\varphi(x)\|_{\mathbf{M}} = \|\varphi(a)\|_{\mathbf{M}}$.

In fact, we will almost always restrict to full $\mathcal{B}$-valued $\mathcal{L}$-structures. As is customary in model theory, we will write that $\mathbf{M}$ is a full $\mathcal{B}$-valued $\mathcal{L}$-structure, suppressing $\|\cdot\|_{\mathbf{M}}$, whenever possible.

Remark 3.3.2. Axiom 6 together with fullness are equivalent to requiring that for all formulas $\varphi(x)$ with parameters from $\mathbf{M}$, $\|\exists x\varphi(x)\|_{\mathbf{M}}$ is the maximum of $\{\|\varphi(b)\|_{\mathbf{M}} : b \in \mathbf{M}\}$, i.e. always $\|\exists x\varphi(x)\|_{\mathbf{M}} \geq \|\varphi(b)\|_{\mathbf{M}}$, and there is some b with equality holding. In particular the axioms for a full $\mathcal{B}$-valued $\mathcal{L}$-structure are finitary.

In the definition of full $\mathcal{B}$-valued $\mathcal{L}$-structures, we only used $\exists, \wedge, \neg$. It is easy to see that one can add $\vee, \rightarrow$ via the usual definitions, and they behave as expected. Universal quantification is also easy, but we isolate it as a lemma:

Lemma 3.3.3. Suppose $\mathbf{M}$ is a full $\mathcal{B}$-valued $\mathcal{L}$-structure, and $\forall x\varphi(x)$ is a formula with parameters from $\mathbf{M}$ (formally, we treat this as $\neg\exists x\neg\varphi(x)$). Then $\|\forall x\varphi(x)\|_{\mathbf{M}}$ is the minimum of $\{\|\varphi(b)\|_{\mathbf{M}} : b \in \mathbf{M}\}$, that is, always $\|\forall x\varphi(x)\|_{\mathbf{M}} \leq \|\varphi(b)\|_{\mathbf{M}}$, and there is some $b \in \mathbf{M}$ with equality holding.

In particular, if $\|\forall\overline{x}\varphi(\overline{x})\|_{\mathbf{M}} = 1$, then for all $\overline{a} \in M^{|\overline{x}|}$, $\|\varphi(\overline{a})\|_{\mathbf{M}} = 1$.

The following theorem will allow us to define the specialization operation. This operation is first considered by Rasiowa and Sikorski [71], although there, in the absence of fullness, one needs the ultrafilter to be sufficiently generic. The general case is implicitly developed by Mansfield [62].

Theorem 3.3.4. Suppose $\mathcal{B}$ is a complete Boolean algebra, $\mathbf{M}$ is a full $\mathcal{B}$-valued $\mathcal{L}$-structure, and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then there is pair (M, π) where:

- M is an ordinary $\mathcal{L}$-structure,
- $\pi : \mathbf{M} \to M$ is a surjection,
- for every $\varphi(\overline{a}) \in \mathcal{L}(\mathbf{M})$, $\|\varphi(\overline{a})\|_{\mathbf{M}} \in \mathcal{U}$ if and only if $M \models \varphi(\pi(\overline{a}))$.

If (M', π') is any other such pair, then there is a unique isomorphism $\sigma : M \cong M'$ such that $\sigma \circ \pi = \pi'$.

Proof. First we construct (M, π).

Define $E \subseteq \mathbf{M} \times \mathbf{M}$ via: aEb if and only if $\|a = b\|_{\mathbf{M}} \in \mathcal{U}$. E is an equivalence relation by condition (3) of the definition of $\mathcal{B}$-valued models, and Lemma 3.3.3.

Claim 1. For all formulas $\varphi(\overline{x})$ with n free variables, and for all $\overline{a}, \overline{b} \in \mathbf{M}^n$, if $a_i E b_i$ for each $i < n$, then $\|\varphi(\overline{a})\|_{\mathbf{M}} \in \mathcal{U}$ if and only if $\|\varphi(\overline{b})\|_{\mathbf{M}} \in \mathcal{U}$.

Proof. This is also by condition (3) of the definition of $\mathcal{B}$-valued models, and Lemma 3.3.3. $\square$

Let the domain of M be $\mathbf{M}/E$, and let $\pi : \mathbf{M} \to M$ be the canonical surjection.

For each n-ary relation symbol R of $\mathcal{L}$, put $R^M = \{\pi(\overline{a}) : \overline{a} \in \mathbf{M}^n, \|R(\overline{a})\|_{\mathbf{M}} = 1\}$. This is well-defined by Claim 1.

Suppose f is an n-ary function symbol of $\mathcal{L}$. I claim that for all $\overline{a} \in \mathbf{M}^n$, there is some $b \in \mathbf{M}$ with $\|f(\overline{a}) = b\|_{\mathbf{M}} \in \mathcal{U}$, and moreover, $\pi(b)$ is uniquely determined by $\pi(\overline{a})$. To see this, note first that by condition (3) in the definition of $\mathcal{B}$-valued model, we have that $\|\forall\overline{x}\exists y f(\overline{x}) = y\|_{\mathbf{M}} = 1$, and so by Lemma 3.3.3, for every $\overline{a}$, $\|\exists y f(\overline{a}) = y\|_{\mathbf{M}} = 1$, and so by fullness, there is some b such that $\|f(\overline{a}) = b\|_{\mathbf{M}} = 1 \in \mathcal{U}$. Thus we have shown existence. For uniqueness, note that by condition (3) and Lemma 3.3.3 again, for all $\overline{a}, \overline{a}', b, c$, if $\|f(\overline{a}) = b\|_{\mathbf{M}} \in \mathcal{U}$ and $\|f(\overline{a}') = c\|_{\mathbf{M}} \in \mathcal{U}$ and $\pi(a_i) = \pi(b_i)$ for all $i < n$, then $\|b = c\|_{\mathbf{M}} \in \mathcal{U}$, i.e. $\pi(b) = \pi(c)$. Thus we can define $f^M(F(\overline{a})) = \pi(b)$ for some or every $b \in \mathbf{M}$ with $\|f(\overline{a}) = b\|_{\mathbf{M}} \in \mathcal{U}$.

We now check by induction on formulas $\varphi(\overline{a}) \in \mathcal{L}(\mathbf{M})$ that $\|\varphi(\overline{a})\|_{\mathbf{M}} \in \mathcal{U}$ if and only if $M \models \varphi(\pi(\overline{a}))$. If $\varphi(t_i(\overline{a}) : i < n)$ is atomic (so each $t_i(\overline{a})$ is a term, and $\varphi(x_i : i < n)$ is either $x_0 = x_1$ or else $R(x_i : i < n)$ for some n-ary relation R), then choose $(b_i : i < n)$ from $\mathbf{M}$ such that each $\|t_i(\overline{a}) = b_i\|_{\mathbf{M}} = 1$ (by condition (3), Lemma 3.3.3 and fullness). Then $\|\varphi(\overline{t}(\overline{a}))\|_{\mathbf{M}} = \|\varphi(\overline{b})\|_{\mathbf{M}}$, and since each $t_i^M(\pi(\overline{a})) = \pi(b_i)$ (by induction on terms), we have $M \models \varphi(\overline{t}^M(\pi(\overline{a})))$ if and only if $M \models \varphi(\pi(\overline{b}))$. So we can replace $\varphi(t_i(\overline{a}) : i < n)$ by $\varphi(b_i : i < n)$. If $\varphi(\overline{b})$ is $b_0 = b_1$ then we conclude by definition of E; if $\varphi(\overline{a})$ is $R(\overline{b})$ then we conclude by definition of R^M.

The rest of the inductive argument is fairly straightforward. To handle negations, we need that $\mathcal{U}$ is ultra, i.e. for all $\mathbf{c} \in \mathcal{B}$, $\mathbf{c} \in \mathcal{U}$ if and only if $\neg\mathbf{c} \notin \mathcal{U}$. To handle the existential stage, we use fullness of $\mathbf{M}$.

Finally, the uniqueness claim is trivial; note that π' must induce a bijection between $\mathbf{M}/E$ and M', and this is the desired isomorphism from M to M'. $\square$

Definition 3.3.5. Suppose $\mathcal{B}$ is a complete Boolean algebra, $\mathbf{M}$ is a full $\mathcal{B}$-valued $\mathcal{L}$-structure, and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Let $(\mathbf{M}/\mathcal{U}, [\cdot]_{\mathbf{M},\mathcal{U}})$ be the pair (M, π) as constructed in Theorem 3.3.4; we call $\mathbf{M}/\mathcal{U}$ the specialization of $\mathbf{M}$ at $\mathcal{U}$, and we call $[\cdot]_{\mathbf{M},\mathcal{U}}$ the canonical surjection. We will only ever use the defining property of $(\mathbf{M}/\mathcal{U}, [\cdot]_{\mathbf{M},\mathcal{U}})$: $[\cdot]_{\mathbf{M},\mathcal{U}}$ is a surjection, and for every $\varphi(\overline{a}) \in \mathcal{L}(\mathbf{M})$, $\|\varphi(\overline{a})\|_{\mathbf{M}} \in \mathcal{U}$ if and only if $\mathbf{M}/\mathcal{U} \models \varphi([\overline{a}]_{\mathbf{M},\mathcal{U}})$.

Usually $\mathbf{M}$ is clear from context, and we omit it in $[\cdot]_{\mathbf{M},\mathcal{U}}$.

Remark 3.3.6. Suppose M is an ordinary $\mathcal{L}$-structure. Then we can define an associated $\{0,1\}$-valued $\mathcal{L}$-structure $(\mathbf{M}, \|\cdot\|_{\mathbf{M}})$ such that $\mathbf{M}$ is the domain of M and $\|\varphi(\overline{a})\|_{\mathbf{M}} = 1$ if and only if $M \models \varphi(\overline{a})$. This sets up an exact correspondence between ordinary $\mathcal{L}$-structures and $\{0,1\}$-valued $\mathcal{L}$-structures, with the inverse map given by specialization at $\mathcal{U}$, where $\mathcal{U}$ is the unique ultrafilter on $\{0,1\}$. Thus, henceforward we identify ordinary $\mathcal{L}$-structures with $\{0,1\}$-valued $\mathcal{L}$-structures, and use the terms interchangeably; we tend to prefer "$\{0,1\}$-valued." (Note that every $\{0,1\}$-valued $\mathcal{L}$-structure is automatically full.) As a convention, lightface M is used for $\{0,1\}$-valued models, and boldface $\mathbf{M}$ is used for general $\mathcal{B}$-valued models.

Note also that whenever $\mathcal{B}_0$ is a subalgebra of $\mathcal{B}_1$, then every full $\mathcal{B}_0$-valued structure is a full $\mathcal{B}_1$-valued structure. The main case we use this is when $\mathcal{B}_0 = \{0,1\}$.

We now aim to prove a compactness theorem for full $\mathcal{B}$-valued $\mathcal{L}$-structures. The reader familiar with forcing can give a rather slicker proof, noting that $\mathcal{B}$-valued $\mathcal{L}$-structures are in correspondence with $\mathcal{B}$-names for models of T; we prefer to avoid forcing machinery in this survey.

Theorem 3.3.7. Suppose $\mathcal{B}$ is a complete Boolean algebra, X is a set, and $F : \mathcal{L}(X) \to \mathcal{B}$. Then the following are equivalent:

(A) There is some full $\mathcal{B}$-valued structure $\mathbf{M}$ and some map $\tau : X \to \mathbf{M}$, such that for all $\varphi(\overline{a}) \in \mathcal{L}(X)$, $F(\varphi(\overline{a})) \leq \|\varphi(\tau(\overline{a}))\|_{\mathbf{M}}$.

(B) For every ultrafilter $\mathcal{U}$ on $\mathcal{B}$, there is some $\{0,1\}$-valued $\mathcal{L}$-structure M and some map $\tau : X \to M$, such that for all $\varphi(\overline{a}) \in \Gamma$, if $F(\varphi(\overline{a})) \in \mathcal{U}$ then $M \models \varphi(\tau(\overline{a}))$.

(C) For every finite $\Gamma_0 \subseteq \Gamma$ and for every $\mathbf{c} \in \mathcal{B}_+$, there is some $\{0,1\}$-valued $\mathcal{L}$-structure M and some map $\tau : X \to M$, such that for every $\varphi(\overline{a}) \in \Gamma_0$, if $\mathbf{c} \leq F(\varphi(\overline{a}))$ then $M \models \varphi(\tau(\overline{a}))$.

Proof. (A) implies (B): suppose (A) holds, and let $\mathcal{U}$ be given. Let $M = \mathbf{M}/\mathcal{U}$, and let $\tau' : X \to M$ be the composition of τ with $[\cdot]_{\mathcal{U}}$. Then this witnesses (B) holds.

(B) implies (C): Suppose (B) holds, and suppose $\mathbf{c}$, Γ_0 are given. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{B}$ containing $\mathbf{c}$; and let $\tau : X \to M$ be as promised by (B). Then this also works for (C).

(C) implies (A): This will be a Henkin construction.

Let $X_* = X \cup Y$, where Y is disjoint from X with $|Y| = |X| + \aleph_0$. Write $\kappa = |X_*| = |X| + \aleph_0$, and write $\Gamma = \mathcal{L}(X)$.

We will be considering pairs (Δ, G), where $\Delta \subseteq \mathcal{L}(X_*)$ and $G : \Delta \to \mathcal{B}$. Given such a pair (Δ, G), and given $\mathbf{c} \in \mathcal{B}_+$, define $T_{\mathbf{c},\Delta,G}$ to be the following theory in $\mathcal{L}(X_*)$ (where we view the elements of X_* as constants). Namely $T_{\mathbf{c},\Delta,G} := \{\varphi(\overline{a}) \in \Gamma : \mathbf{c} \leq F(\varphi(\overline{a}))\} \cup \{\varphi(\overline{a}) \in \Delta : \mathbf{c} \leq G(\varphi(\overline{a}))\} \cup \{\neg\varphi(\overline{a}) : \varphi(\overline{a}) \in \Delta, \mathbf{c} \leq \neg G(\varphi(\overline{a}))\}$.

Let P be the set of all pairs (Δ, G) with $\Delta \subseteq \mathcal{L}(X_*)$ and $G : \Delta \to \mathcal{B}$, such that for every $\mathbf{c} \in \mathcal{B}_+$, $T_{\mathbf{c},\Delta,G}$ is satisfiable. (If we wanted to reprove the standard compactness theorem, we would replace all occurrences of "satisfiable" by "finitely satisfiable," as in [64].) We view P as partially ordered under componentwise $\subseteq$. Note that by hypothesis, $(\emptyset, \emptyset) \in P$.

We plan to find some $G_* : \mathcal{L}(X_*) \to \mathcal{B}$ such that $(\mathcal{L}(X_*), G_*) \in P$, and such that for all formulas $\varphi(x)$ with parameters from X_*, there is some $a \in X_*$ such that

$G_*(\exists x \varphi(x)) = G_*(a)$. We note now how to finish, assuming this. Note first that for all $\varphi(\overline{a}) \in \mathcal{L}(X)$, $G_*(\varphi(\overline{a})) \geq F(\varphi(\overline{a}))$, as otherwise, set $\mathbf{c} = F(\varphi(\overline{a})) \wedge \neg G(\varphi(\overline{a}))$; then $T_{\mathbf{c},\mathcal{L}(X_*),G_*}$ is unsatisfiable. Also, (X_*, G_*) satisfies all the axioms of a full $\mathcal{B}$-valued model, except possibly for condition (7). So let $\mathbf{M} = X/E$ where aEb if and only if $G(a = b) = 1$, and let $\|\varphi([\overline{a}/E])\|_{\mathbf{M}} = G_*(\varphi(\overline{a}))$, and let $\tau : X \to \mathbf{M}$ be defined by $a \mapsto [a/E]$.

So it suffices to find G_*. We break this into claims.

Claim 1. Suppose $(\Delta, G) \in P$, and suppose $|\Delta| < \kappa$. Then for every $\varphi(\overline{a}) \in \mathcal{L}(X_*)$, we can find some $\mathbf{c}_*$ such that $(\Delta \cup \{\varphi(\overline{a})\}, G \cup \{(\varphi(\overline{a}), \mathbf{c}_*)\}) \in P$.

Proof. Let $\mathbf{C}_0$ be the set of all $\mathbf{c} \in \mathcal{B}_+$ such that $T_{\mathbf{c},\Delta,G}$ implies $\varphi(\overline{a})$ (i.e., every model of $T_{\mathbf{c},\Delta,G}$ is a model of $\varphi(\overline{a})$, where we are viewing the elements of X as constants). Let $\mathbf{c}_0 = \bigvee \mathbf{C}_0$.

Let $\mathbf{C}_1$ be the set of all $\mathbf{c} \in \mathcal{B}$ different from 1, such that $T_{\neg\mathbf{c},\Delta,G}$ implies $\neg\varphi(\overline{a})$. Let $\mathbf{c}_1 = \bigwedge \mathbf{C}_1$.

I first of all claim that $\mathbf{c}_0 \leq \mathbf{c}_1$. This amounts to showing that for all $\mathbf{d}_0 \in \mathbf{C}_0$ and for all $\mathbf{d}_1 \in \mathbf{C}_1$, $\mathbf{d}_0 \leq \mathbf{d}_1$. Suppose not; write $\mathbf{c} = \mathbf{d}_0 \wedge \neg\mathbf{d}_1$. Then by definition of $\mathbf{C}_0$ and $\mathbf{C}_1$, we must have that $T_{\mathbf{c},\Delta,G}$ implies both $\varphi(\overline{a})$ and $\neg\varphi(\overline{a})$, i.e. is unsatisfiable, contradicting the compatability of (Δ, G).

Finally, I claim that any $\mathbf{c}_*$ with $\mathbf{c}_0 \leq \mathbf{c}_* \leq \mathbf{c}_1$ will work for the Claim. Indeed, let $\mathbf{c}_*$ be given as such, and write $\Delta' = \Delta \cup \{\varphi(\overline{a})\}$, $G' = G \cup \{(\varphi(\overline{a}), \mathbf{c}_*)\}$.

Let $\mathbf{c} \in \mathcal{B}_+$ be given; we can suppose $\mathbf{c}$ decides $\mathbf{c}_*$. Suppose first $\mathbf{c} \leq \mathbf{c}_*$; thus $\mathbf{c} \leq \mathbf{c}_1$. Note that $T_{\mathbf{c},\Delta',G'} = T_{\mathbf{c},\Delta,G} \cup \{\varphi(\overline{a})\}$; we need to show this is consistent. Suppose towards a contradiction that $T_{\mathbf{c},\Delta,G}$ implies $\neg\varphi(\overline{a})$. Then $\neg\mathbf{c} \in \mathbf{C}_1$ by definition of $\mathbf{C}_1$, so $\mathbf{c}_1 \leq \neg\mathbf{c}$, contradicting $\mathbf{c} \leq \mathbf{c}_1$ is nonzero.

Suppose instead that $\mathbf{c} \leq \neg\mathbf{c}_*$, thus $\mathbf{c} \leq \neg\mathbf{c}_0$. Note that $T_{\mathbf{c},\Delta',G'} = T_{\mathbf{c},\Delta,G} \cup \{\neg\varphi(\overline{a})\}$;

we need to show this is consistent. Suppose towards a contradiction that $T_{\mathbf{c},\Delta,G}$ implies $\varphi(\overline{a})$. Then $\mathbf{c} \in \mathbf{C}_0$ by definition of $\mathbf{C}_0$, so $\mathbf{c} \leq \mathbf{c}_0$, contradicting $\mathbf{c} \leq \neg\mathbf{c}_0$ is nonzero. $\square$

Claim 2. Suppose $(\Delta, G) \in P$, and suppose $|\Delta| < \kappa$. Suppose $\exists x\varphi(x) \in \Delta$, where $\varphi(x)$ has parameters from X_*. Choose $a \in Y$ which does not occur in any formula in Δ (note a cannot occur in any formula of $\Gamma = \mathcal{L}(X)$ either). Write $\Delta' = \Delta \cup \{\varphi(a)\}$, and write $G' = G \cup \{(\varphi(a), G(\exists x\varphi(x))\}$. Then $(\Delta', G') \in P$.

Proof. Suppose $\mathbf{c} \in \mathcal{B}_+$ is given; we can suppose $\mathbf{c}$ decides $G(\exists x\varphi(x))$ (which is equal to $G'(\varphi(a))$). Since a does not appear in $T_{\mathbf{c},\Delta,G}$, we have that $T_{\mathbf{c},\Delta,G} \cup \{\varphi(a)\}$ is satisfiable if and only if $T_{\mathbf{c},\Delta,G} \cup \{\exists x\varphi(x)\}$ is satisfiable, which is the case if and only if $\mathbf{c} \leq G(\exists x\varphi(x))$. Thus, if $\mathbf{c} \leq G'(\varphi(a))$ then $T_{\mathbf{c},\Delta',G'}$ is satisfiable. Finally, if $\mathbf{c} \leq \neg G'(\varphi(a))$, then $T_{\mathbf{c},\Delta,G}$ implies $\neg\varphi(a)$; since $T_{\mathbf{c},\Delta,G}$ is satisfiable, so is $T_{\mathbf{c},\Delta',G'}$. $\square$

Claim 3. Suppose $(\Delta_\alpha, G_\alpha : \alpha < \alpha_*)$ is an increasing chain from P, where α_* is a limit ordinal. Write $\Delta = \bigcup_\alpha \Delta_\alpha$, write $G = \bigcup_\alpha G_\alpha$. Then $(\Delta, G) \in P$.

Proof. Suppose $\mathbf{c} \in \mathcal{B}_+$; then note that $T_{\mathbf{c},\Delta,G} = \bigcup_{\alpha<\alpha_*} T_{\mathbf{c},\Delta_\alpha,G_\alpha}$, so we can apply standard compactness. $\square$

To finish the construction of G_* and hence the proof, note that using Claims 1 through 3 it is now straightforward to find an increasing chain $((\Delta_\alpha, G_\alpha) : \alpha \leq \kappa)$ from P (recall $\kappa = |X_*| = |X| + \aleph_0$) such that:

- For all $\alpha < \kappa$, $|\Delta_\alpha| \leq |\alpha|$;
- For every formula $\varphi(\overline{a}) \in \mathcal{L}(X_*)$, there is $\alpha < \kappa$ with $\varphi(\overline{a}) \in \Delta_\alpha$;
- For every formula $\varphi(x)$ with parameters from X_*, there is $a \in X_*$ and $\alpha < \kappa$, such that $\{\exists x\varphi(x), \varphi(a)\} \subseteq \Delta_\alpha$ and such that $G_\alpha(\exists x\varphi(x)) = G_\alpha(\varphi(a))$.

Then G_κ is visibly as desired. $\square$

The following minor modification will frequently be more convenient in applications:

Corollary 3.3.8. Suppose $\mathcal{B}$ is a complete Boolean algebra, X is a set, $\Gamma \subseteq \mathcal{L}(X)$, and $F_0, F_1 : \Gamma \to \mathcal{B}$ with $F_0(\varphi(\overline{a})) \leq F_1(\varphi(\overline{a}))$ for all $\varphi(\overline{a}) \in \Gamma$. Then the following are equivalent:

(A) There is some full $\mathcal{B}$-valued structure $\mathbf{M}$ and some map $\tau : X \to \mathbf{M}$, such that for all $\varphi(\overline{a}) \in \Gamma$, $F_0(\varphi(\overline{a})) \leq \|\varphi(\tau(\overline{a}))\|_{\mathbf{M}} \leq F_1(\varphi(\overline{a}))$;

(B) For every ultrafilter $\mathcal{U}$ on $\mathcal{B}$, there is some $\{0,1\}$-valued $\mathcal{L}$-structure M and some map $\tau : X \to M$, such that for all $\varphi(\overline{a}) \in \Gamma$, if $F_0(\varphi(\overline{a})) \in \mathcal{U}$ then $M \models \varphi(\tau(\overline{a}))$, and if $F_1(\varphi(\overline{a})) \notin \mathcal{U}$, then $M \models \neg\varphi(\tau(\overline{a}))$;

(C) For every finite $\Gamma_0 \subseteq \Gamma$ and for every $\mathbf{c} \in \mathcal{B}_+$, there is some $\{0,1\}$-valued $\mathcal{L}$-structure M and some map $\tau : X \to M$, such that for every $\varphi(\overline{a}) \in \Gamma$, if $\mathbf{c} \leq F_0(\varphi(\overline{a}))$ then $M \models \varphi(\tau(\overline{a}))$, and if $\mathbf{c} \leq \neg F_1(\varphi(\overline{a}))$ then $M \models \neg\varphi(\tau(\overline{a}))$.

Proof. We can first suppose $\Gamma = \mathcal{L}(X)$, by setting $F_0 = 0$ and $F_1 = 1$ on new formulas. Then define $F : \mathcal{L}(X) \to \mathcal{B}$ via $F(\varphi(\overline{a})) = F_0(\varphi(\overline{a})) \vee \neg F_1(\neg\varphi(\overline{a}))$. Apply Theorem 3.3.7 to F, noting that (A), (B), (C) there are each equivalent to (A), (B), (C) here. $\square$

Historical Remark. $\mathcal{B}$-valued models appear to be first considered by Mostowski [66]. After the advent of forcing, they were independently defined by Scott and Solovay [72], and Vopěnka [91]. We follow the more modern notation of Mansfield [62].

In many of these definitions, the evaluation map $\|\cdot\|_{\mathbf{M}}$ is defined only on the basic atomic formulas. Note that clauses (4), (5) and (6) show that this completely determines $\|\cdot\|_{\mathbf{M}}$, but then one must check that condition (3) holds. Rasiowa and Sikorski prove this in [70], assuming a short list of axioms for equality and function symbols. This is their completeness theorem for $\mathcal{B}$-valued models, and it also follows from the proof of Theorem 3.3.7.

Condition (7) is nonstandard, but tame; if it failed, one should mod out by the equivalence relation E, defined via aEb if $\|a = b\|_{\mathbf{M}} = 1$.

3.4 More on Boolean-Valued Models

In this section, we define the appropriate notion of maps between full $\mathcal{B}$-valued models, and show that λ^+-saturated $\mathcal{B}$-valued models exist. Then we define $\mathcal{B}$-valued ultrapowers.

Definition 3.4.1. Given $\mathbf{M}, \mathbf{N}$ $\mathcal{B}$-valued $\mathcal{L}$-structures, say that $f : \mathbf{M} \preceq \mathbf{N}$ is an elementary map if $f : \mathbf{M} \to \mathbf{N}$, and for every $\varphi(\overline{a}) \in \mathcal{L}(\mathbf{M})$, $\|\varphi(\overline{a})\|_{\mathbf{M}} = \|\varphi(f(\overline{a}))\|_{\mathbf{N}}$. Note this implies f is injective, by condition (7) above (this is the reason I insist on (7)). Say that $\mathbf{M} \preceq \mathbf{N}$ if the inclusion is elementary.

Say that $f : \mathbf{M} \cong \mathbf{N}$ if $f : \mathbf{M} \preceq \mathbf{N}$ is bijective (and so $f^{-1} : \mathbf{N} \preceq \mathbf{M}$).

If $f : \mathbf{M} \preceq \mathbf{N}$ are full and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, then this induces an elementary map $[f]_{\mathcal{U}} : \mathbf{M}/\mathcal{U} \preceq \mathbf{N}/\mathcal{U}$. When f is the inclusion, we pretend $[f]_{\mathcal{U}}$ is also, even though the equivalence classes grow.

The following is a typical application of Corollary 3.3.8.

Example 3.4.2. Suppose $\mathbf{M}$ is a $\mathcal{B}$-valued $\mathcal{L}$-structure. Then there is some full $\mathbf{N} \succeq \mathbf{M}$.

Proof. Write $X = \mathbf{M}$, write $\Gamma = \mathcal{L}(\mathbf{M})$, and write $F_0 = F_1 = \| \cdot \|_{\mathbf{M}}$, and apply Corollary 3.3.8. $\square$

Remark 3.4.3. $\preceq$ has several obvious properties:

- If $f : \mathbf{M} \to \mathbf{N}$ is a function, where $\mathbf{M}, \mathbf{N}$ are full $\mathcal{B}$-valued $\mathcal{L}$-structures, and if $\mathcal{B}$ is a complete subalgebra of $\mathcal{B}_*$, then whether or not $f : \mathbf{M} \preceq \mathbf{N}$ does not depend on whether we consider $\mathbf{M}, \mathbf{N}$ to be $\mathcal{B}$-valued structures or $\mathcal{B}_*$-valued structures.

- Suppose α_* is a limit ordinal, and $(\mathbf{M}_\alpha : \alpha < \alpha_*)$ is an increasing chain of full $\mathcal{B}$-valued $\mathcal{L}$-structures with each $\mathbf{M}_\alpha \preceq \mathbf{M}_{\alpha+1}$. Write $\mathbf{M}_{\alpha_*} = \bigcup_{\alpha<\alpha_*} \mathbf{M}_\alpha$ and write $\| \cdot \|_{\mathbf{M}_{\alpha_*}} = \bigcup_{\alpha<\alpha_*} \| \cdot \|_{\mathbf{M}_\alpha}$. Then $\mathbf{M}_{\alpha_*}$ is a full $\mathcal{B}$-valued $\mathcal{L}$-structure, and for all $\alpha < \alpha_*$, $\mathbf{M}_\alpha \preceq \mathbf{M}$.

Definition 3.4.4. If T is a complete countable theory, we say that $\mathbf{M}$ is a full $\mathcal{B}$-valued model of T if $(\mathbf{M}, \| \cdot \|_{\mathbf{M}})$ is full and $\|\varphi\|_{\mathbf{M}} = 1$ for all $\varphi \in T$. Let $\mathbf{M} \models^{\mathcal{B}} T$ be short-hand for this.

The reader familiar with abstract elementary classes (see, for instance, [1]) will note that the class of full $\mathcal{B}$-valued models of T can be formulated as an abstract elementary class, and the following theorem says it has downward Löwenheim-Skolem number $\aleph_0$, the joint embedding property and the amalgamation property; the existence of λ^+-saturated models follow on general grounds.

Theorem 3.4.5. Suppose T is a complete countable theory and $\mathcal{B}$ is a complete Boolean algebra. Then the following all hold:

1. (Downward Löwenheim-Skolem) Suppose $\mathbf{M} \models^{\mathcal{B}} T$ and $X \subseteq \mathbf{M}$. Then there is $\mathbf{N} \preceq \mathbf{M}$ with $X \subseteq \mathbf{N}$ and $|\mathbf{N}| \leq |X| + \aleph_0$.

2. (Joint Embedding) Suppose $\mathbf{M}_0, \mathbf{M}_1 \models^{\mathcal{B}} T$. Then we can find $\mathbf{N} \models^{\mathcal{B}} T$ such that there exist embeddings $f_i : \mathbf{M}_i \preceq \mathbf{N}$.

3. (Amalgamation Property) Suppose $\mathbf{M}, \mathbf{M}_0, \mathbf{M}_1 \models^{\mathcal{B}} T$ and $\mathbf{M} \preceq \mathbf{M}_0$ and $\mathbf{M} \preceq \mathbf{M}_1$.

Then we can find some $\mathbf{N} \models^{\mathcal{B}} T$ and some $f_i : \mathbf{M}_i \preceq \mathbf{N}$, such that f_0 and f_1 agree on $\mathbf{M}$.

Proof. (1): For each formula $\varphi(x, \overline{y})$ (with no hidden parameters), choose $f_\varphi : \mathbf{M}^{|\overline{y}|} \to \mathbf{M}$ such that always $\|\varphi(f_\varphi(\overline{a}), \overline{a})\|_{\mathbf{M}} = \|\exists x f_\varphi(\overline{a})\|_{\mathbf{M}}$. Choose $\mathbf{N} \subseteq \mathbf{M}$ with $|\mathbf{N}| \leq |X| + \aleph_0$, such that $\mathbf{N}$ is closed under each f_φ; then $\mathbf{N} \preceq \mathbf{M}$.

(2), (3): Use Corollary 3.3.8, and the fact that $\{0, 1\}$-valued models of T have joint embedding and amalgamation. $\square$

Definition 3.4.6. Suppose $\mathbf{M} \models^{\mathcal{B}} T$ where T is a complete countable theory. Then we say that $\mathbf{M}$ is λ-saturated if for all $\mathbf{M}_0 \preceq \mathbf{N}_0 \models^{\mathcal{B}} T$ with $|\mathbf{N}_0| < \lambda$, we have that every $f : \mathbf{M}_0 \preceq \mathbf{M}$ can be extended to some $g : \mathbf{N}_0 \preceq \mathbf{M}$. $\mathbf{M}$ is λ-universal if for all $\mathbf{N} \models^{\mathcal{B}} T$ with $|\mathbf{N}| < \lambda$, we can find some $f : \mathbf{N} \preceq \mathbf{M}$.

As mentioned above, the following is a general fact about AECs with joint embedding and with amalgamation; the proof is exactly the same as in standard model theory.

Theorem 3.4.7. If $\mathbf{M} \models^{\mathcal{B}} T$ is λ-saturated, then it is λ-universal. Also, for every λ, there is some λ-saturated $\mathbf{M} \models^{\mathcal{B}} T$.

We will want the following refinement (which doesn't make sense in general AECs).

Definition 3.4.8. Suppose $\mathbf{N}, \mathbf{M} \models^{\mathcal{B}} T$, and $A \subseteq \mathbf{N}$. Then say that $f : A \to \mathbf{M}$ is partial elementary if or all $\varphi(\overline{a}) \in \mathcal{L}(A)$, $\|\varphi(\overline{a})\|_{\mathbf{N}} = \|\varphi(f(\overline{a}))\|_{\mathbf{M}}$.

Remark 3.4.9. Suppose $\mathbf{M} \models^{\mathcal{B}} T$. Then $\mathbf{M}$ is λ-saturated if and only if whenever $\mathbf{N} \models^{\mathcal{B}} T$ has $|\mathbf{N}| < \lambda$, and whenever $A \subseteq \mathbf{N}$ and $f : A \to \mathbf{M}$ is partial elementary, then there is an extension of f to $\mathbf{N}$ (or equivalently, to $A \cup \{a\}$ for any $a \in \mathbf{N}$). This is because full $\mathcal{B}$-valued models of T actually have amalgamation over sets: if $\mathbf{M}_0, \mathbf{M}_1 \models^{\mathcal{B}} T$ and

$\|\cdot\|_{\mathbf{M}_0}, \|\cdot\|_{\mathbf{M}_1}$ agree on $\mathbf{M}_0 \cap \mathbf{M}_1$, then we can find some $\mathbf{K} \models^{\mathcal{B}} T$ and some $f_i : \mathbf{M}_i \preceq \mathbf{K}$, such that f_0 and f_1 agree on $\mathbf{M} \cap \mathbf{N}$.

We now define Boolean ultrapowers. These are implicit in the work of Scott and Solovay [72], and made explicit by Vopenka [91]. We follow the notation of Mansfield [62]. These will be helpful in some applications, and they are a relatively concrete source of examples of $\mathcal{B}$-valued models.

Definition 3.4.10. Suppose $M \models T$. Let $M^{\mathcal{B}}$ be the set of all partitions of $\mathcal{B}$ by M, namely the set of all functions $\mathbf{a} : M \to \mathcal{B}$, such that for all $a, b \in M$, $\mathbf{a}(a) \wedge \mathbf{a}(b) = 0$, and such that $\bigvee_{a \in M} \mathbf{a}(a) = 1$. Given $\mathbf{a}_i : i < n$ a sequence from $M^{\mathcal{B}}$, put $\|\varphi(\mathbf{a}_i : i < n)\|_{\mathcal{B}} = \bigvee_{M \models \varphi(a_i : i < n)} \bigwedge_{i<n} \mathbf{a}_i(a_i)$. Note that this does not depend on the choice of $\bar{\mathbf{a}}$ (which is always allowed to contain more parameters than φ uses).

Let $\mathbf{i} : M \to M^{\mathcal{B}}$ be the embedding sending $a \in M$ to the function $\mathbf{i}(a) : M \to \mathcal{B}$ which takes the value 1 on a, and 0 elsewhere. We call this the pre-Łoś embedding.

The following theorem is the compilation of Corollary 1.2 and Theorem 1.4 of [62].

Theorem 3.4.11. Suppose M is a $\{0, 1\}$-valued structure and $\mathcal{B}$ is a complete Boolean algebra (so M is also a full $\mathcal{B}$-valued structure). Then $M^{\mathcal{B}}$ is a full $\mathcal{B}$-valued $\mathcal{L}$-structure, and $\mathbf{i} : M \preceq M^{\mathcal{B}}$.

Proof. We first verify axioms (1) through (5), and (7) of a $\mathcal{B}$-valued model. (1) through (3) are trivial.

(4): Suppose $\varphi(\bar{\mathbf{a}})$ is given, write $\bar{\mathbf{a}} = (\mathbf{a}_i : i < n)$. Define $F : M^n \to \mathcal{B}$ via $F(a_i : i < n) = \bigwedge_{i<n} \mathbf{a}_i(a_i)$. Write $\mathbf{c} := \|\varphi(\bar{a}\|_{\mathbf{M}} = \bigvee F[\varphi(M^n)]$ and $\mathbf{d} := \neg\|\varphi(\bar{a})\|_{\mathbf{M}} = \bigvee F[\neg\varphi(M^n)]$. Then $\mathbf{c} \wedge \mathbf{d} = 0$ since $\varphi(M^n)$ and $\neg\varphi(M^n)$ are disjoint, and for every $\bar{a} \neq \bar{b} \in M^n$, $F(\bar{a}) \wedge F(\bar{b}) = 0$. Also, $\mathbf{c} \vee \mathbf{d} = 1$ since $\varphi(M^n) \cup \neg\varphi(M^n) = 1$, and since

$\bigvee_{\overline{a} \in M^n} F(\overline{a}) = 1$. Thus $\mathbf{d} = \neg \mathbf{c}$.

(5) is similar.

(7): If $\mathbf{a} \neq \mathbf{b}$ are elements of $M^{\mathcal{B}}$, we can choose $a \in M$ such that $\mathbf{a}(a) \neq \mathbf{b}(a)$. We can suppose $\mathbf{a}(a) \not\leq \mathbf{b}(a)$. Then $\|\mathbf{a} \neq \mathbf{b}\|_{\mathbf{M}} \geq \mathbf{a}(a) \wedge \neg \mathbf{b}(a) > 0$, so $\|\mathbf{a} = \mathbf{b}\|_{\mathbf{M}} < 1$.

Now we check axiom (6) together with fullness, as in Remark 3.3.2. Suppose $\varphi(x, \mathbf{a}_i : i < n)$ is given. Then for any $\mathbf{b}$, trivially $\|\varphi(\mathbf{b}, \mathbf{a}_i : i < n)\|_{\mathbf{M}} \leq \|\exists x \varphi(x, \mathbf{a}_i : i < n)\|_{\mathbf{M}}$. We need to find $\mathbf{b}$ such that equality holds.

Write $\mathbf{c}_* = \|\exists x \varphi(x, \mathbf{a}_i : i < n)\|_{\mathbf{M}}$ and let D be the set of all nonzero $\mathbf{c} \leq \mathbf{c}_*$ such that for all $i < n$, there is a (necessarily unique) $a_{i,\mathbf{c}} \in M$ such that $\mathbf{c} \leq \mathbf{a}_i(a_{i,\mathbf{c}})$. This is easily dense below $\mathbf{c}_*$, so we can find a maximal antichain $\mathbf{C} \subseteq D$ below $\mathbf{c}_*$. For each $\mathbf{c} \in \mathbf{C}$, choose $b_{\mathbf{c}} \in M$ such that $M \models \varphi(b_{\mathbf{c}}, a_{i,\mathbf{c}} : i < n)$. Let $\mathbf{C}'$ be an extension of $\mathbf{C}$ to a maximal antichain of $\mathcal{B}$, and choose $b_{\mathbf{c}}$ arbitrarily for $\mathbf{c} \in \mathbf{C}' \backslash \mathbf{C}$. For each $b \in M$, define $\mathbf{b}(b) = \bigvee \{\mathbf{c} \in \mathbf{C}' : b_{\mathbf{c}} = b\}$. Then $\mathbf{b} \in M^{\mathcal{B}}$ works.

That $\mathbf{i} : M \preceq M^{\mathcal{B}}$ follows trivially from the definition. $\square$

There is another way of viewing $M^{\mathcal{B}}$, as described in Theorem 1.3 of [62], which is frequently helpful; note that this is what is really going on in the proof of fullness above.

Definition 3.4.12. Suppose $M \models T$. Then define an inverse partition of $\mathcal{B}$ by M to be a pair $(\mathbf{C}, f)$ where $\mathbf{C}$ is a maximal antichain of $\mathcal{B}$ and $f : \mathbf{C} \to M$. Given two inverse partitions $(\mathbf{C}_0, f_0), (\mathbf{C}_1, f_1)$, define $(\mathbf{C}_0, f_0) \sim (\mathbf{C}_1, f_1)$ if there is a common refinement $\mathbf{C}$ of $\mathbf{C}_0, \mathbf{C}_1$, such that for all $\mathbf{c} \in \mathbf{C}$, if $\mathbf{c}_i$ is the unique element of $\mathbf{C}_i$ with $\mathbf{c} \leq \mathbf{c}_i$ (for each $i < 2$), then $f_0(\mathbf{c}_0) = f_1(\mathbf{c}_1)$.

We can identify $M^{\mathcal{B}}$ with the set of all $(\mathbf{C}, f)/\sim$, where $(\mathbf{C}, f)$ is an inverse partition of $\mathcal{B}$ by M; namely associate to $(\mathbf{C}, f)$ the partition $\mathbf{a}$ of M by $\mathcal{B}$, such that $\mathbf{a} = 0$

outside the range of f, and such that $\mathbf{a} \restriction_{f[\mathbf{C}]} = f^{-1}$. Given $((\mathbf{C}_i, f_i) : i < n)$, note that $\|\varphi((\mathbf{C}_i, f_i) : i < n)\|_{\mathbf{M}}$ can be evaluated as follows: by chooing a common refinement of $(\mathbf{C}_i : i < n)$, we can suppose $\mathbf{C}_i = \mathbf{C}_j = \mathbf{C}$ for all $i < n$. Then $\|\varphi((\mathbf{C}, f_i) : i < n)\|_{\mathbf{M}} = \bigvee\{\mathbf{c} \in \mathbf{C} : M \models \varphi(f_i(\mathbf{c}) : i < n)\}$.

This is particularly natural when $\mathcal{B} = \mathcal{P}(\lambda)$, in which we can always suppose $\mathbf{C} = \{\{\alpha\} : \alpha < \lambda\}$; this gives an isomorphism $M^{\mathcal{P}(\lambda)} \cong M^\lambda$. This is the reason for the notation $M^{\mathcal{B}}$.

If $M \models T$ and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, then we can consider the composition $\mathbf{j} := [\cdot]_{\mathcal{U}} \circ \mathbf{i} : M \to M^{\mathcal{B}}/\mathcal{U}$. We call this the Łoś embedding. Łoś's theorem states that this map is elementary in the special case $\mathcal{B} = \mathcal{P}(\lambda)$. Mansfield proves the general case in [62], and seems to credit it to Scott and Vopenka. It follows immediately from what we have done.

Corollary 3.4.13. Suppose $M \models T$ and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then $\mathbf{j} : M \preceq M^{\mathcal{B}}/\mathcal{U}$.

Example 3.4.14. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{P}(\lambda)$, and $M \models T$ is λ^+-saturated (considered as a $\{0,1\}$-valued model of T). Then M^λ is λ^+-saturated (considered as a full $\mathcal{P}(\lambda)$-valued model of T).

Proof. Suppose $\mathbf{N} \models^{\mathcal{P}(\lambda)} T$ with $|\mathbf{N}| \leq \lambda$. It suffices to show that whenever $A \subseteq \mathbf{N}$ and $f : A \to M^\lambda$ is partial elementary, and whenever $a \in \mathbf{N}$, there is some partial elementary $g : A \cup \{a\} \to M^\lambda$ extending f.

So let A, f, a be given. Enumerate $A = \{a_\beta : \beta < \lambda\}$. Write $b_\beta = f(a_\beta)$, so $b_\beta : \lambda \to M$. Fix $\alpha < \lambda$; by λ^+-saturation of M, we can find $b(\alpha) \in M$ such that for every $\varphi(a, a_{\beta_0}, \ldots, a_{\beta_{n-1}}) \in \mathcal{L}(A \cup \{a\})$, $M \models \varphi(b(\alpha), b_{\beta_0}(\alpha), \ldots, b_{\beta_{n-1}}(\alpha))$ if and only if $\alpha \in \|\varphi(a, a_{\beta_0}, \ldots, a_{\beta_{n-1}})\|_{\mathbf{N}}$. Then $b : \lambda \to M$ is such that $g := f \cup \{(a, b)\}$ is partial elementary. $\square$

3.5 Keisler's Order

We now give what we believe is the most natural formulation of Keisler's order, in the general context of $\mathcal{B}$-valued models. Namely, we view the following question as fundamental: suppose $\mathbf{M} \models^{\mathcal{B}} T$ is λ^+-saturated, and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Is $\mathbf{M}/\mathcal{U}$ λ^+-saturated?

We analyze this situation with the following definitions.

Definition 3.5.1. Given $\mathbf{M} \models^{\mathcal{B}} T$, and $p(\overline{x}) \subseteq \mathcal{L}(\mathbf{M} \cup \overline{x})$, say that $p(\overline{x})$ is a partial type over $\mathbf{M}$ if for each finite subset $\Gamma(\overline{x}) \subseteq p(\overline{x})$, $\|\exists \overline{x} \bigwedge \Gamma(\overline{x})\|_{\mathbf{M}} > 0$. By the arity of $p(\overline{x})$ we mean the length of $\overline{x}$ (always finite).

If $p(\overline{x})$ is a partial type over $\mathbf{M}$ and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, let $[p(\overline{x})]_{\mathcal{U}}$ be the image of $p(\overline{x})$ under $[\cdot]_{\mathcal{U}} : \mathbf{M} \to \mathbf{M}/\mathcal{U}$. Say that $p(\overline{x})$ is a partial $\mathcal{U}$-type over $\mathbf{M}$ if $[p(\overline{x})]_{\mathcal{U}}$ is a partial type over $\mathbf{M}/\mathcal{U}$.

Theorem 3.5.2. Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$, suppose T is a complete first order theory in a countable language, and suppose λ is an (infinite) cardinal. Then the following are all equivalent.

(A) Whenever $\mathbf{M} \models^{\mathcal{B}} T$ has $|\mathbf{M}| \leq \lambda$, and whenever $p(x)$ is partial $\mathcal{U}$-type over $\mathbf{M}$ (of arity 1), then there is some $\mathbf{N} \succeq \mathbf{M}$ such that $\mathbf{N}/\mathcal{U}$ realizes $[p(x)]_{\mathcal{U}}$.

(B) Whenever $\mathbf{M} \models^{\mathcal{B}} T$, and whenever $p(\overline{x})$ is a partial $\mathcal{U}$-type over $\mathbf{M}$ of cardinality at most λ, then there is some $\mathbf{N} \succeq \mathbf{M}$ such that $\mathbf{N}/\mathcal{U}$ realizes $[p(\overline{x})]_{\mathcal{U}}$.

(C) Whenever $\mathbf{M} \models^{\mathcal{B}} T$, there is some $\mathbf{N} \succeq \mathbf{M}$ such that $\mathbf{N}/\mathcal{U}$ is λ^+-saturated.

(D) There is a λ^+-universal $\mathbf{M} \models^{\mathcal{B}} T$ such that $\mathbf{M}/\mathcal{U}$ is λ^+-saturated.

(E) Every λ^+-saturated $\mathbf{M} \models^{\mathcal{B}} T$ satisfies that $\mathbf{M}/\mathcal{U}$ is λ^+-saturated.

Proof. (E) implies (D), and (C) implies (B) implies (A) are trivial.

(D) implies (A): Suppose $\mathbf{M}_*$ is λ^+-universal and $\mathbf{M}_*/\mathcal{U}$ is λ^+-saturated. Suppose $\mathbf{M} \models^{\mathcal{B}} T$ has $|\mathbf{M}| \leq \lambda$ and suppose $p(x)$ is a complete $\mathcal{U}$-type over $\mathbf{M}$. Since $\mathbf{M}_*$ is λ^+-universal, after relabeling we can suppose $\mathbf{M} \preceq \mathbf{M}_*$. But then $\mathbf{M}_*/\mathcal{U}$ realizes $[p(x)]_{\mathcal{U}}$, since it is λ^+-saturated.

(A) implies (C) is a standard union of chains argument.

Thus all the upward implications hold.

(A) implies (E): suppose (A) holds, and let $\mathbf{M}_* \models^{\mathcal{B}} T$ be λ^+-saturated. Choose $p(x)$ a partial $\mathcal{U}$-type over $\mathbf{M}$ of cardinality at most λ. We can suppose it is a partial $\mathcal{U}$-type over $\mathbf{M}_0 \preceq \mathbf{M}$, where $|\mathbf{M}_0| \leq \lambda$. Choose $\mathbf{N} \succeq \mathbf{M}_0$ such that $\mathbf{N}/\mathcal{U}$ realizes $[p(x)]_{\mathcal{U}}$; we can suppose $|\mathbf{N}| \leq \lambda$. Since $\mathbf{M}_*$ is λ^+-saturated, we can suppose after relabeling that $\mathbf{N} \preceq \mathbf{M}_*$, and finish. □

We thus feel justified in making the following definition. Previously, this definition was only made in the case when $\mathcal{U}$ is a λ-regular ultrafilter on $\mathcal{P}(\lambda)$.

Definition 3.5.3. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, and T is a complete countable theory. Then say that $\mathcal{U}$ λ^+-saturates T if some or every of the equivalent clauses of Theorem 3.5.2 hold; for instance, if some or every λ^+-saturated $\mathbf{M} \models^{\mathcal{B}} T$ satisfies that $\mathbf{M}/\mathcal{U}$ is λ^+-saturated.

Example 3.5.4. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{P}(\lambda)$ and T is a complete countable theory. Let $M \models T$ be λ^+-saturated. By Example 3.4.14, M^λ is a λ^+-saturated $\mathcal{P}(\lambda)$-valued model of T. Thus, $\mathcal{U}$ λ^+-saturates T if and only if $M^\lambda/\mathcal{U}$ is λ^+-saturated, which is the case if and only if $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-moral. In the case when $\mathcal{U}$ is λ-regular, this agrees with the standard definition of λ^+-saturation, so we have introduced no conflicts.

We will show in the next section, Theorem 3.6.10, that this holds always: if $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, then $\mathcal{U}$ λ^+-saturates T if and only if $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-moral. Also, at the end of the survey, we will prove Corollary 3.16.20, which in particular says the following:

Theorem 3.5.5. Suppose T_0, T_1 are complete countable theories. Then $T_0 \trianglelefteq_\lambda T_1$ if for every complete Boolean algebra with the λ^+-c.c., and for every ultrafilter $\mathcal{U}$ on $\mathcal{B}$, if $\mathcal{U}$ λ^+-saturates T_1 then $\mathcal{U}$ λ^+-saturates T_0.

The proof of Corollary 3.16.20 is involved, and the techniques used are not important for what we wish to do. This is why we defer the proof.

Alert. In the meantime, we take Theorem 3.5.5 as our operating definition of $\trianglelefteq_\lambda$.

We explore several alternative formulations of λ^+-saturation. First, we give a formulation that is most similar to ultrapowers. Frequently, in arguments involving Keisler's order, it is helpful to not only consider ultrapowers of models $M^\lambda/\mathcal{U}$, but also the ultrapower of the universe $\mathbb{V}^\lambda/\mathcal{U}$ in which they live. For a general complete Boolean algebra $\mathcal{B}$, $\mathbb{V}^{\mathcal{B}}$ is typically not saturated enough for this to be helpful, so instead we wish to consider saturated elementary extensions $\mathbf{i} : \mathbb{V} \preceq \mathbf{V}$. Actually, this presents formal difficulties due to quantifying over proper class maps, so instead we consider saturated elementary extensions $\mathbf{i} : V \preceq \mathbf{V}$, for transitive set models V of ZFC^-.

Convention. $\mathbb{V}$ always denotes the universe of sets. V will denote a transitive set model of ZFC^-. $\mathbf{V}$ will denote a full $\mathcal{B}$-valued model of ZFC^-, often associated with an elementary embedding $\mathbf{i} : V \preceq \mathbf{V}$.

Definition 3.5.6. Suppose $\mathcal{B}$ is a complete Boolean algebra, $V \models ZFC^-$ is transitive, and $\mathbf{i} : V \preceq \mathbf{V} \models^{\mathcal{B}} ZFC^-$. Suppose $M \in V$ is a structure in the countable language $\mathcal{L}$. Then let $\mathbf{i}_{\text{std}}(M)$ be the full $\mathcal{B}$-valued $\mathcal{L}$-structure defined as follows. Its domain is $\{\mathbf{a} \in \mathbf{V} : \|\mathbf{a} \in \mathbf{i}(M)\|_{\mathbf{V}} = 1\}$ (this set is also denoted $\mathbf{i}_{\mathrm{std}}(M)$). Given a formula $\varphi(\mathbf{a}_i : i < n) \in \mathcal{L}(\mathbf{i}_{\mathrm{std}}(M))$, let $\|\varphi(\mathbf{a}_i : i < n)\|_{\mathbf{i}_{\mathrm{std}}(M)} = \|\mathbf{i}(M) \models \varphi(\mathbf{a}_i : i < n)\|_{\mathbf{V}}$.

Example 3.5.7. Let $V \models ZFC^-$ and $\mathcal{B}$ is a complete Boolean algebra; write $\mathbf{V} = V^{\mathcal{B}} \models^{\mathcal{B}} ZFC^-$ and let $\mathbf{i} : V \preceq \mathbf{V}$ be the pre-Łoś embedding. Suppose $M \in V$. Then $\mathbf{i}_{\mathrm{std}}(M)$ is naturally isomorphic to $M^{\mathcal{B}}$.

Thus we can view the $\mathbf{i}_{\mathrm{std}}$-operator as a generalization of $M \mapsto M^{\mathcal{B}}$.

The following theorem is a simple application of Corollary 3.3.8.

Theorem 3.5.8. Suppose $\mathcal{B}$ is a complete Boolean algebra, $V \models ZFC^-$ and $\mathbf{i} : V \preceq \mathbf{V}$. Suppose $M \in V$ is a structure. If $\mathbf{V}$ is λ^+-saturated, then so is $\mathbf{i}_{\mathrm{std}}(M)$.

We immediately get the following.

Corollary 3.5.9. Suppose $\mathcal{B}$ is a complete Boolean algebra, $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, λ is a cardinal, and T is a complete countable theory. Then the following are equivalent:

(A) $\mathcal{U}$ λ^+-saturates T.

(B) For some or every transitive $V \models ZFC^-$, and for some or every $\mathbf{i} : V \preceq \mathbf{V}$ with $\mathbf{V}$ λ^+-saturated, and for some or every $M \models T$ with $M \in V$, $\mathbf{i}_{\mathrm{std}}(M)/\mathcal{U}$ is λ^+-saturated.

We now aim for a combinatorial criterion for whether or not $\mathcal{U}$ λ^+-saturates T, which will be helpful for when we want to forget all the model theory. The notion of distribution was already implicit in Keisler's work, but Malliaris was the first to use the word distribution [53]. The term Łoś map is also introduced by Malliaris in [53], in the case of $\mathcal{B} = \mathcal{P}(\lambda)$ and $\mathbf{M} = M^\lambda$ for some $M \models T$.

Definition 3.5.10. Given an index set I, an I-distribution in $\mathcal{B}$ is a function $\mathbf{A} : [I]^{<\aleph_0} \to \mathcal{B}_+$, such that $\mathbf{A}(\emptyset) = 1$, and $s \subseteq t$ implies $\mathbf{A}(s) \geq \mathbf{A}(t)$. If $\mathcal{D}$ is a filter on $\mathcal{B}$, we say that

$\mathbf{A}$ is in $\mathcal{D}$ if $\mathrm{im}(\mathbf{A}) \subseteq \mathcal{D}$. I will often be λ, but at other times it is convenient to let I be a partial type $p(\overline{x})$.

Say that $\mathbf{A}$ is multiplicative if for all $s \in [I]^{<\aleph_0}$, $\mathbf{A}(s) = \bigwedge_{i \in s} \mathbf{A}(\{i\})$. So, multiplicative distributions are in correspondence with maps $\mathbf{A} : I \to \mathcal{B}_+$, such that the image of $\mathbf{A}$ has the finite intersection property.

If $\mathbf{A}, \mathbf{B}$ are I-distributions in $\mathcal{B}$, then say that $\mathbf{B}$ refines $\mathbf{A}$ if $\mathbf{B}(s) \leq \mathbf{A}(s)$ for all $s \in [I]^{<\aleph_0}$.

If $\mathbf{A}$ is an I-distribution in $\mathcal{B}$ and $\mathbf{B}$ is a J-distribution in $\mathcal{B}$, then say that $\mathbf{A} \cong \mathbf{B}$ if there is some $\tau : \mathbf{A} \cong \mathbf{B}$, that is, some bijection $\tau : I \to J$ such that for all $s \in [I]^{<\aleph_0}$, $\mathbf{A}(s) = \mathbf{B}(\tau[s])$.

We now connect the notion of distributions to model theory.

Definition 3.5.11. Suppose T is a theory, suppose $\mathbf{M} \models^{\mathcal{B}} T$, and some $p(\overline{x})$ is a partial type over $\mathbf{M}$. Then the Łoś map of $p(\overline{x})$ is the $p(\overline{x})$-distribution $\mathbf{L}_{p(\overline{x})}$ in $\mathcal{B}$ defined as follows: $\mathbf{L}_{p(\overline{x})}(\Gamma(\overline{x})) = \|\exists \overline{x} \bigwedge \Gamma(\overline{x})\|_{\mathbf{M}}$, for each $\Gamma(\overline{x}) \in [p(\overline{x})]^{<\aleph_0}$. So if $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, then $p(\overline{x})$ is a partial $\mathcal{U}$-type if and only if $\mathbf{L}_{p(\overline{x})}$ is in $\mathcal{U}$.

Generally, $\mathbf{A}$ is an (I, T)-Łoś map in $\mathcal{B}$ if $\mathbf{A}$ is an I-distribution in $\mathcal{B}$, and $\mathbf{A}$ is isomorphic to the Łoś-map of some partial type $p(\overline{x})$ over some $\mathbf{M} \models^{\mathcal{B}} T$. Let the arity of $\mathbf{A}$ be the least possible arity of $p(\overline{x})$ witnessing this.

Say that $\overline{\varphi}$ is an I-sequence of formulas if $\overline{\varphi} = (\varphi_i(\overline{x}, \overline{y}_i) : i \in I)$ for some sequence of formulas $\varphi_i(\overline{x}, \overline{y}_i)$, where all of the $\overline{y}_i$'s are disjoint with each other and with $\overline{x}$. Let the arity of $\overline{\varphi}$ be the length of $\overline{x}$. Say that $\mathbf{A}$ is an $(I, T, \overline{\varphi})$-Łoś map if $\mathbf{A}$ is an $(I, T))$-Łoś map, and we can moreover choose the witnesses $\mathbf{M}, p(\overline{x}), \tau : \mathbf{A} \cong \mathbf{L}_{p(\overline{x})}$ such that for all $i \in I$, $\tau(\{i\})$ is an instance of $\varphi_i(\overline{x}, \overline{y}_i)$.

So $\mathbf{A}$ is an (I, T)-Łoś map if and only if $\mathbf{A}$ is an $(I, T, \overline{\varphi})$-Łoś map for some I-sequence of formulas $\overline{\varphi}$.

The following is a criterion for being a Łoś map that is often easier to evaluate. Essentially this is a special instance of Corollary 3.3.8.

Theorem 3.5.12. Suppose $\mathcal{B}$ is a complete Boolean algebra, $\mathbf{A}$ is an I-distribution, and $\overline{\varphi} = (\varphi_i(\overline{x}, \overline{y}_i) : i \in I)$ is an I-sequence of formulas. Then following are equivalent:

(A) $\mathbf{A}$ is an $(I, T, \overline{\varphi})$-Łoś map.

(B) For every ultrafilter $\mathcal{U}$ on $\mathcal{B}$, there is some $M \models T$ and some sequence $(\overline{a}_i : i \in I)$ from $M^{<\omega}$, such that for every $s \in [I]^{<\aleph_0}$, $M \models \exists \overline{x} \bigwedge_{i \in s} \varphi(\overline{x}, \overline{a}_i)$ if and only if $\mathbf{A}(s) \in \mathcal{U}$;

(C) For every $s \in [I]^{<\aleph_0}$, and for every $\mathbf{c} \in \mathcal{B}_+$ such that $\mathbf{c}$ decides $\mathbf{A}(t)$ for all $t \subseteq s$, there is some $M \models T$ and some sequence $(\overline{a}_i : i \in s)$ from $M^{<\omega}$, such that for each $t \subseteq s$, $M \models \exists \overline{x} \bigwedge_{i \in t} \varphi_i(x, \overline{a}_i : i \in s)$ if and only if $\mathbf{c} \leq \mathbf{A}(t)$.

Proof. Let $\Gamma \subseteq \mathcal{L}(\overline{y}_i : i \in I)$ be $T \cup \{\exists \overline{x} \bigwedge_{i \in s} \varphi_i(\overline{x}, \overline{y}_i) : s \in [I]^{<\aleph_0}\}$. Define $F : \Gamma \to \mathcal{B}$ via $F \restriction_T = 1$ and $F(\exists \overline{x} \bigwedge_{i \in s} \varphi_i(\overline{x}, \overline{y}_i)) = \mathbf{A}(s)$ for each $s \in [I]^{<\aleph_0}$.

Note then that (A) is equivalent to there being some $\mathbf{M} \models^{\mathcal{B}} T$ and some map $\tau : \{\overline{y}_i : i \in I\} \to \mathbf{M}$, such that for all $\psi(\overline{y}) \in \Gamma$, $F(\psi(\overline{y})) = \|\psi(\tau(\overline{y}))\|_{\mathbf{M}}$. Consider Corollary 3.3.8 with $X = \{\overline{y}_i : i \in I\}$, Γ as defined above and $F_0 = F_1 = F$. Then easily, (A), (B), (C) of Corollary 3.3.8 are equivalent to (A), (B), (C) here, and so they are all equivalent. $\square$

The following fundamental theorem explains why we care about distributions.

Theorem 3.5.13. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, and $p(\overline{x})$ is a partial type over $\mathbf{M} \models^{\mathcal{B}} T$. Then the following are equivalent:

(A) There is $\mathbf{N} \succeq \mathbf{M}$ such that $\mathbf{N}/\mathcal{U}$ contains a realization of $[p(\overline{x})]_{\mathcal{U}}$;

(B) $\mathbf{L}_{p(\overline{x})}$ has a multiplicative refinement in $\mathcal{U}$.

Proof. (B) implies (A): let $\mathbf{B}$ be a multiplicative refinement of $\mathbf{L}_{p(\overline{x})}$ in $\mathcal{U}$. We apply Corollary 3.3.8. Let $X = \mathbf{M} \cup \{\overline{x}\}$. Let $\Gamma = \mathcal{L}(\mathbf{M}) \cup p(\overline{x})$, and define $F_0 : \Gamma \to \mathcal{B}$ via $F_0 \restriction_{\mathbf{M}} = F_1 \restriction_{\mathbf{M}} = \|\cdot\|_{\mathbf{M}}$ and, for each $\varphi(\overline{x}) \in p(\overline{x})$, $F_0(\varphi(\overline{x})) = \mathbf{B}(\{\varphi(\overline{x})\})$ and $F_1(\varphi(\overline{x})) = 1$. Multiplicativity of $\mathbf{B}$ and the definition of a Loś map translates into (C) of Corollary 3.3.8 hold, and so by (A) of Corollary 3.3.8 and λ^+-saturation of $\mathbf{M}$, we can find $\overline{a} \in \mathbf{M}$ such that for each $\varphi(\overline{x}) \in p(\overline{x})$, $\|\varphi(\overline{x})\|_{\mathbf{M}} = \mathbf{B}(\{\varphi(\overline{x})\})$. Then $[\overline{a}]_{\mathcal{U}}$ realizes $[p(\overline{x})]_{\mathcal{U}}$.

(A) implies (B): Choose $\mathbf{N} \succeq \mathbf{M}$ and $\overline{b} \in \mathbf{N}$ such that $[\overline{b}]_{\mathcal{U}}$ realizes $[\dot{p}(\overline{x})]_{\mathcal{U}}$. For each $\Gamma(\overline{x}) \in [\mathrm{dom}(\dot{p}(\overline{x}))]^{<\aleph_0}$, put $\mathbf{B}(\Gamma(\overline{x})) = \|\Gamma(\overline{b})\|_{\mathbf{N}}$. Then this is easily a multiplicative refinement of $\mathbf{L}_{p(\overline{x})}$. $\square$

Thus:

Theorem 3.5.14. Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$, and suppose T is a theory. Then the following are equivalent:

(A) $\mathcal{U}$ λ^+-saturates T;

(B) Every (λ, T)-Łoś map in $\mathcal{U}$ has a multiplicative refinement in $\mathcal{U}$;

(C) Every (λ, T)-Łoś map of arity 1 in $\mathcal{U}$ has a multiplicative refinement in $\mathcal{U}$.

Proof. For (A) if and only if (B), use Lemma 3.5.13 and formulation (B) of Theorem 3.5.2. For (B) if and only if (C), use Theorem 3.5.2 (A) if and only if (B). $\square$

3.6 Saturation of Ultrapowers

In this section, we connect our previous definitions with saturation of ultrapowers, in particular we will prove Theorem 3.1.2(A). We continue to proceed in the generality of a complete Boolean algebra $\mathcal{B}$; this necessitates strengthening the notion of regularity. Many of the ideas required to adapt to the general case are motivated by arguments of Mansfield in [62].

Definition 3.6.1. Suppose I is an index set and suppose $\mathcal{B}$ is a complete Boolean algebra. Then $(\mathbf{a}_i : i \in I)$ is an I-regular sequence from $\mathcal{B}$ if $(\mathbf{a}_i : i \in I)$ has the finite intersection property, and for each $J \in [I]^{\aleph_0}$, $\bigwedge_{i \in J} \mathbf{a}_i = 0$. Say that $(\mathbf{a}_i : i \in I)$ is strongly I-regular if additionally, the set of all $\mathbf{b} \in \mathcal{B}_+$ which decide each $\mathbf{a}_i$ is dense.

Suppose $\mathcal{D}$ is a filter on $\mathcal{B}$ and λ is a cardinal. Then $\mathcal{D}$ is (strongly) λ-regular if there is a (strongly) λ-regular sequence $(\mathbf{a}_\alpha : \alpha < \lambda)$ such that each $\mathbf{a}_\alpha \in \mathcal{D}$.

For example, if $\mathcal{U}$ is a λ-regular ultrafilter on $\mathcal{P}(\lambda)$, then $\mathcal{U}$ is also strongly λ-regular. In general this holds whenever $\mathcal{U}$ is λ^+-distributive.

The following easy lemma is proved in many places for $\mathcal{P}(\lambda)$, see e.g. Lemma 1.3 from Chapter 6 of [75].

Lemma 3.6.2. Suppose λ is infinite and $\mathcal{B}$ has an antichain of size λ. Then $\mathcal{B}$ admits a strongly λ-regular sequence, and hence also a strongly λ-regular ultrafilter.

Proof. Let $(\mathbf{c}_s : s \in [\lambda]^{<\aleph_0})$ be an antichain from $\mathcal{B}$. For each $\alpha < \lambda$, let $\mathbf{a}_\alpha = \bigcup_{\alpha \in s} \mathbf{c}_s$. Then $(\mathbf{a}_\alpha : \alpha < \lambda)$ is strongly $(\aleph_0, \lambda)$-regular. Any extension to an ultrafilter is strongly λ-regular. ∎

In fact, the converse holds. We remark that when building ultrafilters with λ^+-

saturation in mind, the main case of interest is in complete Boolean algebras $\mathcal{B}$ with the λ-c.c.; thus, having an antichain of size λ should be viewed as rare.

Theorem 3.6.3. Suppose $\mathcal{B}$ is a complete Boolean algebra. Then $\mathcal{B}$ admits a strongly λ-regular sequence if and only if $\mathcal{B}$ has an antichain of size λ.

Proof. If $\mathcal{B}$ has an antichain of size λ, use Lemma 3.6.2. Conversely, suppose $\mathcal{B}$ admits a strongly λ-regular sequence $(\mathbf{a}_\alpha : \alpha < \lambda)$.

Let $\mathbf{c}_\gamma : \gamma < \kappa$ be a maximal antichain from $\mathcal{B}$ such that each $\mathbf{c}_\gamma$ decides each $\mathbf{a}_\alpha$. For each $\gamma < \kappa$, define $Y_\gamma = \{\alpha < \lambda : \mathbf{c}_\gamma \leq \mathbf{a}_\alpha\}$. So each $|Y_\gamma| < \aleph_0$, but $\bigcup_\gamma Y_\gamma = \lambda$. Thus λ is the union of κ-many finite sets, so $\kappa = \lambda$, and $\mathcal{B}$ has an antichain of size λ. □

We will want the following lemma.

Lemma 3.6.4. Suppose $\mathcal{B}$ is a complete Boolean algebra, and $(\mathbf{a}_i : i \in I)$ is a strongly I-regular sequence from $\mathcal{B}$, and $\mathbf{b}_i \leq \mathbf{a}_i$ for each i, and $(\mathbf{b}_i : i \in I)$ has the finite intersection property. Then $(\mathbf{b}_i : i \in I)$ is strongly I-regular.

Proof. We need to show that the set of all $\mathbf{c} \in \mathcal{B}_+$ which decide each $\mathbf{b}_i$ is dense.

Given $\mathbf{c} \in \mathcal{B}_+$, choose $\mathbf{c}_0 \leq \mathbf{c}$ such that $\mathbf{c}_0$ decides each $\mathbf{a}_i$. Let $X = \{i \in I : \mathbf{c}_0 \leq \mathbf{a}_i\}$, a finite subset of I. Note that $\mathbf{c}_0 \leq \neg\mathbf{b}_i$ for each $i \notin X$. Choose $\mathbf{c}_1 \leq \mathbf{c}_0$ such that $\mathbf{c}_1$ decides $\mathbf{b}_i$ for each $i \in X$; then clearly $\mathbf{c}_1$ decides $\mathbf{b}_i$ for all $i \in I$. □

In order to deduce saturation properties of $M^{\mathcal{B}}/\mathcal{U}$ from regularity of $\mathcal{U}$, we will need a more general notion of (λ, T)-Łoś maps.

Definition 3.6.5. Suppose $\mathbf{A}$ and $\mathbf{B}$ are I-distributions in $\mathcal{B}$. Then say that $\mathbf{B}$ conservatively refines $\mathbf{A}$ if there is a multiplicative I-distribution $\mathbf{C}$ such that each $\mathbf{B}(s) = \mathbf{A}(s) \wedge \mathbf{C}(s)$. Equivalently, for all $s \in [\lambda]^{<\aleph_0}$, $\mathbf{B}(s) = \mathbf{A}(s) \wedge \bigwedge_{i \in s} \mathbf{B}(\{i\})$.

We remark on a simple but important point.

Lemma 3.6.6. Let $\mathcal{B}$ be a complete Boolean algebra and let $\mathcal{U}$ be an ultrafilter on $\mathcal{B}$. If $\mathbf{A}$ is an I-distribution in $\mathcal{U}$ and $\mathbf{B}$ is a conservative refinement of $\mathbf{A}$ in $\mathcal{U}$, then $\mathbf{B}$ has a multiplicative refinement in $\mathcal{U}$ if and only if $\mathbf{A}$ does.

Proof. Since $\mathbf{B}$ refines $\mathbf{A}$, any multiplicative refinement of $\mathbf{B}$ is also one of $\mathbf{A}$.

Suppose $\mathbf{C}$ is a multiplicative refinement of $\mathbf{A}$ in $\mathcal{U}$. Then define $\mathbf{C}'(s) = \mathbf{C}(s) \wedge \bigwedge_{i \in s} \mathbf{A}(\{i\}) \in \mathcal{U}$. This is clearly multiplicative, and since $\mathbf{C}$ refines $\mathbf{B}$, we get $\mathbf{C}'(s) \leq \mathbf{B}(s) \wedge \bigwedge_{i \in s} \mathbf{A}(\{i\}) = \mathbf{A}(s)$ so $\mathbf{C}'$ refines $\mathbf{A}$. $\square$

Definition 3.6.7. Suppose $\mathbf{A}$ is a distribution in $\mathcal{B}$. Then say $\mathbf{A}$ is an (I, T)-possibility if $\mathbf{A}$ is a conservative refinement of an (I, T)-Łoś map. If $\overline{\varphi}$ is an I-sequence of formulas, then say that $\mathbf{A}$ is an $(I, T, \overline{\varphi})$-possibility if $\mathbf{A}$ is a conservative refinement of an $(I, T, \overline{\varphi})$-Łoś map.

So $\mathbf{A}$ is an (I, T)-possibility if and only if $\mathbf{A}$ is an $(I, T, \overline{\varphi})$-possibility for some I-sequence of formulas $\overline{\varphi}$.

The following is analogous to Theorem 3.5.12.

Theorem 3.6.8. Suppose $\mathcal{B}$ is a complete Boolean algebra, $\mathbf{A}$ is an I-distribution, and $\overline{\varphi} = (\varphi_i(\overline{x}, \overline{y}_i) : i \in I)$ is an I-sequence of formulas. Then following are equivalent:

(A) $\mathbf{A}$ is an $(I, T, \overline{\varphi})$-possibility.

(B) For every ultrafilter $\mathcal{U}$ on $\mathcal{B}$, there is some $M \models T$ and some sequence $(\overline{a}_i : i \in I)$ from $M^{<\omega}$, such that for every $s \in [I]^{<\aleph_0}$ with $\mathbf{A}(\{i\}) \in \mathcal{U}$ for all $i \in s$, $M \models \exists \overline{x} \bigwedge_{i \in s} \varphi(\overline{x}, \overline{a}_i)$ if and only if $\mathbf{A}(s) \in \mathcal{U}$;

(C) For every $s \in [I]^{<\aleph_0}$, and for every $\mathbf{c} \in \mathcal{B}_+$ such that $\mathbf{c}$ decides $\mathbf{A}(t)$ for all $t \subseteq s$ and such that $\mathbf{c} \leq \mathbf{A}(\{i\})$ for all $i \in s$, there is some $M \models T$ and some sequence $(\overline{a}_i : i \in s)$ from $M^{<\omega}$, such that for each $t \subseteq s$, $\exists \overline{x} \bigwedge_{i \in t} \varphi_i(x, \overline{a}_i : i \in s)$ is consistent if and only if $\mathbf{c} \leq \mathbf{A}(t)$.

Proof. Let $\Gamma \subseteq \mathcal{L}(\overline{y}_i : i \in I)$ be $T \cup \{\exists \overline{x} \bigwedge_{i \in s} \varphi_i(\overline{x}, \overline{y}_i) : s \in [I]^{<\aleph_0}\}$. Define $F_0 : \Gamma \to \mathcal{B}$ via $F_0 \restriction_T = 1$ and $F_0(\exists \overline{x} \bigwedge_{i \in s} \varphi_i(\overline{x}, \overline{y}_i)) = \mathbf{A}(s)$ for each $s \in [I]^{<\aleph_0}$. Define $F_1 : \Gamma \to \mathcal{B}$ via $F_1 \restriction_T = 1$ and $F_1(\exists \overline{x} \bigwedge_{i \in s} \varphi_i(\overline{x}, \overline{y}_i)) = \mathbf{A}(s) \vee \bigvee_{i \in s} \neg \mathbf{A}_{\{i\}}$ for each $s \in [I]^{<\aleph_0}$.

I claim that (A) is equivalent to there being some $\mathbf{M} \models^{\mathcal{B}} T$ and some map $\tau : \{\overline{y}_i : i \in I\} \to \mathbf{M}$, such that for all $\psi(\overline{y}) \in \Gamma$, $F_0(\psi(\overline{y})) \leq \|\psi(\tau(\overline{y}))\|_{\mathbf{M}} \leq F_1(\psi(\overline{y}))$. Indeed, suppose (A) holds via $\mathbf{B}$, that is $\mathbf{B}$ is an $(I, T, \overline{\varphi})$-Łoś map and $\mathbf{A}$ is a conservative refinement of $\mathbf{B}$. Choose $\tau_* : \mathbf{B} \cong \mathbf{L}_{p(x)}$, where $p(x)$ is a partial type over $\mathbf{M}$. Let $\mathbf{A}' = \mathbf{A} \circ \tau_*^{-1}$, a $p(x)$-distribution with $\tau_* : \mathbf{A} \cong \mathbf{A}'$. Note $\tau_*(\varphi_i(x, \overline{y}_i)) = \varphi_i(x, \overline{a}_i)$ for some $\overline{a}_i \in \mathbf{M}$; let $\tau : \{\overline{y}_i : i \in I\} \to \mathbf{M}$ be given by $\overline{y}_i \mapsto \overline{a}_i$ (this is well-defined since we are assuming the $\overline{y}_i$'s are all disjoint from each other). Clearly this works, and the argument reverses.

Consider Corollary 3.3.8 with $X = \{\overline{y}_i : i \in I\}$, and Γ, F_0, F_1 as defined above. Then easily, (A), (B), (C) of Corollary 3.3.8 are equivalent to (A), (B), (C) here, and so they are all equivalent. □

It is finally convenient to state Malliaris and Shelah's definition of morality from [56]:

Definition 3.6.9. Suppose T is a complete countable theory, $\mathcal{B}$ is a complete Boolean algebra, and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-moral if every (λ, T)-possibility $\mathbf{A}$ in $\mathcal{U}$ has a multiplicative refinement in $\mathcal{U}$.

Then we have the following, as promised:

Theorem 3.6.10. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ λ^+-saturates T if and only if $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-moral.

Proof. By Theorem 3.5.14 and Lemma 3.6.6. □

Henceforward we avoid the terminology "$(\lambda, \mathcal{B}, T)$" moral.

We now integrate the hypothesis of regularity.

Definition 3.6.11. Suppose $\mathbf{A}$ is an I-distribution in $\mathcal{B}$. Then say that $\mathbf{A}$ is strongly I-regular if $(\mathbf{A}(\{i\}) : i \in I)$ is strongly I-regular.

So by Lemma 3.6.4, if $\mathbf{A}$ is strongly I-regular and $\mathbf{B}$ is a refinement of $\mathbf{A}$, then $\mathbf{B}$ is strongly I-regular. Also, easily if $\mathbf{A}$ is strongly I-regular, then the set of all $\mathbf{c} \in \mathcal{B}$ which decide $\mathbf{A}(s)$ for all $s \in [I]^{<\aleph_0}$ is dense, and hence $(\mathbf{A}(s) : s \in [I]^{<\aleph_0})$ is strongly $[I]^{<\aleph_0}$-regular.

We are now finally ready to prove Keisler's original Theorem 3.1.2(A). (Note that we have already proved (B), via Theorem 3.5.2 and Example 3.4.14.)

Theorem 3.6.12. Suppose $\mathcal{B}$ is a complete Boolean algebra, and $\mathcal{U}$ is a strongly λ-regular ultrafilter on $\mathcal{B}$. Then the following are equivalent:

(A) $\mathcal{U}$ λ^+-saturates T;

(B) Every strongly λ-regular (λ, T)-possibility in $\mathcal{U}$ has a multiplicative refinement in $\mathcal{U}$;

(C) For some $M \models T$, $M^{\mathcal{B}}/\mathcal{U}$ is λ^+-saturated;

(D) For every $M \models T$, $M^{\mathcal{B}}/\mathcal{U}$ is λ^+-saturated.

Proof. (A) if and only if (B) is by Theorem 3.6.10 and Lemma 3.6.4.

(B) implies (D): suppose $M \models T$, and $p(x)$ is a partial $\mathcal{U}$-type over $M^{\mathcal{B}}$ of cardinality $\leq \lambda$. Let $\mathbf{A}$ be a conservative, strongly I-regular refinement of $\mathbf{L}_{p(x)}$ in $\mathcal{U}$. Let $\mathbf{B}$ be a multiplicative refinement of $\mathbf{A}$ in $\mathcal{U}$. Let $\mathbf{C}$ be a maximal antichain of $\mathcal{B}$ such that each $\mathbf{c} \in \mathbf{C}$ decides each $\mathbf{B}(\{\varphi(x)\})$. For each $\mathbf{c} \in \mathbf{C}$, let $\Gamma_{\mathbf{c}}(x) = \{\varphi(x) \in p(x) : \mathbf{c} \leq \mathbf{B}(\{\varphi(x)\})\}$. Then $|\Gamma_{\mathbf{c}}(x)| < \aleph_0$, and so we can $f(\mathbf{c}) \in M$ realizing $\Gamma_{\mathbf{c}}(x)$. Then clearly $[(\mathbf{C}, f(\mathbf{c}))]_{\mathcal{U}} \in M^{\mathcal{B}}/\mathcal{U}$ realizes $p(x)$. (Here we are viewing $M^{\mathcal{B}}$ as the set of inverse partitions of $\mathcal{B}$ by M.)

(D) implies (C): trivial.

(C) implies (B): Choose $M \models T$ such that $M^{\mathcal{B}}/\mathcal{U}$ is λ^+-saturated, and let $\mathbf{A}$ be a strongly λ-regular (λ, T)-possibility. Say $\mathbf{A}$ is a $(\lambda, T, \overline{\varphi})$-possibility, where $\overline{\varphi} = (\varphi_\alpha(x, y_\alpha) : \alpha < \lambda)$. Let $\mathbf{C}$ be a maximal antichain of $\mathcal{B}$ such that each $\mathbf{c} \in \mathbf{C}$ decides each $\mathbf{A}(s)$, for $s \in [\lambda]^{<\aleph_0}$. Given $\mathbf{c} \in \mathbf{C}$, let $\Delta_{\mathbf{c}} = \{\alpha < \lambda : \mathbf{c} \leq \mathbf{A}(\{\alpha\})\}$; so $|\Delta_{\mathbf{c}}| < \aleph_0$. Thus we can find $(f_\alpha(\mathbf{c}) : \alpha \in \Delta_{\mathbf{c}})$ from M such that for all $s \in [\Delta_{\mathbf{c}}]^{<\aleph_0}$, $M \models \exists x \bigwedge_{\alpha \in s} \varphi_\alpha(x, f_\alpha(\mathbf{c}))$ if and only if $\mathbf{c} \leq \mathbf{A}(s)$. Let $f_\alpha(\mathbf{c})$ be arbitrary if $\alpha \not\in \Delta_{\mathbf{c}}$.

For each $\alpha < \lambda$, let $a_\alpha \in M^{\mathcal{B}}$ be the element corresponding to the inverse partition $(\mathbf{C}, f_\alpha)$ of M. Let $p(x) = \{\varphi_\alpha(x, a_\alpha) : \alpha < \lambda\}$. Note that given $s \in [\lambda]^{<\aleph_0}$, $\|\exists x \bigwedge_{\alpha \in s} \varphi_\alpha(x, a_\alpha)\|_{M^{\mathcal{B}}} = \bigvee\{\mathbf{c} \in \mathbf{C} : \mathbf{c} \leq \mathbf{A}(s)\} = \mathbf{A}(s) \in \mathcal{U}$. In particular $p(x)$ is a partial $\mathcal{U}$-type over $M^{\mathcal{B}}$. Let $a \in M^{\mathcal{B}}$ be such that $[a]_{\mathcal{U}}$ realizes $p(x)$. After refining $\mathbf{C}$, we can suppose a is represented by the inverse partition $(\mathbf{C}, f)$ of M.

For each $s \in [\lambda]^{<\aleph_0}$, let $\mathbf{B}(s) = \|\bigwedge_{\alpha \in s} \varphi_\alpha(a, a_\alpha)\|$. So $\mathbf{B}(s)$ is a multiplicative distribution in $\mathcal{U}$, and clearly $\mathbf{B}(s)$ refines $\mathbf{A}(s)$. $\square$

3.7 Good Ultrafilters

The following definition is natural, in view of Theorem 3.5.14; it is originally due to Keisler [34] (in the case $\mathcal{B} = \mathcal{P}(\lambda)$), although we drop his requirement that $\mathcal{U}$ be $\aleph_1$-incomplete.

Definition 3.7.1. The ultrafilter $\mathcal{U}$ on the complete Boolean algebra $\mathcal{B}$ is λ^+-good if every λ-distribution in $\mathcal{U}$ has a multiplicative refinement in $\mathcal{U}$.

For example, the unique ultrafilter on $\{0, 1\}$ is λ^+-good for all λ. More generally, principal ultrafilters are λ^+-good for all λ. Theorem 3.15.1 shows the converse as well.

Also, note that if $\mathcal{U}$ is λ-complete then it is λ^+-good: given a λ-distribution $\mathbf{A}$ in $\mathcal{U}$, just let $\mathbf{B}(s) = \bigwedge\{\mathbf{A}(t) : \max(t) \leq \max(s)\}$. Then $\mathbf{B}$ is a multiplicative refinement of $\mathbf{A}$ in $\mathcal{U}$. In particular, every ultrafilter is $\aleph_1$-good. Also note that if $\mathcal{U}$ is λ^+-good and $\kappa \leq \lambda$ then $\mathcal{U}$ is also κ^+-good.

The following theorem of Keisler [34] is the key property of λ^+-good ultrafilters.

Theorem 3.7.2. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then the following are equivalent:

(A) $\mathcal{U}$ is λ^+-good.

(B) For every countable complete theory T, $\mathcal{U}$ λ^+-saturates T.

(C) $\mathcal{U}$ λ^+-saturates $\mathrm{Th}([\omega]^{<\aleph_0}, \subseteq)$.

Thus $\mathrm{Th}([\omega]^{<\aleph_0}, \subseteq)$ is maximal in Keisler's order.

Proof. (A) implies (B): by Theorem 3.5.14.

(B) implies (C): trivial.

(C) implies (A): Let $T = \mathrm{Th}([\omega]^{<\aleph_0}, \subseteq)$.

Suppose $\mathcal{U}$ is an ultrafilter on I, and is not λ^+-good. That means that there is a distribution $\mathbf{A}$ in $\mathcal{U}$ which has no multiplicative refinement in $\mathcal{U}$. Let $\overline{\varphi} = (\varphi_\alpha(x, y_\alpha))$ be defined by: $\varphi_\alpha(x, y_\alpha)$ is $x \subseteq y_\alpha$. Then by Theorem 3.6.10 it suffices to show that $\mathbf{A}$ is a $(\lambda, T, \overline{\varphi})$-Łoś map.

We apply Lemma 3.5.12, using Characterization (C). So let $s \subset \lambda$ be finite and let $\mathbf{c} \in \mathcal{B}$ decide $\mathbf{A}(t)$ for all $t \subseteq s$. Let $\mathcal{J} = \{t \subseteq s : \mathbf{c} \leq \mathbf{A}(\{t\})$. Let $(n_t : t \in \mathcal{J})$ be distinct elements of ω, and for each $i \in s$, let $a_i = \{n_t : i \in t \in \mathcal{J}\}$. Then clearly for any $t \subseteq s$, $\bigcap_{i \in t} a_i$ is nonempty if and only if $t \in \mathcal{J}$, as desired. $\square$

We also remark on the following theorem of Mansfield, namely Theorem 4.1 [62]:

Theorem 3.7.3. Suppose $\mathcal{U}$ is a λ^+-good and $\aleph_1$-incomplete ultrafilter on $\mathcal{B}$. Then for any $\{0, 1\}$-valued structure M, $M^{\mathcal{B}}/\mathcal{U}$ is λ^+-saturated. In particular, if $\mathcal{U}$ is $\aleph_1$-incomplete then $M^{\mathcal{B}}/\mathcal{U}$ is $\aleph_1$-saturated.

The proof of Theorem 3.7.3 proceeds by showing that if $\mathcal{U}$ is λ^+-good and $\aleph_1$-incomplete, then $\mathcal{U}$ is strongly λ-regular, and so we can apply Theorems 3.6.12 and 3.7.2. (Mansfield did not use the terminology of strongly λ-regular, but the concept is implicit, and in fact this motivated our definition of strongly λ-regular.) We will eventually prove the stronger Theorem 3.14.16, which optimizes the hypothesis of λ^+-good; so we defer the proof of Theorem 3.7.3 for now.

We now discuss existence of λ^+-good ultrafilters. In [34], Keisler showed that if $2^\lambda = \lambda^+$, then there is a λ-regular ultrafilter on $\mathcal{P}(\lambda)$. In [45], Kunen removed the hypothesis that $2^\lambda = \lambda^+$. With Theorem 3.16.1, we will prove that if $\mathcal{B}$ is any complete Boolean algebra with an antichain of size λ, then $\mathcal{B}$ admits a strongly λ-regular, λ^+-good ultrafilter. Conversely, it will follow from Theorem 3.15.1 that if $\mathcal{U}$ is a nonprincipal,

λ^+-good ultrafilter on $\mathcal{B}$, then $\mathcal{B}$ has an antichain of size λ.

Constructions of ultrafilters on $\mathcal{P}(\lambda)$ are somewhat complicated, due to the fact that $\mathcal{P}(\lambda)$ is cramped. In this section, we present a considerably simpler proof that there is a strongly λ-regular, λ^+-good ultrafilter on $\mathcal{B}_{2^\lambda \aleph_0 \aleph_0}$. We hope the reader will find this enlightening, although we will not make direct use of it. We recall the definitions:

Definition 3.7.4. For sets X, Y and a regular cardinal θ, let $P_{XY\theta}$ be the forcing notion of all functions partial functions from X to Y of cardinality less than θ, ordered by reverse inclusion. Let $\mathcal{B}_{XY\theta}$ be its Boolean algebra completion.

Theorem 3.7.5. Suppose λ is a cardinal. Then there is a strongly λ-regular, λ^+-good ultrafilter $\mathcal{U}$ on $\mathcal{B}_{2^\lambda \aleph_0 \aleph_0}$.

Proof. For each $\alpha \leq 2^\lambda$, write $\mathcal{B}_\alpha = \mathcal{B}_{\alpha \aleph_0 \aleph_0}$, a complete subalgebra of $\mathcal{B}_{2^\lambda \aleph_0 \aleph_0}$. We construct an increasing chain of ultrafilters $\mathcal{U}_\alpha$ on $\mathcal{B}_\alpha$, by induction on $\alpha \leq 2^\lambda$.

By Theorem 3.6.3, we can find a strongly λ-regular ultrafilter $\mathcal{U}_1$ on $\mathcal{B}_1$. Note that any ultrafilter on $\mathcal{B}_{2^\lambda}$ extending $\mathcal{U}_1$ will also be strongly λ-regular; so now all we have to arrange is λ^+-goodness.

Note that each $\mathcal{B}_\alpha$ has the λ^+-c.c. (by the Δ-system lemma). In particular, every element of $\mathcal{B}_{2^\lambda}$ can be written as the join of λ-many elements from $P_{2^\lambda \aleph_0 \aleph_0}$. Hence $\mathcal{B}_{2^\lambda} = \bigcup_{\alpha < 2^\lambda} \mathcal{B}_\alpha$ (since $\mathrm{cof}(2^\lambda) > \lambda$), and every λ-distribution in $\mathcal{B}_{2^\lambda}$ is in $\mathcal{B}_\alpha$ for some $\alpha < \lambda$. Also, there are only $|\mathcal{B}^\lambda| = 2^\lambda$-many λ-distributions in $\mathcal{B}$.

Thus, by a typical diagonalization argument, it suffices to verify the following:

Claim. Suppose $\mathcal{U}_\alpha$ is an ultrafilter on $\mathcal{B}_\alpha$, and $\mathbf{A}$ is a λ-distribution in $\mathcal{U}_\alpha$. Then there is an ultrafilter $\mathcal{U}_{\alpha+1}$ on $\mathcal{B}_{\alpha+1}$ extending $\mathcal{U}_\alpha$, such that $\mathbf{A}$ has a multiplicative refinement in $\mathcal{U}_{\alpha+1}$.

Proof. Choose a bijection $\rho : [\lambda]^{<\aleph_0} \to \lambda$. For each $s \in [\lambda]^{<\aleph_0}$, let $\mathbf{c}_s = \{(\alpha, \rho(s))\} \in P_{\alpha+1\,\aleph_0\aleph_0} \subseteq \mathcal{B}_{\alpha+1}$; so $(\mathbf{c}_s : s \in [\lambda]^{<\aleph_0})$ is an antichain, and whenever $\mathbf{a} \in \mathcal{B}_\alpha$ is nonzero, then $\mathbf{a} \wedge \mathbf{c}_s$ is nonzero for all s. For each $s \in [\lambda]^{<\aleph_0}$, define $\mathbf{B}(s) = \bigvee\{\mathbf{A}(t) \wedge \mathbf{c}_t : s \subseteq t \in [\lambda]^{<\aleph_0}$.

$\mathbf{B}$ is clearly a λ-distribution.

I claim that $\mathbf{B}$ is multiplicative; let $s \in [\lambda]^{<\aleph_0}$. Suppose towards a contradiction $\mathbf{e} := \left(\bigwedge_{\alpha \in s} \mathbf{B}_{\{\alpha\}}\right) \wedge (\neg \mathbf{B}(s))$ were nonzero. Then we can find $\mathbf{e}' \leq \mathbf{e}$ nonzero, and $(s_\alpha : \alpha \in s)$ a sequence from $[\lambda]^{<\aleph_0}$, such that each $\alpha \in s_\alpha$, and such that $\mathbf{e}' \leq \mathbf{A}(s_\alpha) \wedge \mathbf{c}_{s_\alpha}$ for each $\alpha \in s$. Since $(\mathbf{c}_s : s \in [\lambda]^{<\aleph_0})$ is an antichain this implies $s_\alpha = s_{\alpha'} = t$ say, for all $\alpha, \alpha' \in s$. Visibly then $s \subseteq t$, and so $\mathbf{e}' \leq \mathbf{A}(t) \wedge \mathbf{c}_t$, contradicting that $\mathbf{e}' \wedge \mathbf{B}(s) = 0$.

I claim that $\mathcal{U}_\alpha \cup \{\mathbf{B}(s) : s \in [\lambda]^{<\aleph_0}\}$ has the finite intersection property, which suffices. So suppose towards a contradiction it did not; then we can find $s \in [\lambda]^{<\aleph_0}$ and $\mathbf{a} \in \mathcal{U}_\alpha$ such that $\mathbf{a} \wedge \mathbf{B}(s) = 0$. But then $\mathbf{a} \wedge \mathbf{A}(s) \wedge \mathbf{c}_s = 0$, so $\mathbf{a} \wedge \mathbf{A}(s) = 0$, but $\mathbf{A}(s) \in \mathcal{U}_\alpha$ so this is a contradiction. □

The above ultrafilter construction is extremely typical; in fact, we will always just be trying to construct as generic an ultrafilter as possible on $\mathcal{B}$, and relying on properties of $\mathcal{B}$ to control saturation. Note that if $\mathcal{B}$ has many antichains of size λ then we expect to get a λ^+-good ultrafilter, which is an uninteresting outcome. Hence we will focus on complete Boolean algebras $\mathcal{B}$ with the λ-chain condition, and so our ultrafilters will not be strongly λ-regular.

3.8 The Interpretability Orders

Suppose we are trying to show $T_0 \trianglelefteq_\lambda T_1$. So let $\mathcal{B}$ be a complete Boolean algebra with the λ^+-c.c., let $\mathcal{U}$ be an ultrafilter on $\mathcal{B}$, let $V \models ZFC^-$ be transitive and let

$\mathbf{i} : V \preceq \mathbf{V}$, where $\mathbf{V} \models^{\mathcal{B}} ZFC^-$ is λ^+-saturated. Then it suffices to show that for some or every $M_i \models T_i$ with $M_i \in V$, if $\mathbf{i}_{\text{std}}(M_1)/\mathcal{U}$ is λ^+-saturated, then so is $\mathbf{i}_{\text{std}}(M_0)/\mathcal{U}$.

Write $\hat{V} = \mathbf{V}/\mathcal{U}$, a $\{0,1\}$-valued model of ZFC^-; let $\mathbf{j} : V \preceq \hat{V}$ be the composition $[\cdot]_{\mathcal{U}} \circ \mathbf{i}$. It turns out that most of our arguments really just involve $\hat{V}$, and in fact the presence of $\mathcal{U}, \mathcal{B}, \mathbf{V}$ is distracting notational baggage.

The interpetability orders $\trianglelefteq^*_{\lambda\kappa}$ capture this abstract situation; they were introduced by Shelah [78]. We recall their definitions from Section 3.1. As a convenient piece of notation (following [61]), we say that every structure M is 1-saturated.

Definition 3.8.1. Suppose T, T_* are complete countable theories, and $\mathfrak{C}_*$ is the monster model of T_*. Then an interpretation of T in T_* is given by some definable subset X of $\mathbf{C}^n_*$, and for each m-ary relation symbol $R \in \mathcal{L}_T$, an m-ary definable subset $R_* \subseteq X^m$, and for each m-ary function symbol $f \in \mathcal{L}_T$, an m-ary definable function $f_* : X^m \to X$, such that $(X, \ldots) \models T$. Given $M_* \models T_*$ we always get an interpreted model $M \models T$. We depict interpretations as functions $I : M_* \mapsto M$.

Suppose κ is an infinite cardinal or 1, and $\lambda \geq \aleph_0$. Suppose T_0, T_1 are complete countable theories. Then say that $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$ if there is a complete countable theory $T_*$and interpretations I_0, I_1 of M_* in T_*, such that for all κ-saturated $M_* \models T_*$, if we let $M_i = I_i(M_*)$ be the interpreted model of T_i in M_*, then: if M_1 is λ^+-saturated, then so is M_0. Say that $T_0 \trianglelefteq^*_\kappa T_1$ if $T_0 \trianglelefteq^*_{\lambda\kappa}$ for all λ.

The two main cases of interest are $\kappa = 1$ and $\kappa = \aleph_1$.

We follow the indexing system of [78], which is modeled after Keisler's order. In later papers, e.g. in the recent [61], λ^+-saturation is replaced by λ-saturation, and $T_0 \trianglelefteq^*_\kappa T_1$ is defined to mean $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$ for sufficiently large regular λ. We view this as a strange choice, since under this indexing system it is not known if $\trianglelefteq^*_{\aleph_1} \subseteq \trianglelefteq$.

It is not immediate that $\trianglelefteq^*_\kappa$ is transitive; one has to show one can compose interpretations in a suitable sense. This is stated without proof in [78]. We prove it, for the record:

Lemma 3.8.2. For all $\lambda > \aleph_0$, and for all cardinals κ which are infinite or 1, $\trianglelefteq^*_{\lambda\kappa}$ is transitive.

Proof. Suppose T_0, T_1, T_2 are theories with $T_0 \trianglelefteq^*_{\lambda\kappa} T_1 \trianglelefteq^*_{\lambda\kappa} T_2$. Let T_* and interpretations I_0, I_1 witness that $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$, and let S_* and interpretations J_1, J_2 witness that $T_1 \trianglelefteq^*_{\lambda\kappa} T_2$.

We can suppose that the languages of T_* and S_* are disjoint. Let R_* be the theory in the language $\mathcal{L}(T_*) \cup \mathcal{L}(S_*) \cup \{F\}$, such that $M_* \models R_*$ if and only if $M_* \restriction_{\mathcal{L}(T_*)} \models T_*$ and $M_* \restriction_{\mathcal{L}(S_*)} \models S_*$ and $F^{M_*} : I_1(M_* \restriction_{\mathcal{L}(T_*)}) \cong J_1(M_* \restriction_{\mathcal{L}(S_*)})$. It suffices to show that R_* is consistent, since then it clearly witnesses that $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$.

Since consistency is absolute, we can pass to a forcing extension in which there is some cardinal $\kappa > \aleph_0$ with $\kappa = \kappa^{<\kappa}$. Then we can find some $M_{*,0} \models T_*$ and $M_{*,1} \models S_*$ each of size κ and each κ-saturated; we can suppose both models have domain κ. Then $I_1(M_{*,0})$ is a κ-saturated model of T_1, and of size κ, as is $J_1(M_{*,1})$; thus, we can find an isomorphism $F^{M_*} : I_1(M_{*,0}) \cong J_1(M_{*,1})$. Let M_* be the $\mathcal{L}(T_*) \cup \mathcal{L}(S_*) \cup \{F\}$-structure with universe κ, such that $M_* \restriction_{\mathcal{L}(T_*)} = M_{*,0}$, and $M_* \restriction_{\mathcal{L}(S_*)} = M_{*,1}$, and F^{M_*} is as given. Then $M_* \models R_*$. $\square$

Dealing with the interpretability orders $\trianglelefteq^*_\kappa$ frequently introduces such complications; since our main interest remains in Keisler's order, we introduce our own intepretability orders $\trianglelefteq^\times_\kappa$, which are designed with the goal of being user-friendly. Keisler-order proofs will go through verbatim for $\trianglelefteq^\times_\kappa$, and will be clearer in this context; further, we will get an elegant characterization of $\trianglelefteq^\times_\kappa$ in terms of combinatorial characteristics of models of

ZFC^-.

Convention. $\hat{V}$ will be a model of ZFC^-, typically not well-founded. Frequently $\hat{V}$ will come from an embedding $\mathbf{j} : V \preceq \hat{V}$, where V is transitive.

Whenever $\hat{V} \models ZFC^-$, we will identify HF (the hereditarily finite sets) with its copy in $\hat{V}$. All other elements of $\hat{V}$ will be decorated with a hat (for instance we write $\hat{\omega}$ rather than $(\omega)^{\hat{V}}$), or at least the attempt will be made; sometimes readability takes precedence. For instance, we usually say $\hat{n} < \hat{m}$ rather than $\hat{n} \mathrel{\hat{<}} \hat{m}$ for nonstandard numbers $\hat{m}, \hat{n}$. Given $\hat{X} \in \hat{V}$, we have the associated subset $\{\hat{a} \in \hat{V} : \hat{a} \mathrel{\hat{\in}} \hat{X}\}$ of $\hat{V}$. We will not be careful about distinguishing between these going forth.

We say that $\hat{V}$ is ω-standard, or is an ω-model, if $\hat{\omega} = \omega$ (i.e. every natural number of $\hat{V}$ has finitely many predecessors). We will mainly be interested in the case where this fails.

Definition 3.8.3. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$, and M is an $\mathcal{L}$-structure in V. Then $\mathbf{j}(M)$ is a $\mathbf{j}(\mathcal{L})$-structure, where possibly some of the symbols of $\mathbf{j}(\mathcal{L})$ are nonstandard; let $\mathbf{j}_{\text{std}}(M)$ be the "reduct" to $\mathcal{L}$.

Example 3.8.4. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, and $V \models ZFC^-$ is transitive, and $\mathbf{i} : V \preceq \mathbf{V}$ where $\mathbf{V} \models^{\mathcal{B}} ZFC^-$. Write $\hat{V} = \mathbf{V}/\mathcal{U}$. Let $\mathbf{j} : V \preceq \hat{V}$ be the composition $[\cdot]_{\mathcal{U}} \circ \mathbf{i}$. Then given $M \models T$ with $M \in V$, $\mathbf{j}_{\text{std}}(M) \cong \mathbf{i}_{\text{std}}(M)/\mathcal{U}$. This is because if $\mathbf{a} \in \mathbf{V}$ and $\|\mathbf{a} \in \mathbf{i}(M)\|_{\mathbf{V}} \in \mathcal{U}$, then we can find $\mathbf{b} \in \mathbf{i}_{\text{std}}(\mathbf{M})$ such that $\|\mathbf{a} = \mathbf{b}\|_{\mathbf{V}} \in \mathcal{U}$.

Thus we view the $\mathbf{j}_{\text{std}}$ operator as a generalized ultrapower.

Example 3.8.5. Suppose M, N are $\mathcal{L}$-structures. By the proof of Theorem 3.8.2, the following are equivalent:

- $M \equiv N$.

- For some transitive $V \models ZFC^-$ with $M, N \in V$, there is some $\mathbf{j} : V \preceq \hat{V}$ such that $\mathbf{j}_{\mathrm{std}}(M) \cong \mathbf{j}_{\mathrm{std}}(N)$.

- For every transitive $V \models ZFC^-$ with $M, N \in V$, there is some $\mathbf{j} : V \preceq \hat{V}$ such that $\mathbf{j}_{\mathrm{std}}(M) \cong \mathbf{j}_{\mathrm{std}}(N)$.

This follows from the proof of Lemma 3.8.2. Note this is a baby version of the Keisler-Shelah theorem [77], which says we can in fact arrange $\hat{V} = V^\lambda/\mathcal{U}$, for some λ and some ultrafilter $\mathcal{U}$ on $\mathcal{P}(\lambda)$.

We can phrase $\trianglelefteq^*_{\lambda\kappa}$ in these terms as follows:

Lemma 3.8.6. Suppose κ is regular or 1. Then the following are equivalent:

(A) $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$;

(B) There is some countable transitive $V \models ZFC^-$ with $T_0, T_1 \in V$, and some $M_i \models T_i$ both in V, such that whenever $\mathbf{j} : V \preceq \hat{V}$, if $\hat{V}$ is κ-saturated and $\mathbf{j}_{\mathrm{std}}(M_1)$ is λ^+-saturated, then $\mathbf{j}_{\mathrm{std}}(M_0)$ is λ^+-saturated.

(C) There is some countable transitive $V \models ZFC^-$ with $T_0, T_1 \in V$, and some $M_i \models T_i$ both in V, and some $\mathbf{j}_0 : V \preceq \hat{V}_0$, and some countable expansion $(\hat{V}_0, \ldots)$ of $\hat{V}_0$, such that whenever $\mathbf{j}_1 : (\hat{V}_0, \ldots) \preceq (\hat{V}, \ldots)$, if $(\hat{V}, \ldots)$ is κ-saturated and $(\mathbf{j}_1 \circ \mathbf{j}_0)_{\mathrm{std}}(M_1)$ is λ^+-saturated, then $(\mathbf{j}_1 \circ \mathbf{j}_0)_{\mathrm{std}}(M_0)$ is λ^+-saturated.

Proof. (A) implies (B): Suppose T_* and interpretations I_0, I_1 witness that $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$. Let $M_* \models T_*$. Choose a countable transitive $V \models ZFC^-$ with $T_*, M_*, T_0, T_1 \in V$. This works, because if $\mathbf{j} : V \preceq \hat{V}$ has $\hat{V}$ κ-saturated, then $\mathbf{j}_{\mathrm{std}}(M_*)$ is κ-saturated, and $\mathbf{j}_{\mathrm{std}}(M_i) = I_i(\mathbf{j}_{\mathrm{std}}(M_*))$.

(B) implies (C): clear.

(C) implies (A): Let T_* be the elementary diagram of $(\hat{V}_0, \ldots)$. We get natural interpretations I_i of T_i in T_*, using the constant symbols for $\mathbf{j}_0(M_i)$. Then this witnesses $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$. $\square$

The choices of $M_0, M_1 \in V$ in (B), (C) above are often rather delicate, essentially because λ^+-saturation isn't well-behaved at this level of generality. It is better to restrict to realizing pseudofinite partial types. We define what we mean by pseudofinite:

Definition 3.8.7. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$. Say that $X \subseteq \hat{V}$ is pseudofinite (with respect to $\hat{V}$) if there is some $\hat{X} \in \hat{V}$ finite in the sense of $\hat{V}$, with $X \subseteq \hat{X}$. So if $\hat{X} \in \hat{V}$, then $\hat{X}$ is pseudofinite if and only if it is finite in the sense of $\hat{V}$.

The following characterization may be helpful for understanding our terminology. Note that always, the domain of $\mathbf{j}_{\mathrm{std}}(M)$ is the same as domain of $\mathbf{j}(M)$; we will (try to) use the former when reasoning in $\mathbb{V}$, and the latter when reasoning in $\hat{V}$.

Lemma 3.8.8. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$, and M is an $\mathcal{L}$-structure in V. Suppose $\hat{V}$ is ω-nonstandard, and $p(\overline{x})$ is a partial type over $\mathbf{j}_{\mathrm{std}}(M)$. Then $p(\overline{x})$ is pseudofinite if and only if there is some pseudofinite $X \subseteq \mathbf{j}_{\mathrm{std}}(M)$ such that $p(\overline{x})$ is over X.

Proof. Suppose $p(\overline{x})$ is pseudofinite; let X be the set of all parameters used in $p(\overline{x})$. We wish to show X is pseudofinite. By hypothesis we can find some set $\hat{\Delta} \in \hat{V}$ finite in the sense of $\hat{V}$, with $p(\overline{x}) \subseteq \hat{\Delta}$; then define $\hat{X} \in \hat{V}$ to be the set of all elements of $\mathbf{j}(M)$ which occur as a parameter in a formula in $\hat{\Delta}$. Since $\hat{\Delta}$ is finite in $\hat{V}$, so is $\hat{X}$.

Conversely, suppose $p(\overline{x})$ is a partial type over X with X pseudofinite. Let $\overline{z} = (z_i : i < \omega)$ be variables and let $(\psi_n(\overline{x}, \overline{z}) : n < \omega)$ enumerate all $\mathcal{L}$-formulas in these variables. After rearranging, we can suppose each $\psi_n(\overline{x}, \overline{z})$ only uses the variables $(\overline{x}, \overline{z}_n)$, where $\overline{z}_n = (z_i : i < n)$. Choose $\hat{n} \in \hat{\omega}$ nonstandard, and let $\hat{\Delta} \in \hat{V} = \{\psi_{\hat{m}}(\overline{x}, \overline{a}) : \hat{m} < \hat{n}, \overline{a} \in \mathbf{j}_{\text{std}}(M)^{\hat{m}}\}$. So $\hat{\Delta}$ is finite in $\hat{V}$ and $p(\overline{x}) \subseteq \hat{\Delta}$. $\square$

We make the following key definition:

Definition 3.8.9. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$, and M is an $\mathcal{L}$-structure in V. Say that $\mathbf{j}_{\text{std}}(M)$ is λ^+-pseudosaturated if every pseudofinite partial type $p(\overline{x})$ over a subset of $\mathbf{j}_{\text{std}}(M)$ of size at most λ is realized in $\mathbf{j}_{\text{std}}(M)$.

Note that in the definition of λ-pseudosaturation, it is enough to consider types $p(x)$ of arity 1. Also, it is equivalent to require $|p(\overline{x})| \leq \lambda$.

Example 3.8.10. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$ is ω-standard, and M is an $\mathcal{L}$-structure in V. Then pseudofinite subsets of $\hat{V}$ are the same as finite subsets of $\hat{V}$, so $\mathbf{j}_{\text{std}}(M)$ is always λ^+-pseudosaturated. Thus, the case where $\hat{V}$ is ω-standard will be degenerate in our context. Henceforward we will exclude it, even though some of what we do goes through in a vacuous sense.

The following theorem motivates our interest in λ^+-pseudosaturation. It does not seem to have been articulated before. It is another take on the fundamental phenomenon underlying Keisler's order.

Theorem 3.8.11. Suppose $V \models ZFC^-$ is transitive and $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard, and $M_0 \equiv M_1$ are two elementarily equivalent $\mathcal{L}$-structures in V. Then $\mathbf{j}_{\text{std}}(M_0)$ is λ^+-pseudosaturated if and only if $\mathbf{j}_{\text{std}}(M_1)$ is.

Proof. Suppose M_1 is λ^+-pseudosaturated; we show that M_0 is also. As remarked above, it suffices to consider types of arity 1 (the only effect of this is to increase readability). Let $p(x) = \{\varphi_\alpha(x, \overline{a}_\alpha) : \alpha < \lambda)\}$ be a pseudofinite type over M_0; we show $p(x)$ is realized in M_0. Choose some pseudofinite $\hat{\Delta} \in \hat{V}$, such that $p(x) \subseteq \hat{\Delta}$. By separation in $\hat{V}$, we can suppose $\hat{\Delta} = \hat{\Delta}(x)$ is a set of $\mathbf{j}(\mathcal{L})$-formulas over $\mathbf{j}(M_0)$ in the free variable x.

Since $\hat{V}$ believes $\mathbf{j}(M_0) \equiv \mathbf{j}(M_1)$ (by elementarity of $\mathbf{j} : V \preceq \hat{V}$), we can find a set $\hat{\Gamma}(x)$ of $\mathbf{j}(\mathcal{L})$-formulas over $\mathbf{j}(M_1)$ in the variable x and a bijection $\hat{f} : \hat{\Delta}(x) \to \hat{\Gamma}(x)$, such that the following are true in $\hat{V}$:

(I) For every $\hat{\varphi}(x, \hat{\overline{a}}) \in \hat{\Delta}(x)$, $\hat{f}(\hat{\varphi}(x, \hat{\overline{a}})) = \hat{\varphi}(x, \hat{\overline{b}})$ for some $\hat{\overline{b}} \in \mathbf{j}(M_1)^{|\hat{\overline{a}}|}$;

(II) For every $\hat{\Delta}_0(x) \subseteq \hat{\Delta}(x)$, $\mathbf{j}(M_0) \models \exists x \bigwedge \hat{\Delta}_0(x)$ if and only if $\mathbf{j}(M_1) \models \exists x \bigwedge \hat{f}[\hat{\Delta}_0(x)]$.

Let $q(x)$ be the image of $p(x)$ under $\hat{f}$. By (I), $q(x)$ is a set of $\mathcal{L}$-formulas over $\mathbf{j}_{\mathrm{std}}(M_1)$ in the free variable x. By (II), $p(x)$ is consistent. Visibly $q(x) \subseteq \hat{\Gamma}(x)$ is pseudofinite.

Thus $q(x)$ has a realization $b \in \mathbf{j}_{\mathrm{std}}(M)$. Let $\hat{\Gamma}_0(x)$ be defined in $\hat{V}$ as the set of all $\hat{\varphi}(x) \in \hat{\Gamma}$ such that $\mathbf{j}(M_1) \models \hat{\varphi}(b)$. Let $\hat{\Delta}_0(x) = \hat{f}^{-1}[\hat{\Gamma}_0(x)]$. By (II), in $\hat{V}$, $\mathbf{j}(M_0) \models \exists x \bigwedge \hat{\Delta}_0(x)$, so we can find $a \in \mathbf{j}_{\mathrm{std}}(M_0)$ such that in $\hat{V}$, $\mathbf{j}(M) \models \hat{\Delta}_0(a)$. Since $p(x) \subseteq \hat{\Delta}_0(x)$ we conclude that a realizes $p(x)$. $\square$

One can given an equivalent formulation of the interpretability order $\trianglelefteq^*_\kappa$ in terms of pseudosaturation (see Corollary 3.18.8). However, the formulation is not particularly natural (we must restrict to $\hat{V}$ which admit an expansion to a certain theory ZFC^-_*). We view the following as the most natural interpretability order:

Definition 3.8.12. Suppose $V \models ZFC^-$ is transitive, $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard, and suppose T is a complete countable theory with $T \in V$. Then say that $\hat{V}$ λ^+-pseudosaturates T if for some or every $M \models T$ with $M \in V$, $\mathbf{j}_{\mathrm{std}}(M)$ is λ^+-pseudosaturated. (This also depends on $\mathbf{j}$; if there is ambiguity we would write $(\mathbf{j}, \hat{V})$ λ^+-pseudosaturates T.)

Suppose κ is infinite or 1. Then say that $T_0 \trianglelefteq^{\times}_{\lambda\kappa} T_1$ if there is some countable transitive $V \models ZFC^-$ containing T_0, T_1 such that whenever $\mathbf{j} : V \preceq \hat{V}$, if $\hat{V}$ is κ-saturated and ω-nonstandard, and if $\hat{V}$ λ^+-pseudosaturates T_1, then also it λ^+-pseudosaturates T_0. Say that $T_0 \trianglelefteq^{\times}_{\kappa} T_1$ if $T_0 \trianglelefteq^{\times}_{\lambda\kappa} T_1$ for all λ.

Note that for $\kappa < \kappa'$, $\trianglelefteq^{\times}_{\kappa} \subseteq \trianglelefteq^{\times}_{\kappa'}$. Corollary 3.18.8 states that $\trianglelefteq^{\times}_{\kappa} \subseteq \trianglelefteq^{*}_{\kappa}$ for uncountable κ. Further, Corollary 3.18.7 states that $\trianglelefteq^{\times}_{1} \subseteq \trianglelefteq^{*}_{1}$ also, except perhaps on stable theories. We do not know about the reverse implications. In any case, henceforward we will prove positive reductions in $\trianglelefteq^{\times}_{\kappa}$. Corollaries 3.18.7 and 3.18.8 can be used to get corresponding results for $\trianglelefteq^{*}_{\kappa}$.

Similar ideas are used in Corollary 3.18.9 to show that $\trianglelefteq^{*}_{1}$ and $\trianglelefteq^{*}_{\aleph_1}$ coincide on pairs of theories which are not both stable. This allows the transfer of several nonreducibility results to $\trianglelefteq^{*}_{\aleph_1}$, as described in Corollary 3.18.10.

Remark 3.8.13. Suppose T_0, T_1 are complete countable theories. As far as we are aware, it would be equivalent to drop the countability assumption in $\trianglelefteq^{\times}_{\kappa}$, and also to change the existential quantification for V to a universal quantification. That is, as far as we know, $T_0 \trianglelefteq^{\times}_{\kappa} T_1$ if and only if for some or every transitive $V \models ZFC^-$ with $T_0, T_1 \in V$, for every $\mathbf{j} : V \preceq \hat{V}$ with $\hat{V}$ κ-saturated and ω-nonstandard, and for every λ, if $\hat{V}$ λ^+-pseudosaturates T_1, then it also λ^+-pseudosaturates T_0.

We now connect $\trianglelefteq^{\times}_{\kappa}$ with Keisler's order; namely, we will show that $\trianglelefteq^{\times}_{\aleph_1} \subseteq \trianglelefteq$.

The following theorem is a simple application of Corollary 3.3.8.

Theorem 3.8.14. Suppose $\mathcal{B}$ is a complete Boolean algebra, $V \models ZFC^-$ is transitive and $\mathbf{i} : V \preceq \mathbf{V}$. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$; write $\hat{V} = \mathbf{V}/\mathcal{U}$. If $\mathbf{V}$ is λ^+-saturated, then every subset of $\hat{V}$ of size at most λ is pseudofinite.

We immediately get the following.

Corollary 3.8.15. Suppose $\mathcal{B}$ is a complete Boolean algebra, $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, λ is a cardinal, and T is a complete countable theory. Then the following are equivalent:

(A) $\mathcal{U}$ λ^+-saturates T.

(B) For some or every transitive $V \models ZFC^-$ with $T \in V$, and for some or every $\mathbf{i} : V \preceq \mathbf{V}$ with $\mathbf{V}$ λ^+-saturated, and for some or every $M \models T$ with $M \in V$, $\mathbf{i}_{\text{std}}(M)/\mathcal{U}$ is λ^+-saturated.

(C) For some or every transitive $V \models ZFC^-$, for some or every $\mathbf{i} : V \preceq \mathbf{V}$ with $\mathbf{V}$ λ^+-saturated, $\mathbf{V}/\mathcal{U}$ λ^+-pseudosaturates T.

Proof. (A) if and only if (B) is Theorem 3.5.9. (B) implies (C) is trivial. (C) implies (B) follows from Theorem 3.8.14. $\square$

Corollary 3.8.16. Suppose $T_0 \trianglelefteq^{\times}_{\aleph_1} T_1$. Then $T_0 \trianglelefteq T_1$.

Proof. Since every ultrafilter $\mathcal{U}$ is $\aleph_1$-good, we have that if $\mathbf{V} \models^{\mathcal{B}} T$ is $\aleph_1$-saturated, then so is $\mathbf{V}/\mathcal{U}$. Thus we conclude by Corollary 3.8.15(C). $\square$

Remark 3.8.17. It is also easy to check that if $\mathcal{U}$ is a λ-regular ultrafilter on $\mathcal{P}(\lambda)$, then every subset of $\mathbb{V}^\lambda/\mathcal{U}$ of cardinality at most λ is pseudofinite. This is the reason regularity is helpful in controlling saturation of ultrapowers.

3.9 Maximality of SOP_2

Recall from Theorem 3.7.2 that Keisler proved the existence of a $\trianglelefteq$-maximal class [34], and it contains $\mathrm{Th}([\omega]^{<\aleph_0}, \subseteq)$. In [75], Shelah showed that in fact every SOP theory T is maximal, in particular $\mathrm{Th}(\mathbb{Q}, <)$ is maximal; this is where we can see the beginnings of the interpretability-order viewpoint. Later in [78], Shelah improved this to show that every SOP_3 theory is maximal.

Then, in [54], Malliaris and Shelah proved that in fact every SOP_2 theory is maximal, which is best known so far. This is substantially harder than the SOP_3-result.

In this section, we describe some of the main concepts from [54], deferring the proof of their central theorem until the end of the survey, in Section 3.17. We then describe how this is used to show SOP_2 theories are maximal.

A fascinating outgrowth of the ideas developed in [54] is its application to cardinal invariants of the continuum; in fact, Malliaris and Shelah were able to solve the oldest open problem in the field, by showing $\mathfrak{p} = \mathfrak{t}$. This is also discussed in Section 3.17.

One large difference in our treatment is that in [54], Malliaris and Shelah use cofinality spectrum problems as their base set theory. This is is a weak fragment of arithmetic, not even strong enough for exponentiation. We stick to ZFC^-, and thus avoid many difficulties.

We begin with some definitions.

Definition 3.9.1. If $(L, <)$ is a linear order, and κ, θ are infinite regular cardinals, then a (κ, θ)-pre-cut in L is a pair of sequences $(\overline{a}, \overline{b}) = (a_\alpha : \alpha < \kappa)$, $(b_\beta : \beta < \theta)$ from L, such that for all $\alpha < \alpha'$, $\beta < \beta'$, we have $a_\alpha < a_{\alpha'} < b_\beta < b_{\beta'}$. $(\overline{a}, \overline{b})$ is a cut if there is no $c \in L$ with $a_\alpha < c < b_\beta$ for all α, β. Let the cut spectrum of $(L, <)$ be $\mathcal{C}(L, <) := \{(\kappa, \theta) : L \text{ admits a } (\kappa, \theta) \text{ cut}\}$. Define $\text{cut}(L, <) = \min\{\kappa + \theta : (\kappa, \theta) \in \mathcal{C}(L, <)\}$.

Note that it would be equivalent to drop the requirement of regularity.

Definition 3.9.2. By a tree T we mean a partially ordered set $(T, <)$ with meets and a minimum element 0_T, such that the predecessors of every element are linearly-ordered. In contexts where this conflicts with normal usage (where we want the predecessors to be well-ordered) we could call these trees "model-theoretic trees." Given a tree $(T, <)$ define tree-tops(T) to be the least (necessarily regular) κ such that there is a (strictly) increasing sequence $(s_\alpha : \alpha < \kappa)$ from T with no upper bound in T.

Definition 3.9.3. Suppose $\hat{V}$ is an ω-nonstandard model of ZFC^-. Then define $\mathcal{C}_{\hat{V}} = \mathcal{C}(\hat{\omega}, \hat{<})$, and define $\mathfrak{p}_{\hat{V}} = \text{cut}(\hat{\omega}, \hat{<})$. Also, let $\mathfrak{t}_{\hat{V}}$ be the minimum over all $\hat{n} < \hat{\omega}$ of tree-tops$(\hat{n}^{<\hat{n}}, \hat{\subset})$. Unraveling the definitions, $\mathbf{t}_{\hat{V}}$ is the least κ such that there is some $\hat{n} < \hat{\omega}$ and some increasing sequence $(\hat{s}_\alpha : \alpha < \kappa)$ from $\hat{n}^{<\hat{n}}$, with no upper bound in $\hat{n}^{<\hat{n}}$.

Malliaris and Shelah say that $\hat{V}$ has "κ^+-treetops" if $\kappa < \mathfrak{t}_{\hat{V}}$. Note that $\mathbf{t}_{\hat{V}}$ is can also be defined as the least κ such that there is some $\hat{n} < \hat{\omega}$ and some increasing sequence $(\hat{s}_\alpha : \alpha < \kappa)$ from $\hat{n}^{<\hat{n}}$, with no upper bound in $\hat{\omega}^{<\hat{\omega}}$; this is because if $\hat{s}$ is any upper bound, then $\hat{s} \restriction_{\hat{m}}$ is an upper bound in $\hat{n}^{<\hat{n}}$, where $\hat{m} \leq \hat{n}$ is largest so that $\hat{s} \restriction_{\hat{m}} \in \hat{n}^{<\hat{m}}$.

The following lemma is a component of Shelah's proof in [75] that SOP theories are maximal. Actually, in the context of the cofinality spectrum problems from [54], this lemma need not hold. This is an artifact of cofinality spectrum problems; in Section 10 of [54], Malliaris and Shelah derive the lemma in the context of ultrapower embeddings, following the proof of Shelah's theorem. In fact, in [59], Malliaris and Shelah comment that cofinality spectrum problems with exponentiation are enough for this to go through.

Lemma 3.9.4. Suppose $\hat{V} \models ZFC^-$ is ω-nonstandard. Then $\mathfrak{p}_{\hat{V}} \leq \mathfrak{t}_{\hat{V}}$. In fact, $(\mathfrak{t}_{\hat{V}}, \mathfrak{t}_{\hat{V}}) \in \mathcal{C}_{\hat{V}}$.

Proof. Suppose $(s_\alpha : \alpha < \kappa)$ is an increasing sequence from $\hat{n_*}^{<\hat{n_*}}$ with no upper bound, where κ is regular. We show $(\kappa, \kappa) \in \mathcal{C}_{\hat{V}}$.

Let $\hat{<}_{lex}$ be the lexicographic ordering on $\hat{n}_*^{<\hat{n}_*}$.

Note that if $s \in \hat{T}$, then $s_\alpha{}^\frown(0) \,\hat{\leq}_{lex}\, s \hat{\leq}_{lex} s_\alpha{}^\frown(\hat{n}_* - 1)$ if and only if $s_\alpha \subseteq s$. Since

$(s_\alpha : \alpha < \kappa)$ is unbounded, it follows that $(s_\alpha {}^\frown(0) : \alpha < \kappa)$ and $(s_\alpha {}^\frown(\hat{n}_* - 1) : \alpha < \kappa)$ form a (κ, κ)-cut in $(\hat{n}_*^{<\hat{n}_*}, (\hat{<}_{lex})_*)$.

In $\hat{V}$, let $\hat{\sigma} : (\hat{n}_*^{<\hat{n}_*}, \hat{<}_{lex}) \to (|\hat{n}_*^{<\hat{n}_*}|, \hat{<})$ be the order preserving bijection. Then $(\hat{\sigma}(s_\alpha {}^\frown(0)) : \alpha < \kappa)$ and $(\hat{\sigma}(s_\alpha {}^\frown(\hat{n}_* - 1)) : \alpha < \kappa)$ witness that $(\kappa, \kappa) \in \mathcal{C}(\hat{\omega}, \hat{V})$. □

The following corresponds to Claim 2.14 of [54].

Lemma 3.9.5. Suppose $\hat{V} \models ZFC^-$ is ω-nonstandard. Suppose $(\hat{T}, \hat{<})$ is a finite tree in $\hat{V}$. Then tree-tops$(\hat{T}, \hat{<}) \geq \mathfrak{t}_{\hat{V}}$.

Proof. There is in $\hat{V}$ a subtree of $\hat{\omega}^{<\hat{\omega}}$ which is isomorphic to $\hat{T}$; so we can suppose that $\hat{T}$ is a subtree of $\hat{\omega}^{<\hat{\omega}}$. Then $\hat{T}$ is a subtree of $\hat{n}_*^{<\hat{n}_*}$ for some $\hat{n}_* < \hat{\omega}$.

Now suppose $(s_\alpha : \alpha < \kappa)$ is an increasing sequence from $\hat{T}$ with $\kappa < \mathfrak{t}_{\hat{V}}$; we show there is an upper bound in $\hat{T}$. To see this let s_+ be an upper bound of $(s_\alpha : \alpha < \kappa)$ in $\hat{\omega}^{<\hat{\omega}}$, and let $\hat{n}$ be largest so that $s_+ \restriction_{\hat{n}} \hat{\in} \hat{T}$; and let $s = s_+ \restriction_{\hat{n}}$. □

The following theorem corresponds to Theorem 4.1 of [54], although there the authors must also assume $\lambda < \mathfrak{t}_{\hat{V}}$ in the absence of Lemma 3.9.4.

Theorem 3.9.6. Suppose $\hat{V} \models ZFC^-$ is ω-nonstandard. Suppose $p(x) = (\varphi_\alpha(x, a_\alpha) : \alpha < \lambda)$ is a partial type over $\hat{V}$ of cardinality $\lambda < \mathfrak{p}_{\hat{V}}$. Suppose $\hat{X} \in \hat{V}$ is pseudofinite, and $\varphi_0(x)$ is "$x \in \hat{X}$." Then $p(x)$ is realized in $\hat{V}$.

Proof. Obviously this is true when λ is finite.

Suppose the lemma is true for all $\lambda' < \lambda$; we show it is true for λ. This suffices. Write $\hat{n}_* = |\hat{X}|$.

We choose $(s_\alpha : \alpha \leq \lambda)$ an increasing sequence from $\hat{X}^{<\hat{n}_*}$, such that if we let $\hat{n}_\alpha = \hat{\lg}(s_\alpha)$, then for all $\beta < \alpha < \lambda$ and for all $\hat{n} < \hat{n}_\alpha$, $\hat{V} \models \varphi_\beta(s_\alpha(\hat{n}), a_\beta)$. Obviously then s_λ will be as desired.

Let $s_0 = \emptyset$ say. At successor stage α, just use the hypothesis for $\lambda' = |\alpha| < \lambda$.

Suppose we have defined $(s_\alpha : \alpha < \delta)$ where $\delta \leq \lambda$. Using $|\delta| < \mathfrak{p}_{\hat{V}} \leq \mathfrak{t}_{\hat{V}}$ (by Lemma 3.9.4), we may apply Lemma 3.9.5) to choose $s_+ \in \hat{X}^{<\hat{n}_*}$, an upper bound on $(s_\alpha : \alpha < \delta)$.

Let $\hat{m}_0 = \hat{\lg}(s_+)$. For $\beta \leq \delta$ we will define $\hat{m}_\beta$ so that for all $\alpha < \delta$, and for all $\beta < \beta' < \delta$, $\hat{n}_\alpha < \hat{m}_{\beta'} < \hat{m}_\beta$, and further for every $\beta \leq \delta$, we have that for every $\beta' < \beta$ and for every $\hat{n} < \hat{m}_\beta$, $\hat{V} \models \varphi_{\beta'}(s_+(\hat{n}), a_{\beta'})$. Note once we finish we can set $s_\delta = s_+ \restriction_{\hat{m}_\delta}$.

Having defined $\hat{m}_\beta$ for $\beta < \delta$, let $\hat{m}_{\beta+1}$ be the greatest $\hat{m} < \hat{m}_\beta$ such that for all $\hat{n} < \hat{m}$, $\hat{V} \models \varphi_\beta(s_+(\hat{n}), a_\beta)$; this works. Having defined $\hat{m}_\beta$ for all $\beta < \delta' \leq \delta$, since $\delta' \leq \delta \leq \lambda < \mathfrak{p}_{\hat{V}}$ we can choose $\hat{m}_{\delta'}$ with $\hat{n}_\alpha < \hat{m}_{\delta'} < \hat{m}_\beta$ for all $\alpha < \delta$, $\beta < \delta'$.

This concludes the construction.

□

We will want the following tweak (which perhaps fails for cofinality spectrum problems). It says essentially that if $\lambda < \mathfrak{p}_{\hat{V}}$, then $\hat{V}$ is λ^+-pseudosaturated. Note that when considering types over models of ZFC^-, it always suffices to consider formulas with singleton parameters, since ZFC^- has definable pairing functions.

Theorem 3.9.7. Suppose $\hat{V} \models ZFC^-$ is ω-nonstandard and $\lambda < \mathfrak{p}_{\hat{V}}$ is an infinite cardinal. Suppose $p(x) = \{\varphi_\alpha(x, a_\alpha) : \alpha < \lambda\}$ is a partial type over $\hat{V}$, such that $\{a_\alpha : \alpha < \lambda\}$ is pseudofinite. Then $p(x)$ is realized in $\hat{V}$.

Proof. We first consider some special cases.

Case 1. Suppose each $\varphi_\alpha(x, a_\alpha)$ is $\varphi(x, a_\alpha)$ for some fixed formula $\varphi(x, y)$. Let $\hat{X} \in \hat{V}$ be finite in $\hat{V}$ with each $a_\alpha \in \hat{X}$. By applying collection in $\hat{V}$ (to the finite set $\hat{\mathcal{P}}(\hat{X})$), we can find some $\hat{Y} \in \hat{V}$ such that for every $\hat{X}_0 \subseteq \hat{X}$, if there is some $b \in \hat{V}$ such that $\varphi(b, a)$

holds for every $a \in \hat{X}_0$, then there is some such $b \in \hat{Y}$. By choosing a well-ordering of $\hat{Y}$ and picking least witnesses, we can suppose $\hat{Y}$ is finite in the sense of $\hat{V}$.

Let $q(x) = p(x) \cup \{$"$x \in \hat{Y}$"$\}$. By Theorem 3.9.6, it suffices to note that $q(x)$ is finitely satisfiable. But this is clear, by choice of $\hat{Y}$.

Case 2. Suppose each $\varphi_\alpha(x, a_\alpha)$ is Σ_n, for some fixed $n < \omega$. Let $\psi(x, y, z)$ be a truth predicate for Σ_n formulas; that is, for all Σ_n-formulas $\varphi(x, y)$, ZFC^- proves

$$\forall x \forall y (\psi(x, y, \varphi(x, y)) \leftrightarrow \varphi(x, y))$$

(we can suppose $\varphi(x, y)$ is hereditarily finite, hence 0-definable uniformly in all models of ZFC^-, so it makes sense to plug it in as a parameter). Then we can replace each $\varphi_\alpha(x, a_\alpha)$ by $\psi(x, a_\alpha, \varphi_\alpha(x, y))$, and so we are in the first case.

General case. Write $X = \{a_\alpha : \alpha < \lambda\}$; choose $\hat{X} \in \hat{V}$ finite in the sense of $\hat{V}$, with $X \subseteq \hat{X}$. For each $n < \omega$, let $I_n = \{\alpha < \lambda : \varphi_\alpha(x, a_\alpha) \text{ is } \Sigma_n\}$ and let $p_n(x) = \{\varphi_\alpha(x, a_\alpha) : \alpha \in I_n\}$; so each $p_n(x) \subseteq p_{n+1}(x)$ and $p(x) = \bigcup_n p_n(x)$. By Case 2, we can find a realization b_n of $p_n(x)$. By Case 2 (or even Case 1, and just using $\lambda \geq \aleph_0$), we can find some $\hat{n}_* < \hat{\omega}$ nonstandard and some some function $\hat{f} \in \hat{V}$ with domain $\hat{n}_*$, such that for all $n < \omega$, $\hat{f}(n) = b_n$. For each $\hat{n} < \hat{n}_*$, write $b_{\hat{n}} = \hat{f}(\hat{n})$.

For each $m < \omega$, choose some truth predicate $\psi_m(x, y, z)$ for Σ_m-formulas. For each $\hat{n} < \hat{n}_*$, let $\hat{\Delta}_{m,\hat{n}}$ be the set of all Σ_m-formulas of set theory $\hat{\varphi}(x, a)$ in $\hat{V}$ (so not necessarily a real formula of set theory) such that $a \in \hat{X}$ and $\hat{V} \models \psi_m(b_{\hat{n}}, a, \hat{\varphi}(x, y))$. By Case 2, we can find some $\hat{\Delta}_m$ such that for all $m \leq n < \omega$, $\hat{\Delta}_m \subseteq \hat{\Delta}_{m,n}$, and for all $\alpha \in I_m$, $\varphi(x, a_\alpha) \in \hat{\Delta}_m$. By overflow, we can find some $\hat{n}_m < \hat{\omega}$ nonstandard, such that for all $n \leq \hat{n} \leq \hat{n}_m$, $\hat{\Delta}_m \subseteq \hat{\Delta}_{m,\hat{n}}$.

By Case 2 again, we can find some $\hat{n} < \hat{\omega}$ nonstandard, such that $\hat{n} < \hat{n}_m$ for all

$m < \omega$. Then for all $m < \omega$, $\hat{\Delta}_m \subseteq \hat{\Delta}_{m,\hat{n}}$. But this implies $b_{\hat{n}}$ realizes $p_m(x)$ for all $m < \omega$, hence $b_{\hat{n}}$ realizes $p(x)$.

$\square$

The following corollary follows immediately from Theorem 3.9.7. Similar ideas are implicit in Shelah's proof in [75] that SOP theories are maximal in Keisler's order.

Corollary 3.9.8. Suppose $V \models ZFC^-$ is transitive and $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard and $\lambda < \mathfrak{p}_{\hat{V}}$. Then for all complete, countable theories T, $\hat{V}$ λ^+-pseudosaturates T.

Finally, the following theorem is Central Theorem 9.1 of [54] (except in the context of cofinality spectrum problems, only $\geq$ is necessarily true).

Theorem 3.9.9. Suppose $\hat{V} \models ZFC^-$. Then $\mathfrak{p}_{\hat{V}} = \mathfrak{t}_{\hat{V}}$.

We describe Malliaris and Shelah's proof of Theorem 3.9.9 in Section 3.17.

We can now easily show that SOP_2 theories are maximal in Keisler's order. In fact we will show they are maximal in $\trianglelefteq_1^\times$; this is both more general and more transparent, although it requires no extra argument. Relatedly, Malliaris and Shelah note in [59] that SOP_2 is maximal in $\trianglelefteq_1^*$, although this does require extra argument. Alternatively, one can deduce maximality of SOP_2 in $\trianglelefteq_1^*$ from maximality in $\trianglelefteq_1^\times$ and the general Corollary 3.18.7.

Definition 3.9.10. Suppose T is a complete countable theory and $\varphi(\overline{x}, \overline{y})$ is a formula. Then $\varphi(\overline{x}, \overline{y})$ has SOP_2 if there are $(\overline{a}_s : s \in \omega^{<\omega})$ each of the same length as $\overline{y}$, such that for each $\eta \in \omega^\omega$, $(\varphi(\overline{x}, \overline{a}_{\eta \restriction_n}) : n < \omega)$ is consistent, but whenever $s, t \in \omega^{<\omega}$ are incomparable, $\varphi(\overline{x}, \overline{a}_s) \wedge \varphi(\overline{x}, \overline{a}_t)$ is inconsistent. By compactness, this is equivalent to saying for each $n < \omega$, there are $(\overline{a}_s : s \in n^{<n})$ satisfying the analogous properties. T has SOP_2 if some formula of T does. Otherwise it has $NSOP_2$.

Actually, these are the standard definitions of TP_1; for SOP_2, typically one uses $2^{<\omega}$ instead of $\omega^{<\omega}$. But this is equivalent, see [38].

Theorem 3.9.11. Suppose $V \models ZFC^-$ is transitive and $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard, and suppose $T \in V$ is a countable complete theory with SOP_2, and suppose $\lambda \geq \mathfrak{t}_{\hat{V}}$. Then $\hat{V}$ does not λ^+-pseudosaturate T.

Proof. By decreasing λ we can suppose $\lambda = \mathfrak{t}_{\hat{V}}$. Suppose towards a contradiction that $\hat{V}$ did λ^+-pseudosaturate T. Let $M \models T$ with $M \in V$. Let $\varphi(\overline{x}, \overline{y})$ be a formula of T with SOP_2.

Let $\hat{n}_* < \hat{\omega}$ and let $(\hat{s}_\alpha : \alpha < \lambda)$ be an increasing sequence from $\hat{n}_*^{<\hat{n}_*}$ with no upper bound in $\hat{n}_*^{<\hat{n}_*}$.

By elementary, we can choose $\hat{f} : \hat{n}_*^{<\hat{n}_*} \to \mathbf{j}_{\mathrm{std}}(M)^{\mathrm{lg}(\overline{y})}$ in $\hat{V}$, such that $(\hat{f}(\hat{s}) : \hat{s} \in \hat{n}_*^{<\hat{n}_*})$ is as in the definition of $\varphi(\overline{x}, \overline{y})$ being SOP_2.

By λ^+-pseudosaturation of $\mathbf{j}_{\mathrm{std}}(M)$ we can choose $\overline{a} \in \mathbf{j}_{\mathrm{std}}(M)^{|\overline{x}|}$ such that $\mathbf{j}_{\mathrm{std}}(M) \models \varphi(\overline{a}, \hat{f}(\hat{s}_\alpha))$ for each $\alpha < \kappa$. In $\hat{V}$, let $\hat{s}_*$ be the union of all $\hat{s} \in \hat{n}_*^{<\hat{n}_*}$ such that $\mathbf{j}_{\mathrm{std}}(M) \models \varphi(\overline{a}, \hat{f}(\hat{s}))$. Then $\hat{s} \in \hat{n}_*^{<\hat{n}_*}$ is an upper bound to $(\hat{s}_\alpha : \alpha < \lambda)$, contradiction. $\square$

Corollary 3.9.12. Suppose T has SOP_2. Then T is maximal in $\trianglelefteq_1^\times$, hence also in $\trianglelefteq_{\aleph_1}^\times$ and $\trianglelefteq$.

Malliaris and Shelah use this in [54] to give a characterization of λ^+-goodness among λ-regular ultrafilters on $\mathcal{P}(\lambda)$. The same argument gives a general characterization in our context.

Theorem 3.9.13. Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$ and λ is a cardinal. Then the following are equivalent:

(A) $\mathcal{U}$ is λ^+-good.

(B) For some or every transitive $V \models ZFC^-$, and for some or every $\mathbf{i} : V \preceq \mathbf{V}$ with $\mathbf{V}$ λ^+-saturated, $\lambda < \mathfrak{p}_{\mathbf{V}/\mathcal{U}} = \mathfrak{t}_{\mathbf{V}/\mathcal{U}}$.

Proof. In (B), write $\hat{V} = \mathbf{V}/\mathcal{U}$.

(A) implies (B): by Theorems 3.7.2 and 3.8.15, $\hat{V}$ λ^+-pseudosaturates $\mathrm{Th}(\omega, <)$. Hence $(\hat{\omega}, <)$ is λ^+-pseudosaturated, so $\lambda < \mathfrak{p}_{\hat{V}}$.

(B) implies (A): by Corollary 3.9.8, we know that $\hat{V}$ λ^+-pseudosaturates the theory $\mathrm{Th}([\omega]^{<\aleph_0}, \subseteq)$, thus by Theorems 3.7.2 and 3.8.15, $\mathcal{U}$ is λ^+-good. □

In terms of characterizing the maximal class of Keisler's order, all that is known currently is the following. I proved in [87] that low theories are non-maximal in $\trianglelefteq$. This is an adaptation of a theorem of Malliaris and Shelah [57]: if there is a supercompact cardinal, then simple theories are non-maximal in $\trianglelefteq$. For the interpretability order $\trianglelefteq^*_1$, more is known. Results of [8] (Džamonja and Shelah) and [80] (Shelah and Usvyatsov) together show that if T is $NSOP_2$, and if suitable instances of GCH hold, then T is not maximal in $\trianglelefteq^*_1$. Therefore, consistently SOP_2 characterizes maximality in $\trianglelefteq^*_1$. The pieces for this are all put together in [59]. Finally, in [61], Malliaris and Shelah prove in ZFC that simple theories are not maximal in $\trianglelefteq^*_1$.

By Corollary 3.18.10, the corresponding statements for $\trianglelefteq^*_{\aleph_1}$ hold also.

3.10 Keisler's Order on Stable Theories

Having just finished looking at the top of Keisler's order, we now take a look at the bottom. The situation here is much better understood. Indeed, Shelah proved the following in Chapter VI of [75]:

Theorem 3.10.1. The $\trianglelefteq$-minimal class of theories is the class of stable theories without the finite cover property. The next-least $\trianglelefteq$-class of theories (exists and) is the class of stable theories with the finite cover property. In other words:

(A) If T_0 is stable without the finite cover property, then for all T_1, $T_0 \trianglelefteq T_1$.

(B) If T_1 is stable with the finite cover property, and T_2 is either unstable or is stable with the finite cover property, then $T_1 \trianglelefteq T_2$.

(C) If T_0 is stable without the finite cover property, and T_1 is stable with the finite cover property, and T_2 is unstable, then $T_1 \not\trianglelefteq T_0$ and $T_2 \not\trianglelefteq T_1$.

We describe his proof in this section. We actually prove the positive reductions (A) and (B) in $\trianglelefteq^{\times}_{\aleph_1}$. It follows that Theorem 3.10.1 also holds for $\trianglelefteq^{\times}_{\aleph_1}$. This is really just a matter of presentation; Malliaris and Shelah have previously observed in [61] that these results translate to the interpretability order $\trianglelefteq^{*}_{\aleph_1}$. They also prove there that $\trianglelefteq^{*}_{1}$ properly refines $\trianglelefteq^{*}_{\aleph_1}$ on the stable theories, and promise in a forthcoming paper to show $\trianglelefteq^{*}_{1}$ has exactly six classes on stable theories.

Definition 3.10.2. Let T be a complete countable theory. Then Δ is a set of partitioned formulas if $\Delta = \{\varphi_i(\overline{x}, \overline{y}_i) : i \in I\}$, that is it is a set of partioned formulas with distinguished variables $\overline{x}$. The arity of Δ is $|\overline{x}|$. Suppose Δ is a set of partitioned formulas; then a (positive) Δ-formula is a (positive) boolean combination of formulas of the form $\varphi(\overline{x}, \overline{a})$ for $\varphi(\overline{x}, \overline{y}) \in \Delta$ and parameters $\overline{a} \in \mathfrak{C}$. A (positive) Δ-type is a partial type $p(\overline{x})$, such that every $\varphi(\overline{x}, \overline{a}) \in p(\overline{x})$ is a (positive) Δ-formula. If A is a set we let $S_\Delta(A)$ be the set of all maximal Δ-types over A; so we have the obvious restriction map from $S_n(A)$ to $S_\Delta(A)$, where n is the arity of Δ. If $\overline{a} \in \mathfrak{C}$ then $tp_\Delta(\overline{a}/A) \in S_\Delta(A)$ is the set of all Δ-formulas $\varphi(\overline{x}, \overline{b})$ such that $\overline{b} \in A$ and $\models \varphi(\overline{a}, \overline{b})$. If Δ is the single formula $\varphi(\overline{x}, \overline{y})$ we write $\varphi(\overline{x}, \overline{y})$ instead of $\{\varphi(\overline{x}, \overline{y})\}$.

Suppose I is an index set and Δ is a set of partitioned formulas of T of arity m. For an index set I, a set $\{\overline{a}_i : i \in I\}$ is (Δ, n)-indiscernible if each $\overline{a}_i$ has length m and for every tuples of distinct elements $(i_0, \ldots, i_{n-1})$, $(j_0, \ldots, j_{n-1})$ from I^n, $tp_\Delta(\overline{a}_{i_0}/\overline{a}_{i_1} \ldots \overline{a}_{i_{n-1}}) = tp_\Delta(\overline{a}_{j_0}/\overline{a}_{j_1} \ldots \overline{a}_{j_{n-1}})$.

The formula $\varphi(\overline{x}, \overline{y})$ has the finite cover property if: for every n there exists $m > n$ and $(\overline{a}_i : i < m)$ such that $\{\varphi(\overline{x}, \overline{a}_i) : i < m\}$ is inconsistent, but every n-element subset is consistent. T has the finite cover property if some formula $\varphi(\overline{x}, \overline{y})$ has the finite cover property.

The following equivalents of the finite cover property are proved in [75] Chapter II Theorem 4.4 and Theorem 4.6. One should note that if T does not have the finite cover property then necessarily T is stable ([75] Theorem II.4.2). Actually Shelah only explicitly proves (A) if and only if (B) (Theorem II.4.2) and (C) implies (A) (Theorem II.4.6), but (B) implies (C) is trivial.

Theorem 3.10.3. For T a stable, complete countable theory, the following are equivalent:

(A) T has the finite cover property.

(B) There is a formula $\varphi(x, y, \overline{z})$ such that for every $\overline{c} \in \mathfrak{C}$ of length $|\overline{z}|$, $\varphi(x, y, \overline{c})$ defines an equivalence relation $E_{\overline{c}}$, and for arbitrarily large n there is $\overline{c}_n \in \mathfrak{C}^{|\overline{z}|}$ such that $E_{\overline{c}_n}$ has exactly n classes.

(C) There is some finite set Δ of partitioned formulas and some $M \models T$, such that for arbitrarily large m there is a (Δ, n)-indiscernible set $\{\overline{a}_i : i < m\}$ from M which cannot be extended to an infinite (Δ, n)-indiscernible set from M.

Also, not having the finite cover property implies stability, but we will always just say "stable and not the finite cover property" for emphasis.

We now introduce two combinatorial characteristics of models of ZFC^-, that will control how saturated unstable models can be, and how saturated models of theories with the finite cover property can be, respectively.

Definition 3.10.4. If $(L,<)$ is a linear order with proper initial segment ω, then let $\mathrm{lcf}_{(L,<)}(\omega)$ be the least cardinal κ such that there is a descending sequence $(a_\alpha : \alpha < \kappa)$ from $L \backslash \omega$ which is cofinal above ω (i.e., for every $a \in L$, if $a < a_\alpha$ for each $\alpha < \kappa$, then $a \in \omega$). Also, let $\mu_{(L,<)}$ denote the least cardinality of an initial segment L_0 of L which properly contains ω. So $\mathrm{lcf}_{(L,<)}(\omega) \leq \mu_{(L,<)}$ always.

Suppose $\hat{V} \models ZFC^-$ is ω-nonstandard. Then let $\mathrm{lcf}_{\hat{V}}(\omega) = \mathrm{lcf}_{(\hat{\omega},<)}(\omega)$ (this agrees with our previous definition) and let $\mu_{\hat{V}} = \mu_{(\hat{\omega},<)}$. So $\mathrm{lcf}_{\hat{V}}(\omega) \leq \mu_{\hat{V}}$.

The next three theorems will show (A) and (B) hold from Theorem 3.10.1, and make a fair amount of progress towards (C) as well.

The following is a translation of Theorem VI.4.8 of [75] into the language of interpretability orders.

Theorem 3.10.5. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard, and suppose $T \in V$ is a countable unstable theory. Write $\lambda = \mathrm{lcf}_{\hat{V}}(\omega)$. Then $\hat{V}$ does not λ^+-pseudosaturate T.

Proof. Choose $\varphi(x,\overline{y})$ an unstable T-formula (we can suppose x is a single variable by Theorem II.2.13 of [75], not that it matters). Let $M \models T$ with $M \in V$.

Choose $(\overline{a}^n_m : m < n < \omega)$ from $M^{|\overline{y}|}$ such that for each $m_* < n < \omega$, $M \models \exists x \bigwedge_{m<m_*} \varphi(x,\overline{a}^n_m) \wedge \bigwedge_{m \geq m_*} \neg\varphi(x,\overline{a}^n_m)$. Let $(\hat{\overline{a}}^{\hat{n}}_{\hat{m}} : \hat{m} < \hat{n} < \omega) = \mathbf{j}((\overline{a}^n_m : m < n < \omega))$.

Let $\hat{n} < \hat{\omega}$ be nonstandard. Let $(\hat{c}_\alpha : \alpha < \lambda)$ be a decreasing sequence from $\hat{\omega}$ with $\hat{c}_0 = \hat{n}$, which is cofinal above ω; this is possible by the definition of λ.

Let $p(x)$ be the pseudofinite type over $\mathbf{j}_{\mathrm{std}}(M)$ defined by: $p(x) = \{\varphi(x, \hat{\overline{a}}_i^{\hat{n}}) : i < \omega\} \cup \{\neg\varphi(x, \hat{\overline{a}}_{\hat{c}_\alpha}^{\hat{n}}) : \alpha < \lambda\}$. It suffices to show $p(x)$ is omitted by $\mathbf{j}_{\mathrm{std}}(M)$; so suppose towards a contradiction $b \in \mathbf{j}_{\mathrm{std}}(M)$ realized it. Let $Q(\hat{m}_*)$ be the property: for all $\hat{m} < \hat{m}_*$, $\varphi(b, \hat{\overline{a}}_{\hat{m}}^{\hat{n}})$. $Q(i)$ holds for all $i < \omega$, but since $(\hat{c}_\alpha : \alpha < \lambda)$ is cofinal above ω, $Q(\hat{m}_*)$ fails for all nonstandard $\hat{m}_*$. This contradicts overspill. □

The next two theorems are translations of Theorem VI.5.1 of [75] (except, see the remark before Theorem 3.10.7).

Theorem 3.10.6. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$ is $\aleph_1$-saturated. Suppose $T \in V$ is a countable stable theory with the finite cover property, and suppose $M \models T$ with $M \in V$. Then $\mathbf{j}_{\mathrm{std}}(M)$ is $\mu_{\hat{V}}$-saturated but not $\mu_{\hat{V}}^+$-pseudosaturated.

In particular, if λ is any cardinal, then $\hat{V}$ λ^+-pseudosaturates T if and only if $\lambda < \mu_{\hat{V}}$.

Proof. Suppose $M \models T$ with $M \in V$. We show $\mathbf{j}_{\mathrm{std}}(M)$ is $\mu_{\hat{V}}$-saturated but not $\mu_{\hat{V}}^+$-pseudosaturated.

Clearly, $\mathbf{j}_{\mathrm{std}}(M)$ is $\aleph_1$-saturated. Thus, to show $\mathbf{j}_{\mathrm{std}}(M)$ is $\mu_{\hat{V}}$-saturated, it suffices by Lemma III.3.10 of [75] to verify that whenever $\{a_i : i < \omega\}$ is an indiscernible set from $\mathbf{j}_{\mathrm{std}}(M)$, then it can be extended to an indiscernible sequence of size $\mu_{\hat{V}}$.

Since $\hat{V}$ is $\aleph_1$-compact, we can choose some $\hat{\Delta}$, $\hat{n}$, $\hat{w}$ such that the following hold:

- In $\hat{V}$, $\hat{\Delta}$ is a finite set of partitioned formulas of $\mathbf{j}(T)$;
- Each formula $\varphi(x, \overline{y})$ of T with first variable x is in $\hat{\Delta}$;
- $\hat{n} < \hat{\omega}$ is nonstandard;
- In $\hat{V}$, $\hat{w} = \{\hat{a}_{\hat{m}} : \hat{m} < \hat{n}\}$ is a set of elements from $\mathbf{j}(M)$ which is $(\hat{\Delta}, \hat{n})$-indiscernible, and $\hat{a}_i = a_i$ for $i < \omega$.

(It is clear that each finite fragment of the above conditions is satisfiable.) Then $\{\hat{a}_{\hat{m}} : \hat{m} < \hat{n}\}$ has cardinality at least $\hat{\mu}_{\hat{V}}$, and is an indiscernible set extending $\{a_i : i < \omega\}$, so M is $\mu_{\hat{V}}$-saturated.

Now we show that $\mathbf{j}_{\mathrm{std}}(M)$ is not $\mu_{\hat{V}}^+$-pseudosaturated. Let $\varphi(x, y, \overline{z})$ be a formula witnessing (B) of Theorem 3.10.3 holds. Let $X \subseteq \omega$ be the set of all $n < \omega$ such that for some $\overline{c} \in M^{|\overline{z}|}$, $E_{\overline{c}}^M$ has exactly n classes. Then the nonstandard elements of $\mathbf{j}(X)$ are cofinal above ω; thus we can choose $\hat{n}_* \in \mathbf{j}(X)$ nonstandard such that $|\{\hat{n} : \hat{n} < \hat{n}_*\}| = \mu_{\hat{V}}$ (as computed in $\mathbb{V}$). Let $\overline{c} \in \hat{\omega}^{|\overline{z}|}$ be such that in $\hat{V}$, $E_{\overline{c}}^{\mathbf{j}(M)}$ has $\hat{n}_*$ classes.

Let $\hat{f} : \hat{n}_* \to \mathbf{j}(M)$ choose a representative from each $E_{\overline{c}}$-class. Consider the partial type $p(x)$ over $\mathbf{j}_{\mathrm{std}}(M)$ which says $\neg\varphi(x, \hat{f}(\hat{n}), \overline{c})$ for all $\hat{n} < \hat{n}_*$. $p(x)$ is pseudofinite, and by choice of $\hat{n}_*$, $|p(x)| = \mu_{\hat{V}}$. But $p(x)$ is omitted in $\mathbf{j}_{\mathrm{std}}(M)$ by choice of $\hat{f}$. $\square$

In the following theorem, the fact that $\mathbf{j}_{\mathrm{std}}(M)$ is λ^+-pseudosaturated for all λ, even $\lambda \geq |\mathbf{j}_{\mathrm{std}}(M)|$, is a novelty of pseudosaturation; but its proof is a simple tweak of the first claim as proved by Shelah.

Theorem 3.10.7. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$ is $\aleph_1$-saturated. Suppose $T \in V$ is a countable stable theory without the finite cover property, and suppose $M \models T$ with $M \in V$. Then $\mathbf{j}_{\mathrm{std}}(M)$ is $|\hat{\omega}|$-saturated, and is λ^+-pseudosaturated for all λ.

In particular, for all λ, $\hat{V}$ λ^+-pseudosaturates T.

Proof. Suppose $M \models T$. Write $\kappa = |\hat{\omega}|$. We first show $\mathbf{j}_{\mathrm{std}}(M)$ is κ-saturated.

We know that $\mathbf{j}_{\mathrm{std}}(M)$ is $\aleph_1$-saturated. So it suffices by Lemma III.3.10 of [75] to verify that whenever $\{c_i : i < \omega\}$ is an indiscernible set from $\mathbf{j}_{\mathrm{std}}(M)$, then it can be extended to an indiscernible set of size $|\hat{\omega}|$.

Now, since (C) from Theorem 3.10.3 fails, we can find $f : \omega \to \omega$ such that for all

$n < \omega$, if $\{c_i : i < f(n)\}$ is a (Δ, n)-indiscernible set from M, then it can be extended to an infinite (Δ, n)-indiscernible set. Write $\hat{f} = \mathbf{j}(f)$.

Since $\hat{V}$ is $\aleph_1$-saturated, we can choose some $\hat{\Delta}$, $\hat{n}$, $\hat{w}$ such that the following hold:

- In $\hat{V}$, $\hat{\Delta}$ is a finite set of partitioned formulas of $\mathbf{j}(T)$;
- Each formula $\varphi(x, \overline{y})$ of T is in $\hat{\Delta}$;
- $\hat{n} < \hat{\omega}$ is nonstandard;
- $\hat{w} = \{\hat{c}_i : i < \hat{f}(\hat{n})\}$ is a $(\hat{\Delta}, \hat{n})$-indiscernible set from $\mathbf{j}(M)$ with $\hat{c}_i = c_i$ for $i < \omega$.

But then by choice of f and elementarity, we can find in $\hat{V}$ an infinite $(\hat{\Delta}, \hat{n})$-indiscernible set $\{\hat{c}_{\hat{m}} : \hat{m} < \hat{\omega}\}$. In $\mathbb{V}$, this is an indiscernible set of size κ extending $\{c_i : i < \omega\}$.

Now we check that $\mathbf{j}_{\text{std}}(M)$ is λ^+-pseudosaturated for all λ. This will essentially be a tweak of the proof of Lemma III.3.10 of [75]:

Let $p(x)$ be a pseudofinite type over $\mathbf{j}_{\text{std}}(M)$, say $p(x)$ is over $\hat{X} \in \hat{V}$ with $\hat{X}$ pseudofinite. We wish to show $p(x)$ is realized in $\mathbf{j}_{\text{std}}(M)$.

Let $q(x)$ be a non-forking extension of $p(x)$ to $\mathbf{j}_{\text{std}}(M)$. Let $(a_\alpha : \alpha < \omega + \omega)$ be Morley sequence in $q(x)$ (i.e., each a_α realizes $q(x) \restriction_{a(\beta):\beta<\alpha}$), with each $a_\alpha \in \mathbf{j}_{\text{std}}(M)$; we can find such a sequence by $\aleph_1$-saturation. Note that $q(x)$ is based on $(a_\alpha : \omega \leq \alpha < \omega+\omega)$. By the preceding argument, we can find a set $\{\hat{a}_{\hat{n}} : \hat{n} < \hat{\omega}\}$ in $\hat{V}$ with $\hat{a}_i = a_i$ for all $i < \omega$, which when considered in $\mathbb{V}$ is an indiscernible set.

By overspill, we can find, in $\hat{V}$, a finite set $\hat{\Delta}$ of $\mathbf{j}(\mathcal{L})$-formulas, containing all of the true formulas of $\mathcal{L}$. By Ramsey's theorem in $\hat{V}$, we can find $\hat{Y} \subseteq \hat{\omega}$ infinite, such that $\{\hat{a}_{\hat{n}} : \hat{n} \in \hat{Y}\}$ is $\hat{\Delta}$-indiscernible over $\hat{X}$. Then in $\mathbb{V}$, $\{\hat{\overline{a}}_{\hat{n}} : \hat{n} \in \hat{Y}\}$ is indiscernible over $\hat{X}$.

It suffices to show that for some or every $\hat{n} \in \hat{Y}$, $\hat{a}_{\hat{n}}$ realizes $p(x)$ (the choice of $\hat{n}$

does not matter by indiscernibility). Suppose $\varphi(x, \bar{b}) \in p(x)$ is given; suppose towards a contradiction that $\mathbf{j}_{\mathrm{std}}(M) \models \neg\varphi(\hat{a}_{\hat{n}}, \bar{b})$ for some or every $\hat{n} \in \hat{Y}$.

Write $\psi(\bar{y}, x) = \varphi(x, \bar{y})$. Choose $i_* < \omega$ such that $tp_\psi(\bar{b}/\{a_\alpha : \alpha \in i_*+1 \cup [\omega, \omega+\omega)\}$ does not fork over $\{a_\alpha : \alpha \in i_* \cup [\omega, \omega+\omega)\}$. Since $q(x)$ is based on $\{a_\alpha : \omega \leq \alpha < \omega+\omega\}$ we must have that $\mathbf{j}_{\mathrm{std}}(N) \models \varphi(a_i, \bar{b})$ for all $i \geq i_*$.

Write $J = \{i < \omega : i \geq i_*\} \cup \hat{Y}$. Then $\{a_j : j \in J\}$ is indiscernible, and there are infinitely many $j \in J$ with $\mathbf{j}_{\mathrm{std}}(M) \models \varphi(a_j, \bar{b})$, and there are infinitely many $j \in J$ with $\mathbf{j}_{\mathrm{std}}(M) \models \neg\varphi(a_j, \bar{b})$. This easily contradicts the stability of $\varphi(x, \bar{y})$. □

Thus we have shown (A) and (B) hold of Theorem 3.10.1; we now aim towards showing (C). This will be an ultrafilter construction. Shelah's proof in [75] essentially goes through a precursor to Theorem 3.1.9. We will give a streamlined treatment in terms of complete Boolean algebras.

Suppose $\mathcal{U}$ is an $\aleph_1$-incomplete ultrafilter on the complete Boolean algebra $\mathcal{B}$. Recall as a special case Definition 3.4.10 that if $\mathcal{B}$ is a complete Boolean algebra, then $(\omega, <)^{\mathcal{B}}$ is a full $\mathcal{B}$-valued model of $\mathrm{Th}(\omega, <)$. Its domain is the set of all partitions of $\mathbf{n}$ by ω, that is, the set of all $\mathbf{n} : \omega \to \mathcal{B}$ such that for all $n < m$, $\mathbf{n}(n) \wedge \mathbf{n}(m) = 0$, and such that $\bigvee_n \mathbf{n}(n) = 1$. We have $\|\mathbf{n} < \mathbf{m}\|_{\omega^{\mathcal{B}}} = \bigvee_{n<m} \mathbf{n}(n) \wedge \mathbf{m}(m)$, and similarly for $\leq, =$, etc. If $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, then we view $(\omega, <)^{\mathcal{B}}/\mathcal{U}$ as an elementary extension of $(\omega, <)$.

Definition 3.10.8. Suppose $\mathcal{U}$ is an $\aleph_1$-incomplete ultrafilter on the complete Boolean algebra $\mathcal{B}$. Then define $\mu_{\mathcal{U}} = \mu_{(\omega,<)^{\mathcal{B}}/\mathcal{U}}$ and define $\mathrm{lcf}_{\mathcal{U}}(\omega) = \mathrm{lcf}_{(\omega,<)^{\mathcal{B}}/\mathcal{U}}(\omega)$.

Again $\mathrm{lcf}_{\mathcal{U}}(\omega) \leq \mu_{\mathcal{U}}$.

In order to see that these definitions are significant, we need the following theorem and corollary.

Theorem 3.10.9. Suppose $\mathcal{B}$ is a complete Boolean algebra, and $(\omega, <) \preceq (\mathbf{L}, <)$ is a $\mathcal{B}$-valued elementary extension with $(\mathbf{L}, <)$ $\aleph_1$-saturated. Then there is a unique embedding $\mathbf{i} : (\omega, <)^{\mathcal{B}} \preceq (\mathbf{L}, <)$ which is the identity on ω. The image of $\mathbf{i}$ consists exactly of those $\mathbf{n} \in \mathbf{L}$ such that $\bigvee_{n<\omega} \|\mathbf{n} = n\|_{\mathbf{L}} = 1$. Thus, if $\mathbf{n}$ is in the image of $\mathbf{i}$ and $\|\mathbf{m} \leq \mathbf{n}\|_{\mathbf{L}} = 1$, then $\mathbf{m}$ is in the image of $\mathbf{i}$.

Proof. Given $\mathbf{n} \in (\omega, <)^{\mathcal{B}}$, we can by $\aleph_1$-saturation find $\mathbf{m} \in \mathbf{L}$ such that each $\|\mathbf{m} = n\|_{\mathbf{L}} = \mathbf{n}(n)$. I claim that this specifiies $\mathbf{m}$ uniquely, so that we can set $\mathbf{i}(\mathbf{n}) = \mathbf{m}$. Indeed, suppose $\mathbf{m}'$ were another element of $\mathbf{L}$ with each $\|\mathbf{m}' = n\|_{\mathbf{L}} = \mathbf{n}(n)$. Then $\bigvee_n \|\mathbf{m} = n\|_{\mathbf{L}} = \bigvee_n \|\mathbf{m}' = n\| = 1$, so we can choose a maximal antichain $\mathbf{C}$ of $\mathcal{B}$ such that for each $\mathbf{c} \in \mathbf{C}$, there are $m(\mathbf{c}), m'(\mathbf{c}) < \omega$ with $\mathbf{c} \leq \|\mathbf{m} = m(\mathbf{c}) \wedge \mathbf{m}' = m'(\mathbf{c})\|_{\mathbf{L}}$. But then each $m(\mathbf{c}) = m'(\mathbf{c})$, as otherwise $\mathbf{c} \leq \mathbf{n}(m(\mathbf{c})) \wedge \mathbf{n}(m'(\mathbf{c})) = 0$. Thus each $\mathbf{c} \leq \|\mathbf{m} = \mathbf{m}'\|_{\mathbf{L}}$, and this happens on a maximal antichain. Thus $\|\mathbf{m} = \mathbf{m}'\|_{\mathbf{L}} = 1$, and so $\mathbf{m} = \mathbf{m}'$.

It is simple to see that $\mathbf{i}$ is an elementary embedding, and that we had no choice. Moreover, if $\mathbf{m} \in \mathbf{L}$ has $\bigvee_n \|\mathbf{m} = n\|_{\mathbf{L}} = 1$, then we can define $\mathbf{n} \in (\omega, <)^{\mathcal{B}}$ with each $\mathbf{n}(n) = \|\mathbf{m} = n\|_{\mathbf{L}}$, so $\mathbf{i}(\mathbf{n}) = \mathbf{m}$.

The final claim follows easily, since we have the identity: $\bigvee_n \|\mathbf{n} = n\|_{\mathbf{L}} = 1$ if and only if $\bigvee_n \|\mathbf{n} \leq n\|_{\mathbf{L}} = 1$. □

In future we will always suppose $(\omega, <)^{\mathcal{B}} \preceq \mathbf{V}$, i.e. $\mathbf{i}$ is the inclusion.

Corollary 3.10.10. Suppose $\mathcal{B}$ is a complete Boolean algebra and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Suppose $(\mathbf{L}, <) \succeq (\omega, <)$ is $\aleph_1$-saturated. Then $(\mathbf{L}, <)/\mathcal{U}$ is an end extension of $(\omega, <)^{\mathcal{B}}/\mathcal{U}$.

Proof. By the above convention, we can suppose $(\omega, <)^{\mathcal{B}} \preceq (\mathbf{L}, <)$. Thus we can also view $(\omega, <)^{\mathcal{B}}/\mathcal{U} \preceq (\mathbf{L}, <)/\mathcal{U}$, and so the statement of the corollary makes sense.

Suppose $\mathbf{n} \in \mathbf{L}$ and $\mathbf{m} \in (\omega, <)^{\mathcal{B}}$ with $\|\mathbf{n} \leq \mathbf{m}\|_{\mathbf{L}} \in \mathcal{U}$. Note that $\min(\mathbf{n}, \mathbf{m})$ makes sense by fullness of $\mathbf{L}$, and that $[\min(\mathbf{n}, \mathbf{m})]_{\mathcal{U}} = [\mathbf{n}]_{\mathcal{U}}$. By Corollary 3.10.9, $\min(\mathbf{n}, \mathbf{m}) \in (\omega, <)^{\mathcal{B}}$, so $[\mathbf{n}]_{\mathcal{U}} \in (\omega, <)^{\mathcal{B}}/\mathcal{U}$. $\square$

Corollary 3.10.11. Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$. If T is stable without the finite cover property, then $\mathcal{U}$ λ^+-saturates T for all λ.

Suppose additionally $\mathcal{U}$ is $\aleph_1$-incomplete. If T is stable with the finite cover property, then $\mathcal{U}$ λ^+-saturates T if and only if $\lambda < \mu_{\mathcal{U}}^+$. If T is unstable then $\mathcal{U}$ does not $\mathrm{lcf}_{\mathcal{U}}(\omega)^+$-saturate T.

Proof. Suppose $V \models ZFC^-$ is transitive; choose $\mathbf{i} := V \preceq \mathbf{V}$ with $\mathbf{V}$ $\aleph_1$-saturated. Write $\hat{V} = \mathbf{V}/\mathcal{U}$. Then by Corollary 3.10.10, $\mu_{\mathcal{U}} = \mu_{\hat{V}}$ and $\mathrm{lcf}_{\mathcal{U}} = \mathrm{lcf}_{\hat{V}}$. Thus we conclude by Theorems 3.8.15, 3.10.5, 3.10.6 and 3.10.7. $\square$

One can ask after the situation for $\aleph_1$-complete $\mathcal{U}$; in fact we will later (see Remark 3.14.12) that if $\mathcal{U}$ is $\aleph_1$-complete, then $\mathcal{U}$ λ^+-saturates every stable theory for every λ, so it is reasonable to set $\mu_{\mathcal{U}} = \infty$.

Now to finish the proof of Theorem 3.10.1 it suffices to verify the following theorem. The argument is based on Theorem 3.12 from [75] Chapter 6, although there is a minor gap in that proof. (Claim VI.3.18 (4) from [75] is false.) The problem does not arise in our treatment with complete Boolean algebras, although there are also more direct patches.

Recall that $P_{\alpha\aleph_0\aleph_0}$ is the forcing notion of all finite partial functions from α to $\aleph_0$, ordered by reverse inclusion; $\mathcal{B}_{\alpha\aleph_0\aleph_0}$ is its Boolean algebra completion.

Theorem 3.10.12. Suppose $\aleph_0 < \kappa \leq \mu$ are cardinals with $\mu = \mu^{\aleph_0}$ and κ regular. Let α_* be the ordinal product $\mu\kappa$, or any other ordinal with $\mathrm{cof}(\alpha_*) = \kappa$, such that for all $\alpha < \alpha_*$, $|\alpha_* \backslash \alpha| = \mu$. Let $\mathcal{B} = \mathcal{B}_{\alpha_*, \aleph_0, \aleph_0}$. Then there is an $\aleph_1$-incomplete ultrafilter $\mathcal{U}$ on $\mathcal{B}$

with $\mu_{\mathcal{U}} = \mu$ and $\mathrm{lcf}_{\mathcal{U}}(\omega) = \kappa$.

Proof. For each $\alpha \leq \alpha_*$ let $P_\alpha = P_{\alpha,\aleph_0,\aleph_0}$ and let $\mathcal{B}_\alpha = \mathcal{B}_{\alpha,\aleph_0,\aleph_0}$. Note that for $\alpha < \alpha'$, $(\omega,<)^{\mathcal{B}_\alpha} \preceq (\omega,<)^{\mathcal{B}_{\alpha'}}$, and if $\mathbf{n}, \mathbf{m} \in \mathcal{B}_\alpha$, then $\|\mathbf{n} < \mathbf{m}\|_{\mathcal{B}_\alpha} = \|\mathbf{n} < \mathbf{m}\|_{\mathcal{B}_{\alpha'}}$, etc.

For each $\alpha < \alpha_*$, let $\mathbf{n}_\alpha \in (\omega,<)^{\mathcal{B}_{\alpha+1}}$ be the element defined via $\mathbf{n}_\alpha(m) = \{\langle \alpha, m\rangle\}$ (i.e., the function with domain $\{\alpha\}$, which sends α to m).

I claim we can choose by induction on $\alpha < \alpha_*$ ultrafilters $\mathcal{U}_\alpha$ on $\mathcal{B}_\alpha$ such that:

(I) For $\alpha < \alpha'$, $\mathcal{U}_\alpha \subseteq \mathcal{U}_{\alpha'}$;

(II) For each $\alpha < \alpha_*$, $\mathbf{n}_\alpha$ is $\mathcal{U}_{\alpha+1}$-nonstandard (i.e. for each $m < \omega$, $[\mathbf{n}_\alpha]_{\mathcal{U}_{\alpha+1}} > m$);

(III) Suppose $\alpha < \alpha_*$, and for each $\mathcal{U}_\alpha$-nonstandard $\mathbf{m} \in (\omega,<)^{\mathcal{B}_\alpha}$, $[\mathbf{n}_\alpha]_{\mathcal{U}_{\alpha+1}} < [\mathbf{m}]_{\mathcal{U}_{\alpha+1}}$.

Indeed, suppose towards a contradiction we have chosen $\mathcal{U}_\alpha$ and cannot continue. Then there must be some $m_* < \omega$ and some finite tuple $(\mathbf{m}_i : i < i_*)$ from $(\omega,<)^{\mathcal{B}_\alpha}$ and some $\mathbf{a} \in \mathcal{U}_\alpha$, such that $\mathbf{a} \wedge \|\mathbf{n}_\alpha \geq m_*\|_{\mathcal{B}_{\alpha+1}} \wedge \bigwedge_{i<i_*} \|\mathbf{n}_\alpha < \mathbf{m}_i\|_{\mathcal{B}_{\alpha+1}} = 0$.

Write $f = \{\langle \alpha, m_*\rangle\} = \|\mathbf{n}_\alpha = m_*\|_{\mathcal{B}_{\alpha+1}}$. Then by wedging the preceding equation with f, we get that $f \wedge \mathbf{a} \wedge \bigwedge_{i<i_*} \|m_* < \mathbf{m}_i\|_{\mathcal{B}_{\alpha+1}} = 0$. But $\mathbf{a} \wedge \bigwedge_{i<i_*} \|m_* < \mathbf{m}_i\|_{\mathcal{B}_\alpha} \in \mathcal{U}_\alpha$ is nonzero, so we can find $g \in P_\alpha$ below it; then since g and f have no common domain, $g \cup f = g \wedge f \in P_{\alpha+1}$ is nonzero, a contradiction.

Let $\mathcal{U} = \bigcup_{\alpha<\alpha_*} \mathcal{U}_\alpha$. Since $\mathcal{B}_{\alpha_*}$ has the $\aleph_1$-c.c. and since $\kappa = \mathrm{cof}(\kappa) > \aleph_0$, we have that $\mathcal{U}$ is an ultrafilter on $\mathcal{B}_{\alpha_*}$, and $(\omega,<)^{\mathcal{B}_{\alpha_*}} = \bigcup_{\alpha<\alpha_*}(\omega,<)^{\mathcal{B}_\alpha}$. From this it is clear that $(\mathbf{n}_\alpha : \alpha < \alpha_*)$ is a cofinal sequence above ω in $(\omega,<)^{\mathcal{B}_{\alpha_*}}/\mathcal{U}$, and hence $\mathrm{lcf}_{\mathcal{U}}(\omega) = \mathrm{cof}(\alpha_*) = \kappa$ and $\mu_{\mathcal{U}} \geq \min\{\alpha_* \backslash \alpha : \alpha < \alpha_*\} = \mu$. But also $\mu_{\mathcal{U}} \leq |\mathcal{B}_{\alpha_*}| \leq |P_{\alpha_*}|^{\aleph_0} = \mu$, so we are done. $\square$

3.11 Keisler's Order is Local

We present a theorem due to Malliaris [53], that says that ultrapowers are saturated if and only if they are locally saturated. We present this in the terminology of $\trianglelefteq^{\times}_{\aleph_1}$; the translations are straightforward.

First of all we want the following lemma; it is Lemma 9 from [53].

Lemma 3.11.1. Suppose $\hat{V} \models ZFC^-$ is $\aleph_1$-saturated. Suppose $\theta < \mathrm{lcf}_{\hat{V}}(\omega)$, and $X \subset \hat{V}$ has size at most θ (so X need not be definable). Suppose $(\hat{X}_n : n < \omega)$ is a sequence of elements of $\hat{V}$ with $X \subseteq \hat{X}_n$ for each $n < \omega$. Then there is $\hat{X} \in \hat{V}$ with $X \subseteq \hat{X} \subseteq \hat{X}_n$ for each $n < \omega$.

Proof. By replacing $\hat{X}_n$ with $\bigcap_{m \leq n} \hat{X}_m$ we can suppose $\hat{X}_n \supseteq \hat{X}_m$ for $n < m$. If for some $n < \omega$ we have $\hat{X}_n = \hat{X}_m$ for all $m \geq n$ then we are done, so by discarding duplicates we can suppose $\hat{X}_n \supsetneq \hat{X}_m$ for all $n < m$. Enumerate $X = \{\hat{a}_\alpha : \alpha < \theta\}$.

By $\aleph_1$-saturation, we can choose a linear ordering $\hat{<}_*$ of $\hat{X}_0$ in $\hat{V}$, such that for all $\hat{a} \notin \hat{X}_n$ and for all $\hat{b} \in \hat{X}_n$, we have $\hat{a} \hat{<}_* \hat{b}$. Additionally we can choose some nonstandard $\hat{n}_* < \hat{\omega}$ and some map $\hat{f} : \hat{n}_* \to \hat{X}_0$ such that:

- $\hat{f}(n) \in \hat{X}_n \backslash \hat{X}_{n+1}$ for each $n < \omega$;
- For all $\hat{n} < \hat{m} < \hat{n}_*$, $\hat{f}(\hat{n}) \hat{<}_* \hat{f}(\hat{m})$.

Note that for each nonstandard $\hat{n} < \hat{n}_*$, we have $\hat{f}(\hat{n}) \in \bigcap_{n \in \omega} \hat{X}_n$.

Since $\theta < \mathrm{lcf}_{\hat{V}}(\omega)$, we can find $\hat{n}_\alpha : \alpha \leq \theta$ a descending sequence of nonstandard numbers from $\hat{D}$, so that for all $\alpha < \theta$, $\hat{f}(\hat{n}_{\alpha+1}) \hat{<}_* \hat{a}_\alpha$. Then $\hat{X} := \{\hat{a} \in \hat{X}_0 : \hat{f}(\hat{n}_\theta) \hat{<}_* \hat{a}\}$ is as desired. $\square$

Recall the definition of "set of partitioned formulas," etc. from Section 3.10.1. We will want the following well-known fact:

Lemma 3.11.2. Suppose $\Delta = \{\varphi_i(x, \overline{y}_i) : i < n\}$ is a finite set of partitioned formulas. Then there is a formula $\psi(x, \overline{z})$ such that for every infinite set A, every Δ-formula over A is logically equivalent to a positive ψ-formula over A. In particular every Δ-type over A is logically equivalent to a positive ψ-type over A, and every complete Δ-type over A is equivalent to a conjunction of instances of ψ.

Proof. Let $z_0, \ldots, z_{m-1} = \overline{z}$ be new variables, for m large enough (we need there to be at least $2n$ distinct partitions of m). Let $\sigma_i(\overline{z}) : i < n$, $\tau_i(\overline{z}) : i < n$ be distinct equality types. Then let $\psi(x, \overline{y}_0 \ldots \overline{y}_{n-1}\overline{z})$ be

$$\bigwedge_{i<n} \sigma_i(\overline{z}) \to \varphi_i(x, \overline{y}_i) \wedge \bigwedge_{i<n} \tau_i(\overline{z}) \to \neg\varphi_i(x, \overline{y}_i).$$

□

The following is the main theorem of [53], generalized to the context of $\trianglelefteq^{\times}_{\aleph_1}$.

Theorem 3.11.3. Suppose $V \models ZFC^-$ is transitive, $\mathbf{j} : V \preceq \hat{V}$ is $\aleph_1$-saturated, and $T \in V$ is a complete countable theory. Suppose $M \models T$ with $M \in V$, and λ is a cardinal. Then the following are equivalent.

(A) $\mathbf{j}_{\mathrm{std}}(M)$ is λ^+-pseudosaturated;

(B) For every formula $\varphi(x, \overline{y})$ and for every positive, pseudofinite φ-type $p(x)$ over M of cardinality at most λ, $p(x)$ is realized in M.

Proof. Obviously (A) implies (B).

(B) implies (A): We break into cases, and show that in each of them, (B) implies (A).

First, if T is stable without the finite cover property, then (A) holds.

Second, suppose T is stable with the finite cover property. If $\lambda \geq \mu_{\hat{V}}$, then by the proof of Lemma 3.10.6 we have that (B) fails, and if $\lambda < \mu_{\hat{V}}$ then by Theorem 3.10.6, $\mathbf{j}_{\text{std}}(M)$ is in fact λ^+-saturated, so in particular (A) holds.

Third, if T is unstable and $\lambda \geq \text{lcf}_{\hat{V}}(\omega)$, then by the proof of Lemma 3.10.5 (and applying Lemma 3.11.2 to $\Delta = \{\varphi(x, \overline{y}), \neg\varphi(x, \overline{y})\}$), (B) fails.

Finally we have the key case, where T is unstable and $\lambda < \text{lcf}_{\hat{V}}(\omega)$. Suppose (B) holds, and let $p(x) \in S^1(A)$, where $A \subseteq \mathbf{j}_{\text{std}}(M)$ is pseudofinite. Choose $\hat{A} \in \hat{V}$ pseudofinite with $A \subseteq \hat{A}$ (we can suppose $\hat{A} \subseteq \mathbf{j}_{\text{std}}(M)$). Let $(y_i : i < \omega)$ be variables, and for each $n < \omega$, let $\overline{y}_n = (y_i : i < n)$. As in the proof of Theorem 3.8.11, we can find formulas $(\varphi_n(x, \overline{y}_n) : n < \omega)$ which list all $\mathcal{L}$-formulas in the variables $(x, y_i : i < \omega)$.

For each $n < \omega$, let $X_n = \{\overline{a} \in A^n : \varphi_n(x, \overline{a}) \in p(x)\}$ and let $p_n(x) = \{\varphi_n(x, \overline{a}) : \overline{a} \in X_n\}$, so $p(x)$ is the union of $\{p_n(x) : n < \omega\}$. By hypothesis and Lemma 3.11.2, we can find $a_n \in \mathbf{j}_{\text{std}}(M)$ realizing $\bigcup_{n' \leq n} p_{n'}(x)$.

For each $n < \omega$, write $\hat{X}_n^0 = \hat{A}^n$. For each $n \leq i < \omega$ let $\hat{X}_n^{i+1}$ be the set of all $\overline{b} \in \hat{X}_n^0$ such that for each $n \leq j \leq i$, $M \models \varphi_n(a_j, \overline{b})$. Clearly, for each $n < \omega$, the hypotheses of Lemma 3.11.1 are met for X_n and $(\hat{X}_n^i : n \leq i < \omega)$. Hence we can choose $\hat{X}_n$ from $\hat{V}$ with $X_n \subseteq \hat{X}_n \subseteq \hat{X}_n^i$ for each $i \geq n$.

Thus, for all $m < \omega$, we have that for all $n \leq m$ and for each $\overline{b} \in \hat{X}_n$, $\mathbf{j}_{\text{std}}(M) \models \varphi_n(a_m, \overline{b})$. Write $(\hat{a}_{\hat{m}} : \hat{m} < \hat{\omega}) = \mathbf{j}((a_m : m < \omega))$. By $\aleph_1$-saturation, we can find some nonstandard $\hat{m}$ such that for all $n < \omega$ and for all $\overline{b} \in \hat{X}^n$, $\mathbf{j}_{\text{std}}(M) \models \varphi_n(\hat{a}_{\hat{m}}, \overline{b})$. Then in particular, $\hat{a}_{\hat{m}}$ realizes $p(x)$. $\square$

We now introduce some terminology designed to understand when (B) above holds for a particular $\varphi(x, \overline{y})$. This is closely related to the terminology of characteristic sequences, arrays, and diagrams of [49].

Definition 3.11.4. If I is an index set, then a pattern on I is some $\Delta \subseteq [I]^{<\aleph_0}$ such that Δ is downward-closed (i.e. closed under subsets), and such that $[I]^1 \subseteq \Delta$. A Δ-clique is a subset $X \subseteq I$ with $[X]^{<\aleph_0} \subseteq \Delta$. If Δ is a pattern on I and Δ' is a pattern on J, then say that Δ' is an instance of Δ if for all $s \in [J]^{<\aleph_0}$ there is some map $f : s \to I$ such that for all $t \subseteq s$, $t \in \Delta'$ if and only if $f[t] \in \Delta$. Say that two patterns Δ, Δ' are equivalent if they are instances of each other.

Note that every pattern is equivalent to a pattern on ω. Note also that if $\Delta \subseteq [I]^{<\aleph_0}$ is downward closed, then Δ is always a pattern on $\bigcup \Delta$.

Definition 3.11.5. Suppose $\varphi(\overline{x}, \overline{y})$ is a formula of T and Δ is a pattern on I. Then Δ is a $(T, \varphi(\overline{x}, \overline{y}))$-pattern if for every $s \in [I]^{<\aleph_0}$ there are $M \models T$ and $(\overline{a}_i : i \in s)$ from M, so that for all $t \subseteq s$, $M \models \exists \overline{x} \bigwedge_{i \in t} \varphi(\overline{x}, \overline{a}_i)$ if and only if $t \in I$.

Note that if $\varphi(\overline{x}, \overline{y})$ is a formula of T, then there is a pattern Δ on ω such that if Δ' is any pattern on I, then Δ' is a $(T, \varphi(\overline{x}, \overline{y}))$-pattern if and only if Δ' is an instance of Δ. This is most useful in specific examples, when we can choose an easy-to-understand Δ and then forget about $T, \varphi(x, \overline{y})$.

Definition 3.11.6. If $\mathcal{B}$ is a complete Boolean algebra and $\mathbf{A}$ is an I-distribution and Δ is a pattern on J, then say that $\mathbf{A}$ is an (I, Δ)-distribution if for every $s \in [I]^{<\aleph_0}$ and for every $\mathbf{c} \in \mathcal{B}_+$ such that $\mathbf{c}$ decides $\mathbf{A}_t$ for all $t \subseteq s$, and such that $\mathbf{c} \leq \mathbf{A}_{\{i\}}$ for all $i \in s$, there is some $f : s \to J$ such that for all $t \subseteq s$, $\mathbf{c} \leq \mathbf{A}_t$ if and only if $f[t] \in \Delta$. (Compare this with the characterization of (λ, T)-possibilities in Theorem 3.6.8.)

If $\varphi(\overline{x}, \overline{y})$ is a formula, then say that $\mathbf{A}$ is an $(I, T, \varphi(\overline{x}, \overline{y}))$-Łoś map (possibility) if it is an $(I, T, \overline{\varphi})$-Łoś map (possibility), where $\overline{\varphi} = (\varphi(\overline{x}, \overline{y}_i) : i \in I)$ is obtained from φ by taking I-many disjoint copies $\overline{y}_i$ of $\overline{y}$.

Finally, we define what will be, for our purposes, an exhaustive list of combinatorial invariants of models of ZFC^-.

Definition 3.11.7. Suppose $V \models ZFC^-$ is transitive, $\mathbf{j} : V \preceq \hat{V}$, and suppose $\Delta \in V$ is a pattern on I (so $I \in V$). Then let $\lambda_{\hat{V}}(\Delta)$ be the least λ such that there is some pseudofinite $X \subseteq \mathbf{j}_{\mathrm{std}}(I)$ with $|X| \leq \lambda$, such that $[X]^{<\aleph_0} \subseteq \mathbf{j}(\Delta)$, and such that there is no $\hat{X} \in \mathbf{j}(\Delta)$ with $X \subseteq \hat{X}$.

If T is a complete countable theory in V and $\varphi(\overline{x}, \overline{y})$ is a formula of T, then let $\lambda_{\hat{V}}(\varphi)$ be the minimum of all $\lambda_{\hat{V}}(\Delta)$, for Δ a $(T, \varphi(\overline{x}, \overline{y}))$-pattern on ω (really this depends on T and $\mathbf{j}$ as well; in practice this will not cause confusion). Let $\lambda_{\hat{V}}(T)$ be the minimum over all formulas $\varphi(x, \overline{y})$ (with x a single variable) of $\lambda_{\hat{V}}(\varphi)$.

Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$, and Δ is a pattern on ω (say). Then let $\lambda_{\mathcal{U}}(\Delta)$ be the least λ such that there is some (λ, Δ)-distribution in $\mathcal{U}$ with no multiplicative refinement in $\mathcal{U}$. If $\varphi(\overline{x}, \overline{y})$ is a formula of T then let $\lambda_{\mathcal{U}}(\varphi)$ be the minimum over all $\varphi(\overline{x}, \overline{y})$-patterns Δ on ω of $\lambda_{\mathcal{U}}(\Delta)$. Let $\lambda_{\mathcal{U}}(T)$ be the minimum over all $\lambda_{\mathcal{U}}(\varphi)$, for $\varphi(x, \overline{y})$ a formula of T.

In all of these cases, if no such λ exists, then we let the corresponding value be ∞.

Example 3.11.8. For any pattern Δ on ω, and for any $\mathbf{j} : V \preceq \hat{V}$, we have that $\lambda_{\hat{V}}(\Delta) \geq \mathfrak{p}_{\hat{V}}$ by Theorem 3.9.6, hence each $\lambda_T \geq \mathfrak{p}_{\hat{V}}$. Note that by Theorems 3.9.11 and 3.9.9, if T has SOP_2 then $\lambda_{\hat{V}}(T) = \mathfrak{p}_{\hat{V}}$. Similar statements hold for the ultrafilter versions.

We now connect these notions with the following two simple theorems.

Theorem 3.11.9. Suppose $V \models ZFC^-$ is transitive and $\mathbf{j} : V \preceq \hat{V}$ is $\aleph_1$-saturated and suppose $T \in V$ is a complete countable theory. Then $\hat{V}$ λ^+-pseudosaturates T if and only if $\lambda < \lambda_{\hat{V}}(T)$; and this implies $\lambda < \lambda_{\hat{V}}(\varphi(\overline{x}, \overline{y}))$ for all formulas $\varphi(\overline{x}, \overline{y})$ of T.

Proof. This is basically Theorem 3.11.3, restated. The point is the following: suppose $M \models T$ with $M \in V$, and let $\varphi(\overline{x}, \overline{y})$ be a formula of T. Let $\Delta = \{s \in [M^{|\overline{y}|}]^{<\aleph_0} : M \models \exists \overline{x} \bigwedge_{\overline{b} \in s} \varphi(\overline{x}, \overline{b})\}$ and let $\hat{\Delta} = \mathbf{j}(\Delta)$. Then positive φ-types $p(\overline{x})$ over $\mathbf{j}_{\mathrm{std}}(M)$ correspond exactly to subsets X of $\mathbf{j}_{\mathrm{std}}(M)^{|\overline{y}|}$ with $[X]^{<\aleph_0} \subseteq \hat{\Delta}$, and, assuming $p(\overline{x})$ is pseudofinite, $p(\overline{x})$ is realized in $\mathbf{j}_{\mathrm{std}}(M)$ if and only if there is some $\hat{X} \in \hat{\Delta}$ with $X \subseteq \hat{X}$. Moreover, it suffices for λ^+-pseudosaturation to consider types in a single variable x. $\square$

Theorem 3.11.10. Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$, suppose $V \models ZFC^-$ is transitive, suppose $\mathbf{i} : V \preceq \mathbf{V}$ is λ^+-saturated, and suppose $\Delta \in V$ is a pattern on I. Then $\lambda < \lambda_{\mathcal{U}}(\Delta)$ if and only if $\lambda < \lambda_{\mathbf{V}/\mathcal{U}}(\Delta)$. Thus, for all λ, T, we have that $\mathcal{U}$ λ^+-saturates T if and only if $\lambda < \lambda_{\mathcal{U}}(T)$, i.e. whenever $\mathbf{A}$ is a (λ, T, φ)-Łoś map in $\mathcal{U}$ for some formula $\varphi(x, \overline{y})$ of T, then $\mathbf{A}$ has a multiplicative refinement in $\mathcal{U}$.

Proof. Write $\hat{V} = \mathbf{V}/\mathcal{U}$. We know that $\mathcal{U}$ λ^+-saturates T if and only if $\hat{V}$ λ^+-pseudosaturates T if and only if $\lambda < \lambda_{\hat{V}}(T)$. Thus it suffices to show the first claim, since then $\lambda < \lambda_{\hat{V}}(T)$ if and only if $\lambda < \lambda_{\mathcal{U}}(T)$.

Suppose first $\lambda \geq \lambda_{\mathcal{U}}(\Delta)$. Then we can find $\mathbf{A}$, a (λ, Δ)-distribution in $\mathcal{U}$ with no multiplicative refinement in $\mathcal{U}$. By Corollary 3.3.8 and λ^+-saturation of $\mathbf{V}$, we can find some $\mathbf{X} \in \mathbf{V}$ such that $\|\mathbf{X} \in [\mathbf{I}]^{<\aleph_0}\|_{\mathbf{V}} = 1$, and we can find $(\mathbf{a}_\alpha : \alpha < \lambda)$ with each $\|\mathbf{a}_\alpha \in \mathbf{X}\|_{\mathbf{V}} = 1$, such that for all $s \in [\lambda]^{<\aleph_0}$, $\|\{\mathbf{a}_\alpha : \alpha \in s\} \in \mathbf{i}(\Delta)\|_{\mathbf{V}} = \mathbf{A}(s)$. If $\hat{Y} \in \mathbf{j}(\Delta)$ had each $[\mathbf{a}_\alpha]_{\mathcal{U}} \in \hat{Y}$, then write $\hat{Y} = [\mathbf{Y}]_{\mathcal{U}}$; , and define $\mathbf{B}(s) = \|\{a_\alpha : \alpha \in s\} \subseteq \mathbf{Y}\|_{\mathbf{V}} \wedge \|\mathbf{Y} \in \mathbf{i}(\Delta)\|_{\mathbf{V}}$; this is clearly a multiplicative refinement of $\mathbf{A}$ in $\mathcal{U}$, contradicting choice of $\mathbf{A}$. Thus no such $\hat{Y}$ exists, and this witnesses $\lambda \geq \lambda_{\hat{V}}(\Delta)$.

Conversely, if $\lambda \geq \lambda_{\hat{V}}(\Delta)$, then we can find some pseudofinite $X \subseteq \mathbf{j}(I)$ of size at most λ with $[X]^{<\aleph_0} \subseteq \mathbf{j}(\Delta)$, such that there is no $\hat{X} \in \mathbf{j}(\Delta)$ with $X \subseteq \hat{X}$. Enumerate $X = \{[\mathbf{a}_\alpha]_{\mathcal{U}} : \alpha < \lambda\}$, and for each $s \in [\lambda]^{<\aleph_0}$ define $\mathbf{A}(s) = \|\{\mathbf{a}_\alpha : \alpha \in s\} \in \mathbf{i}_{\mathrm{std}}(\Delta)\|_{\mathbf{V}}$. Then $\mathbf{A}$ is an (I, Δ)-distribution in $\mathcal{U}$. Suppose towards a contradiction that $\mathbf{B}$ were a multiplicative refinement of $\mathbf{A}$ in $\mathcal{U}$. By Corollary 3.3.8 and λ^+-saturation, we can find $\mathbf{Y} \in \mathbf{V}$ such that $\|\mathbf{Y} \in \mathbf{i}(\Delta)\|_{\mathbf{V}} = 1$, and for each $\alpha < \lambda$, $\|\mathbf{a}_\alpha \in \mathbf{Y}\|_{\mathbf{V}} = \mathbf{B}(\{\alpha\})$; then $\hat{Y} = [\mathbf{Y}]_{\mathcal{U}}$ contradicts our choice of X.

□

We can use patterns to measure the complexity of formulas.

Definition 3.11.11. Suppose Δ is a pattern on I, and $\varphi(\overline{x}, \overline{y})$ is a formula of T. Say that $\varphi(\overline{x}, \overline{y})$ admits Δ if Δ is a $(T, \varphi(\overline{x}, \overline{y}))$-pattern. (Really this definition should have T as a parameter; in ambiguous cases we may say "in T.") Say that T admits Δ if some $\varphi(\overline{x}, \overline{y})$ does.

The following corollary is now immediate.

Corollary 3.11.12. Suppose T_0, T_1 are countable complete theories, such that for all patterns Δ (on ω), if T_0 admits Δ, then T_1 admits Δ. Then $T_0 \trianglelefteq^{\times}_{\aleph_1} T_1$.

3.12 A Minimal Unstable Theory

Let T_{rg} be the theory of the random graph. Malliaris proved in [50] that T_{rg} is a $\trianglelefteq$-minimal unstable theory. We present her proof, with routine translations into the terminology of $\trianglelefteq^{\times}_1$. Malliaris and Shelah prove in [61] that T_{rg} is a $\trianglelefteq^*_1$-minimal unstable theory, although that proof has some additional complications that we avoid with the $\times$

version (one can recover their result using Corollary 3.18.7).

The following essentially describes an $(\omega, 2)$-array as in [49].

Definition 3.12.1. Let $\Delta(IP)$ be the pattern on $\omega \times 2$, defined to be the set of all $s \in [\omega \times 2]^{<\aleph_0}$ such that for all $n < \omega$, $\{(n,0),(n,1)\} \not\subseteq s$. (Think of $(n,0)$ as being $\neg(n,1)$.)

And the following is Claim 3.7 from [49].

Lemma 3.12.2. Suppose T is a complete theory and $\varphi(\overline{x}, \overline{y})$ is a formula of T. Define $\theta(\overline{x}, \overline{y}_0, \overline{y}_1) = \varphi(\overline{x}, \overline{y}_0) \wedge \neg\varphi(\overline{x}, \overline{y}_1)$. Then $\theta(\overline{x}, \overline{y}_0\overline{y}_1)$ admits $\Delta(IP)$ if and only if $\theta(\overline{x}, \overline{y}_0\overline{y}_1)$ has the independence property, which is the case if and only if $\varphi(\overline{x}, \overline{y})$ has the independence property.

Thus T has the independence property if and only if it admits $\Delta(IP)$.

Proof. Suppose $(\overline{a}_n : n < \omega)$ witnesses that $\varphi(\overline{x}, \overline{y})$ has the independence property. For each $n < \omega$, write $\overline{b}_{n,0} = (\overline{a}_{2n}, \overline{a}_{2n+1})$, and write $\overline{b}_{n,1} = (\overline{a}_{2n+1}, \overline{a}_{2n})$. Then $(\overline{b}_{n,i} : (n,i) \in \omega \times 2)$ witnesses that $\theta(\overline{x}, \overline{y}_0\overline{y}_1)$ admits $\Delta(IP)$.

Suppose $(\overline{b}_{n,i} : (n,i) \in \omega \times 2)$ witnesses that $\theta(\overline{x}, \overline{y}_0\overline{y}_1)$ admits $\Delta(IP)$; then $(\overline{b}_{n,0} : n < \omega)$ witnesses that $\theta(\overline{x}, \overline{y}_0\overline{y}_1)$ has the independence property.

Finally, suppose that $\theta(\overline{x}, \overline{y}_0\overline{y}_1)$ has the independence property. Then for every n, we can find some $A \subseteq \mathfrak{C}^{|\overline{y}|}$ with $|A| = 2n$, such that $|S_\varphi(A)| \geq 2^n$. This implies there is no polynomial bound on $|S_\varphi(A)|$ in terms of $|A|$; by Theorem II.4.10 of [75], this implies $\varphi(\overline{x}, \overline{y})$ has the independence property. $\square$

The following lemma is helpful in understanding the invariant $\lambda_{\hat{V}}(\Delta(IP))$. It is a translation of remarks in [50] into the context of $\trianglelefteq^{\times}_{\aleph_1}$, for instance see the discussion in Example 2 in Section 3.2.

Lemma 3.12.3. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$. Then $\lambda_{\hat{V}}(\Delta(IP))$ is the least λ such that for some $\hat{n} < \hat{\omega}$, there are disjoint $X_0, X_1 \subseteq \hat{n}$ each of size at most λ, such that there are no disjoint $\hat{X}_0, \hat{X}_1 \in \hat{V}$ with each $X_i \subseteq \hat{X}_i$.

Proof. Given $X \subseteq \hat{n} \times 2$ with $[X]^{<\aleph_0} \subseteq \mathbf{j}(\Delta(IP))$, define $X_i = \{\hat{m} < \hat{n} : (\hat{m}, i) \in X\}$; note that if there exists $\hat{X}_i : i < 2$ with each $X_i \subseteq \hat{X}_i$, then $X \subseteq \hat{X}_0 \times \{0\} \cup \hat{X}_1 \times \{1\} \in \mathbf{j}(\Delta(IP))$.

Conversely, given $X_0, X_1 \subseteq \hat{n}$ disjoint, define $X = X_0 \times \{0\} \cup X_1 \times \{1\}$. Note that if there exists $\hat{X} \in \mathbf{j}(\Delta(IP))$ with $X \subseteq \hat{X}$, then if we set $\hat{X}_i = \{\hat{n} : (\hat{n}, i) \in \hat{X}\}$ for each $i < 2$, then $\hat{X}_0, \hat{X}_1$ are disjoint and each $X_i \subseteq \hat{X}_i$. $\square$

Finally, we give a helpful characterization of $\lambda_{\mathcal{U}}(\Delta(IP))$ (although it will not be immediately used). For the following, note that we really only need $\mathbf{V} \restriction_{\emptyset}$, i.e. the reduct to the language of equality.

Lemma 3.12.4. Suppose $\mathcal{B}$ is a complete Boolean algebra and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Write $I = \lambda \times 2$. Then $\lambda_{\mathcal{U}}(\Delta(IP))$ is the least λ such that there is a I-distribution $\mathbf{A}$ in $\mathcal{U}$ of the following form, with no multiplicative refinement in $\mathcal{U}$. Namely, for some $\mathbf{V} \models^{\mathcal{B}} ZFC^-$ and for some $(\mathbf{a}_{\alpha,i} : (\alpha, i) \in I)$ from $\mathbf{V}$, we have that each $\mathbf{A}(s) = \bigwedge\{\|\mathbf{a}_{\alpha,0} \neq \mathbf{a}_{\beta,1}\|_{\mathbf{V}} : (\alpha, 0), (\beta, 1) \in s\}$.

Proof. Easily, any such $\mathbf{A}$ is a $(\lambda \times 2, \Delta(IP))$-distribution. Conversely, suppose $\mathbf{A}$ is a given $(\lambda, \Delta(IP))$-distribution in $\mathcal{U}$. Choose some transitive $V \models ZFC^-$, for instance $V = \mathrm{HC}$; choose $\mathbf{i} : V \preceq \mathbf{V}$ with $\mathbf{V}$ λ^+-saturated, and choose $(\mathbf{x}_\alpha : \alpha < \lambda)$ a pseudofinite sequence from $\mathbf{V}$ such that each $\|\mathbf{x}_\alpha \in \mathbf{i}(\omega \times 2)\|_{\mathbf{V}} = 1$, and for all $s \in [\lambda]^{<\aleph_0}$, $\|\{\mathbf{x}_\alpha : \alpha \in s\} \in \mathbf{i}(\Delta(IP))\|_{\mathbf{V}} = \mathbf{A}(s)$ (using Corollary 3.3.8).

For each $\alpha < \lambda$, choose $\mathbf{m}_\alpha$ such that $\|\mathbf{x}_\alpha \in \{\mathbf{m}_\alpha\} \times 2\|_{\mathbf{V}} = 1$, and choose $\mathbf{k}_\alpha$ such that $\|\mathbf{x}_\alpha \in \omega \times \{\mathbf{k}_\alpha\}\|_{\mathbf{V}} = 1$; this is possible by fullness of $\mathbf{V}$ (note $\mathbf{k}_\alpha$ is determined by the pair $(\|\mathbf{k}_\alpha = 0\|_{\mathbf{V}}, \|\mathbf{k}_\alpha = 1\|_{\mathbf{V}})$. For each $\alpha < \lambda$, there is a unique $f(\alpha) < 2$ such that $\|\mathbf{k}_\alpha = f(\alpha)\|_{\mathbf{V}} \in \mathcal{U}$. For each $s \in [\lambda]^{<\aleph_0}$, let $\mathbf{A}'(s) = \mathbf{A}(s) \wedge \bigwedge_{\alpha \in s} \|\mathbf{k}_\alpha = f(\alpha)\|_{\mathbf{V}}$; then this is a conservative refinement of $\mathbf{A}$ in $\mathcal{U}$. Thus $\mathbf{A}'$ has a multiplicative refinement in $\mathcal{U}$ if and only if $\mathbf{A}$ does.

For each $i < 2$, let $\{\mathbf{a}_{\alpha,i} : \alpha < \lambda\}$ list $\{\mathbf{m}_\beta : f(\beta) = i\}$, with repetitions if necessary. Let $\mathbf{A}''$ be the I-distribution defined from $(\mathbf{a}_{\alpha,i} : (\alpha, i) \in I)$ as in the statement of the lemma. Note that $\mathbf{A}''$ is in $\mathcal{U}$, since whenever $f(\beta) \neq f(\beta')$, we have $\|\mathbf{m}_\beta \neq \mathbf{m}_{\beta'}\|_{\mathbf{V}} \in \mathcal{U}$. Thus $\mathbf{A}''$ has a multiplicative refinement in $\mathcal{U}$, which easily gives a multiplicative refinement of $\mathbf{A}'$. $\square$

Putting it all together:

Theorem 3.12.5. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard, and λ is a cardinal. Then the following are equivalent:

(A) $\hat{V}$ λ^+-pseudosaturates T_{rg};

(B) $\hat{V}$ λ^+-pseudosaturates some unstable theory;

(C) $\lambda < \lambda_{\hat{V}}(\Delta(IP))$.

Proof. (A) implies (B) is trivial.

(B) implies (C): suppose $T \in V$ is unstable such that $\hat{V}$ λ^+-pseudosaturates T, i.e. $\lambda < \lambda_{\hat{V}}(T)$. Now T either has SOP or else IP; if T has SOP then in particular it has SOP_2, so $\lambda_{\hat{V}}(T) = \mathfrak{p}_{\hat{V}} \leq \lambda_{\hat{V}}(\Delta(IP))$. If on the other hand T has IP, then T admits $\Delta(IP)$ so we get $\lambda_{\hat{V}}(T) \leq \lambda_{\hat{V}}(\Delta(IP))$ in any case.

(C) implies (A): let $M \models T_{rg}$ with $M \in V$, and let $p(x)$ be a pseudofinite partial type over $\mathbf{j}_{\text{std}}(M)$ of cardinality at most λ; say $p(x)$ is over $\hat{n} < \hat{\omega}$. We can suppose

$p(x) \in S^1(A)$ is nonalgebraic for some pseudofinite A. Let $X_0 = \{a \in A : R(x, a) \in p(x)\}$ (defined in $\mathbb{V}$) and let $X_1 = \{a \in A : \neg R(x, a) \in p(x)\}$, and conclude by Lemma 3.12.3. $\square$

We immediately get the following corollaries. The first is Lemma 5.3 of [50] (stated there just for Keisler's order).

Corollary 3.12.6. T_{rg} is a $\trianglelefteq_1^{\times}$-minimal unstable theory. That is, if T is unstable then $T_{rg} \trianglelefteq_1^{\times} T$. Thus, this holds for $\trianglelefteq_{\aleph_1}^{\times}$ and $\trianglelefteq$ as well.

In particular, $\trianglelefteq_{\aleph_1}^{\times}$ and $\trianglelefteq$ both satisfy that the least class is the class of stable theories without the finite cover property, and the next-least class is the class of stable theories with the finite cover property, and the next-least class is the class of T_{rg}.

Corollary 3.12.7. Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$. Then the following are equivalent:

(A) $\mathcal{U}$ λ^+-saturates T_{rg};

(B) $\mathcal{U}$ λ^+-saturates some unstable theory;

(C) $\lambda < \lambda_{\mathcal{U}}(\Delta(IP))$.

It is a major open problem in the subject, see e.g. Problem (1) in the list of open problems in [61], to determine the Keisler class of the random graph model-theoretically. Examples of theories in this class are rather sparse; for instance, one can show n-ary random hypergraphs are equivalent to T_{rg}, but the following concrete question remains open. Let $ACFA$ be the theory of an algebraically closed field of with a generic automorphism. $ACFA$ is incomplete; one must specify the characteristic, and also the isomorphism type of the automorphism restricted to the algebraic closure of the emptyset.

Question. Suppose T is a completion of $ACFA$. Is T equivalent to the random graph in $\trianglelefteq_1^\times$, or at least in $\trianglelefteq$?

3.13 A Minimal Unsimple Theory

It is unclear where exactly the tree properties of the first and second kind were first given these names. Various sources claim they are defined in [79]; the notions do appear there, but are left unnamed. We give the definition of TP_2 as in [38]; TP_1 is equivalent to SOP_2 (as described in [38]).

Definition 3.13.1. The formula $\varphi(\overline{x}, \overline{y})$ has TP_2 (tree property of the second kind) if there are $(\overline{a}_{n,m} : n, m < \omega)$ such that for all $n < \omega$ and for all $m < m' < \omega$, $\exists \overline{x}(\varphi(\overline{x}, \overline{a}_{n,m}) \wedge \varphi(\overline{x}, \overline{a}_{n,m'}))$ is inconsistent, but such that for all $\eta \in \omega^\omega$, $\{\varphi(\overline{x}, \overline{a}_{n,\eta(n)})\}$ is consistent.

Then Theorem 0.2 of [79] states the following (although neither TP_2 nor SOP_2 are given names):

Theorem 3.13.2. If T is unsimple then either T has TP_2 or else T has SOP_2.

In [50], Malliaris proved the existence of a minimal TP_2 theory. In view of Theorem 3.13.2 and Theorem 3.9.12 (due to Malliaris and Shelah [54]), this must also be a minimal unsimple theory; in fact, Malliaris anticipated this would be the case in [50]. We follow her argument now, with the routine translations into the language of $\trianglelefteq_1^\times$. However, we prefer to use a more straightforward theory as our example:

Definition 3.13.3. Let $\mathcal{L}^*_{rg}$ be the language (U, V, f) where U, V are disjoint sorts and $f \subseteq U \times U \times V$ is a ternary relation symbol. Let T^*_{rg} be the model completion of the theory asserting that $f : [U]^2 \to V$. So T^*_{rg} is axiomatized by the following:

- The universe is the disjoint union of U and V, both infinite;

- f is the graph of a symmetric function from $U \times U \backslash \{(u,u) : u \in U\} \to V$, which we denote as a function $f : [U]^2 \to V$ (formally we may define $f(u,u) = u$, or a special garbage value);

- For every $u_0, \ldots, u_{n-1} \in U$, and for every $v_0, \ldots, v_{n-1} \in V$, there is some $u \in U$ such that $f(u, u_i) = v_i$ for each $i < n$.

We think of models of T^*_{rg} as random graphs, except instead of the graph being two-valued (either an edge is in or out), it is V-valued.

We now proceed as in the previous section. The following definition is equivalent to the notion of $(\omega, \omega, 1)$-arrays from [49].

Definition 3.13.4. Let $\Delta(TP_2)$ be the pattern on $\omega \times \omega$ consisting of all $s \in [\omega \times \omega]^{<\aleph_0}$ satisfying: for all $n < \omega$, $|s \cap \{n\} \times \omega| \leq 1$.

The following is then trivial. (This is equivalent to Claim 3.8 of [49], although our definition of TP_2 absorbs some of the work.)

Lemma 3.13.5. Suppose $\varphi(\overline{x}, \overline{y})$ is a formula of T. Then $\varphi(\overline{x}, \overline{y})$ has TP_2 if and only if $\varphi(\overline{x}, \overline{y})$ admits $\Delta(TP_2)$. Thus T has TP_2 if and only if T admits $\Delta(TP_2)$.

Example 3.13.6. T^*_{rg} admits $\Delta(TP_2)$, via the formula $f(x, y_0) = y_1$. Namely, let $M \models T^*_{rg}$; choose $(a_n : n < \omega)$ distinct elements from M^U, choose $(b_m : m < \omega)$ distinct elements from M^V, and for each $(n, m) \in \omega \times \omega$, let $c_{n,m} = (a_n, b_m)$. Then $(c_{n,m} : n, m < \omega)$ witnesses $f(x, y_0) = y_1$ admits $\Delta(TP_2)$.

In particular, T^*_{rg} is unsimple.

The following is essentially Theorem 6.9 of [50].

Lemma 3.13.7. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard, and λ is a cardinal. Then $\lambda_{\hat{V}}(\Delta(TP_2))$ is the least λ such that for some $\hat{n} < \hat{\omega}$, there is some $X \subseteq \hat{n}$ of cardinality at most λ (not necessarily in $\hat{V}$) and some $f : X \to \hat{n}$, such that there is no $\hat{f} : \hat{n} \to \hat{n}$ in $\hat{V}$ extending f.

Proof. Given $f \subseteq \hat{n} \times \hat{n}$ with $[f]^{<\aleph_0} \subseteq \mathbf{j}(\Delta(TP_2))$, define X to be the projection of f onto the first coordinate, and note that f is a function from X to $\hat{n}$. Any extension of f to $\hat{f} : \hat{n} \to \hat{n}$ is itself an element of $\mathbf{j}(\Delta(TP_2))$ with $f \subseteq \hat{f}$. Conversely, suppose $f : X \to \hat{n}$ is given with $|X| \leq \lambda$, and we have $\hat{g} \in \mathbf{j}(\Delta(TP_2))$ with $f \subseteq \hat{g}$. Note that $\hat{g}$ is a partial function from $\hat{\omega}$ to $\hat{\omega}$; define $\hat{f} : \hat{n} \to \hat{n}$ via $\hat{f}(\hat{m}) = \hat{g}(\hat{m})$ if $\hat{m} \in \mathrm{dom}(\hat{g})$, and $\hat{f}(\hat{m}) = 0$ else. $\square$

Lemma 6.8 of [50] is the direction (A) $\leq$ (C) of the following lemma.

Lemma 3.13.8. Suppose $\mathcal{B}$ is a complete Boolean algebra and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then the following cardinals are equal:

(A) $\lambda_{\mathcal{U}}(\Delta(TP_2))$;

(B) The least λ such that there are $\mathbf{V} \models^{\mathcal{B}} ZFC^-$ and $(\mathbf{a}_\alpha : \alpha < \lambda)$ from $\mathbf{V}$, such that there is no multiplicative λ-distribution $\mathbf{B}$ in $\mathcal{U}$ such that each $\mathbf{B}(\{\alpha, \beta\})$ decides $\|\mathbf{a}_\alpha = \mathbf{a}_\beta\|_{\mathbf{V}}$ (necessarily as dictated by $\mathcal{U}$);

(C) The least λ such that there are $\mathbf{V} \models^{\mathcal{B}} ZFC^-$ and $(\mathbf{a}_\alpha : \alpha < \lambda)$ from $\mathbf{V}$, such that $[\mathbf{a}_\alpha]_{\mathcal{U}} \neq [\mathbf{a}_\beta]_{\mathcal{U}}$ for all $\alpha \neq \beta$, and such that there is no multiplicative λ-distribution $\mathbf{B}$ in $\mathcal{U}$ with $\mathbf{B}(\{\alpha, \beta\}) \leq \|\mathbf{a}_\alpha \neq \mathbf{a}_\beta\|_{\mathbf{V}}$ for all $\alpha \neq \beta$.

In [51], Malliaris defines $\mathcal{U}$ to be λ^+-good for equality if λ is less than any or all of these values (under an equivalent formulation in terms of regular ultrapowers).

Proof. Let $\lambda_A, \lambda_B, \lambda_C$ be the cardinals defined in items (A), (B), (C).

$\lambda_C \leq \lambda_B$: suppose $\lambda < \lambda_C$, we show $\lambda < \lambda_B$. Let $\mathbf{V}, (\mathbf{a}_\alpha : \alpha < \lambda)$ be given. Let E be the equivalence relation on λ defined via: $E(\alpha, \beta)$ if $[\mathbf{a}_\alpha]_\mathcal{U} = [\mathbf{a}_\beta]_\mathcal{U}$ (i.e. $\|\mathbf{a}_\alpha = \mathbf{a}_\beta\|_\mathbf{V} \in \mathcal{U}$). Let $I \subseteq \lambda$ be a choice of representative for λ/E, i.e., such that each $\alpha < \lambda$ is E-related to exactly one $\beta \in I$. Let $f : \lambda \to I$ be the map witnessing this, so for all $\alpha < \lambda$ and for all $\beta \in I$, $[\mathbf{a}_\alpha]_\mathcal{U} = [\mathbf{a}_\beta]_\mathcal{U}$ if and only if $\beta = f(\alpha)$. Since $\lambda < \lambda_C$, we can find a multiplicative I-distribution $\mathbf{B}_0$ in $\mathcal{U}$ with each $\mathbf{B}_0(\alpha, \beta) \leq \|\mathbf{a}_\alpha \neq \mathbf{a}_\beta\|_\mathbf{V}$. Define $\mathbf{B}$, a λ-distribution in $\mathcal{U}$, via $\mathbf{B}(s) = \mathbf{B}_0(f[s]) \wedge \bigwedge_{\alpha \in s} \|\mathbf{a}_\alpha = \mathbf{a}_{f(\alpha)}\|_\mathbf{V}$. Then this clearly works to show $\lambda < \lambda_B$.

$\lambda_B \leq \lambda_A$: suppose $\lambda < \lambda_B$, we show $\lambda < \lambda_A$. Suppose $\mathbf{A}$ is a given $(\lambda, \Delta(TP_2))$-distribution in $\mathcal{U}$. Choose some transitive $V \models ZFC^-$, and let $\mathbf{i} : V \preceq \mathbf{V}$ with $\mathbf{V}$ λ^+-saturated, and choose $(\mathbf{x}_\alpha : \alpha < \lambda)$ a sequence from $\mathbf{V}$ so that each $\|\mathbf{x}_\alpha \in \mathbf{i}(\omega \times \omega)\|_\mathbf{V} = 1$, and such that for all $s \in [\lambda]^{<\aleph_0}$, $\|\{\mathbf{x}_\alpha : \alpha \in s\} \in \mathbf{i}(\Delta(TP_2))\|_\mathbf{V} = \mathbf{A}(s)$ (using Corollary 3.3.8).

For each $\alpha < \lambda$, choose $\mathbf{n}_\alpha, \mathbf{m}_\alpha$ such that $\|\mathbf{x}_\alpha = (\mathbf{n}_\alpha, \mathbf{m}_\alpha)\|_\mathbf{V} = 1$ (possible by fullness of $\mathbf{V}$). By two applications of $\lambda < \lambda_B$, we can find a multiplicative distribution $\mathbf{B}$ in $\mathcal{U}$ such that for all $\alpha < \beta < \lambda$, $\mathbf{B}(\{\alpha, \beta\})$ decides $\|\mathbf{n}_\alpha = \mathbf{n}_\beta\|_\mathbf{V}$ and decides $\|\mathbf{m}_\alpha = \mathbf{m}_\beta\|_\mathbf{V}$, from which it follows that $\mathbf{B}$ is a multiplicative refinement of $\mathbf{A}$.

$\lambda_A \leq \lambda_C$: suppose $\lambda < \lambda_A$, we show $\lambda < \lambda_C$. So suppose $\mathbf{V}, (\mathbf{a}_\alpha : \alpha < \lambda)$ are given. Define a λ-distribution $\mathbf{A}$ in $\mathcal{U}$ via $\mathbf{A}(s) = \bigwedge_{\alpha < \beta \in s} \|\mathbf{a}_\alpha \neq \mathbf{a}_\beta\|_\mathbf{V}$. Easily this is a $(\lambda, \Delta(TP_2))$-distribution and thus it has multiplicative refinement $\mathbf{B}$ in $\mathcal{U}$. Then $\mathbf{B}$ will be as desired. $\square$

Putting it all together:

Theorem 3.13.9. Suppose $V \models ZFC^-$ is transitive, and $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard, and λ is a cardinal. Then the following are equivalent:

(A) $\hat{V}$ λ^+-pseudosaturates T^*_{rg};

(B) $\hat{V}$ λ^+-pseudosaturates some unsimple theory;

(C) $\lambda < \lambda_{\hat{V}}(\Delta(TP_2))$.

Proof. (A) implies (B) is trivial.

(B) implies (C): suppose $T \in V$ is unsimple and $\hat{V}$ λ^+-pseudosaturates T, i.e. $\lambda < \lambda_{\hat{V}}(T)$. Now T either has SOP_2 or else TP_2; if T has SOP_2 then $\lambda_{\hat{V}}(T) = \mathfrak{p}_{\hat{V}} \leq \lambda_{\hat{V}}(\Delta(TP_2))$. If on the other hand T has TP_2, then T admits $\Delta(TP_2)$ so we get $\lambda_{\hat{V}}(T) \leq \lambda_{\hat{V}}(\Delta(TP_2))$ in any case.

(C) implies (A): let $M \models T^*_{rg}$ with $M \in V$ and let $p(x)$ be a pseudofinite partial type over $\mathbf{j}_{\mathrm{std}}(M)$ of cardinality less than λ; say $p(x) \in S^1(A)$, with A pseudofinite and $|A| \leq \lambda$. We can suppose $p(x)$ is nonalgebraic.

Note that if $V(x) \in p(x)$, then it is isolated by asserting $x \neq v$ for all $v \in V^M \cap A$. Thus, any $v \in V^{\mathbf{j}_{\mathrm{std}}(M)} \backslash A$ realizes $p(x)$.

So suppose instead $U(x) \in p(x)$. Write $A = A_0 \cup A_1$ where $A_0 = A \cap U^{\mathbf{j}_{\mathrm{std}}(M)}$ and $A_1 = A \cap V^{\mathbf{j}_{\mathrm{std}}(M)}$. After extending A and $p(x)$, we can suppose that for all $u \in A_0$, there is some $v \in A_1$ such that $f(x, u) = v \in p(x)$. Choose some pseudofinite $\hat{A} \in \hat{V}$ with $A \subseteq \hat{A}$. Let $g \subseteq A \times A$ be the set of all pairs (u, v) such that $p(x) \models f(x, u) = v$. Then g is indeed a partial function from $\hat{A}$ to $\hat{A}$, and hence can be extended to a total function $\hat{g} : \hat{A} \to \hat{A}$ with $\hat{g} \in \hat{V}$ (by Lemma 3.13.7 and applying a bijection between $\hat{A}$ and its cardinality in $\hat{V}$). Then we can find $u_* \in U^{\mathbf{j}_{\mathrm{std}}(M)}$ such that for all $u \in \hat{A} \cap U^{\mathbf{j}_{\mathrm{std}}(M)}$, if $\hat{g}(u) \in V^{\mathbf{j}_{\mathrm{std}}(M)}$, then $\mathbf{j}_{\mathrm{std}}(M) \models f(u_*, u) = \hat{g}(u)$. Then u_* clearly realizes $p(x)$. $\square$

We immediately get the following corollaries. The first is Lemma 5.3 of [50] (stated there just for Keisler's order).

Corollary 3.13.10. T_{rg}^* is a $\trianglelefteq_1^\times$-minimal unsimple theory. That is, if T is unsimple then $T_{rg}^* \trianglelefteq_1^\times T$. Thus, this holds for $\trianglelefteq_{\aleph_1}^\times$ and $\trianglelefteq$ as well.

Corollary 3.13.11. Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$. Then the following are equivalent:

(A) $\mathcal{U}$ λ^+-saturates T_{rg}^*;

(B) $\mathcal{U}$ λ^+-saturates some unsimple theory;

(C) $\lambda < \lambda_{\mathcal{U}}(\Delta(TP_2))$.

A similar proof shows that T_{feq}^* is also a minimal unsimple theory, although we view that example as unnatural. A better example is the following:

Definition 3.13.12. Let T_{rf} be the theory of the random binary function. That is, T_{rf} is the model completion of the empty theory in the language containing a single binary function symbol F.

T_{rf} is shown to be $NSOP_1$ in [43] (and by an easier proof, one can also show that T_{rg}^* is $NSOP_1$). In particular, T_{rf} is $NSOP_2$. Further, T_{rf} is TP_2 via the formula $f(x, y_0) = y_1$, exactly as for T_{rg}^*.

Theorem 3.13.13. T_{rf} is also a minimal unsimple theory in $\trianglelefteq_1^\times$.

Proof. Suppose $V \models ZFC^-$ is transitive (we just need to check the countable case), and $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard, and suppose $\lambda < \lambda_{\hat{V}}(\Delta(TP_2))$. It suffices to show $\hat{V}$ λ^+-pseudosaturates T_{rf}.

So let $F : \omega^2 \to \omega$ be such that $(\omega, F) \models T_{rf}$, and write $\hat{F} = \mathbf{j}(F)$ (we also use F to denote the symbol in the language). Let $p(x)$ be a pseudofinite partial type over $(\hat{\omega}, \hat{F})$, say $p(x)$ is over $X \subseteq \hat{n}$ with $|X| \leq \lambda$. Since $\lambda < \mu_{\hat{V}}$, we can find $Y \subseteq \hat{n}$ with $X \subseteq Y$ and $|Y \backslash X| = \lambda$. Extend $p(x)$ to a complete type $q(x)$ over Y such that for all $a \in Y$, there is $b \in Y$ with $q(x) \models F(x, a) = b$, and there is c with $q(x) \models F(a, x) = c$, and there is $c_* \in Y$ with $q(x) \models F(x, x) = c_*$. Thus $q(x)$ induces a function $f : Y \to Y$ via $f(a) =$ the unique $b \in Y$ with $q(x) \models F(x, a) = b$, and similarly a function $g : Y \to Y$ via $g(a) =$ the unique $c \in Y$ with $q(x) \models F(a, x) = c$.

We can suppose $q(x)$ is nonalgebraic; then note that $q(x)$ is isolated by the formulas $\{x \neq a : a \in Y\}$,$\{F(x, a) = f(a) : a \in Y\}$, $\{F(a, x) = g(a) : a \in Y\}$, and $\{F(x, x) = c_*\}$. Since $\lambda < \lambda_{\hat{V}}(\Delta(TP_2))$, we can find some function $\hat{f} : \hat{n} \to \hat{n}$ extending f and some function $\hat{g} : \hat{n} \to \hat{n}$ extending g. Thus we can find $a_* \geq \hat{n}$ such that $\hat{F}(a_*, a) = \hat{f}(a)$ and $\hat{F}(a, a_*) = \hat{g}(a)$ for all $a < \hat{n}$, and such that $\hat{F}(a_*, a_*) = c_*$; then a_* realizes $q(x)$ and hence $p(x)$. $\square$

3.14 A Minimal Nonlow Theory

In this section, we proceed similarly to Sections 3.12 and 3.13 to show that there is a minimal nonlow theory in Keisler's order. I first proved this result in [87], although we will be translating the results into the terminology of $\trianglelefteq_1^\times$.

First, we define what we mean by low:

Definition 3.14.1. The complete countable theory T is low if it is simple and for every formula $\varphi(\overline{x}, \overline{y})$, there is some k such that for all $\overline{b}$, if $\varphi(\overline{x}, \overline{b})$ does not k-divide over $\emptyset$ then it does not divide over $\emptyset$.

This is the standard definition of low, for instance it is equivalent to the definition in [5], where the concept of lowness is introduced. Malliaris defined low slightly differently starting in [52], namely not requiring T to be simple (this definition is then also used in later papers by Malliaris and Shelah, for instance in [56]). To clarify:

Definition 3.14.2. T has the *finite dividing property* (FDP) if there is some formula $\varphi(\overline{x}, \overline{y})$ such that for every k there is some indiscernible sequence $(\overline{b}_n : n < \omega)$ over the emptyset such that $\{\varphi(\overline{x}, \overline{b}_n) : n < \omega\}$ is k-consistent but not consistent.

What Malliaris calls low is what we call not having the finite dividing property; and we say that a theory T is low if it is simple and does not have the finite dividing property.

Note that one can easily check that $(\mathbb{Q}, <)$ does not have the finite dividing property, and so the finite dividing property is not a dividing line in Keisler's order. Lowness, on the other hand, is [87].

T_{Cas} was introduced by Casanovas [6] and was in fact the first example of a simple non-low theory. The language $\mathcal{L}_{Cas}$ is $(R, P, Q, Q_n : 1 \leq n < \omega)$, where P, Q, Q_n are each unary relation symbols and R is binary. We adopt the convention that $a, a', \ldots$ are elements of P, $b, b', \ldots$ are elements of Q.

1. The universe is the disjoint union of P and Q, both infinite;
2. Each $Q_n \subseteq Q$, and the Q_n's are infinite and disjoint;
3. $R \subseteq P \times Q$;
4. For each $a \in P$ and for each $n < \omega$, there are exactly n elements $b \in Q_n$ such that $R(a, b)$;
5. Whenever B_0, B_1 are finite disjoint subsets of Q such that each $|B_1 \cap Q_n| \leq n$, there is $a \in P$ such that $R(a, b)$ for all $b \in B_1$ and $\neg R(a, b)$ for all $b \in B_0$.
6. For all A_0, A_1 finite disjoint subsets of P, there is $b \in Q$ such that $R(a, b)$ for all $a \in A_1$ and $\neg R(a, b)$ for all $a \in A_0$.

Actually, if in the definition of $\mathcal{L}_{Cas}$ we allow $n = 0$ then Q_0 will be completely harmless, so for notational convenience we let $\mathcal{L}_{Cas}$ be $(R, P, Q, Q_n : n < \omega)$.

In [6] it is shown that T_{Cas} is complete, and is the model completion of the theory axiomatized by the first four items above. In particular, it is shown that T_{Cas} has quantifier elimination in an expanded language, where we add predicates $S_{\dots}$ that express, given $A_0, A_1 \subset P$ finite and disjoint with $A_1 \neq \emptyset$, how many $b \in Q_n$ are there such that $R(a, b)$ for all $a \in A_1$ and $\neg R(a, b)$ for all $a \in A_0$. It follows that the algebraic closure of a set X is $X \cup \bigcup\{b \in \bigcup_n Q_n :$ there is $a \in X \cap P$ such that $R(a, b)\}$, and every formula over a set X is equivalent to a quantifier-free formula over $\operatorname{acl}(X)$.

Casanovas also shows that T_{Cas} is simple with the following forking relation: $X \mathop{\smile\!\!\!\!\!\!\mid}_Z Y$ if and only if $\operatorname{acl}(X) \cap \operatorname{acl}(Y) \subseteq \operatorname{acl}(Z)$. Finally, the formula $R(x, y)$ witnesses that T_{Cas} is not low.

The following lemma is immediate from the quantifier elimination in the expanded language discussed above:

Lemma 3.14.3. Let $M \models T_{Cas}$ and let $C \subseteq M$ be algebraically closed. Write $C = A_* \cup B_*$ where $A_* = C \cap P^M$ and $B_* = C \cap Q^M$. As notation let Q_ω denote $Q \setminus \bigcup_n Q_n$.

(I) For each $n < \omega$, there is a unique nonalgebraic type $p(x)$ over C with $Q_n(x) \in p(x)$. It is isolated by the formulas $Q_n(x)$ together with $\neg R(a, x)$ for each $a \in A_*$.

(II) For each $A \subseteq A_*$ let $p_A(x)$ be the partial type over C that says $Q_\omega(x)$ holds, $x \neq b$ for each $b \in B_*$, and finally for each $a \in A_*$, $R(a, x)$ holds if and only if $a \in A$. Then $p_A(x)$ generates a complete type over C that does not fork over $\emptyset$. Moreover, all nonalgebraic complete types over M extending $\{Q(x)\} \cup \bigcup_n \{\neg Q_n(x)\}$ are of this form.

(III) Suppose $B \subseteq B_*$ is such that each $|B \cap Q_n^M| \leq n$. Let $p_B(x)$ be the type over C_* that says $P(x)$ holds, and $x \neq a$ for each $a \in A_*$, and for each $b \in B_*$, $R(x, b)$ holds if and only if $b \in B$. Then $p_B(x)$ generates a complete type over C, and moreover every complete nonalgebraic type over C extending $P(x)$ is of this form. Further, given $C_0 \subseteq C$, we have that $p(x)$ does not fork over C_0 if and only if for each $n < \omega$, $B \cap Q_n^M \cap C_0 = B \cap Q_n^M$.

We now introduce the relevant patterns.

Definition 3.14.4. Given $I \subseteq \omega \backslash \{0\}$ infinite, let $\Delta_I(FDP)$ be the pattern on $I \times \omega$ defined by: $s \in \Delta(FDP)$ if $s \in \{m\} \times [\omega]^{\leq m}$ for some $m \in I$. Let $\Delta_I^*(FDP)$ be the pattern on $I \times \omega$, defined to be all s with each $|s \cap \{m\} \times \omega| \leq m$.

Write $\Delta(FDP) = \Delta_{\omega \backslash \{0\}}(FDP)$.

The following is then straightforward. On the other hand, one would like to know which Δ we actually need to check.

Lemma 3.14.5. Suppose $\varphi(\overline{x}, \overline{y})$ is a formula of T. Then $\varphi(\overline{x}, \overline{y})$ has the finite dividing property if and only if for some infinite $I \subseteq \omega \backslash \{0\}$, $\varphi(\overline{x}, \overline{y})$ admits some Δ with $\Delta_I(FDP) \subseteq \Delta \subseteq \Delta_I^*(FDP)$. Hence T has the finite dividing property if and only if T admits some such Δ.

Proof. Suppose $\varphi(\overline{x}, \overline{y})$ admits some such Δ, via $(\overline{a}_{m,n} : (m, n) \in I \times \omega)$. Then by compactness and Ramsey's theorem, for each $m \in I$ we get an indiscernible sequence $(\overline{b}_{m,n} : n < \omega)$ such that $\{\varphi(\overline{x}, \overline{b}_{m,n}) : n < \omega\}$ is m-consistent but $m+1$-inconsistent. Hence $\varphi(\overline{x}, \overline{y})$ has the finite dividing property.

Conversely, suppose $\varphi(\overline{x}, \overline{y})$ has the finite dividing property; choose $I \subseteq \omega$ infinite, and indiscernible sequences $((\overline{b}_n^m : n < \omega) : m \in I)$ witnessing this, so each $\{\varphi(\overline{x}, \overline{b}_n^m) :$

$m \in I\}$ is m-consistent but $m+1$-inconsistent. Let Δ be the set of all $s \in [\omega \times \omega]^{<\aleph_0}$ such that $\{\varphi(\overline{x}, \overline{b}^m_n) : (m,n) \in s\}$ is consistent. Clearly, $\Delta_I(FDP) \subseteq \Delta \subseteq \Delta^*_I(FDP)$. $\square$

We now wish to compare the various $\lambda_{\hat{V}}(\Delta_I(FDP))$, $\lambda_{\hat{V}}(\Delta^*_I(FDP))$. The following holds unconditionally:

Lemma 3.14.6. Suppose V is a transitive model of ZFC^-, and $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard. Suppose $I \subseteq \omega$ is infinite, and $\Delta_I(FDP) \subseteq \Delta \subseteq \Delta^*_I(FDP)$ satisfies $\Delta \in V$. If $\lambda < \lambda_{\hat{V}}(\Delta)$, then for every $\hat{m}_* < \hat{n}_* < \hat{\omega}$ with $\hat{m}_*$ nonstandard, and for every $X \subseteq \hat{n}_*$ of cardinality at most λ, there is $\hat{X} \in [\hat{n}_*]^{\leq \hat{m}_*}$ in $\hat{V}$ with $X \subseteq \hat{X}$. If $\Delta = \Delta_I(FDP)$, then the converse is true as well.

Proof. Suppose first $\lambda < \lambda_{\hat{V}}(\Delta)$, and $\hat{m}_* < \hat{n}_*$, X are as above. By decreasing $\hat{m}_*$, we can suppose $\hat{m}_* \in \mathbf{j}(I)$ while keeping $\hat{m}_*$ nonstandard. Write $Y = \{(\hat{m}_*, \hat{n}) : \hat{n} \in X\}$. Since $\hat{m}_*$ is nonstandard we have $[Y]^{<\aleph_0} \subseteq \mathbf{j}(\Delta)$, thus we can find $\hat{Y} \in \mathbf{j}(\Delta)$ with $Y \subseteq \hat{Y}$. Let $\hat{X} = \{\hat{n} < \hat{n}_* : (\hat{m}_*, \hat{n}) \in \hat{Y}\}$; then $\hat{X} \in [\hat{n}_*]^{\leq \hat{m}_*}$ with $X \subseteq \hat{X}$.

Next, suppose $\Delta = \Delta_I(FDP)$ let $Y \subseteq \hat{n}_* \times \hat{n}_*$ be of size at most λ with $[Y]^{<\aleph_0} \subseteq \mathbf{j}(\Delta_I(FDP))$. Then $Y \subseteq \{\hat{m}_*\} \times \hat{n}_*$ for some $\hat{m}_* < \hat{n}_*$ with $\hat{m}_* \in \mathbf{j}(I)$, so let $X = \{\hat{n} < \hat{n}_* : (\hat{m}_*, \hat{n}) \in Y\}$. By hypothesis we can find $\hat{X} \supseteq X$ with $\hat{X} \in [\hat{n}_*]^{\leq \hat{m}_*}$. Then $\hat{Y} = \{\hat{m}_*\} \times \hat{X}$ is as desired. $\square$

Remark 3.14.7. It follows that for all $\Delta_I(FDP) \subseteq \Delta \subseteq \Delta^*_I(FDP)$, $\lambda_{\hat{V}}(\Delta) \leq \lambda_{\hat{V}}(\Delta_I(FDP))$, and $\lambda_{\hat{V}}(\Delta_I(FDP)) = \lambda_{\hat{V}}(\Delta(FDP))$. Also, we easily get that $\lambda_{\hat{V}}(\Delta(TP_2)) \leq \lambda_{\hat{V}}(\Delta(FDP)) \leq \mu_{\hat{V}}$.

Under the additional assumption that $\lambda < \mathrm{lcf}_{\hat{V}}(\omega)$, we get more. We use Theorem 3.17.1, which is proved later, in Section 3.17. There is no circularity, in fact the proof of Theorem 3.17.1 can be read now.

Lemma 3.14.8. Suppose V is a transitive model of ZFC^-, and $\mathbf{j} : V \preceq \hat{V}$ is ω-nonstandard. Suppose $\lambda < \mathrm{lcf}_{\hat{V}}(\omega)$. Then for every infinite $I \subseteq \omega$, $\lambda < \lambda_{\hat{V}}(\Delta(FDP))$ if and only if $\lambda < \lambda_{\hat{V}}(\Delta^*_I(FDP))$.

Proof. Suppose $\lambda < \lambda_{\hat{V}}(\Delta(FDP))$; it suffices to show $\lambda < \lambda_{\hat{V}}(\Delta^*_I(FDP))$. Suppose $X \subseteq \hat{n}_* \times \hat{n}_*$ has size at most λ, and $[X]^{<\aleph_0} \subseteq \mathbf{j}(\Delta^*_I(FDP))$. For each $\hat{m} \in \mathbf{j}(I) \cap \hat{n}_*$, let $X_{\hat{m}} = \{\hat{n} < \hat{n}_* : (\hat{m}, \hat{n}) \in X\}$. Our hypothesis on X just says that for each $m \in I$, $|X_m| \leq m$.

Let $Y = X \backslash \bigcup_{m \in I} X_m$. First we handle Y. Define $Y_{\hat{m}} = \{\hat{n} < \hat{n}_* : (\hat{m}, \hat{n}) \in Y\}$ for each $\hat{m} < \hat{n}_*$; so $Y_{\hat{m}} = X_{\hat{m}}$ for $\hat{m} \in \mathbf{j}(I)$ nonstandard, and for all other $\hat{m}$, $Y_{\hat{m}} = \emptyset$.

Let $Z = \{\hat{m} < \hat{n}_* : Y_{\hat{m}} \neq \emptyset\}$, so $Z \subseteq \mathbf{j}(I) \backslash \omega$. Since $\lambda < \mathrm{lcf}_{\hat{V}}(\omega)$, Z is not cofinal above ω, so we can find $\hat{m}_*$ nonstandard with $\hat{m} \geq \hat{m}_*$ for each $\hat{m} \in Z$. By decreasing $\hat{m}_*$, we can suppose $\hat{m}_* \in \mathbf{j}(I)$, while keeping $\hat{m}_*$ nonstandard. We can choose $\hat{Y} \in [\hat{n}_* \times \hat{n}_*]^{\leq \hat{m}_*}$ with $Y \subseteq \hat{Y}$, by hypothesis and Lemma 3.14.6 (and using a pairing function $\hat{\omega} \times \hat{\omega} \to \hat{\omega}$). We can suppose $\hat{Y} \cap \hat{m}_* \times \hat{n}_* = \emptyset$, and so $\hat{Y} \in \mathbf{j}(\Delta^*_I(FDP))$.

So we have found $\hat{Y} \in \mathbf{j}(\Delta^*_I(FDP))$ with $Y \subseteq \hat{Y}$. We need to find some $\hat{X} \in \mathbf{j}(\Delta^*_I(FDP))$ with $X \subseteq \hat{X}$.

Note that since $\lambda < \mathrm{lcf}_{\hat{V}}(\omega)$, we in particular have that $\mathrm{lcf}_{\hat{V}}(\omega) \geq \aleph_1$. By Theorem 3.17.1, $(\aleph_0, \aleph_0) \notin \mathcal{C}_{\hat{V}}$; thus $\mathfrak{p}_{\hat{V}} \geq \aleph_1$. Thus, by Theorem 3.9.7, countable pseudofinite partial types over $\hat{V}$ are realized, so we can find $\hat{X} \in \mathbf{j}(\Delta^*_I(FDP))$ with $\hat{X} \cap [\hat{m}_*, \hat{n}_*) \times \hat{n}_* = \hat{Y} \cap [\hat{m}_*, \hat{n}_*) \times \hat{n}_*$, and with $X_m \subseteq \hat{X}$ for all $m \in I$. Then $\hat{X}$ is as desired. $\square$

The corresponding invariants for ultrafilters have been studied under various guises. λ-OK was first defined by Kunen [46], and λ-flexibility was first defined by Malliaris in [52]. Previously these definitions were made only in the case of $\mathcal{B} = \mathcal{P}(\lambda)$.

Definition 3.14.9.

- The ultrafilter $\mathcal{U}$ on the complete Boolean algebra $\mathcal{B}$ is λ-OK if whenever $\mathbf{A}$ is a λ-distribution in $\mathcal{U}$ such that for all $s, t \in [\lambda]^n$, $\mathbf{A}(s) = \mathbf{A}(t)$, then $\mathbf{A}$ has a multiplicative refinement in $\mathcal{U}$. (Note in this case that $\mathbf{A}$ is determined by the descending sequence $(\mathbf{A}(n) : n < \omega)$.)

- The ultrafilter $\mathcal{U}$ on $\mathcal{B}$ is λ-flexible if $\mathcal{U}$ is $\aleph_1$-incomplete and, for every $\mathcal{U}$-nonstandard $\mathbf{n} \in (\omega, <)^{\mathcal{B}}/\mathcal{U}$, there is some multiplicative λ-distribution $\mathbf{B}$ in $\mathcal{U}$, such that for all $s \in [\lambda]^n$, $\mathbf{B}(s) \leq \|\mathbf{n} \geq n\|_{(\omega,<)^{\mathcal{B}}}$.

So obviously, if $\mathcal{U}$ is $\aleph_1$-complete, then $\mathcal{U}$ is λ-OK for all λ.

We first remark that this definition of λ-flexibility coincides with the one given by Malliaris. In the case $\mathcal{B} = \mathcal{P}(\lambda)$ (which is the only case in which λ-flexibility was previously defined), strong λ-regularity coincides with λ-regularity, which is easier to verify. To get strongness, we adapt Mansfield's argument in Theorem 4.1 of [62].

Theorem 3.14.10. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, and $[\mathbf{n}]_{\mathcal{U}} \in (\omega, <)^{\mathcal{B}}/\mathcal{U}$ is nonstandard, and $\mathbf{B}$ is a multiplicative λ-distribution in $\mathcal{U}$. Then the following are equivalent:

(A) For every $s \in [\lambda]^n$, $\mathbf{B}(s) \leq \|\mathbf{n} \geq n\|_{(\omega,<)^{\mathcal{B}}}$.

(B) $\mathbf{B}$ is strongly λ-regular, and for every $\mathbf{c} \in \mathbf{B}$, if $\mathbf{c}$ decides $\mathbf{B}(s)$ for each $s \in [\lambda]^{<\aleph_0}$ (or equivalently, $\mathbf{c}$ decides $\mathbf{B}(\{\alpha\})$ for each $\alpha < \lambda$) and if we write $n := |\{\alpha < \lambda : \mathbf{c} \leq \mathbf{B}(\{\alpha\})|$, then $\mathbf{c} \leq \|\mathbf{n} \geq m\|_{(\omega,<)^{\mathcal{B}}}$.

In particular, if $\mathcal{U}$ is λ-flexible, then $\mathcal{U}$ is strongly λ-regular.

Proof. (A) implies (B): recall that $\mathbf{B}$ is strongly λ-regular if and only if $(\mathbf{B}(\{\alpha\}) : \alpha < \lambda)$ is a strongly λ-regular sequence, and this implies $(\mathbf{B}(s) : s \in [\lambda]^{<\aleph_0})$ is a strongly $[\lambda]^{<\aleph_0}$-regular sequence (this holds for all distributions, although for multiplicative distributions it is trivial). We show that for every $\mathbf{c} \in \mathcal{B}$ nonzero, there is $\mathbf{c}' \leq \mathbf{c}$ nonzero such that $\mathbf{c}'$ decides each $\mathbf{B}(\{\alpha\})$, and there is $m_* < \omega$ such that $\mathbf{c}' \leq \|\mathbf{n} = m_*\|_{(\omega,<)^\mathcal{B}}$ and $|\{\alpha < \lambda : \mathbf{c} \leq \mathbf{B}(\{\alpha\})\}| \leq m_*$. This clearly suffices.

So let $\mathbf{c} \in \mathcal{B}$ be nonzero. Choose $\mathbf{c}_0 < \mathbf{c}$ nonzero, such that there is some $m_* < \omega$ with $\mathbf{c}_0 \leq \|\mathbf{m} = m_*\|_{(\omega,<)^\mathcal{B}}$. Try to find, by induction on $m \leq m_* + 1$, a descending sequence $(\mathbf{c}_m : m \leq m_* + 1)$ such that for each m, $|\{\alpha < \lambda : \mathbf{c}_m \leq \mathbf{B}(\{\alpha\})\}| \geq m$. There must be some stage $m < m_* + 1$ at which we cannot continue, since for any $s \in [\lambda]^{m_*+1}$, $\mathbf{c}_0 \wedge \mathbf{B}(s) = 0$ (since $\mathbf{B}(s) \leq \|\mathbf{n} > m_*\|_{(\omega,<)^\mathcal{B}}$). So we get some $m < m_* + 1$ such that if we set $s = \{\alpha < \lambda : \mathbf{c}_m \leq \mathbf{B}(\{\alpha\})\}$, then for all $\alpha \notin s$, $\mathbf{c}_m \leq \neg\mathbf{B}(\{\alpha\})$. So $\mathbf{c}_m$ is desired.

(B) implies (A): given $s \in [\lambda]^n$, choose $\mathbf{c} \leq \mathbf{B}(s)$ nonzero such that $\mathbf{c}$ decides each $\mathbf{B}(\{\alpha\})$. Necessarily then $\mathbf{c} \leq \|\mathbf{n} \geq n\|_{(\omega,<)^\mathcal{B}}$. □

We have the following theorem connecting all of these notions. It is a translation of Observation 9.9 of [55] into our terminology.

Theorem 3.14.11. Suppose $\mathcal{U}$ is an ultrafilter on the complete Boolean algebra $\mathcal{B}$. Then $\lambda_\mathcal{U}(\Delta(FDP))$ is the least λ such that $\mathcal{U}$ is not λ-OK. Additionally, if $\mathcal{U}$ is $\aleph_1$-incomplete, then this is the least λ such that $\mathcal{U}$ is not λ-flexible.

Proof. Choose some transitive $V \models ZFC^-$, and some $\mathbf{i} : V \preceq \mathbf{V}$ with $\mathbf{V}$ λ^+-saturated. Write $\hat{V} = \mathbf{V}/\mathcal{U}$ and let $\mathbf{j} : V \preceq \hat{V}$ be the usual embedding.

Suppose first that $\lambda < \lambda_\mathcal{U}(\Delta(FDP))$, and $\mathbf{A}$ is a λ-distribution with $\mathbf{A}(s) = \mathbf{A}(t)$ for all $|s| = |t|$. Then $\mathbf{A}$ is a $(\lambda, \Delta(FDP))$-distribution, so by hypothesis $\mathbf{A}$ has a multplicative refinement in $\mathcal{U}$; thus $\mathcal{U}$ is λ-OK. Conversely, suppose $\mathcal{U}$ is λ-OK; we show $\lambda < \lambda_{\hat{V}}(\Delta(FDP))$. Indeed, suppose $\mathbf{m}_*, \mathbf{n}_* \in \mathbf{i}(\omega)$ and $\{\mathbf{n}_\alpha : \alpha < \lambda\} \subseteq \mathbf{i}(\omega)$ are given with $[\mathbf{m}_*]_\mathcal{U}$ nonstandard and $[\mathbf{m}_*]_\mathcal{U} < [\mathbf{n}_*]_\mathcal{U}$, and each $[\mathbf{n}_\alpha]_\mathcal{U} < [\mathbf{n}_*]_\mathcal{U}$. We can suppose each

$\|\mathbf{m}_* < \mathbf{n}_*\|_\mathbf{V}, \|\mathbf{n}_\alpha < \mathbf{n}_*\|_\mathbf{V} = 1$. Define a λ-distribution $\mathbf{A}$ in $\mathcal{U}$, via $\mathbf{A}(s) = \|\mathbf{m} \geq n\|_\mathbf{V}$ for each $s \in [\lambda]^n$. By λ-OK-ness we can find a multiplicative refinement $\mathbf{B}$ of $\mathbf{A}$ in $\mathcal{U}$. By λ^+-saturation of $\mathbf{V}$, we can find $\mathbf{X} \in [\mathbf{n}_*]^{\leq \mathbf{m}_*}$ (i.e. $\mathbf{X} \in \mathbf{V}$ and $\|\mathbf{X} \in [\mathbf{n}_*]^{\leq \mathbf{m}_*}\|_\mathbf{V} = 1$) so that for all $\alpha < \lambda$, $\|\mathbf{n}_\alpha \in \mathbf{X}\|_\mathbf{V} = \mathbf{B}(\{\alpha\})$. Then $\hat{X} := [\mathbf{X}]_\mathcal{U}$ is as desired.

Suppose next that $\mathcal{U}$ is λ-flexible. Suppose $\mathbf{A}$ is a distribution in $\mathcal{U}$ such that for all $s, t \in [\lambda]^n$, $\mathbf{A}(s) = \mathbf{A}(t)$. If $\bigwedge_n \mathbf{A}(n) = \mathbf{a} \in \mathcal{U}$ then obviously the constant distribution with value $\mathbf{a}$ is a refinement in $\mathcal{U}$. Otherwise, we can suppose $\bigwedge_n \mathbf{A}(n) = 0$ (by intersecting each $\mathbf{A}(s)$ with $\neg\mathbf{a}$). Define $\mathbf{m} \in (\omega, <)^\mathcal{B}$ via $\mathbf{m}(n) = \mathbf{A}(n) \wedge \neg\mathbf{A}(n+1)$. Then since $\bigwedge_n \mathbf{A}(n) = 0$ and since $\mathbf{A}(0) = \mathbf{A}(\emptyset) = 1$ (by definition of distribution), we really have $\mathbf{m} \in (\omega, <)^\mathcal{B}$. $\mathbf{m}$ is $\mathcal{U}$-nonstandard since each $\|\mathbf{m} \geq n\|_{(\omega,<)^\mathcal{B}} = \mathbf{A}(n) \in \mathcal{U}$. Thus we can find a multiplicative distribution $\mathbf{B}$ in $\mathcal{U}$, such that for all $s \in [\lambda]^n$, $\mathbf{B}(s) \leq \|\mathbf{m} \geq n\| = \mathbf{A}(n)$.

Finally, suppose $\mathcal{U}$ is λ-OK and $\aleph_1$-incomplete, and let $[\mathbf{m}]_\mathcal{U}$ be a $\mathcal{U}$-nonstandard element of $(\omega, <)^\mathcal{B}/\mathcal{U}$. Define $\mathbf{A}(s) = \|\mathbf{m} \geq n\|_{(\omega,<)^\mathcal{B}}$ for each $s \in [\lambda]^n$ and let $\mathbf{B}$ be a multiplicative refinement of $\mathbf{A}$ in $\mathcal{U}$. $\square$

Remark 3.14.12. It follows then that if $\mathcal{U}$ is λ-OK, then $\mu_\mathcal{U} > \lambda$, and so $\mathcal{U}$ λ^+-saturates every stable theory. In the case $\mathcal{B} = \mathcal{P}(\lambda)$ this was proved similarly by Malliaris and Shelah in [60]. We see then that if $\mathcal{U}$ is $\aleph_1$-complete, then $\mathcal{U}$ λ^+-saturates every stable theory, for every λ.

We can now wrap up the proof that T_{Cas} is a minimal nonlow theory. (B) implies (C) is due to Malliaris [52] in the context of regular ultrafilters on $\mathcal{P}(\lambda)$; (C) implies (A) is from [87].

Theorem 3.14.13. Suppose $V \models ZFC^-$ is transitive, $\mathbf{j} : V \preceq \hat{V}$ with $\hat{V}$ ω-nonstandard, and suppose λ is given. Then the following are equivalent:

(A) $\hat{V}$ λ^+-pseudosaturates T_{Cas}.

(B) $\hat{V}$ λ^+-pseudosaturates some nonlow theory.

(C) $\lambda < \lambda_{\hat{V}}(\Delta(IP))$ and $\lambda < \lambda_{\hat{V}}(\Delta(FDP))$.

Proof. (A) implies (B) is trivial.

(B) implies (C): suppose $T \in V$ is nonlow; (B) is equivalent to $\lambda < \lambda_T(\hat{V})$. Now T is unstable, so $\lambda_{\hat{V}}(T) \leq \lambda_{\hat{V}}(\Delta(IP))$. If T is unsimple, then $\lambda_{\hat{V}}(T) \leq \lambda_{\hat{V}}(\Delta(TP_2)) \leq \lambda_{\hat{V}}(\Delta(FDP))$, and if T has the finite dividing property then $\lambda_{\hat{V}}(T) \leq \lambda_{\hat{V}}(\Delta(FDP))$ by Lemma 3.14.6. Hence $\lambda_{\hat{V}}(T) \leq \lambda_{\hat{V}}(\Delta(FDP))$ in any case, and (C) holds.

(C) implies (A): suppose $\lambda < \lambda_{\hat{V}}(\Delta(IP))$ and $\lambda < \lambda_{\hat{V}}(\Delta(FDP))$, and let $M \models T_{Cas}$ have universe ω (say), with $M \in V$. Write $M = (\omega, R, P, Q, Q_n : n < \omega)$ and write $\mathbf{j}(M) = (\hat{\omega}, \hat{R}, \hat{P}, \hat{Q}, \hat{Q}_{\hat{n}} : \hat{n} < \hat{\omega})$ (so $\mathbf{j}_{\mathrm{std}}(M) = (\hat{\omega}, \hat{R}, \hat{P}, \hat{Q}, \hat{Q}_n : n < \omega)$). We show that $\mathbf{j}_{\mathrm{std}}(M)$ is λ^+-pseudosaturated.

So let $p(x)$ be a pseudofinite partial type over $\mathbf{j}_{\mathrm{std}}(M)$ of cardinality at most λ. I first of all claim that we can suppose $p(x)$ is a type over an algebraically closed set (this is a general fact, in fact we could arrange over a model). Indeed, choose $\hat{n}_0 < \hat{\omega}$ such that $p(x)$ is over $\hat{n}_0$; we can suppose $\hat{n}_0$ is nonstandard. In $\hat{V}$, choose $(\hat{n}_{\hat{m}} : \hat{m} \leq \hat{n}_0)$ such that for all $\hat{m} < \hat{n}_0$ we have: for all $\hat{n} < \hat{n}_0$ and for all $\hat{a} < \hat{n}_{\hat{m}}$ with $\hat{a} \in \hat{P}$, $\{\hat{b} \in \hat{Q}_{\hat{n}} : \hat{R}(\hat{a}, \hat{b})\} \subseteq \hat{n}_{\hat{m+1}}$. Then clearly $\hat{n}_* := \hat{n}_{\hat{n}_0}$ is algebraically closed (in $\mathbf{j}_{\mathrm{std}}(M)$). Thus we can find some algebraically closed set $C \subseteq \hat{n}_*$ of cardinality at most λ, and suppose that $p(x)$ is a complete type over C.

We can suppose $p(x)$ is nonalgebraic. Then $p(x)$ must be in one of the forms from Lemma 3.14.3. Write $A_* = C \cap P^M$ and $B_* = C \cap Q^M$.

If $p(x)$ is of form (I), then clearly any $\hat{b} \in \hat{Q}_n$ with $\hat{b} \geq \hat{n}_*$ realizes $p(x)$; and if $p(x)$

is of form (II), use that $\lambda < \lambda_{\hat{V}}(\Delta(IP))$.

So we can suppose $p(x)$ is of form (III), say $p(x) = p_B(x)$ for some $B \subseteq B_*$ with each $|B \cap \hat{Q}_n| \leq n$. Let X be the set of all pairs $(\hat{m}, \hat{b})$ such that $\hat{m} < \hat{\omega}$ and $\hat{b} \in \hat{Q}_{\hat{m}}$ and $R(x, \hat{b}) \in p(x)$. By possibly increasing $\hat{n}_*$ we can suppose $X \subseteq \hat{n}_* \times \hat{n}_*$. Clearly $[X]^{<\aleph_0} \subseteq \mathbf{j}(\Delta^*_{\omega \backslash \{0\}}(FDP))$. Also, since $\lambda < \lambda_{\hat{V}}(\Delta(IP)) \leq \mathrm{lcf}_{\hat{V}}(\omega)$ (by Theorems 3.10.5 and 3.12.5), we can apply Lemma 3.14.8 to find some $\hat{X} \in \mathbf{j}(\Delta^*_{\omega \backslash \{0\}}(FDP))$ with $X \subseteq \hat{X}$. Also, since $\lambda < \lambda_{\hat{V}}(\Delta(IP))$, we can find disjoint $\hat{X}_0, \hat{X}_1 \subseteq \hat{n}_*$ such that $B \subseteq \hat{X}_0$ and $B_* \backslash B \subseteq \hat{X}_1$.

Let $\hat{q}(x)$ be the partial type over $\hat{n}_*$ defined in $\hat{V}$, as follows:

- $P(x) \in \hat{q}(x)$;
- For every $\hat{b} < \hat{n}_*$, if for some $\hat{m} < \hat{\omega}$, $(\hat{m}, \hat{b}) \in \hat{X}$ and $\hat{b} \in \hat{X}_0 \cap \hat{Q}_{\hat{m}}$, then put $R(x, \hat{b}) \in \hat{q}(x)$;
- For every $\hat{b} < \hat{n}_*$, if $\hat{b} \in \hat{Q}_{\hat{\omega}}$ (i.e. $\hat{b} \in \hat{Q}_{\hat{m}}$ for each $\hat{m} < \hat{\omega}$), and if $\hat{b} \in \hat{X}_0$, then put $R(x, \hat{b}) \in \hat{q}(x)$;
- For every $\hat{b} \in \hat{Q} \cap \hat{X}_1$, put $\neg R(x, \hat{b}) \in \hat{q}(x)$.

Clearly $\hat{V}$ believes $\hat{q}(x)$ is a consistent finite type, and so it must have a realization $\hat{a}$. But $p(x) \subseteq \hat{q}(x)$ so we are done. $\square$

We immediately get the following corollaries.

Corollary 3.14.14. T_{Cas} is a minimal nonlow theory in $\trianglelefteq^\times_1$, and hence also in $\trianglelefteq^\times_\kappa$ for any κ, and in $\trianglelefteq$.

Corollary 3.14.15. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$ and λ is given. Then the following are equivalent:

(A) $\mathcal{U}$ λ^+-saturates T_{Cas};

(B) $\mathcal{U}$ λ^+-saturates some nonlow theory;

(C) $\mathcal{U}$ λ^+-saturates T_{rg} and $\mathcal{U}$ is λ-OK.

We can finally state the following theorem, which was promised earlier. It follows immediately from Corollary 3.14.15 and Theorem 3.14.11.

Theorem 3.14.16. Suppose $\mathcal{U}$ is an $\aleph_1$-incomplete ultrafilter on $\mathcal{B}$. If $\mathcal{U}$ λ^+-saturates some nonlow theory, then $\mathcal{U}$ is strongly λ-regular; thus for any $M \models T$, $\mathcal{U}$ λ^+-saturates T if and only if $M^{\mathcal{B}}/\mathcal{U}$ is λ^+-saturated.

3.15 The Chain Condition and Saturation

In this section, we prove two theorems relating the chain condition on $\mathcal{B}$ with the existence of ultrafilters $\mathcal{U}$ on $\mathcal{B}$ which λ^+-saturate various T.

The following is a generalization of Claim 5.11 of [57]. There, only the case of $\mathcal{B} = \mathcal{B}_{\lambda\mu\theta}$ was considered (so c.c.$(\mathcal{B}) = (\mu^{<\theta})^+$), and the conclusion of their theorem held only for ultrafilters extending a special filter $\mathcal{D}$.

Theorem 3.15.1. Suppose $\mathcal{B}$ is a complete Boolean algebra; write $\lambda =$ c.c.$(\mathcal{B})$. Suppose $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ does not λ^+-saturate any nonsimple theory. In fact, we can find a $(\lambda, \Delta(TP_2))$-distribution $\mathbf{A}$ in $\mathcal{U}$, such that if $\mathcal{B}$ is a complete subalgebra of $\mathcal{B}_*$ where $\mathcal{B}_*$ has the λ-c.c., then $\mathbf{A}$ has no multiplicative refinement in $\mathcal{B}_*$.

Proof. It suffices to show the second claim. Note that $\lambda > \aleph_0$, as otherwise $\mathcal{B}$ would be finite by Corollary 2.1.8, and so would not admit any nonprincipal ultrafilters. Thus, λ is regular by Theorem 2.1.5.

Let σ be the completeness of $\mathcal{U}$; so by Lemma 2.2.1, $\sigma < \lambda$, and we can find an antichain $\mathbf{C}$ of $\mathcal{B}$ of size σ such that for every $X \in [\mathbf{C}]^{<\sigma}$, $\bigvee X \notin \mathcal{U}$. Enumerate $\mathbf{C} = (\mathbf{c}_\gamma : \gamma < \sigma)$.

Let $S \subseteq \lambda$ be the set of all $\alpha < \lambda$ with $\mathrm{cof}(\alpha) = \sigma$, so S is stationary in λ. For each $\alpha \in S$, let $L_\alpha : \sigma \to \alpha$ be a cofinal, increasing map, and let $\underline{\delta}_\alpha \in (\lambda, <)^{\mathcal{B}}$ be the element such that for all $\gamma < \sigma$, $\|\underline{\delta}_\alpha = L_\alpha(\gamma)\|_{(\lambda,<)^{\mathcal{B}}} = \mathbf{c}_\gamma$. This determines $\underline{\delta}_\alpha$, since $\mathbf{C}$ is a maximal antichain. In particular, we have that $\|\underline{\delta}_\alpha < \alpha\|_{(\lambda,<)^{\mathcal{B}}} = 1$, and for all $\beta < \alpha$, $\|\underline{\delta}_\alpha > \beta\|_{(\lambda,<)^{\mathcal{B}}} \in \mathcal{U}$. In particular, for all $\alpha < \beta$ both in S, $\|\underline{\delta}_\alpha < \underline{\delta}_\beta\|_{(\lambda,<)^{\mathcal{B}}} \in \mathcal{U}$.

For each $s \in [\lambda]^{<\aleph_0}$, put $\mathbf{A}(s) = \bigwedge_{\alpha \neq \beta \in s} \|\underline{\delta}_\alpha \neq \underline{\delta}_\beta\|_{(\lambda,<)^{\mathcal{B}}}$; so $\mathbf{A}$ is a $(\lambda, \Delta(TP_2))$-distribution in $\mathcal{U}$. Suppose $\mathcal{B}$ is a complete subalgebra of $\mathcal{B}_*$ where $\mathcal{B}_*$ has the λ-c.c. We show that $\mathbf{A}$ has no multiplicative refinement in $\mathcal{B}_*$, i.e. there is no multiplicative λ-distribution $\mathbf{B}$ in $\mathcal{B}_*$ such that for all $\alpha < \beta$, $\mathbf{B}(\{\alpha, \beta\}) \leq \|\underline{\delta}_\alpha \neq \underline{\delta}_\beta\|_{(\lambda,<)^{\mathcal{B}}}$.

Suppose there were. For each $\alpha < \lambda$ there is some $f(\alpha) < \sigma$ with $\mathbf{B}(\{\alpha\}) \wedge \mathbf{c}_{f(\alpha)}$ nonzero, i.e. with $\mathbf{B}(\{\alpha\}) \wedge \|\underline{\delta}_\alpha = L_\alpha(f(\alpha))\|_{(\lambda,<)^{\mathcal{B}}/\mathcal{U}}$ nonzero. Write $g(\alpha) = L_\alpha(f(\alpha)) < \alpha$. By Fodor's Lemma (using that λ is regular), we can find a stationary set $S' \subseteq S$ on which g is constant, say with value γ. Since $\mathcal{B}_*$ has λ-c.c., $(\mathbf{B}(\{\alpha\}) \wedge \|\underline{\delta}_\alpha = \gamma\|_{(\lambda,<)^{\mathcal{B}}/\mathcal{U}} : \alpha \in S')$ is not an antichain, so we can choose $\alpha < \beta$ both in S' such that $\mathbf{B}(\{\alpha\}) \wedge \mathbf{B}(\{\beta\}) \wedge \|\underline{\delta}_\alpha = \gamma\|_{(\lambda,<)^{\mathcal{B}}/\mathcal{U}} \wedge \|\underline{\delta}_\beta = \gamma\|_{(\lambda,<)^{\mathcal{B}}/\mathcal{U}}$ is nonzero. But $\mathbf{B}(\{\alpha, \beta\}) \leq \|\underline{\delta}_\alpha \neq \underline{\delta}_\beta\|_{(\lambda,<)^{\mathcal{B}}/\mathcal{U}}$, a contradiction. $\square$

We remark that Malliaris and Shelah have shown in [57] that if there is a supercompact cardinal, then Theorem 3.15.1 is sharp. That is, we can find $\mathcal{B}$ with the λ-c.c., and an ultrafilter $\mathcal{U}$ on $\mathcal{B}$, which λ^+-saturates every simple theory. In particular, simplicity is a principal dividing line in Keisler's order. We give a streamlined proof of this in the next chapter.

A special case of the following theorem is proved by Malliaris and Shelah in [56] (namely, the special case when $\mathcal{B}$ is of the form $\mathcal{B}_{2^\lambda \mu\theta}$).

Theorem 3.15.2. Suppose $\mathcal{B}$ is a complete Boolean algebra; write $\lambda = \text{c.c.}(\mathcal{B})$. Suppose $\mathcal{U}$ is an $\aleph_1$-incomplete ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ does not λ^+-saturate any nonlow theory. In fact, there is a $(\lambda, \Delta(FDP))$-distribution $\mathbf{A}$ in $\mathcal{U}$ such that if $\mathcal{B}$ is a complete subalgebra of $\mathcal{B}_*$ and $\mathcal{B}_*$ has the λ-c.c., then $\mathbf{A}$ has no multiplicative refinement in $\mathcal{B}_*$.

Proof. It suffices to show the second claim. Note that $\lambda > \aleph_0$, as otherwise $\mathcal{B}$ would be finite by Corollary 2.1.8, and so would not admit any nonprincipal ultrafilters.

Let $\mathcal{U}$ be an $\aleph_1$-incomplete ultrafilter on $\mathcal{B}$; then we can choose a descending sequence $(\mathbf{c}_n : n < \omega)$ from $\mathcal{U}$ such that $\mathbf{c}_0 = 1$ and $\bigwedge_n \mathbf{c}_n = 0$. Let $\mathbf{A}$ be the distribution in $\mathcal{U}$, defined by $\mathbf{A}(s) = \mathbf{c}_{|s|}$. Then $\mathbf{A}$ is a $(\lambda, \Delta(FDP))$-distribution. Suppose $\mathcal{B}_*$ has the λ-c.c. and $\mathcal{B}$ is a complete subalgebra of $\mathcal{B}_*$. If $\mathbf{A}$ has a multiplicative refinement $\mathbf{B}$ in $\mathcal{U}$, then by the proof of Theorem 3.14.10, $\mathbf{B}$ would be a strongly λ-regular distribution. But this is impossible since $\mathcal{B}_*$ has the λ-c.c. $\square$

Thus we see that if we wish to construct an ultrafilter on a complete Boolean algebra $\mathcal{B}$ with the λ-c.c. which λ^+-saturates some nonlow theory, then we need at least a measurable cardinal. Generally, if we wish to construct an ultrafilter $\mathcal{U}$ on any $\mathcal{B}$ which λ^+-saturates some nonlow theories but not every theory, then we need $\mathcal{U}$ to be λ-OK and we need $\mathcal{U}$ to λ^+-saturate T_{rg}, and we need $\mathcal{U}$ to not be λ^+-good. It is open if this can be done in ZFC, even if we drop the requirement that $\mathcal{U}$ λ^+-saturate T_{rg}.

On a related note, we remark that Malliaris and Shelah construct in [55] an ultrafilter $\mathcal{U}$ on $\mathcal{P}(\lambda)$ which is λ-OK, but does not λ^+-saturate T_{rg}, starting with a measurable cardinal.

We remark that we have shown in [87] that Theorem 3.15.2 is sharp. That is, we can find a complete Boolean algebra $\mathcal{B}$ with the λ-c.c., and an $\aleph_1$-incomplete ultrafilter $\mathcal{U}$ on $\mathcal{B}$, which λ^+-saturates every low theory. We give a streamlined proof of this in the next chapter.

Finally, for context we remark on the following theorem, which we will prove later (Corollary 3.16.18). Kunen proved the special case of $\mathcal{B} = \mathcal{P}(\lambda)$ in [45].

Theorem 3.15.3. Suppose $\mathcal{B}$ is a complete Boolean algebra with an antichain of size λ. Then there is a strongly λ-regular, λ^+-good ultrafilter on $\mathcal{B}$.

Thus, Theorems 3.15.1 and 3.15.2 are also sharp with respect to $\mathcal{B}$.

3.16 Ultrafilter Pullbacks

Suppose $\mathcal{B}_0, \mathcal{B}_1$ are complete Boolean algebras, and $\mathcal{U}_1$ is an ultrafilter on $\mathcal{B}_1$. When can we find $\mathcal{B}_0$, an ultrafilter on $\mathcal{U}_0$, which λ^+-saturates the same theories as $\mathcal{U}_1$? The following is the best we know currently:

Theorem 3.16.1. Suppose $\mathcal{B}_0, \mathcal{B}_1$ are complete Boolean algebras such that c.c.$(\mathcal{B}_0) > \lambda$ (i.e. $\mathcal{B}_0$ has an antichain of size λ) and $2^{<\text{c.c.}(\mathcal{B}_1)} \leq 2^\lambda$. Suppose $\mathcal{U}_1$ is an ultrafilter on $\mathcal{B}_1$. Then there is a strongly λ-regular ultrafilter $\mathcal{U}_0$ on $\mathcal{B}_0$ such that for all complete countable theories T, $\mathcal{U}_0$ λ^+-saturates T if and only if $\mathcal{U}_1$ does.

Remark 3.16.2. Note that, for instance, if $\mathcal{B}_1$ has the λ^+-c.c., or if $|\mathcal{B}_1| \leq 2^\lambda$, then $\mathcal{B}_1$ satisfies the hypothesis of the theorem.

Theorem 3.15.1 shows that the hypothesis on $\mathcal{B}_0$ is necessary, but possibly the chain condition hypothesis on $\mathcal{B}_1$ can be dropped.

In the next two subsections, we will prove two theorems of Malliaris and Shelah, which we overview now, and which together handle the case when $|\mathcal{B}_1| \leq 2^\lambda$. In the third subsection, we wrap up the proof of Theorem 3.16.1.

First of all, we want some definitions:

Definition 3.16.3. Let $\mathcal{L}_{\mathbb{B}} = (0, 1, \leq, \wedge, \vee, \neg)$ be the language of Boolean algebras (so the operations $\wedge, \vee$ are binary).

If $\mathcal{B}_0, \mathcal{B}_1$ are complete Boolean algebras, then a Boolean algebra homomorphism $\mathbf{j} : \mathcal{B}_0 \to \mathcal{B}_1$ is a homomorphism of $\mathcal{L}_{\mathbb{B}}$-structures; we do not require it preserve infinite meets and joins.

A filter $\mathcal{D}$ on $\mathcal{B}$ is λ^+-good if every λ-distribution in $\mathcal{D}$ has a multiplicative refinement in $\mathcal{D}$. $\mathcal{D}$ is strongly λ-regular if $\mathcal{D}$ contains a strongly λ-regular sequence. (This generalizes the definitions of λ^+-good and strongly λ-regular ultrafilters.)

In Section 3.16.1, prove the following theorem of Malliaris and Shelah; it is essentially Theorem 5.11 of [56], and they term it "Separation of Variables." (In addition to differences in terminology, Malliaris and Shelah introduce a notion of λ^+-excellence, and use it in place of λ^+-good. But they then prove it is equivalent to λ^+-goodness, and we avoid the extra notion. Also, Malliaris and Shelah just prove the case where $\mathcal{B}_0 = \mathcal{P}(\lambda)$, and the case when $\mathcal{D}$ is λ-regular, but the general case is the same.)

Theorem 3.16.4. Suppose $\mathcal{B}_0, \mathcal{B}_1$ are complete Boolean algebras, and $\mathbf{j} : \mathcal{B}_0 \to \mathcal{B}_1$ is a surjective homomorphism. Write $\mathcal{D} = \mathbf{j}^{-1}(1_{\mathcal{B}_1})$; suppose $\mathcal{D}$ is λ^+-good. Suppose $\mathcal{U}_1$ is an ultrafilter on $\mathcal{B}_1$; let $\mathcal{U}_0 = \mathbf{j}^{-1}(\mathcal{U}_1)$, so $\mathcal{U}_0$ is an ultrafilter extending $\mathcal{D}$. Then for every complete countable theory T, $\mathcal{U}_0$ λ^+-saturates T if and only if $\mathcal{U}_1$ λ^+-saturates T.

In Section 3.16.2, we show that the setup described in Theorem 3.16.4 can occur, and moreover we can arrange $\mathcal{D}$ to be strongly λ-regular. Malliaris and Shelah call this the Existence Theorem; it is Theorem 7.1 of [56] (it was also Excercise VI.3.11(2) of [75]). They just consider the special case where $\mathcal{B}_0 = \mathcal{P}(\lambda)$ and $\mathcal{B}_1$ has the λ^+-c.c., but the general case is the same.

Theorem 3.16.5. Suppose $\mathcal{B}_0, \mathcal{B}_1$ are complete Boolean algebras, such that $\mathcal{B}_0$ has an antichain of size λ, and $|\mathcal{B}_1| \leq 2^\lambda$. Then there is a surjective Boolean algebra homomorphism $\mathbf{j} : \mathcal{B}_0 \to \mathcal{B}_1$, such that $\mathbf{j}^{-1}(1_{\mathcal{B}_1})$ is λ^+-good and strongly λ-regular.

3.16.1 Separation of Variables

We establish some very useful terminology.

Definition 3.16.6. If $\mathcal{D}$ is a filter on the complete Boolean algebra $\mathcal{B}$, then by $\mathcal{B}/\mathcal{D}$ we mean the Boolean algebra whose elements are equivalence classes of $=_\mathcal{D}$, where $\mathbf{a} =_\mathcal{D} \mathbf{b}$ if $\neg(\mathbf{a} \triangle \mathbf{b}) \in \mathcal{D}$. (Here $\mathbf{a} \triangle \mathbf{b} = (\mathbf{a} \wedge (\neg \mathbf{b})) \vee ((\neg \mathbf{a}) \wedge \mathbf{b})$ is symmetric difference.) Thus if $\mathbf{j} : \mathcal{B}_0 \to \mathcal{B}_1$ is a surjective homomorphism, then $\mathcal{B}_0/\mathbf{j}^{-1}(1_{\mathcal{B}_1}) \cong \mathcal{B}_1$. (It would be more algebraically natural to mod out by an ideal; however, since we are concerned with ultrafilters instead of maximal ideals, we use this notation, following Malliaris and Shelah [56].)

If $\varphi(x_i : i < n)$ is a quantifier-free $\mathcal{L}_\mathbb{B}$-formula, and $\mathcal{D}$ is a filter on $\mathcal{B}$, then say that $\varphi(\mathbf{a}_i : i < n)$ holds mod $\mathcal{D}$ if $\varphi([\mathbf{a}_0/\mathcal{D}], \ldots, [\mathbf{a}_{n-1}/\mathcal{D}])$ holds in $\mathbf{B}/\mathcal{D}$.

In the rest of the subsection we prove Theorem 3.16.4, following [56]. We fix the setup: suppose $\mathcal{B}_0, \mathcal{B}_1$ are complete Boolean algebras, and $\mathbf{j} : \mathcal{B}_0 \to \mathcal{B}_1$ is a surjective homomorphism. Write $\mathcal{D} = \mathbf{j}^{-1}(1_{\mathcal{B}_1})$; suppose $\mathcal{D}$ is λ^+-good. Suppose $\mathcal{U}_1$ is an ultrafilter on $\mathcal{B}_1$; let $\mathcal{U}_0 = \mathbf{j}^{-1}(\mathcal{U}_1)$, so $\mathcal{U}_0$ is an ultrafilter extending $\mathcal{D}$. We wish to show that for every complete, countable T, $\mathcal{U}_0$ λ^+-saturates T if and only if $\mathcal{U}_1$ does.

We will prove three lemmas, after which we will essentially be done via Theorem 3.5.14.

Lemma 3.16.7. Suppose $\mathbf{A}_0$ is a λ-distribution in $\mathcal{B}_0$; write $\mathbf{A}_1 = \mathbf{j} \circ \mathbf{A}_0$, a λ-distribution in $\mathcal{B}_1$. Suppose T is a complete countable theory. Then $\mathbf{A}_0$ is a (λ, T)-Łoś map if and only if $\mathbf{A}_1$ is a (λ, T)-Łoś map.

Proof. In this lemma, we actually won't need the hypothesis that $\mathcal{D}$ is λ^+-good.

Let $\overline{\varphi} = (\varphi_\alpha(x, y_\alpha) : \alpha < \lambda)$ be a sequence of formulas. It suffices to show that $\mathbf{A}_0$ is a $(\lambda, T, \overline{\varphi})$-Łoś map if and only if $\mathbf{A}_1$ is a $(\lambda, T, \overline{\varphi})$-Łoś map.

We apply Theorem 3.5.12, using characterization (C).

First suppose $\mathbf{A}_0$ is a $(\lambda, T, \overline{\varphi})$-possibility. Let $s \in [\lambda]^{<\aleph_0}$, and let $\mathbf{c}_1 \in \mathcal{B}_1$ be nonzero, such that $\mathbf{c}_1$ decides $\mathbf{A}_1(t)$ for all $t \subseteq s$. Let $\mathcal{J} = \{t \subseteq s : \mathbf{c}_1 \leq \mathbf{A}_1(t)\}$, and let $\mathbf{c}_0 = \bigwedge_{t \in \mathcal{J}} \mathbf{A}_0(t) \wedge \bigwedge_{t \in \mathcal{P}(s) \setminus \mathcal{J}} \neg \mathbf{A}_0(t)$. Then since $\mathbf{j}$ is a homomorphism and each $\mathbf{j}(\mathbf{A}_0(t)) = \mathbf{A}_1(t)$ we get that $\mathbf{j}(\mathbf{c}_0) \geq \mathbf{c}_1$. In particular $\mathbf{c}_0$ is nonzero. Since $\mathbf{A}_0$ is a $(\lambda, T, \overline{\varphi})$-Łoś map, we can find $M \models T$ and $(a_\alpha : \alpha \in s)$ from M, such that for each $t \subseteq s$, $\exists x \bigwedge_{\alpha \in t} \varphi_\alpha(x, a_\alpha)$ is consistent if and only if $t \in \mathcal{J}$, as desired.

Next, suppose $\mathbf{A}_1$ is a $(\lambda, T, \overline{\varphi})$-Łoś map. Let $s \in [\lambda]^{<\aleph_0}$, and let $\mathbf{c}_0 \in \mathcal{B}_0$ be nonzero, such that $\mathbf{c}_0$ decides $\mathbf{A}_0(t)$ for all $t \subseteq s$. Let $\mathbf{c}_1 = \mathbf{j}(\mathbf{c}_0)$. Then $\mathbf{c}_1$ is nonzero, and $\mathbf{c}_1$ decides each $\mathbf{A}_1(t)$, in the same way that $\mathbf{c}_0$ decides $\mathbf{A}_0(t)$. Thus, since $\mathbf{A}_1$ is a $(\lambda, T, \overline{\varphi})$-Łoś map, we conclude as above. $\square$

The next two lemmas say that distributions in $\mathcal{U}_0$ correspond to distributions in $\mathcal{U}_1$, in a way that preserves the existence of multiplicative refinements.

Lemma 3.16.8. Suppose $\mathbf{A}_0$ is a λ-distribution in $\mathcal{U}_0$, so $\mathbf{A}_1 := \mathbf{j} \circ \mathbf{A}_0$ is a multiplicative refinement in $\mathcal{U}_1$. Then: $\mathbf{A}_0$ has a multiplicative refinement in $\mathcal{U}_0$ if and only if $\mathbf{A}_1$ has a multiplicative refinement in $\mathcal{U}_1$.

Proof. Clearly, if $\mathbf{B}_0$ is a multiplicative refinement of $\mathbf{A}_0$ in $\mathcal{U}_0$, then $\mathbf{j}\circ\mathbf{B}_0$ is a multiplicative refinement of $\mathbf{A}_1$ in $\mathcal{U}_1$.

So suppose $\mathbf{B}_1$ is a multiplicative refinement of $\mathbf{A}_1$ in $\mathcal{U}_1$. For each $\alpha < \lambda$, choose $\mathbf{B}_0'(\{\alpha\}) \in \mathcal{B}_0$ such that $\mathbf{j}(\mathbf{B}_0'(\{\alpha\})) = \mathbf{B}_1(\{\alpha\})$; for each $s \in [\lambda]^{<\aleph_0}$, define $\mathbf{B}_0'(s) = \bigwedge_{\alpha\in s}\mathbf{B}_0'(\{\alpha\})$. Note that $\mathbf{B}_0'$ is a multiplicative λ-distribution in $\mathcal{U}_0$, and for all $s \in [\lambda]^{<\aleph_0}$, $\mathbf{A}_0(s) \leq \mathbf{B}_0'(s)$ mod $\mathcal{D}$.

For each $s \in [\lambda]^{<\aleph_0}$, let $\mathbf{C}(s) = \bigwedge_{t\subseteq s}(\neg\mathbf{A}_0(t) \vee \mathbf{B}_0'(t))$, so $\mathbf{C}$ is a λ-distribution in $\mathcal{D}$, and we can think of it as measuring where $\mathbf{B}_0'$ is a refinement of $\mathbf{A}_0$. Let $\mathbf{D}$ be a multiplicative refinement of $\mathbf{C}$ in $\mathcal{D}$, possible by λ^+-goodness of $\mathcal{D}$. For each $s \in [\lambda]^{<\aleph_0}$, define $\mathbf{B}_0(s) = \mathbf{B}_0'(s) \wedge \mathbf{D}(s)$. Clearly $\mathbf{B}_0$ is a multiplicative λ-distribution in $\mathcal{U}_0$; but since $\mathbf{D}$ refines $\mathbf{C}$, we clearly get that $\mathbf{B}_0$ refines $\mathbf{A}_0$. $\square$

Lemma 3.16.9. Suppose $\mathbf{A}_1$ is a λ-distribution in $\mathcal{U}_1$. Then there is a λ-distribution $\mathbf{A}_0$ in $\mathcal{U}_0$ such that $\mathbf{j} \circ \mathbf{A}_0 = \mathbf{A}_1$.

Proof. Choose $\mathbf{A}_0' : [\lambda]^{<\aleph_0} \to \mathcal{B}_0$ such that $\mathbf{j} \circ \mathbf{A}_0' = \mathbf{A}_1$ (possible since $\mathbf{j}$ is surjective). We have that each $\mathbf{A}_0'(s) \in \mathcal{U}_0$, and further, for all $t \subseteq s \in [\lambda]^{<\aleph_0}$, $\mathbf{A}_0(s) \leq \mathbf{A}_0(t)$ mod $\mathcal{D}$.

Define $\mathbf{C}(s) = \bigwedge_{t\subseteq t'\subseteq s}\mathbf{A}_0'(t) \vee (\neg\mathbf{A}_0'(t')) \in \mathcal{D}$; this is a λ-distribution in $\mathcal{D}$, and we can think of it as measuring where $\mathbf{A}_0'$ is a distribution. Let $\mathbf{D}(s)$ be a multiplicative refinement of $\mathbf{C}(s)$ in $\mathcal{D}$, and define $\mathbf{A}_0(s) = \mathbf{A}_0'(s) \wedge \mathbf{D}(s)$ for each $s \in [\lambda]^{<\aleph_0}$. Clearly $\mathbf{j} \circ \mathbf{A}_0 = \mathbf{A}_1$ still, and so $\mathbf{A}_0(s) \in \mathcal{U}_0$ for all s. But also, since $\mathbf{D}$ refines $\mathbf{C}$, we get easily that $\mathbf{A}_0$ is a distribution, i.e. $t \subseteq s$ implies $\mathbf{A}_0(t) \subseteq \mathbf{A}_0(s)$. Thus $\mathbf{A}_0$ works. $\square$

We can now finish the proof of Theorem 3.16.4 by chasing implications. Indeed, suppose first that $\mathcal{U}_0$ λ^+-saturates T, and let $\mathbf{A}_1$ be a (λ, T)-Łoś map in $\mathcal{U}_1$. Choose a λ-distribution $\mathbf{A}_0$ in $\mathcal{U}_0$ with $\mathbf{j} \circ \mathbf{A}_0 = \mathbf{A}_1$, by Lemma 3.16.9. Then $\mathbf{A}_0$ is a (λ, T)-Łoś map by Lemma 3.16.7, so we can find a multiplicative refinement $\mathbf{B}_0$ of $\mathbf{A}_0$ in $\mathcal{U}_0$. Then $\mathbf{B}_1 := \mathbf{j} \circ \mathbf{A}_1$ is a multiplicative refinement of $\mathbf{A}_1$ in $\mathcal{U}_1$, by the trivial direction of Lemma 3.16.8.

Conversely, suppose $\mathcal{U}_1$ λ^+-saturates T, and let $\mathbf{A}_0$ be a (λ, T)-Łoś map in $\mathcal{U}_0$. Then $\mathbf{A}_1 := \mathbf{j} \circ \mathbf{A}_0$ is a (λ, T)-Łoś map in $\mathcal{U}_1$, by Lemma 3.16.7, so we can find a multiplicative refinement $\mathbf{A}_1$ in $\mathcal{U}_1$. Thus we can find a mulitplicative refinement of $\mathbf{A}_0$ in $\mathcal{U}_0$, by the nontrivial direction of Lemma 3.16.8.

3.16.2 The Existence Theorem

In this subsection, we prove Theorem 3.16.5. So let $\mathcal{B}_0$ be a complete Boolean algebra with an antichain of size λ, and let $\mathcal{B}_1$ be a complete Boolean algebra with $|\mathcal{B}_1| \leq 2^\lambda$. We aim to find a surjective homomorphism $\mathbf{j} : \mathcal{B}_0 \to \mathcal{B}_1$, such that $\mathbf{j}^{-1}(1_{\mathcal{B}_1})$ is λ^+-good and strongly λ-regular. Write $\mu = \text{c.c.}(\mathcal{B}_1)$, so μ is regular and $2^{<\mu} \leq 2^\lambda$.

The history of this proof is long; previously only the case where $\mathcal{B}_0 = \mathcal{P}(\lambda)$ and where $\mathcal{B}_1$ has the λ^+-c.c. have been considered. The first iteration was due to Keisler [34], who proved that if $2^\lambda = \lambda^+$ then $\mathcal{P}(\lambda)$ admits a λ-regular, λ^+-good ultrafilter; this is the special case of the Existence Theorem where $\mathcal{B}_1 = \{0, 1\}$. Next, Kunen [45] removed the hypothesis that $2^\lambda = \lambda^+$, replacing it with the notion of independent families of functions (which for us, will be independent families of maximal antichains). Then Shelah listed the general case as Exercise VI.3.11(2) of [75], but in the absence of Separation of Variables, its significance was unclear. Finally, the Existence Theorem is proved as Theorem 7.1 of [56]. We follow their proof; the generalizations to any $\mathcal{B}_0$ with an antichain of size λ, and any $\mathcal{B}_1$ with $|\mathcal{B}_1| \leq 2^\lambda$, both require only minor adjustments.

By Theorem 2.5.4, we can choose an independent family $\mathbb{C}_*$ of 2^λ-many maximal antichains of $\mathcal{B}_0$, such that each $\mathbf{C} \in \mathbb{C}_*$ has size λ. Write $\mathbb{C}_*$ as the disjoint union of $\mathbb{C}_{-1}$ and $\mathbb{C}'$, each having size 2^λ. Enumerate $\mathbb{C}' = (\mathbf{C}_\alpha : \alpha < 2^\lambda)$, and choose $\mathbf{c}_\alpha \in \mathbf{C}_\alpha$ for each α. Replace $\mathbf{C}_\alpha$ with $\{\mathbf{c}_\alpha, \neg\mathbf{c}_\alpha\}$; note that $\mathbb{C}_*$ is still independent. Also, let $(\mathbf{c}'_\alpha : \alpha < 2^\lambda)$ be an enumeration of $\mathcal{B}_1$, with repetitions if necessary.

$\mathbf{j}$ will be our eventual surjective homomorphism $\mathbf{j} : \mathcal{B}_0 \to \mathcal{B}_1$. We plan to arrange that $\mathbf{j}(\mathbf{c}_\alpha) = \mathbf{c}'_\alpha$ for all $\alpha < 2^\lambda$. Subject to this condition, note that $\mathbf{j}$ is determined by $\mathcal{D} = \mathbf{j}^{-1}(1_{\mathcal{B}_1})$, since necessarily we must have that for all $\mathbf{a} \in \mathcal{B}_0$, there is some $\alpha < 2^\lambda$ with $\mathbf{a} = \mathbf{c}_\alpha \mod \mathcal{D}$, and so $\mathbf{j}(\mathbf{a}) = \mathbf{c}'_\alpha$. Thus our problem has turned into constructing a filter $\mathcal{D}$ on $\mathcal{B}_0$. In fact, we can conveniently describe what properties we need $\mathcal{D}$ to have:

Lemma 3.16.10. Suppose $\mathcal{D}$ is a filter on $\mathcal{B}_0$ satisfying the following:

(A) For every $\mathbf{a} \in \mathcal{B}_0$, there is some $\alpha < 2^\lambda$ such that $\mathbf{a} = \mathbf{c}_\alpha \mod \mathcal{D}$;

(B) For every $\mathcal{L}_{\mathbb{B}}$-term $\tau(x_i : i < n)$ and for every $\alpha_0 < \ldots < \alpha_{n-1}$ from 2^λ, $\tau(\mathbf{c}'_{\alpha_i} : i < n) = 1_{\mathcal{B}_1}$ if and only if $\tau(\mathbf{c}_{\alpha_i} : i < n) \in \mathcal{D}$.

Define $\mathbf{j} : \mathcal{B}_0 \to \mathcal{B}_1$ via $\mathbf{j}(\mathbf{a}) = \mathbf{c}'_\alpha$, for some or every $\alpha < 2^\lambda$ such that $\mathbf{a} = \mathbf{c}_\alpha \mod \mathcal{D}$. Then $\mathbf{j}$ is a well-defined surjective homomorphism from $\mathcal{B}_0$ to $\mathcal{B}_1$, with $\mathbf{j}^{-1}(1_{\mathcal{B}_1}) = \mathcal{D}$.

Proof. $\mathbf{j}$ is trivially well-defined. (Such α must exist by (A); the choice of α does not matter by (B).) Also trivially, $\mathbf{j}$ is surjective.

We check that $\mathbf{j}^{-1}(1_{\mathcal{B}_1}) = \mathcal{D}$. Choose $\alpha < 2^\lambda$ so that $\mathbf{c}'_\alpha = 1_{\mathcal{B}_1}$. Then $\mathbf{c}_\alpha \in \mathbf{j}^{-1}(1_{\mathcal{B}_1})$. But since $\mathbf{c}'_\alpha = 1_{\mathcal{B}_1}$, we get by (B) that $\mathbf{c}_\alpha = 1_{\mathcal{B}_0} \mod \mathcal{D}$, i.e. $\mathbf{c}_\alpha \in \mathcal{D}$. Thus, given $\mathbf{a} \in \mathcal{B}_0$, $\mathbf{j}(\mathbf{a}) = 1_{\mathcal{B}_1}$ if and only if $\mathbf{a} = \mathbf{c}_\alpha \mod \mathcal{D}$ if and only if $\mathbf{a} \in \mathcal{D}$.

Next we show that $\mathbf{j}(\neg\mathbf{a}) = \neg\mathbf{j}(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{B}_1$. Choose α_0, α_1 such that $\mathbf{a} = \mathbf{c}_{\alpha_0} \mod \mathcal{D}$ and $\neg\mathbf{a} = \mathbf{c}_{\alpha_1} \mod \mathcal{D}$. Then $\mathbf{c}_{\alpha_0} \vee (\neg\mathbf{c}_{\alpha_1})$ and $(\neg\mathbf{c}_{\alpha_0}) \vee \mathbf{c}_{\alpha_1}$ are both in $\mathcal{D}$, thus

$\mathbf{c}'_{\alpha_0} \vee (\neg \mathbf{c}'_{\alpha_1})$ and $(\neg \mathbf{c}'_{\alpha_0}) \vee \mathbf{c}'_{\alpha_1}$ are both $1_{\mathcal{B}_1}$, thus $\mathbf{j}(\neg \mathbf{a}) = \mathbf{c}'_{\alpha_1} = \neg \mathbf{c}'_{\alpha_0} = \neg \mathbf{j}(\mathbf{a})$ as desired.

In particular, $\mathbf{j}(0_{\mathcal{B}_0}) = \neg \mathbf{j}(1_{\mathcal{B}_0}) = 0_{\mathcal{B}_1}$, so it suffices (by De Morgan's laws) to show that $\mathbf{j}(\mathbf{a} \wedge \mathbf{b}) = \mathbf{j}(\mathbf{a}) \wedge \mathbf{j}(\mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathcal{B}_0$. Choose $\alpha_0, \alpha_1, \alpha_2$ such that $\mathbf{a} = \mathbf{c}_{\alpha_0}$ mod $\mathcal{D}$ and $\mathbf{b} = \mathbf{c}_{\alpha_1}$ mod $\mathcal{D}$ and $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}_{\alpha_2}$ mod $\mathcal{D}$. Then reasoning as above gives that $\mathbf{c}'_{\alpha_0} \wedge \mathbf{c}'_{\alpha_1} = \mathbf{c}'_{\alpha_2}$, as desired.

□

Thus, it suffices to find a λ-regular, λ^+-good filter $\mathcal{D}$ on $\mathcal{B}_0$ which satisfies conditions (A) and (B) of Lemma 3.16.10.

As some convenient notation, let Σ be the set of all $\mathcal{L}_{\mathbb{B}}$-terms $\sigma(\overline{x})$ in the variables $\overline{x} = (x_\alpha : \alpha < 2^\lambda)$ (so of course $\sigma(\overline{x})$ only uses finitely much of $\overline{x}$). Let Σ_1 be the set of all of $\sigma(\overline{x}) \in \Sigma$ such that $\sigma(\overline{\mathbf{c}'}) = 1_{\mathcal{B}_1}$, and let Σ_+ be the set of all $\sigma(\overline{x}) \in \Sigma$ such that $\sigma(\overline{\mathbf{c}'})$ is nonzero.

Then Condition (B) of Lemma 3.16.10 can be reformulated as: for all $\sigma(\overline{x}) \in \Sigma_1$, $\sigma(\overline{\mathbf{c}}) \in \mathcal{D}$, and for all $\sigma(\overline{x}) \in \Sigma_+$, $\sigma(\overline{\mathbf{c}})$ is nonzero mod $\mathcal{D}$.

We will be approximating our desired filter $\mathcal{D}$ as a union of filters $(\mathcal{D}_\alpha : \alpha < 2^\lambda)$. To make sure we don't run out of space, at each stage $\alpha < 2^\lambda$ we will also have a large subset $\mathbb{C}_\alpha$ of $\mathbb{C}_{-1}$ which is independent mod $\mathcal{D}_\alpha$, in the strong sense described below.

We describe the requirements we will place each $(\mathcal{D}_\alpha, \mathbb{C}_\alpha)$:

Definition 3.16.11. Given $\mathbb{C} \subseteq \mathbb{C}_{-1}$, let $P_{\mathbb{C}}$ be the set of all functions f, such that $\mathrm{dom}(f)$ is a finite subset of $\mathbf{C}$, and f is a choice function on $\mathrm{dom}(f)$. Let $\mathbf{x}_f = \bigwedge_{\mathbf{C} \in \mathrm{dom}(f)} f(\mathbf{C})$.

Suppose $\mathbb{C} \subseteq \mathbb{C}_{-1}$ has size 2^λ and $\mathcal{D}$ is a filter on $\mathcal{B}_0$. Then say that $(\mathcal{D}, \mathbb{C})$ is a pre-good pair the following two conditions hold: first, for each $\sigma(\overline{x}) \in \Sigma^1$, $\sigma(\overline{\mathbf{c}}) \in \mathcal{D}$. Second, for each $f \in P_{\mathbb{C}}$ and for each $\sigma(\overline{x}) \in \Sigma_+$, $\mathbf{x}_f \wedge \sigma(\overline{\mathbf{c}})$ is nonzero.

Say that $(\mathcal{D}, \mathbb{C})$ is a good pair if $\mathcal{D}$ is maximal subject to the condition that $(\mathcal{D}, \mathbb{C})$

is a pre-good pair; i.e. $(\mathcal{D}, \mathbb{C})$ is a pre-good pair, but whenever $\mathcal{D}'$ properly extends $\mathcal{D}$ then $(\mathcal{D}', \mathbb{C})$ is not.

Our plan then is to build a sequence $(\mathcal{D}_\alpha, \mathcal{C}_\alpha : \alpha < 2^\lambda)$ where each $(\mathcal{D}_\alpha, \mathcal{C}_\alpha)$ is a good pair; $\mathcal{D} := \bigcup_\alpha \mathcal{D}_\alpha$ will be our desired filter.

Note that it is clear that if $(\mathcal{D}, \mathbb{C})$ is a pre-good pair, we can find $\mathcal{D}'$ extending $\mathcal{D}$ such that $(\mathcal{D}', \mathbb{C})$ is a good pair, since the conditions involved are finitary.

Lemma 3.16.12. There is a filter $\mathcal{D}_{-1}$ such that $(\mathcal{D}_{-1}, \mathcal{C}_{-1})$ is a good pair.

Proof. First, let $\mathcal{D}'_{-1}$ be the filter generated by $\{\sigma(\overline{\mathbf{c}}) : \sigma(\overline{x}) \in \Sigma_1\}$. We want to show $(\mathcal{D}'_{-1}, \mathbb{C}_{-1})$ is a pre-good pair; that is for every $f \in P_{\mathbb{C}_{-1}}$ and for every $\tau(\overline{x}) \in \Sigma_+$, $\mathbf{x}_f \wedge \tau(\overline{\mathbf{c}})$ is nonzero mod $\mathcal{D}'_{-1}$ (in particular $\mathcal{D}'_{-1}$ is in fact a filter, i.e. is proper).

So let $\sigma_0(\overline{x}), \ldots, \sigma_{n-1}(\overline{x}) \in \Sigma_1$, let $f \in P_{\mathbf{C}_{-1}}$ and let $\tau(\overline{x}) \in \Sigma_+$. We want to check that $\sigma_0(\overline{\mathbf{c}}) \wedge \ldots \wedge \sigma_{n-1}(\overline{\mathbf{c}}) \cap \tau(\overline{\mathbf{c}}) \cap \mathbf{c}_f$ is nonzero.

To see this, let $\sigma(\overline{x})$ be $\sigma_0(\overline{x}) \wedge \ldots \wedge \sigma(\overline{x}) \wedge \sigma(\overline{x})$ in disjunctive normal form (so as a disjoint of conjunctions of $x_\alpha^{\pm 1}$'s). Note that $\sigma(\overline{\mathbf{c}}') = \tau(\overline{\mathbf{c}}') \neq 0_{\mathcal{B}}$. Hence we can choose a conjunct $\sigma^*(\overline{x}) = x_{\alpha_0}^{i_0} \cap \ldots \cap x_{\alpha_{m-1}}^{i_{m-1}}$ of $\sigma(\overline{x})$, such that $\sigma^*(\overline{\mathbf{c}}') \neq 0_{\mathcal{B}}$. (Here, each $i_j \in 2$; for $i = 1$ we let $x^i = x$ and for $i = 0$ we let $x^i = \neg x$.) Since $\mathbb{C}_* = \mathbb{C}_{-1} \cup \mathbb{C}'$ is independent, we have that $\mathbf{x}_f \wedge \bigwedge_{j<m} \mathbf{c}_{\alpha_j}^{i_j} \neq 0$.

Now choose an extension $\mathcal{D}_{-1}$ of $\mathcal{D}'_{-1}$ such that $(\mathcal{D}_{-1}, \mathbb{C}_{-1})$ is a good pair. $\square$

We now handle strong λ-regularity, once and for all.

Lemma 3.16.13. There is a good pair $(\mathcal{D}_0, \mathbb{C}_0)$ with $\mathcal{D}_{-1} \subseteq \mathcal{D}_0$, such that $\mathcal{D}_0$ is strongly λ-regular.

Proof. Pick $\mathbf{C} \in \mathbb{C}_{-1}$ and let $\mathbb{C}_0 = \mathbb{C}_{-1} \backslash \{\mathbf{C}\}$. Enumerate $\mathbf{C} = (\mathbf{a}_s : s \in [\lambda]^{<\aleph_0})$, and for each $\alpha < \lambda$ let $\mathbf{b}_\alpha = \bigvee\{\mathbf{a}_s : \alpha \in s \in [\lambda]^{<\aleph_0}\}$. As in Lemma 3.6.2, $(\mathbf{b}_\alpha : \alpha < \lambda)$ is a strongly λ-regular sequence. Let $\mathcal{D}_0'$ be the filter generated by $\mathcal{D}_{-1}$ and $(\mathbf{b}_\alpha : \alpha < \lambda)$.

I claim that $(\mathcal{D}_0', \mathbb{C}_0)$ is a pre-good pair, which suffices, since then we can extend it to a good pair $(\mathcal{D}_0, \mathbb{C}_0)$. So suppose $f \in P_{\mathbb{C}_0}$ and $\sigma(\overline{x}) \in \Sigma_+$; it suffices to show that $\mathbf{x}_f \wedge \sigma(\overline{\mathbf{c}})$ is nonzero mod $\mathcal{D}_0'$. But given $s \in [\lambda]^{<\aleph_0}$, we have that $\mathbf{a}_s \wedge \mathbf{x}_f \wedge \sigma(\overline{\mathbf{c}})$ is nonzero mod $\mathcal{D}_{-1}$ since $(\mathcal{D}_{-1}, \mathbb{C}_{-1})$ is a good pair (applied to the function $g = f \cup \{(\mathbf{C}, \mathbf{a}_s)\} \in P_{\mathbb{C}_{-1}}$). Since $\mathbf{a}_s \leq \bigwedge_{\alpha \in s} \mathbf{b}_\alpha$ we conclude. □

The following lemma is a key property of good pairs.

Lemma 3.16.14. If $(\mathcal{D}, \mathbb{C})$ is a good pair and $\mathbf{a} \in \mathcal{B}_0$ is nonzero mod $\mathcal{D}$, then there is some $f \in P_{\mathbb{C}}$ and some $\sigma(\overline{x}) \in \Sigma_+$ such that $\mathbf{x}_f \wedge \sigma(\overline{\mathbf{c}}) \leq \mathbf{a}$ mod $\mathcal{D}$. In fact, we can choose $\sigma(\overline{x})$ to be of the form x_α for some $\alpha < 2^\lambda$ (necessarily with $\mathbf{c}'_\alpha \neq 0$).

Hence, $|\mathcal{B}_0/\mathcal{D}| \leq 2^\lambda$.

Proof. If there were no such $f, \sigma(\overline{x})$, then let $\mathcal{D}'$ be the filter generated by $\mathcal{D}$ and $\neg\mathbf{a}$; we are exactly assuming that $(\mathcal{D}', \mathbb{C})$ is a pre-good pair, contradicting the maximality of $\mathcal{D}$. We can arrange $\sigma(\overline{x})$ to be of the form x_α for some α, because we can choose $\alpha < 2^\lambda$ with $\mathbf{c}'_\alpha = \sigma(\overline{\mathbf{c}}')$; then $\mathbf{c}_\alpha = \sigma(\overline{\mathbf{c}})$ mod $\mathcal{D}$, so this works.

For the hence claim: let $\{(f_\gamma, \alpha_\gamma) : \gamma < \kappa\}$ satisfy:

- Each $f_\gamma \in P_{\mathbb{C}}$, and each $\alpha_\gamma < 2^\lambda$ satisfies $\mathbf{c}'_{\alpha_\gamma} \neq 0$;
- Each $\mathbf{x}_{f_\gamma} \wedge \mathbf{c}_{\alpha_\gamma} \leq \mathbf{a}$ mod $\mathcal{D}$;
- For $\gamma \neq \gamma'$, $(\mathbf{x}_{f_\gamma} \wedge \mathbf{c}_{\alpha_\gamma}) \wedge (\mathbf{x}_{f_{\gamma'}} \wedge \mathbf{c}_{\alpha_{\gamma'}}) = 0$ mod $\mathcal{D}$;
- $\{(f_\gamma, \alpha_\gamma) : \gamma < \kappa\}$ is maximal subject to the preceding conditions.

I claim that $[\mathbf{a}/\mathcal{D}] = \bigvee_{\gamma<\kappa}[\mathbf{x}_{f_\gamma} \wedge \mathbf{c}_{\alpha_\gamma}/\mathcal{D}]$ in $\mathcal{B}_0/\mathcal{D}$ (note $\mathcal{B}_0/\mathcal{D}$ is not necessarily complete, so the join on the right-hand-side is not required to exist in general). Clearly $[\mathbf{a}/\mathcal{D}]$ is an upper bound to $\{[\mathbf{x}_{f_\gamma} \wedge \mathbf{c}_{\alpha_\gamma}/\mathcal{D}] : \gamma < \kappa\}$. If it were not the least upper bound, then we could find $\mathbf{b} \leq \mathbf{a}$ mod $\mathcal{D}$ with $\mathbf{b}$ nonzero mod $\mathcal{D}$, such that each $\mathbf{b} \wedge \mathbf{x}_{f_\gamma} \wedge \mathbf{c}_{\alpha_\gamma} = 0$ mod $\mathcal{D}$. But then choose f, α with $\mathbf{x}_f \wedge \mathbf{c}_\alpha \leq \mathbf{b}$ mod $\mathcal{D}$, contradicting maximality.

Thus it suffices to show that $2^\kappa \leq 2^\lambda$, since then there are only $|P_{\mathbb{C}}|^\kappa \cdot |\mathcal{B}_1|^\kappa \leq 2^\lambda$-many possibilities for $\{(f_\gamma, \alpha_\gamma) : \gamma < \kappa\}$ and this determines $\mathbf{a}$ mod $\mathcal{D}$.

We can suppose $\kappa > \lambda$; then it suffices to show that $\mathcal{B}_1$ has an antichain of size κ. By Theorem 2.3.5, it suffices to show that $\mathcal{B}_1$ has an antichain of size μ, for every regular $\mu \leq \kappa$.

By the Δ-system lemma we can find $I \in [\kappa]^\mu$ such that for all $\gamma, \gamma' \in I$, f_γ and $f_{\gamma'}$ are compatible. Then I claim that $(\mathbf{c}'_{\alpha_\gamma} : \gamma \in I)$ is an antichain of $\mathcal{B}_1$, as desired. Indeed, suppose we had $\gamma < \gamma'$ both in I, with $\mathbf{c}'_{\alpha_\gamma} \wedge \mathbf{c}'_{\alpha_{\gamma'}} \neq 0$. We know that $(\mathbf{x}_{f_\gamma} \wedge \mathbf{c}_{\alpha_\gamma}) \wedge (\mathbf{x}_{f_{\gamma'}} \wedge \mathbf{c}_{\alpha_{\gamma'}}) = 0$ mod $\mathcal{D}$. But this contradicts that $(\mathcal{D}, \mathbb{C})$ is a good pair, since $x_{\alpha_\gamma} \wedge x_{\alpha_{\gamma'}} \in \Sigma_+$ and $\mathbf{x}_{f_\gamma} \wedge \mathbf{x}_{f_{\gamma'}} = \mathbf{x}_{f_\gamma \cup f_{\gamma'}}$, where $f_\gamma \cup f_{\gamma'} \in P_{\mathbb{C}}$. $\square$

Let $\mathbf{X} \subseteq \mathcal{B}_0$ be a choice of representatives for $\mathcal{B}_0/\mathcal{D}_0$; so by Lemma 3.16.14, $|\mathbf{X}| \leq 2^\lambda$.

The following lemma describes our strategy for finishing:

Lemma 3.16.15. It suffices to find good pairs $((\mathcal{D}_\alpha, \mathbb{C}_\alpha) : \alpha < 2^\lambda)$ such that:

1. $(\mathcal{D}_0, \mathbb{C}_0)$ is as already defined;
2. For $\alpha < \beta < 2^\lambda$, $\mathcal{D}_\alpha \subseteq \mathcal{D}_\beta$ and $\mathbb{C}_\beta \subseteq \mathbb{C}_\alpha$;
3. If $\mathbf{A}$ is a λ-distribution in $\mathcal{D}_\alpha$, then for some $\beta > \alpha$, $\mathbf{A}$ has a multiplicative refinement in $\mathcal{D}_\beta$;
4. For each $\mathbf{a} \in \mathbf{X}$, there are some $\alpha, \beta < 2^\lambda$ such that $\mathbf{a} = \mathbf{c}_\beta$ mod $\mathcal{D}_\alpha$.

Proof. Write $\mathcal{D} = \bigcup_{\alpha < 2^\lambda} \mathcal{D}_\alpha$. Since $\mathcal{D}_0$ is strongly λ-regular, so is $\mathcal{D}$. By condition (3),

$\mathcal{D}$ is λ^+-good. By condition (4) and the definition of $\mathbf{X}$, $\mathcal{D}$ satisfies condition (A) of Lemma 3.16.10. Since each $\mathcal{D}_\alpha$ satisfies condition (B) of Lemma 3.16.10, so does $\mathcal{D}$. $\square$

Note that there are only 2^λ many λ-distributions in $\mathcal{B}$; since $|\mathbf{X}| \leq 2^\lambda$, there are only 2^λ-many challenges we have to handle in total. Thus, it suffices to show that we can handle each challenge individually, without using up to much of $\mathbb{C}_0$. Formally, the following two lemmas will finish. Recall that $\mu = \text{c.c.}(\mathcal{B}_1) \leq 2^\lambda$ is regular, so in the terminology of Lemma 3.16.15, if each $\mathbb{C}_\alpha \backslash \mathbb{C}_{\alpha+1}$ has size less than μ, then at limit stages δ we can set $\mathbb{C}_\delta = \bigcap_{\alpha<\delta} \mathbb{C}_\alpha$ and still have $|\mathbb{C}_\delta| \geq 2^\lambda$.

Lemma 3.16.16. Suppose $(\mathcal{E}, \mathbb{D})$ is a good pair, and suppose $\mathbf{a} \in X$. Then there is a good pair $(\mathcal{E}', \mathbb{D}')$ with $\mathcal{E} \subseteq \mathcal{E}'$, $\mathbb{D}' \subseteq \mathbb{D}$ and $|\mathbb{D} \backslash \mathbb{D}'| < \mu$, and such that there is some $\alpha^* < 2^\lambda$ with $\mathbf{a} = \mathbf{c}_{\alpha_*} \bmod \mathcal{D}'$.

Proof. We will try to define by induction on $\gamma < \mu$ filters $\mathcal{E}_\alpha \supseteq \mathcal{E}$, subsets $\mathbb{D}_\alpha \subseteq \mathbb{D}$, and ordinals $\alpha_\gamma < 2^\lambda$ such that:

(a) Each $(\mathcal{E}_\gamma, \mathbb{D}_\gamma)$ is a good pair, and each $\mathbf{c}'_{\alpha_\gamma}$ is nonzero.

(b) $(\mathcal{E}_0, \mathbb{D}_0) = (\mathcal{E}, \mathbb{D})$, and $\gamma < \gamma'$ implies $\mathcal{E}_\gamma \subseteq \mathcal{E}_{\gamma'}$ and $\mathbb{D}_\gamma \supseteq \mathbb{D}_{\gamma'}$ and $\mathbf{c}'_{\alpha_\gamma} \wedge \mathbf{c}'_{\alpha_{\gamma'}} = 0_{\mathcal{B}_1}$.

(c) For limit $\delta < \mu^+$, $\mathbb{D}_\delta = \bigcap_{\gamma<\delta} \mathbb{D}_\gamma$.

(d) For each γ, $\mathbb{D}_\gamma \backslash \mathbb{D}_{\gamma+1}$ is finite.

(e) For each γ, either $\mathbf{c}_{\alpha_\gamma} \leq \mathbf{a} \bmod \mathcal{E}_{\gamma+1}$ or else $\mathbf{c}_{\alpha_\gamma} \wedge \mathbf{a} = 0_{\mathcal{B}_0} \bmod \mathcal{E}_{\gamma+1}$.

Since $\mathcal{B}_1$ has the μ-c.c. we must eventually reach a stage at which we cannot continue. Clearly this must happen at a successor stage, i.e. for some $\gamma_* < \mu$, we have constructed $(\mathcal{E}_\gamma, \mathbb{D}_\gamma : \gamma \leq \gamma_*)$ and $(\alpha_\gamma : \gamma < \gamma_*)$, and we cannot find $(\mathcal{E}_{\gamma_*+1}, \mathbb{D}_{\gamma_*+1}, \alpha_{\gamma_*})$.

Let

$$\mathbf{c}' = \bigvee\{\mathbf{c}'_{\alpha_\gamma} : \gamma < \gamma_* \text{ and } \mathbf{c}_{\alpha_\gamma} \leq \mathbf{a} \bmod \mathcal{E}_{\gamma_*}\}.$$

(We define that the empty join is $0_{\mathcal{B}_1}$.)

Choose α with $\mathbf{c}' = \mathbf{c}'_\alpha$. There are three cases a priori, but only the first will be possible:

Case 1: $\mathbf{a} = \mathbf{c}_\alpha \bmod \mathcal{E}_{\gamma_*}$. Then we are done, with $\mathcal{E}' = \mathcal{E}_{\gamma_*}$, $\mathbb{D}' = \mathbb{D}_{\gamma_*}$, and $\alpha^* = \alpha$.

Case 2: $\mathbf{a} \wedge (\neg \mathbf{c}_\alpha)$ is nonzero mod $\mathcal{E}_{\gamma_*}$. By Lemma 3.16.14 we can choose $f \in P_{\mathbb{D}_{\gamma_*}}$ and $\alpha_{\gamma_*} < 2^\lambda$ so that $\mathbf{c}'_{\alpha_{\gamma_*}}$ is nonzero and $\mathbf{x}_f \cap \mathbf{c}_{\alpha_{\gamma_*}} \leq \mathbf{a} \wedge (\neg \mathbf{c}_\alpha) \bmod \mathcal{E}_{\gamma_*}$. Let $\mathbb{D}_{\gamma_*+1} = \mathbb{D}_{\gamma_*} \backslash \mathrm{dom}(f)$, and let $\mathcal{E}'_{\gamma_*+1}$ be the filter generated by $\mathcal{E}_{\gamma_*}$ and $\mathbf{x}_f$. Then $(\mathcal{E}'_{\gamma_*+1}, \mathbb{D}_{\gamma_*+1})$ is a pre-good pair so we can choose $\mathcal{E}_{\gamma_*+1} \supseteq \mathcal{E}_{\gamma_*+1}$ such that $(\mathcal{E}_{\gamma_*+1}, \mathbb{G}_{\gamma_*+1})$ is a good pair. Then $\mathcal{E}_{\gamma_*+1}, \mathbb{D}_{\gamma_*+1}, \gamma_\alpha$ contradicts that we could not continue.

Case 3: $\mathbf{c}_\alpha \wedge (\neg \mathbf{a})$ is nonzero mod $\mathcal{E}_{\gamma_*}$. Choose $f \in P_{\mathbb{D}_{\gamma_*}}$ and $\alpha_{\gamma_*} < 2^\lambda$ with $\mathbf{x}_f \cap \mathbf{c}_{\alpha_{\gamma_*}} \leq \mathbf{c}_\alpha \wedge (\neg \mathbf{a}) \bmod \mathcal{E}_{\gamma_*}$ and proceed as in Case 2 to get a contradiction. $\square$

We have saved the crux of the argument until the end:

Lemma 3.16.17. Suppose $(\mathcal{E}, \mathbb{D})$ is a good pair, and suppose $\mathbf{A}$ is a λ-distribution in $\mathcal{E}$. Then there is a good pair $(\mathcal{E}', \mathbb{D}')$ with $\mathcal{E} \subseteq \mathcal{E}'$, $\mathbb{D}' \subseteq \mathbb{D}$ and $|\mathbb{D} \backslash \mathbb{D}'| = 1$, and such that $\mathbf{A}$ has a multiplicative refinement in $\mathcal{E}'$.

Proof. Pick $\mathbf{D} \in \mathbb{D}$ and let $\mathbb{D}' = \mathbb{D} \backslash \{\mathbf{D}\}$. Enumerate $\mathbf{D} = \{\mathbf{d}_s : s \in [\lambda]^{<\aleph_0}\}$. For each $s \in [\lambda]^{<\aleph_0}$, let $\mathbf{B}(s) := \bigvee\{\mathbf{A}(t) \wedge \mathbf{d}_t : s \subseteq t \in [\lambda]^{<\aleph_0}\}$. Clearly this is a λ-distribution in $\mathcal{B}_0$, refining $\mathbf{A}$.

I claim that $\mathbf{B}$ is multiplicative; let $s \in [\lambda]^{<\aleph_0}$. Suppose towards a contradict $\mathbf{e} := \left(\bigwedge_{\alpha \in s} \mathbf{B}_{\{\alpha\}}\right) \wedge (\neg \mathbf{B}(s))$ were nonzero. Then we can find $\mathbf{e}' \leq \mathbf{e}$ nonzero, and $(s_\alpha : \alpha \in s)$ a sequence from $[\lambda]^{<\aleph_0}$, such that each $\alpha \in s_\alpha$, and such that $\mathbf{e}' \leq \mathbf{A}(s_\alpha) \wedge \mathbf{d}_{s_\alpha}$ for each

$\alpha \in s$. Since $\mathbf{D}$ is an antichain this implies $s_\alpha = s_{\alpha'} = t$ say, for all $\alpha, \alpha' \in s$. Visibly then $s \subseteq t$, and so $\mathbf{e}' \leq \mathbf{A}(t) \wedge \mathbf{d}_t$, contradicting that $\mathbf{e}' \wedge \mathbf{B}(s) = 0$.

Let $\mathcal{E}'_0$ be the filter generated by $\mathcal{E}$ and $(\mathbf{B}(s) : s \in [\lambda]^{<\aleph_0})$. I claim that $(\mathcal{E}'_0, \mathbb{D}')$ is pre-good, which suffices. So suppose towards a contradiction it were not; then we could find $f \in P_{\mathbb{D}'}$ and $\sigma(\overline{x}) \in \Sigma_+$ such that $\mathbf{x}_f \wedge \sigma(\overline{\mathbf{c}}) = 0 \mod \mathcal{E}'_0$. Then we can find $s \in [\lambda]^{<\aleph_0}$ such that $\mathbf{x}_f \wedge \sigma(\overline{\mathbf{c}}) \wedge \mathbf{B}(s) = 0 \mod \mathcal{E}$. Thus $\mathbf{x}_f \wedge \sigma(\overline{\mathbf{c}}) \wedge \mathbf{A}(s) \wedge \mathbf{d}_s = 0 \mod \mathcal{E}$. But since $\mathbf{A}(s) \in \mathcal{E}$, if we set $g = f \cup \{(\mathbf{D}, \mathbf{d}_s)\} \in P_{\mathbb{D}}$, then this implies that $\mathbf{x}_g \wedge \sigma(\overline{\mathbf{c}}) = 0 \mod \mathcal{E}$, contradicting that $(\mathcal{E}, \mathbb{D})$ is a good pair. $\square$

As mentioned above, using Lemmas 3.16.16 , 3.16.17 it is now straightforward to meet the requirements of Lemma 3.16.15; thus this finishes the proof of Theorem 3.16.5.

We remark on the following consequence of what we have done so far. As mentioned above, Kunen proved this in the special case of $\mathcal{B}_0 = \mathcal{P}(\lambda)$ in [45].

Corollary 3.16.18. Suppose $\mathcal{B}_0$ is a complete Boolean algebra with an antichain of size λ. Then there is a strongly λ-regular, λ^+-good ultrafilter on $\mathcal{B}_0$.

Proof. Apply Theorems 3.16.5 to the special case where $\mathcal{B}_0 = \mathcal{P}(\lambda)$, $\mathcal{B}_1 = \{0, 1\}$, and $\mathcal{U}_1$ is the unique ultrafilter on $\{0, 1\}$ (which is in particular λ^+-good). To see that the obtained $\mathcal{U}_0$ is λ^+-good, apply Theorem 3.7.2 and Theorem 3.16.4. $\square$

3.16.3 Downward Löwenheim-Skolem for Saturation

We now finish the proof of Theorem 3.16.1.

The following is the only genuinely new piece of the proof:

Theorem 3.16.19. Suppose λ is a cardinal, $\mathcal{B}$ is a complete Boolean algebra such that $2^{<\mathrm{c.c.}(\mathcal{B})} \leq 2^\lambda$. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then there is a complete subalgebra $\mathcal{B}_*$

of $\mathcal{B}$ with $|\mathcal{B}_*| \leq 2^\lambda$, such that $\mathcal{U} \cap \mathcal{B}_*$ λ^+-saturates exactly the same theories as $\mathcal{U}$.

Proof. We use the characterization of λ^+-saturation given by Theorem 3.5.14.

Write $\mu = \text{c.c.}(\mathcal{B})$. Note that $\text{cof}(2^\lambda) \geq \mu$: to see this, consider three cases. First, if $2^{<\mu} < 2^\lambda$, then necessarily $\mu \leq \lambda$, so conclude by König's theorem (always $\text{cof}(2^\lambda) > \lambda$). If for all $\kappa < \mu$, $2^\kappa < 2^\lambda$, then visibly $(2^\kappa : \kappa < \mu)$ is a cofinal sequence in 2^λ, and since μ is regular we conclude $\text{cof}(2^\lambda) \geq \mu$. Finally, suppose there is some $\kappa < \mu$ with $2^\kappa = 2^\lambda$. Then for all κ' with $\kappa \leq \kappa' < \mu$, $2^{\kappa'} = 2^\lambda$; thus for each such κ', $\text{cof}(2^\lambda) = \text{cof}(2^{\kappa'}) > \kappa'$; thus $\text{cof}(2^\lambda) \geq \mu$.

Note that for every $X \subseteq \mathcal{B}$ with $|X| \leq 2^\lambda$, if we let $\mathcal{B}(X)$ be the complete subalgebra of $\mathcal{B}$ generated by X, then $|\mathcal{B}(X)| \leq |X|^{<\mu} \leq 2^\lambda$. Also, since $\text{cof}(2^\lambda) \geq \mu$, whenever $(\mathcal{B}_\alpha : \alpha < 2^\lambda)$ is an increasing sequence of complete subalgebras of $\mathcal{B}$, we have that $\bigcup_{\alpha < 2^\lambda} \mathcal{B}_\alpha$ is a complete subalgebra of $\mathcal{B}$.

Choose $X_0 \subseteq \mathcal{B}$ with $|X_0| \leq 2^\lambda$ such that for every complete theory T in a countable language, if $\mathcal{U}$ does not λ^+-saturate T, then there is some (λ, T)-Łoś map $\mathbf{A}$ in $\mathcal{U}$ with no multiplicative refinement in $\mathcal{U}$, such that $\text{im}(\mathbf{A}) \subseteq X_0$. Then choose an increasing sequence $(\mathcal{B}_\alpha : \alpha < 2^\lambda)$ of complete subalgebras of $\mathcal{B}$, such that $X_0 \subseteq \mathcal{B}_0$, and each $|\mathcal{B}_\alpha| \leq 2^\lambda$, and for each $\alpha < 2^\lambda$ and each λ-distribution $\mathbf{A}$ in $\mathcal{U} \cap \mathcal{B}_\alpha$, if $\mathbf{A}$ has a multiplicative refinement in $\mathcal{U}$ then it has one in $\mathcal{U} \cap \mathcal{B}_{\alpha+1}$. (There are only $|\mathcal{B}_\alpha|^\lambda \leq 2^\lambda$-many distributions to handle.)

Write $\mathcal{B}_* = \bigcup_{\alpha < 2^\lambda} \mathcal{B}_\alpha$. Then $\mathcal{B}_*$ is a complete subalgebra of $\mathcal{B}$ and and $|\mathcal{B}_*| \leq 2^\lambda$. Write $\mathcal{U}_* = \mathcal{U} \cap \mathcal{B}_*$. Since $X_0 \subseteq \mathcal{B}_*$, we get that if $\mathcal{U}$ does not λ^+-saturate T, then neither does $\mathcal{U}_*$. Suppose on the other hand that $\mathcal{U}$ λ^+-saturates T, and that $\mathbf{A}$ is a (λ, T)-Łoś map in $\mathcal{U}_*$. Then $\mathbf{A}$ is in some $\mathcal{U} \cap \mathcal{B}_\alpha$ for some $\alpha < 2^\lambda$, since $\text{cof}(2^\lambda) > \lambda$. Thus $\mathbf{A}$ has a multiplicative refinement in $\mathcal{U} \cap \mathcal{B}_{\alpha+1} \subseteq \mathcal{U}_*$. $\square$

Note then that Theorem 3.16.1 follows immediately from Theorems 3.16.4, 3.16.5

and 3.16.19.

As a corollary, we get the following:

Corollary 3.16.20. Let λ be a cardinal and let T_0, T_1 be theories. The following are equivalent:

(A) For every λ-regular ultrafilter on $\mathcal{P}(\lambda)$, if $\mathcal{U}$ λ^+-saturates T_1 then $\mathcal{U}$ λ^+-saturates T_0.

(B) There is a complete Boolean algebra $\mathcal{B}$ with an antichain of size λ, such that for every strongly λ-regular ultrafilter $\mathcal{U}$ on $\mathcal{B}$, if $\mathcal{U}$ λ^+-saturates T_1 then $\mathcal{U}$ λ^+-saturates T_0.

(C) For every complete Boolean algebra $\mathcal{B}$ with $2^{<\mathrm{c.c.}(\mathcal{B})} \leq 2^\lambda$, and for every ultrafilter $\mathcal{U}$ on $\mathcal{B}$, if $\mathcal{U}$ λ^+-saturates T_1 then $\mathcal{U}$ λ^+-saturates T_0;

(D) For every complete Boolean algebra $\mathcal{B}$ with the λ^+-c.c., and for every ultrafilter $\mathcal{U}$ on $\mathcal{B}$, if $\mathcal{U}$ λ^+-saturates T_1 then $\mathcal{U}$ λ^+-saturates T_0.

Note that (A) is the definition of $\trianglelefteq_\lambda$, so Theorem 3.5.5 holds as promised.

3.17 Cuts and Treetops

In this section, we will survey the key theorem of Malliaris and Shelah from [54]. In particular, we will show that for all ω-nonstandard $\hat{V} \models ZFC^-$, $\mathfrak{p}_{\hat{V}} = \mathfrak{t}_{\hat{V}}$; then we will discuss combinatorial characteristics of the continuum, and prove that $\mathfrak{p} = \mathfrak{t}$.

3.17.1 $\mathfrak{p}_{\hat{V}} = \mathfrak{t}_{\hat{V}}$

Fix, for the entirety of this subsection, some ω-nonstandard $\hat{V} \models ZFC^-$. We aim to show $\mathfrak{p}_{\hat{V}} = \mathfrak{t}_{\hat{V}}$.

We begin with the following theorem; it corresponds to Theorem 3.1 of [54].

Theorem 3.17.1. Suppose $\kappa < \min(\mathfrak{p}^+_{\hat{V}}, \mathfrak{t}_{\hat{V}})$ is regular. Then there is a unique regular cardinal λ with $(\kappa, \lambda) \in \mathcal{C}_{\hat{V}}$; moreover this λ is also unique with the property that $(\lambda, \kappa) \in \mathcal{C}_{\hat{V}}$.

Note that if $\kappa < \mathfrak{p}_{\hat{V}}$ then necessarily $\lambda \geq \mathfrak{p}_{\hat{V}}$.

Proof. We first show that there exist λ_0, λ_1 regular cardinals with $(\kappa, \lambda_0) \in \mathcal{C}_{\hat{V}}$ and $(\lambda_1, \kappa) \in \mathcal{C}_{\hat{V}}$. We will then show that $\lambda_0 = \lambda_1$, which suffices to prove the theorem.

For λ_0: pick $\hat{n}_*$ nonstandard, and note that by Lemma 3.9.5 applied to the tree $(\hat{n}_*, \hat{<})$ we can choose $(\hat{n}_\alpha : \alpha < \kappa)$ a strictly increasing sequence below $\hat{n}_*$. Let $(\hat{m}_\beta : \beta < \beta_*)$ be any strictly decreasing sequence in $(\hat{n}_*, \hat{<})$, cofinal above $(\hat{n}_\alpha : \alpha < \kappa)$, and then discard elements to replace β_* by $\mathrm{cof}(\beta_*) =: \lambda_0$.

For λ_1: I claim that we can define $(\hat{n}'_\alpha : \alpha < \kappa)$, a strictly decreasing sequence of nonstandard numbers from $\hat{\omega}$. To see that we can do this: first let $\hat{n}'_0$ be an arbitrary nonstandard natural number. Having defined $\hat{n}'_\alpha$, let $\hat{n}'_{\alpha+1} = \hat{n}'_\alpha - 1$. Having defined $\hat{n}'_\alpha$ for all $\alpha < \delta$ where $\delta < \kappa$ is a limit, consider the pre-cut $(n : n < \omega), (\hat{n}'_\alpha : \alpha < \delta)$. Since $\omega + \delta < \kappa \leq \mathfrak{p}_{\hat{V}}$ this cannot be a cut, so choose $\hat{n}'_\delta$ in the gap. Having constructed $\hat{n}'_\alpha$ for each $\alpha < \kappa$, we can as in the previous paragraph choose a regular λ_1 and a strictly increasing sequence $(\hat{m}'_\gamma : \gamma < \lambda_1)$, cofinal below $(\hat{n}'_\alpha : \alpha < \kappa)$.

Now to show $\lambda_0 = \lambda_1$: first, by possibly increasing $\hat{n}_*$, we can suppose $\hat{n}_* > \hat{n}'_0$, and thus each $\hat{n}_\alpha, \hat{n}'_\alpha, \hat{m}_\beta, \hat{m}'_\beta < \hat{n}_*$. Let $(\hat{T}, \hat{<})$ be the tree of all sequences $s \in (\hat{n}_* \times \hat{n}_*)^{<\hat{n}_*}$, such that that for all $\hat{n} < \hat{m} < \hat{\mathrm{lg}}(s)$, $s(\hat{n})(0) < s(\hat{m})(0) < s(\hat{m})(1) < s(\hat{n})(0)$.

We now choose a strictly increasing sequence $(s_\alpha : \alpha < \kappa)$ from $\hat{T}$ such that for each $\alpha < \kappa$, if we set $\hat{a}_\alpha = \hat{\mathrm{lg}}(s_\alpha)$, then $s_\alpha(\hat{a}_\alpha - 1) = (\hat{n}_\alpha, \hat{n}'_\alpha)$. Let $s_0 = \emptyset$; having defined s_α, let $s_{\alpha+1} = s_\alpha{}^\frown(\hat{n}_{\alpha+1}, \hat{n}'_{\alpha+1})$. Finally, having defined s_α for each $\alpha < \delta$ for $\delta < \kappa$ limit, since $|\delta| < \mathfrak{t}_{\hat{V}}$ we can choose s_+ an upper bound for $(s_\alpha : \alpha < \delta)$. Let $\hat{n}$ be greatest so that $s_+(\hat{n})(0) < \hat{n}_\delta$ and $s_+(\hat{n})(1) > \hat{n}'_\delta$; let $s_\delta = s_+ \restriction_{\hat{n}} {}^\frown(\hat{n}_\delta, \hat{n}'_\delta)$.

Since $\kappa < \mathfrak{t}_{\hat{V}}$ we can choose s an upper bound on $(s_\alpha : \alpha < \kappa)$. Let $\hat{b}_0 = \hat{\lg}(s)$, and choose λ regular and $(\hat{b}_i : i < \lambda)$ a strictly decreasing sequence from $\hat{\omega}$, cofinal above $(\hat{a}_\alpha : \alpha < \kappa)$.

Then the sequences $(\hat{m}_\alpha : \alpha < \lambda_0)$ and $(s(\hat{b}_i, 0) : i < \lambda)$ are cofinal in each other, so $\lambda_0 = \lambda$; and the sequences $(\hat{m}'_\alpha : \alpha < \lambda_0)$ and $(s(\hat{b}_i, 1) : i < \lambda)$ are cofinal in each other, so $\lambda_1 = \lambda$. $\square$

For the following definition, we will eventually be proving that $\min(\mathfrak{p}^+_{\hat{V}}, \mathfrak{t}_{\hat{V}}) = \mathfrak{p}_{\hat{V}} = \mathfrak{t}_{\hat{V}}$.

Definition 3.17.2. For $\kappa < \min(\mathfrak{p}^+_{\hat{V}}, \mathfrak{t}_{\hat{V}})$ regular, define $\mathrm{lcf}_{\hat{V}}(\kappa)$ to be the unique regular λ with $(\kappa, \lambda) \in \mathcal{C}_{\hat{V}}$ (which is also the unique regular λ with $(\lambda, \kappa) \in \mathcal{C}_{\hat{V}}$).

We can now describe our strategy for showing $\mathfrak{p}_{\hat{V}} = \mathfrak{t}_{\hat{V}}$. Namely, we will show that whenever $\kappa \leq \mathfrak{p}_{\hat{V}} < \mathfrak{t}_{\hat{V}}$, then $(\kappa, \mathfrak{p}_{\hat{V}}) \notin \mathcal{C}(\hat{\omega}, \hat{V})$. This will suffice for a contradiction since if $\mathfrak{p}_{\hat{V}} < \mathfrak{t}_{\hat{V}}$ then necessarily $\kappa = \mathrm{lcf}_{\hat{V}}(\mathfrak{p}_{\hat{V}}) \leq \mathfrak{p}_{\hat{V}}$ and $(\kappa, \mathfrak{p}_{\hat{V}}) \in \mathcal{C}_{\hat{V}}$.

The following easy case corresponds to Lemma 6.1 of [54].

Lemma 3.17.3. Suppose $\mathfrak{p}_{\hat{V}} < \mathfrak{t}_{\hat{V}}$. Write $\kappa = \mathfrak{p}_{\hat{V}}$. Then $(\kappa, \kappa) \notin \mathcal{C}_{\hat{V}}$.

Proof. Suppose it were, say via $(\hat{a}_\alpha : \alpha < \kappa)$, $(\hat{b}_\alpha : \alpha < \kappa)$. Write $\hat{n}_* = \hat{b}_0 + 1$. Let $(\hat{T}, \hat{<})$ be the tree of all sequences s in $(\hat{n}_* \times \hat{n}_*)^{<\hat{n}_*}$ such that for all $\hat{n} < \hat{m} < \hat{\lg}(s)$, $s(\hat{n})(0) < s(\hat{n})(1) < s(\hat{m})(1) < s(\hat{m})(0)$. Using the techniques of the previous proofs it is easy to define $(s_\alpha : \alpha < \kappa)$ an increasing sequence from $\hat{T}$ such that if we set $\hat{n}_\alpha = \hat{\lg}(s_\alpha)$, then $s_\alpha(\hat{n}_\alpha - 1) = (\hat{a}_\alpha, \hat{b}_\alpha)$. Then since $\kappa < \mathfrak{t}_{\hat{V}}$ is regular we can choose an upper bound s for $(s_\alpha : \alpha < \kappa)$. Then $s(\hat{\lg}(s) - 1)(0)$ is in the gap $(\hat{a}_\alpha : \alpha < \kappa), (\hat{b}_\alpha : \alpha < \kappa)$; but this was supposed to be a cut. $\square$

It will be helpful if we articulate the following lemma.

Lemma 3.17.4. Suppose $\gamma_* < \mathfrak{p}_{\hat{V}}$, and $(n_\gamma : \gamma < \hat{\gamma}_*)$ is a sequence of distinct elements from $\hat{\omega}$. Suppose $\hat{e}$ is an element of $\hat{\omega}$. Suppose $\hat{D} \subseteq \hat{\omega}$ is finite in $\hat{V}$, with each $\hat{n}_\gamma \in \hat{D}$, and suppose $\hat{g} : [\hat{D}]^2 \to \hat{\omega}$ is such that for all $\gamma < \gamma' < \gamma_*$, $\hat{g}(\hat{n}_\gamma, \hat{n}_{\gamma'}) < \hat{e}$.

Then there is some $\hat{D}' \hat{\subseteq} \hat{D}$ such that each $\hat{n}_\gamma \in \hat{D}'$, and for all $\hat{n} < \hat{m} < \hat{D}'$, $\hat{g}(\hat{n}, \hat{m}) < \hat{e}$.

Proof. By Theorem 3.9.6, applied to $X = \mathcal{P}(\hat{D})$. □

The following simple remark is Fact 8.4 of [54].

Lemma 3.17.5. For each κ, there is a function $g : [\kappa^+]^2 \to \kappa$, such that whenever $X \subset \kappa^+$ is unbounded, the image of $[X]^2$ under g is unbounded in κ.

Proof. For each $\alpha < \kappa^+$ let $g_\alpha : \alpha \to |\alpha|$ be a bijection; for $\alpha' < \alpha < \kappa^+$ let $g(\alpha', \alpha) = g_\alpha(\alpha')$. □

Finally we show that whenever $\kappa < \lambda = \mathfrak{p}_{\hat{V}} < \mathfrak{t}_{\hat{V}}$, then $(\kappa, \lambda) \notin \mathcal{C}_{\hat{V}}$. This is Theorem 8.1 of [54].

Lemma 3.17.6. Suppose $\mathfrak{p}_{\hat{V}} < \mathfrak{t}_{\hat{V}}$. Write $\lambda = \mathfrak{p}_{\hat{V}}$ and suppose $\kappa < \lambda$. Then (κ, λ) is not in $\mathcal{C}_{\hat{V}}$.

Proof. Let $(\hat{a}_\alpha : \alpha < \kappa)$, $(\hat{b}_\beta : \beta < \lambda)$ be a (κ, λ)-cut. Also, choose $g : [\kappa^+]^2 \to \kappa$ as in Lemma 3.17.5. Let $\hat{N}_* = \hat{b}_0 + 1$, and choose a sequence of distinct elements $(\hat{n}_\gamma : \gamma < \kappa^+)$ below $\hat{N}_*$.

We define a special tree $\hat{T}$.

First let $\hat{Q}$ be the set of all triples $(\hat{e}, \hat{D}, \hat{g})$ satisfying:

- $\hat{e} < \hat{N}_*$;

- $\hat{D} \subseteq \hat{N}_*$;

- $\hat{g} : [\hat{D}]^2 \to \hat{e}$.

So $\hat{Q}$ is finite in $\hat{V}$. Let $\hat{T}$ be the subtree of $\hat{Q}^{<\hat{N}_*}$ consisting of all s such that, if we set $\hat{n}^* = \hat{\lg}(s)$, and if we set $s(\hat{n}) = (\hat{e}^{\hat{n}}, \hat{D}^{\hat{n}}, \hat{g}^{\hat{n}})$ for $\hat{n} < \hat{n}^*$, then:

- For all $\hat{n} < \hat{m} < \hat{n}^*$, $\hat{e}^{\hat{m}} < \hat{e}^{\hat{n}}$;

- if $\hat{n} + 1 < \hat{n}^*$, and if $\hat{a} < \hat{b}$ are both in $\hat{D}^{\hat{n}} \cap \hat{D}^{\hat{n}+1}$, then $g^{\hat{n}}(\hat{a}, \hat{b}) = g^{\hat{n}+1}(\hat{a}, \hat{b})$.

We now break into two cases, depending on whether $\kappa^+ = \lambda$. The first case can be viewed as a warm-up to the second case.

Case 1. $\kappa^+ < \lambda$.

We choose an increasing sequence $(s_\beta : \beta < \lambda)$ from $\hat{T}$ such that if we set $\hat{d}_\beta = \hat{\lg}(s_\beta)$ and if, for $\hat{n} < \hat{d}_\beta$, we set $s_\beta(\hat{n}) = (\hat{e}^{\hat{n}}, \hat{D}^{\hat{n}}, \hat{g}^{\hat{n}})$ (which doesn't depend on the choice of β), then:

- For all $\beta < \lambda$, $\hat{e}^{\hat{d}_\beta - 1} = \hat{b}_\beta$;

- For all $\beta < \lambda$ and for all $\hat{n} < \hat{d}_\beta$ and for all $\gamma < \kappa^+$, $\hat{n}_\gamma \in \hat{D}^{\hat{n}}$;

- For all $\beta < \lambda$ and for all $\hat{n} < \hat{d}_\beta$ and for all $\gamma < \gamma' < \kappa^+$, $\hat{g}^{\hat{n}}(\hat{n}_\gamma, \hat{n}'_\gamma) = \hat{a}_{g(\gamma,\gamma')}$.

For $\beta = 0$: by Theorem 3.9.6 we can choose some $\hat{D}, \hat{g}$ such that $\hat{D} \subseteq \hat{N}_*$ with each $\hat{n}_\gamma \in \hat{D}$, and $\hat{g} : [\hat{D}]^2 \to \hat{b}_0$, and for each $\gamma < \gamma' < \kappa^+$, $\hat{g}(\hat{n}_\gamma, \hat{n}_{\gamma'}) = \hat{a}_{g(\gamma,\gamma')}$. Let s_0 be the sequence of length 1 with $s_0(1) = (\hat{b}_0, \hat{D}, \hat{g})$.

Suppose we have defined s_β; write $s_\beta(\hat{d}_\beta - 1) = (\hat{b}_\beta, \hat{D}, \hat{g})$. Apply Lemma 3.17.4 to get $\hat{D}' \hat{\subseteq} \hat{D}$ so that if we set $\hat{g}' := \hat{g} \restriction_{[\hat{D}']^2}$, then $\hat{g}' : [\hat{D}']^2 \to \hat{b}_{\beta+1}$. Let $s_{\beta+1} = s_\beta{}^\frown(\hat{a}_{\beta+1}, \hat{D}', \hat{g}')$.

Suppose we have defined s_β for all $\beta < \delta$, δ limit; since $|\delta| < \mathfrak{t}_{\hat{V}}$ we can choose an upper bound s for $(s_\beta : \beta < \delta)$. For $\hat{n} < \hat{\lg}(s)$ write $s(\hat{n}) = (\hat{e}^{\hat{n}}, \hat{D}^{\hat{n}}, \hat{g}^{\hat{n}})$. Using that $\mathfrak{p}_{\hat{V}} > \kappa^+$ we can arrange that for all $\hat{n} < \hat{\lg}(s)$, $\hat{e}^{\hat{n}} > \hat{b}_\delta$ and for all $\hat{n} < \hat{\lg}(s)$, for all $\gamma < \kappa^+$, $\hat{n}_\gamma \in \hat{D}^{\hat{n}}$. Let $\hat{n} = \hat{\lg}(s) - 1$; it follows by induction in $\hat{V}$ that $\hat{g}^{\hat{n}}$ takes on the right values on $\hat{n}_\gamma : \gamma < \kappa^+$. So we can now proceed as in the successor case.

Case 2. $\kappa^+ = \lambda$.

We choose an increasing sequence $(s_\beta : \beta < \lambda)$ from $\hat{T}$ such that if we set $\hat{d}_\beta = \hat{\lg}(s_\beta)$ and if, for $\hat{n} < \hat{d}_\beta$, we set $s_\beta(\hat{n}) = (\hat{e}^{\hat{n}}, \hat{D}^{\hat{n}}, \hat{g}^{\hat{n}})$ (which doesn't depend on the choice of β), then:

- For all $\beta < \lambda$, $\hat{e}^{\hat{d}_\beta - 1} = \hat{b}_\beta$;
- For all $\beta < \lambda$ and for all $\hat{n} < \hat{d}_\beta$ and for all $\gamma < \lambda$, if $\hat{n} \geq \hat{d}_\gamma - 1$ then $\hat{n}_\gamma \in \hat{D}^{\hat{n}}$;
- For all $\beta < \lambda$ and for all $\hat{n} < \hat{d}_\beta$ and for all $\gamma < \gamma' < \lambda$, if $\hat{n} \geq \hat{d}_{\gamma'} - 1$ then $\hat{g}^{\hat{n}}(\hat{n}_\gamma, \hat{n}'_\gamma) = \hat{a}_{g(\gamma,\gamma')}$.

Getting this sequence is essentially the same as in Case 1. For instance, in the successor case: suppose we have defined s_β. Write $s_\beta(\hat{d}_\beta - 1) = (\hat{b}_\beta, \hat{D}, \hat{g})$. We can suppose $\hat{n}_{\beta+1}$ is not in $\mathcal{D}$, since if $\hat{n}_{\beta+1}$ is in $\hat{D}$ then we can just revise our choice of s_β so that $s_\beta(\hat{d}_\beta - 1) = (\hat{b}_\beta, \hat{D} \backslash \{\hat{n}_{\beta+1}\}, \hat{g} \restriction_{[\hat{D} \backslash \{\hat{n}_{\beta+1}\}]^2})$. Then let $\hat{D}' = \hat{D} \cup \{\hat{n}_{\beta+1}\}$; we can choose $\hat{g}' : [\hat{D}']^2 \to \hat{b}_\beta$ with the correct values on $[\{\hat{n}_\gamma : \gamma \leq \beta + 1\}]^2$ by Theorem 3.9.6, and then we can proceed as in the successor case of Case 1.

The cases now rejoin (really we could have done it in one case, but this seems clearer); we have defined $(s_\alpha : \alpha < \lambda)$ so that if we set $\hat{d}_\beta = \hat{\lg}(s_\beta)$ and if, for $\hat{n} < \hat{d}_\beta$, we set $s_\beta(\hat{n}) = (\hat{e}^{\hat{n}}, \hat{D}^{\hat{n}}, \hat{g}^{\hat{n}})$, then:

- For all $\beta < \lambda$, $\hat{e}^{\hat{d}_\beta - 1} = \hat{b}_\beta$;

- For all $\beta < \lambda$ and for all $\hat{n} < \hat{d}_\beta$ and for all $\gamma < \kappa^+$, if $\hat{n} \geq \hat{d}_\gamma - 1$ then $\hat{n}_\gamma \in \hat{D}^{\hat{n}}$;

- For all $\beta < \lambda$ and for all $\hat{n} < \hat{d}_\beta$ and for all $\gamma < \gamma' < \kappa^+$, if $\hat{n} \geq \hat{d}_{\gamma'} - 1$ then $\hat{g}^{\hat{n}}(\hat{n}_\gamma, \hat{n}'_\gamma) = \hat{a}_{g(\gamma,\gamma')}$.

By λ-tree-tops we can choose an upper bound $s \in \hat{T}$ for $(s_\alpha : \alpha < \lambda)$. Let $\hat{k}_0 = \hat{\lg}(s)$ and for $\hat{n} < \hat{k}_0$ write $s(\hat{n}) = (\hat{e}^{\hat{n}}, \hat{D}^{\hat{n}}, \hat{g}^{\hat{n}})$ (this doesn't conflict with the previous definitions).

Choose a decreasing sequence $(\hat{k}_\alpha : \alpha < \kappa)$ from $\hat{N}_*$ so that $(\hat{d}_\beta : \beta < \lambda), (\hat{k}_\alpha : \alpha < \kappa)$ is a cut; this is possible by uniqueness of $\mathrm{lcf}_{\hat{V}}(\lambda) = \kappa$. For each $\alpha < \kappa$, let Γ_α be the set of all $\gamma < \kappa^+$ such that for every $\hat{n} < \hat{k}_\alpha$ with $\hat{n} \geq \hat{d}_\gamma - 1$, $\hat{n}_\gamma \in \hat{D}^{\hat{n}}$. Then $\Gamma_\alpha : \alpha < \kappa$ is an increasing sequence of subsets of κ^+ with union κ^+; hence there must be some $\alpha < \kappa$ with $|\Gamma_\alpha| = \kappa^+$. Let $\alpha' < \kappa$ be large enough so that $\hat{e}^{\hat{k}_\alpha} \leq \hat{a}_{\alpha'}$ (if there were no such α' then $(\hat{a}_\alpha : \alpha < \kappa), (\hat{b}_\beta : \beta < \lambda)$ wouldn't be a cut).

Now by choice of g, there are $\gamma < \gamma' \in \Gamma_\alpha$ with $g(\gamma, \gamma') \geq \alpha'$. But then $\hat{g}^{\hat{k}_\alpha}(\hat{n}_\gamma, \hat{n}_{\gamma'}) = \hat{a}_{g(\gamma,\gamma')} \geq \hat{a}_{\alpha'} \geq \hat{e}^{\hat{k}_\alpha}$, a contradiction. □

This concludes the proof that $\mathfrak{p}_{\hat{V}} = \mathfrak{t}_{\hat{V}}$, and hence of Theorem 3.9.9.

3.17.2 $\mathfrak{p} = \mathfrak{t}$

In [54], Malliaris and Shelah apply their results on cofinality spectrum problems to solve the oldest open problem on cardinal invariants of the continuum; namely, they showed that $\mathfrak{p} = \mathfrak{t}$. We give their argument now; we streamline matters slightly so as to avoid reference to a hard theorem of Shelah [76].

We begin with the relevant definitions:

Definition 3.17.7. • Given $f, g \in \omega^\omega$ say that $f \leq_* g$ if $\{n : f(n) \leq g(n)\}$ is cofinite and say that $f <_* g$ if $\{n : f(n) < g(n)\}$ is cofinite. Warning: $f <_* g$ is stronger than saying $f \leq_* g$ and $g \not\leq_* f$, although in the arguments below this will not matter.

- Let $\mathfrak{b}$ be the least cardinality of a $<_*$-unbounded subset of ω^ω (which is the same as the least cardinality of a $\leq_*$-unbounded subset of ω^ω).
- Given $X, Y \subset \omega$, say that $X \subseteq_* Y$ if $X \backslash Y$ is finite.
- Given $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ say that $\mathcal{B}$ has the strong finite intersection property if the intersection of finitely many elements from $\mathcal{B}$ is infinite. Say that $\mathcal{B}$ has a pseudo-intersection if there is some infinite $X \subset \omega$ with $X \subseteq_* B_\alpha$ for each $\alpha < \kappa$.
- Let $\mathfrak{p}$ be the least cardinality of a familiy $\mathcal{B}$ of subsets of ω with the strong finite intersection property but without an infinite pseudo-intersection.
- Say that $(X_\alpha : \alpha < \kappa)$ is a tower if $\alpha < \beta < \kappa$ implies $X_\alpha \supseteq_* X_\beta$.
- Let $\mathfrak{t}$ be the least cardinality of a tower with no pseudo-intersection.

Obviously $\mathfrak{p} \leq \mathfrak{t}$; and it is well-known that $\mathfrak{t} \leq \mathfrak{b}$. See [90] for a survey on the classical theory.

We will want the following definition.

Definition 3.17.8. Suppose $\hat{V} \models ZFC^-$ is ω-nonstandard. Then let $\mathfrak{b}_{\hat{V}}$ be the least cardinality of a unbounded subset of $\hat{\omega}$ (or equivalently, of a strictly increasing cofinal sequence in $\hat{\omega}$; thus $\mathfrak{b}_{\hat{V}}$ is regular).

The following is the connection between $\mathfrak{b}$ and $\mathfrak{b}_{\hat{V}}$.

Lemma 3.17.9. Suppose $\mathcal{U}$ is a nonprincipal ultrafilter on ω and suppose $V \models ZFC^-$ is transitive; let $\hat{V} = V^\omega/\mathcal{U}$. Then $\mathfrak{b} \leq \mathfrak{b}_{\hat{V}}$. Hence $\mathfrak{t} \leq \mathfrak{b}_{\hat{V}}$.

Proof. Suppose $\lambda < \mathfrak{b}$ and $\{[f_\alpha]_{\mathcal{D}} : \alpha < \lambda\}$ is a sequence from $\hat{\omega}$. Choose a $<_*$-upper bound $f \in \omega^\omega$ for $\{f_\alpha : \alpha < \lambda\}$. Then clearly $[f_\alpha]_{\mathcal{U}} < [f]_{\mathcal{U}}$ for each $\alpha < \lambda$.

The latter part follows since $\mathfrak{t} \leq \mathfrak{b}$; see [90]. $\square$

To begin making connections with the previous section we observe the following lemma.

Lemma 3.17.10. Suppose $\hat{V} \models ZFC^-$ is ω-nonstandard. Then the following are equivalent:

(A) $\lambda < \mathfrak{p}_{\hat{V}}$.

(B) $\lambda < \mathfrak{t}_{\hat{V}}$.

(C) Whenever $\hat{n} < \hat{\omega}$ and $(\hat{a}_\alpha : \alpha < \lambda)$ is a family from $\mathcal{P}(\hat{n})$ with the finite intersection property (i.e. given $\alpha_0, \ldots, \alpha_{n-1} \in \lambda$, $\hat{a}_{\alpha_0} \hat{\cap} \ldots \hat{\cap} \hat{a}_{\alpha_{n-1}}$ is nonempty), if each $|\hat{a}_\alpha|$ is nonstandard, then there is $\hat{a} \subseteq \hat{n}$ with $|\hat{a}|$ nonstandard, such that $\hat{a} \subseteq \hat{a}_\alpha$ for each $\alpha < \lambda$.

(D) Whenever $(\hat{a}_\alpha : \alpha < \lambda)$ is a descending sequence of nonempty sets from $[\hat{\omega}]^{<\hat{\aleph}_0}$, there is some $\hat{m} < \hat{\omega}$ such that $\hat{m} \in \hat{a}_\alpha$ for each $\alpha < \lambda$.

Proof. (A) and (B) are equivalent by Theorem 3.9.9, and they imply the other items by Theorem 3.9.6. Also, clearly (C) implies (D). So it suffices to show that (D) implies (B).

Suppose $(s_\alpha : \alpha < \lambda)$ is an increasing sequence from $\hat{n}^{<\hat{n}}$. Let $\hat{a}_\alpha = \{s \in \hat{n}^{\hat{n}-1} : s_\alpha \subseteq s\}$. Then by (D) (and applying an injection from $\hat{n}^{\hat{n}-1}$ to $\hat{n}'$ for large enough $\hat{n}'$) we can choose $s \in \hat{n}^{\hat{n}}$ with $s \in \hat{a}_\alpha$ for each $\alpha < \lambda$. Then s is an upper bound on $(s_\alpha : \alpha < \lambda)$. $\square$

We need one more lemma.

Definition 3.17.11. Suppose $f, g : \omega \to [\omega]^{<\aleph_0}$ and $A \subseteq \omega$ is infinite. Then say that $f \leq_A g$ if $\{n \in A : f(n) \not\subseteq g(n)\}$ is finite.

Lemma 3.17.12. Suppose $\lambda < \mathfrak{t}$ is an infinite cardinal and $A \subseteq \omega$ is infinite and $(f_\alpha : \alpha < \lambda)$ is a sequence from $([\omega]^{<\aleph_0})^\omega$ with $f_\alpha \geq_A f_\beta$ for all $\alpha < \beta < \lambda$. Suppose further that for each $\alpha < \lambda$, $\{m \in A : f_\alpha(m) = \emptyset\}$ is finite. Then there is some infinite $B \subseteq_* A$ and some $f : \omega \to [\omega]^{<\aleph_0}$ such that $f \leq_B f_\alpha$ for each $\alpha < \lambda$, and further $\{m \in B : f(m) = \emptyset\}$ is finite.

Proof. For each $\alpha < \lambda$ define $X_\alpha := \{\langle m, n \rangle : m \in A \text{ and } n \in f_\alpha(m)\}$; so X_α is an infinite subset of $\omega \times \omega$. Suppose $\alpha < \beta$; then there is m_* so that for all $m \in A \backslash m_*$, $f_\alpha(m) \subseteq f_\beta(m)$. Hence $X_\alpha \backslash X_\beta \subseteq \bigcup_{m \in A \cap m_*} f_\alpha(m)$ is finite, so $X_\alpha \supseteq_* X_\beta$. Hence $(X_\alpha : \alpha < \lambda)$ is a tower; by hypothesis on λ we can choose an infinite $X \subseteq \omega \times \omega$ such that $X \subseteq_* X_\alpha$ for each $\alpha < \lambda$. Define $f : \omega \to [\omega]^{<\aleph_0}$ by $f(m) = \{n : \langle m, n \rangle \in X\}$. (Each $f(m)$ is finite because $X \subseteq_* X_0$.) Let $B = \{m : f(m) \neq \emptyset\}$. Clearly this works. □

Theorem 3.17.13. $\mathfrak{p} = \mathfrak{t}$.

Proof. We know that $\mathfrak{p} \leq \mathfrak{t}$; suppose towards a contradiction that $\mathfrak{p} < \mathfrak{t}$. We can suppose that $\mathfrak{t} = 2^{\aleph_0} = 2^{<\mathfrak{t}}$ since if we force by the Levy collapse of $2^{<\mathfrak{t}}$ to $\mathfrak{t}$, this adds no new sequences of reals of length less than $\mathfrak{t}$, and so does not affect the values of $\mathfrak{p}$ and $\mathfrak{t}$. So henceforth we assume this. I switch between the symbols $\mathfrak{t}$ or $2^{\aleph_0}$ depending on the role they are playing.

Our aim is to build a special ultrafilter $\mathcal{U}$ on ω, such that if we set $\hat{V} = V^\omega/\mathcal{U}$ for some or any countable transitive $V \models ZFC^-$, then $\mathfrak{p}_{\hat{V}} \leq \mathfrak{p}$ and $\mathfrak{t}_{\hat{V}} = 2^{\aleph_0}$. In view of $\mathfrak{p}_{\hat{V}} = \mathfrak{t}_{\hat{V}}$ this clearly suffices to show $\mathfrak{p} = 2^{\aleph_0} = \mathfrak{t}$, a contradiction.

Enumerate $\mathcal{P}(\omega) = (Y_\gamma : \gamma < 2^{\aleph_0})$. Enumerate $(([\omega]^{<\aleph_0})^\omega)^{<\mathfrak{t}} = (\overline{f}^\gamma : \gamma < 2^{\aleph_0})$, where $\overline{f}^\gamma = (f^\gamma_\alpha : \alpha < \lambda_\gamma)$ for some $\lambda_\gamma < \mathfrak{t}$, so that each sequence occurs cofinally often.

Inductively choose a tower $(A_\gamma : \gamma < 2^{\aleph_0})$ so that:

(1) (This is the definition of tower) Each A_γ is infinite and $\gamma < \gamma' < 2^{\aleph_0}$ implies $A_{\gamma'} \subseteq_* A_\gamma$.

(2) For each $\gamma < 2^{\aleph_0}$, either $A_{\gamma+1} \subseteq X_\gamma$ or else $A_{\gamma+1} \cap X_\gamma = \emptyset$.

(3) Suppose for every $\alpha < \beta < \lambda_\gamma$, $f^\gamma_\alpha \geq_{A_\gamma} f^\gamma_\beta$, and for every $\alpha < \lambda_\gamma$, $\{m \in A_\gamma : f^\gamma_\alpha(m) = \emptyset\}$ is finite. Then there is some $f : \omega \to [\omega]^{<\aleph_0}$ such that $\{m \in A_{\gamma+1} : f(m) = \emptyset\}$ is finite and $f \leq_{A_{\gamma+1}} f_\alpha$ for each $\alpha < \lambda_\gamma$.

This is straightforward, using Lemma 3.17.12 at successor steps, and using $\mathfrak{t} = 2^{\aleph_0}$ at limit stages. Let $\mathcal{U}$ be the set of all $A \subset \omega$ such that $A_\gamma \subseteq_* A$ for some $\gamma < 2^{\aleph_0}$. Then $\mathcal{U}$ is a nonprincipal ultrafilter, by (1) and (2). Let V be a countable transitive model of ZFC^- and let $\hat{V} = V^\omega/\mathcal{U}$. Note that $|\hat{V}| = 2^{\aleph_0}$, so in particular we have $\mathfrak{p}_{\hat{V}}, \mathfrak{t}_{\hat{V}}, \mathfrak{b}_{\hat{V}} \leq 2^{\aleph_0}$. Also, we know $\mathfrak{t} \leq \mathfrak{b}_{\hat{V}}$ by Lemma 3.17.9, so $\mathfrak{b}_{\hat{V}} = 2^{\aleph_0}$.

Claim 1. $\mathbf{p}_{\hat{V}} \leq \mathfrak{p}$.

Proof of Claim 1. Suppose $\lambda < \mathfrak{p}_{\hat{V}}$; we show $\lambda < \mathfrak{p}$. Note that $\lambda < \mathfrak{b}_{\hat{V}}$ since $\mathfrak{b}_{\hat{V}} = 2^{\aleph_0} \geq \mathfrak{p}_{\hat{V}}$, as remarked above. Let $\{B_\alpha : \alpha < \lambda\}$ be a family of subsets of ω with the strong finite intersection property. Define $f_\alpha : \omega \to [\omega]^{<\aleph_0}$ by $f_\alpha(m) = B_\alpha \cap m$; let $\hat{a}_\alpha = [f_\alpha]_\mathcal{U}$. So each $\hat{a}_\alpha \in [\hat{\omega}]^{<\hat{\aleph}_0}$. Since $\lambda < \mathfrak{b}_{\hat{V}}$, we can choose $\hat{n}$ such that each $\hat{a}_\alpha \subseteq \hat{n}$. Then $\{\hat{a}_\alpha : \alpha < \lambda\}$ satisfies the hypothesis of Lemma 3.17.10 part (C); since $\lambda < \mathfrak{p}_{\hat{V}}$ there is $\hat{a} \subseteq \hat{n}$ with $|\hat{a}|$ nonstandard, and $\hat{a} \subseteq \hat{a}_\alpha$ for each $\alpha < \omega$.

Write $\hat{a} = [f]_\mathcal{U}$. For each $\alpha < \lambda$ there is some $\gamma < 2^{\aleph_0}$ such that $f \leq_{A_\gamma} f_\alpha$. Since $2^{\aleph_0} = \mathfrak{t}$ is regular, we can choose γ_* large enough so that $f \leq_{A_{\gamma_*}} f_\alpha$ for each $\alpha < \lambda$. Define $B \subseteq \omega$ by $B = \bigcup_{m \in A_{\gamma_*}} f(m)$. B is infinite since $\{m \in A_{\gamma_*} : |f(m)| \geq n\} \in \mathcal{D}$

for each $n < \omega$. Also, suppose $\alpha < \lambda$; choose m_* large enough so that $f(m) \subseteq f_\alpha(m)$ for every $m \in A_{\gamma_*} \backslash m_*$.Then $B \backslash B_\alpha \subseteq \bigcup_{m \in A_{\gamma_*} \cap m_*} f(m)$ is finite, so $B \subseteq_* B_\alpha$. This shows that $\lambda < \mathfrak{p}$, concluding the proof of the claim.

Claim 2. $\mathfrak{t}_{\hat{V}} = 2^{\aleph_0}$.

Proof of Claim 2. We know $\mathfrak{t}_{\hat{V}} \leq 2^{\aleph_0}$. Let $\lambda < 2^{\aleph_0}$ be given; we show $\lambda < \mathfrak{t}_{\hat{V}}$. Note that $\lambda < 2^{\aleph_0} = \mathfrak{b}_{\hat{V}}$ as remarked above. So it suffices to show (D) from Lemma 3.17.10 holds. So let $(\hat{a}_\alpha : \alpha < \lambda)$ be a descending sequence of nonempty sets from $[\hat{\omega}]^{<\hat{\aleph}_0}$; write $\hat{a}_\alpha = [f_\alpha]_{\mathcal{U}}$.

Note that for each $\alpha < \beta < \lambda$ there is some γ with $f_\alpha \geq_{A_\gamma} f_\beta$, and for each $\alpha < \lambda$ there is some γ with $\{m \in A_\gamma : f_\alpha(m) = \emptyset\}$ finite. Since $2^{\aleph_0}$ is regular we can choose γ_* large enough so that $f_\alpha \geq_{A_{\gamma_*}} f_\beta$ for all $\alpha < \beta < 2^{\aleph_0}$, and such that $\{m \in A_{\gamma_*} : f_\alpha(m) = \emptyset\}$ is finite for each $\alpha < \lambda$.

Choose $\gamma \geq \gamma_*$ so that $\lambda_\gamma = \lambda$ and $f^\gamma_\alpha = f_\alpha$ for each $\alpha < \lambda$. By item (3) of the construction we can choose $f : \omega \to [\omega]^{<\aleph_0}$ such that $\{m \in A_{\gamma+1} : f(m) = \emptyset\}$ is finite and $f \leq_{A_{\gamma+1}} f_\alpha$ for each $\alpha < \lambda$. Let $\hat{a} = [f]_{\mathcal{U}}$; then $\hat{a}$ is nonempty, so any $\hat{m} \in \hat{a}$ is as desired. $\square$

3.18 $\trianglelefteq^*_{\lambda\kappa}$ and pseudosaturation

In this section, we connect $\trianglelefteq^*_{\lambda\kappa}$ with pseudosaturation.

We will want a different definition of $\mathfrak{b}_{\hat{V}}$ than in Section 3.17:

Definition 3.18.1. Suppose $\hat{V} \models ZFC^-$. Then let $\mathfrak{b}^*_{\hat{V}}$ be the least cardinality of a non-pseudofinite subset of $\hat{V}$.

So of course, $\mathfrak{b}^*_{\hat{V}} \leq \mathfrak{b}_{\hat{V}}$. The following is easy after the results of Section 3.10.

Lemma 3.18.2. Suppose $V \models ZFC^-$ is transitive and $\mathbf{j} : V \preceq \hat{V}$, and suppose $\mathfrak{b}^*_{\hat{V}} \geq \lambda_{\hat{V}}(\Delta(IP))$. Suppose $T \in V$ is a complete countable unstable theory. Then for every $M \models T$ with $M \in V$, and for every cardinal λ, the following are equivalent:

(A) $\hat{V}$ λ^+-pseudosaturates T;

(B) $\mathbf{j}_{\mathrm{std}}(M)$ is λ^+-saturated.

Proof. If $\lambda < \mathfrak{b}^*_{\hat{V}}$ then this is clear, since every partial type over M of cardinality at most λ is pseudofinite. So suppose $\lambda \geq \mathfrak{b}^*_{\hat{V}}$; then $\lambda \geq \lambda_{\hat{V}}(\Delta(IP))$, so (A) is false by Theorem 3.10.5. So (B) is also false. □

Further, by the results in Section 3.10, we know exactly what the situation is for stable T, provided that $\hat{V}$ is $\aleph_1$-saturated.

Thus we are led to consider the relationship between $\mathfrak{b}^*_{\hat{V}}$ and $\lambda_{\hat{V}}(\Delta(IP))$ in models $\hat{V} \models ZFC^-$. In general we do not know how to prove an inequality, but under some hypotheses on $\hat{V}$, we can. The arguments here are inspired by Malliaris and Shelah's proof in [61] that T_{rg} is the $\trianglelefteq^*_1$-minimal unstable theory.

Definition 3.18.3. Let $\mathcal{L}_* = \{\in, I, F\}$ where I is a unary relation symbol and F is a unary function symbol. Let ZFC^-_* be the $\mathcal{L}_*$ theory, such that $(\hat{V}, \hat{\in}, I^{\hat{V}}, F^{\hat{V}}) \models ZFC^-_*$ if:

- $(\hat{V}, \hat{\in}) \models ZFC^-$;
- $I^{\hat{V}}$ is a bounded subset of $(\omega)^{\hat{V}}$;
- $F^{\hat{V}}$ is a bijection from $I^{\hat{V}}$ onto $\hat{V}$, and for every $\hat{n} \in I^{\hat{V}}$, $F^{\hat{V}} \restriction \{\hat{m} \in I^{\hat{V}} : \hat{m} \leq \hat{n}\} \in \hat{V}$;
- For every $\hat{a} \in \hat{V}$, either $\{\hat{n} \in I^{\hat{V}} : \hat{n} \mathrel{\hat{\in}} \hat{a}\}$ is bounded in $I^{\hat{V}}$, or its complement is.

Given $\hat{V} \models ZFC^-$, say that $\hat{V} \models ZFC^-_{\mathrm{pre}}$ if $\hat{V}$ can be expanded to a model $(\hat{V}, I^{\hat{V}}, F^{\hat{V}})$ of ZFC^-_*. In particular this implies $\hat{V}$ is ω-nonstandard.

Lemma 3.18.4. Suppose $V \models ZFC^-$ is transitive. Then there is some $\mathbf{j} : V \preceq \hat{V}$ with $\hat{V} \models ZFC^-_{\mathrm{pre}}$.

Proof. We can suppose, by compactness, that V is countable. Let $\mathbf{j} : V \preceq \hat{V}$ with $\hat{V}$ ω-nonstandard and countable. We show that $\hat{V} \models ZFC^-_{\mathrm{pre}}$, that is, we find an expansion to a model of ZFC^-_*.

Enumerate $\hat{V} = \{a_m : m < \omega\}$ and let $\mathcal{U}$ be a nonprincipal ultrafilter on ω. For each $m < \omega$, let Y_m be either $\{n \in \omega : n \in a_m\}$ or else $\{n \in \omega : n \notin a_m\}$, whichever is in $\mathcal{U}$. Define $(b_n : n < \omega)$ inductively on n; having defined $b_n : n < n'$, note that $\bigcap_{n \leq n'} Y_n \in \mathcal{U}$ is infinite, so we can find $b_{n'} \in \bigcap_{n \leq n'} Y_n$, such that $b_{n'} > b_n$ for all $n < n'$.

Define $I^{\hat{V}} := \{b_n : n < \omega\}$, an infinite subset of $\omega \subseteq \hat{\omega}$. Choose a bijection $F^{\hat{V}} : I^{\hat{V}} \to \hat{V}$. Then $(\hat{V}, I^{\hat{V}}, F^{\hat{V}})$ works (using that $[\hat{V}]^{<\aleph_0} \subseteq \hat{V}$ to see that $\hat{V}$ contains the initial approximations to $F^{\hat{V}}$). □

The definition of ZFC^-_* was rigged to make the following work.

Theorem 3.18.5. Suppose $V \models ZFC^-$ is transitive and suppose $\mathbf{j} : V \preceq \hat{V} \models ZFC^-_{\mathrm{pre}}$. Then $\mathfrak{b}^*_{\hat{V}} \geq \lambda_{\hat{V}}(\Delta(IP))$.

Proof. Let $(\hat{V}, I^{\hat{V}}, F^{\hat{V}})$ be an expansion of $\hat{V}$ to a model of ZFC^-_*.

Suppose $X \subseteq \hat{V}$ is not pseudofinite; it suffices to show that $|X| \geq \lambda_{\hat{V}}(\Delta(IP))$. Look at $Y := (F^{\hat{V}})^{-1}(X)$; since X is not pseudofinite, Y must be unbounded in $I^{\hat{V}}$. Thus it suffices to show that whenever $(\hat{n}_\alpha : \alpha < \lambda)$ is a strictly increasing, cofinal sequence in $I^{\hat{V}}$, then $\lambda \geq \lambda_{\hat{V}}(\Delta(IP))$.

Let I_0, I_1 be two unbounded, disjoint subsets of λ each of size λ; for instance, we could let $I_0 = \{2\alpha : \alpha < \lambda\}$ and $I_1 = \{2\alpha + 1 : \alpha < \lambda\}$. Write $X_i = \{\hat{n}_\alpha : \alpha \in I_i\}$. Note that whenever $\hat{X}_0 \in \hat{V}$ contains X_0, we must have that $\hat{X}_0$ contains an end segment of $I^{\hat{V}}$, by definition of ZFC^-_*; in particular $\hat{X}_0$ contains elements of X_1. Thus (X_0, X_1) witness that $\lambda \geq \lambda_{\hat{V}}(\Delta(IP))$. $\square$

On a related note, the following theorem is the key observation in showing that $\trianglelefteq^*_1$ and $\trianglelefteq^*_{\aleph_1}$ coincide on unstable theories.

Theorem 3.18.6. Suppose $V \models ZFC^-$, and $T \in V$ is unstable. Suppose $\mathbf{j} : V \preceq \hat{V} \models ZFC^-_{\text{pre}}$. If $\hat{V}$ $\aleph_1$-pseudosaturates T, then $\hat{V}$ is $\aleph_1$-saturated.

Proof. Choose $M \models T$ with $M \in V$. We know $\mathbf{j}_{\text{std}}(M)$ is not $\text{lcf}_{\hat{V}}(\omega)^+$-pseudosaturated, by Theorem 3.10.5, so $\text{lcf}_{\hat{V}}(\omega) \geq \aleph_1$. By Theorem 3.17.1, $(\aleph_0, \aleph_0) \notin \mathcal{C}_{\hat{V}}$; thus $\mathfrak{p}_{\hat{V}} \geq \aleph_1$. Now also by Theorem 3.18.5 (and Theorem 3.12.5), we have that $\mathfrak{b}^*_{\hat{V}} \geq \aleph_1$. Thus by Theorem 3.9.7, $\hat{V}$ is $\aleph_1$-saturated. $\square$

We can now get several corollaries. First of all, we have the promised characterizations of $\trianglelefteq^*_{\lambda\kappa}$ in terms of pseudosaturation:

Corollary 3.18.7. Suppose T_0, T_1 are complete countable theories, suppose κ is infinite or 1. If either T_0 or T_1 is unstable, then the following are equivalent:

(A) $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$;

(B) There is some countable transitive $V \models ZFC^-$ with $T_0, T_1 \in V$, such that whenever $\mathbf{j} : V \preceq \hat{V} \models ZFC^-_{\text{pre}}$ with $\hat{V}$ κ-saturated, if $\hat{V}$ λ-pseudosaturates T_1, then it λ-pseudosaturates T_0.

In particular, $\trianglelefteq^\times_\kappa \subseteq \trianglelefteq^*_\kappa$, except perhaps on pairs of stable theories.

Proof. We break into cases. Note that if T_1 is stable and T_0 is not, then (A) and (B) both fail by Corollary 3.10.11 and Theorem 3.10.12. So we can suppose T_1 is unstable.

(B) implies (A): choose an embedding $\mathbf{j}_0 : V \preceq \hat{V}_0 \models ZFC^-_{\text{pre}}$ with $\hat{V}_0$ countable, and let $(\hat{V}_0, I^{\hat{V}_0}, F^{\hat{V}_0})$ be an expansion of $\hat{V}_0$ to a model of ZFC^-_*. Choose $M_i \models T_i$ with $M_i \in V$.

I claim that this setup witnesses Lemma 3.8.6(C) holds, and hence that $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$. Indeed, suppose $\mathbf{j}_1 : (\hat{V}_0, I^{\hat{V}_0}, F^{\hat{V}_0}) \preceq (\hat{V}, I^{\hat{V}}, F^{\hat{V}})$, with $(\hat{V}, I^{\hat{V}}, F^{\hat{V}})$ being κ-saturated. Write $\mathbf{j} = \mathbf{j}_1 \circ \mathbf{j}_0$. Then in particular, $\hat{V}$ is κ-saturated and $\hat{V} \models ZFC^-_{\text{pre}}$. Suppose $\mathbf{j}_{\text{std}}(M_1)$ is λ^+-saturated. Then in particular $\mathbf{j}_{\text{std}}(M_1)$ is λ^+-pseudosaturated, hence $\mathbf{j}_{\text{std}}(M_0)$ is λ^+-pseudosaturated. Since T_1 is unstable, we also get that $\lambda < \lambda_{\hat{V}}(\Delta(IP))$; hence $\lambda < \mathfrak{b}^*_{\hat{V}}$ by Theorem 3.18.5. Thus $\mathbf{j}_{\text{std}}(M_0)$ is λ^+-saturated.

(A) implies (B): Suppose first that T_0 is unstable; choose some transitive countable $V \models ZFC^-$ and $M_i \models T_i$ with $M_i \in V$, as in Lemma 3.8.6(B). Now suppose $\mathbf{j} : V \preceq \hat{V} \models ZFC^-_{\text{pre}}$ satisfies that $\hat{V}$ is κ-saturated, and $\mathbf{j}_{\text{std}}(M_1)$ is λ^+-pseudosaturated. Then by Lemma 3.18.2 and Theorem 3.18.5, $\mathbf{j}_{\text{std}}(M_1)$ is λ^+-saturated, so $\mathbf{j}_{\text{std}}(M_0)$ is λ^+-saturated, in particular it is λ^+-pseudosaturated.

Finally, suppose T_0 is stable; we show that (B) holds (in fact, we won't need (A).) let $V \models ZFC^-$ be countable with $T_0, T_1 \in V$, and suppose $\mathbf{j} : V \preceq \hat{V} \models \mathrm{ZFC}^-_{\text{pre}}$ with $\hat{V}$ ω-nonstandard. Suppose $\hat{V}$ λ^+-pseudosaturates T_1. This implies by Theorem 3.18.6 that $\hat{V}$ is $\aleph_1$-saturated. Note $\mu_{\hat{V}} \geq \mathrm{lcf}_{\hat{V}}(\omega) > \lambda$; thus by Theorem 3.10.6, $\hat{V}$ λ^+-pseudosaturates T_0. $\square$

We can replace the condition on T_0, T_1 by a condition on κ:

Corollary 3.18.8. Suppose T_0, T_1 are complete countable theories, suppose $\kappa \geq \aleph_1$. Then the following are equivalent:

(A) $T_0 \trianglelefteq^*_{\lambda\kappa} T_1$;

(B) There is some countable transitive $V \models ZFC^-$ with $T_0, T_1 \in V$, such that whenever $\mathbf{j} : V \preceq \hat{V} \models ZFC^-_{\text{pre}}$ with $\hat{V}$ κ-saturated, if $\hat{V}$ λ-pseudosaturates T_1, then it λ-pseudosaturates T_0.

In particular, $\trianglelefteq^\times_\kappa \subseteq \trianglelefteq^*_\kappa$.

Proof. By Corollary 3.18.7, it suffices to consider the case where T_0, T_1 are both stable. If T_0 is stable without the finite cover property and T_1 is stable with the finite cover property, then (A) and (B) fail by Corollary 3.10.11 and Theorem 3.10.12. So it suffices to consider the case where if T_0 fails the finite cover property, then so does T_1. But by Theorems 3.10.6 and 3.10.7, in these cases both (A) and (B) hold. □

As mentioned before, Malliaris and Shelah prove in [61] that $\trianglelefteq^*_1$ properly refines $\trianglelefteq^*_{\aleph_1}$ on the stable theories, and promise in forthcoming work to show that $\trianglelefteq^*_1$ has exactly six classes on the stable theories. Depending on whether this goes through for $\trianglelefteq^\times_1$ as well, perhaps $\kappa = 1$ in Corollary 3.18.8 is also true.

We have the following further consequence of Theorem 3.18.6, which shows that the stable theories are the only theories on which $\trianglelefteq^*_1$ and $\trianglelefteq^*_{\aleph_1}$ differ:

Corollary 3.18.9. Suppose T_0, T_1 are complete countable theories, not both stable. Then $T_0 \trianglelefteq^*_{\lambda 1} T_1$ if and only if $T_0 \trianglelefteq^*_{\lambda\aleph_1} T_1$; thus $T_0 \trianglelefteq^*_1 T_1$ if and only if $T_0 \trianglelefteq^*_{\aleph_1} T_1$. In other words, $\trianglelefteq^*_1$ and $\trianglelefteq^*_{\aleph_1}$ coincide on pairs of theories which are not both stable.

This immediately gives some new results on $\trianglelefteq^*_{\aleph_1}$:

Corollary 3.18.10. $NSOP_2$ theories are nonmaximal in $\trianglelefteq^*_{\aleph_1}$, assuming instances of GCH. Simplicity is a dividing line in $\trianglelefteq^*_{\aleph_1}$ (even without a supercompact cardinal).

Proof. This is because of the corresponding statements for $\trianglelefteq_1^*$. Nonmaximality of $NSOP_2$ in $\trianglelefteq_1^*$ is proved in [8] and [80]; the dividing line for simple theories in $\trianglelefteq_1^*$ is constructed in [61]. $\square$

Chapter 4: Amalgamation Properties and Keisler's Order

In [57], Mallairis and Shelah show that if there is a supercompact cardinal, then simplicity is a dividing line in Keisler's order. In [58], they use similar arguments to show (in ZFC) that Keisler's order has infinitely many classes, and in [87], I use similar arguments to show that lowness is a dividing line in Keisler's order.

In this chapter, we give a uniform treatment of these ultrafilter constructions, and we investigate the model-theoretic properties detected by the ultrafilters of [58].

The fundamental examples at play here are the hypergraph examples $T_{n,k}$: namely, for $n > k \geq 2$, $T_{n,k}$ is the random k-ary n-clique free hypergraph. These were introduced by Hrushovksi in [26], and used by Malliaris and Shelah in [58] to show that Keisler's order has infinitely many classes. (They subtract 1 from both indices in $T_{n,k}$.) Each $T_{n,2}$ has SOP_2, and thus is maximal in $\trianglelefteq$; so we are only really interested in the case $k \geq 3$. Also, it suffices for our present needs to consider the case $n = k + 1$.

In Section 4.2, we define a pattern Δ_k, and prove that $T_{k+1,k}$ is the $\trianglelefteq$-minimal theory admitting Δ_k. Thus, if T admits Δ_k, then $T_{k+1,k} \trianglelefteq T$. However, to conclude $T_{k+1,k} \not\trianglelefteq T$, we presently need stronger hypotheses than not admitting Δ_k. We discuss a large class of amalgamation properties along these lines, by trying to abstract properties that $T_{k+1,k}$ fails.

Here is one property of $T_{k+1,k}$ that seems key: let $k \geq 3$, and consider $T_{k+1,k}$. Let $T^{\forall}_{k+1,k}$ be the universal theory of $T_{k+1,k}$, i.e. the theory of (not necessarily random) k-ary

$k+1$-clique free hypergraphs. Suppose $(A_u : u \subsetneq n)$ is a system of models of $T^{\forall}_{k+1,k}$, not necessarily in any monster model, such that each $A_u \cap A_v = A_{u \cap v}$. If $n \leq k$, then we can always find some $A \models T^{\forall}_{k+1,k}$ with each $A_u \subseteq A$. This is because we cannot have created a $k+1$-clique, and so we can define A by not putting in any new relations. However, if $n > k$, then this may fail, since there may be $a_i \in A_{\{i\}}$ for each $i \leq k$, such that $A_u \models R((a_i : i \in u))$ for each $u \in [k+1]^k$.

This generalized amalgamation seems like a promising property to study. Indeed, in [39], Kim, Kolesnikov and Tsuboi introduced the notion of n-simplicity for each $n \geq 1$, building off of work of Kolesnikov in [40]. For example, $T_{k+1,k}$ is $k-2$-simple but not $k-1$-simple; 1-simplicity is the same as simplicity. Skipping some technicalities, n-simplicity says that independent systems of types of boundedly closed sets $(p_u(\overline{x}_u) : u \subsetneq n+2)$ have solutions. The hypotheses of independence and bounded closures are necessary to avoid certain trivial amalgamation failures. See [37] for a definition of bounded closure, although it will not be used in what we do.

However, n-simplicity cannot be the right definition for Keisler's order: in [39], the authors give an example of a theory U_n for each even $n \geq 4$, which in particular is not $n-1$-simple. But it is easily checked that U_n is bi-interpretable with the random graph T_{rg}, and so U_n is Keisler-equivalent to T_{rg}. Even worse, in [18], Goodrick, Kim and Kolesnikov give examples of totally categorical theories which are not 2-simple.

These problems are solved if we replace boundedly closed sets by models. In particular, we make the following definition in Section 4.4, where $\mathcal{P}^-(k)$ denotes the proper subsets of k: T has $\mathcal{P}^-(k)$ amalgamation models if every independent system of models $(M_u : u \subsetneq k)$ can be amalgamated. As examples, $T_{k+1,k}$ has $\mathcal{P}^-(k)$-amalgamation of models but not $\mathcal{P}^-(k+1)$-amalgamation of models, and T_{rg} has $\mathcal{P}^-(k)$-amalgamation of models for all k, as does each U_n, and every stable theory.

We also define the property of having Λ_k-type amalgamation, for each $k \geq 3$. This is somewhat more technical than having $\mathcal{P}^-(k)$-amalgamation of models, but it is more finely tuned to Keisler's order. These were first studied in [81], joint with Shelah, with the aim of analyzing $\leq_{SP}$.

We now discuss our ultrafilter construction. The idea is as follows: we will have, by various means, a class $\mathbb{P}$ of forcing notions, which is closed under $< \theta$-support forcing iterations, and such that each $P \in \mathbb{P}$ has the λ^+-c.c. We are not actually interested in forcing with elements of $\mathbb{P}$; rather, we are interested in building ultrafilters on their Boolean algebra completions. Nonetheless, the combinatorics of building these ultrafilters turns out to be intertwined with combinatorics in the associated forcing extensions.

The class of forcing notions $\mathbb{P}$ will be controlled by a sequence of cardinals $(\lambda, \kappa, \theta, \sigma)$, which satisfy the following constraints (similarly to the situation in [57]):

Definition 4.0.1. $(\lambda, \kappa, \theta, \sigma)$ is a suitable sequence of cardinals if:

- $\aleph_0 \leq \sigma \leq \theta < \kappa \leq \lambda$;
- θ, κ are regular, and for all $\mu < \kappa$, $\mu^{<\theta} < \kappa$;
- σ is either $\aleph_0$ or else supercompact.

In Section 4.6, given a suitable sequence of cardinals $\mathbf{s}$, and given an amalgamation parameter $3 \leq k \leq \theta$, we define the class of forcing notions $\mathbb{P}_{\mathbf{s},k}$. Every $P \in \mathbb{P}_{\mathbf{s},k}$ has the κ-c.c. and is θ-closed, and $\mathbb{P}_{\mathbf{s},k}$ is closed under $< \theta$-support forcing iterations. σ indicates the completeness of the ultrafilters we will be constructing, and λ is the level of saturation we are interested in.

In Section 4.7, for every suitable sequence $\mathbf{s}$ and for every $3 \leq k \leq \theta$, we define two properties of theories, namely: the $\mathbb{P}_{\mathbf{s},k}$-amalgamation property, and the smooth $\mathbb{P}_{\mathbf{s},k}$-amalgamation property. These detect some sort of k-dimensional type amalgamation properties. In Theorem 4.7.8, we show that there is some $P \in \mathbb{P}_{\mathbf{s},k}$ and some σ-complete ultrafilter $\mathcal{U}$ on $\mathcal{B}(P)$, such that $\mathcal{U}$ λ^+-saturates every theory with the smooth $\mathbb{P}_{\mathbf{s},k}$-amalgamation property, and does not λ^+-saturate any theory without the $\mathbb{P}_{\mathbf{s},k}$-amalgamation property.

In the subsequent sections, we give model-theoretic necessary and sufficient conditions for a theory T to have the (smooth) $\mathbb{P}_{\mathbf{s},k}$-amalgamation property: suppose $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence of cardinals, and $3 \leq k < \omega$. Then as described in Section 4.10, we show the following:

(A) If T is unsimple, or if T is nonlow and $\sigma = \aleph_0$, then T will fail the $\mathbb{P}_{\mathbf{s},k}$-amalgamation property;

(B) If T admits $\Delta_{k'}$ for some $k' < k$, and if $\lambda \geq \kappa^{+\omega}$, then T will fail the $\mathbb{P}_{\mathbf{s},k}$-amalgamation property;

(C) If T has $\Lambda_{k'}$ type-amalgamation for all $k' < k$, and either T is low or else T is simple and $\sigma > \aleph_0$, and finally if $\theta > \aleph_0$, then T satisfies the smooth $\mathbb{P}_{\mathbf{s},k}$-amalgamation.

Collectively, these results uniformize many of the recent ultrafilter constructions for Keisler's order. In particular, we will get that if there is a supercompact cardinal σ, then simplicity is a dividing line, with $\theta = \sigma$, $\kappa = \sigma^+$, $\lambda = \kappa^{+\omega}$, and $k = 3$; this was first proved by Malliaris and Shelah [57]. Also, we will get that low is a dividing line with $\sigma = \aleph_0$, $\theta = \aleph_1$, $\kappa = (2^{\aleph_0})^+$, $\lambda = \kappa^{+\omega}$ and $k = 3$; this was first proved by myself in [87].

Finally, by varying k, we will get that there is a principal dividing line in $\trianglelefteq$ between the theories with $\Lambda_{k'}$-type amalgamation for all $k' < k$, and the theories which admit $\Delta_{k'}$ for some $k' < k$. In particular, for all $k < k'$, $T_{k+1,k} \not\trianglelefteq T_{k'+1,k'}$. This improves the theorem

Malliaris and Shelah in [58], stating that for $k < k' - 1$, $T_{k+1,k} \not\trianglelefteq T_{k'+1,k'}$.

We now have many notions of k-dimensional amalgamation; possibly they are all equivalent, but we can prove this only in rather special cases. In Section 4.11, we introduce the *well-behaved* simple theories. As examples, each $T_{n,k}$ is well-behaved, as is T_{rg}, or any stable theory. The following is a partial list of the equivalences proved in Theorem 4.11.12:

Theorem 4.0.2. Suppose T is a well-behaved simple theory and $k \geq 3$. Then the following are equivalent:

(A) T does not admit $\Delta_{k'}$ for any $k' < k$;

(B) T has $\mathcal{P}^-(k)$-amalgamation of models;

(C) T has $\Lambda_{k'}$-type amalgamation for all $k' < k$.

As far as we know, every simple theory is well-behaved, and so perhaps this theorem holds for all simple theories. In any case, the following is an immediate corollary:

Corollary 4.0.3. $\mathcal{P}^-(k)$-amalgamation of models is a principal dividing line in Keisler's order among well-behaved low theories. If there is a supercompact cardinal, then this is also a principal dividing line among well-behaved simple theories.

4.1 A Characterization of Low Theories

In this technical section, we prove a key theorem about low theories, which says that forking is type-definable in a strong way.

Recall that by a theorem of Kim [36], in any simple theory T, forking is the same as dividing; that is, $\varphi(\overline{x}, \overline{a})$ forks over A if and only if it divides over A. We thus use the terms *forking* and *dividing* interchangeably.

We give the following equivalents for T being low. (C) is Buechler's original definition of lowness from [5]; equivalently it states that for every formula $\varphi(\overline{x},\overline{y})$, $D(\overline{x} = \overline{x}, \varphi(\overline{x},\overline{y})) < \omega$, where D is the D-rank for low theories; in the same paper he proved the equivalence of that with our definition in terms of dividing. Thus (A) if and only if (C) is already known.

Theorem 4.1.1. Suppose T is simple. Then the following are equivalent:

(A) T is low.

(B) Suppose $\varphi(\overline{x},\overline{b})$ does not fork over A. Then there is some $\overline{c} \in A$ and some $\psi(\overline{y},\overline{z}) \in \mathrm{tp}(\overline{b},\overline{c})$ such that whenever $(\overline{b}',\overline{c}') \models \psi(\overline{y},\overline{z})$, then $\varphi(\overline{x},\overline{b}')$ does not fork over $\overline{c}'$.

(C) For every formula $\varphi(\overline{x},\overline{y})$, there is some k such that there is no sequence $(\overline{b}_i : i < k)$ such that $\bigwedge_{i<k} \varphi(\overline{x},\overline{b}_i)$ is consistent, and such that for each $i < k$, $\varphi(\overline{x},\overline{b}_i)$ forks over $\{\overline{b}_j : j < i\}$.

Proof. (A) implies (B): Choose k such that if $\varphi(\overline{x},\overline{b}')$ does not k-divide over $\emptyset$ then it does not divide over $\emptyset$. It follows that if A' is any set and $\varphi(\overline{x},\overline{b}')$ does not k-divide over A' then $\varphi(\overline{x},\overline{b}')$ does not divide over A'. Since $\varphi(\overline{x},\overline{b})$ does not divide over A, $\varphi(\overline{x},\overline{b})$ does not k-divide over A; by a compactness argument we can choose $\overline{c} \in A$ and $\psi(\overline{y},\overline{z}) \in \mathrm{tp}(\overline{b},\overline{c})$ such that whenever $\models \psi(\overline{b}',\overline{c}')$ then $\varphi(\overline{x},\overline{b}')$ does not k-divide over $\overline{c}'$. But then by choice of k, $\varphi(\overline{x},\overline{b}')$ does not divide over $\overline{c}'$.

(B) implies (C): Suppose (C) holds, and let $\varphi(\overline{x},\overline{y})$ be given. Let Γ be the partial type in the variables $(\overline{y}_\alpha : \alpha < \omega_1)$ asserting:

- For each $s \in [\omega_1]^{<\omega}$, $\exists \overline{x} \bigwedge_{\alpha \in s} \varphi(\overline{x},\overline{y}_\alpha)$;

- For each $\alpha < \omega_1$, $\varphi(\overline{x},\overline{y}_\alpha)$ forks over $(\overline{y}_\beta : \beta < \alpha)$.

The second item is possible to express by hypothesis.

Suppose towards a contradiction that Γ were consistent; choose $(\bar{b}_\alpha : \alpha < \omega_1)$ a realization of Γ. Let $p(\overline{x})$ be the type over $(\bar{b}_\alpha : \alpha < \omega_1)$ asserting that $\varphi(\overline{x}, \bar{b}_\alpha)$ holds for each $\alpha < \omega_1$. Then $p(\overline{x})$ is consistent but forks over every countable subset of its domain, contradicting simplicity of T.

Thus Γ is inconsistent; by symmetry we can choose n such that $\Gamma \restriction_{(\overline{y}_i : i < n)}$ is inconsistent. This just says that (C) holds.

(C) implies (A): let $\varphi(\overline{x}, \overline{y})$ be given, and let k be as in (C). We claim that if $\varphi(\overline{x}, \bar{b})$ does not $k+1$-divide over $\emptyset$ then $\varphi(\overline{x}, \bar{b})$ does not divide over $\emptyset$. Indeed, suppose towards a contradiction that we had an indiscernible sequence $(\bar{b}_i : i < \omega)$ such that $\varphi(\overline{x}, \bar{b}_i)$ were k-consistent but $k+1$-inconsistent. Then $(\bar{b}_i : i < k)$ is a counterexample to the choice of k.

$\square$

The following is the major application of Theorem 4.1.1. First we need some definitions.

Definition 4.1.2. Suppose T is simple, and $\mathbf{M} \models^{\mathcal{B}} T$ (recall this means $\mathbf{M}$ is a full $\mathcal{B}$-valued model of T) and $A \subseteq \mathbf{M}$. Then say that $p(x)$ does not $\mathcal{U}$-fork over A if $[p(x)]_{\mathcal{U}}$ does not fork over $[A]_{\mathcal{U}}$.

Suppose $\mathbf{M} \models^{\mathcal{B}} T$. Let $\mathbb{V}[G]$ be a forcing extension by $\mathcal{B}_+$. Then G is an ultrafilter on $\mathcal{B}$ in $\mathbb{V}[G]$; now $\mathcal{B}$ is typically not complete in $\mathbb{V}[G]$, but the definition of specializations did not require completeness, and so we can still form the specialization $(\mathbf{M}/G, [\cdot]_G)$. Thus, in $\mathbb{V}$, $\check{\mathbf{M}}/\dot{G}$ is a $\mathcal{B}$-name for a model of T, and $[\cdot]_{\dot{G}}$ is a name for a surjection from $\check{\mathbf{M}} \to \check{\mathbf{M}}/\dot{G}$. We have that for every $\varphi(\overline{a}) \in \mathbf{L}(\mathbf{M})$, $\|\varphi(\overline{a})\|_{\mathbf{M}} = \|\check{\mathbf{M}}/\dot{G} \models \varphi([\overline{a}]_{\dot{G}})\|_{\mathcal{B}}$. We call $(\check{\mathbf{M}}/\dot{G}, [\cdot]_{\dot{G}})$ the generic specialization of $\mathbf{M}$.

Theorem 4.1.3. Suppose $\mathcal{B}$ is a complete Boolean algebra, and T is low. Suppose $\mathbf{M} \models^{\mathcal{B}} T$, $\mathbf{M}_0 \preceq \mathbf{M}$ is countable, and $\varphi(x)$ is a formula over $\mathbf{M}$. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. If $\varphi(x)$ does not $\mathcal{U}$-fork over $\mathbf{M}_0$, then $\|\varphi(x)$ does not fork over $\check{\mathbf{M}}_0/\dot{G}$ in $\check{\mathbf{M}}/\dot{G}\|_{\mathcal{B}} \in \mathcal{U}$.

Proof. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{B}$ such that $\varphi(x)$ does not $\mathcal{U}$-fork over $\mathbf{M}_0$. Suppose $\varphi(x)$ is over $\overline{a} \in \mathbf{M}^{<\omega}$. Choose $\psi_\varphi(\overline{a}, \overline{a}_0)$ such that $\overline{a}_0 \in \mathbf{M}_0$ and $\mathbf{M}/\mathcal{U} \models \psi_\varphi([\overline{a}]_{\mathcal{U}}, [\overline{a}_0]_{\mathcal{U}})$, such that whenever $\mathfrak{C} \models \psi_\varphi(\overline{b}, \overline{b}_0)$, then $\varphi(x, \overline{b}, \overline{b}_0)$ does not fork over $\overline{b}_0$. Put $\mathbf{c} = \|\psi_\varphi(\overline{a}, \overline{a}_0)\|_{\mathbf{M}}$. Then $\mathbf{c} \in \mathcal{U}$, but clearly $\mathbf{c} \leq \|\varphi(x)$ does not fork over $\check{\mathbf{M}}_0/\dot{G}$ in $\check{\mathbf{M}}/\dot{G}\|_{\mathcal{B}}$. □

This is false for nonlow theories; in general, we need to restrict to $\aleph_1$-complete ultrafilters.

Theorem 4.1.4. Suppose $\mathcal{B}$ is a complete Boolean algebra, and T is simple. Suppose $\mathbf{M} \models^{\mathcal{B}} T$, $\mathbf{M}_0 \preceq \mathbf{M}$ is countable, and $\varphi(x)$ is a formula over $\mathbf{M}$. Suppose $\mathcal{U}$ is an $\aleph_1$-complete ultrafilter on $\mathcal{B}$. If $\varphi(x)$ does not $\mathcal{U}$-fork over $\mathbf{M}_0$, then $\|\varphi(x)$ does not fork over $\check{\mathbf{M}}_0/\dot{G}$ in $\check{\mathbf{M}}/\dot{G}\|_{\mathcal{B}} \in \mathcal{U}$.

Proof. Let $\mathcal{U}$ be an $\aleph_1$-complete ultrafilter on $\mathcal{B}$ such that $\varphi(x)$ does not $\mathcal{U}$-fork over $\mathbf{M}_0$. Suppose $\varphi(x)$ is over $\overline{a} \in \mathbf{M}^{<\omega}$. Let $\mathbf{c} = \bigwedge\{\|\psi(\overline{a}, \overline{a}_0)\|_{\mathbf{M}} : \overline{a}_0 \in \mathbf{M}_0 \text{ and } \mathbf{M}/\mathcal{U} \models \psi([\overline{a}]_{\mathcal{U}}, [\overline{a}_0]_{\mathcal{U}})\}$. Then $\mathbf{c} \in \mathcal{U}$, but clearly $\mathbf{c} \leq \|\varphi(x)$ does not fork over $\check{\mathbf{M}}_0/\dot{G}$ in $\check{\mathbf{M}}/\dot{G}\|_{\mathcal{B}}$. □

4.2 Patterns and Hypergraphs Omitting Cliques

In this section, we introduce the major class of examples of simple theories with interesting amalgamation properties.

Definition 4.2.1. For each $2 \leq k < n < \omega$, let $T_{n,k}$ be the theory of the random k-ary, n-clique free hypergraph.

These were introduced by Hrushovksi [26], who proved that each $T_{n,2}$ is unsimple, in fact it has SOP_2 and so is maximal in Keisler's order. We shall mainly be interested in the case $T_{n,k}$ for $k \geq 3$; these are simple, with forking given by equality. In fact, we will only be interested in the case $n = k+1$.

The following are the relevant patterns:

Definition 4.2.2. Suppose $R \subseteq [I]^k$ for some k. Then let $\Delta(R)$ be the pattern on $[I]^{k-1}$, consisting of all $s \in [[I]^{k-1}]^{<\aleph_0}$ such that there is no $v \in R$ with $[v]^{k-1} \subseteq s$.

Clearly, then, if $R \subseteq [I]^k$ is $k+1$-clique free, then $R(x, \overline{y})$ admits $\Delta(R)$ in $T_{n,k}$.

To relate the various $\lambda_{\hat{V}}(\Delta(R))$'s, we need the following fact.

Definition 4.2.3. Suppose Δ is a pattern on I. For each $n < \omega$, let Δ^n be the pattern on $[I]^{\leq n}$ consisting of all $s \in [[I]^{\leq n}]^{<\aleph_0}$ such that $\bigcup s \in \Delta$.

Theorem 4.2.4. Suppose Δ is a pattern on I and $n < \omega$.

1. If T is a complete countable theory, then T admits Δ if and only if T admits Δ^n.

2. If $V \models ZFC^-$ is transitive and $\mathbf{j} : V \preceq \hat{V}$ with $\hat{V}$ not ω-standard, then $\lambda_{\hat{V}}(\Delta) = \lambda_{\hat{V}}(\Delta^n)$.

3. If $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, then $\lambda_{\mathcal{U}}(\Delta) = \lambda_{\mathcal{U}}(\Delta^n)$.

Proof. (1): Note that Δ is an instance of Δ^n (using $[I]^1 \subseteq [I]^{\leq n}$), so it suffices to show that if T admits Δ then T admits Δ^n. Suppose $\varphi(x, y)$ admits Δ (really x, y could be tuples). Let $\overline{y} = (y_i : i < n)$ and let $\psi(x, \overline{y}) = \bigwedge_{i<n} \varphi(x, y_i)$. Easily then $\psi(x, \overline{y})$ admits Δ^n.

(2), (3): Similar. $\square$

The following lemma clearly remains true if we replace ω by an infinite set X, by compactness.

Lemma 4.2.5. Suppose $k \geq 2$. Let R be any k-ary graph on ω and let R_* be a random k-ary $k+1$-clique free graph on ω. Then $\Delta(R)$ is an instance of $\Delta(R_*)^k$.

Proof. Let R' be the k-ary graph on $k \times \omega$ consisting of the graphs of increasing functions from k to ω whose range is in R. Since R' is $k+1$-clique free, it suffices to show that $\Delta(R)$ is an instance of $\Delta(R')^k$. In fact, we will embed all of $\Delta(R)$ into $\Delta(R')^k$ at once.

Indeed, given $v \in [\omega]^{k-1}$, define $F(v) \in [[k \times \omega]^{k-1}]^k$ to be the set of all order-preserving bijections from u to v, for some $u \in [k]^{k-1}$. Easily, then, for every $s \subseteq [\omega]^{k-1}$ finite, $s \in \Delta(R)$ if and only if $F[s] \in \Delta(R')^k$, i.e. $\bigcup F[s] \in \Delta(R')$.

$\square$

We now specialize to the situation we are really interested in.

Definition 4.2.6. For each $k \geq 2$, let R_k be a random k-ary graph on ω, and let $\Delta_k = \Delta(R_k)$.

So each $T_{k+1,k}$ admits Δ_k. Admitting Δ_k is a strong way of failing k-dimensional type amalgamation. Note that admitting Δ_2 in particular implies SOP_2, hence maximality in Keisler's order. So the main case of interest is in $k \geq 3$, although the $k = 2$ case fits into our theorems without problems.

We now aim to prove that $T_{k+1,k}$ is the $\trianglelefteq$-minimal theory admitting Δ_k. As a preliminary case, we have to show that if T admits Δ_k then T is unstable:

Lemma 4.2.7. Suppose $k \geq 2$. Then $\Delta(IP)$ is an instance of $(\Delta_k)^{k-1}$.

Proof. Let u_* be a $k-2$ element set (so $u_* = k-2$ works, but this would confuse the notation).

Let R be the k-ary graph on $\omega \times 2 \cup u_*$ consisting of all $w \in [(\omega \times 2) \cup u_*]^k$ of the form $u_* \cup \{(n,0),(n,1)\}$, for some $n < \omega$. Then $\Delta(R)$ is an instance of Δ_k, so it suffices to show that $\Delta(IP)$ is an instance of $\Delta(R)^{k-1}$.

Given $(n,i) \in \omega \times 2$, define $F(n,i)$ to be the set of all $v \in [u_* \cup \{(n,0),(n,1)\}]^{k-1}$ other than $u_* \cup \{(n,i)\}$. Then clearly, for any $s \subseteq \omega \times 2$ finite, $s \in \Delta(IP)$ if and only if $F[s] \in \Delta(R)^{k-1}$. □

Thus we get the following:

Theorem 4.2.8. Suppose $3 \leq k < \omega$, suppose $V \models ZFC^-$ is transitive, and suppose $\mathbf{j} : V \preceq \hat{V}$. Then $\hat{V}$ λ^+-pseudosaturates $T_{k+1,k}$ if and only if $\lambda < \lambda_{\hat{V}}(\Delta_k)$. In particular, $T_{k+1,k}$ is a $\trianglelefteq_1^\times$-minimal theory admitting Δ_k (so this is also true for $\trianglelefteq_\kappa^\times$ and $\trianglelefteq$).

Proof. Obviously if $\hat{V}$ λ^+-pseudosaturates $T_{k+1,k}$ then $\lambda < \lambda_{\hat{V}}(\Delta_k)$.

So suppose $\lambda < \lambda_{\hat{V}}(\Delta_k)$. Let $M \models T_{k+1,k}$ have universe ω, and let $p(x)$ be a pseudofinite partial type over $\mathbf{j}_{\mathrm{std}}(M)$ of cardinality at most λ, say $p(x)$ is over $\hat{n} < \hat{\omega}$. Write $M = (\omega, R)$, write $\mathbf{j}_{\mathrm{std}}(M) = \mathbf{j}(M) = (\hat{\omega}, \hat{R})$. (We also use R for the symbol in the language.)

Let $X_0 = \{\overline{a} \in \hat{n}^{k-1} : R(x, \overline{a}) \in p(x)\}$ and let $X_1 = \{\overline{a} \in \hat{n}^{k-1} : \neg R(x, \overline{a}) \in p(x)\}$. Since $[X_0]^{<\aleph_0} \subseteq \mathbf{j}(\Delta(R))$ we can find $\hat{X}_0' \in \mathbf{j}(\Delta(R))$ with $X_0 \subseteq \hat{X}_0'$. By Lemma 4.2.7 we can find disjoint $\hat{X}_0, \hat{X}_1 \subseteq \hat{n}$ with $X_0 \subseteq \hat{X}_0$ and $X_1 \subseteq \hat{X}_1$; we can suppose $\hat{X}_0 \subseteq \hat{X}_0'$.

Let $q(x) \in \hat{V}$ be the pseudofinite partial type, defined in $\hat{V}$ via: $q(x) = \{R(x, \overline{a}) : \overline{a} \in \hat{X}_0\} \cup \{\neg R(x, \overline{a}) : \overline{a} \in \hat{X}_1\}$. Clearly $p(x) \subseteq q(x)$ and $\hat{V}$ believes $q(x)$ is consistent, so we are done. □

4.3 Independent Systems

This technical section is as in [81] (joint with Shelah), with a few minor strengthenings. We define what we mean by independent systems of sets and models, and prove some facts we will need later.

The following definition is similar to the definition of stable system in Shelah [75] for stable theories, see Section XII.2. In fact we are modeling our definition after Fact 2.5 there (we cannot take the definition from [75] because we allow P to contain infinite subsets of I).

Alert: in the context of independent systems and amalgamation properties, we do not always work within the monster model $\mathfrak{C}$. We may say that a model $M \models T$ is floating if it is not an elementary substructure of $\mathfrak{C}$.

Definition 4.3.1. Let T be simple.

Suppose $\Delta \subseteq \mathcal{P}(I)$ is closed under finite intersections, and suppose $M \models T$. Say that $(A_s : s \in \Delta)$ is a system of subsets of M if each $A_s \subseteq M$ and $s \subseteq t$ implies $A_s \subseteq A_t$, and each $A_s \cap A_t = A_{s \cap t}$. Say that $(A_s : s \in \Delta)$ is an independent system if for all $s_i : i < n, t \in \Delta$, $\bigcup_{i<n} A_{s_i}$ is free from A_t over $\bigcup_{i<n} A_{s_i \cap t}$. If each A_s is an elementary submodel of M, we say that $(A_s : s \in \Delta)$ is a system of submodels of M.

Say that $(M_s : s \in \Delta)$ is a system of models if each $M_s \models T$ (possibly floating) and for each $s \in I$, $(M_t : t \in \Delta, t \subseteq s)$ is a system of submodels of M_s, and for all $s, t \in \Delta$, $M_s \cap M_t = M_{s \cap t}$. Say that $(M_s : s \in \Delta)$ is independent if for each $s \in P$, $(M_t : t \in \Delta, t \subseteq s)$ is independent. Finally, say that M is a solution to $(M_s : s \in \Delta)$ if M is a model of T and $(M_s : s \in \Delta)$ is an independent system of submodels of M.

If $(M_s : s \in \Delta)$ is a system of models, but not necessarily submodels of some model

M, then sometimes for emphasis we say that $(M_s : s \in \Delta)$ is a floating system of models.

The terminology "non-forking diagrams" is used in [81], but we prefer "independent systems" to align with [39]. Typically we will deal with the case where Δ is closed under subsets, i.e. is a pattern; the general case is harder, and we manage to avoid it in applications. We proceed in generality for now.

We need some technical lemmas.

Note the following has a corresponding statement for floating systems of models:

Lemma 4.3.2. Suppose $(A_s : s \in \Delta)$ is a system of subsets of M, where $\Delta \subseteq \mathcal{P}(I)$ is closed under finite intersections. Then the following are equivalent:

(A) For all downward-closed subsets $\Delta_0, \Delta_1 \subseteq \Delta$, $\bigcup_{s \in \Delta_0} A_s$ is free from $\bigcup_{s \in \Delta_2} A_s$ over $\bigcup_{s \in \Delta_0 \cap \Delta_1} A_s$.

(B) For all $s_i : i < n, t_j : j < m$ from Δ, $\bigcup_{i<n} A_{s_i}$ is free from $\bigcup_{j<m} A_{t_j}$ over $\bigcup_{i<n, j<m} A_{s_i \cap t_j}$.

(C) $(A_s : s \in \Delta)$ is independent.

Proof. (A) implies (B) implies (C) is trivial. For (B) implies (A), use local character of nonforking and monotonicity.

We show (C) implies (B). So suppose $(A_s : s \in \Delta)$ is non-forking. By induction on m, we show that for all n, if $s_i : i < n, t_j : j < m$ are from Δ, then $\bigcup_{i<n} A_{s_i}$ is free from $\bigcup_{j<m} A_{s_j}$ over $\bigcup_{i<n, j<m} A_{s_i \cap t_j}$. $m = 1$ is the definition of non-forking diagrams. Suppose true for all $m' \leq m$ and we show it holds at $m+1$; so we have $s_i : i < n, t_j : j < m+1$. Let $A_* = \bigcup_{i<n} A_{s_i}$ and let $B_* = \bigcup_{j<m} A_{t_j}$. By inductive hypothesis applies at $(s_i : i < n, t_m), (t_j : j < m)$, we get that $A_* \cup A_{t_m}$ is free from B_* over $(A_* \cup A_{t_m}) \cap B_*$. By monotonicity, A_* is free from $B_* \cup A_{t_m}$ over $(A_* \cap B_*) \cup A_{t_m}$. By the inductive hypothesis applied at $(s_i : i < n), t_m$, we get that A_* is free from A_{t_m} over $A_* \cap A_{t_m}$, so by monotonicity we get that A_* is free from $(A_* \cap B_*) \cup A_{t_m}$ over $A_* \cap (B_* \cup A_{t_m})$. $\square$

The following says that to find a solution to an amalgamation problem of models, it is enough to look at finite subproblems.

Lemma 4.3.3. Suppose $\Delta \subseteq \mathcal{P}(I)$ is closed under finite intersections, and suppose $(M_s : s \in \Delta)$ is an independent system of (floating) models. Suppose for every finite $\Delta' \subseteq \Delta$ closed under finite intersections, $(M_s : s \in \Delta')$ has a solution. Then $(M_s : s \in \Delta)$ has a solution.

Proof. Let $\mathcal{F}$ be the set of all tuples $((s_i : i < n), t, (\varphi_i(\overline{a}_i, \overline{b}) : i < n))$ where for some $n < \omega$, each $s_i, t \in \Delta$, and $\overline{b} \in M_t$, and each $\overline{a}_i \in M_{s_i}$, and $\bigwedge_{i<n} \varphi_i(\overline{x}_i, \overline{b})$ forks over $\bigcup_{i<n} M_{t \cap s_i}$. Let Γ be the following theory, where we allow the elements of the various M_s as constants: assert the elementary diagram of each M_s, and for each $((s_i : i < n), t, (\varphi_i(\overline{a}_i, \overline{b}) : i < n)) \in \mathcal{F}$, put in $\bigvee_{i<n} \neg\varphi_i(\overline{a}_i, \overline{b})$. By hypothesis, Γ is finitely satisfiable, and thus Γ is satisfiable. But this just means $(M_s : s \in \Delta)$ has a solution. $\square$

The following lemma is similar to Lemma 2.3 from [75] Section XII.2.

Lemma 4.3.4. Suppose $\Delta \subseteq \mathcal{P}(I)$ is closed under finite intersections, $M \models T$, and $(A_s : s \in \Delta)$ is a system of subsets of M. Suppose there is a well-ordering $<_*$ of $\bigcup_s A_s$ such that for all $a \in \bigcup_s A_s$, and for all $s_* \in \Delta$ with $a \in A_{s_*}$, a is free from $\{b \in \bigcup_s A_s : b <_* a\}$ over $\{b \in A_{s_*} : b <_* a\}$. Then $(A_s : s \in \Delta)$ is independent.

Proof. Let $(a_\alpha : \alpha < \alpha_*)$ be the $<_*$-increasing enumeration of $\bigcup_s A_s$. For each $\alpha \leq \alpha_*$ and for each $s \in \Delta$ let $A_{s,\alpha} = A_s \cap \{a_\beta : \beta < \alpha\}$. We show by induction on α that $(A_{s,\alpha} : s \in \Delta)$ is independent. In fact we show (B) holds of Lemma 4.3.2 (due to symmetry it is easier).

Limit stages are clear. So suppose we have shown $(A_{s,\alpha} : s \in \Delta)$ is independent. Let $(s_i : i < n), (t_j : j < m) \in \Delta$ be given. We wish to show that $\bigcup_{i<n} A_{s_i,\alpha+1}$ is free from $\bigcup_{j<m} A_{t_j \cap \alpha+1}$ over $\bigcup_{i<n,j<n} A_{s_i \cap t_j,\alpha+1}$. If $a_\alpha \notin s_i$ and $a_\alpha \notin t_j$ for each $i < n$ then we conclude by the inductive hypothesis. Finally, suppose $a_\alpha \in s_{i_*} \cap t_{j_*}$ for some $i_* < n$, $j_* < m$, then we conclude by the inductive hypothesis and the fact that a_α is free from $\bigcup_{i<n} A_{s_i,\alpha} \cup \bigcup_{j<m} A_{t_j,\alpha}$ over $A_{s_{i_*} \cap t_{j_*},\alpha}$, since $a_\alpha \in A_{s_{i_*} \cap t_{j_*}}$. If $a_\alpha \in s_i$ for some $i < n$ and $a_\alpha \notin t_j$ for any $j < m$, then reindex so that there is $0 < i_* \leq n$ so that $a_\alpha \in s_i$ if and only if $i < i_*$. Now a_α is free from $\{a_\beta : \beta < \alpha\}$ over $\bigcup_{i<n} A_{s_i,\alpha}$, so by monotonicity, $\bigcup_{i<n} A_{s_i,\alpha+1}$ is free from $\bigcup_{j<m} A_{s_j,\alpha+1}$ over $\bigcup_{i<n} A_{s_i,\alpha}$; use transitivity and the inductive hypothesis to finish.

□

And the following is a tweak (with the same proof). Note that such wellorderings $<_*$ exist exactly when $(\Delta, \subseteq)$ is well-founded, for example, $\Delta \subseteq [I]^{<\aleph_0}$.

Lemma 4.3.5. Suppose $\Delta \subseteq \mathcal{P}(I)$ is closed under finite intersections, $M \models T$, and $(A_s : s \in \Delta)$ is a system of subsets of M. Suppose there is a well-ordering $<_*$ of Δ such that for all for all $s \subseteq t \in \Delta$, $s \leq_* t$. Suppose for every $s \in \Delta$, A_s is free from $\bigcup_{t<_* s} A_t$ over $\bigcup_{t<_* s} A_{t \cap s} = \bigcup_{t \subsetneq s} A_t$. Then $(A_s : s \in \Delta)$ is independent.

In the supersimple case, we would always be able to restrict to considering $\Delta \subseteq [I]^{<\aleph_0}$. For general simple theories we cannot, but we can still get similar behavior:

Definition 4.3.6. Suppose $\mathcal{A} \subseteq \mathcal{P}(I)$. Then say that $\mathcal{A}$ is a frame if $\mathcal{A}$ is closed under finite unions, and $(\mathcal{A}, \subseteq)$ is well-founded, and for every $s \subseteq \lambda$, there are at most $|s| + \aleph_0$-many $t \in \mathcal{A}$ with $t \subseteq s$.

For example, $[I]^{<\aleph_0}$ is a frame. Frames will be useful for various inductive constructions, for instance:

Lemma 4.3.7. Suppose Δ is a pattern on I, $M \models T$ and $(A_s : s \in \Delta)$ is an independent system of subsets of Δ. Suppose there is a frame $\mathcal{A}$ such that for every $s \in P$, $A_s = \bigcup\{A_t : t \subseteq s, t \in \mathcal{A}\}$. If M is sufficiently saturated, then we can find an independent system $(M_s : s \in [\Delta]^{\leq \aleph_0})$ of submodels of M such that:

- For all $s \in [\Delta]^{\leq \aleph_0}$, $M_s = \bigcup\{M_t : t \in \mathcal{A}, t \subseteq s\}$;
- For all $s \in [\Delta]^{\leq \aleph_0}$, $A_s \subseteq M_s$ and $|M_s| = |A_s| + \aleph_0$;
- If for some $s \in [\Delta]^{\leq \aleph_0}$, we have that $A_t \preceq M$ for all $t \subseteq s$ with $t \in \mathcal{A}$ (in particular each such $A_t \models T$), then $M_s = M_t$.

Proof. Note that we can suppose $\Delta = [I]^{\leq \aleph_0}$, since we can define $A_s = \bigcup\{A_t : t \subseteq s, t \in \mathcal{A} \cap P\}$ for all $s \in [I]^{\leq \aleph_0}$.

Let $<_*$ be a well-ordering of $\mathcal{A}$ such that for all $s, t \in \mathcal{A}$, if $s \subseteq t$ then $s <_* t$. Now by induction on $<_*$, choose models $(M_s : s \in \mathcal{A})$ so that $M_s \supseteq A_s$ and such that M_s is free from $\bigcup_{t \in \Delta} A_t \cup \bigcup\{M_t : t \in \mathcal{A}, t <_* s\}$ over $A_s \cup \bigcup\{M_t : t \in \mathcal{A}, t \subsetneq s\}$. Note that we can choose $M_s \leq |A_s| + \aleph_0$, and if $A_t \preceq M$ for all $t \subseteq s$ with $t \in \mathcal{A}$, inductively we will have $M_t = A_t$ for each $t \subseteq s$ with $t \in \mathcal{A}$, and so we can choose $M_s = A_s$. Finally, given $s \in [I]^{\leq \aleph_0}$, let $M_s := \bigcup\{M_t : t \in \mathcal{A}, t \subseteq s\}$. This is an elementary submodel of M, since it is a direct limit of elementary submodels of M, since $\mathcal{A}$ is closed under unions. By Lemma 4.3.4, $(M_s : s \in [\lambda]^{\leq \aleph_0})$ is independent. $\square$

Corollary 4.3.8. Suppose Δ is a pattern on the finite index set I, $M \models T$ and $(M_s : s \in \Delta)$ is an independent system of (countable) submodels of M. If M is sufficiently saturated, then we can extend $(M_s : s \in \Delta)$ to an independent system $(M_s : s \subseteq I)$ of (countable) submodels of M.

Theorem 4.3.9. Suppose T is a simple theory in a countable language, and suppose $M_* \models T$ is sufficiently saturated, and suppose $A_* \subseteq M_*$ has size at most λ. Then we can find an independent system of countable submodels $(M_s : s \in [\lambda]^{\leq \aleph_0})$ of M_*, with $A_* \subseteq \bigcup_s M_s$. Further, we can find a frame $\mathcal{A} \subseteq [\lambda]^{\leq \aleph_0}$ such that for all $s \in [\lambda]^{\leq \aleph_0}$, $M_s = \bigcup \{M_t : t \in \mathcal{A}, t \subseteq s\}$.

For example, if T is supersimple then we could take $\mathcal{A} = [\lambda]^{<\aleph_0}$, by the same proof.

Proof. Enumerate $A_* = (a_\alpha : \alpha < \lambda)$.

We define $(\mathrm{cl}(\{\alpha\}) : \alpha < \lambda)$ inductively as follows, where each $\mathrm{cl}(\{\alpha\})$ is a countable subset $\alpha + 1$ with $\alpha \in \mathrm{cl}(\{\alpha\})$. Suppose we have defined $(\mathrm{cl}(\{\beta\}) : \beta < \alpha)$. Choose a countable set $\Gamma \subseteq \alpha$ such that a_α is free from $\{a_\beta : \beta < \alpha\}$ over $\bigcup_{\beta \in \Gamma} a_\beta$; put $\mathrm{cl}(\{\alpha\}) = \{\alpha\} \cup \bigcup_{\beta \in \Gamma} \mathrm{cl}(\{\beta\})$.

Now, for each $s \subseteq \lambda$, let $\mathrm{cl}(s) := \bigcup_{\alpha \in s} \mathrm{cl}(\{\alpha\})$. Say that $A \subseteq \lambda$ is closed if $\mathrm{cl}(A) = A$; this satisfies the usual properties of a set-theoretic closure operation, that is $\mathrm{cl}(A) \supseteq A$, and $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$, and $\mathrm{cl}^2(A) = \mathrm{cl}(A)$, and cl is finitary: in fact $\mathrm{cl}(A) = \bigcup_{\alpha \in A} \mathrm{cl}(\{\alpha\})$, which is even stronger. Finally, $|\mathrm{cl}(A)| \leq |A| + \aleph_0$.

For each $s \in [\lambda]^{\leq \omega}$, let $A_s = \{a_\alpha : \alpha < \lambda \text{ and } \mathrm{cl}(\{\alpha\}) \subseteq s\}$. Since each $a_\alpha \in A_{\mathrm{cl}(\{\alpha\})}$, clearly $\bigcup_s A_s = A_*$. I claim that $(A_s : s \in [\lambda]^{\leq \omega})$ is an independent system of subsets of M_*. But this follows from Lemma 4.3.4, since each a_α is free from $\{a_\beta : \beta < \alpha\}$ over $A_{\mathrm{cl}(\{\alpha\})} \cap \{a_\beta : \beta < \alpha\}$.

Write $\mathcal{A} = \{\mathrm{cl}(s) : s \in [\lambda]^{<\aleph_0}\}$. Note that for all $s \in [\lambda]^{\leq \aleph_0}$, $A_s = \bigcup \{A_t : t \subseteq s, t \in \mathcal{A}\}$. So by Lemma 4.3.7 it suffices to show that $\mathcal{A}$ is a frame. Clearly, for all $s \subseteq \lambda$, there are at most $|s^{<\aleph_0}| \leq |s| + \aleph_0$-many $t \subseteq s$ with $t \in \mathcal{A}$.

For each $\alpha \leq \lambda$, let $\mathcal{A}_\alpha = \{\mathrm{cl}(s) : s \in [\alpha]^{<\omega}\}$; so $\mathcal{A}_\lambda = \mathcal{A}$. I show by induction on $\alpha \leq \lambda$ that $(\mathcal{A}_\alpha, \subset)$ is well-founded. Since $\mathcal{A}_\alpha$ is an end extension of $\mathcal{A}_\beta$ for $\alpha > \beta$, the limit stage is clear. So suppose we have shown $(\mathcal{A}_\alpha, \subset)$ is well-founded.

Write $X = \mathrm{cl}(\{\alpha\}) \cap \alpha$; note that $\mathrm{cl}(X) = X$. Now suppose $s, t \in [\alpha]^{<\omega}$. I claim that $\mathrm{cl}(s \cup \{\alpha\}) \subseteq \mathrm{cl}(t \cup \{\alpha\})$ if and only if $\mathrm{cl}(s \cup X) \subseteq \mathrm{cl}(t \cup X)$. But this is clear, since $\mathrm{cl}(s \cup \{\alpha\}) = \mathrm{cl}(s) \cup X \cup \{\alpha\}$, and $\mathrm{cl}(t \cup \{\alpha\}) = \mathrm{cl}(t) \cup X \cup \{\alpha\}$, and $\mathrm{cl}(s \cup X) = \mathrm{cl}(s) \cup X$, and $\mathrm{cl}(t \cup X) = \mathrm{cl}(t) \cup X$.

Thus it follows from the inductive hypothesis that $(\{\mathrm{cl}(s \cup \{\alpha\}) : s \in [\alpha]^{<\omega}\}, \subset)$ is well-founded, and hence that $\mathcal{A}_{\alpha+1}$ is well-founded; hence $\mathcal{A}$ is well-founded. $\square$

4.4 Amalgamation properties

Suppose T is a simple theory in a countable language. We introduce a slew of k-ary amalgamation properties for T. In Section 4.11, we will show that if T is well-behaved then they are all equivalent; conjecturally, every simple theory is well-behaved.

Definition 4.4.1. Let $\mathbf{\Delta}$ be the class of all finite patterns (i.e. patterns on finite index sets).

Given $\Delta \in \mathbf{\Delta}$, say that T has Δ-amalgamation of models if every independent system of models $(M_s : s \in \Delta)$ has a solution.

If X is a set then let $\mathcal{P}^-(X)$ be the set of proper subsets of X.

Note that in determining whether or not T has Δ-amalgamation of models, it is enough to consider just countable models. Also, we warn the reader that there is no connection between whether or not T admits Δ, and whether or not T has Δ-amalgamation of models; indeed, every theory T admits every finite pattern.

Example 4.4.2. T_{rg}, the theory of the random graph, has Δ-amalgamation of models for all $\Delta \in \mathbf{\Delta}$. For each $3 \leq k < n$, $T_{\ell,k}$ has $\mathcal{P}^-(k)$-amalgamation of models but not

$\mathcal{P}^-(k+1)$-amalgamation of models. (If $k = 2$ then $T_{\ell,k}$ is not simple.) Every simple theory has $\mathcal{P}^-(3)$-amalgamation of models.

Example 4.4.3. It follows from Conclusion XII.2.12 of [75] that every stable theory has Δ-amalgamation of models for all $\Delta \in \mathbf{\Delta}$.

Lemma 4.4.4. Suppose $k < k'$, and T has $\mathcal{P}^-(k')$-amalgamation of models. Then T has $\mathcal{P}^-(k)$-amalgamation of models.

Proof. Suppose $(M_u : u \subsetneq k)$ is an independent system of models with no solution. For each $u \subsetneq k'$, define $N_u = M_\emptyset$ if $k' \backslash k \not\subseteq u$, and otherwise $N_u = M_{u \cap k}$. Easily this works. $\square$

We give two measures of complexity of patterns $\Delta \in \mathbf{\Delta}$.

Definition 4.4.5. Suppose $\Delta \in \mathbf{\Delta}$. Then let $\dim(\Delta)$ be the largest ℓ so that there is some $t \in [n]^\ell \backslash \Delta$ with $\mathcal{P}^-(t) \subseteq \Delta$. Let $\dim^*(\Delta)$ be the number of maximal elements of Δ.

Example 4.4.6. Each $\dim(\mathcal{P}^-(k)) = \dim^*(\mathcal{P}^-(k)) = \dim([N]^{<k}) = k$, but $\dim^*([N]^{<k})$ is large.

The following amalgamation notion was first introduced in [81]. It is hand-tailored to the $\leq_{SP}$-ordering, and in fact this amounts to being hand-tailored to the Keisler ordering as well. Given natural numbers n, m, we write ${}^n m$ for the set of functions from n to m, to avoid ambiguity with exponentiation m^n.

Definition 4.4.7. Given $\Lambda \subseteq {}^n m$, let Δ_Λ be the set of all partial functions from n to m which can be extended to an element of Λ; so Δ_Λ is a pattern on $n \times m$, and Λ is the set of maximal elements of Δ_λ.

By a Λ-array, we mean an independent system $(N_s : s \in \Delta_\Lambda)$ of submodels of $\mathbb{C}$ (or generally, any model M; the point is, not floating), together with maps $(\pi_{\eta,\eta'} : \eta, \eta' \in P_\Lambda)$ such that:

- Each $\pi_{\eta\eta'} : N_\eta \to N_{\eta'}$ is an isomorphism,

- For all η, η', η'', $\pi_{\eta',\eta''} \circ \pi_{\eta',\eta} = \pi_{\eta,\eta''}$;

- For all η, η', if we put $u = \{i < n : \eta(i) = \eta(i')\}$, and if we put $s = \eta \restriction_u = \eta' \restriction_u$, then $\pi_{\eta,\eta'} \restriction_{N_s}$ is the identity.

If $(\overline{N}, \overline{\pi})$ is a Λ-array, then $\overline{p}(x) = (p_\eta(x) : \eta \in \Lambda)$ is a coherent system of types over $(\overline{N}, \overline{\pi})$ if each $p_\eta(x)$ is a type over N_η which does not fork over N_0, and each $\pi_{\eta\eta'}[p_\eta(x)] = p_{\eta'}(x)$.

Definition 4.4.8. Suppose $\Lambda \subseteq {}^n m$. Then T has Λ-type amalgamation if, whenever $(N_s : s \in \Delta_\Lambda), (\pi_{\eta,\eta'} : \eta, \eta' \in \Lambda)$ is a Λ-array, and $(p_\eta(x) : \eta \in \Lambda)$ is a coherent system of types over $(\overline{N}, \overline{\pi})$, then $\bigcup_{\eta \in \Lambda} p_\eta(x)$ does not fork over N_0 (as computed in $\mathbb{C}$).

Let $\mathbf{\Lambda}$ be the set of all $\Lambda \subseteq {}^n m$, for varying $n, m < \omega$.

For each $2 \leq k < \omega$, let $\Lambda_k \subseteq {}^k 2$ be the set of all $\eta : k \to 2$ such that there is exactly one $i < k$ with $\eta(i) = 1$.

The following lemma is straightforward.

Lemma 4.4.9. Suppose $\Lambda \subseteq {}^n m$. Then in the definition of Λ-type amalgamation, the following changes would not matter:

(A) We could restrict to just countable models N_s.

(B) We could allow $p_\eta(x)$ to be any partial type, or insist it is a single formula. Also, we could replace x by a tuple $\overline{x}$ of arbitrary cardinality.

Example 4.4.10. If $|\Lambda| \leq 2$, then every simple theory has Λ-type amalgamation. T_{rg} has Λ-type amalgamation for all $\Lambda \in \mathbf{\Lambda}$.

Example 4.4.11. Suppose $\ell > k \geq 3$. Then $T_{\ell,k}$ has Λ-type amalgamation for all $|\Lambda| < k$, but fails Λ_k-type amalgamation.

Proof. This will follow from the fact that $T_{\ell,k}$ is well-behaved, and $T_{\ell,k}$ has $\mathcal{P}^-(k')$-amalgamation of models if and only if $k' \leq k$; see Example 4.11.2 and Theorem 4.11.12. $\square$

Example 4.4.12. Easily, every stable theory has Λ-type amalgamation for all Λ, since each $p_\eta(x)$ is the unique nonforking extension of $p_\eta(x) \upharpoonright_{N_\emptyset}$.

The following fact is the only unconditional implication we can prove between Δ-amalgamation of models and Λ-type amalgamation, although Theorem 4.11.12 suggests that there is more to say.

Theorem 4.4.13. If T has Δ-amalgamation of models for all $\Delta \in \boldsymbol{\Delta}$ with $\dim^*(\Delta) \leq k+1$ ($\dim(\Delta) \leq k+1$), then T has Λ-type amalgamation for all $\Lambda \in \boldsymbol{\Lambda}$ with $|\Lambda| \leq k$ ($\dim(\Delta_\Lambda) \leq k$).

Proof. Suppose $\Lambda \in \boldsymbol{\Lambda}$ is such that T failed Λ-type amalgamation, say via $(N_s : s \in \Delta_\Lambda), (\pi_{\eta,\eta'} : \eta, \eta' \in \Lambda), (p_\eta(x) : \eta \in \Lambda)$. We find $\Delta \in \boldsymbol{\Delta}_*$ such that $\dim(\Delta) = \dim(\Delta_\Lambda)+1$, and $\dim^*(\Delta) = |\Lambda| + 1$, and such that T fails Δ-amalgamation of models; this suffices.

Pick some $\eta \in \Lambda$, and extend $(N_s : s \subseteq \eta)$ to an independent system $N_s : s \subseteq \eta \cup \{*\}$ such that some element of $N_{\{*\}}$ realizes $p(x)$. Let $\Delta = \mathcal{P}(n \times m) \cup \{s \cup \{*\} : s \in \Delta_\Lambda\}$, a pattern on $n \times m \cup \{*\}$. Then it it not hard to find a floating independent system of models $(N_s : s \in \Delta)$ extending $(N_s : s \in \Delta_\Lambda)$ and $(N_s : s \subseteq \eta \cup \{*\})$, such that moreover the isomorphisms $\pi_{\eta,\eta'}$ lift to maps $\tau_{\eta,\eta'}$ such that:

- Each $\pi_{\eta\eta'} : N_{\eta \cup \{*\}} \to N_{\eta' \cup \{*\}}$ is an isomorphism,
- For all η, η', η'', $\pi_{\eta',\eta''} \circ \pi_{\eta',\eta} = \pi_{\eta,\eta''}$;

- For all η, η', if we put $u = \{i < n : \eta(i) = \eta(i')\}$, and if we put $s = \eta \restriction_u = \eta' \restriction_u$, then $\pi_{\eta,\eta'} \restriction_{N_{s\cup\{*\}}}$ is the identity.

Note that clearly, $\dim^*(\Delta) = |\Lambda| + 1$ (the maximal elements are $\{\eta \cup \{*\} : \eta \in \Lambda\}$ together with $n \times m$). Also, easily $\dim(\Delta) \geq \dim(\Delta_\Lambda) + 1$ (this is the unimportant direction).

We show $\dim(\Delta) \leq \dim(\Delta_\Lambda) + 1$. Suppose $\ell \leq \dim(\Delta)$. Then there is $t \in [n \times m \cup \{*\}]^\ell \backslash \Delta$ with $\mathcal{P}^-(t) \subseteq \Delta$. Necessarily $* \in T$ as otherwise we would have $t \subseteq n \times m$ and so $t \in \Delta$. Then easily $t \backslash \{*\}$ witnesses $\dim(\Delta_\Lambda) \geq \ell - 1$, i.e. $\ell \leq \dim(\Delta_\Lambda) + 1$, as desired. □

4.5 Forcing Iterations

In this section, we fix notation for forcing iterations and observe some basic facts about them; essentially we follow [44]. Recall that while we are interested in building forcing notions P, we will not be primarily interested in forcing extensions by P; rather, we will be interested in constructing ultrafilters on $\mathcal{B}(P)$. Nonetheless, using the language of forcing (and passing to forcing extensions) will simplify several arguments. First, a convenient definition:

Definition 4.5.1. Suppose $\dot{X}$ is a nice P-name (see Definition 2.3.1). Then a partition of $\mathcal{B}(P)$ by $\dot{X}$ is a map $\dot{A} : \mathrm{dom}(\dot{X}) \to \mathcal{B}(P)$ (so $\dot{A}$ is a nice P-name) such that:

- P forces that $\dot{A}$ has a single element $\bigcup \dot{A}$, which is in $\dot{X}$, and
- For all $\dot{a} \in \mathrm{dom}(\dot{X})$, $\dot{A}(\dot{a}) = \| \bigcup \dot{A} = \dot{a} \|_{\mathcal{B}}$, or equivalently for all $\dot{a}, \dot{b} \in \mathrm{dom}(\dot{X})$, $\dot{A}(\dot{a}) \geq \dot{A}(\dot{b}) \wedge \|\dot{a} = \dot{b}\|_{\mathcal{B}}$.

Define $\mathbb{N}_P(\dot{X})$, the set of nice names for elements of $\dot{X}$, to be the set of all $\bigcup \dot{A}$, for $\dot{A}$ a partition of $\mathcal{B}(P)$ by $\dot{X}$. The point is that when considering names for elements of $\dot{X}$, it is enough to consider just names in $\mathbb{N}_P(\dot{X})$, and the latter is in particular a set.

Easily, if $\dot{a}$ is a P-name such that some $p \in P$ forces $\dot{a} \in \dot{X}$, then p forces $\dot{a} = \dot{b}$ for some $\dot{b} \in \mathbb{N}_P(\dot{X})$.

Definition 4.5.2. Suppose $\alpha_* > 0$ is an ordinal. By a $< \theta$-support forcing iteration of length α_*, we mean sequences $(P_\alpha : \alpha \leq \alpha_*)$, $(\dot{Q}_\alpha : \alpha < \alpha_*)$, where:

- Each P_α is a forcing notion consisting of α-sequences, so $P_0 = \{0\}$ is the trivial forcing notion;

- For each $\alpha < \alpha_*$, $\dot{Q}_\alpha$ is a nice P_α-name for a forcing notion; we can always suppose P_α decides what $1^{\dot{Q}_\alpha}$ is;

- For each $\alpha < \alpha_*$, $P_{\alpha+1}$ is the set of all $\alpha + 1$-sequences p such that $p \restriction_\alpha \in P_\alpha$ and $p(\alpha) \in \mathbb{N}_{P_\alpha}(\dot{Q}_\alpha)$, and where $p \leq^{P_{\alpha+1}} q$ if $p \restriction_\alpha \leq^{P_\alpha} q \restriction_\alpha$ and p forces that $p(\alpha) \leq^{\dot{Q}_\alpha} q(\alpha)$.

- For all $\alpha \leq \alpha_*$ limit, P_α is the set of all α-sequences p such that for all $\beta < \alpha$, $p \restriction_\beta \in P_\beta$, and further, $\mathrm{supp}(p)$ has cardinality less than θ, where $\mathrm{supp}(p)$ is $\{\beta < \alpha : p(\beta) = 1^{\dot{Q}_\beta}\}$; put $p \leq^{P_\alpha} q$ if for all $\beta < \alpha$, $p \restriction_\beta \leq^{P_\beta} q \restriction_\beta$.

Note that $\dot{Q}_0$ is really just a forcing notion in $\mathbb{V}$, so we write it as Q_0. In the case $\alpha_* = 2$, we write $P_2 = Q_0 * \dot{Q}_1$.

Note that under our definitions, if P, Q are forcing notions, then $P * \dot{Q}$ is larger than $P \times Q$ (although they both have the same Boolean algebra completions). We also remind the reader of our notational deceit of always identifying forcing notions with their separative quotients. Indeed, forcing iterations $P * \dot{Q}$ are almost never separative, since as long as P and $\dot{Q}$ are nontrivial, then we can find $p \in P$ and distinct $\dot{q}_0, \dot{q}_1 \in \mathbb{N}_P(\dot{Q})$

such that $p \Vdash \dot{q}_0 = \dot{q}_1$. We just ignore this going forward; the concerned reader should take separative quotients everywhere.

It is a standard fact that if each P_α forces that $\dot{Q}_\alpha$ is θ-closed, or $< \theta$-distributive, then P_{α_*} is θ-closed or $< \theta$-distributive, respectively; see [44].

If $(P_\alpha : \alpha \leq \alpha_*), (\dot{Q}_\alpha : \alpha < \alpha_*)$ is a forcing iteration, then for all $\alpha < \beta \leq \alpha_*$, $\mathcal{B}(P_\alpha)$ is a complete subalgebra of $\mathcal{B}(P_\beta)$. It turns out we get projection maps in this scenario. These maps will be very helpful later:

Definition 4.5.3. Suppose $\mathcal{B}_0$ is a complete subalgebra of $\mathcal{B}_1$. Then define $\pi = \pi_{\mathcal{B}_1\mathcal{B}_0} : \mathcal{B}_1 \to \mathcal{B}_0$ as follows. Suppose $\mathbf{a} \in \mathcal{B}_1$; then let $\pi(\mathbf{a})$ be the meet of all $\mathbf{b} \in \mathcal{B}_0$ with $\mathbf{b} \geq \mathbf{a}$.

We now have a couple of lemmas exploring this notion.

Lemma 4.5.4. (A) Suppose $\mathcal{B}_0$ is a complete subalgebra of $\mathcal{B}_1$ and $\mathbf{a} \in \mathcal{B}_1$. Then $\pi(\mathbf{a}) \geq \mathbf{a}$, and is the least element of $\mathcal{B}_0$ satisfying this.

(B) Each $\pi_{\mathcal{B},\mathcal{B}}$ is the identity of $\mathcal{B}$. If $\mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2$ are complete subalgebras, then $\pi_{\mathcal{B}_1\mathcal{B}_0} \circ \pi_{\mathcal{B}_2\mathcal{B}_1} = \pi_{\mathcal{B}_2\mathcal{B}_0}$.

(C) Suppose $\mathcal{B}_0$ is a complete subalgebra of $\mathcal{B}_1$, and $\mathbf{a} \in \mathcal{B}_1$. Write $\pi = \pi_{\mathcal{B}_1\mathcal{B}_0}$. Then for every $\mathbf{b} \in \mathcal{B}_0$, $\mathbf{b} \wedge \pi(\mathbf{a})$ is nonzero if and only if $\mathbf{b} \wedge \mathbf{a}$ is nonzero. This characterizes $\pi(\mathbf{a})$.

Proof. (A): Let X be the set of all $\mathbf{b} \in \mathcal{B}_0$ with $\mathbf{b} \geq \mathbf{a}$. Since $\mathbf{a}$ is a lower bound to X, we get that $\mathbf{a} \leq \bigwedge X = \pi(\mathbf{a})$. The second statement is clear.

(B): Clearly, $\pi_{\mathcal{B}\mathcal{B}}$ is the identity. For the second part, suppose $\mathbf{a}_2 \in \mathcal{B}_2$ is given. Let X_{21} be the set of all $\mathbf{a} \in \mathcal{B}_1$ with $\mathbf{a} \geq \mathbf{a}_2$, and write $\mathbf{a}_{21} = \bigwedge X_{21}$. Similarly, let X_{20} be the set of all $\mathbf{a} \in \mathcal{B}_0$ with $\mathbf{a} \geq \mathbf{a}_0$, and write $\mathbf{a}_{20} = \bigwedge X_{20}$; and let X_{210} be the set of all $\mathbf{a} \in \mathcal{B}_0$ with $\mathbf{a} \geq \mathbf{a}_{21}$, and write $\mathbf{a}_{210} = \bigwedge X_{210}$. We wish to show that $\mathbf{a}_{210} = \mathbf{a}_{20}$; for this it suffices to show that $X_{210} = X_{20}$. That is, given $\mathbf{a} \in \mathcal{B}_0$, we show that $\mathbf{a} \geq \mathbf{a}_{21}$ if and only if $\mathbf{a} \geq \mathbf{a}_2$. By part (A) we have that $\mathbf{a}_{21} \geq \mathbf{a}_2$, so suppose $\mathbf{a} \geq \mathbf{a}_2$; we show $\mathbf{a} \geq \mathbf{a}_{21}$. But this is clear, since $\mathbf{a}$ must be in X_{21}.

(C): Since $\pi(\mathbf{a}) \geq \mathbf{a}$, we have that if $\mathbf{b} \wedge \mathbf{a}$ is nonzero, then so is $\mathbf{b} \wedge \pi(\mathbf{a})$. On the other hand, if $\mathbf{b} \wedge \mathbf{a} = 0$, then $\neg\mathbf{b} \geq \mathbf{a}$, so $\pi(\mathbf{a}) \geq \neg\mathbf{b}$, so $\mathbf{b} \wedge \pi(\mathbf{a}) = 0$. Uniqueness is clear. $\square$

We relate this to forcing:

Lemma 4.5.5. Suppose $(P, \dot{Q})$ is a two-step forcing iteration. Write $\mathcal{B}_1 = \mathcal{B}(P * \dot{Q})$, and write $\mathcal{B}_0 = \mathcal{B}(P)$, and write $\pi = \pi_{\mathcal{B}_1\mathcal{B}_0}$.

(A) Suppose $(q, \dot{q}) \in P * \dot{Q} \subseteq \mathcal{B}_1$; then $\pi(q, \dot{q}) = q$.

(B) More generally, if $\mathbf{a} = \bigvee_{\delta<\delta_*} (q_\delta, \dot{q}_\delta) \in \mathcal{B}_1$, then $\pi(\mathbf{a}) = \bigvee_{\delta<\delta_*} q_\delta$.

(C) If $\mathbf{a} \in \mathcal{B}_1$, then $\pi(\mathbf{a})$ is the join of all $q \in P$ such that for some $\dot{q} \in \mathbb{N}_P(\dot{Q})$, we have $(q, \dot{q}) \leq \mathbf{a}$.

(D) If $\mathbf{a} \in \mathcal{B}_1$, and $q \in P$, then $q \leq \pi(\mathbf{a})$ if and only if there is some $\dot{q} \in \mathbb{N}_P(\dot{Q})$ such that $(q, \dot{q}) \leq \mathbf{a}$. This characterizes $\pi(\mathbf{a})$.

Proof. (A) follows from (B).

(B): Write $\mathbf{a}_0 = \bigvee_{\delta<\delta_*} q_\delta$. We show that for all $\mathbf{b} \in \mathcal{B}_0$, $\mathbf{b} \wedge \mathbf{a}_0$ is nonzero if and only if $\mathbf{b} \wedge \mathbf{a}$ is nonzero; this suffices, by Lemma 4.5.4. Suppose $\mathbf{b} \wedge \mathbf{a}_0$ is nonzero; then we can find $\delta < \delta_*$ such that $\mathbf{b} \wedge q_\delta$ is nonzero. Choose $q \in P$ with $q \leq \mathbf{b} \wedge q_\delta$; then $(q, \dot{q}_\delta) \leq \mathbf{b} \wedge \mathbf{a}$ is nonzero, as desired. Conversely, if $\mathbf{b} \wedge \mathbf{a}_0$ is nonzero, then we can find $\delta < \delta_*$ such that $\mathbf{b} \wedge (q_\delta, \dot{q}_\delta)$ is nonzero; thus $\mathbf{b} \wedge q$ is nonzero. (We are identifying $q \in P$ with $(q, 0)$ in $P * \dot{Q}$, naturally.)

(C) follows from (B) (let $(q_\delta, \dot{q}_\delta) : \delta < \delta_*$ list all elements of $P * \dot{Q}$ below $\mathbf{a}$).

(D): let X be the set of all $q \in P$ such that there is some $\dot{q} \in \mathbb{N}_P(\dot{Q})$ with $(q, \dot{q}) \leq \mathbf{a}$. By (C) (or (B)), $\pi(\mathbf{a}) = \bigvee X$. Thus, whenever $q \in X$ then $q \leq \pi(\mathbf{a})$. (D) asks for the converse. So suppose $q \leq \pi(\mathbf{a})$. Let C be a maximal antichain of P below q, such that $C \subseteq X$ (this is possible since $\pi(\mathbf{a}) = \bigvee X$). For each $p \in C$ choose $\dot{p}(p) \in \dot{Q}$ such that $(p, \dot{p}(p)) \leq \mathbf{a}$. Let $\dot{q} \in \mathbb{N}_P(\dot{Q})$ be the unique P-name for the element of $\dot{Q}$, such that for each $p \in C$, $p \Vdash \dot{q} = \dot{p}(p)$. I claim that $(q, \dot{q}) \leq \mathbf{a}$). It suffices to show that if $(r, \dot{r}) \leq (q, \dot{q})$, then $(r, \dot{r})$ is compatible with $(p, \dot{p}(p))$ for some $p \in C$; but this is clear, since we can find some $p \in C$ such that r is compatible with p (since $q = \bigvee C$), so choose $r' \leq r \wedge p$. Then $r' \Vdash \dot{p}(p) = \dot{q}$, so $(\dot{r}, \dot{p}(p))$ and $(\dot{r}, \dot{q})$ are compatible (recalling our convention that we really always work in the separative quotient). Uniqueness is clear. $\square$

4.6 Coloring Properties of Forcing Notions

This section is as in [81], a joint work with Shelah, although we change the notation somewhat. In that paper, we are concerned with constructing dividing lines associated to $\leq_{SP}$. It turns out the combinatorial questions at stake with $\leq_{SP}$ are deeply intertwined with those for Keisler's order, although we will not discuss $\leq_{SP}$ further in this thesis.

The following definition will be key. The main case of interest is when $k < \aleph_0$ (and this is the reason for the choice of letter), but nothing is gained from this assumption in general.

Definition 4.6.1. Suppose P, R are forcing notions and $k \geq 3$ is a cardinal (usually finite). Say that $f : P \to R$ is a weak (R, k)-coloring of P if for every sequence $(p_i : i < i_*)$ from P of length less $i_* < k$, if $(f(p_i) : i < i_*)$ is compatible in R, then $(p_i : i < i_*)$ is compatible in P. Say that P is weakly (R, k)-colorable if there there is some such f.

One can view this as a generalization of chromatic numbers. Specifically, given a forcing notion P and given an integer n, one can form the hypergraph $H_n := \{s \in [P]^n : s \text{ has no lower bound in } P\}$. In the case when $R = (\mu, |)$ (i.e. the partial order with domain μ in which all $\alpha, \beta < \mu$ are incomparable), we have that P is weakly $(R, n+1)$-colorable if and only if $\chi(H_n) \leq \mu$, where χ is the chromatic number.

We will mainly be interested in the following examples.

Example 4.6.2. Suppose T is a countable simple theory, $M \models T$ has $|M| \leq \lambda$, and $M_0 \preceq M$ is countable. Then let Γ^θ_{M,M_0} be the forcing notion of all partial types $p(x)$ over M of cardinality less than θ, which do not fork over M_0; we order Γ^θ_{M,M_0} by reverse inclusion.

We will be interested in when Γ^θ_{M,M_0} is weakly (R, k)-colorable for various R. Note that Γ^θ_{M,M_0} is always θ-closed. Further, Γ^θ_{M,M_0} has the greatest lower bounds property: any subset of Γ^θ_{M,M_0} with a lower bound has a greatest such bound (namely, take the union).

The following will follow from Theorem 4.6.5 (and has a somewhat easier proof):

Theorem. Suppose θ is a regular cardinal, $(P_\alpha : \alpha \leq \alpha_*, \dot{Q}_\alpha : \alpha < \alpha_*)$ is a $< \theta$-support forcing iteration, and suppose R is a forcing notion. Suppose $3 \leq k \leq \theta$, and each P_α forces that $\dot{Q}_\alpha$ is θ-closed, has the greatest lower bounds property, and is weakly $(\check{R}, k)$-colorable. Then P_{α_*} is weakly $(\prod_{\alpha_*} R, k)$-colorable, where $\prod_{\alpha_*} R$ is the $< \theta$-support product of α_*-many copies of R.

In fact, this would be enough for our applications, but we find it unsatisfying that the hypotheses are not fully preserved. Namely, the greatest lower bound property is not necessarily preserved under $< \theta$-forcing iterations. We find the following sweet spot, intermediate between being weakly (R, k) colorable, and being weakly (R, k)-colorable and

θ-closed and having the greatest lower bounds property.

Definition 4.6.3. Suppose P, R are forcing notions and $3 \leq k \leq \theta$. Then say that P is (R, k)-colorable if for some dense subset P_0 of P, there is some $f : P_0 \to R$, such that for every sequence $(p_i : i < i_*)$ from P_0 of length less $i_* < k$, if $(f(p_i) : i < i_*)$ is compatible in R, then $(p_i : i < i_*)$ has a greatest lower bound in P. We also say that $f : (P, P_0) \to R$ is an (R, k)-coloring of P. Say that P is (R, k)-colorable if there exist some such P_0, f.

Say that P has greatest lower bounds for $< \theta$-chains if whenever $(p_\alpha : \alpha < \alpha_*)$ is a descending chain from P of length $\alpha_* < \theta$, then $(p_\alpha : \alpha < \alpha_*)$ has a greatest lower bound in P. (In particular, this implies P is θ-closed.)

The following lemma sums up some immediate facts.

Lemma 4.6.4. Suppose θ is a regular cardinal, $3 \leq k \leq \theta$, and P, R are forcing notions.

1. If P is weakly (R, k)-colorable, then so is every dense subset of $\mathcal{B}(P)_+$.

2. If P is (R, k)-colorable then P is weakly (R, k)-colorable.

3. If P is (R, k)-colorable and R is weakly (R', k)-colorable, then P is (R', k)-colorable.

4. If P has the greatest lower bound property, then P is (R, k)-colorable if and only if P is weakly (R, k)-colorable.

Proof. (1): Suppose $Q \subseteq \mathcal{B}(P)_+$ is dense, and suppose $f : P \to R$ is an (R, k)-coloring of P. Choose a function $g : Q \to R$ so that for all $q \in Q$, there is $p \leq q$ with $p \in P$ and $g(q) = f(p)$. Then g is clearly an (R, k)-coloring of P.

(2): Suppose $f : (P, P_0) \to R$ is an (R, k)-coloring of P. Then clearly f is a weak (R, k)-coloring of P; since P_0 is dense in P, we can use (1) to get a weak (R, k)-coloring of P.

(3): Immediate, by composing the maps.

(4): Immediate. $\square$

Now we prove that k-colorability is preserved under $< \theta$-support forcing iterations, if we add the requirement of greatest lower bounds for $< \theta$-chains. The case $\theta > \aleph_0$ is proven in [81], although the case $\theta = \aleph_0$ is strictly easier.

Theorem 4.6.5. Suppose θ is a regular cardinal, $(P_\alpha : \alpha \leq \alpha_*, \dot{Q}_\alpha : \alpha < \alpha_*)$ is a $< \theta$-support forcing iteration, and suppose R is a forcing notion. Suppose $3 \leq k \leq \theta$, and each P_α forces that $\dot{Q}_\alpha$ has greatest lower bounds for $< \theta$-chains, and is $(\check{R}, k)$-colorable. Then P_{α_*} has greatest lower bounds for $< \theta$-chains, and is $(\prod_{\alpha_*} R, k)$-colorable, where $\prod_{\alpha_*} R$ is the $< \theta$-support product of α_*-many copies of R.

Proof. Suppose $(P_\alpha : \alpha \leq \alpha_*), (\dot{Q}_\alpha : \alpha < \alpha_*)$ and R are given. We can suppose inductively that each P_α has greatest lower bounds for $< \theta$-chains and is $(\check{R}, k)$-colorable; in particular, each P_α is θ-closed, and thus does not add sequences of length less than θ. Actually, we won't formally need this inductive hypothesis.

For each $\alpha < \alpha_*$, P_α forces there is some $(\check{R}, k)$-coloring $\dot{F}_\alpha : (\dot{Q}_\alpha, \dot{Q}^0_\alpha) \to \check{R}$ of $\dot{Q}_\alpha$ (so $\dot{Q}^0_\alpha$ is forced to be a dense subset of $\dot{Q}_\alpha$, and $\dot{F}_\alpha : \dot{Q}^0_\alpha \to \check{R}$). We can suppose P_α forces that $\dot{F}_\alpha(1) = 1$. Since each $\dot{Q}^0_\alpha$ is forced by P_α to be dense in $\dot{Q}_\alpha$, the sequence $(\dot{Q}^0_\alpha : \alpha < \alpha_*)$ induces a forcing iteration $(P^0_\alpha : \alpha \leq \alpha_*), (\dot{Q}^0_\alpha : \alpha < \alpha_*)$, with each P^0_α dense in P_α (so these are equivalent forcing iterations).

We now split into cases depending on whether $\theta = \aleph_0$.

Case 1. Suppose $\theta = \aleph_0$. Note then that the greatest lower bounds for $< \theta$-chains property is vacuous.

Let $R' = \prod_{\alpha_*} R$ be the finite support product of α_*-many copies of R; we show that

P_{α_*} is (R', k)-colorable.

Let $P^0 \subseteq P^0_{\alpha_*}$ be the set of all p such that for each $\alpha < \alpha_*$, $p \restriction_\alpha$ decides $\dot{F}_\alpha(p(\alpha))$. I claim that P^0 is dense in $P^0_{\alpha_*}$ (and hence in P_{α_*}). Suppose towards a contradiction $p \in P^0_{\alpha_*}$ had no extension in P^0. Write $p_0 = p$. Having defined $p_n \leq p$, let $\alpha_n < \alpha_*$ be largest so that $p_n \restriction_{\alpha_n}$ does not decide $\dot{F}_{\alpha_n}(p_n(\alpha_n))$ (this is possible since $\mathrm{supp}(p_n)$ is finite). Choose $q_n \leq p_n \restriction_{\alpha_n}$ in $P^0_{\alpha_n}$ which decides $\dot{F}_{\alpha_n}(p_n(\alpha_n))$. Let $p_{n+1} \in P^0_{\alpha_*}$ be defined by: $p_{n+1}(\alpha) = p_n(\alpha)$ for all $\alpha \geq \alpha_n$, and $p_{n+1}(\alpha) = q_n(\alpha)$ for all $\alpha < \alpha_n$. Then $p_{n+1} < p_n$ and we can continue. But this will give an infinite decreasing sequence of ordinals $(\alpha_n : n < \omega)$.

Thus P^0 is dense in $P^0_{\alpha_*}$. We now find an (R', k)-coloring $F : (P_{\alpha_*}, P^0) \to R'$. Given $p \in P^0$ and $\alpha < \alpha_*$, let $r_\alpha(p) \in R$ be such that $p \restriction_\alpha$ forces that $\dot{F}_\alpha(p(\alpha)) = \check{r}_\alpha(\check{p})$. Let $r = (r_\alpha : \alpha < \alpha_*)$; since $r_\alpha = 1$ whenever $\alpha \notin \mathrm{supp}(p)$, we have that $r \in R'$. Define $F(p) = r$.

Now suppose $(p_i : i < i_*)$ is a sequence from P^0 with $i_* < k$, such that $(F(p_i) : i < i_*)$ are compatible in R'. Write $\Gamma = \bigcup_{i<i_*} \mathrm{supp}(p_i)$.

By induction $\alpha \leq \alpha_*$, we construct a greatest lower bound s_α to $(p_i \restriction_\alpha : i < i_*)$ in P_α, such that $\mathrm{supp}(s_\alpha) \subseteq \Gamma \cap \alpha$, and for $\alpha < \alpha'$, $s_{\alpha'} \restriction_\alpha = s_\alpha$.

Limit stages of the induction are clear. So suppose we have constructed s_α. If $\alpha \notin \Gamma$ clearly we can let $s_{\alpha+1} = s_\alpha{}^\frown(1^{\dot{Q}_\alpha})$; so suppose instead $\alpha \in \Gamma$. s_α forces that each $\dot{F}_\alpha(p_i(\alpha)) = \check{r}_\alpha(\check{p}_i)$, and $(r_\alpha(p_i) : i < i_*)$ are compatible in R_α, thus we can choose $\dot{q} \in \mathbb{N}_{P_\alpha}(\dot{Q}_\alpha)$, such that s_α forces $\dot{q}$ is the greatest lower bound to $(p_i(\alpha) : i < i_*)$ in $\dot{Q}_\alpha$. Let $s_{\alpha+1} = s_\alpha{}^\frown(\dot{q})$.

Thus the induction goes through, and s_{α_*} is a lower bound to $(p_i : i < i_*)$.

Case 2. Suppose $\theta > \aleph_0$.

We begin by proving the following easy claim.

Claim. P_{α_*} has greatest lower bounds for $< \theta$-chains. In fact, suppose $\gamma_* < \theta$, and $(p_\gamma : \gamma < \gamma_*)$ is a descending chain from P_{α_*}; then it has a greatest lower bound p in P_{α_*}, such that $\mathrm{supp}(p) \subseteq \bigcup_{\gamma<\gamma_*} \mathrm{supp}(p_\gamma)$.

Proof. By induction on $\alpha \leq \alpha_*$, we construct $(q_\alpha : \alpha \leq \alpha_*)$ such that each $q_\alpha \in P_\alpha$ with $\mathrm{supp}(q_\alpha) \subseteq \bigcup_{\gamma<\gamma_*} \mathrm{supp}(p_\gamma) \cap \alpha$, and for $\alpha < \beta \leq \alpha_*$, $q_\beta \restriction_\alpha = q_\alpha$, and for each $\alpha \leq \alpha_*$, q_α is a greatest lower bound to $(p_\gamma \restriction_\alpha : \gamma < \gamma_*)$ in P_α. At limit stages there is nothing to do; so suppose we have defined q_α. If $\alpha \notin \bigcup_{\gamma<\gamma_*} \mathrm{supp}(p_\gamma)$ then let $q_{\alpha+1} = q_\alpha {}^\frown (1^{\dot{Q}_\alpha})$. Otherwise, since q_α forces that $(p_\gamma(\alpha) : \gamma < \gamma_*)$ is a descending chain from $\dot{Q}_\alpha$, we can find $\dot{q}$, a P_α-name for an element of $\dot{Q}_\alpha$, such that q_α forces $\dot{q}$ is the greatest lower bound. Let $q_{\alpha+1} = q_\alpha {}^\frown (\dot{q})$. $\square$

Now, if $\alpha_* < \aleph_0$, then we can finish as in Case 1 (since finite iterations are also finite support iterations). Thus we can suppose $\alpha_* \geq \aleph_0$. Let $R' = \prod_{\omega\times\alpha_*} R$ be the finite support product of $\omega \times \alpha_*$-many copies of R; we show that P_{α_*} is (R', k)-colorable. (The only reason we need $\alpha_* \geq \aleph_0$ is to get $R' \cong \prod_{\alpha_*} R$.)

Fix some $p \in P^0_{\alpha_*}$ for a while. Note that $\mathrm{supp}(p) \in [\alpha_*]^{<\theta}$.

It is easy to find, for each $n < \omega$, elements $\mathbf{q}_n(p) \in P^0_{\alpha_*}$ with $\mathbf{q}_0(p) = p$, so that for all $n < \omega$:

- $\mathbf{q}_{n+1}(p) \leq \mathbf{q}_n(p)$;
- For all $\alpha < \alpha_*$, $\mathbf{q}_{n+1}(p) \restriction_\alpha$ decides $\dot{F}_\alpha(\mathbf{q}_n(\alpha))$.

For each $n > 0$ and for each $\alpha < \alpha_*$, we can find $r_{n-1,\alpha}(p) \in R$ such that $q_n \restriction_\alpha$ forces that $\dot{F}_\alpha(q_{n-1}(\alpha)) = \check{r}_{n-1,\alpha}(p)$. (Whenever $\alpha \notin \mathrm{supp}(\mathbf{a}_{n-1})$, we have $r_{n-1,\alpha}(p) = 0$.)

Let $\mathbf{q}_\omega(p) \in P_{\alpha_*}$ be the greatest lower bound of $(\mathbf{q}_n(p) : n < \omega)$; this exists by the claim.

Define $P^0 = \{\mathbf{q}_\omega(p) : p \in P^0_{\alpha_*}\}$. For each $q \in P^0$, choose $\mathbf{p}(q) \in P^0_{\alpha_*}$ such that $q = \mathbf{q}_\omega(\mathbf{p}(q))$. For each $n < \omega$, let $\mathbf{p}_n(q) = \mathbf{q}_n(\mathbf{p}(q))$, and for each $\alpha < \alpha_*$, let $r_{n,\alpha}(q) = r_{n,\alpha}(\mathbf{p}(q))$.

We have arranged that for all $q \in P^0$, q is the greatest lower bound of $(\mathbf{p}_n(q) : n < \omega)$, and for all $n < \omega$ and $\alpha < \alpha_*$, $\mathbf{p}_{n+1}(q) \restriction_\alpha$ forces that $\dot{F}_\alpha(\mathbf{p}_n(q)(\alpha)) = \check{r}_{n,\alpha}(\check{q})$.

Define $F : P^0 \to R'$ via $F(q) = (r_{n,\alpha}(q) : \alpha < \alpha_*, n < \omega)$. I claim that $F : (P, P_0) \to R'$ is an (R', k)-coloring.

So suppose $(q_i : i < i_*)$ is a sequence from P^0 with $i_* < k$, such that $(F(q_i) : i < i_*)$ are compatible. Write $\Gamma = \bigcup_{i<i_*,n<\omega} \mathrm{supp}(\mathbf{p}_n(q_i))$.

By induction $\alpha \leq \alpha_*$, we construct a greatest lower bound s_α to $(\mathbf{p}_n(q_i) \restriction_\alpha : i < i_*, n < \omega)$ in P_α, such that $\mathrm{supp}(s_\alpha) \subseteq \Gamma \cap \alpha$, and for $\alpha < \alpha'$, $s_{\alpha'} \restriction_\alpha = s_\alpha$.

Limit stages of the induction are clear. So suppose we have constructed s_α. If $\alpha \notin \Gamma$ clearly we can let $s_{\alpha+1} = s_\alpha{}^\frown(1^{\dot{Q}_\alpha})$; so suppose instead $\alpha \in \Gamma$. Let $n < \omega$ be given. Then $(r_{n,\alpha}(q_i) : i < i_*)$ are compatible, and s_α forces that $\dot{F}_\alpha(\mathbf{p}_n(q_i)(\alpha)) = \check{r}_{n,\alpha}(\check{r}_i)$ for each $i < i_*$, since $\mathbf{p}_{n+1}(q_i) \restriction_\alpha$ does. Thus s_α forces that $(\mathbf{p}_n(q_i)(\alpha) : i < i_*)$ has the greatest lower bound $\dot{s}_n$ in $\dot{Q}_\alpha$. Now s_α forces that $(\dot{s}_n : n < \omega)$ is a descending chain in $\dot{Q}_\alpha$, and hence has the greatest lower bound $\dot{s}$. Let $s_{\alpha+1} = s_\alpha{}^\frown(\dot{s})$.

Thus the induction goes through, and s_{α_*} is a greatest lower bound $(q_i : i < i_*)$ in P_{α_*}. $\square$

Definition 4.6.6. Suppose X is a set and $\overline{\mu} = (\mu_x : x \in X)$ is a sequence of ordinals indexed by X (usually but not necessarily cardinals). Then let $P_{X\overline{\mu}\theta}$ be the set of all partial functions $f : X \to \kappa$ of cardinality less than θ, such that for all $x \in X$, $f(x) < \mu_x$. Given a suitable sequence $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$, let $\mathbb{P}_{\mathbf{s},\infty}$ be the class of all forcing notions of the form $P_{X\overline{\mu}\theta}$, for some set X and some sequence $\overline{\mu}$ with each $\mu_x < \kappa$.

So in the special case when $\overline{\mu}$ is constant with value $\mu < \kappa$, then $P_{X\overline{\mu}\theta} = P_{X\mu\theta}$, the set of all partial functions from X to μ of cardinality less than θ.

In [56] and the sequel [57], Malliaris and Shelah obtain dividing lines in Keisler's order by constructing sufficiently generic ultrafilters on the Boolean-algebra completion of $P_{2^\lambda\mu\theta}$, so they were essentially dealing with $\mathbb{P}_{\mathbf{s},\infty}$ (in the case where $\kappa = \mu^+$ is a successor cardinal). In order to detect various amalgamation properties of theories, they varied $\kappa \leq \lambda \leq \kappa^{+\omega}$. The set theory for this is delicate; working with the following class of forcing notions allows us to avoid these difficulties and obtain sharper results.

Definition 4.6.7. Suppose $3 \leq k \leq \theta$. Then let $\mathbb{P}_{\mathbf{s},k}$ be the class of all forcing notions P which have greatest lower bounds for $< \theta$-chains, and are (R, k)-colorable for some $R \in \mathbb{P}_{\mathbf{s},\infty}$.

We view it as unlikely that there is any model-theoretic information to be gained by varying $k \geq \aleph_0$, but we cannot prove this.

We now some key properties of $\mathbb{P}_{\mathbf{s},k}$.

Theorem 4.6.8. Suppose $3 \leq k \leq \theta$. Then:

(A) For every $P \in \mathbb{P}_{\mathbf{s},k}$, P is θ-closed and has the κ-c.c.

(B) Suppose $P, Q \in \mathbb{P}_{\mathbf{s},k}$. Then P forces that $\check{Q} \in \mathbb{P}_{\mathbf{s},k}^{\mathbb{V}[\dot{G}]}$.

(C) Suppose $(P_\alpha : \alpha \leq \alpha_*), (\dot{Q}_\alpha : \alpha < \alpha_*)$ is a $< \theta$-support forcing iteration, such that each P_α forces $\dot{Q}_\alpha \in \mathbb{P}_{\mathbf{s},k}^{\mathbb{V}[\dot{G}_\alpha]}$, where $\dot{G}_\alpha$ is the P_α-generic name. Then $P_{\alpha_*} \in \mathbb{P}_{\mathbf{s},k}$.

(D) $\mathbb{P}_{\mathbf{s},k}$ is closed under $< \theta$-support products.

Proof. (A): P is θ-closed by definition of $\mathbb{P}_{\mathbf{s},k}$. For the κ-c.c.: we can find some $R \in \mathbb{P}_{\mathbf{s},k}$, and some weak $(R, 3)$-coloring $F : P \to R$. Now R has the κ-c.c. by the Δ-system lemma, so it immediately follows that P has the κ-c.c.

(B): If $F : (P, P_0) \to R$ is an (R, k)-coloring, then this will continue to work in forcing extensions, because $k \leq \theta$ and P is θ-closed.

(C): We want to apply Theorem 4.6.5. To do so, we need to find some $R \in \mathbb{P}_{\mathbf{s},\infty}$ such that each P_α forces $\dot{Q}_\alpha$ is (R, k)-colorable. Let $R_0 = P_{\kappa\overline{\mu}\theta}$ where $\overline{\mu} = (\mu_\alpha : \alpha < \kappa)$ is defined by $\mu_\alpha = |\alpha|$. Then for χ large enough, we can take R to be the $< \theta$-support product of χ-many copies of R_0.

(D) follows immediately from (B) and (C). $\square$

4.7 The Ultrafilter Constructions

In this section, we give a streamlined construction of the perfect and optimal ultrafilters of Malliaris and Shelah from [56], [57], using the ideas of Section 4.6.

Suppose $\mathbf{s}$ is a suitable sequence and $3 \leq k \leq \theta$. Our goal is to build a long forcing iteration $(P_\alpha : \alpha \leq \alpha_*, \dot{Q}_\alpha : \alpha < \alpha_*)$ from $\mathbb{P}_{\mathbf{s},k}$, then build a sufficiently generic ultrafilter on $\mathcal{B}(P_{\alpha_*})$, and then check which theories it λ^+-saturates. Our treatment differs from that of Malliaris and Shelah in two respects. First, Malliaris and Shelah use $\mathbb{P}_{\mathbf{s},\infty}$ rather than $\mathbb{P}_{\mathbf{s},k}$; our approach allows us to circumvent some ingenious but ad-hoc coding methods (e.g. "collision detection") and obtain sharper bounds on our final dividing lines. Second, in our construction of $(P_\alpha : \alpha \leq \alpha_*)$, $(\dot{Q}_\alpha : \alpha < \alpha_*)$, we will be anticipating not only (λ, T)-Łoś maps in $\mathcal{B}(P_\alpha)$, but also entire ultrafilters on $\mathcal{B}(P_\alpha)$. This is a relatively minor change; the upshot is that we get a better handle on which theories our eventual ultrafilter will λ^+-saturate. On the other hand, we lose control over the length α_* of the forcing iteration; Mallairis and Shelah always arranged $\alpha_* = 2^\lambda$. In view of Theorem 3.16.19, this is not a serious drawback; so long as our eventual Boolean algebra $\mathcal{B}$ has the λ^+-c.c., then every ultrafilter on $\mathcal{B}$ is Keisler-equivalent to one on a complete subalgebra of $\mathcal{B}$ of size at most 2^λ.

Looking beneath these differences, our construction is really the same as Malliaris and Shelah's.

The following definition will be convenient:

Definition 4.7.1. If $\mathbf{A}$ is a λ-distribution in $\mathcal{B}$, then for $S \subseteq \lambda$, define $\mathbf{A}(S) = \bigwedge_{s \in [S]^{<\aleph_0}} \mathbf{A}(s)$. This need not be nonzero, but if $\mathbf{A}$ is in a σ-complete ultrafilter $\mathcal{U}$ and $|S| < \sigma$, then it will be.

The following lemma describes the situation we will be interested in while building our generic ultrafilter $\mathcal{U}$ on $\mathcal{B}(P_{\alpha_*})$.

Lemma 4.7.2. Suppose $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence, and $3 \leq k \leq \theta$, and T is a complete first order theory. Suppose $P \in \mathbb{P}_{\mathbf{s},k}$, and $\mathcal{U}$ is a σ-complete ultrafilter on $\mathcal{B}(P)$, and $\mathbf{A}$ is a λ-distribution in $\mathcal{U}$. Then the following are equivalent:

(A) There is some $\dot{Q} \in \mathbb{P}_{\mathbf{s},k}^{\mathbb{V}[\dot{G}]}$ and some multiplicative refinement $\mathbf{B}$ of $\mathbf{A}$ in $\mathcal{B}(P * \dot{Q})$, such that for every $S \in [\lambda]^{<\sigma}$, $\pi(\mathbf{B}(S)) \in \mathcal{U}$. (Here $\dot{G}$ is the P-generic name, and $\pi = \pi_{\mathcal{B}(P*\dot{Q}),\mathcal{B}(P)} : \mathcal{B}(P * \dot{Q}) \to \mathcal{B}(P)$ is the projection map.)

(B) There is some $\dot{Q} \in \mathbb{P}^{\mathbb{V}[\dot{G}]}$ and some σ-complete ultrafilter $\mathcal{V}$ on $\mathcal{B}(P * \dot{Q})$ extending $\mathcal{U}$, such that $\mathbf{A}$ has a multiplicative refinement $\mathbf{B}$ in $\mathcal{V}$.

Proof. (A) implies (B): for each $S \in [\lambda]^{<\sigma}$, we have that $\pi(\mathbf{B}(S)) \in \mathcal{U}$. By Lemma 4.5.4(C), it follows that $\mathcal{U} \cup \{\mathbf{A}(S) : S \in [\lambda]^{<\sigma}\}$ generates a σ-complete filter on $\mathcal{B}(P * \dot{Q})$. Since either $\sigma = \aleph_0$ or else is supercompact (and in particular strongly compact), by Lemma 2.2.2 we can find a σ-complete ultrafilter $\mathcal{V}$ on $\mathcal{B}(P * \dot{Q})$ extending $\mathcal{U}$ such that $\mathbf{B}$ is in $\mathcal{V}$.

(B) implies (A): trivial. $\square$

We turn the lemma into a definition.

Definition 4.7.3. Suppose $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence, and $3 \leq k \leq \theta$, and T is a complete first order theory. Write $\mathbb{P} = \mathbb{P}_{\mathbf{s},k}$. Then say that $(P, \mathcal{U}, \mathbf{A})$ is a $(T, \mathbb{P})$-problem if $P \in \mathbb{P}$, and $\mathcal{U}$ is a σ-complete ultrafilter on $\mathcal{B}(P)$, and $\mathbf{A}$ is a (λ, T)-Łoś map in $\mathcal{U}$. $(P * \dot{Q}, \mathbf{B})$ is a $\mathbb{P}$-solution to $(P, \mathcal{U}, \mathbf{A})$ if $\dot{Q} \in \mathbb{P}^{\mathbb{V}[\dot{G}]}$, and $\mathbf{B}$ is a multiplicative refinement of $\mathbf{A}$ in $\mathcal{B}(P * \dot{Q})$, and for every $S \in [\lambda]^{<\sigma}$, $\pi(\mathbf{B}(S)) \in \mathcal{U}$. Say that T has the $\mathbb{P}$-amalgamation property if every $(T, \mathbb{P})$-problem has a $\mathbb{P}$-solution.

The $\mathbb{P}$-amalgamation property will be a lower bound for our eventual principal dividing line in Keisler's order. In fact, if $\sigma = \aleph_0$ then it will exactly be a principal dividing line in Keisler's order—given a sufficiently generic iteration sequence $(P_\alpha : \alpha \leq \alpha_*), (\dot{Q}_\alpha : \alpha < \alpha_*)$, we will build our desired ultrafilter $\mathcal{U}$ on $\mathcal{B}(P_{\alpha_*})$ as a union of a chain of ultrafilters $\mathcal{U}_\alpha$ on $\mathcal{B}(P_\alpha)$, and the $\mathbb{P}$-amalgamation property for T will be exactly what we need to arrange a multiplicative refinement for a given (λ, T)-Łoś map at stage α.

For $\sigma > \aleph_0$, we cannot construct σ-complete ultrafilters so naïvely, and we will need the following technical strengthening. This is essentially the difference between perfect and optimal ultrafilters in [57].

Definition 4.7.4. Suppose $\mathbf{s}, k, \mathbb{P}, T$ are as above. Suppose $(P, \mathcal{U}, \mathbf{A})$ is a $(T, \mathbb{P})$-problem. Then say that $(P * \dot{Q}, \mathbf{B})$ is a smooth $\mathbb{P}$-solution to $\mathbf{A}$ if it is a $\mathbb{P}$-solution to $\mathbf{A}$, and for each $S \in [\lambda]^{<\sigma}$, $\pi(\mathbf{B}(S)) = \bigwedge_{s \in [S]^{<\aleph_0}} \pi(\mathbf{B}(s))$. Say that T has the smooth $\mathbb{P}$-amalgamation property if every $(T, \mathbb{P})$-problem has a smooth solution.

Note that if $\sigma = \aleph_0$, then every solution is smooth.

Remark 4.7.5. In practice, when we know how to show T fails the $\mathbb{P}$-amalgamation property, we can actually arrange a problem $(P, \mathcal{U}, \mathbf{A})$, such that $\mathbf{A}$ has no multiplicative refinement at all in any $\mathcal{B}(P * \dot{Q})$, for $\dot{Q} \in \mathbb{P}^{\mathbb{V}[\dot{G}]}$.

The following example explains why we require $\lambda \geq \kappa$ in the definition of suitable sequence.

Example 4.7.6. Suppose, in the definition of suitable sequences $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$, we had allowed $\lambda < \kappa$. Then for any such suitable sequence $\mathbf{s}$, and for every $3 \leq k \leq \theta$, we would have that every theory T has the smooth $\mathbb{P}_{\mathbf{s},k}$-amalgamation property.

Proof. Write $\mathbb{P} = \mathbb{P}_{\mathbf{s},k}$. Choose $P_0 \in \mathbb{P}$ that has an antichain of size λ, and let P_1 be the $< \theta$-support product of θ-many copies of P_0. Easily, P_1 has an antichain of size $\lambda^{<\theta}$, and hence of size $\lambda^{<\sigma}$. Let $(\mathbf{c}_s : s \in [\lambda]^{<\sigma})$ be a maximal antichain of P_1.

We follow the proof of Theorem 3.9.13. Suppose $(P, \mathcal{U}, \mathbf{A})$ is a $(T, \mathbb{P})$-problem for some T. Let $\dot{Q} = \check{P}_1$ (so we can identify $P * \dot{Q}$ with $P \times P_1$). For each $s \in [\lambda]^{<\aleph_0}$, let $\mathbf{B}(s) = \bigvee_{s \subseteq S \in [\lambda]^{<\sigma}} (\mathbf{A}(S), \mathbf{c}_S) \in \mathcal{B}(P) \times P_1 \subseteq \mathcal{B}(P \times P_1)$. As in Theorem 3.9.13, $\mathbf{B}$ is a multiplicative refinement of $\mathbf{A}$, and in fact $\mathbf{B}(S) = \bigvee_{S \subseteq S' \in [\lambda]^{<\sigma}} (\mathbf{A}(S'), \mathbf{c}_{S'})$ for all $S \in [\lambda]^{<\sigma}$.

We check that each $\pi(\mathbf{B}(S)) = \mathbf{A}(S)$, which shows that $P * (\dot{Q}, \mathbf{B})$ is a smooth solution to $(P, \mathcal{U}, \mathbf{A})$. It suffices to show that for all $\mathbf{a} \in \mathcal{B}(P)$, $\mathbf{a} \wedge \mathbf{A}(S)$ is nonzero if and only if $\mathbf{a} \wedge \mathbf{B}(S)$ is nonzero. Since $\mathbf{B}(s) \leq \mathbf{A}(s)$, the reverse direction is trivial, so suppose $\mathbf{a} \wedge \mathbf{A}(S)$ is nonzero. Then $(\mathbf{a} \wedge \mathbf{A}(S), \mathbf{c}_S) \leq \mathbf{B}(S)$ is nonzero, as desired.

□

On the other hand, this never happens when $\lambda \geq \kappa$. Indeed, we can establish at once the following baselines for the $\mathbb{P}$-amalgamation properties:

Theorem 4.7.7. Suppose $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence, and $3 \leq k \leq \theta$, and T is a complete first order theory. Write $\mathbb{P} = \mathbb{P}_{\mathbf{s},k}$.

(A) If T is nonsimple, then T fails the $\mathbb{P}$-amalgamation property.

(B) If $\sigma = \aleph_0$ and T is nonlow, then T fails the $\mathbb{P}$-amalgamation property.

Proof. (A): Let $P \in \mathbb{P}$ have c.c.$(P) = \kappa$. Then Theorem 3.15.1 together with Lemma 2.2.2 gives a $(T^*_{rg}, \mathbb{P})$-problem $(P, \mathcal{U}, \mathbf{A})$ with no $\mathbb{P}$-solution (namely, let $\mathcal{U}$ be a nonprincipal, σ-complete ultrafilter on $\mathcal{B}(P)$, and let $\mathbf{A}$ be as given in Theorem 3.15.1).

(B): Similarly, by Theorem 3.15.2. $\square$

And the following shows that we are doing is relevant. The case $\sigma = \aleph_0$ is similar to the construction of perfect ultrafilters in [57], and the case $\sigma > \aleph_0$ is similar to the construction of optimal ultrafilters there.

Theorem 4.7.8. Suppose $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence, and $3 \leq k \leq \theta$, and T is a complete first order theory. Write $\mathbb{P} = \mathbb{P}_{\mathbf{s},k}$. Then for some $P \in \mathbb{P}$, there is an ultrafilter $\mathcal{U}$ on $\mathcal{B}(P)$ which λ^+-saturates every theory with the smooth $\mathbb{P}$-amalgamation property, and does not λ^+-saturate any theory without the $\mathbb{P}$-amalgamation property.

Proof. As a convenient abbreviation, say that $(P, \mathcal{U}, \mathbf{A})$ is a problem if it is a $(T, \mathbb{P})$-problem for some theory T with the smooth $\mathbb{P}$-amalgamation property; and say that $(\dot{Q}, \mathbf{B})$ is a smooth solution if it is a smooth $\mathbb{P}$-solution. So trivially, every problem has a smooth solution. Let $(T_\delta : \delta < 2^{\aleph_0})$ enumerate all complete first order theories which fail the $\mathbb{P}$-amalgamation property. For each $\delta < \aleph_0$, let $(Q_{0,\delta}, \mathcal{U}_{0,\delta}, \mathbf{A}_{0,\delta})$ be a $(T_\delta, \mathbb{P})$-problem with no $\mathbb{P}$-solution. Let Q_0 be the $< \theta$-support product of $(Q_{0,\delta} : \delta < 2^{\aleph_0})$; so $Q_0 \in \mathbb{P}$. Let $\mathcal{V}_1$ be a σ-complete ultrafilter on $\mathcal{B}(Q_0)$ extending each $\mathcal{U}_{0,\delta}$.

The following setup is straightforward to arrange:

1. α_* is an ordinal, and $(\chi_\delta : \delta < \lambda^+)$ is a cofinal sequence of cardinals in α_* (it follows that α_* is a limit cardinal of cofinality λ^+);

2. $(P_\alpha : \alpha \leq \alpha_*)$, $(\dot{Q}_\alpha : \alpha < \alpha_*)$ is a $< \theta$-support forcing iteration, and each P_α forces that $\dot{Q}_\alpha \in \mathbb{P}$;

3. Q_0 is as defined above;

4. For each $1 \leq \alpha < \alpha_*$, $(P_{\beta_\alpha}, \mathcal{U}_\alpha, \mathbf{A}_\alpha)$ is a problem, with the smooth solution $(P_{\beta_\alpha} * \dot{Q}_\alpha, \mathbf{B}_\alpha)$ (in particular, $\dot{Q}_\alpha$ is a P_{β_α}-name);

5. For each $1 \leq \delta < \lambda^+$, for each $\beta \leq \chi_\delta$, and for each problem $(P_\beta, \mathcal{U}, \mathbf{A})$, there is some $\alpha < \chi_{\delta+1}$ such that $\beta_\alpha = \beta$, $\mathcal{U}_\alpha = \mathcal{U}$ and $\mathbf{A}_\alpha = \mathbf{A}$.

By Corollary 4.6.8, each $P_\alpha \in \mathbb{P}$ is θ-closed and has the λ^+-c.c. It suffices to find a σ-complete ultrafilter $\mathcal{V}$ on $\mathcal{B}(P_{\alpha_*})$ which λ^+-saturates every theory with the smooth $\mathbb{P}$-amalgamation property, and no theory without the $\mathbb{P}$-amalgamation property.

I claim that it suffices to find a σ-complete ultrafilter $\mathcal{V}$ on $\mathcal{B}(P_{\alpha_*})$ extending $\mathcal{V}_1$, such that for all $1 \leq \alpha < \alpha_*$, if $\mathcal{V}$ extends $\mathcal{U}_\alpha$ then $\mathbf{B}_\alpha$ is in $\mathcal{V}$ (i.e. $\mathbf{B}_\alpha(s) \in \mathcal{U}_\alpha$ for all $s \in [\lambda]^{<\aleph_0}$, or equivalently for all $S \in [\lambda]^{<\sigma}$). Indeed, suppose $\mathcal{V}$ is given as such.

First suppose $\delta < 2^{\aleph_0}$; we verify that $\mathcal{V}$ does not λ^+-saturate T_δ. Suppose towards a contradiction that $\mathbf{B}$ were a multiplicative refinement of $\mathbf{A}_{0,\delta}$ in $\mathcal{V}$. Let $\dot{Q}'$ be the $Q_{0,\delta}$-name for the forcing iteration $(\dot{P}_\alpha : \alpha \leq \alpha_*), (\dot{Q}'_\alpha : \alpha < \alpha_*)$, where $\dot{Q}_0$ is the $< \theta$-support product of $(Q_{0,\delta'} : \delta' < 2^{\aleph_0}, \delta' \neq \delta)$, and $\dot{Q}'_\alpha = \dot{Q}_\alpha$ for $\alpha > 0$. Then $\mathcal{B}(Q_{0,\delta} * \dot{Q}')$ is naturally isomorphic to $\mathcal{B}(P_{\alpha_*})$, and this witnesses that $(Q_{0,\delta}, \mathcal{U}_{0,\delta}, \mathbf{A})$ has a $\mathbb{P}$-solution (by Lemma 4.7.2).

Next, we show that $\mathcal{V}$ λ^+-saturates every T with the smooth $\mathbb{P}$-amalgamation property. Indeed, suppose $\mathbf{A}$ is a $(T, \mathcal{B}(P_{\alpha_*}), \lambda)$-possibility in $\mathcal{V}$. Since $\mathrm{cof}(\alpha_*) = \lambda^+$ and since $\mathcal{B}(P_{\alpha_*})$ has the λ^+-c.c., we have that $\mathcal{B}(P_{\alpha_*}) = \bigcup_{\alpha < \alpha_*} \mathcal{B}(P_\alpha)$, and so $\mathbf{A}$ is in $\mathcal{B}_\beta$ for some $\beta < \alpha_*$. Let $\mathcal{U} = \mathcal{V} \cap \mathcal{B}(P_\beta)$, and let $\delta < \lambda^+$ be least with $\chi_\delta \geq \beta$. Choose $\alpha < \chi_{\delta+1}$ such that $\beta_\alpha = \beta, \mathcal{U}_\alpha = \mathcal{U}$ and $\mathbf{A}_\alpha = \mathbf{A}$. Since $\mathcal{V}$ extends $\mathcal{U}_\alpha$ we must have that $\mathcal{V}$ extends $\mathbf{B}_\alpha$, but $\mathbf{B}_\alpha$ is a multiplicative refinement to $\mathbf{A}$ so we are done.

So it remains to find $\mathcal{V}$. If $\sigma = \aleph_0$ then this is fairly trivial; having constructed $\mathcal{V} \cap \mathcal{B}(P_\alpha)$, if $\mathcal{V} \cap \mathcal{B}(P_\alpha)$ extends $\mathcal{U}_\alpha$, then note that for all $s \in [\lambda]^{<\aleph_0}$, $\pi_{\mathcal{B}(P_{\alpha+1}),\mathcal{B}(P_{\beta_\alpha})}(\mathbf{B}_\alpha(s)) \in \mathcal{U}_\alpha$, and so $(\mathcal{V} \cap \mathcal{B}(P_\alpha)) \cup \{\mathbf{B}(s) : s \in [\lambda]^{<\aleph_0}\}$ has the finite intersection property. So we can find $\mathcal{V}_{\alpha+1}$.

Finally, suppose $\sigma > \aleph_0$. We cannot adopt a straightforward construction as above, since we cannot preserve σ-completeness through limit stages. Thus we take a different approach. The remainder of the argument mirrors Theorem 5.9 of [57].

Let $\mathcal{E}$ be a normal, σ-complete ultrafilter on $[H(\chi)^{<\sigma}]$ where χ is large enough. Let Ω be the set of all $N \in [H(\chi)]^{<\sigma}$ such that $N \preceq (H(\chi), \in, \ldots)$ where $\ldots$ is the list of finitely many relevant parameters. Then $\Omega \in \mathcal{E}$.

Fix $N \in \Omega$ for a while. Let $(\alpha_\gamma : \gamma < \gamma_*)$ enumerate $N \cap \alpha_*$ in the increasing order, so $\gamma_* < \sigma$. By induction on $\gamma < \gamma_*$ we construct $(p_\gamma : \gamma < \gamma_*)$ with each $p_\gamma \in P_{\alpha_\gamma}$, such that:

- For $\gamma < \gamma'$, $p_\gamma \geq p'_\gamma$;
- For each $\gamma < \gamma_*$, and for each $\alpha \in N \cap \alpha_*$, p_γ decides every element of $\mathcal{B}(P_{\alpha_\gamma})$;
- $p_1 \leq \bigwedge(\mathcal{V}_1 \cap N)$;
- If $p_\gamma \leq \bigwedge(\mathcal{U}_{\alpha_\gamma} \cap N)$ then $p_{\gamma+1} \leq \mathbf{B}_{\alpha_\gamma}(N \cap \lambda)$.

The base case is easy. If $\delta < \gamma_*$ is a limit ordinal, then when constructing p_δ we just need to handle the first and second conditions. We can do this because P_{α_δ} is θ-closed, and hence σ-closed.

The key point is the following. Suppose p_γ is defined; write $\alpha = \alpha_\gamma$. Suppose $p_\gamma \leq \bigwedge(\mathcal{U}_\alpha \cap N)$. We need to show that $p_\gamma \wedge \mathbf{B}_\alpha(N \cap \lambda)$ is nonzero. Write $S = N \cap \lambda$, and let $\pi = \pi_{\mathcal{B}(P_{\alpha+1}),\mathcal{B}(P_{\beta_\alpha})}$ be the projection map. It suffices to show that $p_\gamma \leq \pi(\mathbf{B}_\alpha(S))$. But $\mathbf{B}_\alpha(S) = \bigwedge_{s \in [S]^{<\aleph_0}} \mathbf{B}_\alpha(s)$, and each $\pi(\mathbf{B}_\alpha(s)) \in \mathcal{U}_\alpha$, by definition of smooth solution. But then each $\pi(\mathbf{B}_\alpha(s)) \in N \cap \mathcal{U}_\alpha$, since $[S]^{<\aleph_0} \subseteq N$. Thus $p_\gamma \leq \pi(\mathbf{B}_\alpha(s))$ for each $s \in [S]^{<\aleph_0}$, and so we can satisfy the fourth condition above for $p_{\gamma+1}$. The other conditions can be gotten as in the limit case, using $P_{\alpha_{\gamma+1}}$ is σ-closed.

Let $p_N \in P_{\alpha_*}$ be a lower bound to $(p_\gamma : \gamma < \gamma_*)$. To sum up, for each $N \in \Omega$, we have defined $p_N \in P_{\alpha_*}$, so that $p_N \leq \bigwedge(N \cap \mathcal{V}_1)$, and p_N decides every element of $N \cap P_{\alpha_*}$, and for all $\alpha \in \alpha_* \cap N$, either $p_N \leq \mathbf{B}_\alpha(N \cap \lambda)$, or else p_N contradicts some element of $\mathcal{U}_\alpha \cap N$.

Define $\mathcal{V}$ to be the set of all $\mathbf{a} \in \mathcal{B}(P_{\alpha_*})$ such that $\{N \in \Omega : p_N \leq \mathbf{a}\} \in \mathcal{E}$. I claim that $\mathcal{V}$ is as desired. $\mathcal{V}$ is obviously a filter. Given $\mathbf{a} \in \mathcal{B}(P_{\alpha_*})$, we have that $\{N \in \Omega : \mathbf{a} \in N\} \in \mathcal{E}$ since $\mathcal{E}$ is fine, thus $\mathcal{V}$ is an ultrafilter. Since $\mathcal{E}$ is σ-complete, so is $\mathcal{V}$.

Finally, suppose $\alpha < \alpha_*$; we need to show that either $\mathbf{B}_\alpha$ is in $\mathcal{V}$ or else $\mathcal{V}$ does not extend $\mathcal{U}_\alpha$. Let $C_1 := \{N \in \Omega : p_N \leq \bigwedge \mathcal{U}_\alpha \cap N\}$, and let $C_2 = \{N \in \Omega : p_N \not\leq \bigwedge \mathcal{U}_\alpha \cap N\}$. Either $C_1 \in \mathcal{E}$ or else $C_2 \in \mathcal{E}$. Suppose first that $C_1 \in \mathcal{E}$. Then for each $N \in C_1$ and for each $s \in [\lambda]^{<\aleph_0} \cap N$, $p_N \leq \mathbf{B}_\alpha(s_0)$; since $\mathcal{E}$ is fine, it follows that each $\mathbf{B}_\alpha(s) \in \mathcal{V}$, so $\mathbf{B}$ is in $\mathcal{V}$. Next, suppose $C_2 \in \mathcal{E}$; for each $N \in C_2$, we can choose $f(N) \in \mathcal{U}_\alpha \cap N$ such that $p_N \not\leq f(N)$. Since $\mathcal{E}$ is normal, we can find $C \subseteq C_2$ with $C \in \mathcal{E}$, such that f is constant on C, say with value $\mathbf{a} \in \mathcal{U}_\alpha$. Then $\mathbf{a} \notin \mathcal{V}$, so $\mathcal{V}$ does not extend $\mathcal{U}_\alpha$. Thus, in either case, either $\mathbf{B}_\alpha$ is in $\mathcal{V}$ or else $\mathcal{V}$ does not extend $\mathcal{U}_\alpha$. $\square$

Corollary 4.7.9. Suppose $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence, and $3 \leq k \leq \theta$, and T

is a complete first order theory. Write $\mathbb{P} = \mathbb{P}_{\mathbf{s},k}$. Then there is a principal dividing line in Keisler's order between the $\mathbb{P}$-amalgamation property and the smooth $\mathbb{P}$-amalgamation property. If $\sigma = \aleph_0$ then the $\mathbb{P}$-amalgamation property is itself a principal dividing line.

Proof. This follows from Theorem 4.7.8 since for every $P \in \mathbb{P}$, $\mathcal{B}(P)$ has the λ^+-c.c. (in fact, the λ-c.c.). $\square$

4.8 The Saturation Condition

In this section, we show that if $\mathbf{s}$ is a suitable sequence, and $3 \leq k \leq \theta$, and T is a simple theory with Λ-type amalgamation for all $|\Lambda| < k$, then T has the smooth $\mathbb{P}_{\mathbf{s},k}$-amalgamation property (see Definition 4.7.4). The argument for this is inspired by the saturation argument in [81].

In practice, verifying directly that some T has the smooth $\mathbb{P}$-amalgamation property may be difficult. We first describe a sufficient condition.

Definition 4.8.1. Suppose $\mathbf{s}$ is suitable, and $3 \leq k \leq \theta$; write $\mathbb{P} = \mathbb{P}_{\mathbf{s},k}$. Say that T has the concrete $\mathbb{P}$-amalgamation property if for every $M \models T$ with $|M| \leq \lambda$, and for every $M_0 \preceq M$ countable, $\Gamma^\theta_{M,M_0} \in \mathbb{P}$. Say that T has the absolute concrete $\mathbb{P}$-amalgamation property if for every $P \in \mathbb{P}$, P forces that $\check{T}$ has the concrete $\mathbb{P}$-amalgamation property.

In the following theorem, we show that the absolute concrete $\mathbb{P}$-amalgamation property implies the smooth $\mathbb{P}$-amalgamation property, except in cases where the latter fails due to Theorem 4.7.7.

Theorem 4.8.2. Suppose $\mathbf{s}$ is suitable, and $3 \leq k \leq \theta$; write $\mathbb{P} = \mathbb{P}_{\mathbf{s},k}$. Suppose T is simple, and either $\sigma > \aleph_0$ or else T is low. If T has the absolute concrete $\mathbb{P}$-amalgamation property, then T has the smooth $\mathbb{P}$-amalgamation property.

Proof. Let $(P, \mathcal{U}, \mathbf{A})$ be a $(T, \mathbb{P})$-problem. By definition of a Łoś-map, we can choose $\mathbf{M} \models^{\mathcal{B}} T$, and a partial type $p(x) = \{\varphi_\alpha(x, \overline{a}_\alpha) : \alpha < \lambda\}$ over $\mathbf{M}$, such that for all $s \in [\lambda]^{<\aleph_0}$, $\|\exists x \bigwedge_{\alpha \in s} \varphi_\alpha(x, \overline{a}_\alpha)\|_{\mathbf{M}} = \mathbf{A}(s)$; we can arrange $|\mathbf{M}| \leq \lambda$. Choose $\mathbf{M}_0 \preceq \mathbf{M}$ countable such that $p(x)$ does not $\mathcal{U}$-fork over $\mathbf{M}_0$.

Let $\dot{Q} = \Gamma^\theta_{\check{\mathbf{M}}/\dot{G}, \check{\mathbf{M}}_0/\dot{G}}$, so P forces that $\dot{Q} \in \mathbb{P}^{\mathbb{V}[\dot{G}]}$. For each $s \in [\lambda]^{<\aleph_0}$, define

$$\mathbf{C}(s) = \bigwedge_{\alpha \in s} \|\varphi_\alpha(x, [\overline{a}_\alpha]_{\dot{G}}) \text{ does not fork over } \check{\mathbf{M}}_0/\dot{G} \text{ in } \check{\mathbf{M}}/\dot{G}\|_{\mathcal{B}}$$

so $\mathbf{C}(s) \in \mathcal{U}$, by Theorem 4.1.3 or else Theorem 4.1.4.

For each $s \in [\lambda]^{<\aleph_0}$, let $\mathbf{B}(s) = (\mathbf{C}(s), \{\varphi_\alpha(x, [\overline{a}_\alpha]_{\dot{G}}) : \alpha \in s\}) \in \mathcal{B}(P) * \dot{Q} \subseteq \mathcal{B}(P * \dot{Q})$.

Clearly $\mathbf{B}$ is a multiplicative refinement of $\mathbf{A}$. Moreover, letting $\pi : \mathcal{B}(P * \dot{Q}) \to \mathcal{B}(P)$ be the projection, note that whenever $(\mathbf{c}, \dot{p}(x)) \in \mathcal{B}(P) * \dot{Q}$ we have that $\pi(\mathbf{c}, \dot{p}(x)) = \mathbf{c}$. Hence $\pi(\mathbf{B}(s)) = \mathbf{C}(s)$ for each $s \in [\lambda]^{<\aleph_0}$. But moreover, given $S \in [\lambda]^{<\sigma}$, $\mathbf{B}(S) = \bigwedge_{s \in [S]^{<\aleph_0}} \mathbf{B}(s) = (\mathbf{C}(S), \{\varphi_\alpha(x, [\overline{a}_\alpha]_{\dot{G}}) : \alpha \in S\})$, so $\pi(\mathbf{B}(S)) = \mathbf{C}(S) = \bigwedge_{s \in [S]^{<\aleph_0}} \mathbf{C}(s) = \pi(\mathbf{B}(s))$ as desired.

Thus $(\dot{Q}, \mathbf{B})$ is a smooth $\mathbb{P}$-solution to $(P, \mathcal{U}, \mathbf{A})$. $\square$

We now apply this:

Theorem 4.8.3. Suppose $\mathbf{s}$ is suitable, and $3 \leq k \leq \theta$; write $\mathbb{P} = \mathbb{P}_{\mathbf{s},k}$. Suppose $\theta > \aleph_0$, and T is a simple theory with Λ-type amalgamation for all $|\Lambda| < k$. Then T has the absolute concrete $\mathbb{P}$-amalgamation property.

Proof. It suffices to show T has the concrete $\mathbb{P}$-amalgamation property, since the same argument will run in any forcing extension. So suppose $M \models T$ has $|M| \leq \lambda$, and $M_0 \preceq M$ is countable. We show $\Gamma^\theta_{M,M_0} \in \mathbb{P}_{\mathbf{s},k}$. It suffices to show that Γ^θ_{M,M_0} is weakly (R, k)-colorable for some $R \in \mathbb{P}_{\mathbf{s},k}$.

By Theorem 4.3.9 and reindexing, we can suppose $M = \bigcup_{s \in [\lambda]^{\leq \aleph_0}} M_s$, where $(M_s : s \in [\lambda]^{\leq \aleph_0})$ is an independent system of countable submodels of M (we are possibly increasing M_0). Moreover we can suppose there is a frame $\mathcal{A} \subseteq [\lambda]^{\leq \aleph_0}$ such that for all $s, M_s = \bigcup\{M_t : t \subseteq s, t \in \mathcal{A}\}$. We can further suppose that the universe of each M_s is $\bigcup\{\{t\} \times \omega : t \subseteq s, t \in \mathcal{A}\}$.

For each $s \in [\lambda]^{<\theta}$ define $M_s := \bigcup_{t \in [s]^{\leq \aleph_0}} M_t$. So $(M_s : s \in [\lambda]^{<\theta})$ is still independent.

Let P be the set of all $p(x) \in \Gamma^\theta_{M,M_0}$ such that for some $s \in [\lambda]^{<\theta}$, $p(x)$ is a complete type over M_s; we write $p(x, M_s)$ to indicate this. P is dense in Γ^θ_{M,M_0}. Note that for every set X, $P_{X\theta\theta} \in \mathbb{P}_{\mathbf{s}\infty}$. Thus it suffices to find some set X and some weak $(P_{X\theta\theta}, k)$-coloring $F : P \to P_{X\theta\theta}$ of P. Write $X = [\theta]^{<\omega} \times \omega \cup (\lambda + 2)$ (which we suppose is a disjoint union).

Suppose $p(x, M_s)$ is given. Enumerate $s = \{\alpha_\gamma : \gamma < \gamma^0_*\}$ in the increasing order. Also, enumerate $\{t \subseteq s : t \in \mathcal{A}\} = \{t_\gamma : \gamma < \gamma^1_*\}$. So $\gamma^0_*, \gamma^1_* < \theta$. Let $\pi_p : M_s \to \gamma^1_* \times \omega$ be the bijection sending (t_γ, i) to (γ, i). Let N_p be the model with universe $\gamma^1_* \times \omega$ such that π_p is an isomorphism.

Let the domain of $F(p)$ be $[\gamma^1_*]^{<\omega} \times \omega \cup s \cup \{\lambda, \lambda + 1\}$. Define $F \restriction^{<\omega}_{[\gamma^1_*]}$ to encode N_p and $\pi_p[p(x)]$. For each $\gamma < \gamma^0_*$ define $F(\alpha_\gamma) = \gamma$. Finally, define $F(\lambda) = \gamma^0_*, F(\lambda+1) = \gamma^1_*$.

I claim this works.

So suppose $p_i(x, M_{s_i}) : i < i_*$ is a sequence from P where $i_* < k$, such that $(F(p_i) : i < i_*)$ is compatible in $P_{\lambda\theta\theta}$.

Let γ^0_* be the order-type of some or any s_i. Enumerate each $s_i = \{\alpha_{i,\gamma} : \gamma < \gamma_*\}$ in increasing order. Let E be the equivalence relation on γ_* defined by: $\gamma E \gamma'$ if and only if for all $i, i' < i_*$, $\alpha_{i,\gamma} = \alpha_{i',\gamma}$ if and only if $\alpha_{i,\gamma'} = \alpha_{i',\gamma'}$. Let $(E_j : j < n)$ enumerate the equivalence classes of E. For each $i < i_*$, and for each $j < n$, let $X_{i,j} = \{\alpha_{i,\gamma} : \gamma \in E_j\}$.

Thus s_i is the disjoint union of $X_{i,j}$ for $j < n$. Moreover, $X_{i,j} \cap X_{i',j'} = \emptyset$ unless $j = j'$; and if $X_{i,j} \cap X_{i',j} \neq \emptyset$ then $X_{i,j} = X_{i',j}$. For each $j < n$, enumerate $\{X_{i,j} : i < i_*\} = (Y_{\ell,j} : \ell < m_i)$ without repetitions. Let $m = \max(m_j : j < n)$; and for each $i < i_*$, define $\eta_i \in {}^n m$ via: $\eta_i(j) =$ the unique $\ell < m_i$ with $X_{i,j} = Y_{\ell,j}$.

Let $\Lambda = \{\eta_i : i < i_*\}$. For each $s \in P_\Lambda$, let $N_s = M_{t_s}$ where $t_s = \bigcup_{(j,\ell)\in s} Y_{\ell,j}$. For each $i, i' < i_*$, let $\pi_{\eta_i,\eta_{i'}} = \pi_{p_{i'}}^{-1} \circ \pi_{p_i}$, using that $N_{p_i} = N_{p_{i'}}$. Also, define $p_{\eta_i}(x) = p_i(x)$. Then $(\overline{N}, \overline{\pi})$ is a Λ array, and $|\Lambda| < k$, so we are done by $< k$-type amalgamation. $\square$

4.9 The Nonsaturation Condition

In this section, we prove the following.

Theorem 4.9.1. Suppose $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence. Suppose $3 \leq k_* < \omega$, and $\lambda \geq [\kappa]^{+(k_*-1)}$. If T admits Δ_k for some $k < k_*$, then T fails the $(\mathbb{P}_{\mathbf{s},k_*}, \mathbf{s})$-amalgamation property.

Before we begin the proof, we will need some combinatorial lemmas. The notion $\rightarrow$ has a long history, see [11].

Definition 4.9.2. Suppose $F : [\lambda]^k \to [\lambda]^{<\kappa}$. Then $w \in [\lambda]^n$ is independent with respect to F if for each $u \in [w]^k$, $F(u) \cap w \subseteq u$.

Given cardinals $\lambda \geq \kappa$ and numbers $n > k$, say that $(\lambda, k, \kappa) \to n$ if whenever $F : [\lambda]^k \to [\lambda]^{<\kappa}$, there is some $w \in [\lambda]^n$ which is independent with respect to F.

The following is Theorem 46.1 of [11]:

Theorem 4.9.3. Suppose $\lambda \geq \kappa^{+\ell}$. Then $(\lambda, \ell, \kappa) \to \ell + 1$.

Proof. We prove by induction on ℓ that $(\kappa^{+\ell}, \ell, \kappa) \to \ell + 1$. This clearly suffices.

If $\ell = 0$: we have $F : [\kappa]^0 \to [\kappa]^{<\kappa}$. Note that $[\kappa]^0 = \{\emptyset\}$. Let $A = F(\emptyset)$ and choose $\alpha \in \kappa - A$. Then $\{\alpha\} \in [\kappa]^1$ is independent with respect to F.

Suppose we are at case $\ell + 1$; write $\lambda = \kappa^{+\ell}$. We are supposing $(\lambda, \ell, \kappa) \to \ell + 1$. Suppose $F : [\lambda^+]^{\ell+1} \to [\lambda^+]^{<\kappa}$ is given. Choose $A \subseteq \lambda^+$ such that $|A| = \lambda$ and A is closed under F. Pick $\alpha \in \lambda^+ \backslash A$ and define $F' : [A]^\ell \to [A]^{<\kappa}$ by: $F'(u) = F(u \cup \{\alpha\})$. By the induction hypothesis we can choose $w \in [A]^{\ell+1}$ which is independent with respect to F'. Then $w \cup \{\alpha\} \in [\lambda^+]^{\ell+2}$ is independent with respect to F. $\square$

Actually, this theorem is sharp, but we won't have a use for the reverse direction.

We can now prove Theorem 4.9.1. This proof mirrors the proof of Claim 5.1 of [58].

Proof. Let $P = P_{[\lambda]^k \theta\theta} \in \mathbb{P}_{\mathbf{s},k_*}$. Write $\mathcal{B} = \mathcal{B}(P)$; for each $v \in [\lambda]^k$, write $\mathbf{c}_v = \{(v, 0)\} \in P \subseteq \mathcal{B}$. Let $\mathbf{A}$ be the $[\lambda]^{k-1}$-distribution in $\mathcal{B}$, defined by putting $\mathbf{A}(s) = \bigwedge\{\mathbf{c}_v : v \in [\lambda]^k, [v]^{k-1} \subseteq s\}$. Easily, $\mathbf{A}$ is a $([\lambda]^{k-1}, \Delta_k)$-distribution.

Suppose $\dot{Q} \in \mathbb{P}^{\mathbb{V}[\dot{G}]}_{\mathbf{s},k_*}$ were given; it suffices to show that $\mathbf{A}$ has no multiplicative refinement in $\mathcal{B}(P * \dot{Q})$. Suppose towards a contradiction that there were, say $\mathbf{B}$. We can find $R \in \mathbb{P}_{\mathbf{s},\infty}$ and a $\mathcal{B}$-name $\dot{F}$ such that $\mathcal{B}$ forces $\dot{F} : \dot{Q} \to \check{R}$ is a weak $(\check{R}, k)$-coloring. Write $R = P_{X\overline{\mu}\theta}$ for some set X and some sequence of cardinals $(\mu_x : x \in X)$ below κ.

For each $v \in [\lambda]^k$ choose $(p_v, \dot{q}_v) \in P * \dot{Q}$ such that $(p_v, \dot{q}_v) \leq \mathbf{B}([v]^{k-1})$ and p_v decides $\dot{F}(\dot{q}_v)$, say p_v forces that $\dot{F}(\dot{q}_v) = \check{f}_v$ for some $f_v \in R$.

Let $\mathcal{B}_*$ be the Boolean-algebra completion of $P \times R$. For each $u \in [\lambda]^{k-1}$, let $\mathbf{b}_u$ be the least upper bound in $\mathcal{B}_*$ of $((p_v, f_v) : u \subseteq v \in [\lambda]^k)$. Since $\mathcal{B}_*$ has the κ-c.c., we can find $S(u) \in [\lambda]^{<\kappa}$ such that $\mathbf{b}_u$ is also the greatest lower bound in $\mathcal{B}_*$ of $((p_v, f_v) : u \subseteq v \in [S(u)]^k)$. By expanding $S(u)$, we can suppose that for all $v \in [S(u)]^k$, $\bigcup \mathrm{dom}(p_v) \subseteq [S(u)]^k$.

By Theorem 4.9.3, $(\lambda, k-1, \kappa) \to k$, so we can find some $v \in [\lambda]^k$ such that for all $u \in [v]^{k-1}$, $S(u) \cap v = u$. Now $(\mathbf{b}_u : u \in [v]^{k-1})$ has a greatest lower bound in $\mathcal{R}$, namely (p_v, f_v); thus we can find $(v_u : u \in [v]^{k-1})$ such that each $u \subseteq v_u \in [S(u)]^k$, and $((p_{v_u}, f_{v_u}) : u \in [w]^{k-1})$ are all compatible in $P \times R$. Thus $(p_{v_u} : u \in [v]^{k-1})$ are all compatible; write $p = \bigcup_{u \in [v]^{k-1}} p_{v_u} \in P$ (recall $P = P_{[\lambda]^k \theta \theta}$). Also, p forces that each $\dot{F}(\dot{q}_{v_u}) = \hat{f}_{v_u}$; so we can choose a P-name $\dot{q}$ for an element of $\dot{Q}$ such that p forces $\dot{q}$ is a lower bound to $(\dot{q}_{v_u} : u \in [v]^{k-1})$ (this is where we use $k < k_*$). Then $(p, \dot{q})$ is a lower bound in $P * \dot{Q}$ to $((p_{v_u}, \dot{q}_{v_u}) : u \in [v]^{k-1})$. Note that $v \notin \mathrm{dom}(p)$, since if $v \in \mathrm{dom}(p_{v_u})$ say, then $v \subseteq \bigcup \mathrm{dom}(p_{v_u}) \subseteq S(u)$, contradicting that $S(u) \cap v = u$. Thus we can choose $p' \leq p$ in P with $p'(v) = 1$; note than $(p', \dot{q}) \in P * \dot{Q}$.

Now for each $u \in [v]^{k-1}$, $(p', \dot{q}) \leq (p_{v_u}, \dot{q}_{v_u}) \leq \mathbf{B}([v_u]^{k-1}) \leq \mathbf{B}(\{u\})$. Thus, by multiplicativity, $(p', \dot{q}) \leq \mathbf{B}([v]^{k-1}) \leq \mathbf{A}([v]^{k-1}) = \{(v, 0)\}$, contradicting the choice of p'. $\square$

4.10 Putting It All Together

We now reel off consequences of what we have done. The following was proven by Malliaris and Shelah in [57].

Theorem 4.10.1. Suppose there is a supercompact cardinal. Then simplicity is a principal dividing line in Keisler's order.

Proof. Let σ be supercompact. Write $\theta = \sigma$ and $\kappa = \sigma^+$; let $\lambda \geq \kappa$ be arbitrary. Then $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence. By Theorems 4.8.3 and 4.8.2, every simple theory has the smooth $\mathbb{P}_{\mathbf{s},3}$-amalgamation property, and by Theorem 4.7.7, every unsimple theory fails the $\mathbb{P}_{\mathbf{s},3}$-amalgamation property. Hence we conclude by Theorem 4.7.8. $\square$

I proved the following in [87].

Theorem 4.10.2. Lowness is a principal dividing line in Keisler's order.

Proof. Let $\sigma = \aleph_0$, let $\theta = \aleph_1$, let $\kappa = (2^{\aleph_0})^+$, and let $\lambda \geq \kappa$ be arbitrary. Then $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence. By Theorems 4.8.2 and 4.8.3, every low theory has the smooth $\mathbb{P}_{\mathbf{s},3}$-amalgamation property, and by Theorem 4.7.7, every nonlow theory fails the $\mathbb{P}_{\mathbf{s},3}$-amalgamation property. Hence we conclude by Theorem 4.7.8. $\square$

The first iteration of the following theorem was proven by Malliaris and Shelah in [58], although the conclusion there was weaker: they only obtained that for all $3 \leq k < k' - 1$, $T_{k'+1,k'} \not\trianglelefteq T_{k+1,k}$. We eliminate this gap and get more model-theoretic information.

Theorem 4.10.3. Suppose $3 \leq k_* \leq \aleph_0$. Then there is a principal dividing line $\mathbf{T}$ in Keisler's order, which includes every low complete countable theory T that has Λ-type amalgamation for all $\Lambda \in \mathbf{\Lambda}$ with $|\Lambda| < k_*$, but does not include any theory which admits Δ_k for some $k < k_*$, nor any nonlow theory. In particular, $T_{k'+1,k'} \not\trianglelefteq T_{k+1,k}$ for $k' > k$.

Proof. Let $\sigma = \aleph_0$, let $\theta = \aleph_1$, let $\kappa = (2^{\aleph_0})^+$, and let $\lambda \geq \kappa^{+\omega}$. Then $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence. By Theorems 4.8.2 and 4.8.3, every low theory with Λ-type amalgamation for all $\Lambda \in \mathbf{\Lambda}$ with $|\Lambda| < k_*$ has the smooth $(\mathbb{P}_{\mathbf{s},k_*}, \mathbf{s})$-amalgamation property. By Theorem 4.7.7, every nonlow theory fails the $(\mathbb{P}_{\mathbf{s},k_*}, \mathbf{s})$-amalgamation property, and by Theorem 4.9.1, if T admits Δ_k for some $k < k_*$, then T fails the $(\mathbb{P}_{\mathbf{s},k_*}, \mathbf{s})$-amalgamation property. Hence we conclude by Theorem 4.7.8. $\square$

We also get the following, under the presence of a supercompact cardinal.

Theorem 4.10.4. Suppose $3 \leq k_* \leq \aleph_0$, and suppose there is a supercompact cardinal. Then there is a principal dividing line $\mathbf{T}$ in Keisler's order, which includes every simple complete countable theory T that has Λ-type amalgamation for all $\Lambda \in \mathbf{\Lambda}$ with $|\Lambda| < k_*$, but does not include any theory T which admits Δ_k for some $k < k_*$, nor any unsimple theory.

Proof. Let σ be supercompact. Write $\theta = \sigma$ and $\kappa = \sigma^+$; and let $\lambda \geq \kappa^{+\omega}$. Then $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence. By Theorems 4.8.2 and 4.8.3, every simple theory with Λ-type amalgamation for all $\Lambda \in \mathbf{\Lambda}$ with $|\Lambda| < k_*$ has the smooth $(\mathbb{P}_{\mathbf{s},k_*}, \mathbf{s})$-amalgamation property. By Theorem 4.7.7, every nonsimple theory fails the $(\mathbb{P}_{\mathbf{s},k_*}, \mathbf{s})$-amalgamation property, and by Theorem 4.9.1, if T admits Δ_k for some $k < k_*$, then T fails the $(\mathbb{P}_{\mathbf{s},k_*}, \mathbf{s})$-amalgamation property. Hence we conclude by Theorem 4.7.8. □

The following corollary is close to an immediate consequence of results of Malliaris and Shelah in [57] and [58] (the extra piece is the existence of T_{Cas}), and in fact they were aware of the result, but did not disseminate it. I was the first to observe it in publication [88]. Malliaris and Shelah have since recorded their independent (and morally identical) proof in [61].

Corollary 4.10.5. If there is a supercompact cardinal, then Keisler's order is not linear.

Proof. Compare T_{Cas} with $T_{4,3}$, say. T_{Cas} has Λ-type amalgamation for all $\Lambda \in \mathbf{\Lambda}$, but is not low; $T_{4,3}$ is low, but fails Λ_3-type amalgamation. Thus we conclude by Theorems 4.10.2 and 4.10.4. □

4.11 Well-behaved Simple Theories

Suppose T is a countable simple theory. We have difficulty arguing that various versions of amalgamation are equivalent, but in the examples of which we are aware, everything works out. The following is an ad-hoc list of principles that isolate the good behavior.

Definition 4.11.1. T has extendible solutions if the following holds. Suppose $\Delta \in \mathbf{\Delta}$, and Δ_0 is the closure of the maximal elements of Δ under intersections. Suppose $(M_s : s \in \Delta)$ is an independent system of models. Then $(M_s : s \in \Delta)$ has a solution if and only if $(M_s : s \in \Delta_0)$ has a solution.

T has canonical amalgamation if, whenever $\Delta_0 \subseteq \Delta_1$ are both in $\mathbf{\Delta}$, and whenever $(M_s : s \in \Delta_0)$ is an independent system of models, if for every $t \in \Delta_1$ we have that $(M_{s\cap t} : s \in \Delta_0)$ has a solution, then we can extend $(M_s : s \in \Delta_0)$ to an independent system of models $(M_t : t \in \Delta_1)$.

T has a surprise amalgamation problem if for some $3 \leq k < \omega$, T has $\mathcal{P}^-(k)$-amalgamation of models, and there is some independent system of models $(M_u : u \subsetneq k+1)$ with no solution, such that if we let $p_u(\overline{x}) = tp_{M_{u\cup\{k\}}}(M_{\{k\}}/M_u)$ for each $u \subsetneq k$, then $\bigcup_{u\subsetneq k} p_u(\overline{x})$ does not fork over $M_\emptyset$, as evaluated in M_k.

T is well-behaved if T has extendible solutions, canonical amalgamation and no surprise amalgamation problems.

Trivially, if T has Δ-amalgamation of models for all $\Delta \in \mathbf{\Delta}$, then T is well-behaved. In particular, every stable theory is well-behaved.

Question 1. Is every simple theory well-behaved?

We have the following class of examples of well-behaved theories,.

Example 4.11.2. For $3 \leq k < \omega$, each $T_{n,k}$ is well-behaved. More generally, suppose T is a complete countable simple theory, with forking given by equality (in particular, supersimple of rank 1). Suppose further that T admits quantifier elimination in a relational language $\mathcal{L}$, such that the following holds: suppose $M \models T$ and N is an $\mathcal{L}$-structure with the same universe as M. Suppose for every relation symbol R of $\mathcal{L}$ of arity at least two, $R^N \subseteq R^M$, and for every unary predicate P of $\mathcal{L}$, $P^M = P^N$. Then $N \models T^\forall$ (i.e. N can be extended to a model of T).

Then T is well-behaved.

Proof. T clearly has extendible solutions, since the only obstacle is that there could be extra forking in $(M_s : s \in \Delta)$; but T has forking given by equality. So we check canonical amalgamation and no surprise amalgamation problems.

We speak of systems $(A_s : s \in I)$ of (possibly floating) models of $T^\forall$, defined like systems of models of T. Note that since forking for T is given by equality, independence is always vacuous.

I first of all claim that if $(A_s : s \in \Delta)$ is a system of models of $T^\forall$ for some $\Delta \in \mathbf{\Delta}$, then it can be extended to a system of models $(M_s : s \in \Delta)$ of T (i.e. with each $A_s \subseteq M_s$); moreover, if for some $s \in \Delta$ we have that $A_t \models T$ for all $t \subseteq s$, then we can arrange $M_s = A_s$. It suffices to show that given $s \in \Delta$, we can extend $(A_s : s \in \Delta)$ to a system $(B_s : s \in \Delta)$ of models of $T^\forall$, such that $A_t = B_t$ for all t with $s \not\subseteq t$, and with $B_s \models T$. Indeed, choose some $B_s \models T$ extending A_s, and such that $B_s \cap \bigcup_{t \in I} A_t = A_s$. It suffices to show we can coherently define $(B_t : s \subseteq t)$, so that each B_t is an amalgam of A_t and B_s with universe $|A_t| \cup |B_s|$. But we can just let B_t be the free amalgam, where we add no new relations. Since models of $T^\forall$ have disjoint amalgamation, this works.

T has canonical amalgamation: suppose $\Delta_0 \subseteq \Delta_1 \in \mathbf{\Delta}$, and $(M_s : s \in \Delta_0)$ is an (independent) system of models, such that for each $t \in \Delta_1$, $(M_{s \cap t} : s \in \Delta_0)$ has a solution. For each $t \in \Delta_1$, let A_t be the structure obtained by freely amalgamating $(M_{s\cap t} : s \in \Delta_0)$ (i.e., adding no new relations). Since $(M_{s \cap t} : s \in \Delta_0)$ has a solution, by hypothesis we get that $A_t \in T^\forall$. Then $(A_t : t \in \Delta_0)$ is a system of models of $T^\forall$, so by the preceding, we can extend it to an (independent) system of models of T without changing $(M_s : s \in \Delta_0)$, as desired.

T has no surprise amalgamation problems: suppose T has $\mathcal{P}^-(k)$-amalgamation of models, and $(M_u : u \subsetneq k+1)$ is an (independent) system of models of T. For each $u \subsetneq k$, let $p_u(\overline{x}) = tp_{M_{u\cup\{k\}}}(M_{\{k\}}/M_u)$; suppose that $\bigcup_u p_u(\overline{x})$ does not fork over $M_\emptyset$ (i.e. is consistent). We construct a solution to $(M_u : u \subsetneq k+1)$.

For each $u \subsetneq k+1$, define A_u as follows: let $A_k = M_k$, and for $u \not\subseteq k$, let $A_u = M_{u\cap k} \cup \{M_k\}$. So $(A_u : u \subsetneq k+1)$ is a system of models of $T^\forall$, and it has a solution by hypothesis. Let $A_* \in T^\forall$ be the free amalgam, where we add no new relations (and where the universe of A_* is $|A_*| = \bigcup_{u\subsetneq k+1} |A_u|$). Next, for each $u \subsetneq k$, let B_u be the free amalgam of $M_{u\cup\{k\}}$ and A_* over $A_{u\cup\{k\}} = M_u \cup M_{\{k\}}$. Then $(B_u : u \subsetneq k)$ is a system of models of $T^\forall$. By the claim at the beginning, we can extend this to an (independent) system of models $(N_u : u \subsetneq k)$ of T. Since T has $\mathcal{P}^-(k)$-amalgamation, this has a solution N. But then N is a solution to our original problem $(M_u : u \subsetneq k+1)$.

□

We put together an omnibus theorem, showing that for well-behaved T, "everything is equivalent." To begin, we recap some theorems of [39] in our context.

The following is similar to Theorem 4.6 of [39].

Theorem 4.11.3. Suppose T has $\mathcal{P}^-(k)$-amalgamation of models and has extendible solutions. Then whenever Δ with $\dim^*(\Delta) \leq k$, T has Δ-amalgamation of models.

Proof. It suffices to consider the case where Δ is a pattern on n; we proceed by induction on n. If $n \leq 1$ the claim is clear. So suppose the claim is true for all $n' < n$, and suppose $\{s_i : i < k\}$ is given with each $s_i \subseteq n$; let Δ be the closure of $\{s_i : i < k\}$ under subsets.

We check that T has Δ-amalgamation of models.

We can suppose each $s_i \subsetneq n$. For each $i < k$, choose $t_i \in [n]^{n-1}$ with $s_i \subseteq t_i$. For each $i_* \leq k$, let Δ_{i_*} be the closure of $\{t_i : i < i_*\} \cup \{s_i : i_* \leq i < k\}$ under subsets. So $\Delta_0 = \Delta$. I show by induction on $i_* \leq k$ that there is an extension of $(M_s : s \in \Delta_0)$ to an independent system of models $(M_s : s \in \Delta_{i_*})$.

So suppose we have extended $(M_s : s \in \Delta_0)$ to an independent system of models $(M_s : s \in \Delta_{i_*})$ where $i_* < k$. For each $i < i_*$, define $r_i = t_i \cap t_{i_*}$; for each $i \geq i_*$, define $r_i = s_i \cap t_{i_*}$. So each $r_i \subseteq t_{i_*}$. Let Δ' be the closure of $\{r_i : i < k\}$ under subsets. By the inductive hypothesis, there is a solution M to $(M_s : s \in \Delta')$. If we choose M to be sufficiently saturated, then we can extend $(M_s : s \in \Delta')$ to an independent system of models $(M_s : s \in \mathcal{P}(t_{i_*}))$ by Corollary 4.3.8. Thus we have defined $(M_s : s \in \Delta_{i_*+1})$.

Thus we have $(M_s : s \in \Delta_k)$, where k is the downward closure of $\{t_i : i < k\}$, where each $t_i \in [n]^{n-1}$. By discarding repetitions (and so possibly decreasing k) we can suppose that $t_i \neq t_j$ for all $i \neq j < k$. After relabeling, we can suppose each $t_i = n \backslash \{i\}$. Since T has $\mathcal{P}^-(k)$-amalgamation of models, $(M_s : n \backslash k \subseteq s \subsetneq n)$ has a solution; since T has extendible solutions, so must $(M_s : s \in \Delta)$. $\square$

We now proceed to prove the various implications in our omnibus theorem. Actually, the first does not use well-behavedness at all:

Lemma 4.11.4. Suppose T is simple, and $k \geq 3$, and T has Δ-amalgamation of models for some $\Delta \in \boldsymbol{\Delta}$ with $\dim(\Delta) \geq k$. Then T has $\mathcal{P}^-(k)$-amalgamation of models.

Proof. We show the contrapositive. Let $\Delta \in \boldsymbol{\Delta}$ have $\dim(\Delta) \geq k$. Say $\Delta \subseteq \mathcal{P}(n)$. Choose $\ell \geq k$ and $t \in [n]^\ell \backslash \Delta$ such that $\mathcal{P}^-(t) \subseteq \Delta$. By hypothesis and Lemma 4.4.4, we can find an independent system of models $(M_u : u \subsetneq t)$ with no solution. For each $s \in \Delta$, define $N_s = M_{s\cap t}$; note that this is defined, since $t \notin \Delta$. Clearly then, $(N_s : s \in \Delta)$ is an independent system of models with no solution. $\square$

Lemma 4.11.5. Suppose T is simple with canonical amalgamation, and $k \geq 3$, and T has $\mathcal{P}^-(k)$-amalgamation of models. Then T has Δ-amalgamation of models for every $\Delta \in \boldsymbol{\Delta}$ with $\dim(\Delta) \leq k$.

Proof. Suppose T has $\mathcal{P}^-(k)$-amalgamation of models. We proceed by induction on n to show that whenever $\Delta \in \boldsymbol{\Delta}$ has $\dim(\Delta) \leq k$ and $\Delta \subseteq \mathcal{P}(n)$, then M has Δ-amalgamation of models. Indeed, suppose we have verified for all $n' < n$, and $\Delta \subseteq \mathcal{P}(n)$ is given. Choose $t_* \in \mathcal{P}(n)\backslash\Delta$, such that $s \in I$ for all $s \subsetneq t_*$. Then $|t_*| \leq k$. Let Δ' be the set of all $t \subseteq n$ such that $t_* \not\subseteq t$.

Easily, for each $t \in \Delta'$, if we set $\Delta_t = \{s \cap t : s \in \Delta\}$, then $\dim(\Delta_t) \leq k$. Thus, for each $t \in \Delta'$, $(M_{s\cap t} : s \in \Delta)$ has a solution, by the inductive hypothesis. Thus, by canonical amalgamation, we can find an extension of $(M_s : s \in \Delta)$ to an independent system of models $(M_s : s \in \Delta')$. By hypothesis, $(M_s : s \in \Delta')$ has a solution, which must also be a solution to $(M_s : s \in I)$. $\square$

Lemma 4.11.6. Suppose T has no surprise amalgamation problems and has extendible solutions, and $k \geq 2$, and T has $\Lambda_{k'}$-type amalgamation for all $k' \leq k$. Then T has $\mathcal{P}^-(k+1)$-amalgamation of models.

Proof. We can suppose inductively that T has $\mathcal{P}^-(k)$-amalgamation of models.

Suppose $(M_u : u \subsetneq k+1)$ is an independent system of models with no solution. By Theorem 4.11.3 we can find a solution $\hat{M}_{k+1}$ to $(M_u : u \subsetneq k+1, u \neq k)$. Define $\hat{M}_u = M_u$ for each $u \subsetneq k+1$ with $u \neq k$. If we choose $\hat{M}_{k+1}$ sufficiently saturated then we can extend $(\hat{M}_u : u \subsetneq k+1, u \neq k)$ to an independent system $(\hat{M}_u : u \subseteq k+1)$ of submodels of $\hat{M}_{k+1}$, by Corollary 4.3.8.

Let $\overline{0} : k \to \{0\}$ be the constant 0 function. Let $\Delta \subseteq \mathcal{P}(k \times 2)$ be the closure of $\Lambda_k \cup \{\overline{0}\}$ under subsets (so each $s \in \Delta$ is a partial function from k to 2, which is equal to 1 in at most one coordinate). Let $(N_s : s \in \Delta)$ be the following system of models. For each $s \subseteq \overline{0}$, $N_s = M_{\text{dom}(s)}$. Suppose $s \not\subseteq \overline{0}$. Say $i < k$ is such that $i \in \text{dom}(s)$ and $s(i) = 1$. Write $u = \text{dom}(s)$ and write $u' = u \backslash \{i\}$; note that $N_{s \cap \overline{0}} = M_{u'} = \hat{M}_{u'}$. Define $N_s = \hat{M}_{u'} \cup (\hat{M}_u \backslash \hat{M}_{u'}) \times \{i\}$, where N_s is made into a model of T so that the obvious bijection between N_s and $\hat{M}_u$ is an isomorphism. Thus, for each $\eta \in \Lambda_k$, $(N_s : s \subseteq \eta)$ is a copy of $(\hat{M}_u : u \subseteq k)$.

Given $\eta \in \Lambda_k$, let $N_{\eta \cup \overline{0}}$ be an independent amalgam of N_η and $N_{\overline{0}}$ over $N_{\eta \cap \overline{0}}$. Choosing $N_{\eta \cap \overline{0}}$ to be sufficiently saturated, we can extend $(N_s : s \subseteq \eta \text{ or } s \subseteq \overline{0})$ to an independent system $(N_s : s \subseteq \eta \cup \{0\})$ of submodels of $N_{\eta \cap \overline{0}}$, by Corollary 4.3.8. We can do this for all $\eta \in \Lambda_k$ without introducing conflicts.

Hence we get $(N_{\eta \cup \overline{0}} : \eta \in \Gamma_k)$ (recall Γ_k is the set of all functions from k to 2 which take on the value 1 at exactly one coordinate). Let $N_{k \times 2}$ be an independent amalgam of $(N_{\eta \cup \overline{0}} : \eta \in \Gamma_k)$ over $N_{\overline{0}}$ (by the Independence Theorem for simple theories).

For each $\eta, \eta' \in \Lambda_k$, let $\pi_{\eta,\eta'} : N_\eta \cong N_{\eta'}$ be the canonical isomorphism, using that both are canonically isomorphic with $\hat{M}_k$. Clearly this turns $(N_s : s \in \Delta_{\Lambda_k}), (\pi_{\eta,\eta'} : \eta, \eta' \in \Lambda_k)$ into a Λ_k-array.

Now, since T has no surprise amalgamation problems (and by the finite character of nonforking), we can choose $a_* \in M_{\{k\}}$ (possibly a tuple) such that if we let $p_u(x) = tp_{M_{u \cup \{k\}}}(a_*/M_u)$ for each $u \subsetneq k$, then $\bigcup_u p_u(x)$ forks over $M_\emptyset$ as computed in M_k. But by choice of $\hat{M}_k$, $\bigcup_u p_u(x)$ does not fork over $\hat{M}_\emptyset = M_\emptyset$ as computed in $\hat{M}_k$. Let $p(x)$ be some complete extension of $\bigcup_u p_u(x)$ to $\hat{M}_k$ which does not fork over $\hat{M}_\emptyset$.

For each $\eta \in \Lambda_k$, let $p_\eta(x)$ be the canonical copy of $p(x)$ over N_η. Then $(p_\eta(x) : \eta \in \Lambda_k)$ is a coherent system of types over $(\overline{N}, \overline{\pi})$. But for each $u \in [k]^{k-1}$, if we let $\eta \in \Lambda_k$ be the unique element with $\eta \restriction_u = \overline{0} \restriction_u$, then $p_u(x) \subseteq p_\eta(x)$. Thus $\bigcup_\eta p_\eta(x)$ forks over $N_0 = M_0$. $\square$

Lemma 4.11.7. Suppose T has no surprise amalgamation problems. Suppose $k \geq 3$ is such that T has Δ-amalgamation of models for all $\Delta \in \mathbf{\Delta}$ with $\dim(\Delta) \leq k$, and such that T fails $\mathcal{P}^-(k+1)$ amalgamation of models. Then T admits Δ_k.

Proof. Let $R \subseteq [\omega]^k$ be given. We show that T admits $\Delta(R)$, which suffices.

Let $(M_u : u \subsetneq k+1)$ be an independent system with no solution, and since T has no surprise amalgamation problems, we can choose $a_* \in M_{\{k\}}$ (possibly a tuple) such that if we set $p_u(x) = tp_{M_{u \cup \{k\}}}(a_*/M_u)$ for each $u \subsetneq k$, then $\bigcup_u p_u(x)$ forks over $M_\emptyset$ as computed in M_k.

For each $w \in [\omega]^k$, let $\rho^w : k \to w$ be the increasing enumeration. Also, for each $v \in [\omega]^{k-1}$ and for each $u \in [k]^{k-1}$, let $\rho^v_u : u \to v$ be the increasing enumeration.

Let Δ be the pattern on $k \times \omega \cup \{*_S : S \in \Delta(R)\}$ defined as follows: $s \in \Delta$ if and only if either $s \subseteq \rho^w$ for some $w \in R$, or else $s \subseteq \rho^v_u \cup \{*_S\}$ for some $u \in [k]^{k-1}, v \in [\omega]^{k-1}$ and some $S \in \Delta(R)$ with $v \in S$. Note that the maximal elements Δ are of the form ρ^w for $w \in R$, or $\rho^v_u \cup \{*_S\}$ with $v \in S$.

Note that $\dim(\Delta) \leq k$: suppose towards a contradiction we had $t \in [k \times \omega \cup \{*_S : S \in \mathcal{S}\}]^{k+1} \backslash \Delta$ such that $\mathcal{P}^-(t) \subseteq \Delta$. Clearly t can contain at most one $*_S$ (using $k \geq 3$). Write $t' = t \cap k \times \omega$. Clearly t' is the graph of an increasing function from k to ω; in particular, $t \backslash t' = \{*_S\}$ is nonempty. Write $w = t'[k]$ (i.e. the image of t'). Since $t' \in \Delta$ we must have that $w \in R$. Thus there is some $v \in [w]^{k-1}$ such that $v \notin S$. Let $u = (t')^{-1}[v]$; then $\rho^v_u \cup \{*_S\} \notin \Delta$, a contradiction.

Given $s \in \Delta \cap \mathcal{P}(k \times \omega)$, we define N_s, inductively on $|s|$. Let $N_\emptyset = M_\emptyset$. Having defined $N_{s'}$ for all $s' \subsetneq s$, let $\pi : s \to k$ be projection onto the first factor, and let $N_s = (\bigcup_{s' \subsetneq s} N_{s'}) \cup (M_{\pi(s)} \setminus \bigcup_{u \subsetneq \pi(s)} M_u) \times \{s\}$; so there is an obvious bijection $\tau_s : M_{\pi(s)} \to N_s$. We turn N_s into a model of T so that τ_s is an isomorphism. The point is that $(N_{s'} : s' \subseteq s)$ will be a copy of $(M_{\pi(s')} : s' \subseteq s)$.

Now, given $s \cup \{*_S\} \in \Delta$, we define N_s, inductively on $|s|$. Having defines $N_{s'}$ for all $s' \subsetneq s \cup \{*_S\}$, let $\pi : s \to k$ be projection onto the first factor. Let $A = \bigcup_{s' \subsetneq s \cup \{*_S\}} N_{s'}$, and let $B = \bigcup_{u \subsetneq \pi(s) \cup \{k\}} M_u$. Define $N_{s \cup \{*_S\}} = A \cup (M_{\pi(s) \cup \{k\}} \setminus B) \times \{s \cup \{*_S\}\}$, so there is an obvious bijection $\tau_{s \cup \{*_S\}} : M_{\pi(s) \cup \{k\}} \to N_{s \cup \{*_S\}}$. We turn $N_{s \cup \{*_S\}}$ into a model of T so that $\tau_{s \cup \{*_S\}}$ is an isomorphism. The point is that $(N_{s'} : s' \subseteq s \cup \{*_S\})$ is a copy of $(M_u : u \subseteq \pi(s) \cup \{k\})$.

Then $(N_s : s \in \Delta)$ is an independent system of models. Now dim is hereditary, in the sense the if we take $X \subseteq k \times \omega \cup \{*_S : S \in \mathcal{S}\}$ finite, then $\dim(\Delta \cap \mathcal{P}(X)) \leq k$. In particular, for each such X, $(N_s : s \in \Delta \cap \mathcal{P}(X))$ has a solution. By Lemma 4.3.3, this implies that $(N_s : s \in \Delta)$ has a solution, say N.

Now, for each $v \in [\omega]^{k-1}$, and for each $u \in [k]^{k-1}$, let $q_u^v(x) = \tau_{\rho_u^v}[p_u(x)]$. Let $q^v(x) = \bigcup_{u \in [k]^{k-1}} q_u^v(x)$.

I claim that for all $S \subseteq [\omega]^{k-1}$ finite, $\bigcup_{v \in S} q^v(x)$ forks over N_0 if and only $S \in \Delta(R)$.

First suppose $S \notin \Delta(R)$, so there is some $w \in R$ with $[w]^{k-1} \subseteq S$; then $\bigcup_{v \in S} q^v(x) \supseteq \tau_{\rho^w}[\bigcup_{u \subsetneq k} p_u(x)]$ forks over $\tau_{\rho^w}[M_0] = N_0$. On the other hand, if $S \in \Delta(R)$, then look at $a_S = \tau_{\{*_S\}}(a_*) \in M_{*_S}$; by construction, a_S realizes $\bigcup_{v \in S} q^v(x)$, and so we conclude by independence of $(N_s : s \in \Delta)$.

Now, choose $\varphi_u(x, b_u) \in p_u(x)$ for each $u \in [k]^{k-1}$, such that $\bigwedge_{u \in [k]^{k-1}} \varphi_u(x, b_u)$ forks over M_0 (as computed in M_k). Choose $\ell_* < \omega$ so that $\bigwedge_{u \in [k]^{k-1}} \varphi_u(x, b_u)$ ℓ_*-divides over M_0.

Let $N_\ell : \ell < \ell_*$ be a Morley sequence in N over N_0 (we can suppose $N \preceq \mathbb{C}$). Let $\sigma_\ell : N \cong N_\ell$ be isomorphisms.

Let $\overline{y} = (y_{u,\ell} : u \in [k]^{k-1}, \ell < \ell_*)$. For each $v \in [\omega]^{k-1}$, for each $u \in [k]^{k-1}$ and for each $\ell < \ell_*$, let $b^v_{u,\ell} = \sigma_\ell(\tau_{\rho^v_u}(b_u))$. Let $\overline{b}^v = (b^v_{u,\ell} : u \in [k]^{k-1}, \ell < \ell_*)$. Then clearly, $(\overline{b}^v : v \in [\omega]^{k-1})$ witness that $\psi(x, \overline{y}) = \bigwedge_{u \in [k]^{k-1}, \ell < \ell_*} \varphi_u(x, y_{u,\ell})$ admits $\Delta(R)$. □

Finally, we show the following: if T is simple and has Λ-type amalgamation for all $\Lambda \in \mathbf{\Lambda}$ with $|\Lambda| < K$, then T does not admit $\Delta_{k'}$ for any $k' < k$ (note that there is no hypothesis of well-behavedness on T). Our strategy is as follows: by Theorem 4.8.3, we know that T must have the absolute concrete $\mathbb{P}_{\mathbf{s},k}$ amalgamation property for appropriate $\mathbf{s}$, so it will suffice to show that if T admits $\Delta_{k'}$ then T fails this. Note that if either T is low or else there is a supercompact cardinal, we are done by Theorem 4.9.1 and Theorem 4.8.2. Otherwise, we need to do some work—we will need to show that if T admits Δ_k, then it does so in a particularly nice way. The following is a first approximation to this, and is rather general. Here, q.f.tp is quantifier-free type.

Lemma 4.11.8. Suppose T is a complete countable theory, and T admits Δ_k via the formula $\varphi(x, y)$ (where x, y are possibly tuples). Suppose λ is a cardinal and $R \subseteq [\lambda]^k$. Then we can find $(b_u : u \in [\lambda]^{k-1})$ from $\mathfrak{C}$ witnessing that T admits $\Delta(R)$, and such that for every $w, w' \in [\lambda]^{<\aleph_0}$ of the same length, if $q.f.tp_{(\lambda,<,R)}(w) = q.f.tp_{(\lambda,<,R)}(w')$, then $tp_{\mathfrak{C}}(b_u : u \in [w]^{k-1}) = tp_{\mathfrak{C}}(b_u : u \in [w']^{k-1})$.

Proof. We first of all verify the following combinatorial claim.

Claim. Suppose $n < \omega$ and $R \subseteq [n]^k$ and $c < \omega$. Then there is some $n_* \geq n$ and some $R_* \subseteq [n_*]^k$ such that whenever $f : [n_*]^{\leq n} \to c$, there is some $w \in [n_*]^n$, such that

$(w, <, R_* \cap [w]^k) \cong (n, <, R)$, and for all $u \subseteq n$, if $g_0, g_1 : (u, <, R \cap [u]^k) \to (w, <, R_* \cap [w]^k)$ are embeddings, then $f(g_0(u)) = f(g_1(u))$.

Proof. A theorem of Nešitřil and Rödl [67] states that this is possible when we fix $u \subseteq n$. To get the full claim, iterate their theorem 2^n-times, once for each subset of n. $\square$

With the claim, the lemma follows easily, by compactness. $\square$

Thus, we can get the following equivalence of admitting Δ_k.

Lemma 4.11.9. Suppose T is simple, and T admits Δ_k via $\varphi(x, y)$ (where x, y are possibly tuples). Then for any λ and for any $R \subseteq [\lambda]^k$, there is some countable $M_0 \preceq \mathbb{C}$, such that we can find $(b_u : u \in [\lambda]^{k-1})$, such that for every $s \in \Delta(R)$, $\{\varphi(x, b_u) : u \in s\}$ does not fork over M_0, and for every $s \in [[\lambda]^{k-1}]^{<\aleph_0} \backslash \Delta(R)$, $\{\varphi(x, b_u) : u \in s\}$ is inconsistent.

Proof. Choose some $R_* \subseteq [\lambda]^k$ such that:

- If we set $R_0 = R_* \cap [\omega]^k$, then $(\omega, <, R_0)$ is a random k-ary hypergraph;
- $\alpha \mapsto \alpha + \omega$ gives an isomorphism from $(\lambda, <, R)$ to $(\lambda \backslash \omega, <, R_* \cap [\lambda \backslash \omega]^k)$;
- For every $u \in R$, either $u \subseteq \omega$ or else $u \cap \omega = \emptyset$.

By Lemma 4.11.8, we can find $(b_u : u \in [\lambda]^{k-1})$ witnessing that T admits $\Delta(R_*)$, such that further, for every $w, w' \in [\lambda]^{<\aleph_0}$ of the same length, if $q.f.tp_{(\lambda,<,R_*)}(w) = q.f.tp_{(\lambda,<,R_*)}(w')$, then $tp_{\mathfrak{C}}(b_u : u \in [w]^{k-1}) = tp_{\mathfrak{C}}(b_u : u \in [w']^{k-1})$. Choose $M_0 \preceq \mathbb{C}$ countable such that $b_u \in M_0$ for each $u \in [\omega]^{k-1}$. We claim this works.

First of all, I claim that for every $s \in \Delta(R_*)$ with $\bigcup s \cap \omega = \emptyset$, $\{\varphi(x, b_u) : u \in s\}$ does not fork over M_0. Indeed, suppose towards a contradiction that $\{\varphi(x, b_u) : u \in s\}$ forks over M_0. Choose $\ell < \omega$ such that $\{\varphi(x, b_u) : u \in s\}$ ℓ-forks over M_0. Write $w = \bigcup s \in [\lambda \backslash \omega]^{<\aleph_0}$, and choose $(w_i : i < \omega)$ a sequence from $[\omega]^{|w|}$, such that:

- Each $\max(w_i) < \min(w_{i+1})$;

- each $q.f.tp_{(\lambda,<,R_*)}(w_i) = q.f.tp_{(\lambda,<,R_*)}(w)$;

- For every $u \in R_*$, there is at most one $i < \omega$ with $u \cap w_i$ nonempty.

This is easily possible.

For each i, let $s_i \subseteq [w_i]^{k-1}$ be the subset corresponding to $s \subseteq [w]^{k-1}$, under the unique order-preserving bijection. Let $p(x) = \{\varphi(x, b_u) : u \in \bigcup_i s_i\}$. Clearly, $p(x)$ is consistent, so there must be some $i < \omega$ such that $p(x)$ does not ℓ-divide over $\bigcup_{j<i}\{b_u : u \in s_i\}$. Then $\{\varphi(x, b_u) : u \in s_i\}$ does not ℓ-divide over $\bigcup_{j<i}\{b_u : u \in s_j\}$. But this contradicts that $tp_{\mathfrak{C}}(b_u : u \in s_i/(b_u : u \in \bigcup_{j<i} s_j)) = tp_{\mathfrak{C}}(b_u : u \in s/(b_u : u \in \bigcup_{j<i} s_j))$.

For each $u \in [\lambda]^k$, define $b'_u = \{b_{\alpha+\omega} : \alpha \in u\}$; then $(b'_u : u \in [\lambda]^k)$ is as desired. $\square$

We then obtain the following:

Theorem 4.11.10. Suppose $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence. Suppose $3 \leq k_* < \omega$, and $\lambda \geq [\kappa]^{+(k_*-1)}$. If T admits Δ_k for some $k < k_*$, then T fails the absolute concrete $(\mathbb{P}_{\mathbf{s},k_*}, \mathbf{s})$-amalgamation property.

Proof. Say $\varphi(x, y)$ admits Δ_k (where possibly x, y are tuples).

Let $P = P_{[\lambda]^k\theta\theta} \in \mathbb{P}_{\mathbf{s},k_*}$. Write $\mathcal{B} = \mathcal{B}(P)$; for each $v \in [\lambda]^k$, write $\mathbf{c}_v = \{(v, 0)\} \in P \subseteq \mathcal{B}$. Let $\mathbf{A}$ be the $[\lambda]^{k-1}$-distribution in $\mathcal{B}$, defined by putting $\mathbf{A}(s) = \bigwedge\{\mathbf{c}_v : v \in [\lambda]^k, [v]^{k-1} \subseteq s\}$. By the proof of Theorem 4.9.1, whenever $\dot{Q} \in \mathbb{P}_{\mathbf{s},k_*}^{\mathbb{V}[\dot{G}]}$, then $\mathbf{A}$ has no multiplicative refinement in $\mathcal{B}(P * \dot{Q})$.

Let G be P-generic over $\mathbb{V}$; we show that T fails the concrete $\mathbb{P}_{\mathbf{s},k_*}$-amalgamation property in $\mathbb{V}[G]$, which suffices. Indeed, in $\mathbb{V}[G]$, let $f : [\lambda]^k \to \theta$ be the generic function adding by G (so $f = \bigcup G$). Let $R = \{v \in [\lambda]^k : f(v) \neq 0\}$. By Lemma 4.11.9, we can find $M \models T$ and $M_0 \preceq M$ countable, and $(b_u : u \in [\lambda]^k)$ from $\mathfrak{C}$, such that for every $s \in \Delta(R)$, $\{\varphi(x, b_u) : u \in s\}$ does not fork over M_0, and for every $s \in [[\lambda]^{k-1}]^{<\aleph_0} \backslash \Delta(R)$, $\{\varphi(x, b_u) : u \in s\}$ is inconsistent.

Suppose towards a contradiction that $\Gamma^\theta_{M,M_0} \in \mathbb{P}_{\mathbf{s}_*,k}$. Pull everything back to $\mathbb{V}$ to get names $\dot{f}, \dot{R}, \dot{M}, \dot{M}_0, (\dot{b}_u : u \in [\lambda]^k)$. Then $\Gamma^\theta_{\dot{M},\dot{M}_0} \in \mathbb{P}^{\mathbb{V}[\dot{G}]}_{\mathbf{s},k_*}$, and by the proof of Theorem 4.8.2, this implies $\mathbf{A}$ has a multiplicative refinement in $P * \Gamma^\theta_{\dot{M},\dot{M}_0}$, contradiction.

$\square$

Corollary 4.11.11. Suppose T is simple, and T has Λ-type for all $\Lambda \in \mathbf{\Lambda}$ with $|\Lambda| < k$. Then T does not admit $\Delta_{k'}$ for any $k' < k$.

Proof. Suppose T contradicted this. Let $\sigma = \aleph_0$, let $\theta = \aleph_1$, let $\kappa = (2^{\aleph_0})^+$, and let $\lambda \geq \kappa^{+\omega}$. Then $\mathbf{s} = (\lambda, \kappa, \theta, \sigma)$ is a suitable sequence. But T would have the absolute concrete $\mathbb{P}_{\mathbf{s},k}$-amalgamation property by Theorem 4.8.3, and would also fail it by Theorem 4.11.10.

$\square$

We have now proved all the required equivalences in our omnibus theorem:

Theorem 4.11.12. Suppose T is well-behaved and $3 \leq k < \aleph_0$. Then the following are all equivalent.

(A) T does not admit $\Delta_{k'}$ for all $k' < k$.

(B) T has $\Lambda_{k'}$-type amalgamation for all $k' < k$.

(C) T has Λ-type amalgamation for all $\Lambda \in \mathbf{\Lambda}$ with $\dim(\Delta_\Lambda) < k$.

(D) T has Δ-amalgamation of models for some $\Delta \in \mathbf{\Delta}$ with $\dim(\Delta) \geq k$.

(E) T has $\mathcal{P}^-(k)$-amalgamation of models.

(F) T has Δ-amalgamation of models for every $\Delta \in \mathbf{\Delta}$ with $\dim(\Delta) \leq k$.

Proof. Clearly, (F) implies (E) implies (D). (D) implies (E) is by Lemma 4.11.4. (E) implies (F) is by Lemma 4.11.5. So (D), (E), (F) are all equivalent.

(F) implies (C) is by Theorem 4.4.13. (C) implies (B) is clear. (B) implies (E) is by Lemma 4.11.6 and Theorem 4.11.3. Thus, (B) through (F) are all equivalent.

(C) implies (A) is by Corollary 4.11.11. An easy inductive argument together with Lemma 4.11.7 gives (A) implies (E)/(F).

□

By taking the conjunction of Theorem 4.11.12 over all $k < \aleph_0$, we obtain the following:

Corollary 4.11.13. Suppose T is well-behaved. Then (A) through (F) are all equivalent.

(A) T does not admit Δ_k for all k.

(B) T has Λ_k-type amalgamation for all k.

(C) T has Λ-type amalgamation for all $\Lambda \in \mathbf{\Lambda}$.

(D) T has Δ-amalgamation of models for $\Delta \in \mathbf{\Delta}$ with $\dim(\Delta)$ arbitrarily large.

(E) T has $\mathcal{P}^-(k)$-amalgamation of models for all k.

(F) Every independent system of models of T has a solution.

Finally, we mention a consequence for Keisler's order:

Corollary 4.11.14. Among well-behaved low theories, for each $3 \leq k < \omega$, the property of having $\mathcal{P}^-(k)$-amalgamation of models is a principal dividing line in Keisler's order, as is the property of having $\mathcal{P}^-(k)$-amalgamation of models for all k. If there is a supercompact cardinal, we can replace low by simple.

Proof. By Theorems 4.10.4, 4.10.3 and 4.11.12. ∎

Chapter 5: Borel Complexity and Potential Canonical Scott Sentences

We now move on from Keisler's order and consider Borel complexity. The motivation here is to find interesting dividing lines for *countable* model theory. Here are some important examples.

- $\mathrm{Th}(\mathbb{Q}, <)$, or any other $\aleph_0$-categorical theory.

- $T := \mathrm{Th}(\mathbb{Z}, S)$, where S is the successor relation. T has countably many countable models (up to isomorphism), namely we can say how many S-chains there are (some number $1 \leq n \leq \aleph_0$).

- $T := \mathrm{Th}(\mathbb{Q}, <, c_r : r \in \mathbb{Q})$, where we add constants for the elements of $\mathbb{Q}$. The countable models of these are easy to understand. Namely, suppose $M \models T$ is countable. For each cut in $\mathbb{Q}$ (formally, a partition $\mathbb{Q} = I_0 \cup I_1$ where every element of I_0 is below every element of I_1), we have to give the order-type isomorphism of $M_{I_0,I_1} := \{a \in : a > c_r \text{ for all } r \in I_0 \text{ and } a < c_r \text{ for all } r \in I_1\}$. This is a dense linear order, so the only possibilities are $(\mathbb{Q}, <)$, $(\mathbb{Q} \cup \{\infty\}, <)$, $(\mathbb{Q} \cup \{-\infty\}, <)$, $(\mathbb{Q} \cup \{\pm\infty\}, <)$, $(\{0\}, <)$, and $\emptyset$. (Not all of these are always possible; for instance, if I_0 contains a maximal element then $(\mathbb{Q} \cup \{-\infty\}, <)$, $(\mathbb{Q} \cup \{\pm\infty\}, <)$ and $(\{0\}, <)$ are impossible). Note that there are only finitely many possibilities for M_{I_0,I_1} up to isomorphism, and we can code partitions (I_0, I_1) of $\mathbb{Q}$ with $I_0 < I_1$ as elements of $\mathbb{R}$. Further, there are only countably many (I_0, I_1) such that M_{I_0,I_1} is nonempty (since

M is countable). Thus we can code the isomorphism class of M as a function from a countable subset of $\mathbb{R}$ to ω, or equivalently as a countable subset of $\mathbb{R}$.

- Recall that an abelian group $(G, +)$ is torsion if for every $a \in G$, there is some $n > 0$ such that $na = 0$ (this is true for all groups, not just abelian groups, but nonabelian groups would not be written additively). If p is a prime, then G is a p-group if we can alway choose n to be a power of p. Ulm classified countable abelian p-groups in [86], with what are now known as Ulm invariants; these are essentially elements of $2^{<\omega_1}$. Let TAG_1 be the sentence of $\mathcal{L}_{\omega_1\omega}$ describing torsion abelian groups. A countable model of TAG_1 can be written uniquely as a direct sum over primes p of abelian p-groups, and thus can also be coded by elements of $2^{<\omega_1}$.

- Let Graphs be the first order theory in the language $\{R\}$ of graphs (i.e. saying R is symmetric and irreflexive). Theorem 5.5.1 of [21] states that graphs can interpret any theory, and the proof indicates that the countable models of Graphs are as complicated as all countable structures.

We wish to have a notion of complexity which captures our impression that $\mathrm{Th}(\mathbb{Q}, <)$ is less complicated than $\mathrm{Th}(\mathbb{Z}, S)$, which is less complicated than $\mathrm{Th}(\mathbb{Q}, <, c_r : r \in Q)$ and TAG_1. Moreover, these last two theories are incomparable (since $2^{<\omega_1}$ seems neither larger nor smaller than $\mathcal{P}_{\aleph_1}(\mathbb{R})$), and both are less complicated than Graphs.

The naïve method of just counting the number of countable models of isomorphism does work for $\mathrm{Th}(\mathbb{Q}, <)$ and $\mathrm{Th}(\mathbb{Z}, S)$, since these have only one countable model and only countably many countable models, respectively (always, up to isomorphism). But the interesting case is when there are the maximum number of countable models, namely, continuum many; for instance $\mathrm{Th}(\mathbb{Q}, <, c_r : r \in Q)$, TAG_1 and Graphs. These are not distinguished by straightforward counting.

But note that there is no natural bijection between the isomorphism types of countable models of $\mathrm{Th}(\mathbb{Q}, <, c_r : r \in Q)$ and the isomorphisms types of countable models of TAG_1 (there is no natural bijection between $\mathcal{P}_{\aleph_1}(\mathbb{R})$ and $2^{<\omega_1}$). In [12], Friedman and Stanley introduce the notion of Borel complexity, with the intent of restricting to definable cardinality. This turns out to do a reasonable job of capturing our intuitive notion of complexity of countable models.

The setup is as follows. We are interested in sentences $\Phi \in \mathcal{L}_{\omega_1\omega}$; it turns out that for countable model theory, first-order theories are no easier to work with. We could actually be more general than sentences of $\mathcal{L}_{\omega_1\omega}$. For instance, in most applications we could allow a single second-order quantification, but $\mathcal{L}_{\omega_1\omega}$ is general enough for what we wish to do.

Given a sentence $\Phi \in \mathcal{L}_{\omega_1\omega}$, we can form $\mathrm{Mod}(\Phi)$, the set of models of Φ with universe ω. This is naturally a standard Borel space, where the Borel sets are taken to be solution sets to formulas of $\mathcal{L}_{\omega_1\omega}$. In [12], Friedman and Stanley make the following definition.

Definition 5.0.1. Suppose Φ, Ψ are sentences of $\mathcal{L}_{\omega_1\omega}$. Then $f : \Phi \leq_B \Psi$ is a Borel reduction if $f : \mathrm{Mod}(\Phi) \to \mathrm{Mod}(\Psi)$ is a Borel map, such that for all $M, N \in \mathrm{Mod}(\Phi)$, $M \cong N$ if and only if $f(M) \cong f(N)$. We say that $\Phi \leq_B \Psi$ if there is some $f : \Phi \leq_B \Psi$. Say that $\Phi \sim_B \Psi$ (Φ and Ψ are Borel bireducible) if $\Phi \leq_B \Psi \leq_B \Phi$.

By a Borel map, we mean a function whose graph is Borel; this is equivalent to requiring the preimage of a Borel set is Borel. Note that the condition that f preserve isomorphism is equivalent to requiring that f induce an injection from $\mathrm{Mod}(\Phi)/\cong$ to $\mathrm{Mod}(\Psi)/\cong$. Thus, if $\Phi \leq_B \Psi$ then $|\mathrm{Mod}(\Phi)/\cong| \leq |\mathrm{Mod}(\Psi)/\cong|$, but moreover this holds via a definable injection.

There is a more general definition of Borel reductions given independently in [19] by Harrington, Kechris and Louveau. Namely, suppose X and X' are standard Borel spaces, E is an equivalence relation on X, and E' is an equivalence relation on X'. Then say that $f : (X, E) \leq_B (X', E')$ is a Borel reduction if $f : X \to X'$ is a Borel map, and for all $x, y \in X$, xEy if and only if $f(x)E'f(y)$. The connection to the previous definition is as follows: suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$. Let $\cong_\Phi$ be the equivalence relation of isomorphism on $\mathrm{Mod}(\Phi)$. Then $\Phi \leq_B \Psi$ if and only if $(\mathrm{Mod}(\Phi), \cong_\Phi) \leq_B (\mathrm{Mod}(\Psi), \cong_\Psi)$.

It is easily checked that whenever Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$, then $\cong_\Phi$ is analytic, but sometimes $\cong_\Phi$ is additionally Borel. Classically, most of the techniques developed for $\leq_B$ only applied in this latter case. Note that if $\Phi \leq_B \Psi$ and $\cong_\Psi$ is Borel, so is $\cong_\Phi$.

We describe some of the initial results on $\leq_B$ obtained by Friedman and Stanley in [12].

First, Friedman and Stanley showed that there is a maximal class of sentences under $\leq_B$, namely the *Borel complete sentences*. For example, Graphs is Borel complete, as are the theories of groups, rings, linear orders, and trees. This notion provides a way to answer the question "Is it possible to classify the countable models of Φ" negatively in a precise sense: if Φ is Borel complete, then classifying the countable models of Φ is as hard as classifying arbitrary countable structures. Friedman and Stanley also showed in [12] that if Φ is Borel complete, then $\cong_\Phi$ is not a Borel subset of $\mathrm{Mod}(\Phi) \times \mathrm{Mod}(\Phi)$.

Also, Friedman and Stanley introduced the Friedman-Stanley tower. There are many equivalent formulations of this; we will find a certain family $(\Phi_\alpha : \alpha < \omega_1)$ of sentences of $\mathcal{L}_{\omega_1\omega}$ to be the most convenient to work with. The countable models of Φ_α up to isomorphism are, in a precise sense, identifiable with $\mathrm{HC}_{\omega+\alpha}$, the hereditarily countable sets of foundation rank less than $\omega + \alpha$. It is easy to see that $\Phi_\alpha \leq_B \Phi_\beta$ for $\alpha \leq \beta$; using sophisticated methods of descriptive set theory, Friedman and Stanley show that when $\alpha < \beta$, then $\Phi_\alpha <_B \Phi_\beta$ (i.e. $\Phi_\alpha \leq_B \Phi_\beta$ but $\Phi_\beta \not\leq_B \Phi_\alpha$).

$(\Phi_\alpha : \alpha < \omega_1)$ is a natural heirarchy in Borel complexity: by Corollary 12.2.8 of [14], if Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$, then $\cong_\Phi$ is Borel if and only if $\Phi \leq_B \Phi_\alpha$ for some $\alpha < \omega_1$. Also, we note that Φ_2 in arises naturally in many contexts, for instance, $\Phi_2 \sim_B \mathrm{Th}(\mathbb{Q}, <, c_r : r \in \mathbb{Q})$ (countable models of Φ_2 can be coded by elements $\mathrm{HC}_{\omega+2}$, which in turn can be coded as countable sets of reals, and conversely).

Finally, in [12], Friedman and Stanley prove that $\cong_{\mathrm{TAG}_1}$ is not Borel, and yet nonetheless $\Phi_2 \not\leq_B \mathrm{TAG}_1$. Thus, $\mathrm{Th}(\mathbb{Q}, <, c_r : r \in \mathbb{Q})$ and TAG_1 are $\leq_B$-incomparable as desired, and neither is Borel complete. At the time, TAG_1 was essentially the only example of a sentence of $\mathcal{L}_{\omega_1\omega}$ whose isomorphism relation was non-Borel, but which was not Borel complete; in particular there was no known first-order example of this phenomenon.

We view $\leq_B$ as a source of semantic dividing lines, and are interested if we can find syntactic equivalents. There are few results in this direction so far, the main difficulty being that it is difficult to compute $\leq_B$ at all. In particular, it is difficult to show non-reducibilities $\Phi \not\leq_B \Psi$, especially when $\cong_\Psi$ is non-Borel. Our main contribution is the introduction of some machinery for doing this, namely the machinery of potential canonical Scott sentences.

5.1 Chapter Overview

In the rest of this chapter, we include the results of [89], joint work with Richard Rast and Chris Laskowski, although we make some small changes to notation.

One of the central ideas of [89] is the following. Given a structure M, let $\mathrm{css}(M)$

denote its canonical Scott sentence; this is a canonical sentence of $\mathcal{L}_{|M|^+\omega}$ characterizing M up to back-and-forth equivalence. In particular, if M is countable, then $\mathrm{css}(M) \in \mathcal{L}_{\omega_1\omega}$ characterizes M up to isomorphism. Given $\Phi \in \mathcal{L}_{\omega_1,\omega}$, let $\mathrm{CSS}(\Phi) \subseteq \mathrm{HC}$ denote the set $\{\mathrm{css}(M) : M \in \mathrm{Mod}(\Phi)\}$. Then any Borel map $f : \mathrm{Mod}(\Phi) \to \mathrm{Mod}(\Psi)$ induces an HC-definable injection $f^* : \mathrm{CSS}(\Phi) \to \mathrm{CSS}(\Psi)$. This leads us to the investigation of definable subclasses of HC and definable maps between them. We begin by restricting our notion of classes to those definable by *HC-forcing invariant* formulas (see Definition 5.2.1). Three straightforward consequences of the Product Forcing Lemma show that these classes are well-behaved. It is noteworthy that we do not define HC-forcing invariant formulas syntactically. Whereas it is true that every Σ_1-formula is HC-forcing invariant, determining precisely which classes are HC-forcing invariant depends on our choice of $\mathbb{V}$.

The second ingredient of the development in [89] is that every set A in $\mathbb{V}$ is *potentially in HC*, i.e., there is a forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$ (indeed a Levy collapse suffices) such that $A \in \mathrm{HC}^{\mathbb{V}[G]}$. Given an HC-forcing invariant $\varphi(x)$, we define φ_{ptl} – that is, the *potential solutions to* φ – to be those $A \in \mathbb{V}$ for which $(\mathrm{HC}^{\mathbb{V}[G]}, \in) \models \varphi(A)$ whenever $A \in \mathrm{HC}^{\mathbb{V}[G]}$. The definition of HC-forcing invariance makes this notion well-defined. Thus, given a sentence Φ, one can define $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$, which should be read as the class of 'potential canonical Scott sentences of Φ.' This is the class of all $\varphi \in \mathcal{L}_{\infty,\omega}$ such that in some forcing extension $\mathbb{V}[G]$, φ is the canonical Scott sentence of some countable model of Φ. We define Φ to be *short* if $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ is a **set** as opposed to a proper class and define the *potential cardinality* of Φ, denoted $\|\Phi\|$, to be the (usual) cardinality of $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ if Φ is short, or ∞ otherwise.

Putting these two notions together, we get a reducibility notion $\leq_{\mathrm{HC}}$ on sentences of $\mathcal{L}_{\omega_1\omega}$, which coarsens $\leq_B$. Namely, put $\Phi \leq_{\mathrm{HC}} \Psi$ if there is some HC-forcing invariant formula $\varphi(x)$ which, in every forcing extension $\mathbb{V}[G]$, defines an injection from $\mathrm{CSS}(\Phi)^{\mathbb{V}[G]}$ to $\mathrm{CSS}(\Psi)^{\mathbb{V}[G]}$. By tracing all of this through, we see that with if $f : \Phi \leq_B \Psi$, then this induces some $f_* : \Phi \leq_{\mathrm{HC}} \Psi$, which in turn induces an injection $(f_*)_{\mathrm{ptl}} : \mathrm{CSS}(\Phi)_{\mathrm{ptl}} \to \mathrm{CSS}(\Psi)_{\mathrm{ptl}}$. This is the content of Theorem 5.3.11:

If $\|\Psi\| < \|\Phi\|$, then $\Phi \not\leq_{\mathrm{HC}} \Psi$, and thus $\Phi \not\leq_B \Psi$.

The advantage of this is that the potential cardinality $\|\Phi\|$ is, in applications, something we can calculate; thus, this gives an important new method for proving nonreducibilities.

As a particular example: we define the Friedman-Stanley tower $(\Phi_\alpha : \alpha < \omega_1)$ in Section 5.3.4 (handling the limit case slightly differently than in [89]), and show that each $\|\Phi_\alpha\| = \beth_\alpha$. This gives a much simpler proof that $\Phi_\alpha <_B \Phi_\beta$ for $\alpha < \beta$ than the one in [12].

Another issue raised in [89] is a comparison of the class the potential canonical Scott sentences $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ with the class $\mathrm{CSS}(\Phi)_{\mathrm{sat}}$, consisting of all sentences of $\mathcal{L}_{\infty,\omega}$ that are canonical Scott sentences of some model $M \models \Phi$ with $M \in \mathbb{V}$. Clearly, the latter class is contained in the former, and we call Φ *grounded* (see Definition 5.3.9) if equality holds. We show that the incomplete theory REF of refining equivalence relations is grounded. By contrast, the theory TK, defined in Section 5.6, is a complete, ω-stable theory for which $|\mathrm{CSS}(\mathrm{TK})_{\mathrm{sat}}| = \beth_2$, while $\mathrm{CSS}(\mathrm{TK})_{\mathrm{ptl}}$ is a proper class.

This machinery is applied in [89] as follows. Section 5.4 discusses continuous actions by compact groups on Polish spaces, proving a key theorem to be used in Section 5.6. Sections 5.5 and 5.6 discuss four complete first-order theories that are not very complicated stability-theoretically, yet the isomorphism relation is properly analytic in each case. In particular, we obtain the following:

REF(inf) is the theory of 'infinitely splitting, refining equivalence relations'. Its language is $\mathcal{L} = \{E_n : n \in \omega\}$. It asserts that each E_n is an equivalence relation, E_0 consists of a single class, each E_{n+1} refines E_n, and each E_n-class is partitioned into infinitely many E_{n+1}-classes. REF(inf) is one of the standard examples of a stable, unsuperstable theory. We show that REF(inf) is Borel complete, in fact, it is λ-Borel complete for all infinite λ (see Definition 5.3.17). On the other hand, we also show that REF(inf) is grounded, i.e., $\text{CSS}(\text{REF}(\text{inf}))_{\text{sat}} = \text{CSS}(\text{REF}(\text{inf}))_{\text{ptl}}$.

REF(bin) is the theory of 'binary splitting, refining equivalence relations'. The language is also $\mathcal{L} = \{E_n : n \in \omega\}$. The axioms of REF(bin) assert that each E_n is an equivalence relation, E_0 is trivial, each E_{n+1} refines E_n, and each E_n-class is partitioned into exactly two E_{n+1}-classes. REF(bin) is superstable (in fact, weakly minimal) but is not ω-stable. We show that $\Phi_2 \leq_B$ REF(bin), but $\|\text{REF}(\text{bin})\| = \beth_2$. In particular, REF(bin) is short, and $\Phi_3 \not\leq_B$ REF(bin). On the other hand, we show that REF(bin) does not have Borel isomorphism relation, in particular, we have produced a non-Borel complete first order theory whose isomorphism relation is not Borel. We also show REF(bin) is grounded.

K is the Koerwien theory, originating in [42] and defined in Section 5.6. Koerwien proved that K is complete, ω-stable, eni-NDOP, and of eni-depth 2. Koerwien proved K does not have Borel isomorphism relation, but left open whether or not K is Borel complete. We show that $\|\text{K}\| = \beth_2$. Thus K is short, and $\Phi_3 \not\leq_B$ K (in particular, K is not Borel complete). Whether K is grounded or not remains open.

TK is a 'tweaked version of K' and is also defined in Section 5.6. TK is also complete, ω-stable, eni-NDOP, of eni-depth 2, and so is very much like the theory K; however, the automorphism groups of models of TK induce a more complicated group of elementary permutations of $\mathrm{acl}(\emptyset)$ than do the automorphism groups of models of K. We show that TK is Borel complete, hence not short, hence its isomorphism relation is not Borel. On the other hand, TK is not grounded; in fact $\mathrm{CSS}(\mathrm{TK})_{\mathrm{ptl}}$ is a proper class, whereas $|\mathrm{CSS}(\mathrm{TK})_{\mathrm{sat}}| = \beth_2$.

The ideas for HC-forcing invariant formulas and potential canonical Scott sentences came from absolutely Δ^1_2-formulas in Chapter 9 of [22]. An alternative approach to canonical Scott sentences would be to use 'pinned names' as surveyed in e.g., [30], although for sentences of $\mathcal{L}_{\omega_1\omega}$, canonical Scott sentences are much more convenient. The whole of [89] was written independently of the development there.

5.2 A notion of cardinality for HC-forcing invariant sets

We develop a reducibility notion on well-behaved definable subsets of HC. Behind the scenes, we rely heavily on the fact that all sets A in $\mathbb{V}$ are 'potentially' elements of HC, the set of hereditarily countable sets. For example, if κ is the cardinality of the transitive closure of A and we take P be the Levy collapsing poset $\mathrm{Coll}(\kappa^+, \omega_1)$ that collapses κ^+ to ω_1, then for any choice G of a generic filter, $A \in \mathrm{HC}^{\mathbb{V}[G]}$.

5.2.1 HC-forcing invariant formulas

We begin with our principal definitions.

Definition 5.2.1. Suppose $\varphi(x)$ is any formula of set theory, possibly with a hidden parameter from HC.

- $\varphi(\mathrm{HC}) = \{a \in \mathrm{HC} : (\mathrm{HC}, \in) \models \varphi(a)\}$.

- If $\mathbb{V}[G]$ is a forcing extension of $\mathbb{V}$, then $\varphi(\mathrm{HC})^{\mathbb{V}[G]} = \{a \in \mathrm{HC}^{\mathbb{V}[G]} : \mathbb{V}[G] \models$

'$a \in \varphi(\mathrm{HC})$'}.

- $\varphi(x)$ is *HC-forcing invariant* if, for every twice-iterated forcing extension $\mathbb{V}[G][G']$,

$$\varphi(\mathrm{HC})^{\mathbb{V}[G][G']} \quad \cap \quad \mathrm{HC}^{\mathbb{V}[G]} \quad = \quad \varphi(\mathrm{HC})^{\mathbb{V}[G]}$$

The reader is cautioned that when computing $\varphi(\mathrm{HC})^{\mathbb{V}[G]}$, the quantifiers of φ range over $\mathrm{HC}^{\mathbb{V}[G]}$ as opposed to the whole of $\mathbb{V}[G]$. Visibly, the class of HC-forcing invariant formulas is closed under boolean combinations. Note that by Shoenfield's Absoluteness Theorem, e.g., Theorem 25.20 of [27], any Σ^1_2 subset of $\mathbb{R}$ is HC-forcing invariant. There is also the closely related Lévy Absoluteness Principle, which has various forms (e.g., Theorem 9.1 of [4] or Section 4 of [29]); we give a version more convenient to us. For a proof, see [89] (it is also standard).

Lemma 5.2.2. If $\mathbb{V}[G]$ is any forcing extension, and if $\varphi(x)$ is a Σ_1 formula of set theory, then for every $a \in \mathrm{HC}$, $\mathrm{HC} \models \varphi(a)$ if and only if $\mathrm{HC}^{\mathbb{V}[G]} \models \varphi(a)$. In particular, Σ_1-formulas are HC-forcing invariant.

For more complicated formulas, whether or not $\varphi(x)$ is HC-forcing invariant or not may well depend on the choice of set-theoretic universe. For example, consider the formula $\varphi(x) := (x = \emptyset) \vee (\mathbb{V} \neq \mathbb{L})$. Then $\varphi(\mathrm{HC})$ is equal to $\{\emptyset\}$ if $\mathrm{HC} \subseteq \mathbb{L}$, and $\varphi(\mathrm{HC}) = \mathrm{HC}$ otherwise. Because '$\mathrm{HC} \not\subseteq \mathbb{L}$' holding in $\mathbb{V}$ implies that it holds in any forcing extension $\mathbb{V}[H]$, it follows that the formula $\varphi(x)$ is HC-forcing invariant if and only if $\mathrm{HC} \not\subseteq \mathbb{L}$.

Before continuing, we state three set-theoretic lemmas that form the lynchpin of our development. Lemma 5.2.3 is a simple consequence of our definitions. Lemma 5.2.4 is well-known. It is mentioned in the proof of Theorem 9.4 in [22]; a full proof is given in the more recent [32]. Lemma 5.2.5 is just a rephrasing of Lemma 5.2.4. The key tool for all of these lemmas is the Product Forcing Lemma, see e.g., Lemma 15.9 of [27], which states that given any $P_1 \times P_2$-generic filter G, if G_ℓ is the projection of G onto P_ℓ, then $G = G_1 \times G_2$, each G_ℓ is P_ℓ-generic, and $\mathbb{V}[G] = \mathbb{V}[G_1][G_2] = \mathbb{V}[G_2][G_1]$ (i.e., G_ℓ meets every dense subset of P_ℓ in $\mathbb{V}[G_{3-\ell}]$).

Lemma 5.2.3. Suppose $P_1, P_2 \in \mathbb{V}$ are notions of forcing and $\varphi(x)$ is HC-forcing invariant (possibly with a hidden parameter from HC). If $A \in V$ and, for $\ell = 1, 2$, H_ℓ is P_ℓ-generic and $V[H_\ell] \models A \in \mathrm{HC}$, then $\mathbb{V}[H_1] \models A \in \varphi(\mathrm{HC})$ if and only if $\mathbb{V}[H_2] \models A \in \varphi(\mathrm{HC})$. (The filters H_1 and H_2 are not assumed to be mutually generic.)

Proof. Assume this were not the case. By symmetry, choose $p_1 \in H_1$ such that $p_1 \Vdash \check{A} \in \varphi(\mathrm{HC})$, and choose $p_2 \in H_2$ such that $p_2 \Vdash \check{A} \in \mathrm{HC} \wedge \check{A} \notin \varphi(\mathrm{HC})$. Let G be a $P_1 \times P_2$-generic filter with $(p_1, p_2) \in G$. Write $G = G_1 \times G_2$, hence $\mathbb{V}[G] = \mathbb{V}[G_1][G_2] = \mathbb{V}[G_2][G_1]$. As $p_1 \in G_1$ and $p_2 \in G_2$, we have $\mathbb{V}[G_1] \models A \in \varphi(\mathrm{HC})$ and $\mathbb{V}[G_2] \models A \in \mathrm{HC} \wedge A \notin \varphi(\mathrm{HC})$. But applying the HC-forcing invariance of φ twice, we get that $A \in \varphi(\mathrm{HC})^{\mathbb{V}[G_1]}$ if and only if $A \in \varphi(\mathrm{HC})^{\mathbb{V}[G_1][G_2]}$ if and only if $A \in \varphi(\mathrm{HC})^{\mathbb{V}[G_2]}$, a contradiction. $\square$

Lemma 5.2.4. Suppose P_1 and P_2 are both notions of forcing in $\mathbb{V}$. If G is a $P_1 \times P_2$-generic filter and $G = G_1 \times G_2$, then $\mathbb{V} = \mathbb{V}[G_1] \cap \mathbb{V}[G_2]$.

Proof. This is Corollary 2.3 of [32]. $\square$

Lemma 5.2.5. Let $\theta(x)$ be any formula of set theory, possibly with hidden parameters from $\mathbb{V}$, and let $\mathbb{V}[G]$ be any forcing extension of $\mathbb{V}$. Suppose that there is some $b \in \mathbb{V}[G]$ such that for every forcing extension $\mathbb{V}[G][H]$ of $\mathbb{V}[G]$,

$$\mathbb{V}[G][H] \models \theta(b) \wedge \exists^{=1} x \theta(x)$$

Then $b \in \mathbb{V}$.

Proof. Fix $\theta(x), \mathbb{V}[G]$ and b as above. Let $P \in \mathbb{V}$ be the forcing notion for which G is P-generic. Let τ be a P-name such that $b = \mathrm{val}(\tau, G)$. Choose $p \in G$ such that

$$p \Vdash \text{"for all forcing notions } \mathbb{Q}, \Vdash_{\mathbb{Q}} \theta(\check{\tau}) \wedge \exists^{=1} x\theta(x)\text{"}$$

Let H be $\mathbb{P}$-generic over $\mathbb{V}[G]$ with $p \in H$. So $G \times H$ is $\mathbb{P} \times \mathbb{P}$-generic over $\mathbb{V}$. Let $i_1, i_2 : \mathbb{P} \to \mathbb{P} \times \mathbb{P}$ be the canonical injections. Then, since $(p, p) \in G \times H$, we have that

$$\mathbb{V}[G][H] \models \theta(\mathrm{val}(i_1(\tau), G \times H)) \wedge \theta(\mathrm{val}(i_2(\tau)), G \times H) \wedge \exists^{=1} x\theta(x).$$

Hence $\mathbb{V}[G][H] \models \mathrm{val}(i_1(\tau), G \times H) = \mathrm{val}(i_2(\tau), G \times H)$ and so by Lemma 5.2.4, $\mathrm{val}(i_1(\tau), G \times H) \in \mathbb{V}$. But $\mathrm{val}(i_1(\tau), G \times H) = b$ so we are done. $\square$

Lemma 5.2.3 lends credence to the following definition.

Definition 5.2.6. Suppose that $\varphi(x)$ is HC-forcing invariant. Then φ_{ptl} is the class of all sets A such that $A \in \mathbb{V}$ and, for some (equivalently, for every) forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$ with $A \in \mathrm{HC}^{\mathbb{V}[G]}$, we have $A \in \varphi(\mathrm{HC})^{\mathbb{V}[G]}$.

As motivation for the notation used in the definition above, φ_{ptl} describes the class of all $A \in \mathbb{V}$ that are *potentially* in $\varphi(\mathrm{HC})$. We are specifically interested in those HC-forcing invariant φ for which φ_{ptl} is a **set** as opposed to a proper class.

Definition 5.2.7. An HC-forcing invariant formula $\varphi(x)$ is *short* if φ_{ptl} is a set.

We list some trivial observations; for fully fleshed out proofs, see [89]. As notation, if C is a subclass of $\mathbb{V}$, then define $\mathcal{P}(C)$ to be all sets A in $\mathbb{V}$ such that every element of A is in C. (This definition is only novel when C is a proper class.) Similarly, $P_{\aleph_1}(C)$ is the class of all sets $A \in \mathcal{P}(C)$ that are countable (in $\mathbb{V}$!). Let $\delta(x)$ be the formula

$$\delta(x) := \exists h[h : x \to \omega \text{ is 1-1}]$$

Given a formula $\varphi(x)$, let $\mathcal{P}(\varphi)(y)$ denote the formula $\forall x(x \in y \to \varphi(x))$ and let $\mathcal{P}_{\aleph_1}(\varphi)$ denote $\mathcal{P}(\varphi)(y) \wedge \delta(y)$.

Lemma 5.2.8. 1. The ptl-operator commutes with boolean combinations, i.e., if φ and ψ are both HC-forcing invariant, then $(\varphi \wedge \psi)_{\mathrm{ptl}} = \varphi_{\mathrm{ptl}} \cap \psi_{\mathrm{ptl}}$ and $(\neg\varphi)_{\mathrm{ptl}} = \mathbb{V} \setminus \varphi_{\mathrm{ptl}}$.

2. $\delta(\mathrm{HC}) = \mathrm{HC}$. In particular $\delta(x)$ is HC-forcing invariant and $\delta_{\mathrm{ptl}} = \mathbb{V}$.

3. If φ is HC-forcing invariant, then so are both $\mathcal{P}(\varphi)$ and $\mathcal{P}_{\aleph_1}(\varphi)$. Moreover, $\mathcal{P}(\varphi)(\mathrm{HC}) = \mathcal{P}_{\aleph_1}(\varphi)(\mathrm{HC})$ and $\mathcal{P}(\varphi_{\mathrm{ptl}}) = (\mathcal{P}(\varphi))_{\mathrm{ptl}} = (\mathcal{P}_{\aleph_1}(\varphi))_{\mathrm{ptl}}$.

4. Suppose $s : \omega \to \mathrm{HC}$ is any map such that for each n, $\varphi(x, s(n))$ is HC-forcing invariant. (Recall that HC-forcing invariant formulas are permitted to have a parameter from HC.) Then $\psi(x) := \exists n(n \in \omega \wedge \varphi(x, s(n)))$ is HC-forcing invariant and $\psi_{\mathrm{ptl}} = \bigcup_{n\in\omega} \varphi(x, s(n))_{\mathrm{ptl}}$.

5.2.2 HC-forcing invariant sets

We dislike dealing with formulas and prefer to deal with sets. In [89], a complicated notion of HC-forcing invariant families was developed to get around this. We prefer to just abuse notation and denote formulas $\varphi(x)$ as sets $X \subseteq \mathrm{HC}$, with the understanding that we have also specified the defining formula. Thus

Convention. When we say $X \subseteq \mathrm{HC}$ is an HC-forcing invariant set, we mean it is a definable subset of HC and in fact we have fixed a defining formula $\varphi(x)$ (with parameters from HC), such that $\varphi(x)$ is HC-forcing invariant. If $\mathbb{V}[G]$ is a forcing extension, then by $X^{\mathbb{V}[G]}$ we mean $(\varphi(\mathrm{HC}))^{\mathbb{V}[G]}$. Let $X_{\mathrm{ptl}} = \varphi(x)_{\mathrm{ptl}}$. Say that X is short if $\varphi(x)$ is.

If X is HC-forcing invariant via $\varphi(x)$, then by $\mathcal{P}_{\aleph_1}(X)$ be mean the subset of HC defined either by $\mathcal{P}(\varphi)(y)$ or $\mathcal{P}_{\aleph_1}(\varphi)(y)$ (the choice does not matter).

We enumerate several examples and easy observations that help establish our notation.

Example 5.2.9. 1. $\omega \subseteq \mathrm{HC}$ is HC-forcing invariant via any reasonable definition (e.g. use ω as a parameter and say "$x \in \omega$"), and $\omega_{\mathrm{ptl}} = \omega$ (a similar remark holds for any $a \in \mathrm{HC}$). In particular, ω is short.

2. ω_1 is HC-forcing invariant via any reasonable definition (e.g., "x is an ordinal"). Then $(\omega_1)_{\mathrm{ptl}} = ON$, the class of all ordinals. Thus, ω_1 is not short.

3. The set of reals, $\mathbb{R} = \mathcal{P}_{\aleph_1}(\omega)$, is HC-forcing invariant, and $\mathbb{R}_{\mathrm{ptl}} = \mathcal{P}(\omega) = \mathbb{R}$, hence $\mathbb{R}$ is short.

4. $\mathcal{P}_{\aleph_1}(\mathbb{R})$, the set of countable sets of reals, is HC-forcing invariant and $(\mathcal{P}_{\aleph_1}(\mathbb{R}))_{\mathrm{ptl}} = \mathcal{P}(\mathcal{P}(\omega))$, hence is short.

5. More generally, if X is short, then it follows from Lemma 5.2.8(3) that $\mathcal{P}_{\aleph_1}(X)_{\mathrm{ptl}} = \mathcal{P}(X_{\mathrm{ptl}})$ and so $\mathcal{P}_{\aleph_1}(X)$ is short.

6. For any $\alpha < \omega_1$, let HC_α denote the sets in HC whose foundation rank is less than α. Then each HC_α is HC-forcing invariant. Also, each $(\mathrm{HC}_\alpha)_{\mathrm{ptl}} = \mathbb{V}_\alpha$, so each HC_α is short.

7. HC is HC-forcing invariant, and $\mathrm{HC}_{\mathrm{ptl}} = \mathbb{V}$, in particular HC is not short.

The following deviates slightly from [89], in that we evaluate ψ in $\mathrm{HC}^{\mathbb{V}[G]}$ rather than $\mathbb{V}[G]$.

Notation 5.2.10. Suppose that $X_1, \ldots, X_n$ are HC-forcing invariant subsets of HC. Suppose $\psi(U_1, \ldots, U_n)$ is a formula of set theory with n extra unary relations $U_1, \ldots, U_n$. We say $\psi(X_1, \ldots, X_n)$ *holds persistently* if, for every forcing extension $\mathbb{V}[G]$, we have

$$\mathrm{HC}^{\mathbb{V}[G]} \models \psi({X_1}^{\mathbb{V}[G]}, \ldots, {X_n}^{\mathbb{V}[G]}).$$

We list three examples of this usage in the definition below.

Definition 5.2.11. Suppose that f, X and Y are HC-forcing invariant subsets of HC.

- The notation $f : X \to Y$ *persistently* means that $f^{\mathbb{V}[G]} : X^{\mathbb{V}[G]} \to Y^{\mathbb{V}[G]}$ for all forcing extensions $\mathbb{V}[G]$ of $\mathbb{V}$.
- The notation $f : X \to Y$ *is persistently injective* means that $f : X \to Y$ persistently and additionally, for all forcing extensions $\mathbb{V}[G]$ of $\mathbb{V}$, $f^{\mathbb{V}[G]} : X^{\mathbb{V}[G]} \to Y^{\mathbb{V}[G]}$ is 1-1.
- The notion $f : X \to Y$ *is persistently bijective* means $f^{-1} : X \to Y$ is HC-forcing invariant and both $f : X \to Y$ and $f^{-1} : Y \to X$ are persistently injective.

The reader is cautioned that when $f : X \to Y$ persistently (or is persistently injective), the image of f need not be HC-forcing invariant. Indeed the "image" of an HC-forcing invariant function is not well-behaved in many respects, including the lack of a surjectivity statement in the following proposition.

Proposition 5.2.12. Suppose that f, X, and Y are HC-forcing invariant.

1. Suppose $f : X \to Y$ persistently. Then $f_{\mathrm{ptl}} : X_{\mathrm{ptl}} \to Y_{\mathrm{ptl}}$, i.e., f_{ptl} is a class function with domain X_{ptl} and image contained in Y_{ptl}.
2. If $f : X \to Y$ is persistently injective, then $f_{\mathrm{ptl}} : X_{\mathrm{ptl}} \to Y_{\mathrm{ptl}}$ is injective as well.
3. If $f : X \to Y$ is persistently bijective, then $f_{\mathrm{ptl}} : X_{\mathrm{ptl}} \to Y_{\mathrm{ptl}}$ is bijective.

We close this subsection with a characterization of surjectivity. Its proof is simply an unpacking of the definitions. However, $f : X \to Y$ being persistently surjective need not imply that the induced map $f_{\mathrm{ptl}} : X_{\mathrm{ptl}} \to Y_{\mathrm{ptl}}$ is surjective.

Lemma 5.2.13. Suppose that f, X, and Y are each HC-forcing invariant, via the HC-forcing invariant formulas $\theta(x, y)$, $\varphi(x)$, and $\gamma(y)$, respectively and that $f : X \to Y$ persistently. Then $f : X \to Y$ is persistently surjective if and only if the formula $\rho(y) := \exists x\theta(x, y)$ is HC-forcing invariant and persistently equivalent to $\gamma(y)$.

5.2.3 Potential Cardinality

Definition 5.2.14. Suppose X and Y are HC-forcing invariant. We say that X *is HC-reducible to* Y, written $X \leq_{\mathrm{HC}} Y$, if there is an HC-forcing invariant f such that $f : X \to Y$ is persistently injective. As notation, we write $X <_{\mathrm{HC}} Y$ if $X \leq_{\mathrm{HC}} Y$ but $Y \not\leq_{\mathrm{HC}} X$. We also write $X \sim_{\mathrm{HC}} Y$ if $X \leq_{\mathrm{HC}} Y$ and $Y \leq_{\mathrm{HC}} X$; this is weaker than X and Y being in persistent bijection.

The following notion will be very useful for our applications, as it can often be computed directly. With this and Proposition 5.2.16 we can prove otherwise difficult non-embeddability results for $\leq_{HC}$.

Definition 5.2.15. Suppose X is HC-forcing invariant. The *potential cardinality* of X, denoted $\|X\|$, refers to $|X_{\mathrm{ptl}}|$ if X is short, or ∞ otherwise. By convention we say $\kappa < \infty$ for any cardinal κ.

Proposition 5.2.16. Suppose X and Y are both HC-forcing invariant.

1. If Y is short and $X \leq_{\mathrm{HC}} Y$, then X is short.

2. If $X \leq_{\mathrm{HC}} Y$, then $\|X\| \leq \|Y\|$.

3. If X is short, then $X <_{\mathrm{HC}} \mathcal{P}_{\aleph_1}(X)$.

Proof. (1) follows immediately from (2).

(2) Choose an HC-forcing invariant f such that $f : X \to Y$ is persistently injective. Then by Proposition 5.2.12(2), $f_{\mathrm{ptl}} : X_{\mathrm{ptl}} \to Y_{\mathrm{ptl}}$ is an injective class function. Thus, $|X_{\mathrm{ptl}}| \leq |Y_{\mathrm{ptl}}|$.

(3) Note that for any HC-forcing invariant X, $X \leq_{\mathrm{HC}} \mathcal{P}_{\aleph_1}(X)$ is witnessed by the HC-forcing invariant map $x \mapsto \{x\}$. For the other direction, suppose by way of contradiction that X is short, but $\mathcal{P}_{\aleph_1}(X) \leq_{\mathrm{HC}} X$. Note $(\mathcal{P}_{\aleph_1}(X))_{\mathrm{ptl}} = \mathcal{P}(X_{\mathrm{ptl}})$ by Lemma 5.2.8(3). Thus, by (2), we would have that $|\mathcal{P}(X_{\mathrm{ptl}})| \leq |X_{\mathrm{ptl}}|$, which contradicts Cantor's theorem since X_{ptl} is a set. □

Using the fact that $(\mathrm{HC}_\beta)_{\mathrm{ptl}} = \mathbb{V}_\beta$, the following Corollary is immediate.

Corollary 5.2.17. If X is HC-forcing invariant and $\|X\| \leq \beth_\alpha$ for some $\alpha < \omega_1$, then $\mathrm{HC}_{\omega+\alpha+1} \not\leq_{\mathrm{HC}} X$.

5.2.4 Quotients

We begin with the obvious definition.

Definition 5.2.18. A pair (X, E) is an *HC-forcing invariant quotient* if both X and E are HC-forcing invariant and persistently, E is an equivalence relation on X.

There is an immediate way to define a reduction of two quotients:

Definition 5.2.19. Let (X, E) and (Y, F) be HC-forcing invariant quotients. Say $(X, E) \leq_{\mathrm{HC}} (Y, F)$ if there is an HC-forcing invariant f such that all of the following hold persistently:

- f is a subclass of $X \times Y$.
- The E-saturation of $\mathrm{dom}(f)$ is X. That is, for every $x \in X$, there is an $x' \in X$ and $y' \in Y$ where xEx' holds and $(x', y') \in f$.

- f induces a well-defined injection on equivalence classes. That is, if (x, y) and (x', y') are in f, then xEx' holds if and only if yFy' does.

Define $(X, E) <_{\mathrm{HC}} (Y, F)$ and $(X, E) \sim_{\mathrm{HC}} (Y, F)$ in the natural way.

It is more common in reducibility theory to require that f be an actual map from X to Y, but we will find this more convenient.

We wish to define the potential cardinality $\|(X, E)\|$. It turns out that $|X_{\mathrm{ptl}}/E_{\mathrm{ptl}}|$ is oftentimes too small. For our purposes, we can restrict to more well-behaved quotients.

Definition 5.2.20. A *representation* of an HC-forcing invariant quotient (X, E) is a pair f, Z of HC-forcing invariant sets such that $f : X \to Z$ is persistently surjective and persistently,

$$\forall a, b \in X[E(a, b) \Leftrightarrow f(a) = f(b)]$$

We say that (X, E) is *representable* if it has a representation.

In the case that (X, E) is representable, the set of E-classes is HC-forcing invariant in a sense – we equate it with the representation. For this reason we will also say Z is a representation of (X, E). Note that if $f_1 : X \to Z_1$ and $f_2 : X \to Z_2$ are two representations of (X, E), then there is a persistently bijective, HC-forcing invariant $h : Z_1 \to Z_2$. This observation implies the following definition is well-defined.

Definition 5.2.21. If (X, E) is a representable HC-forcing invariant quotient, then define $\|(X, E)\| = \|Z\|$ for some (equivalently, for all) Z such that there is a representation $f : (X, E) \to Z$.

The following lemma can be proved by a routine composition of maps.

Lemma 5.2.22. Suppose (X, E) and (Y, E') are HC-forcing invariant quotients, with representations $f : (X, E) \to Z$ and $g : (Y, E') \to Z'$. Suppose h is a witness to $(X, E) \leq_{\mathrm{HC}}$

(Y, E'). Then the induced function $h^* : Z \to Z'$ is HC-forcing invariant, persistently injective, and witnesses $Z \leq_{\mathrm{HC}} Z'$.

We close this section with an observation about restrictions of representations.

Lemma 5.2.23. Suppose $f : (X, E) \to Z$ is a representation and $Y \subseteq X$ is HC-forcing invariant and persistently E-saturated. Let E' and $g = f\restriction_Y$ be the restrictions of E and f, respectively, to Y. Then the image $g(Y)$ is HC-forcing invariant, and so $g : (Y, E') \to g(Y)$ is a representation.

All of the examples we work with will be representable, where the representations are Scott sentences. Therefore this simple definition of $\|(X, E)\|$ will suffice completely for our purposes. In the absence of a representation, one can still define $\|(X, E)\|$ using the notion of *pins*; see for instance [30] for a thorough discussion.

5.3 Connecting Potential Cardinality with Borel Reducibility

The standard framework for Borel reducibility of invariant classes is the following. Let $\mathcal{L}$ be a countable langauge and let $X_{\mathcal{L}}$ be the set of $\mathcal{L}$-structures with universe ω. Endow $X_{\mathcal{L}}$ with the usual logic topology (with clopen sets being solution sets of formulas); then $X_{\mathcal{L}}$ becomes a Polish space. Moreover, if Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ then $\mathrm{Mod}(\Phi)$ is a Borel subset of $X_{\mathcal{L}}$; hence $\mathrm{Mod}(\Phi)$ is a standard Borel space. The relation $\cong_\Phi$ is the restriction of the isomorphism relation to $\mathrm{Mod}(\Phi) \times \mathrm{Mod}(\Phi)$. When no ambiguity arises we omit the Φ. If $\mathcal{L}'$ is another countable language and Φ' is a a sentence of $\mathcal{L}'_{\omega_1\omega}$, then a Borel reduction from $(\mathrm{Mod}(\Phi), \cong) \to (\mathrm{Mod}(\Phi'), \cong)$ is a Borel map $f : \mathrm{Mod}(\Phi) \to \mathrm{Mod}(\Phi')$ such that, for all $M, N \in \mathrm{Mod}(\Phi)$, $M \cong N$ if and only if $f(M) \cong f(N)$.

We want to apply the machinery of the previous section to this setup. First, recall that we are working entirely in ZFC; thus a language $\mathcal{L}$ is just a set with an arity function, and an $\mathcal{L}$-structure with universe ω is just a function $f : \mathcal{L} \to \bigcup_n \mathcal{P}(\omega^n)$ respecting the arities. Since our languages are countable we can suppose that they are elements of HC. We will presently show that for any sentence Φ of $\mathcal{L}_{\omega_1\omega}$, $(\mathrm{Mod}(\Phi), \cong)$ is an HC-forcing invariant quotient. We will also show that $(\mathrm{Mod}(\Phi), \cong)$ is representable, and that Borel reductions are in particular HC-reductions.

5.3.1 Canonical Scott sentences

For what follows, we need the notion of a canonical Scott sentence of any infinite $\mathcal{L}$-structure, regardless of cardinality. The definition below is in both Barwise [4] and Marker [64].

Definition 5.3.1. Suppose $\mathcal{L}$ is countable and M is any infinite $\mathcal{L}$-structure, say of power κ. For each $\alpha < \kappa^+$, define an $\mathcal{L}_{\kappa^+,\omega}$ formula $\varphi_\alpha^{\overline{a}}(\overline{x})$ for each finite $\overline{a} \in M^{<\omega}$ as follows:

- $\varphi_0^{\overline{a}}(\overline{x}) := \bigwedge\{\theta(\overline{x}) : \theta$ atomic or negated atomic and $M \models \theta(\overline{a})\}$;
- $\varphi_{\alpha+1}^{\overline{a}}(\overline{x}) := \varphi_\alpha^{\overline{a}}(\overline{x}) \;\wedge\; \bigwedge\left\{\exists y\, \varphi_\alpha^{\overline{a},b}(\overline{x}, y) : b \in M\right\} \;\wedge \forall y \bigvee\left\{\varphi_\alpha^{\overline{a},b}(\overline{x}, y) : b \in M\right\}$;
- For α a non-zero limit, $\varphi_\alpha^{\overline{a}}(\overline{x}) := \bigwedge\left\{\varphi_\beta^{\overline{a}}(\overline{x}) : \beta < \alpha\right\}$.

Next, let $\alpha^*(M) < \kappa^+$ be least ordinal α such that for all finite $\overline{a}$ from M,

$$\forall \overline{x}[\varphi_\alpha^{\overline{a}}(\overline{x}) \to \varphi_{\alpha+1}^{\overline{a}}(\overline{x})].$$

Finally, put $\mathrm{css}(M) := \varphi_{\alpha^*(M)}^{\emptyset} \wedge \bigwedge\left\{\forall\, \overline{x}[\varphi_{\alpha^*(M)}^{\overline{a}}(\overline{x}) \to \varphi_{\alpha^*(M)+1}^{\overline{a}}(\overline{x})] : \overline{a} \in M^{<\omega}\right\}$.

For what follows, it is crucial that the choice of $\mathrm{css}(M)$ really is canonical. In particular, in the infinitary clauses forming the definition of $\mathrm{tp}_{\alpha+1}^{\overline{a}}(\overline{x})$, we consider the conjunctions and disjunctions be taken over *sets* of formulas, as opposed to sequences. By our conventions about working wholly in ZFC, countable languages and sentences of $\mathcal{L}_{\infty,\omega}$ are sets, and in particular $\operatorname{css}(M)$ is a set.

We summarize the well-known, classical facts about canonical Scott sentences with the following:

Fact 5.3.2. Fix a countable language $\mathcal{L}$.

1. For every $\mathcal{L}$-structure M, $M \models \operatorname{css}(M)$; and for all $\mathcal{L}$-structures N, $M \equiv_{\infty,\omega} N$ if and only if $\operatorname{css}(M) = \operatorname{css}(N)$ if and only if $N \models \operatorname{css}(M)$.

2. If M is countable, then $\operatorname{css}(M) \in \mathrm{HC}$.

3. css is absolute between transitive models of ZFC^-. (Recall that $\mathrm{HC} \models ZFC^-$.)

4. If M and N are both countable, then $M \cong N$ if and only if $\operatorname{css}(M) = \operatorname{css}(N)$ if and only if $N \models \operatorname{css}(M)$.

Our primary interest in canonical Scott sentences is that they give rise to representations of classes of $\mathcal{L}$-structures. The key to the representability is Karp's Completeness Theorem for sentences of $\mathcal{L}_{\omega_1,\omega}$, see e.g., Theorem 3 of Keisler [35], which says that if a sentence σ of $\mathcal{L}_{\omega_1\omega}$ is consistent, then it has a countable model. It quickly follows that if σ is a sentence of $\mathcal{L}_{\omega_1\omega}$, and σ has a model in a forcing extension, then σ already has a countable model in $\mathbb{V}$.

We begin by considering $\mathrm{CSS}(\mathcal{L})$, the set of all canonical Scott sentences of structures in $X_{\mathcal{L}}$, the set of $\mathcal{L}$-structures with universe ω.

Lemma 5.3.3. Fix a countable language $\mathcal{L}$. Then:

1. $\mathrm{CSS}(\mathcal{L})$ is HC-forcing invariant via the formula $\varphi(y) := \exists M(M \in X_{\mathcal{L}} \wedge \operatorname{css}(M) = y)$;

2. The HC-forcing invariant function $\operatorname{css} : X_{\mathcal{L}} \to \mathrm{CSS}(\mathcal{L})$ is persistently surjective;

3. $\mathrm{css} : X_{\mathcal{L}} \to \mathrm{CSS}(\mathcal{L})$ is a representation of the HC-forcing invariant quotient $(X_{\mathcal{L}}, \cong)$.

Proof. Note that $\mathcal{L}_{\omega_1\omega}$ is HC-forcing invariant, clearly.

(1) We need to verify that $\varphi(y)$ is HC-forcing invariant. Suppose $\sigma \in \mathrm{HC}$ and $\mathbb{V}[G]$ is a forcing extension of $\mathbb{V}$. If $\sigma \in \varphi(\mathrm{HC})^{\mathbb{V}[G]}$, then there is some $M \in X_{\mathcal{L}}^{\mathbb{V}[G]}$ such that $\mathrm{css}(M) = \sigma$. Hence by the preceding discussion, σ has a countable model $N \in (X_{\mathcal{L}})^{\mathbb{V}}$. In $\mathbb{V}[G]$, $N \models \sigma$; but $\sigma = \mathrm{css}(M)$. So $\mathrm{css}(N) = \sigma$. As this argument readily relativizes to any forcing extension, φ is HC-forcing invariant.

(2) follows from (1) and Lemma 5.2.13.

(3): $(X_{\mathcal{L}}, \cong)$ is an HC-forcing invariant quotient since css is HC-forcing invariant. Thus we conclude by (2) and Fact 5.3.2(4). □

In most of our applications, we are interested in HC-forcing invariant subclasses of $X_{\mathcal{L}}$ that are closed under isomorphism. For any sentence Φ of $\mathcal{L}_{\omega_1,\omega}$, because $\mathrm{Mod}(\Phi)$ is a Borel subset of $X_{\mathcal{L}}$, it follows from Shoenfield's Absoluteness Theorem that both $\mathrm{Mod}(\Phi)$ and the restriction of css to $\mathrm{Mod}(\Phi)$ (also denoted by css) $\mathrm{css} : \mathrm{Mod}(\Phi) \to \mathrm{HC}$ are HC-forcing invariant.

Definition 5.3.4. For Φ any sentence of $\mathcal{L}_{\omega_1,\omega}$, $\mathrm{CSS}(\Phi) = \{\mathrm{css}(M) : M \in \mathrm{Mod}(\Phi)\} \subseteq \mathrm{HC}$.

Proposition 5.3.5. Fix any sentence $\Phi \in \mathcal{L}_{\omega_1,\omega}$ in a countable vocabulary. Then $\mathrm{css} : \mathrm{Mod}(\Phi) \to \mathrm{CSS}(\Phi)$ is a representation of the quotient $(\mathrm{Mod}(\Phi), \cong)$. In particular the latter is HC-forcing invariant.

Proof. As $\mathrm{Mod}(\Phi)$ is HC-forcing invariant, this follows immediately from Lemmas 5.3.3 and 5.2.23. □

Thus the following definition makes sense.

Definition 5.3.6. Suppose Φ, Ψ are sentences of $\mathcal{L}_{\omega_1\omega}$. Then say that $\Phi \leq_{\mathrm{HC}} \Psi$, or Φ is HC-reducible to Ψ, if $\mathrm{CSS}(\Phi) \leq_{\mathrm{HC}} \mathrm{CSS}(\Psi)$, or equivalently if $(\mathrm{Mod}(\Phi), \cong) \leq_{\mathrm{HC}} (\mathrm{Mod}(\Psi), \cong)$. Similarly, if X is an HC-forcing invariant set and $\Phi \in \mathcal{L}_{\omega_1\omega}$, say that $X \leq_{\mathrm{HC}} \Phi$ if $X \leq_{\mathrm{HC}} \mathrm{CSS}(\Phi)$, and say that $\Phi \leq_{\mathrm{HC}} X$ if $\mathrm{CSS}(\Phi) \leq_{\mathrm{HC}} X$.

Definition 5.3.7. Let Φ be any sentence of $\mathcal{L}_{\omega_1,\omega}$ in a countable vocabulary. We say that Φ is *short* if $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ is a set (as opposed to a proper class). If Φ is short, let the *potential cardinality* of Φ, denoted $\|\Phi\|$, be the (usual) cardinality of $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$; otherwise let it be ∞.

It follows from Proposition 5.3.5 and Definitions 5.2.14 and 5.2.21 that

$$\|\Phi\| = \|(\mathrm{Mod}(\Phi), \cong)\| = \|\mathrm{CSS}(\Phi)\| = |\mathrm{CSS}(\Phi)_{\mathrm{ptl}}|.$$

In order to understand the class $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$, note that if $\varphi \in \mathrm{CSS}(\Phi)_{\mathrm{ptl}}$, then $\varphi \in \mathbb{V}$ and is a sentence of $\mathcal{L}_{\infty,\omega}$. If we choose any forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$ for which $\varphi \in \mathrm{HC}^{\mathbb{V}[G]}$, then $\mathbb{V}[G] \models$ '$\varphi \in \mathcal{L}_{\omega_1,\omega}$' and there is some $M \in \mathrm{HC}^{\mathbb{V}[G]}$ such that $V[G] \models$ '$M \in \mathrm{Mod}(\Phi)$ and $\mathrm{css}(M) = \varphi$'. Thus, we refer to elements of $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ as being *potential canonical Scott sentences* of a model of Φ. In particular, every element of $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ is *potentially satisfiable* in the sense that it is satisfiable in some forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$. There is a proof system for sentences of $\mathcal{L}_{\infty,\omega}$ for which a sentence is consistent if and only if it is potentially satisfiable as defined above.[1] When we say 'φ implies ψ', we mean with respect to this proof system; equivalently, in any forcing extension $\mathbb{V}[G]$, every model of φ is a model of ψ.

[1]In Chapter 4 of [35], Keisler gives a proof system for $\mathcal{L}_{\omega_1\omega}$, and shows in Theorem 3 that it is complete, i.e. if φ is unprovable then $\neg\varphi$ has a model. The natural generalization of this proof system to $\mathcal{L}_{\infty\omega}$ works: the proof of Theorem 3 shows that whenever φ is unprovable, then $\neg\varphi$ lies in a consistency property. Forcing on the consistency property gives a model of $\neg\varphi$.

We can also ask what is the image of the class function css when restricted to the class of models of Φ. As notation, let $\mathrm{CSS}(\Phi)_{\mathrm{sat}}$ denote the class of satisfiable canonical Scott sentences $\{\mathrm{css}(M) : M \in \mathbb{V} \text{ and } M \models \Phi\}$. This choice of notation is clarified by the following easy lemma.

Lemma 5.3.8. $\mathrm{CSS}(\Phi)_{\mathrm{sat}} \subseteq \mathrm{CSS}(\Phi)_{\mathrm{ptl}}$.

Proof. Choose any $\varphi \in \mathrm{CSS}(\Phi)_{\mathrm{sat}}$ and choose any $M \in \mathbb{V}$ such that $M \models \Phi$ and $\mathrm{css}(M) = \varphi$. Choose a forcing extension $\mathbb{V}[G]$ in which M is countable. Then, in $\mathbb{V}[G]$, there is some $M' \in \mathrm{Mod}(\Phi)$ (i.e., where the universe of M' is ω) such that $M' \cong M$. Then $(\mathrm{css}(M'))^{\mathbb{V}[G]} = \varphi$ and so $\varphi \in \mathrm{CSS}(\Phi)_{\mathrm{ptl}}$. $\square$

This suggests the following property of the sentence Φ.

Definition 5.3.9. A sentence $\Phi \in \mathcal{L}_{\omega_1,\omega}$ (or a complete first-order theory T) is *grounded* if $\mathrm{CSS}(\Phi)_{\mathrm{sat}} = \mathrm{CSS}(\Phi)_{\mathrm{ptl}}$, i.e., if every potential canonical Scott sentence is satisfiable.

As a trivial example, if T is $\aleph_0$-categorical, then as all models of T are back-and-forth equivalent, $\mathrm{CSS}(T)_{\mathrm{ptl}}$ is a singleton, hence T is grounded. In Section 5.5 we show that both of the theories REF(bin) and REF(inf) are grounded, but in Section 5.6 we prove that the theory TK is not grounded.

This concept was previously and independetly investigated in the language of pins by Kaplan and Shelah [32], where they prove that not every sentence Φ is grounded. Kaplan and Shelah ask in [32] if linear orders are grounded; this remains open.

Next, we show that a Borel reduction between invariant classes yields an HC-forcing invariant map between the associated canonical Scott sentences.

Fact 5.3.10. Suppose Φ and Ψ are sentences of $\mathcal{L}_{\omega_1,\omega}$. If $\Phi \leq_B \Psi$ then $\Phi \leq_{\mathrm{HC}} \Psi$.

Proof. It is a standard theorem, see e.g., [33] Proposition 12.4, that the graph of f is Borel. So f is HC-forcing invariant. By Lemma 5.2.22, it suffices to show that $f : (\mathrm{Mod}(\Phi), \cong) \leq_{\mathrm{HC}} (\mathrm{Mod}(\Phi'), \cong)$, which amounts to showing that f remains well-defined and injective on isomorphism classes in every forcing extension. But this is a $\mathbf{\Pi^1_2}$ statement in codes for f, Φ, Φ', and thus is absolute to forcing extensions by Shoenfield's Absoluteness Theorem. $\square$

The following Theorem is simply a distillation of our previous results.

Theorem 5.3.11. Let Φ and Ψ be sentences of $\mathcal{L}_{\omega_1,\omega}$. If $\|\Psi\| < \|\Phi\|$, then $\Phi \not\leq_{\mathrm{HC}} \Psi$, hence $\Phi \not\leq_B \Psi$. In particular, if Ψ is short and Φ is not, then this holds.

5.3.2 Consequences of $\cong_\Phi$ being Borel

Although our primary interest is classes $\mathrm{Mod}(\Phi)$ where $\cong_\Phi$ is not Borel, in this brief subsection we see the consequences of $\cong_\Phi$ being Borel.

Theorem 5.3.12. The following are equivalent for a sentence $\Phi \in \mathcal{L}_{\omega_1,\omega}$ in countable vocabulary.

1. The relation of $\cong$ on $\mathrm{Mod}(\Phi)$ is a Borel subset of $\mathrm{Mod}(\Phi) \times \mathrm{Mod}(\Phi)$;
2. For some $\alpha < \omega_1$, $\mathrm{CSS}(\Phi) \subseteq \mathrm{HC}_\alpha$;
3. For some $\alpha < \omega_1$, $\mathrm{CSS}(\Phi)$ is persistently contained in HC_α;
4. $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ is contained in $\mathbb{V}_\alpha$ for some $\alpha < \omega_1$.

Proof. To see the equivalence of (1) and (2), first note that in both conditions we are only considering models of Φ with universe ω and the canonical Scott sentence of such objects. In particular, neither condition involves passing to a forcing extension. However, it is a classical result (see for instance [14], Theorem 12.2.4) that $\cong$ is Borel if and only if the Scott ranks of countable models are bounded below ω_1, which is equivalent to stating that there is a bound on the canonical Scott sentences in the HC_α hierarchy.

For (2) implies (3), note that the formula $\exists M : M \models \Phi \wedge \mathrm{css}(M) \notin \mathrm{HC}_\alpha$ is a Σ_1 formula in the parameters $\Phi, \alpha \in HC$ and so is absolute to forcing extensions by Lemma 5.2.2.

That (3) implies (4) and (4) implies (2) follow directly from Example 5.2.9(6). $\square$

We obtain an immediate corollary to this. Let $I_{\infty,\omega}(\Phi)$ denote the cardinality of a maximal set of pairwise $\equiv_{\infty,\omega}$-inequivalent models $M \in \mathbb{V}$ (of any cardinality) with $M \models \Phi$. If no maximal set exists, we write $I_{\infty,\omega}(\Phi) = \infty$. By Fact 5.3.2 and Lemma 5.3.8, $I_{\infty,\omega}(\Phi) = |\mathrm{CSS}(\Phi)_{\mathrm{sat}}| \leq \|\Phi\|$.

Corollary 5.3.13. Let Φ be any sentence in $\mathcal{L}_{\omega_1,\omega}$ in a countable vocabulary such that $\cong$ is a Borel subset of $\mathrm{Mod}(\Phi) \times \mathrm{Mod}(\Phi)$. Then

1. Φ is short; and

2. $I_{\infty,\omega}(\Phi) < \beth_{\omega_1}$. (In fact $\|\Phi\| < \beth_{\omega_1}$.)

Proof. Assume that $\cong$ is a Borel subset of $\mathrm{Mod}(\Phi) \times \mathrm{Mod}(\Phi)$. By Theorem 5.3.12(3), $\mathrm{CSS}(\Phi)_{\mathrm{ptl}} \subseteq \mathbb{V}_\alpha$ for some $\alpha < \omega_1$ and hence is a set. Thus, Φ is short, and $I_{\infty,\omega}(\Phi) = |\mathrm{CSS}(\Phi)_{\mathrm{sat}}| \leq |\mathrm{CSS}(\Phi)_{\mathrm{ptl}}| \leq |\mathbb{V}_\alpha| < \beth_{\omega_1}$. $\square$

We remark that the implication in Corollary 5.3.13 does not reverse. In Sections 5.5 and 5.6 we show that both of the complete theories REF(bin) and K have potential cardinality $\beth_2$, but on countable models of either theory, $\cong$ is not Borel.

5.3.3 Maximal Complexity

In this subsection, we recall two definitions of maximality. The first, Borel completeness, is from Friedman-Stanley [12].

Definition 5.3.14. Let $\Phi \in \mathcal{L}_{\omega_1,\omega}$. Then Φ is *Borel complete* if for every $\Psi \in \mathcal{L}_{\omega_1\omega}$, we have $\Psi \leq_B \Phi$. Also, Φ is HC-*complete* if $\Psi \leq_{\mathrm{HC}} \Phi$ for all $\Psi \in \mathcal{L}_{\omega_1\omega}$.

So, if Φ is Borel complete then Φ is $\leq_{\mathrm{HC}}$-complete; there is no known example of the reverse implication failing.

Proposition 5.3.15. Φ is $\leq_{\mathrm{HC}}$-complete if and only if $\mathrm{HC} \leq_{\mathrm{HC}} \Phi$.

Proof. If $\mathrm{HC} \leq_{\mathrm{HC}} \Phi$ then $\mathrm{CSS}(\Psi) \leq_{\mathrm{HC}} \Phi$ for all Ψ. Conversely, it suffices to show that $\mathrm{HC} \leq_{\mathrm{HC}} \mathrm{CSS}(\mathcal{L})$, where $\mathcal{L}$ is the language $\{\in\}$. To see this, given $a \in \mathrm{HC}$, let $\varphi = \mathrm{css}(M)$, where M is the structure with universe $\mathrm{tcl}(a) \cup \{a\}$, and where $\in$ is interpreted naturally. $\square$

Corollary 5.3.16. If Φ is $\leq_{\mathrm{HC}}$-complete, then Φ is not short. In particular, if Φ is Borel complete, then Φ is not short.

If one is only interested in classes of countable models, then the Borel complete classes are clearly maximal with respect to Borel reducibility. As any invariant class of countable structures has a natural extension to a class of uncountable structures, one can ask for more. The following definitions from [47] generalize Borel completeness to larger cardinals λ. To see that it is a generalization, recall that among countable structures, isomorphism is equivalent to back-and-forth equivalence, and that for structures of size λ, $\equiv_{\lambda^+,\omega}$-equivalence is also equivalent to back-and-forth equivalence. Consequently, 'Borel complete' in the sense of Definition 5.3.14 is equivalent to '$\aleph_0$-Borel complete' in

Definition 5.3.17. So, 'Φ is λ-Borel complete for all infinite λ' implies Φ Borel complete. However, in Section 5.6 we will see that the theory TK is Borel complete, but is not λ-Borel complete for large λ.

Definition 5.3.17. Let Φ be a sentence of $\mathcal{L}_{\omega_1,\omega}$.

- For $\lambda \geq \aleph_0$, let $\mathrm{Mod}_\lambda(\Phi)$ denote the class of models of Φ with universe λ.
- Toplogize $\mathrm{Mod}_\lambda(\Phi)$ by declaring that $\mathcal{B} := \{U_{\theta(\overline{\alpha})} : \theta(\overline{x})$ is quantifier free and $\overline{\alpha} \in \lambda^{<\omega}\}$ is a sub-basis, where $U_{\theta(\overline{\alpha})} = \{M \in \mathrm{Mod}_\lambda(\Phi) : M \models \theta(\overline{\alpha})\}$.
- A set is λ-*Borel* if it is in the λ^+-algebra generated by the sub-basis $\mathcal{B}$.
- A function $f : \mathrm{Mod}_\lambda(\Phi) \to \mathrm{Mod}_\lambda(\Psi)$ is a λ-*Borel embedding* if
  - the inverse image of every (sub)-basic open set is λ-Borel; and
  - For $M, N \in \mathrm{Mod}_\lambda(\Phi)$, $M \equiv_{\infty,\omega} N$ if and only if $f(M) \equiv_{\infty,\omega} f(N)$.
- $(\mathrm{Mod}_\lambda(\Phi), \equiv_{\infty,\omega})$ is λ-*Borel reducible* to $(\mathrm{Mod}_\lambda(\Psi), \equiv_{\infty,\omega})$ if there exists a λ-Borel embedding $f : \mathrm{Mod}_\lambda(\Phi) \to \mathrm{Mod}_\lambda(\Psi)$.
- Φ is λ-*Borel complete* if for every sentence Ψ of $\mathcal{L}_{\omega_1\omega}$, $(\mathrm{Mod}_\lambda(\Psi), \equiv_{\infty,\omega})$ is λ-Borel reducible to $(\mathrm{Mod}_\lambda(\Phi), \equiv_{\infty,\omega})$.

For example, the class of graphs (directed or undirected) is λ-Borel complete for all infinite λ. This is a standard coding argument. Although we are not aware of any direct reference, Theorem 5.5.1 of [21] states that graphs can interpret any theory. It is easily checked that the map constructed in the proof of Theorem 5.5.1 is in fact a λ-Borel reduction for every λ.

Also, in [47] it is proved that the class of subtrees of $\lambda^{<\omega}$ is λ-Borel complete, and more recently the second author has proved that the class of linear orders is λ-Borel complete for all λ.

5.3.4 The Friedman-Stanley Tower

In this subsection we define the Friedman Stanley tower. There are many versions of these in circulation; for instance the $\mathcal{I}_\alpha$ in [12], the $\cong_\alpha$ in [25], the $=^\alpha$ in [14], and the T_α in [41]. In [89] we used the tower $(T_\alpha : \alpha < \omega_1)$ from [41]. The advantage of this is that it is a tower of complete first order theories. For this thesis we prefer to use a tower $(\Phi_\alpha : \alpha < \omega_1)$ of sentences of $\mathcal{L}_{\omega_1\omega}$. We will show that $T_n \sim_B \Phi_n$ for each $n < \omega$, and $T_\alpha \sim_B \Phi_{\alpha+1}$ for all $\alpha \geq \omega$.

The following is as defined by [12].

Definition 5.3.18. Suppose $\mathcal{L}$ is a countable relational language and $\Phi \in \mathcal{L}_{\omega_1,\omega}$. The *jump of* Φ, written $J(\Phi)$, is a sentence of $\mathcal{L}'_{\omega_1\omega}$ defined as follows, where $\mathcal{L}' = \mathcal{L} \cup \{E\}$ is obtained by adding a new binary relation symbol E to $\mathcal{L}$. Namely $J(\Phi)$ states that E is an equivalence relation with infinitely many classes, each of which is a model of Φ. If $R \in \mathcal{L}$ and $\overline{x}$ is a tuple not all from the same E-class, then $R(\overline{x})$ is defined to be false, so that the models are independent.

There is a corresponding notion of jump that can be defined directly on equivalence relations: Given an equivalence relation E on X, its jump is the equivalence relation $J(E)$ on X^ω, defined by setting $(x_n : n \in \omega)J(E)(y_n : n \in \omega)$ if there is some $\sigma \in S_\infty$ with $x_{\sigma(n)}Ey_n$ for all $n \in \omega$. Then the previous definition of the jump can be viewed as the special case where (X, E) is $(\mathrm{Mod}(\Phi), \cong)$.

The notion of a jump was investigated in [12], where it was shown that if E is a

Borel equivalence relation on a Polish space X with more than one class, then $E <_B J(E)$. We give a partial generalization of this in Corollary 5.3.22 – if $\Phi \in \mathcal{L}_{\omega_1,\omega}$ is short, then $\|\Phi\| < \|J(\Phi)\|$, so $\Phi <_B J(\Phi)$. Using the theory of pins, one can use essentially the same proof to give a true generalization: if (X, E) is HC-forcing invariant, short, and has more than one E-class, then $\|(X, E)\| < \|(X^\omega, J(E))\|$, so in particular $E <_B J(E)$.

We wish to iterate the Friedman-Stanley jump. At limit stages we must explain what we will do. In [89] we took products, but it is more natural to take disjoint unions:

Definition 5.3.19. Suppose I is a countable set and for each i, Φ_i is a sentence of $\mathcal{L}_{\omega_1,\omega}$ in the countable relational language $\mathcal{L}_i$. The *disjoint union of the* Φ_i, denoted $\sqcup_i \Phi_i$, is a sentence of $\mathcal{L}_{\omega_1\omega}$, where $\mathcal{L} = \{U_i : i \in I\} \cup \bigcup_i \mathcal{L}_i$ is the disjoint union of the $\mathcal{L}_i$'s together with new unary predicates $\{U_i : i \in I\}$.

Namely $\sqcup_i \Phi_i$ states that the U_i are disjoint and exhaustive, and that exactly one U_i is nonempty, and that this U_i forms a model of Φ_i when viewed as an $\mathcal{L}_i$-structure.

We now define the tower $(\Phi_\alpha : \alpha < \omega_1)$. Actually, we proceed more generally, starting with any base theory.

Definition 5.3.20. Suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ and $\alpha < \omega_1$. Then we define the α'th jump, $J^\alpha(\Phi)$, of Φ as follows. Let $J^0(\Phi) = \Phi$. Having defined $J^\alpha(\Phi)$, let $J^{\alpha+1}(\Phi) = J(J^\alpha(\Phi))$. For limit stages, let $J^\delta(\Phi) = \sqcup_{\alpha<\delta} J^\alpha(\Phi)$.

Let $\Phi_\alpha = J^\alpha(\mathrm{Th}(\mathbb{Z}, S))$.

Then we have the following straightforward inductive argument:

Proposition 5.3.21. Suppose $\Phi \in \mathcal{L}_{\omega_1\omega}$ and $\alpha < \omega_1$. Then:

- Φ is grounded if and only if $J^\alpha(\Phi)$ is grounded.

- If the isomorphism relation for Φ is Borel, then so is the isomorphism relation for each $J^\alpha(\Phi)$.

- If Φ has infinitely many countable models, then $\|J^\alpha(\Phi)\| = \beth_\alpha(\|J^\alpha(\Phi)\|)$.

Corollary 5.3.22. Suppose $\Phi \in \mathcal{L}_{\omega_1\omega}$. Then for all $\alpha < \beta$, $\Phi_\alpha \leq_B \Phi_\beta$. If Φ is short with more than one countable model, then $J(\Phi) \not\leq_{\mathrm{HC}} \Phi$. Thus for all $\alpha < \beta$, $\Phi_\beta \not\leq_{\mathrm{HC}} \Phi_\alpha$.

We now relate this to complete first order theories. We remark that by the proof of this proposition, each $T_\alpha \sim_B \Phi_{\alpha+1}$, where $(T_\alpha : \alpha < \omega_1)$ is the tower from [89]. So in [89] we are skipping the limit stages.

Theorem 5.3.23. Suppose T is a first order theory, and $\alpha < \omega_1$. Then there is a first order theory S_α such that $S_\alpha \sim_B J^\alpha(T)$. If α is not a limit ordinal, and if T is complete, then we can arrange S_α to be complete.

Proof. First of all, note we can suppose T has infinitely many countable models. Indeed, if T has only one countable model, then $J^n(T) \sim_B T$ for each $n < \omega$, and $J^\omega(T) \sim_B \mathrm{Th}(\mathbb{Z}, S)$. Also, if T has finitely many but more than one countable model, then $J(T) \sim_B \mathrm{Th}(\mathbb{Z}, S)$.

We show the first claim.

Note that if α is a such that we have found a first-order theory S_α with $S_\alpha \sim_B J^\alpha(T)$, then we can set $S_{\alpha+1} = J(S_\alpha)$. Thus it suffices to show the following: suppose δ is a limit, and for all $\alpha < \delta$, we have found $S_\alpha \sim_B J^\alpha(T)$. Then we can find $S_\delta \sim_B J^\delta(T)$. Note that $J^\delta(T) \sim_B \sqcup_{\alpha<\delta} S_\alpha$, so it suffices to find $S_\delta \sim_B \sqcup_{\alpha<\delta} S_\alpha$.

We let S_δ be the theory in the same language as $\sqcup_{\alpha<\delta} S_\alpha$, i.e. the disjoint union of the languages of S_α for $\alpha < \delta$; let S_δ assert that at most one U_α is nonempty, and if U_α is nonempty then everything is in U_α. Then S_δ is first order, and a weakening of $\sqcup_{\alpha<\delta} S_\alpha$. Further, there is up to isomorphism only one countable (infinite) model of S_δ which is not a model of $\sqcup_{\alpha<\delta} S_\alpha$, namely the model with infinitely many unsorted elements. So trivially $\sqcup_{\alpha<\delta} S_\alpha \leq_B S_\delta$; for the reverse, let $(M_n : n < \omega)$ be infinitely many pairwise-nonisomorphic models in $\mathrm{Mod}(\sqcup_{\alpha<\delta} S_\alpha)$. Given $M \in \mathrm{Mod}(S_\delta)$, if M is the model where each U_α is empty then let $f(M) = M_0$. If $M \cong M_n$ for some $n < \omega$, then let $f(M) = M_{n+1}$ (this is a Borel condition, because the isomorphism class of any structure is Borel). Otherwise, let $f(M) = M$.

The second claim is proved by a separate induction on α.

Note that if α is a such that we have found a complete first-order theory S_α with $S_\alpha \sim_B J^\alpha(T)$, then we can set $S_{\alpha+1} = J(S_\alpha)$. Thus it suffices to show the following: suppose δ is a limit, and for all $\alpha < \delta$ non-limit, we have found $S_\alpha \sim_B J^\alpha(T)$. Then we can find $S_{\delta+1} \sim_B J^{\delta+1}(T)$. Write $I = \{\alpha < \delta : \alpha \text{ is not a limit}\}$.

We let $S_{\delta+1} = \prod_{\alpha \in I} S_\alpha$; that is, there is a sort U_α for each $\alpha \in I$, and $S_{\delta+1}$ says each $U_\alpha \models S_\alpha$. Thus we can view models of $S_{\delta+1}$ as sequences $(X, M_\alpha : \alpha \in I)$, where X is the set of unsorted elements (i.e. any elements not in any U_α). Note that there is no structure on X, so all we need to know about it is its cardinality (finite or $\aleph_0$). It is easily checked that $S_{\delta+1}$ is a complete first order theory.

We wish to show $S_{\delta+1} \sim_B J^{\delta+1}(T)$. To do this, note first that if we let $T_* = \prod_{\alpha \in I} J^\alpha(T)$, then easily $T_* \sim_B S_{\delta+1}$ (since each $S_\alpha \sim_B J^{\delta+1}(T)$), so it suffices to show that $T_* \sim_B J^{\delta+1}(T) = J(\sqcup_{\alpha<\delta} J^\alpha(T))$.

First we informally describe the reduction $g : T_* \leq_B J^{\delta+1}(T)$. Given $(X, M_\alpha : \alpha \in I) \models T_*$ (so each $M_\alpha \models J^\alpha(T)$, define $(N_\alpha : \alpha \in I)$ via $N_\alpha = M_\alpha$ for $\alpha > 0$, and for $\alpha = 0$, M_α is a model of $\mathrm{Th}(\mathbb{Z}, S)$ with n-many S-chains, where n encodes $(|X|, m)$, where m is the number of S-chains in M_0. Then each N_α can be naturally viewed as a model of $\sqcup_{\alpha<\delta} J^\alpha(T)$, so $(N_\alpha : \alpha \in I)$ can be viewed as a model of $J^{\delta+1}(T)$, after fixing some bijection between α and ω (which will not affect the isomorphism type).

Next, we describe the reduction $f : J^{\delta+1}(T) \leq_B T_*$. First, for each $\alpha < \delta$, let $N_{\alpha,k} : k < \omega$ be infinitely many pairwise nonisomorphic models of $J^\alpha(T)$. Now, suppose we are given $(M_n : n \in I) \models J^{\delta+1}(T)$. For each $n < \omega$, let $\alpha_n < \delta$ be such that $M_n \models J^{\alpha_n}(T)$. Let $M_n^* = M_n$ if M_n is not isomorphic to any $N_{\alpha_n,k}$, otherwise let let k_n be the unique $k < \omega$ with $M_n \cong N_{\alpha_n,k}$, and let $M_n^* = N_{\alpha_n,k_n+1}$. (This can be done in a Borel fashion, since the isomorphism class of any structure is always Borel.) Now, for each $\alpha < \delta$, let $R_\alpha \models J(\Phi_\alpha)$ be $(M_n : n \in \omega, \alpha_n = \alpha)$, along with infinitely many copies of $N_{\alpha,0}$. Let $R_0 = (\mathbb{Z}, S)$. Then $f(M_n : n < \omega) := (R_\alpha : \alpha < \delta)$ works. $\square$

We also remark on the following nice viewpoint of Φ_α.

Proposition 5.3.24. For all $\alpha < \omega_1$, $\Phi_\alpha \sim_{\mathrm{HC}} \mathrm{HC}_{\omega+\alpha}$.

Proof. That $\mathrm{HC}_{\omega+\alpha} \leq_{\mathrm{HC}} \Phi_\alpha$ is a routine inductive argument. To show that $\Phi_\alpha \leq_{\mathrm{HC}} \mathrm{HC}_{\omega+\alpha}$ we need to handle multiplicities; for this we need to show that $\mathrm{HC}_{\omega+\alpha} \times \omega \leq_{\mathrm{HC}} \mathrm{HC}_{\omega+\alpha}$. For each $n < \omega$, define $f_n : \mathrm{HC}_{\omega+\alpha} \to \mathrm{HC}_{\omega+\alpha}$ inductively. First, if $a \in \mathrm{HC}_\omega$, then define $f_n(a)$ to be the ordered pair (a, n). Next, having defined $f_n(b)$ for all $b \in \mathrm{HC}_{\omega+\beta}$, where $\beta < \alpha$, and given a of rank β, define $f_n(a) = \{f_n(b) : b \in a\}$.

Then we can define our pairing function as $(a, n) \mapsto f_n(a)$. $\square$

5.4 Compact group actions

In this section we use the technology of canonical Scott sentences and representability to analyze the effect of a continuous action of a compact group on a Polish space X. In particular, we show that the quotient of $\mathcal{P}_{\aleph_1}(X)$ by the diagonal action of G is representable. We also show that if the group is abelian, we can bound the potential cardinality of the representation. In Section 5.6 we use these results to analyze the models of the theory K and to contrast K with TK.

Suppose we have a Polish group G acting on a Polish space X. To apply our machinery to this situation we need to say what it means for the objects involved to be HC-forcing invariant:

Definition 5.4.1.

- an HC-forcing invariant Polish space is a sequence (X, d, D, i) of HC-forcing invariant sets, where persistently: d is a complete metric on X, $D \subset X$ is dense and $i : \omega \to D$ is a bijection.

- an HC-forcing invariant Polish group is a sequence $(G, d', D', i', \times)$ where (G, d', D', i') is an HC-forcing invariant Polish space and persistently, $\times$ is a compatable group operation on G.

- Suppose $(G, d', D', i', \times)$ is an HC-forcing invariant Polish group, (X, d, D, i) is an HC-forcing invariant Polish space. Then an HC-forcing invariant continuous action of G on X is an HC-forcing invariant set $\cdot$ such that persistently, $\cdot \subset G \times X \times X$ is a continuous action of G on X.

Throughout this subsection, we fix an HC-forcing invariant Polish space (X, d, D, i), an HC-forcing invariant, persistently compact Polish group $(G, d', D', i', \times)$, and an HC-forcing invariant continuous action $\cdot$ of G on X.

We also fix HC-forcing invariant sets

$$\mathcal{B}_n = \{U^n_i : i \in \omega\}$$

such that persistently, each $\mathcal{B}_n$ is a basis for the topology on X^n. (For instance, take $\mathcal{B}_1$

to be the balls with rational radius and center in D, using the enumeration of D given by i.)

The action of G on X naturally gives diagonal actions on both X^n and $\mathcal{P}(X)$ defined by $g \cdot \overline{a} = \langle g \cdot a : a \in \overline{a} \rangle$ and $g \cdot A = \{g \cdot a : a \in A\}$, respectively. Clearly, the diagonal action of G takes countable subsets of X to countable subsets. For all of these spaces, let $\sim^G$ be the equivalence relation induced by G.

In order to understand the quotient $(\mathcal{P}_{\aleph_1}(X), \sim^G)$, we begin with one easy lemma that uses the fact that G is compact. This lemma is the motivation for the language we define below.

Lemma 5.4.2. If $A, B \in \mathcal{P}_{\aleph_1}(X)$, then $A \sim^G B$ if and only if there is a bijection $\sigma : A \to B$ satisfying $\overline{a} \sim^G h(\overline{a})$ for all $\overline{a} \in A^{<\omega}$.

Proof. If $g \cdot A = B$, then $\sigma := g\restriction_A$ is as desired. For the converse, fix such a σ; we will show there is $g \in G$ inducing σ. Let $\{a_n : n \in \omega\}$ be an enumeration of A, and for each n, let $\overline{a}_n$ be the tuple $a_0 \dots a_{n-1}$ and let $C_n \subseteq G$ be the set of all $g \in G$ with $g \cdot \overline{a}_n = \sigma(\overline{a}_n)$. C_n is closed since the action is continuous and C_n is nonempty by hypothesis. Since G is compact, $C = \bigcap_n C_n$ is nonempty, and clearly any $g \in C$ has $g \cdot A = B$. $\square$

We define a language $\mathcal{L}$ and a class of $\mathcal{L}$-structures that encode this information. Put $\mathcal{L} := \{R_i^n : i \in \omega, n \geq 1\}$, where each R_i^n is an n-ary relation. Let M_X be the $\mathcal{L}$-structure with universe all of X, with each R_i^n interpreted by

$$M_X \models R_i^n(\overline{a}) \quad \text{if and only if} \quad G \cdot \overline{a} \cap U_i^n = \emptyset$$

As notation, let $\mathrm{qf}_n(\overline{a})$ denote the quantifier-free type of $\overline{a} \in X^n$. It is easily seen that to specify $\mathrm{qf}_n(\overline{a})$ it is enough to specify the set of $i \in \omega$ such that $M_X \models R_i^n(\overline{a})$. Also,

$$\mathrm{qf}_n(\overline{a}) = \mathrm{qf}_n(\overline{b}) \quad \text{if and only if} \quad G \cdot \overline{a} = G \cdot \overline{b}$$

As well, note that every $g \in G$ induces an $\mathcal{L}$-automorphism of M_X given by $a \mapsto g \cdot a$. These two observations imply that M_X has a certain homogeneity – For $\overline{a}, \overline{b} \in X^n$, $\mathrm{qf}_n(\overline{a}) = \mathrm{qf}_n(\overline{b})$ if and only if there is an automorphism of M_X taking $\overline{a}$ to $\overline{b}$.

For $\overline{a}, \overline{b} \in X^n$ the relation $\overline{a} \sim^G \overline{b}$ is absolute between $\mathbb{V}$ and any forcing extension $\mathbb{V}[H]$. To see this, note that it suffices to check that qf_n is absolute; and in turn it suffices to check that each R_i^n is absolute. But $\overline{a} \in R_i^n$ if and only if for some or any sequence $(\overline{d}_m : m \in \omega)$ from D^n converging to $\overline{a}$, we have that for large enough m, $D' \cdot \overline{d}_m \cap U_i^n = \emptyset$.

It is not hard to check that the range of qf_n is analytic ($\Gamma(\overline{x})$ is in the range of qf_n if and only if there is a convergent sequence $(\overline{d}_m : m < \omega)$ from D^n satisfying various Borel properties). Hence by Shoenfield Absoluteness, the range of qf_n is absolute.

As notation, call an $\mathcal{L}$-structure $N \in \mathrm{HC}$ *nice* if it is isomorphic to a substructure of M_X. Let $\mathcal{W}$ consist of all nice $\mathcal{L}$-structures.

Lemma 5.4.3. An $\mathcal{L}$-structure $N \in \mathrm{HC}$ is nice if and only if for every $n \geq 1$, every quantifier-free n-type realized in N is realized in M_X.

Proof. Left to right is obvious. For the converse, choose any $N \in HC$ for which every quantifier-free n type realized in N is realized in M_X. We construct an $\mathcal{L}$-embedding of N into M_X via a "forth" construction using the homogeneity of M_X. Enumerate the universe of $N = \{a_n : n \in \omega\}$ and let $\overline{a}_n$ denote $\langle a_i : i < n\rangle$. Assuming $f_n : \overline{a}_n \to M_X$ has been defined, choose any $\overline{b} \in X^{n+1}$ such that $\mathrm{qf}_{n+1}(\overline{a}_{n+1}) = \mathrm{qf}_{n+1}(\overline{b})$. Write $\overline{b}$ as $\overline{b}_n b^*$. As $\mathrm{qf}_n(\overline{b}_n) = \mathrm{qf}_n(f_n(\overline{a}_n))$, there is an automorphism σ of M_X with $\sigma(\overline{b}_n) = f_n(\overline{a}_n)$. Then define f_{n+1} to extend f_n and satisfy $f_{n+1}(a_n) = \sigma(b^*)$. $\square$

Define a map $f : \mathcal{P}_{\aleph_1}(X) \to \mathcal{W}$ by $A \mapsto M_A$, the substructure of M_X with universe A.

Our first goal is the following Theorem.

Theorem 5.4.1. Suppose (X, d, D, i) is an HC-forcing invariant Polish space, $(G, d', D', i', \times)$ is an HC-forcing invariant, persistently compact Polish group, and $\cdot$ is an HC-forcing invariant continuous action of G on X. Then:

1. Both $\mathcal{W}$ and f are HC-forcing invariant;

2. Persistently, for all $A, B \in \mathcal{P}_{\aleph_1}(X)$, $A \sim^G B$ if and only if $f(A) \cong f(B)$;

3. The canonical Scott sentence map css : $(\mathcal{W}, \cong) \to CSS(\mathcal{W})$ is a representation, where $\mathrm{CSS}(\mathcal{W}) = \{\mathrm{css}(N) : N \in \mathcal{W}\}$;

4. The quotient $(\mathcal{P}_{\aleph_1}(X), \sim^G)$ is representable via the composition map $\mathrm{css} \circ f$ that takes $A \mapsto \mathrm{css}(M_A)$.

Proof. (1) It is obvious that f is HC-forcing invariant.

That $\mathcal{W}$ is HC-forcing invariant follows from Lemma 5.4.3 and the absoluteness results mentioned above. In particular, for any $\mathcal{L}$-structure $N \in \mathrm{HC}$ that is not nice, there is some n and $\overline{a} \in N^n$ such that $\Gamma := \mathrm{qf}_n(\overline{a})$ is not realized in M_X. But then, in any forcing extension $\mathbb{V}[H]$, $(M_X)^{\mathbb{V}[H]}$ does not realize Γ, so N is not nice in $\mathbb{V}[H]$. As this argument relativizes to any forcing extension, $\mathcal{W}$ is HC-forcing invariant.

For (2), if $A \sim^G B$, then any $g \in G$ that satisfies $g \cdot A = B$ induces a bijection between A and B such that $\overline{a} \sim^G g \cdot \overline{a}$ for all $\overline{a} \in A^{<\omega}$. As this implies $G \cdot \overline{a} = G \cdot (g \cdot \overline{a})$, $\mathrm{qf}_n(\overline{a})$ in M_A is equal to $\mathrm{qf}_n(g \cdot \overline{a})$ in M_B. Thus, the action by g induces an isomorphism of the $\mathcal{L}$-structures M_A and M_B. Conversely, suppose $\sigma : M_A \to M_B$ is an $\mathcal{L}$-isomorphism. Then $\sigma(\overline{a}) \sim^G \overline{a}$ for every $\overline{a} \in A^{<\omega}$, so $A \sim^G B$ by Lemma 5.4.2.

(3) As $\mathcal{W}$ is HC-forcing invariant, this follows immediately from Lemmas 5.3.3 and 5.2.23.

(4) follows immediately from (1), (2), and (3). $\square$

As a consequence of Theorem 5.4.1, $\|(\mathcal{P}_{\aleph_1}(X), \sim^G)\|$ is defined. For an arbitrary compact group action, this quotient need not be short. Indeed, Theorem 5.6.4 gives an example where it is not. However, if we additionally assume that G is abelian, then we will see below that $\|(\mathcal{P}_{\aleph_1}(X), \sim^G)\| \leq \beth_2$. The reason for this stark discrepancy is due to the comparative simplicity of abelian group actions. In particular, if an abelian group G acts transitively on a set S, then S is essentially an affine copy of $G/\mathrm{Stab}(a)$, where $\mathrm{Stab}(a)$ is the subgroup of G stabilizing any particular $a \in S$. The following Lemma is really a restatement of this observation.

Lemma 5.4.4. Suppose that an abelian group G acts on a set X. Then for every $n \geq 1$, if three n-tuples $\overline{a}, \overline{b}, \overline{c} \in X^n$ satisfy $\overline{a} \sim^G \overline{b} \sim^G \overline{c}$ and $\overline{ab} \sim^G \overline{ac}$, then $\overline{b} = \overline{c}$.

Proof. Let $g \in G$ be such that $g\overline{a} = \overline{b}$. Choose $h \in G$ such that $h(\overline{ab}) = \overline{ac}$. Then in particular, $h\overline{a} = \overline{a}$ and $h\overline{b} = \overline{c}$. From this, $gh\overline{a} = \overline{b}$ and $hg\overline{a} = \overline{c}$. But $gh = hg$, so $\overline{b} = \overline{c}$, as desired. $\square$

We will show that $\|(\mathcal{W}, \cong)\| \leq \beth_2$ by showing that each Scott sentence in the representation is from $\mathcal{L}_{\beth_1^+\omega}$, and then using the fact that there are at most $\beth_2$ such sentences. We will accomplish this complexity bound by a type-counting argument; here is the notion of type we will use.

Definition 5.4.5. If φ is a canonical Scott sentence – that is, $\varphi \in \mathrm{CSS}(\mathcal{L})_{\mathrm{ptl}}$ – then let $S^n_\infty(\varphi)$ be the set of all canonical Scott sentences in the language $\mathcal{L}' = \mathcal{L} \cup \{c_0, \ldots, c_{n-1}\}$ which imply φ. We will refer to elements of $S^n_\infty(\varphi)$ as types – infinitary formulas with free variables $x_0, \ldots, x_{n-1}$, resulting from replacing each c_i with a new variable x_i not otherwise appearing in the formula. It is equivalent to define $S^n_\infty(\varphi)$ by forcing – if $\mathbb{V}[H]$ makes φ hereditarily countable and $M \in \mathbb{V}[H]$ is the unique countable model of φ, then $S^n_\infty(\varphi)$ is the set $\{\mathrm{css}(M, \overline{a}) : \overline{a} \in M^n\}$. Evidently this set depends only on the isomorphism class of M, so by the usual argument with Lemma 5.2.5, this set is in $\mathbb{V}$.

Suppose φ is a potential canonical Scott sentence; we use the precise syntactic definition of Scott formulas from Definition 5.3.1. For a moment, pass to a forcing extension $\mathbb{V}[G]$ in which φ is hereditarily countable, and let M be its unique countable model. For each ordinal α, let $S^n_\alpha(\varphi)$ be the set $\{\varphi^{\overline{a}}_\alpha(\overline{x}) : \overline{a} \in M^n\}$. By Lemma 5.2.5, $S^n_\alpha(\varphi)$ is in $\mathbb{V}$ and depends only on φ. Moreover, there is a natural surjection $\pi^n_\alpha : S^n_\infty(\varphi) \to S^n_\alpha(\varphi)$ taking $\mathrm{css}(M, \overline{a})$ to $\varphi^{\overline{a}}_\alpha(\overline{x})$; each π^n_α is in $\mathbb{V}$. (If there is possible ambiguity, we will write $\pi^n_{\alpha,\varphi}$.)

Define the Scott rank of φ to be the Scott rank of M. Again, this is invariant under isomorphism, so depends only on φ. Write this ordinal as α_*. For any two distinct sentences $\psi, \tau \in S^n_\infty(\varphi)$, let $d(\psi, \tau)$ be the least $\alpha < \alpha^*$ where $\pi^n_{\alpha+1}(\psi) \neq \pi^n_{\alpha+1}(\tau)$. If $\psi = \tau$ then let $d(\psi, \tau) = \alpha^*$. d depends only on φ, so by Lemma 5.2.5, $d \in \mathbb{V}$. (If there is ambiguity we will write d_φ.)

Proposition 5.4.6. Suppose φ is a canonical Scott sentence in a language of size at most κ, and for all n, $|S^n_\infty(\varphi)| \leq \kappa$, where κ is an infinite cardinal. Then φ is a sentence of $\mathcal{L}_{\kappa^+\omega}$.

Proof. Let α^* be the Scott rank of φ. Note that it is immediate from the construction of Scott formulas that if $\alpha \leq \alpha^*$, there is some n and some pair ψ, τ from $S^n_\infty(\varphi)$, such that $d(\psi, \tau) = \alpha$; hence $d : \bigcup_n (S^n_\infty(\varphi))^2 \to \alpha^* + 1$ is surjective.

Since $|S^n_\infty(\varphi)| \leq \kappa$ and $\pi^n_\alpha : S^n_\infty(\varphi) \to S^n_\alpha(\varphi)$ is surjective, $|S^n_\alpha(\varphi)| \leq \kappa$ for all α (in particular, for all $\alpha \leq \alpha_*$). Similarly, since $d : \bigcup_n (S^n_\infty(\varphi))^2 \to \alpha^* + 1$ is surjective,

$|\alpha^*+1| \leq \kappa$. By induction we show that for all $\alpha \leq \alpha^*+1$, $S^n_\alpha(\varphi) \subseteq \mathcal{L}_{\kappa^+\omega}$.

The base case is trivial, since there are only κ atomic formulas. The step follows from the fact that $|S^n_\alpha(\varphi)| \leq \kappa$, and the limit follows from the fact that $\alpha^* < \kappa^+$, so in both cases we need only take conjunctions and disjunctions of κ formulas at a time.

Observe that φ is precisely the following:

$$\pi^0_{\alpha^*}(\varphi) \wedge \bigwedge \left\{\forall \overline{x} \left(\pi^n_{\alpha^*}(\varphi^*)(\overline{x}) \to \pi^n_{\alpha^*+1}(\varphi^*)(\overline{x})\right) : n \in \omega, \varphi^* \in S^n_\infty(\varphi)\right\}$$

Since $S^n_\alpha(\varphi) \subseteq \mathcal{L}_{\kappa^+\omega}$ for all α and n, and since they all have size at most κ, φ is in $\mathcal{L}_{\kappa^+\omega}$, as desired. □

The following holds by a straightforward induction on the complexity of formulas:

Lemma 5.4.7. For all infinite cardinals κ and languages $\mathcal{L}$ of size at most κ, there are exactly 2^κ different $\mathcal{L}_{\kappa^+\omega}$ formulas (up to relabeling variables).

Now we can prove our theorem. Recall that $\sim_G$ is the diagonal equivalence relation on $\mathcal{P}_{\aleph_1}(X)$, induced by the diagonal action of G.

Theorem 5.4.8. Let X be an HC-forcing invariant Polish space, let G be an HC-forcing invariant, persistently compact abelian group, and suppose $\cdot$ is an HC-forcing invariant continuous action of G on X. Suppose all this holds persistently. Then $\|(\mathcal{P}_{\aleph_1}(X), \sim_G)\| \leq \beth_2$.

Proof. We use Proposition 5.4.6 to show that $\mathrm{CSS}(\mathcal{W})_{\mathrm{ptl}} \subseteq \mathcal{L}_{\beth_1^+\omega}$; then by Lemma 5.4.7, we have that $|\mathrm{CSS}(\mathcal{W})_{\mathrm{ptl}}| \leq \beth_2$, as desired. So let $\varphi \in \mathrm{CSS}(\mathcal{W})_{\mathrm{ptl}}$ be arbitrary; it is enough to show that $|S^\infty_n(\varphi)| \leq \beth_1$.

For each n, let $\mathrm{qf}_n(\varphi)$ be the set of quantifier-free n-types which are consistent with φ. We have a surjective map $\pi_n : S^\infty_n(\varphi) \to \mathrm{qf}_n(\varphi)$ sending $\psi(\overline{x})$ to the set of quantifier-free formulas in $\overline{x}$ which it implies. For any $p \in \mathrm{qf}_n(\varphi)$, let $S^\infty_n(\varphi, p)$ be $\pi^{-1}(p)$, the set of $\psi \in S_n^\infty(\varphi)$ where $\pi_n(\psi) = p$. (All of these definitions have taken place in $\mathbb{V}$.) Since the language is countable, $|\mathrm{qf}_n(\varphi)| \leq \beth_1$. Thus it is sufficient to show that for all p, $|S_n^\infty(\varphi, p)| \leq \beth_1$.

Now we take advantage of the fact that G is abelian:

Claim: Suppose $p^* \in \mathrm{qf}_{2n}(\varphi)$ is such that $p^*\restriction_{[0,n)} = p^*\restriction_{[n,2n)} = p$. Suppose that $\psi^*, \tau^* \in S_{2n}^\infty(\varphi)$ both complete p^*. Further, suppose $\psi^*\restriction_{[0,n)} = \tau^*\restriction_{[0,n)}$. Then $\psi^* = \tau^*$.

Proof: Pass to a forcing extension $\mathbb{V}[H]$ in which φ is hereditarily countable, and let M be its unique countable model. By Theorem 5.4.1, we may assume $M = M_A$ for some $A \in \mathcal{P}_{\aleph_1}(X)^{\mathbb{V}[H]}$. Choose some tuples $(\overline{a_0 a_1})$ and $(\overline{b_0 b_1})$ from M^{2n} where $\mathrm{css}(M, \overline{a_0 a_1}) = \psi^*$ and $\mathrm{css}(M, \overline{b_0 b_1}) = \tau^*$. By assumption $\mathrm{css}(M, \overline{a}_0) = \mathrm{css}(M, \overline{b}_0)$, so we may assume $\overline{a}_0 = \overline{b}_0$. Since all of the tuples $\overline{b}_0$, $\overline{a}_1$, and $\overline{b}_1$ have the same quantifier-free type, they are in the same G-orbit, and similarly with $\overline{b}_0\overline{a}_1$ and $\overline{b}_0\overline{b}_1$. Thus Lemma 5.4.4 applies directly to the triple $(\overline{b}_0, \overline{b}_1, \overline{a}_1)$, so in particular $\overline{b}_1 = \overline{a}_1$. Thus $\psi^* = \mathrm{css}(M, \overline{b_0 a_1}) = \mathrm{css}(M, \overline{b_0 b_1}) = \tau^*$, as desired. $\square$

Fix some $\psi \in S_n^\infty(\varphi, p)$, and define $\Gamma(\psi)$ to be the set of all $p^* \in \mathrm{qf}_{2n}(\varphi)$ such that $p^*\restriction_{[0,n)} = p^*\restriction_{[n,2n)} = p$ and such that for some $\psi^* \in S_{2n}^\infty(\varphi, p^*)$, $\psi^*\restriction_{[0,n)} = \psi$.

By the Claim, if $p^* \in \Gamma(\psi)$, there is a unique $\psi^* \in S_{2n}^\infty(\varphi)$ where $\pi_{2n}(\psi^*) = p^*$ and $\psi^*\restriction_{[0,n)} = \psi$. So define $F(p^*)$ to be $\psi^*\restriction_{[n,2n)}$. Evidently $|\Gamma(\psi)| \leq \beth_1$, so it is enough to show that $F : \Gamma(\psi) \to S_n^\infty(\varphi, p)$ is surjective.

But this is almost immediate. Fix any $\tau \in S_n^\infty(\varphi, p)$ and let $\mathbb{V}[H]$ be a forcing extension in which φ is hereditarily countable, and let M be its unique countable model; as before, we may assume $M = M_A$ for some $A \in \mathcal{P}_{\aleph_1}(X)^{\mathbb{V}[H]}$. Choose any $\overline{a} \in A^n$ where $\mathrm{css}(M, \overline{a}) = \psi$ and any $\overline{b} \in A^n$ where $\mathrm{css}(M, \overline{a}) = \tau$. Finally, let p^* be the quantifier-free type of $\overline{ab}$ in M. Clearly $p^* \in \Gamma(\psi)$ and $F(p^*) = \tau$. $\square$

This theorem will be crucial in Section 5.6.

5.5 Refining Equivalence Relations

We begin by defining an incomplete first-order theory REF. Its language is $\mathcal{L} = \{E_n : n \in \omega\}$ and its axioms posit:

- Each E_n is an equivalence relation;
- E_0 has a single equivalence class;
- For all n, E_{n+1} refines E_n; that is, every E_n-class is a union of E_{n+1}-classes.

The theory REF is very weak, which makes the generality of the following proposition surprising.

Proposition 5.5.1. REF is grounded.

Proof. We begin with an analysis of an arbitrary model M of REF. As notation, for any $a \in M$ and $n \in \omega$, let $[a]_n$ denote the equivalence class of a, i.e., $\{b \in M : M \models E_n(a, b)\}$. As the equivalence relations refine each other, the classes $T(M) = \{[a]_n : a \in M, n \in \omega\}$ form an ω-tree, ordered by $[a]_n \leq [b]_m$ if and only if $n \leq m$ and $[b]_m \subseteq [a_n]$. Next, let E_∞ be the equivalence relation given by $E_\infty(a, b)$ if and only if $E_n(a, b)$ for every $n \in \omega$. Let $[a]_\infty$ be the E_∞-class of a. Then M/E_∞ can be construed as a subset of the branches $[T(M)]$ of $T(M)$. As we are interested in determining models up to back-and-forth equivalence (as opposed to isomorphism), the following definition is natural.

For each $a \in M$, let the *color of* a, $c(a) \in (\omega + 1) \setminus \{0\}$ be given by

$$c(a) = \begin{cases} |[a]_\infty| & \text{if } [a]_\infty \text{ is finite} \\ \omega & \text{if } [a]_\infty \text{ is infinite} \end{cases}$$

Next, we describe some expansions of M to larger languages. For each $n \in \omega$, let $\mathcal{L}_n = \mathcal{L} \cup \{U_i : i \leq n\}$, where the U_i's are distinct unary predicates. Given any $M \models \mathrm{REF}$, $n \in \omega$, and $a \in M$, let $M_n(a)$ denote the $\mathcal{L}_n$-structure $(M, [a]_0, \ldots, [a]_n)$, i.e., where each predicate U_i is interpreted as $[a]_i$.

We now exhibit some invariants, which we term the *data of* M, written $D(M)$ which we will see only depend on the $\equiv_{\infty,\omega}$-equivalence class of M.

For each $n \in \omega$, let

$$I_n(M) = \{\mathrm{css}(M_n(a)) : a \in M\}.$$

We combine the sets $I_n(M)$ into a tree $(I(M), \leq)$ where $I(M) = \bigcup_{n\in\omega} I_n(M)$ and, for $\sigma_n \in I_n(M)$ and $\psi_m \in I_m(M)$, we say $\sigma_n \leq \psi_m$ if and only if $n \leq m$ and $\psi_m \vdash \sigma_n$. That is, if in any forcing extension the reduct of any model of ψ_m to $\mathcal{L}_n$ is a model of σ_n. Then clearly $(I(M), \leq)$ is an ω-tree.

Continuing, for each $n > 0$ and $\sigma_n \in I_n(M)$, let the multiplicity of σ_n, $\mathrm{mult}_M(\sigma_n) \in (\omega+1) \setminus \{0\}$, be given by: $\mathrm{mult}_M(\sigma_n) = k < \omega$ if k is maximal such that there are elements $\{b_i : i < k\} \subseteq M$ such that

$$\bigwedge_{i<j<k} \Big[E_{n-1}(b_i, b_j) \wedge \neg E_n(b_i, b_j) \Big] \wedge \bigwedge_{i<k} \mathrm{css}(M_n(b_i)) = \sigma_n$$

and let $\mathrm{mult}_M(\sigma_n) = \omega$ if there is an infinite family $\{b_i : i < \omega\}$ as above.

Now, each $a \in M$ induces a canonical sequence $\mathit{Seq}_M(a) := \langle \mathrm{css}(M_n(a)) : n \in \omega \rangle$, which is clearly a branch through the tree $I(M)$, and depends only on $\mathrm{css}(M, a)$. Let $\mathit{Seq}(M) = \{\mathit{Seq}_M(a) : a \in M\}$. So $\mathit{Seq}(M) \subseteq [I(M)]$, the set of branches of $I(M)$. Finally, for any $s \in \mathit{Seq}(M)$, we define the color spectrum of s as $\mathit{Sp}_M(s) := \{c(a) : \mathit{Seq}_M(a) = s\}$. Thus, each $\mathit{Sp}_M(s)$ is a non-empty subset of $(\omega + 1) \setminus \{0\}$.

Define the *data of* M, $D(M) := \langle (I(M), \leq), \mathrm{mult}_M, \mathit{Seq}(M), \mathit{Sp}_M \rangle$.

Claim 1: For any $M, N \models \mathrm{REF}$, $M \equiv_{\infty,\omega} N$ if and only if $D(M) = D(N)$.

Proof: First, note that if $D(M) = D(N)$, then as the trees $(I(M), \leq)$ and $(I(N), \leq)$ are equal, they have the same root, so $\mathrm{css}(M_0(a)) = \mathrm{css}(N_0(b))$ for some/every $a \in M, b \in N$. So $M \equiv_{\infty,\omega} N$.

For the forward direction, it is easy to check that $D(M)$ only depends on the isomorphism type of M, and also that D is absolute to forcing extensions. Hence if $M \equiv_{\infty\omega} N$, then pass to a forcing extension $\mathbb{V}[G]$ in which $M \cong N$; then we get $(D(M))^{\mathbb{V}} = (D(M))^{\mathbb{V}[G]} = (D(N))^{\mathbb{V}[G]} = (D(N))^{\mathbb{V}}$. $\square$

To begin the proof of groundedness, choose any $\sigma \in \mathrm{CSS}(\mathrm{REF})_{\mathrm{ptl}}$. Choose any forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$ in which $\sigma \in \mathrm{HC}^{\mathbb{V}[G]}$ and hence $\sigma \in \mathrm{CSS}(\mathrm{REF})^{\mathbb{V}[G]}$. Choose any model $M \in \mathbb{V}[G]$ with $\mathrm{css}(M) = \sigma$. Working in $\mathbb{V}[G]$, compute $D(M)$, the data of M. However, in light of Claim 1, $D(M)$ only depends on σ, and so by Lemma 5.2.5 $D(M) \in \mathbb{V}$. As σ is fixed, for the remainder of the argument we write

$$D = \langle (I, \leq), \mathrm{mult}, Seq, Sp \rangle.$$

To complete the proof of the Proposition, we work in $\mathbb{V}$ and 'unpack' the data D to construct an $\mathcal{L}$-structure $N \in \mathbb{V}$ such that in $\mathbb{V}[G]$, $M \equiv_{\infty,\omega} N$. Once we have this, as $\sigma = \mathrm{css}(M)$, it follows that $N \models \sigma$ and so N witnesses that $\sigma \in \mathrm{CSS}(\mathrm{REF})_{\mathrm{sat}}$. That is, the proof of groundedness will be finished once we establish the following Claim.

Claim 2: There is an $\mathcal{L}$-structure $N \in \mathbb{V}$ such that $\mathbb{V}[G] \models$ '$N \equiv_{\infty,\omega} M$'.

Proof: Before beginning the 'unpacking' of D, we note some connections between M and D that are not part of the data. First, there is a surjective tree homomorphism $h : T(M) \cup M/E_\infty \to I \cup Seq$ given by $h([a]_n) = \mathrm{css}(M_n(a))$ for $n \in \omega$ and $h([a]_\infty) = \langle \mathrm{css}(M_n(a)) : n \in \omega \rangle$. Note that for each $s \in Seq$ and each $k \in Sp(s)$, $\{[a]_\infty : h(a) =$

s and $c(a) = k\}$ is dense in $h^{-1}(s)$. The following relationship between M and h follows quickly:

> $(\star)_{M,h}$: For every $n \geq 1$, $s \in Seq$, $k \in Sp(s)$, and $a \in M$ such that $h([a]_{n-1}) = s(n-1)$, there are pairwise E_n-inequivalent $\{d_i : i < \text{mult}(s(n))\} \subseteq M$ such that
>
> $$\bigwedge_{i<\text{mult}(s(n))} E_{n-1}(d_i, a) \wedge h([d_i]_\infty) = s \wedge c(d_i) = k$$

We also identify two species of elements of Seq. Call $s \in Seq$ of *isolated type* if there is $n \in \omega$ such that $\text{mult}(s(m)) = 1$ for every $m \geq n$ and of *perfect type* otherwise. The latter name is apt, as $h^{-1}(s)$ is perfect (has no isolated points) whenever s is not of isolated type. We argue that if $s \in Seq$ is of isolated type, then $Sp(s)$ is a singleton. Indeed, choose n such that $\text{mult}(s(m)) = 1$ for every $m \geq n$ and choose $a, b \in M$ such that $h([a]_\infty) = h([b]_\infty) = s$. We will show that $c(a) = c(b)$. To see this, by applying $(\star)_{M,h}$ at level $n+1$ with $k = c(b)$, get $d \in M$ such that $E_n(a, d)$, $h([d]_\infty) = s$, and $c(d) = c(b)$. But now, as $h([a]_\infty) = h([d]_\infty) = s$, the choice of n implies that $E_\infty(a, d)$. Thus, $c(a) = c(d) = c(b)$ as required.

We begin 'unpacking' D by inductively constructing an ω-tree $(J, \leq)$ and a surjective tree homomorphism $h' : (J, \leq) \to (I, \leq)$. Begin the construction of $J = \bigcup_{n\in\omega} J_n$ by taking $J_0 = \{\rho_0\}$ to be a singleton and defining $h'(\rho_0) = \sigma$. Suppose the nth level J_n has been defined, together with $h' : \bigcup_{j\leq n} J_j \to \bigcup_{j\leq n} I_j$. For each $\rho_n \in J_n$, we define its immediate successors $Succ_J(\rho_n)$ as follows. Look at $Succ_I(h'(\rho_n)) \subseteq I_{n+1}$. For each $\sigma_{n+1} \in Succ_I(h'(\rho_n))$, choose a set $A_{n+1}(\sigma_{n+1})$ of cardinality $\text{mult}(\sigma_{n+1}) \in (\omega+1) \setminus \{0\}$ such that the sets $A_{n+1}(\sigma_{n+1})$ are pairwise disjoint. Let

$$Succ_J(\rho_n) := \bigcup\{A_{n+1}(\sigma_{n+1}) : \sigma_{n+1} \in Succ_I(h(\rho_n))\}$$

and put $J_{n+1} := \bigcup\{Succ_J(\rho_n) : \rho_n \in J_n\}$. We extend h' by $h'(\rho) = \sigma_{n+1}$ for every $\rho \in A_{n+1}(\sigma_{n+1})$.

Now, having completed the construction of $(J, \leq)$ and the tree homomorphism $h' : (J, \leq) \to (I, \leq)$, there is a unique extension (which we also call h') $h' : [J] \to [I]$ from the branches of J to the branches of I such that $h'(\eta) = s$ if and only if $h'(\eta\restriction_n) = s(n-1)$ for every $n \in \omega$.

The universe of the $\mathcal{L}$-structure N we are building will be a subset of $(h')^{-1}(Seq) \times (\omega + 1)$ and for $(\eta, i), (\nu, j) \in N$, we will interpret E_n by

$$E_n((\eta, i), (\nu, j)) \quad \text{if and only if} \quad \eta\restriction_n = \nu\restriction_n.$$

In particular, we will have $[(\eta, i)]_\infty = \{(\eta, j) : (\eta, j) \in N\}$. To finish our description of N, we must assign a 'color' to the elements of $(h')^{-1}(s)$, for each $s \in Seq$. Fix $s \in Seq$. First, if s is of isolated type, then from above, we know that $Sp(s) = \{k\}$ for a single color $k \leq \omega$. Accordingly, put elements $\{(\eta, i) : i < k\}$ into the universe of N for every η satisfying $h'(\eta) = s$. For each $s \in Seq$ that is not of isolated type, note that $(h')^{-1}(s)$ has no isolated points. Thus, we can choose a partition $(h')^{-1}(s) = \bigcup D_k(s)$ into disjoint dense subsets indexed by colors $k \in Sp(s)$. Then, for each $\eta \in D_k(s)$ put elements $\{(\eta, i) : i < k\}$ into the universe of N. This completes our construction of the $\mathcal{L}$-structure $N \in \mathbb{V}$, and it is easily verified that this construction entails $(\star)_{N,h'}$.

We now work in $\mathbb{V}[G]$ and demonstrate that $M \equiv_{\infty,\omega} N$. Indeed, all that we need for this is that in $\mathbb{V}[G]$, both $(\star)_{M,h}$ and $(\star)_{N,h'}$ hold. Let $\mathcal{F}$ consist of all $(\overline{a}, \overline{b})$ such that $\lg(\overline{a}) = \lg(\overline{b})$, $\overline{a}$ from M, and $\overline{b}$ from N that satisfy for each $i < \lg(\overline{a})$, $c(a_i) = c(b_i)$ and $h([a_i]_\infty) = h'([b_i]_\infty)$; and for each $n \in \omega$, $i < j < \lg(\overline{a})$, $M \models E_n(a_i, a_j)$ if and only if $N \models E_n(b_i, b_j)$ and $a_i = a_j$ if and only if $b_i = b_j$.

To see that $\mathcal{F}$ is a back-and-forth system, choose any $(\overline{a}, \overline{b}) \in \mathcal{F}$ and choose any

$a^* \in M$. We will find $b^* \in N$ such that $(\overline{a}a^*, \overline{b}b^*) \in \mathcal{F}$, and the argument in the other direction is symmetric. If $\lg(\overline{a}) = 0$, or if $a^* \in \overline{a}$, it is obvious what to do, so assume this is not the case. If $E_\infty(a^*, a_i)$ for some i, then as $c(a_i) = c(b_i)$, we can find $b^* \notin \overline{b}$ such that $E_\infty(b^*, b_i)$ which suffices.

Now assume that $\neg E_\infty(a^*, a_i)$ holds for each i. Let $k = c(a^*)$ and $s = h([a^*]_\infty)$. Let $n > 0$ be least such that $\neg E_n(a^*, a_i)$ for all i. Let $A_1 = \{a_i : E_{n-1}(a^*, a_i)\}$ and let B_1 be the associated subset of $\overline{b}$. By the axioms of REF it suffices to find $b^* \in N$ such that $c(b^*) = k$, $h'([b^*]_\infty) = s$, $E_{n-1}(b^*, b)$ for some/every $b \in B_1$, but $\neg E_n(b^*, b)$ for every $b \in B_1$. To find such an element, let

$$A_2 = \{a \in A_1 : \text{ there is some } a' \in [a]_n \text{ such that } c(a') = k \text{ and } h([a']_\infty) = s\}$$

Let $A_3 \subseteq A_2$ be any maximal, pairwise E_n-inequivalent subset of A_2 and let $\ell = |A_3|$. The set $\{a^*\} \cup A_3$ witnesses that $\mathrm{mult}(s(n)) > \ell$. [More precisely, for each $a \in A_3$, choose $a' \in [a]_n$ with $c(a') = k$ and $h([a']_\infty) = s$. Then $\{a^*\} \cup \{a' : a \in A_3\}$ witnesses $\mathrm{mult}(s(n)) > \ell$.] Let B_3 be the associated subset of $\overline{b}$; so $|B_s| = \ell$.

Choose $a_i \in A_1$. Then by $(\star)_{N,h'}$, applied at b_i (noting that $[b_i]_{n-1} = s(n-1)$), choose a family $\{d_i : i < \mathrm{mult}(s(n))\}$ as there. By pigeon-hole choose an $i^* < \mathrm{mult}(s(n))$ such that $\neg E_n(d_{i^*}, b)$ holds for all $b \in B_3$. It is easily checked that d_{i^*} is a possible choice for b^*. As noted above, this completes the proof of the Claim. $\square$

In particular, $N \models \sigma$, establishing groundedness. $\square$

We now turn our attention to two classical complete theories extending REF. These are often given as first examples in stability theory. We denote them by REF(inf) and REF(bin), respectively. REF(bin) is the extension of REF asserting that for every n, E_{n+1} partitions each E_n-class into two E_{n+1}-classes, while REF(inf) asserts that for all n, E_{n+1}

partitions each E_n-class into infinitely many E_{n+1}-classes.

The following facts are well known.

Fact 5.5.2. Both REF(bin) and REF(inf) are complete theories that admit quantifier elimination.

- REF(bin) is superstable but not ω-stable; and
- REF(inf) is stable, but not superstable.

These examples are similar in that the isomorphism relation $\cong$ is not Borel on either of them. However, it turns out that REF(inf) is Borel complete, and indeed, is λ-Borel complete for every λ. On the other hand, $\|\text{REF(bin)}\| = \beth_2$, and hence is far from being Borel complete.

5.5.1 Finite Branching

In this subsection we show that $\Phi_2 \leq_B$ REF(bin), and $\|\text{REF(bin)}\| = \beth_2$, and the isomorphism relation of REF(bin) is not Borel.

For the following, it would be inconvenient to work with Φ_2 directly. Instead, let F_2 be the equivalence relation on $(2^\omega)^\omega$ defined by: $(x_n : n \in \omega) F_2 (y_n : n \in \omega)$ if and only if $\{x_n : n \in \omega\} = \{y_n : n \in \omega\}$. Then the quotient $(2^\omega)^\omega / F_2$ is in natural bijection with $\mathcal{P}_{\aleph_1}(2^\omega) \backslash \{\emptyset\}$, so we think of F_2 as representing countable sets of reals. It is not hard to check that $(\text{Mod}(\Phi_2), \cong)$ is Borel bireducible with $((2^\omega)^\omega, F_2)$. So for $\Phi \in \mathcal{L}_{\omega_1\omega}$, showing that $\Phi_2 \leq_B \Phi$ is the same thing as showing $F_2 \leq_B \Phi$.

Theorem 5.5.1. $\Phi_2 \leq_B$ REF(bin), i.e. $F_2 \leq_B \Phi$.

Proof. Begin by building a special countable model M of REF(bin). Let S be the set of sequences from 2^ω which are eventually zero, and fix a bijection $c : S \to \mathbb{N}$. Let M be the set of all (η, n) where $\eta \in S$ and $n < c(\eta)$. As usual, say $(\eta_1, n_1)E_m(\eta_2, n_2)$ holds if and only if η_1 and η_2 agree on the first m places. Clearly M is a model of REF(bin), and the color of (η, n) is exactly $c(\eta)$; observe that no element has color $\aleph_0$. (Recall that the color of a is the cardinality of $[a]_\infty$.) We will construct our models as superstructures of M, whose new elements all have color $\aleph_0$ and are not E_∞-equivalent to any element of M.

Let $X \subseteq (2^\omega)^\omega$ be the set of all $(x_n : n \in \omega)$ such that each $x_n \notin S$. Then $(X, F_2 \restriction_X) \cong_B ((2^\omega)^\omega, F_2)$, via any Borel bijection between 2^ω and $2^\omega \backslash S$. (By Corollary 13.4 and Theorem 4.6 of [33], any two uncountable Borel sets are in Borel bijection.) So it suffices to show that $(X, F_2 \restriction_X) \leq_B$ REF(bin). Given $I \subseteq 2^\omega \backslash S$ countable, let M_I be the $\mathcal{L}$-structure extending M with universe $M \cup (I \times \omega)$, where again, $(\eta_1, n_1)E_m(\eta_2, n_2)$ holds if and only if $\eta_1 \restriction_m = \eta_2 \restriction_m$.

It is not hard to check that one can define a Borel map $f : X \to \mathrm{Mod}(\mathrm{REF}(\mathrm{bin}))$, such that for all $\overline{x} = (x_n : n \in \omega) \in X$, $f(\overline{x}) \cong M_{\{x_n : n \in \omega\}}$. Given that, it suffices to show that for all distinct $I, J \subseteq 2^\omega \backslash S$ countable, $M_I \ncong M_J$.

So suppose $M_I \cong M_J$, say via $g : M_I \to M_J$. I aim to show that for all $(\eta, n) \in M_I$, $g(\eta, n) = (\eta, n')$ for some $n' < \omega$. This suffices to show $I = J$, since then $I = \{\eta : (\eta, n) \in M_I \text{ for all } n\} = \{\eta : (\eta, n) \in M_J \text{ for all } n\} = J$. So let $(\eta, n) \in M_I$; write $g(\eta, n) = (\tau, n')$. I show for each $m < \omega$ that $\eta \restriction_m = \tau \restriction_m$. Indeed, pick $\nu \in S$ such that $\nu \restriction_m = \eta \restriction_m$. Then $g(\nu, 0) = (\nu, k)$ for some $k < c(\tau)$, since $g([(\nu, 0)]_\infty)$ is the unique E_∞-class of M_J of size $c(\nu)$. Then since $((\eta, n)E_m(\tau, 0))^{M_I}$, we have $((\tau, n')E_m(\nu, k))^{M_J}$. Hence $\tau \restriction_m = \nu \restriction_m = \eta \restriction_m$. $\square$

We now proceed to show $\|\mathrm{REF}(\mathrm{bin})\| = \beth_2$. Actually we show more: let REF(fin) denote the sentence of $\mathcal{L}_{\omega_1,\omega}$ extending REF, asserting additionally that every E_n-class is partitioned into finitely many E_{n+1}-classes.

Lemma 5.5.3. Every model M of REF(fin) has an $\equiv_{\infty\omega}$-equivalent submodel $N \subseteq M$ of size at most $\beth_1$.

Proof. For each E_∞-class $[a]_\infty \subseteq M$, let

$$B([a]_\infty) = \begin{cases} [a]_\infty & \text{if } [a]_\infty \text{ is countable} \\ \text{any countably infinite subset of } [a]_\infty & \text{if } [a]_\infty \text{ is uncountable} \end{cases}$$

and let N be the substructure of M with universe $\bigcup\{B([a]_\infty) : a \in M\}$. It is easily seen that $N \equiv_{\infty,\omega} M$. That N has size at most continuum follows from the finite splitting at each level. $\square$

Combined with groundedness, this gives us the nonembedding result we wanted:

Theorem 5.5.2. $\|\mathrm{REF(bin)}\| = \|\mathrm{REF(fin)}\| = I_{\infty,\omega}(\mathrm{REF(bin)}) = I_{\infty,\omega}(\mathrm{REF(fin)}) = \beth_2$. In particular, both REF(bin) and REF(fin) are short and $\Phi_3 \not\leq_{\mathrm{HC}} \mathrm{REF(bin)}, \mathrm{REF(fin)}$.

Proof. Recall that each $\|\Phi_\alpha\| = \beth_\alpha$. Since $\Phi_2 \leq_B \mathrm{REF(bin)}$, $\beth_2 = \|\Phi_2\| \leq \|\mathrm{REF(bin)}\|$. On the other hand, since REF is grounded, $\|\mathrm{REF(fin)}\| = I_{\infty,\omega}(\mathrm{REF(fin)})$ but the latter cardinal is bounded above by $\beth_2$ by Lemma 5.5.3. Thus, all four cardinals are equal to $\beth_2$. So, by definition, both REF(bin) and REF(fin) are short. As $\|\Phi_3\| = \beth_3$, the nonembeddability of Φ_3 into either class follows from Theorem 5.3.11(2). $\square$

Finally, we show that isomorphism for REF(bin) is not Borel.

Theorem 5.5.3. Isomorphism on REF(bin) is not Borel.

Proof. It is commonly known – see for example Theorem 12.2.4 of [14] – that the isomorphism relation of a sentence $\Phi \in \mathcal{L}_{\omega_1\omega}$ is Borel if and only if, for some $\alpha < \omega_1$, $\equiv_\alpha$ is sufficient to decide isomorphism on models of Φ (the $\equiv_\alpha$'s are defined shortly). Since

$\equiv_0$ is implied by $\equiv$ and REF(bin) is a complete theory with more than one countable model, $\equiv_0$ does not decide isomorphism. We proceed by induction with a combined step and limit induction step. So suppose $\alpha_0 \leq \alpha_1 \leq \cdots$ are such that each $\equiv_{\alpha_n}$ does not decide isomorphism. That is, for each n, there is a pair A_n, B_n of countable models of REF(bin) which are nonisomorphic but where $A_n \equiv_{\alpha_n} B_n$. Let $\alpha = \sup\{\alpha_n + 1 : n \in \omega\}$. We will construct a pair (indeed, a large family) of countable models of REF(bin) that are pairwise $\equiv_\alpha$-equivalent but not isomorphic. This is sufficient.

Recall that among countable models M of REF(bin), the *color* of an element $a \in M$ is the size of its E_∞-class $[a]_\infty$. By adding an element to each finite E_∞ class occurring in A_n, B_n, respectively, we can suppose the color "1" does not occur in any of the A_n's, B_n's.

Let $\mathfrak{C} \models$ REF(bin) be the model with universe $2^\omega \times \omega$, where as usual $(\eta, n)E_k(\tau, m)$ if $\eta \restriction_k = \tau \restriction_k$. $\mathfrak{C}$ will serve as a 'monster model' of sorts; in particular we can suppose each A_n, B_n are (elementary) substructures of $\mathfrak{C}$.

We begin by forming a single countable model $M \preceq \mathfrak{C}$ that encodes all of the complexity of the models A_n, B_n. For $s \in 2^{<\omega}$, let $(A_n)^s$ be a 'shift' of A_n by s. Formally, $(A_n)^s = \{(s \frown \eta, j) : (\eta, j) \in A_n\}$ and we define $(B_n)^s$ analogously. Whereas the substructures A_n and $(A_n)^s$ of $\mathfrak{C}$ are certainly not elementarily equivalent, the relationships between A_n and B_n are maintained. That is, if $\lg(s) = \lg(t)$, then for any n, $(A_n)^s \equiv_{\alpha_n} (B_n)^t$, but because $A_n \not\cong B_n$, there is no elementary bijection $f : (A_n)^s \to (B_n)^t$.

As notation, for $i \in \{0, 1\}$ and $n \in \omega$, let S_n^i be the subset of 2^{2n+2} satisfying

- $s(j) = 0$ for every odd $j < 2n$;
- $s(2n) = i$; and

- $s(2n+1) = 1$.

Note that not only are the sets S^i_n disjoint, but in fact, $S^* := \bigcup\{S^i_n : i \in \{0,1\}, n \in \omega\}$ is a maximal antichain of $(2^{<\omega}, \subseteq)$ (with respect to incomparability). Let

$$M := \bigcup_{n<\omega, s \in S^0_n} (A_n)^s \cup \bigcup_{n<\omega, s \in S^1_n} (B_n)^s$$

It is readily checked that $M \preceq \mathfrak{C}$. Because every element of every A_n, B_n has color distinct from 1, no element of M has color 1 either. As notation, we refer to the subsets $(A_n)^s$ and $(B_n)^s$ as the *s-bubbles* of M. Obviously, for a specific choice of s, M contains only one of $(A_n)^s$ or $(B_n)^s$. We write $M(s)$ for this s-bubble.

For each $x \in 2^\omega$, let $x^* \in 2^\omega$ be defined by $x^*(j) = 0$ if j is odd and $x^*(j) = x(j/2)$ if j is even. For each countable, dense subset $X \subseteq 2^\omega$, let

$$M_X = M \cup \{(x^*, 0) : x \in X\} \quad \text{and let} \quad S^*_X = S^* \cup \{x^* : x \in X\}$$

Clearly, $M \preceq M_X \preceq \mathfrak{C}$ and an element $c \in M_X$ has color 1 if and only if $c \notin M$. Write $M_X(s) = M(s)$ for $s \in S^i_n$.

<u>Claim 1:</u> Let $X, Y \subset 2^\omega$ be countable and dense. Then $M_X \cong M_Y$ if and only if $X = Y$.

<u>Proof:</u> If $X = Y$ then $M_X = M_Y$. On the other hand, suppose X and Y are dense and $f : M_X \cong M_Y$. We claim that for all $\eta \in X$, $f(\eta^*, 0) = (\eta^*, 0)$. This suffices, since then $X = \{\eta : (\eta^*, n) \in M_X \text{ iff } n = 0\} \subseteq \{\eta : (\eta^*, n) \in M_Y \text{ iff } n = 0\} = Y$ and by symmetry $Y \subseteq X$.

So fix $\eta \in X$ and write $f(\eta^*, 0) = (\tau^*, 0)$ where $\tau \in Y$ ($f(\eta^*, 0)$ must be of this form since it is of color 1 in M_Y). Suppose towards a contradiction that $\eta \neq \tau$; let n be least such that $\eta(n) \neq \tau(n)$. Let $s = \eta^* \restriction_{2n+1} {}^\frown(1)$ and let $t = \tau^* \restriction_{2n+1} {}^\frown(1)$. Then our purported isomorphism f would induce an elementary bijection between $(A_n)^s$ and $(B_n)^t$ (or between $(A_n)^t$ and $(B_n)^s$), which is impossible since $A_n \ncong B_n$.

$\square$

By contrast, we have:

<u>Claim 2:</u> Let $X, Y \subset 2^\omega$ be countable and dense. Then $M_X \equiv_\alpha M_Y$.

<u>Proof:</u> We recall that $M_X \equiv_\alpha M_Y$ if and only if Player II has a winning strategy in the following game $\mathcal{G}(M_X, M_Y, \alpha)$:

Players I and II alternate moves. On Player I's n-th turn, he either plays a pair (a_n, β_n) where $a_n \in M_X$ and β_n is an ordinal with $\alpha > \beta_0 > \ldots > \beta_n$, or else he plays a pair (b_n, β_n), where $b_n \in M_Y$ and β_n is an ordinal with $\alpha > \beta_0 > \ldots > \beta_n$. (Really Player I should also specify which of M_X, M_Y he is playing in, but we suppress this). On Player II's n-th turn, she plays either $b_n \in M_Y$ or $a_n \in M_X$, depending on Player I's move; she is required to make sure that $(a_0, \ldots, a_n) \mapsto (b_0, \ldots, b_n)$ is partial elementary from M_X to M_Y. This specifies the game, since Player I cannot survive indefinitely (in other words, the first person to have no legal moves loses).

Now for each $n < \omega$, we are assuming that $A_n \equiv_{\alpha_n} B_n$, where $(\alpha_n : n < \omega)$ is increasing (possibly not strictly), and $\alpha = \sup\{\alpha_n + 1 : n < \omega\}$. Fix a winning strategy Γ_n for Player II in the game $\mathcal{G}(A_n, B_n, \alpha_n)$. Given $s \in S^0_n, t \in S^1_n$, let $\Gamma_{s,t} = \Gamma_{t,s}$ be the corresponding strategy for the game $\mathcal{G}((A_n)^s, (B_n)^t, \alpha_n)$. For $s, t \in S^0_n$, $(A_n)^s \cong (A_n)^t$; use this to get $\Gamma_{s,t}$, a winning strategy for Player II in the game $\mathcal{G}((A_n)^s, (A_n)^t, \infty)$. Similarly define $\Gamma_{s,t}$ for $s, t \in S^1_n$.

We now describe a winning strategy Γ for Player II in the game $\mathcal{G}(M_X, M_Y, \alpha)$.

Case 1: suppose Player I plays (a_0, β_0) where $a_0 = (\eta^*, 0)$ for some $\eta \in X$. Choose n large enough so that $\alpha_n \geq \beta_0$. By back-and-forth, we can choose a tree isomorphism $F : (2^{<\omega} \cup X, \subseteq) \cong (2^{<\omega} \cup Y, \subseteq)$ such that F is the identity on 2^n. Note that F induces a tree isomorphism $F^* : S^*_X \to S^*_Y$ defined by $F^*(s^*) = F(s)^*$. On the first move, Player II plays $(F^*(\eta^*), 0)$.

On subsequent moves:

If Player I plays $((\nu^*, 0), \beta)$ where $\nu \in X$, then Player II plays $(F^*(\nu^*), 0)$.

If Player I plays $((\nu^*, 0), \beta)$ where $\nu \in Y$, then Player II plays $((F^*)^{-1}(\nu^*), 0)$.

If Player I plays $((\nu, k), \beta)$, where $(\nu, k) \in M_X(s)$ for some $s \in S^0_m \cup S^1_m$, then Player II plays according to $\Gamma_{s,F^*(s)}$, where we take as input all the previous moves that took place in $M_X(s)$ and $M_Y(F^*(s))$. This will be valid, since either $m \leq n$, in which case $\Gamma_{s,F^*(s)}$ actually describes an isomorphism, or else $m > n$, and so the ordinals involved in the relevant previous moves will all be less than $\beta_0 \leq \alpha_n$.

If Player I plays $((\nu, k), \beta)$, where $(\nu, k) \in M_Y(s)$ for some $s \in S^0_n \cup S^1_n$, then Player II plays according to $\Gamma_{(F^*)^{-1}(s),s}$, where we take as input all the previous moves that took place in $M_X((F^*)^{-1}(s))$ and $M_Y(s)$.

Case 2: Suppose Player I plays (a_0, β_0) where $a_0 \in M_X(s)$ for some $s \in S^0_N \cup S^1_N$. Choose $n \geq N$ such that $\alpha_n \geq \beta_0$. By back-and-forth, we can choose a tree isomorphism $F : (2^{<\omega} \cup X, \subseteq) \cong (2^{<\omega} \cup Y, \subseteq)$ such that F is the identity on 2^n. From F we obtain $F^* : S^*_X \to S^*_Y$ as in Case 1. On the first move, Player II plays according to $\Gamma_{s,s}$, and afterwards plays as in Case 1.

The remaining cases where Player I starts in M_Y are the same, just interchange the roles of X and Y. $\square$

With both claims finished, let $X \subset 2^\omega$ be the set of sequences which are eventually zero, and $Y \subset 2^\omega$ be the set of sequences which are eventually one. Then $M_X \equiv_\alpha M_Y$ and $M_X \ncong M_Y$. This completes the induction and the proof. $\square$

This gives the first known example of the following behavior:

Corollary 5.5.4. There is a complete first-order theory for whom isomorphism is neither Borel nor Borel complete.

Here, the example is REF(bin), the paradigmatic example of a superstable, non-ω-stable theory. Thus we might informally expect this behavior to be extremely common for such theories. Since isomorphism is not Borel, we cannot truly consider REF(bin) to be especially tame. However, the theory is relatively simple in the sense that it cannot code much infinitary behavior. We end with the following class of examples which follow naturally from this one:

Corollary 5.5.5. For any ordinal α with $2 \leq \alpha < \omega_1$, there is a first-order theory S_α whose isomorphism relation is not Borel, and where $\Phi_\beta \leq_B S_\alpha$ if and only if $\beta \leq \alpha$. If α is not a limit ordinal, we can arrange that S_α is complete.

For α not a limit ordinal, each of these theories is grounded, superstable, but not ω-stable.

Proof. By Theorem 5.3.23 (and the proof, noting what we get is grounded, superstable but not ω-stable). □

In [89], the following question was asked:

Question 2. Let α be 0 or 1. Is there a first-order theory S_α whose isomorphism relation is not Borel, and where $T_\beta \leq_B S_\alpha$ if and only if $\beta \leq \alpha$?

The instance of the above question for $\alpha = 0$ is precisely Vaught's conjecture for first-order theories. (A theory T has a perfect set of nonisomorphic models if and only if $T_1 \leq_B T$.) For $\alpha = 1$, abelian p-groups are an infinitary counterexample. Soon after the publication of [89], Mathew Harrison-Trainor answered the question positively [20] by giving a first-order presentation of abelian p-groups.

5.5.2 Infinite Branching

We now turn our attention to REF(inf) specifically, and prove the following theorem:

Theorem 5.5.4. REF(inf) is Borel complete. Indeed, for each infinite cardinal λ, REF(inf) is λ-Borel complete.

Proof. Let Φ be the $\mathcal{L}_{\omega_1\omega}$ sentence in the language $\{\leq\}$ describing ω-trees. By Theorem 3.11 of [47], Φ is λ-Borel complete for each λ, so it is enough to produce a λ-Borel reduction f from $\mathrm{Mod}_\lambda(\Phi)$ to $\mathrm{Mod}_\lambda(\mathrm{REF(inf)})$.

Call a subtree $S \subset \lambda^{<\omega}$ is *reasonable* if for every element $s \in S$, $\{\alpha < \lambda : s^\frown(\alpha) \notin S\}$ is infinite. We describe an operation $S \mapsto M_S$ sending reasonable subtrees of $\lambda^{<\omega}$ of size λ, to models of REF(inf) of size λ, such that $S \equiv_{\infty\omega} S'$ if and only if $M_S \equiv_{\infty\omega} M_{S'}$. It will then be routine to define a λ-Borel map $f : \mathrm{Mod}_\lambda(\Phi) \to \mathrm{Mod}_\lambda(\mathrm{REF(inf)})$, such that given $S' \in \mathrm{Mod}_\lambda(\Phi)$ there is some subtree $S \subset \lambda^{<\omega}$ reasonable with $S \cong S'$ and $f(S') \cong M_S$. Then f will be the desired reduction.

Let $I \subset \lambda^\omega$ be the set of all ω-sequences from λ which are eventually zero. For any set M satisfying

$$I \times \{0\} \subseteq M \subseteq I \times \{0, 1\}$$

if we construe M as an $\mathcal{L} = \{E_n : n \in \omega\}$-structure by the rule $E_n((\eta, i), (\nu, j))$ if and only if $\eta\restriction_n = \nu\restriction_n$, then M is a model of REF(inf).

So, given a reasonable subtree $S \subset \lambda^{<\omega}$ of size λ, let M_S be the $\mathcal{L}$-structure whose universe is

$$(I \times \{0\}) \cup \{(\eta, 1) : t \frown (1) \subset \eta \text{ for some } t \in \lambda^{<\omega} \setminus S\}$$

We check that the mapping $S \mapsto M_S$ works.

To see this, we describe an inverse operation. Given any $\mathcal{L}$-structure M whose universe satisfies $I \times \{0\} \subseteq M \subseteq I \times \{0,1\}$, let

$$Tr(M) = \{s \in \lambda^{<\omega} : \forall \alpha < \lambda \exists \eta \in \lambda^{\omega} \, [s \frown (\alpha) \subset \eta \text{ and } (\eta, 1) \notin M]\}$$

We first argue that for any subtree $S \subseteq \lambda^{<\omega}$, we have $Tr(M_S) = S$. Indeed, suppose $s \in S$. Choose $\alpha \in \lambda$ arbitrarily. Then the element $\eta := s \frown (\alpha) \frown \overline{0}$ of I witnesses that $s \in Tr(M_S)$. Conversely, if $s \notin S$ then as $(\eta, 1) \in M_S$ for every $\eta \supset s \frown (1)$, $s \notin Tr(M_S)$.

Thus, in particular, $Tr(M_S)$ is a subtree of $\lambda^{<\omega}$ whenever S is.

Claim: For any subtrees S, T of $\lambda^{<\omega}$, if the $\mathcal{L}$-structures $M_S \equiv_{\infty,\omega} M_T$, then $(S, \subseteq) \equiv_{\infty,\omega} (T, \subseteq)$.

Proof: Assume $M_S \equiv_{\infty,\omega} M_T$. Pass to a forcing extension $\mathbb{V}[G]$ in which $\lambda^{\mathbb{V}}$ is countable. Choose an $\mathcal{L}$-isomorphism $f : M_S \to M_T$. This induces a tree isomorphism $f^* : (Tr(M_S), \subseteq) \to (Tr(M_T), \subseteq)$. Combined with the computation above, $(S, \subseteq)$ and $(T, \subseteq)$ are isomorphic in $\mathbb{V}[G]$, so they are back-and-forth equivalent in $\mathbb{V}$. □

To complete the proof, suppose two reasonable subtrees satisfy $(S, \subseteq) \equiv_{\infty\omega} (T, \subseteq)$. Pass to a forcing extension wherein λ is countable, so that $S \cong T$. Then, since S and T are reasonable, we can choose a tree automorphism $f : (\lambda^{<\omega}, \subseteq) \cong (\lambda^{<\omega}, \subseteq)$ that carries S to T. Then clearly f induces an $\mathcal{L}$-isomorphism from M_S to M_T. This implies that the $\mathcal{L}$-structures M_S and M_T are back-and-forth equivalent in the ground model.

□

The following Corollary follows immediately from Corollary 5.3.16, Proposition 5.5.1, and Theorem 5.5.4.

Corollary 5.5.6. REF(inf) is not short. Indeed, REF(inf) has class-many $\equiv_{\infty\omega}$-inequivalent models in $\mathbb{V}$.

5.6 ω-Stable Examples

Here we discuss two more first-order theories whose isomorphism relations are not Borel, but where one is Borel complete, and the other does not embed Φ_3. Interestingly, both are extremely similar model-theoretically. Both are ω-stable with quantifier elimination, and have ENI-NDOP and eni-depth 2, which together give a strong structure theorem in terms of stability theory.[2]

Let us define the theories. The first, K, is due to Koerwien and constructed in [42]. The language has unary sorts U, V_i, and C_i, as well as unary functions S_i and π_i^j for $i \in \omega$ and $j \leq i+1$. The axioms are as follows:

- The sorts U, V_i, and C_i are all disjoint. U and each of the V_i are infinite, but each C_i has size 2.
- π_i^{i+1} is a function from V_i to U; π_i^j is a function from V_i to C_j when $j \leq i$.
- For each tuple $\bar{c} = (c_0, \ldots, c_i)$ and each $u \in U$, $\pi_i^{-1}(\bar{c}, u)$ is nonempty. Here π_i refers to the product map $\pi_i^0 \times \cdots \times \pi_i^{i+1} : V_i \to C_0 \times \cdots C_i \times U$.
- S_i is a unary successor function from V_i to itself, and $\pi_i \circ S_i = \pi_i$.

We have a few remarks. Typically we will drop the subscript on π_i and S_i if it is clear from context. There is a slight ambiguity about the sorts, whether one works in traditional first-order logic (and thus there may be "unsorted" elements) or in multisorted

[2] In [47], an attempt is made to characterize which first-order ω-stable theories are Borel complete, using the dividing lines: ENI-DOP vs ENI-NDOP, and eni-deep vs eni-shallow. In particular, it is shown that any ω-stable theory which either has ENI-DOP or is eni-deep is Borel complete; and if an ω-stable theory has both ENI-NDOP and is eni-shallow, then it has fewer then $\beth_{\omega_1}$-many models up to back-and-forth equivalence.

logic (where there will not be). Since the unsorted elements never have any effect other than to complicate notation, we work in multisorted logic.

The properties of K have been well studied by Koerwien in [42]; we summarize his findings here:

Theorem 5.6.1. K is complete with quantifier elimination. It is ω-stable, has ENI-NDOP, and is eni-shallow of eni-depth 2. Furthermore, the isomorphism relation for K is not Borel.

The proof in [42] that isomorphism for K is not Borel is rather involved, but one can imitate Theorem 5.5.3 to give a much simpler proof.

Our other theory is a tweak of K, so we call it TK (T is for "tweaked"). The language is slightly different; we have unary sorts U, V_i, and C_i as before, but have unary functions S_i, π_i^0, π_i^1, and τ_{i+1} for $i \in \omega$. The axioms are as follows:

- The sorts U, V_i, and C_i are all disjoint. U and each of the V_i are infinite, but each C_i has size 2^i.
- τ_{i+1} is a surjection from C_{i+1} to C_i where, for all $c \in C_i$, $|\tau_{i+1}^{-1}(c)| = 2$.
- π_i^1 is a function from V_i to U; π_i^0 is a function from V_i to C_i.
- For each tuple $c \in C_i$ and each $u \in U$, $\pi_i^{-1}(c, u)$ is nonempty. Here π_i refers to the product map $\pi_i^0 \times \pi_i^1 : V_i \to C_i \times U$.
- S_i is a unary successor function from V_i to itself, and $\pi_i \circ S_i = \pi_i$.

The preceding notes also apply to K. The behavior is extremely similar, and essentially the same proofs of basic properties of K apply to TK. We summarize this now:

Theorem 5.6.2. TK is complete with quantifier elimination. It is ω-stable, has ENI-NDOP, and is eni-shallow of eni-depth 2.

We can easily see that both K and TK have relatively few models up to back-and-forth equivalence:

Proposition 5.6.1. $I_{\infty,\omega}(K) = I_{\infty,\omega}(TK) = \beth_2$.

Indeed, every model M of either theory has a submodel N where $M \equiv_{\infty\omega} N$ and $|N| \leq \beth_1$.

Proof. Let T be either K or TK. For the proof of the proposition we can restrict attention to models of T with a fixed algebraic closure of the empty set $\bigcup_i C_i$. If $T = K$, then let C be all finite sequences $(a_j : j < i)$ with $i > 0$ and with each $a_j \in C_j$; if $T = TK$ then let $C = \bigcup_i C_i$.

We first show $I_{\infty,\omega}(T) \geq \beth_2$. For each $\eta \in 2^\omega$, let u_η be some element which will eventually be part of U in some model of T. For any $n \in \omega$ and any $c \in C$ where $\pi_n^{-1}(c, u_\eta)$ is nonempty, we insist the S_n-dimension of $\pi_n^{-1}(c, u_\eta)$ be infinite if $\eta(n) = 1$, or equal to one otherwise. (If $T = K$, then $\pi_n^{-1}(c, u_\eta)$ is nonempty if and only if $\lg(c) = n$; if $T = TK$ then $\pi_n^{-1}(c, u_\eta)$ is nonempty if and only if $c \in C_n$.) For any infinite $X \subseteq 2^\omega$, define M_X to have $U^{M_X} = \{u_\eta : \eta \in X\}$ with the described behavior of the V_i and S_i. Evidently if $Y \subseteq 2^\omega$ is infinite and $X \neq Y$, then for any $\eta \in X \setminus Y$, there is no $\nu \in Y$ where $(M_X, u_\eta) \equiv_{\infty\omega} (M_Y, u_\nu)$, and symmetrically. Thus, $M_X \not\equiv_{\infty\omega} M_Y$. Since there are $\beth_2$ infinite subsets of 2^ω, $I_{\infty\omega}(T) \geq \beth_2$.

That $I_{\infty\omega}(T) \leq \beth_2$ follows immediately from the second claim. So let M be some model of T, of any particular cardinality. We begin by stripping down the V_i. For every $u \in U$ and $c \in C$, if $\pi^{-1}(c, u)$ is uncountable, drop all but a countable S-closed subset of infinite S-dimension. Do this for all pairs (c, u). The result is $\equiv_{\infty\omega}$-equivalent to the original by an easy argument, and $\pi^{-1}(c, u)$ is now always countable.

Next we need only ensure that U has size at most continuum. So put an equivalence relation E on U, where we say uEu' holds if and only if, for all $c \in C$, the dimensions of $\pi^{-1}(c, u)$ and $\pi^{-1}(c, u')$ are equal. If any E-class is uncountable, drop all but a countably infinite subset; the resulting structure is $\equiv_{\infty\omega}$-equivalent to the original again. Further, each E-class is now countable, and there are only $|C^\omega| = \beth_1$ possible E-classes, so the structure now has size at most $\beth_1$. This completes the proof. $\square$

Any additional complexity of either theory comes from elementary permutations of the algebraic closure of the empty set. In any model M of either K or TK, $\mathrm{acl}_M(\emptyset) = \bigcup_{i \in \omega} C_i(M)$. In models M of TK, the projection functions $\{\tau_i\}$ naturally induce a tree structure, so we think of $\mathrm{acl}_M(\emptyset)$ as being a copy of $(2^{<\omega}, \leq)$. In models M of K, as each $C_i(M)$ has exactly two elements, so one can think of $\mathrm{acl}_M(\emptyset)$ as being indexed by $2 \times \omega$. Note, however, there is some freedom with all this; for our purposes, $\mathrm{acl}_M(\emptyset)$ could equally well be viewed as any subset of $\mathrm{acl}_{M^{eq}}(\emptyset)$ whose definable closure contains $\mathrm{acl}_M(\emptyset)$ (here M^{eq} is the result of eliminating imaginaries from M). In the case $M \models K$ it is most convenient to say that $\mathrm{acl}_M(\emptyset)$ is all finite sequences $\langle a_j : j < i \rangle$, where each $a_j \in C_j(M)$. These finite sequences, when ordered by initial segment, also give a natural correspondence of $\mathrm{acl}_M(\emptyset)$ with the tree $(2^{<\omega}, \leq)$. Henceforth, when discussing models M of either K or TK, we will view $\mathrm{acl}_M(\emptyset)$ as being indexed by the tree $(2^{<\omega}, \leq)$.

Next, we discuss the group G of elementary permutations of $\mathrm{acl}_M(\emptyset)$ (which only depends on the theory). For K, the relevant group is $(2^\omega, \oplus)$, the direct product of ω copies of the two-element group. Indeed, in any model of K, any elementary permuation of $\mathrm{acl}_M(\emptyset)$ is determined by the sequence of permutations of $C_i(M)$. In TK, as elementary permutations just have to respect the τ_i structure, the relevant group of elementary permutations is $\mathrm{Aut}(2^{<\omega}, \leq)$. Both of these groups are compact (in fact this is true for all first order theories), but only the group for K is abelian. It turns out that being abelian is enough to produce relative simplicity, while being nonabelian leaves enough room to allow TK to be Borel complete.

For the next proposition we need some setup.

Let X be the Polish space of all $f : 2^{<\omega} \to (\omega + 1 \backslash \{\emptyset\})$. Let T be either K or TK, and let G be either $(2^\omega, \oplus)$ or $\mathrm{Aut}(2^{<\omega}, \leq)$, respectively. G acts on $2^{<\omega}$ naturally: if $G = (2^\omega, \oplus)$, then $g \cdot \sigma = g\restriction_{|\sigma|} \oplus \sigma$. If $G = \mathrm{Aut}(2^{<\omega}, \leq)$, then $g \cdot \sigma = g(\sigma)$. From this we get an action of G on X: namely for $f \in X, g \in G$, $(g \cdot f)(\sigma) = f(g^{-1} \cdot \sigma)$. This is an HC-forcing invariant, continuous action in the sense of Definition 5.4.1. Let E_G be the equivalence relation on X induced by the action, as well as the equivalence relation on $\mathcal{P}_{\aleph_1}(X)$ induced by the diagonal action.

Now G acts diagonally on X^ω also; this action commutes with the permuation action of S_∞ on X^ω. So $G \times S_\infty$ acts naturally on X^ω; let $E_{G \times S_\infty}$ be the equivalence relation induced by this action.

Proposition 5.6.2. Let T be either K or TK. Then:

- $(\mathrm{Mod}(T), \cong) \sim_B (X^\omega, E_{G \times S_\infty})$.
- $(\mathrm{Mod}(T), \cong) \sim_{\mathrm{HC}} (\mathcal{P}_{\aleph_1}(X), E_G)$.

Proof. For the various codings below, fix a pairing function $\langle \cdot, \cdot \rangle : (\omega + 1 \backslash \emptyset)^2 \to (\omega + 1 \backslash \emptyset)$. Note that one difference between K and TK that frequently affects the coding is: $\pi^{-1}(\emptyset, u)$ is only nonempty for models of TK.

To show $(\mathrm{Mod}(T), \cong) \leq_B (X^\omega, E_{G \times S_\infty})$, first let $M \in \mathrm{Mod}(T)$ be arbitrary. We may choose an indexing of $\mathrm{acl}_M(\emptyset)$ by $2^{<\omega}$, and of U^M by ω, using the original indexing of the universe of M by ω. Then each element $u \in U^M$ induces a function $c_u \in X$, where $c_u(\sigma)$ is the S-dimension of $\pi^{-1}(\sigma, u)$. (In the case of $T = $ K, define $c_u(\emptyset) = 1$.) Then take M to the sequence $(c_{u_n} : n \in \omega)$, where u_n is the n-th element of U. It is clear that this works.

To show $(X^\omega, E_{G\times S_\infty}) \leq_B (\mathrm{Mod}(T), \cong)$, fix a sequence $\overline{x} = (x_n : n \in \omega)$. We describe the case for the theory K. We define $M_{\overline{x}}$ to have $U^{M_{\overline{x}}} = \{4n : n \in \omega\}$, and have $C_i^{M_{\overline{x}}} = \{4i+1, 4i+2\}$. Then, using the infinitely many remaining elements, we arrange that for each $\sigma \in 2^{<\omega}\backslash\emptyset$ and each $n < \omega$, the S-dimension of $\pi^{-1}(\sigma_*, 4n)$ is $\langle x_n(\sigma), x_n(\emptyset)\rangle$, where $\sigma_* = (4i+1+\sigma(i) : i < \lg(\sigma))$. The case for TK is similar.

We have shown that $(\mathrm{Mod}(T), \cong) \sim_B (X^\omega, E_{G\times S_\infty})$. It follows that they are $\leq_{\mathrm{HC}}$-biembeddable; so to conclude the proof of the proposition it suffices to show that $(X^\omega, E_{G\times S_\infty}) \sim_{\mathrm{HC}} (\mathcal{P}_{\aleph_1}(X), E_G)$.

To show $(X^\omega, E_{G\times S_\infty}) \leq_{\mathrm{HC}} (\mathcal{P}_{\aleph_1}(X), E_G)$, we just need to handle multiplicities. So fix $\overline{x} = (x_n : n \in \omega) \in X^\omega$, and for each n, define $m_{\overline{x}}(n)$ to be $|\{m : x_m = x_n\}|$. For each $n \in \omega$ let $y_n \in X$ be defined by: $y_n(\sigma) = \langle x_n(\sigma), m_{\overline{x}}(n)\rangle$. Then $\overline{x} \mapsto \{y_n : n < \omega\}$ works.

We define a reverse embedding $f : (\mathcal{P}_{\aleph_1}(X), E_G) \leq_{\mathrm{HC}} (X^\omega, E_{G\times S_\infty})$ as follows (where recall that we do not require f to be single-valued). Namely, given $A \subset X$ countable and given $\overline{x} \in X^\omega$, put $(A, \overline{x}) \in f$ whenever $\overline{x}$ is an infinite-to-one enumeration of A. Also put $(\emptyset, \overline{x}) \in f$ for some fixed injective $\overline{x} \in X^\omega$. $\square$

5.6.1 Koerwien's Example

For this subsection, we show that $\Phi_2 \leq_B$ K, and that $\|\Phi_2\|_{\mathrm{ptl}} = \beth_2$. The former is quite straightforward:

Proposition 5.6.3. $\Phi_2 \leq_B$ K, i.e. $F_2 \leq_B K$.

Proof. Let $X \subset 2^\omega$ be countable; we describe a model $M_X \models$ K from which X can be easily recovered. Let U be the set $A \cup X$, where A is some countable infinite set which is disjoint from X. Let $C_i = \{c_0^i, c_1^i\}$. For each tuple $(a, \overline{c})$ with $a \in A$, arrange that $\pi^{-1}(a, \overline{c})$ has S-dimension 1. For each tuple $(x, \overline{c})$ with $x \in X$, arrange that $\pi^{-1}(x, \overline{c})$ has S-dimension $x(|\overline{c}|) + 2$. Clearly $M_X \cong M_Y$ if and only if $X = Y$.

Now it is not hard, given $\overline{x} = (x_n : n \in \omega) \in (2^\omega)^\omega$, to produce in a Borel fashion a model $M_{\overline{x}} \models \mathrm{K}$ with universe ω, such that $M_{\overline{x}} \cong M_{\{x_n : n \in \omega\}}$. This gives a Borel reduction from $((2^\omega)^\omega, F_2)$ to $(\mathrm{Mod}(\mathrm{K}), \cong)$, which suffices (see the discussion preceding Theorem 5.5.1). $\square$

Theorem 5.6.3. $\|K\| = \beth_2$. Therefore, K is not Borel complete; indeed, there is no Borel embedding of Φ_3 into Mod(K).

Proof. That $\|K\| \geq \beth_2$ follows immediately from Proposition 5.6.1. Since $G = (2^\omega, \oplus)$ is compact and abelian, $\|(\mathcal{P}_{\aleph_1}(X), E_G)\| \leq \beth_2$ by Theorem 5.4.8. As $\|\mathrm{K}\| = \|(\mathcal{P}_{\aleph_1}(X), E_G)\|$ by Proposition 5.6.2, we conclude that $\|K\| = \beth_2$. That there is no Borel embedding of Φ_3 into Mod(K) is immediate from Theorem 5.3.11(2) and Proposition 5.3.21. $\square$

Once we have one such example, we can apply the usual constructions to get a large class of ω-stable examples:

Corollary 5.6.4. For each non-limit ordinal α with $2 \leq \alpha < \omega_1$, there is an ω-stable theory S_α whose isomorphism relation is not Borel and where $T_\beta \leq_B S_\alpha$ if and only if $\beta \leq \alpha$.

Proof. By Theorem 5.3.23 (and its proof). $\square$

There is no such example when $\alpha = 0$, since Vaught's Conjecture holds for ω-stable theories. (T has a perfect set of nonisomorphic models if and only if $T_1 \leq_B T$, and so whenever T is ω-stable, either $T \leq_B T_0$ or $T_1 \leq_B T$.) It is open if there is an example when $\alpha = 1$ (the example provided by Harrison-Trainor in [20] is not ω-stable).

5.6.2 A New ω-Stable Theory

We now consider TK, with the aim of showing it is Borel complete. Indeed with Proposition 5.6.2 we have already shown $(\mathrm{Mod}(\mathrm{TK}), \cong)$ is Borel equivalent to $(X^\omega, E_{G\times S_\infty})$, where X is the space of all $c : 2^{<\omega} \to \omega$. (We are replacing $\omega+1\backslash\emptyset$ with ω, which is harmless.) Recall that $G = \mathrm{Aut}(2^{<\omega}, \leq)$ acts on X by permuting the fibers; that is, for any $c : 2^{<\omega} \to \omega$, any $g \in G$, and any $\sigma \in 2^{<\omega}$, $(g \cdot c)(\sigma) = c(g^{-1} \cdot \sigma)$. Then G acts on X^ω diagonally, while S_∞ acts on X^ω by permuting the fibers, so these actions commute with one another and induce an action of the product group $G \times S_\infty$.

Thus, to show TK is Borel complete, it is enough to show $(X^\omega, E_{G\times S_\infty})$ is Borel complete, which we do directly.

Theorem 5.6.4. $(\mathrm{Graphs}, \cong) \leq_B (X^\omega, E_{G\times S_\infty})$.

Proof. To simplify notation, for the whole of this proof we write E in place of $E_{G\times S_\infty}$. We need some setup first. Observe that G naturally acts on 2^ω, the set of branches of $(2^{<\omega}, \leq)$, by $g \cdot \sigma = \bigcup_n g \cdot \sigma\restriction_n$; this is a well-defined sequence precisely because g is a tree automorphism. Let $\{D_i : i \in \omega\}$ be a countable set of dense, disjoint, countable subsets of 2^ω, and let $D = \bigcup_i D_i$. We need one claim, where we use the relative complexity of G (it would not go through if we replaced TK with K):

<u>Claim:</u> For any $\sigma \in S_\infty$, there is some $g \in G$ where for all $i \in \omega$, $g \cdot D_i = D_{\sigma(i)}$ as sets.

<u>Proof:</u> We construct g by a back-and-forth argument. So let $\mathcal{F}$ be the set of finite partial functions from D to itself, satisfying all the following:

- For each $f \in \mathcal{F}$ and each $\eta \in \mathrm{dom}(f)$, if $\eta \in D_i$, then $f(\eta) \in D_{\sigma(i)}$.

- For each $f \in \mathcal{F}$ and each $\eta, \nu \in \mathrm{dom}(f)$, $\mathrm{lg}(\eta \wedge \nu) = \mathrm{lg}(f(\eta) \wedge f(\nu))$, where $\eta \wedge \nu$ denotes the longest common initial segment of η and ν.

- The previous conditions, but with f^{-1} and σ^{-1} instead of f and σ.

Suppose we have established that $\mathcal{F}$ is a back-and-forth system, then $\mathcal{F}$ defines $g \in G$ with the desired property. Choose a bijection $f : D \to D$ such that the finite restrictions of f all lie in $\mathcal{F}$. If $s \in 2^n$, let $g(s)$ be $f(\eta)\restriction_n$ for any η extending s; because of the consistency properties of $\mathcal{F}$, and because D is dense, this is well-defined. Then clearly $g \in G$ has the desired property with respect to σ.

So we need only show that $\mathcal{F}$ is a back-and-forth system. Of course the empty function is in $\mathcal{F}$. So say $f \in \mathcal{F}$ and $\eta \in 2^\omega$; we want $f' \supset f$ in $\mathcal{F}$ with $\eta \in \mathrm{dom}(f')$. The case where f is empty is easy, so suppose f is nonempty. We also assume $\eta \notin \mathrm{dom}(f)$ already. Let n be maximal among $\{\lg(\eta \wedge \nu) : \nu \in \mathrm{dom}(f)\}$, and let $\nu \in \mathrm{dom}(f)$ be such that $\lg(\eta \wedge \nu) = n$. We then pick an element $f(\eta)$ of 2^ω which extends $f(\nu)\restriction_n {}^\frown (1 - f(\nu)(n))$. That is, $f(\eta)$ agrees with $f(\nu)$ before stage n, but disagrees with it at n. If $\eta \in D_i$, choose this element from $D_{\sigma(i)}$, which is possible by density. This clearly satisfies the desired properties, and the other direction is symmetric, proving the claim. $\square$

Given $\eta, \tau \in 2^\omega$ and $k \in \{1, 2, 3\}$, let $c^k_{\eta,\tau} : 2^{<\omega} \to \omega$ be the coloring which sends $s \in 2^{<\omega}$ to k, if $s \subset \eta$ or $s \subset \tau$, or 0 otherwise. Also, fix a bijection $\rho : \omega \to \bigcup_{i \leq j} D_i \times D_j$. We have now fixed enough notation and can describe our map $f : \mathrm{Graphs} \to X^\omega$.

Let R be a graph on ω – that is, R is a binary relation on ω which is symmetric and irreflexive. For each $n \in \omega$, $\rho(n)$ is a pair $(\eta, \tau) \in D_i \times D_j$ for some $i \leq j$. If $i = j$, define $c_n = c^1_{\eta,\tau}$. If $i < j$ and $\{i, j\} \in R$, then let $c_n = c^2_{\eta,\tau}$. Otherwise let $c_n = c^3_{\eta,\tau}$. Then put $f(R) := (c_n : n \in \omega)$. $f(R)$ is a visibly element of X^ω, and clearly f is Borel. Note also that f is injective.

Suppose $\sigma : (\omega, R) \cong (\omega, R')$ is a graph isomorphism. We show that $f(R) E f(R')$. By the claim, there is a $g \in G$ where for all $i \in \omega$, $g \cdot D_i = D_{\sigma(i)}$. Let A be the range of

$f(R)$ and let A' be the range of $f(R')$. We show that $g \cdot A = A'$ setwise. First suppose $c^1_{\eta,\tau} \in A$. Let i be such that $\eta, \tau \in D_i$, so $g(\eta), g(\tau) \in D_{\sigma(i)}$. Then $g \cdot c^1_{\eta,\tau} = c^1_{g(\eta),g(\tau)} \in A'$. Similarly if $c^2_{\eta,\tau} \in A$, there is some $i < j$ where $\eta \in D_i$, $\tau \in D_j$, and $\{i, j\} \in R$. Since $\sigma : (\omega, R) \to (\omega, R')$ is a graph isomorphism, $\{\sigma(i), \sigma(j)\} \in R'$, so $c^2_{g(\eta),g(\tau)} \in A'$ (this uses $c^2_{g(\eta),g(\tau)} = c^2_{g(\tau),g(\eta)}$). The case $c^3_{\eta,\tau} \in A$ is the same. Thus $g \cdot A \subset A'$; by a symmetric argument $g \cdot A = A'$. Since $g \cdot f(R)$ and $f(R')$ are both injective and they have the same range, some permutation of $g \cdot f(R)$ is equal to $f(R')$, i.e. $f(R)Ef(R')$.

It only remains to show that if $f(R)Ef(R')$, then $(\omega, R) \cong (\omega, R')$. So suppose $f(R)Ef(R')$. Let A be the range of $f(R)$ and let A' be the range of $f(R')$, and choose $g \in G$ such that $g \cdot A = A'$. Let $i < \omega$; then since for all $\eta, \tau \in D_i$, $c^1_{g(\eta),g(\tau)} \in A'$, we have that $g \cdot D_i = D_{\sigma(i)}$ for some $\sigma(i) < \omega$. I claim that $\sigma : (\omega, R) \cong (\omega, R')$. Indeed, for $i < j$, $(i, j) \in R$ if and only if there are $\eta \in D_i, \tau \in D_j$ with $c^2_{\eta,\tau} \in A$, which is the case if and only if there are $\eta \in D_{\sigma(i)}, \tau \in D_{\sigma(j)}$ with $c^2_{\eta,\tau} \in A'$, which is the case if and only if $(i, j) \in R'$. □

We have now shown:

Theorem 5.6.5. TK is Borel complete.

Proof. By Theorem 5.6.4, together with the fact that graphs are Borel complete. □

This resolves a few open questions, raised in [47]:

Corollary 5.6.5. The ω-stable theory TK is Borel complete, but does not have ENI-DOP and is not eni-deep. Indeed TK is not λ-Borel complete for any λ with $2^\lambda > \beth_2$, as $|\mathrm{CSS}(\mathrm{TK})_{\mathrm{sat}}| = \beth_2$.

Chapter 6: Borel Complexity, Thickness, and the Schröder-Bernstein Property

One limitation of potential cardinality is that there exist sentences Φ which are not short (i.e. $\|\Phi\| = \infty$) and yet Φ is not Borel complete. For example, let $\mathrm{TAG}_1 \in \mathcal{L}_{\omega_1\omega}$ describe torsion abelian groups. Using Ulm's classification of countable torsion abelian groups [86], we can identify $\mathrm{CSS}(\mathrm{TAG}_1)_{\mathrm{ptl}}$ with $\mathcal{P}(\mathrm{ON})$ (the class of all sets of ordinals). In particular TAG_1 is not short. But as mentioned above, Friedman and Stanley showed in [12] that TAG_1 is not Borel complete, and in fact that $\Phi_2 \not\leq_B \mathrm{TAG}_1$.

We can identify $\mathrm{CSS}(\Phi_2)_{\mathrm{ptl}}$ with $\mathcal{P}(\mathbb{R})$ (i.e., with $\mathcal{P}(\mathcal{P}(\omega))$). We wish to be able to apply a counting argument based to see that $\Phi_2 \not\leq_B \mathrm{TAG}_1$. The motivating idea is that $|\mathcal{P}(\mathbb{R}) \cap \mathbb{V}_{\omega_1}| = \beth_2$, but $|\mathcal{P}(\mathrm{ON}) \cap \mathbb{V}_{\omega_1}| = \beth_1$, and we should be able to conclude something from this.

The need for this becomes more pressing if we consider the Friedman-Stanley jumps of TAG_1. Namely, in Section 5.3.4 (of the previous chapter), for any sentence Φ of $\mathcal{L}_{\omega_1\omega}$ and for any $\alpha < \omega_1$, we defined the tower of Friedman-Stanley jump $J^\alpha(\Phi)$; the special case where $\Phi = \mathrm{Th}(\mathbb{Z}, S)$ gives the Friedman-Stanley tower $(\Phi_\alpha : \alpha < \omega_1)$. In general, provided Φ has infinitely many countable models, we can identify $\mathrm{CSS}(J^\alpha(\Phi))_{\mathrm{ptl}}$ with $\mathcal{P}^\alpha(\mathrm{CSS}(\Phi))_{\mathrm{ptl}}$, where $\mathcal{P}^\alpha$ is the powerset operation iterated α-many times, taking unions at limits.

In particular, for each $\alpha < \omega_1$, we can identify $\mathrm{CSS}(\Phi_\alpha)_{\mathrm{ptl}}$ with $\mathcal{P}^\alpha(\omega)$. Let $\mathrm{TAG}_{1+\alpha} = J^\alpha(\mathrm{TAG}_1)$ for each α; then we can identify each $\mathrm{CSS}(\mathrm{TAG}_\alpha)_{\mathrm{ptl}}$ with $\mathcal{P}^\alpha(\mathrm{ON})$.

Note that for $\alpha \geq 1$, $|\mathcal{P}^\alpha(\omega) \cap \mathbb{V}_{\omega_1}| = |\mathcal{P}^\alpha(\mathrm{ON}) \cap \mathbb{V}_{\omega_1}| = \beth_\alpha$, so we are led to suspect that $\Phi_{\alpha+1} \not\leq_B \mathrm{TAG}_\alpha$ for all $\alpha < \omega_1$. The proof in [12] that $\Phi_2 \not\leq_B \mathrm{TAG}_1$ does not generalize, and it seems that a counting argument is the most natural way to proceed.

In this chapter, I introduce a new invariant called thickness that captures these counting arguments. Namely, for each sentence $\Phi \in \mathcal{L}_{\omega_1\omega}$, we get the thickness spectrum $\tau(\Phi, \cdot)$ of Φ, a function from cardinals to cardinals; $\tau(\Phi, \cdot)$ is closely related to $|\mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap \mathbb{V}_{\lambda^+}|$, but might be slightly smaller. It follows immediately from the definition that for every λ, $\tau(\Phi, \lambda) \leq |\mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap \mathbb{V}_{\lambda^+}| \leq \beth_{\lambda^+}$, and $\tau(\Phi, \cdot)$ is monotonically increasing, and $\lim_{\lambda \to \infty} \tau(\Phi, \lambda) = \|\Phi\|$.

Recall that if T is a complete countable theory, then $I(T, \cdot)$ is the spectrum of T, where $I(T, \lambda)$ is the number of models of T of size λ, up to isomorphism. For classification theory of uncountable models, this turns out to be a very fruitful source of dividing lines, as exposed by Shelah in [75]. We view $\tau(\Phi, \cdot)$ as an analogue of this for countable model theory.

The definition of thickness is arranged so that it is a Borel-reducibility invariant:

Theorem 6.0.1. Suppose $\Phi \leq_B \Psi$. Then for every cardinal λ, $\tau(\Phi, \lambda) \leq \tau(\Psi, \lambda)$.

Thickness is not necessarily an $\leq_{\mathrm{HC}}$-reducibility invariant; we could have modified the definition of τ to make this so, but this would introduce technical complications at various stages. It is more convenient to refine $\leq_{\mathrm{HC}}$ slightly. In Section 6.1, we introduce a large family of reducibility notions between $\leq_B$ and $\leq_{\mathrm{HC}}$. These will be parametrized by robust Γ; these are countable subsets of the elementary diagram of $(\mathrm{HC}, \in)$ with some additional nice properties. In particular, if Γ is robust then Γ extends ZFC^-. Given some robust Γ, we will say that $\Phi \leq_\Gamma \Psi$ if there is some HC-definable injection $f : \mathrm{CSS}(\Phi) \to \mathrm{CSS}(\Psi)$ which is absolute to countable transitive models of Γ. Now, Borel reductions are absolute to transitive models of ZFC^-, and hence if $\Phi \leq_B \Psi$ then $\Phi \leq_{ZFC^-} \Psi$, and thus $\Phi \leq_\Gamma \Psi$ for all robust Γ. Further, it is easy to check that each $\leq_\Gamma \subseteq \leq_{\mathrm{HC}}$. The definition of thickness is arranged so that if $\Phi \leq_\Gamma \Psi$ for some robust Γ, then for every cardinal κ, $\tau(\Phi, \lambda) \leq \tau(\Psi, \lambda)$.

In fact, there is another important notion of reducibility along these lines, namely absolute Δ^1_2-reducibility, as first discussed in Chapter 9 of [22]. We will also have that if $\Phi \leq_{a\Delta^1_2} \Psi$, then for every cardinal λ, $\tau(\Phi, \lambda) \leq \tau(\Psi, \lambda)$.

As a first application of the definition of thickness, I show the following in Section 6.5 (using technical lemmas from Sections 6.3 and 6.4):

(I) For every $\alpha < \omega_1$ and for every cardinal λ, $\tau(\Phi_\alpha, \lambda) = \beth_\alpha$;

(II) For every $\alpha < \omega_1$ and for every regular strong limit λ, $\tau(\mathrm{TAG}_\alpha, \lambda) = \beth_\alpha(\lambda)$;

(III) For every Borel complete Φ and for every regular strong limit λ, $\tau(\Phi, \lambda) = \beth_{\lambda^+}$;

(IV) Moreover, (II) and (III) can be arranged to hold at every regular cardinal λ in a class-forcing extension of $\mathbb{V}$, without adding any reals.

Note that a regular strong limit cardinal is either $\aleph_0$ or inaccessible. In particular, $\tau(\Phi_\alpha, \aleph_0) = \tau(\mathrm{TAG}_\alpha, \aleph_0) = \beth_\alpha$; thus, we obtain that $\Phi_{\alpha+1} \not\leq_B \mathrm{TAG}_\alpha$ for all $\alpha < \omega_1$, generalizing the theorem of Friedman and Stanley in [12] that $\Phi_2 \not\leq_B \mathrm{TAG}_1$.

We present another application of the thickness machinery, namely to the Schröder-Bernstein property:

Say that a complete first order theory T has the Schröder-Bernstein property in the class of all models if whenever $M, N \models T$ are elementarily bi-embeddable, then they are isomorphic. This notion was originally introduced by Nurmagambetov [69], [68] (without the phrase "in the class of all models"), and further studied by Goodrick in several papers, including his thesis [17]. There, he proves that if T has the Schröder-Bernstein property, then T is classifiable of depth 1, i.e. $I(T, \aleph_\alpha) \leq |\alpha + \omega|^{2^{\aleph_0}}$ for all α.

We deviate from this set-up in two ways. First, we are interested in Schröder-Bernstein properties for countable structures (or generally for potential canonical Scott sentences). Second, it is convenient in applications to use the following notion of embedding.

Definition 6.0.2. Suppose $\mathcal{L}$ is a language, and M, N are $\mathcal{L}$-structures. Then say that $f : M \leq N$ is an embedding if it is a homomorphism; that is, f commutes with the function symbols, and if R is an n-ary relation, then $f[R^M] \subseteq R^N$.

This allows the most freedom. If one wanted to look at elementary embedding, then just Morleyize; generally, we can pass to $\mathcal{L}_{\omega_1\omega}$-definable expansions to get whatever notion we wanted.

Definition 6.0.3. Say that Φ has the Schröder-Bernstein property if for all $M, N \models \Psi$ countable, if $M \sim N$ then $M \cong N$.

Note that this is rather sensitive to the choice of language.

Some initial properties of the Schröder-Bernstein property are developed in Section 6.6.

In Section 6.7, we prove the following. $\kappa(\omega)$, the *ω'th Erdös cardinal*, be the least cardinal satisfying $\kappa \rightarrow (\omega)^{<\omega}_2$; $\kappa(\omega)$ cannot be proven to exist in ZFC, but it is relatively low in the hierarchy of large cardinal axioms.

Theorem 6.0.4. Assume $\kappa(\omega)$ exists, and suppose Φ has the Schröder-Bernstein property. Then for every cardinal λ, $\tau(\Phi, \lambda) \leq \lambda^{<\kappa(\omega)}$, so in particular $\mathrm{TAG}_1 \not\leq_B \Phi$.

We view this as a striking analogy to Goodrick's theorem, that if T has the Schröder-Bernstein property in the class of all models, then for all κ, $I(T, \aleph_\alpha) \leq |\alpha + \omega|^{2^{\aleph_0}}$.

In fact, in Section 6.6, we introduce the α-ary Schröder-Bernstein property for a given ordinal α, where the 0-ary Schröder-Bernstein property is the same as the Schröder-Bernstein property. In Section 6.7, we actually prove the following:

Theorem 6.0.5. Assume $\kappa(\omega)$ exists, and suppose Φ has the α-ary Schröder-Bernstein property. Then for every cardinal λ, $\tau(\Phi, \lambda) \leq \beth_\alpha(\lambda^{<\kappa(\omega)})$. Thus, if $\alpha < \omega_1$, then $\mathrm{TAG}_{\alpha+1} \not\leq_B \Phi$.

Motivated by this, we formulate the following conjecture. It is analogous to Shelah's Main Gap theorem for complete countable theories, which says that for all T, either $I(T, \lambda) = 2^\lambda$ for all λ, or else there is $\alpha < \omega_1$ such that always $I(T, \aleph_\beta) \leq \beth_\alpha(|\beta|)$.

Conjecture. Suppose there are sufficient large cardinals; for example, suppose there is a supercompact cardinal σ. Suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$. Then the following are equivalent:

1. There is some $\alpha < \omega_1$ such that for every cardinal λ, $\tau(\Phi, \lambda) \leq \beth_\alpha(\lambda^{<\sigma})$;
2. There is some $\alpha < \omega_1$ such that $\tau(\Phi, \sigma) \leq \beth_\alpha(\sigma)$;
3. $\tau(\Phi, \sigma) < \beth_{\sigma^+}$;
4. There is some inaccessible cardinal λ such that $\tau(\Phi, \lambda) < \beth_{\lambda^+}$;
5. Φ is not Borel complete.

We have shown all of the downward implications (i.e. $1 \to 2 \to 3 \to 4 \to 5$) but none of the upward implications.

The overall strategy of Shelah's proof of the Main Gap theorem was to find a syntactic equivalent to having $I(T, \lambda) = 2^\lambda$ for all λ; at present, we lack a syntactic understanding of any of the above conditions, and this seems to be the current barrier to progress.

6.1 A Plethora of Reducibility Notions

HC refers to the hereditarily countable sets. When speaking of definability properties of subsets of HC, it is customary practice of descriptive set theory to work in the codes. That is, we would fix a canonical coding of HC by the reals (i.e. a surjection from HC onto $\mathbb{R}$), and then say that a subset of HC is absolutely Δ^1_2 in the codes if its preimage under the coding operation is absolutely Δ^1_2. However, we will find it much more convenient to stay in HC.

We have seen one way of approaching this, with $\leq_{\mathrm{HC}}$-reducibility, as in [89] (see Chapter 5). In this chapter, we will be using somewhat more delicate arguments, for which $\leq_{\mathrm{HC}}$ is too general. We thus introduce a family of reducibility notions intermediate between $\leq_B$ and $\leq_{\mathrm{HC}}$, which are a priori better behaved. Actually, we have no examples of sentences Φ, Ψ of $\mathcal{L}_{\omega_1\omega}$ witnessing a distinction between any of these notions.

Given a cardinal κ, recall that $H(\kappa)$ denotes the set of sets of hereditary cardinality less than κ; so $\mathrm{HC} = H(\aleph_1)$.

Definition 6.1.1. Suppose Γ is a countable set of formulas of set theory with parameters from HC. Then Γ is robust if:

(I) For every regular cardinal κ, $H(\kappa) \models \Gamma$;

(II) Whenever $V \models \Gamma$ is transitive and $P \in V$ is a forcing notion, then P forces $V[\dot{G}] \models \Gamma$;

(III) Whenever $V \models \Gamma$ is transitive and $\kappa \in V$ is regular in V, then $(H(\kappa))^V \models \Gamma$;

(IV) $ZFC^- \subseteq \Gamma$.

For example, ZFC^- is robust. Also, note that if $V \models ZFC^-$, then either $(\mathrm{HC})^V = V$, or else ω_1^V exists and is regular and $(\mathrm{HC})^V = (H(\omega_1^V))^V$; so using (III), we have that in any case $(\mathrm{HC})^V \models \Gamma$.

It would be slightly nicer if we could replace (I) by $\mathbb{V} \models \Gamma$; but this causes problems with definability of truth.

We now prove several lemmas about robustness. First of all, we need the following:

Lemma 6.1.2. Suppose λ is a regular cardinal and $P \in H(\lambda)$ is a forcing notion. Suppose G is P-generic over $\mathbb{V}$. Then $H(\lambda)[G] = H(\lambda)^{\mathbb{V}[G]}$.

Proof. Clearly, $H(\lambda)[G] \subseteq H(\lambda)^{\mathbb{V}[G]}$. Conversely, suppose $a \in H(\lambda)^{\mathbb{V}[G]}$; we need to find a name for a in $H(\lambda)$. Let b be the transitive closure of $a \cup \{a\}$. Let rnk be foundation rank. Let $\gamma_* = \mathrm{rnk}(b) < \lambda^+$, and choose a surjection $f : \gamma_* \times \lambda \to b$, such that for all $(\gamma, \alpha) \in \gamma_* \times \lambda$, $\mathrm{rnk}(f(\gamma, \alpha)) \leq \gamma$.

Choose nice P-names $\dot{a}, \dot{b}, \dot{f}$ (not necessarily in $H(\lambda)$) such that $\mathrm{val}(\dot{a}, G) = a$, $\mathrm{val}(\dot{b}, G) = b$, and $\mathrm{val}(\dot{f}, G) = f$, and such that P forces the preceding holds.

The remainder of the argument takes place in $\mathbb{V}$. Note that we can construe $\mathcal{B}(P)$ as a subset of the powerset of P, and hence as a subset of $H(\lambda)$.

By induction on $\gamma < \gamma_*$, define nice P-names $(\dot{c}_{\alpha,\gamma} : \alpha < \lambda)$ in $H(\lambda)$. Namely, $\dot{c}_{\alpha,\gamma}$ has domain $\{\dot{c}_{\beta,\gamma'} : \beta < \lambda, \gamma' < \gamma\}$, and each $\dot{c}_{\alpha,\gamma}(\dot{c}_{\beta,\gamma'}) = \|\dot{f}(\beta, \gamma') \in \dot{f}(\alpha, \gamma)\|_{\mathcal{B}(P)}$, an element of $\mathcal{B}(P)$.Then $P \Vdash \dot{c}_{\gamma_*,0} = \dot{a}$, and $\dot{c}_{\gamma_*,0} \in H(\lambda)$, so we are done. □

Lemma 6.1.3. Suppose Γ is robust and $\mathbb{V}[G]$ is a forcing extension. Then Γ remains robust in $\mathbb{V}[G]$.

Proof. Say $\mathbb{V}[G]$ is a forcing extension by P.

We verify (I) holds in $\mathbb{V}[G]$: suppose κ is a regular cardinal in $\mathbb{V}[G]$. Choose $\lambda > \kappa$ regular, such that $P \in H(\lambda)$. Then $H(\lambda) \models \Gamma$, so $H(\lambda)[G] \models \Gamma$. Also, $H(\lambda)[G] = H(\lambda)^{\mathbb{V}[G]}$, by Lemma 6.1.2. Finally κ is regular in $\mathbb{V}[G]$, hence also in $H(\lambda)[G]$, so $(H(\kappa))^{H(\lambda)[G]} = (H(\kappa))^{\mathbb{V}[G]} \models \Gamma$.

(II), (III): for both, by downward Lowenheim-Skolem it is enough to check countable transitive models V, and so we can use Levy's absoluteness principle. □

The following key proposition follows immediately.

Proposition 6.1.4. Suppose Γ is robust. Then whenever $\mathbb{V}[G]$ is a forcing extension of $\mathbb{V}$, we have that $(\mathrm{HC})^{\mathbb{V}[G]} \models \Gamma$.

Proof. Let P be a forcing notion, let $\mathbb{V}[G]$ be a forcing extension of $\mathbb{V}$ by P. Then by Lemma 6.1.3, (I) in the definition of robustness holds in $\mathbb{V}[G]$, so $(\mathrm{HC})^{\mathbb{V}[G]} \models \Gamma$. □

Our plan for the rest of the section is to define, for every robust Γ, a reducibility notion $\leq_\Gamma$ on sentences of $\mathcal{L}_{\omega_1\omega}$. $\leq_\Gamma$ will refine $\leq_{\mathrm{HC}}$ and it will coarsen $\leq_B$. This notion will be particularly well-suited to the technology we develop. Our development will be highly analogous to the development of $\leq_{\mathrm{HC}}$.

Definition 6.1.5. Suppose $X \subseteq \mathrm{HC}$ is definable and Γ is robust. Then say that X is Γ-absolute if there is some formula $\varphi(x, a)$ defining X, such that whenever V is a countable transitive model of Γ with $a \in V$, then $\varphi(V, a) = X \cap V$. We say that $\varphi(x, a)$ witnesses that X is Γ-absolute.

Example 6.1.6. Suppose X is Γ-absolute, say via $\varphi(x, a)$. Then $\mathcal{P}_{\aleph_1}(X)$ is Γ-absolute, via $\psi(y, a) :=$ "$\forall x \in y(\varphi(x, a))$." The formula $\psi'(y, a) :=$ "y is countable and $\psi(y, a)$" does not necessarily work, since there may be $V \models \Gamma$ countable and transitive, such that V does not believe every subset of $\varphi(V, a)$ is countable.

With this caveat, all of the examples in Section 5.2.2 go through for $\leq_\Gamma$, where Γ is robust.

Lemma 6.1.7. Suppose Γ is robust, and suppose $X \subseteq \mathrm{HC}$ is Γ-absolute via $\varphi(x, a)$, and $\mathbb{V}[G]$ is a forcing extension of $\mathbb{V}$. Then in $\mathbb{V}[G]$: $\varphi(\mathrm{HC}^{\mathbb{V}[G]}, a)$ is Γ-absolute in $\mathbb{V}[G]$, and $\varphi(\mathrm{HC}^{\mathbb{V}[G]}, a) \cap \mathrm{HC}^{\mathbb{V}} = X$. Moreover (back in $\mathbb{V}$), suppose Γ' is robust, and X is also Γ'-absolute via $\psi(x, b)$. Then for every forcing extension $\mathbb{V}[G]$, $\varphi(\mathrm{HC}^{\mathbb{V}[G]}, a) = \psi(\mathrm{HC}^{\mathbb{V}[G]}, b)$.

The moreover clause is convenient notationally, as contrasted with the situation for $\leq_{\mathrm{HC}}$; it allows us to literally deal with Γ-absolute sets rather than formulas, without any ambiguity.

Proof. Let $\mathbb{V}[G][G']$ be a further forcing extension in which $\mathrm{HC}^{\mathbb{V}[G]}$ is countable. Note that in $\mathbb{V}[G][G']$, we have that for all countable transitive models $a \in V \subseteq V'$ of Γ, we have that $\varphi(a, V) = \varphi(a, V') \cap V$, by Lévy's absoluteness principle. Applied to $V' = \mathrm{HC}^{\mathbb{V}[G]}$ and $V \subseteq V'$ yields that $\varphi(\mathrm{HC}^{\mathbb{V}[G]}, a)$ is Γ-absolute in $\mathbb{V}[G]$, and the special case where $V = \mathrm{HC}^{\mathbb{V}[G]}$ yields $\varphi(\mathrm{HC}^{\mathbb{V}[G]}, a) \cap \mathrm{HC}^{\mathbb{V}} = X$.

Finally, suppose $\psi(x, b)$ witnesses that X is Γ'-absolute. Then again by Lévy's absoluteness principle, in $\mathbb{V}[G][G']$, for every countable transitive $V \models ZFC^-$, and for every $V_0, V_1 \subseteq V$ with $a \in V_0 \models \Gamma$ and $b \in V_1 \models \Gamma'$, and for every $x \in V_0 \cap V_1$, we have that $\varphi(a, x)^{V_0}$ holds if and only if $\psi(b, x)^{V_1}$ holds; apply this to $V = \mathrm{HC}^{\mathbb{V}[G]}$. $\square$

In light of Lemma 6.1.7 the following definition makes sense.

Definition 6.1.8. Suppose Γ is robust, and $X \subseteq \mathrm{HC}$ is Γ-absolute.

If A is a set, then we say that X is Γ-definable over A if there is some $\varphi(x, a)$ witnessing that X is Γ-absolute, such that $a \in A$ (so necessarily $a \in \mathrm{HC}$).

Suppose V is a transitive model of Γ, possibly in a forcing extension, and suppose X is Γ-definable over V. Then define $(X)^V = \{a \in V : V \models \varphi(a, a_0)\}$, for some or any $\varphi(x, a_0)$ witnessing X is Γ-definable over V.

Thus for all $V \subseteq V'$ both models of Γ, if X is Γ-definable over V, then $(X)^{V'} \cap V = (X)^V$.

We now define what we mean by persistence; this is analogous to the definition of persistence in [89].

Definition 6.1.9. Suppose $X_i : i < n$ are Γ-absolute and $\psi(U_i : i < n)$ is a sentence of set theory with n new unary predicates, and possibly with parameters from HC. Then say that $\psi(X_i : i < n)$ holds Γ-persistently if there is some $a \in \mathrm{HC}$ containing the parameters for ψ, such that whenever V is a countable transitive model of Γ with $a \in V$, if each X_i is Γ-definable over V, then $(V, (X_i)^V : i < n) \models \psi$.

By an argument similar to Lemma 6.1.7 we get that if $\psi(X_i : i < n)$ holds Γ-persistently and $\mathbb{V}[G]$ is a forcing extension, then $\psi(X_i^{\mathbb{V}[G]} : i < n)$ still holds persistently.

We can now define $\leq_\Gamma$. As for $\leq_{\mathrm{HC}}$, we do not require that the reduction f is itself a function, but rather just that it induces a function on equivalence classes.

Definition 6.1.10. A Γ-quotient space is a pair (X, E) of Γ-absolute subsets of HC, such that persistently, E is an equivalence relation on X. Given Γ-quotient spaces (X, E) and (X', E'), say that $f : (X, E) \leq_\Gamma (X', E')$ (f is a Γ-reduction) if $f \subseteq X \times Y$ is Γ-absolute and persistently, f induces an injection from X/E to X'/E'. (That is, persistently: for every $x \in X$ there are $x' \in X, y \in Y$ such that xEx' and $(x', y) \in f$, and moreover whenever xEx' and $(x', y') \in f$, and xEx'' and $(x'', y'') \in f$, then $y'Ey''$.)

If (X_i, E_i) are Γ-absolute quotient spaces, say that $(X_1, E_1) \sim_\Gamma (X_2, E_2)$ if $(X_1, E_1) \leq_\Gamma$

(X_2, E_2) and $(X_2, E_2) \leq_\Gamma (X_1, E_1)$. Also, say that $f : (X_1, E_1) \cong_\Gamma (X_2, E_2)$ if Γ-persistently, f induces a bijection from X_1/E_1 to X_2/E_2 (in which case $(X_1, E_1) \sim_\Gamma (X_2, E_2)$).

Note that whenever X is Γ-absolute, then $(X, =)$ is a Γ-quotient space; in this case we omit $=$. Note then that $X \leq_\Gamma Y$ if and only if there is some Γ-persistent injection $f : X \to Y$.

We will also want the following definition.

Definition 6.1.11. By a Γ-absolute complete separable metric space P we mean a structure (X, d, D, i) where X, d, D, i are Γ-absolute, and Γ-persistently: (X, d) is a complete separable metric space, $D \subseteq X$ is dense and $i : \omega \to D$ is a bijection. (We want D and i so that the various theorems from descriptive set theory hold Γ-persistently.)

X is a Γ-absolute standard Borel space if X is the Borel σ-algebra of a Γ-absolute complete separable metric space. All Borel spaces one normally deals with are of this form (provided the elements of X are hereditarily countable; otherwise we just need to code e.g. closed sets by reals.)

We are mainly interested in Γ-quotients that are either of the form $(X, =)$ for some arbitrary X, or else of the form (X, E) where X is a Γ-standard Borel space.

As an example, note that if X is a Γ-absolute standard Borel space and $Y \subseteq X$ is analytic or co-analytic, then Y is Γ-absolute. In particular, if Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$, then the isomorphism relation on $\mathrm{Mod}(\Phi)$ (being analytic) is Γ-absolute. Thus $(\mathrm{Mod}(\Phi), \cong)$ is a Γ-quotient space.

Lemma 6.1.12. Suppose (X_1, E_1), (X_2, E_2) are Γ-quotient spaces, such that additionally each X_i is a Γ-absolute standard Borel space. Suppose $f : (X_1, E_1) \leq_B (X_2, E_2)$. Then

$f : (X_1, E_1) \leq_\Gamma (X_2, E_2)$, so in particular $\leq_\Gamma$ coarsens $\leq_B$.

Proof. We can suppose $X_1, X_2 = 2^\omega$. For each $n < \omega$ let B_n be the Borel set of all $x \in 2^\omega$ such that $x(n) = 0$. Suppose V is a countable transitive model of Γ containing codes for all the relevant parameters (including a code for the sequence $(B_n : n < \omega)$). We want to check that $(f : (X_1, E_1) \leq_B (X_2, E_2))^V$. Note that f^V is a function from X_1 to X_2, since if $x \in (2^\omega)^V$, then $f(x)$ is definable in V via $f(x)(n) = 0$ if and only if $x \in B_n$. Further, $f^V = f \restriction_V$, since Borel sets are absolute to transitive models of ZFC^-. Finally, since each E_i is absolute to V, we have that for all $x, y \in X_1$, $xE_1^V y$ if and only if xE_1y if and only if $f(x)E_2f(y)$ if and only if $f^V(x)E_2^V f^V(y)$, as desired. □

We now turn to countable model theory, the main source of examples we are interested in. Note that if Γ is robust, and Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$, then CSS($\Phi$) is Γ-absolute, and css : (Mod(Φ), $\cong$) $\leq_\Gamma$ CSS(Φ), although there is rarely a reduction in the other direction.

Definition 6.1.13. Suppose Γ is robust. and Φ, Ψ are sentences of $\mathcal{L}_{\omega_1\omega}$. Then define $\Phi \leq_\Gamma \Psi$ to mean CSS(Φ) $\leq_\Gamma$ CSS(Ψ).

The following is a key consequence of robustness (note that up until now, we have not used all of the definition of robustness—instead we have just used Proposition 6.1.4). We will use this equivalence of $\Phi \leq_\Gamma \Psi$ interchangeably with the above definition.

Theorem 6.1.14. Suppose Γ is robust, and Φ, Ψ are sentences of $\mathcal{L}_{\omega_1\omega}$. Then $\Phi \leq_\Gamma \Psi$ if and only if (Mod(Φ), $\cong$) $\leq_\Gamma$ (Mod(Ψ), $\cong$).

Hence, if $\Phi \leq_B \Psi$ then $\Phi \leq_\Gamma \Psi$.

Proof. Clearly CSS(Φ) is ZFC^--absolute, and the map css : Mod(Φ) $\to$ CSS(Φ) is ZFC^--persistently an injection. Hence (since $\Gamma \supseteq ZFC^-$), the theorem makes sense.

Given $f : (\mathrm{Mod}(\Phi), \cong) \leq_\Gamma (\mathrm{Mod}(\Psi), \cong)$, define $f_* : \mathrm{CSS}(\Phi) \leq_\Gamma \mathrm{CSS}(\Psi)$ via the following formula: $f_*(\varphi) = \psi$ if and only if in some or any forcing extension $\mathbb{V}[G]$ in which φ, ψ become hereditarily countable, and for some or any $M \in \mathrm{Mod}(\Phi)^{\mathbb{V}[G]}, N \in \mathrm{Mod}(\Psi)^{\mathbb{V}[G]}$ with $(M, N) \in (f)^{\mathbb{V}[G]}$, if $\mathrm{css}(M) = \varphi$ then $\mathrm{css}(N) = \psi$.

Given $f : \mathrm{CSS}(\Phi) \leq_\Gamma \mathrm{CSS}(\Psi)$, define $f_* : (\mathrm{Mod}(\Phi), \cong) \leq_\Gamma (\mathrm{Mod}(\Psi), \cong)$ via $(M, N) \in f_*$ if and only if $f(\mathrm{css}(M)) = \mathrm{css}(N)$. Note that, working in a model $V \models \Gamma$, we have that for all $M \in \mathrm{Mod}(\Phi)^V$, $f(\mathrm{css}(M)) \in (\mathrm{HC})^V$, since $(\mathrm{HC})^V \models \Gamma$. Thus f_* works. $\square$

Now we relate the various $\leq_\Gamma$ to the previously studied $\leq_{a\Delta^1_2}$. This notion (read: absolute Δ^1_2-reduction) was introduced by Hjorth in [22].

Definition 6.1.15. Suppose Φ, Ψ are sentences of $\mathcal{L}_{\omega_1\omega}$. Say that $\Phi \leq_{a\Delta^1_2} \Psi$ if there is some function $f : \mathrm{Mod}(\Phi) \to \mathrm{Mod}(\Psi)$ with Δ^1_2 graph, such that for all $M, N \in \mathrm{Mod}(\Phi)$, $M \cong N$ if and only if $f(M) \cong f(N)$, and such that further, this continues to hold in any forcing extension. Explicitly, we require that f has a Π^1_2-definition $\sigma(x, y)$, and a Σ^1_2-definition $\tau(x, y)$, such that if $\mathbb{V}[G]$ is any forcing extension, then $\sigma(x, y)$ and $\tau(x, y)$ coincide on $\mathrm{Mod}(\Phi)^{\mathbb{V}[G]} \times \mathrm{Mod}(\Psi)^{\mathbb{V}[G]}$ and define the graph of a function $f^{\mathbb{V}[G]}$, such that for all $M, N \in \mathrm{Mod}(\Phi)^{\mathbb{V}[G]}$, $M \cong N$ if and only if $f^{\mathbb{V}[G]}(M) \cong f^{\mathbb{V}[G]}(N)$.

For context, we will also want to make the following definition:

Definition 6.1.16. Define $\Phi \leq^*_{a\Delta^1_2} \Psi$ in the same way as $\Phi \leq_{a\Delta^1_2} \Psi$, except we just require that $f \subseteq \mathrm{Mod}(\Phi) \times \mathrm{Mod}(\Psi)$ induces an injection from $\mathrm{Mod}(\Phi)/\cong$ to $\mathrm{Mod}(\Psi)/\cong$, i.e.: for all $M, M' \in \mathrm{Mod}(\Phi)$ and for all $N, N' \in \mathrm{Mod}(\Psi)$, if (M, N) and (M', N') are in f then $M \cong M'$ if and only if $N \cong N'$; and for all $M \in \mathrm{Mod}(\Phi)$ there is $M' \in \mathrm{Mod}(\Phi)$ sand $N \in \mathrm{Mod}(\Psi)$ such that $M \cong M'$ and $(M', N) \in f$.

Clearly then, $\leq^*_{a\Delta^1_2}$ is a coarsening of $\leq_{a\Delta^1_2}$. The following is why we care about $\leq^*_{a\Delta^1_2}$:

Theorem 6.1.17. Suppose Φ, Ψ are sentences of $\mathcal{L}_{\omega_1\omega}$. Then $\Phi \leq^*_{a\Delta^1_2} \Psi$ if and only if $\Phi \leq_\Gamma \Psi$ for some robust Γ. In particular, if $\Phi \leq_{a\Delta^1_2} \Psi$ then $\Phi \leq_\Gamma \Psi$ for some robust Γ.

Proof. Clearly if $\Phi \leq_\Gamma \Psi$ for some robust Γ, then $\Phi \leq^*_{a\Delta^1_2} \Psi$.

Conversely, suppose $f \subseteq \mathrm{Mod}(\Phi) \times \mathrm{Mod}(\Psi)$ witnesses that $\Phi \leq^*_{a\Delta^1_2} \Psi$. Let $\varphi(x, y)$ be the Σ^1_2-definition of f and let $\psi(x, y)$ be the Π^1_2-definition of f, witnessing that f is absolutely Δ^1_2. Let σ be the formula of set theory (with parameters Φ, Ψ, and the parameter for f) asserting that $\varphi(x, y)$, $\psi(x, y)$ describe the same subset f of $\mathrm{Mod}(\Phi) \times \mathrm{Mod}(\Psi)$, and f induces an injection from $\mathrm{Mod}(\Phi)/\cong$ to $\mathrm{Mod}(\Psi)/\cong$. By hypothesis, $\mathrm{HC}^{\mathbb{V}[G]} \models \sigma$, for every forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$.

Let Γ assert that ZFC^- holds, and in every forcing extension $\mathbb{V}[G]$, $\mathrm{HC}^{\mathbb{V}[G]} \models \sigma$. We must check that Γ is robust.

Axiom (I): suppose κ is a regular cardinal and $P \in H(\kappa)$ is a forcing notion. Then $H(\kappa) \models ZFC^-$. Let G be P-generic over $\mathbb{V}$. Then $H(\kappa)[G] = H(\kappa)^{\mathbb{V}[G]}$ by Lemma 6.1.2, so $(\mathrm{HC})^{H(\kappa)[G]} = (\mathrm{HC})^{\mathbb{V}[G]}$; also, $(\mathrm{HC})^{\mathbb{V}[G]} \models \sigma$. Thus $H(\kappa) \models \Gamma$.

Axiom (II): suppose $V \models \Gamma$ is transitive. Suppose $V[G]$ is a forcing extension of V; then $V[G] \models ZFC^-$. Also, every forcing extension $V[G][G']$ of $V[G]$ is a forcing extension of V, and hence $(\mathrm{HC})^{V[G][G']} \models \sigma$, so $V[G] \models \Gamma$.

Axiom (III): suppose $V \models \Gamma$ is transitive and $\kappa \in V$ is regular in V. Write $V' = (H(\kappa))^V$. Then $V' \models ZFC^-$. Suppose $P \in V'$ and $V'[G]$ is a P-generic forcing extension; it suffices (by definability of forcing) to consider the case where G is also P-generic over V. But then, $V'[G] = H(\kappa)^{V[G]}$ by Lemma 6.1.2, so $(\mathrm{HC})^{V'[G]} = (\mathrm{HC})^{V[G]} \models \sigma$.

Axiom (IV): by fiat, Γ extends ZFC^-.

Now we finish, by showing $f : (\mathrm{Mod}(\Phi), \cong) \leq_\Gamma (\mathrm{Mod}(\Psi), \cong)$.

Let $\varphi_*(x, y)$ be the formula of set theory (over the relevant parameters) asserting: $x \in \mathrm{Mod}(\Phi)$ and $y \in \mathrm{Mod}(\Psi)$, $\mathrm{Mod}(\Phi) \times \mathrm{Mod}(\Psi) \models \varphi(x, y)$, and similarly define $\psi_*(x, y)$. I claim that f is Γ-absolute, as witnessed by $\varphi_*(x, y)$ (or $\psi_*(x, y)$). Suppose $V \models \Gamma$ is a countable transitive model containing the relevant parameters, and choose $(M, N) \in (\mathrm{Mod}(\Phi) \times \mathrm{Mod}(\Psi))^V$. If $V \models \varphi_*(M, N)$, then $\mathrm{HC} \models \varphi_*(M, N)$ and so $(M, N) \in f$, using that Σ^1_2-sentences are upwards absolute between transitive models of ZFC^-. If $\mathrm{HC} \models \varphi_*(M, N)$, then $\mathrm{HC} \models \psi_*(M, N)$, so $V \models \psi_*(M, N)$, so $V \models \varphi_*(M, N)$. $\square$

Finally, the following trivial observation relates what we have done to $\leq_{\mathrm{HC}}$:

Theorem 6.1.18. Suppose Γ is robust.

- Suppose $X \subseteq \mathrm{HC}$ is Γ-absolute. Then X is HC-forcing invariant, via any definition of X witnessing X is Γ-absolute. (Recall that HC-forcing invariant subsets of HC must come equipped with a defining formula.)

- Suppose $X_i : i < n$ are Γ-absolute and $\psi(X_i : i < n)$ holds Γ-persistently. Then $\psi(X_i : i < n)$ holds persistently.

- Suppose $(X_i, E_i) : i < 2$ are Γ-quotient spaces and $(X_0, E_0) \leq_\Gamma (X_1, E_1)$. Then $(X_0, E_0) \leq_{\mathrm{HC}} (X_1, E_1)$.

- Suppose Φ, Ψ are sentences of $\mathcal{L}_{\omega_1\omega}$. If $\Phi \leq_\Gamma \Psi$ then $\Phi \leq_{\mathrm{HC}} \Psi$.

6.2 Thickness

In this section we define the key technical concept of the paper.

First, we discuss potential cardinality in the context of Γ-absoluteness. Note that by Theorem 6.1.18, if Γ is robust and $X \subseteq \mathrm{HC}$ is Γ-absolute, then X_{ptl} and $\|X\|$ make sense, and if $X \leq_\Gamma Y$ then $\|X\| \leq \|Y\|$; and this also holds for sentences of $\mathcal{L}_{\omega_1\omega}$.

In fact, in the context of Γ-absoluteness, we have a nicer characterization of X_{ptl}:

Theorem 6.2.1. Suppose Γ is robust, and $X \subseteq \mathrm{HC}$ is Γ-absolute. Suppose V is a transitive model of Γ. Then $X_{\mathrm{ptl}} \cap V = (X)^V$.

Proof. Choose κ regular such that $V \in H(\kappa)$, and let $\mathbb{V}[G]$ be the a forcing extension collapsing κ to ω_1. By Theorem 6.1.7, we are done. $\square$

If we wanted to define $(X, E)_{\mathrm{ptl}}$ for general Γ-absolute quotients we should use the notion of *pins* (developed in [30] previously to but independently of [89]), and much of what we could do could generalize.

We would like to use counting arguments to characterize $\leq_\Gamma$-completeness, i.e., to characterize which sentences Φ are $\leq_\Gamma$-maximal. Potential cardinality is not enough: there are examples of relatively nice Φ that are not short, so potential cardinality says nothing about them. For instance, note that $\mathrm{TAG}_1 \sim_{ZFC^-} \mathcal{P}_{\aleph_1}(\omega_1)$ by Ulm invariants [86], so we can identify $\mathrm{CSS}(\mathrm{TAG}_1)_{\mathrm{ptl}}$ with $\mathcal{P}(\mathrm{ON})$. Now $\|\mathrm{TAG}_1\| = \infty$, so this gives no upper bounds on the complexity of TAG_1, but we note that $\mathcal{P}(\mathrm{ON})$ still seems much thinner than $\mathbb{V}$, so a counting argument seems reasonable. More specifically note that for each cardinal λ, $|\mathrm{CSS}(\mathrm{TAG}_1)_{\mathrm{ptl}} \cap \mathbb{V}_{\lambda^+}| = 2^\lambda$ which is much less than the maximum possible value of $\beth_{\lambda^+}$.

Actually, in [12], Friedman and Stanley give a fairly simple proof that TAG_1 is not Borel complete (and the same proof shows it is not HC-complete.) The need for a counting argument is more acute when we consider the jumps of TAG_1. We recall their definition from the introduction:

Definition 6.2.2. For each $\alpha < \omega_1$, write $\mathrm{TAG}_{1+\alpha} = \mathcal{J}^\alpha(\mathrm{TAG}_1)$, the α'th jump of torsion abelian groups.

If $X \subseteq \mathrm{HC}$, and $\alpha < \omega_1$, then let $\mathcal{P}^\alpha_{\aleph_1}(X)$ be the countable powerset operation iterated α-many times on X, where we take unions at limit stages. If X is any class and α is any ordinal, let $\mathcal{P}^\alpha(X)$ be the powerset operation iterated α-many times on X, where we take unions at limit stages. Here, the powerset of a proper class X is the class of all sub*sets* of X.

Note that whenever X is Γ-absolute or HC-forcing invariant, so is $\mathcal{P}^\alpha_{\aleph_1}(X)$; and $(\mathcal{P}^\alpha_{\aleph_1}(X))_{\mathrm{ptl}} = \mathcal{P}^\alpha(X_{\mathrm{ptl}})$.

Now, for all $\alpha \geq 1$, $\mathrm{TAG}_\alpha \sim_{ZFC^-} \mathcal{P}^\alpha_{\aleph_1}(\omega_1)$, so $(\mathrm{TAG}_\alpha)_{\mathrm{ptl}} = \mathcal{P}^\alpha(\mathrm{ON})$, which again seems much thinner than $\mathbb{V}$. However, the simple proof that TAG_1 is not Borel complete does not carry through here, and as far as we know the machinery we develop is necessary to show this.

Our first attempt at capturing this notion with counting would be to directly count $\mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap \mathbb{V}_{\lambda^+}$ for various cardinals λ. However, we would need to show this is a $\leq_B$-reducibility invariant; while we have no counterexample to this, the following example prevents any straightforward proof:

Example 6.2.3. Let $\mathcal{L}_0 = \{R_0\}$ and let $\mathcal{L}_1 = \{R_0, R_1\}$, where R_0, R_1 are binary relation symbols. Let $f : \mathrm{Mod}(\mathcal{L}_1) \to \mathrm{Mod}(\mathcal{L}_0)$ be the reduct map. Let $f_* : \mathrm{CSS}(\mathcal{L}_1) \to \mathrm{CSS}(\mathcal{L}_0)$ be the induced map on Scott sentences. Then for every cardinal λ and for every $\kappa < \beth_{\lambda^+}$, there is some $\varphi \in \mathrm{CSS}(\mathcal{L}_1)_{\mathrm{ptl}} \cap \mathbb{V}_{\lambda^+}$, such that $(f_*)_{\mathrm{ptl}} \not\in \mathbb{V}_\kappa$—in particular (choosing $\kappa = \lambda^+$), we can arrange $(f_*)_{\mathrm{ptl}} \not\in \mathbb{V}_{\lambda^+}$.

Proof. Choose $\alpha < \lambda^+$ such that $\kappa^+ < \beth_\alpha$. We define an $\mathcal{L}_1$-structure (M, R_0^M, R_1^M) as follows: let $(M, R_1^M) = (\mathbb{V}_\alpha, \in)$, and let R_0^M be a well-ordering of $\mathbb{V}_\alpha$. Note that $(\mathbb{V}_\alpha, \in)$ is rigid and has Scott rank approximately α, so $\mathrm{css}(M, R_0^M, R_1^M) \in \mathbb{V}_{\lambda^+}$. On the other hand, (M, R_0^M) is a well-ordering of length longer than κ^+, and so its canonical Scott sentence cannot be in $\mathbb{V}_\kappa$. □

The idea for getting around this is to count $|\mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap A|$ for sufficiently closed $A \in \mathbb{V}_{\lambda^+}$, instead of all of $|\mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap \mathbb{V}_{\lambda^+}|$.

Definition 6.2.4. Suppose Γ is robust. Then let $\mathbb{F}_\Gamma$ be the set of all Γ-absolute f such that Γ-persistently, $f : \mathrm{HC} \to \mathrm{HC}$. Let $\mathbb{F} = \bigcup_\Gamma \mathbb{F}_\Gamma$.

Suppose $\overline{f} = (f_i : i < n) \in \mathbb{F}^{<\omega}$. Then say that A is $\overline{f}$-closed if A is a transitive set with $A^{<\omega} \subseteq A$, and for each $i < n$ $(f_i)_{\mathrm{ptl}}[A] \subseteq A$. We do not require $A \models \Gamma$ or any reasonable fragment of set theory, or that $(f_i)_{\mathrm{ptl}}$ be definable within A in any way.

If f is a Γ-persistent map defined on some Γ-absolute $X \subseteq \mathrm{HC}$, we identify f with $f' \in \mathbb{F}_\Gamma$ which is defined to be $\emptyset$ off of X.

The following simple lemma will be used implicitly henceforth:

Lemma 6.2.5. Suppose $f_i : i < n$ is any sequence from $\mathbb{F}$. Define $f : \mathrm{HC} \to \mathrm{HC}$ to be $\prod_{i<n} f_i$, that is $f(a) = (f_i(a) : i < n)$. Then $f \in \mathbb{F}$, and for every set A, then A is f-closed if and only if A is $\overline{f}$-closed.

Proof. First, we check that $f \in \mathbb{F}$. For each $i < n$, choose some robust Γ_i with $f \in \Gamma_i$. Then $\Gamma = \bigcup_i \Gamma_i$ is robust and $f \in \mathbb{F}_\Gamma$.

To finish, since we are requiring A to be transitive and $A = A^{<\omega}$, we have that $(f_i(a) : i < n) \in A$ if and only if each $f_i(a) \in A$. □

The following fundamental observation will be the motivation for our definition of thickness:

Theorem 6.2.6. Suppose Γ is robust. Suppose X, Y are Γ-absolute, such that for every $f \in \mathbb{F}_\Gamma$, there is an f-closed set A with $|X_{\text{ptl}} \cap A| > |Y_{\text{ptl}} \cap A|$. Then $X \not\leq_\Gamma Y$.

Proof. We prove the contrapositive. Suppose $f : X \leq_\Gamma Y$. As just mentioned, we can view $f \in \mathbb{F}$ by defining $f(a) = \emptyset$ for $a \notin X$. Suppose A is f-closed. Then f_{ptl} clearly witnesses that $|X_{\text{ptl}} \cap A| \leq |Y_{\text{ptl}} \cap A|$. $\square$

We could view all of our results on thickness through these lens. We find it convenient to introduce a cardinal invariant capturing much of the information available.

Definition 6.2.7. Suppose X is Γ-absolute, for some robust Γ. Suppose λ is a cardinal.

Then define $\tau(X, \lambda)$, the thickness of X at λ, to be the least cardinal κ such that there is some $f \in \mathbb{F}$, such that whenever $A \in \mathbb{V}_{\lambda^+}$ is f-closed, we have $|X_{\text{ptl}} \cap A| \leq \kappa$. Alternatively, we have that $\tau(X, \lambda) > \kappa$ if and only if for every $f \in \mathbb{F}$, there is some f-closed $A \in \mathbb{V}_{\lambda^+}$ with $|X_{\text{ptl}} \cap A| > \kappa$.

If Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ then define $\tau(\Phi, \lambda) = \tau(\text{CSS}(\Phi), \lambda)$.

The reader may wonder why we define $\tau(X, \lambda)$ in terms of λ^+ rather than λ, and why we insist that $|X_{\text{ptl}} \cap A| \leq \kappa$ rather than $< \kappa$. This is for cosmetic reasons; we believe our results are more readable this way. We do not seem to be losing any important information.

Some simple observations: $\tau(X, \lambda) \leq |X_{\text{ptl}} \cap \mathbb{V}_{\lambda^+}| \leq \beth_{\lambda^+}$, and $\tau(X, \lambda)$ is monotone in λ, with $\lim_{\lambda \to \infty} \tau(X, \lambda) = \|X\|$.

The following theorem is a simple twist to the idea of Theorem 6.2.6, just packaged in terms of the τ function.

Theorem 6.2.8. If $X_1 \leq_\Gamma X_2$ for some robust Γ, then $\tau(X_1, \lambda) \leq \tau(X_2, \lambda)$ for every cardinal λ.

Proof. Choose $f : X_1 \leq_\Gamma X_2$. Let λ be given. Suppose towards a contradiction that $\tau(X_1, \lambda) > \tau(X_2, \lambda) = \kappa$. Choose $g \in \mathbb{F}$ witnessing that $\tau(X_2, \lambda) = \kappa$, that is, whenever $A \in \mathbb{V}_{\lambda^+}$ is f-closed, we have $|(X_2)_{\mathrm{ptl}} \cap A| \leq \kappa$.

By hypothesis (and Lemma 6.2.5), we can find some (f, g)-closed $A \in \mathbb{V}_{\lambda^+}$ such that $|(X_1)_{\mathrm{ptl}} \cap A| > \kappa$; by choice of g, $|(X_2)_{\mathrm{ptl}} \cap A| \leq \kappa$. But since A is also f-closed, we have that f_{ptl} restricts to an injection from $(X_1)_{\mathrm{ptl}} \cap A$ to $(X_2)_{\mathrm{ptl}} \cap A$, a contradiction. $\square$

The following theorem is also straightforward.

Theorem 6.2.9. For all Φ, λ, α, if Φ has infinitely many countable models, then $\tau(\mathcal{J}^\alpha(\Phi), \lambda) \leq \beth_\alpha(\tau(\Phi, \lambda))$.

Proof. Write $\kappa = \tau(\Phi, \lambda)$; choose $f \in \mathbb{F}$ such that whenever $A \in \mathbb{V}_{\lambda^+}$ is f-closed, then $|\mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap A| \leq \kappa$. Then clearly also $|\mathrm{CSS}(\mathcal{J}^\alpha(\Phi))_{\mathrm{ptl}} \cap A| \leq \beth_\alpha(\kappa)$ as desired. $\square$

We do not now how to prove the reverse inequality in general, although we suspect that at least for $\lambda = \aleph_0$, it should be true. Instead we focus on special cases, where Φ is either some Φ_α or some TAG_α. Our task boils down to constructing thick transitive sets in $\mathbb{V}_{\lambda^+}$, as the following proposition indicates.

Proposition 6.2.10. There is some $f \in \mathbb{F}$, such that for every f-closed A, $|\mathrm{CSS}(\mathrm{Graphs})_{\mathrm{ptl}} \cap A| = |A|$, and for every $\alpha < \omega_1$, $|\mathrm{CSS}(\Phi_\alpha)_{\mathrm{ptl}} \cap A| = |\mathcal{P}^\alpha(\omega) \cap A|$, and $|\mathrm{CSS}(\mathrm{TAG}_\alpha)_{\mathrm{ptl}} \cap A| = |\mathcal{P}^\alpha(\mathrm{ON}) \cap A|$.

Proof. I claim we can choose $f \in \mathbb{F}_{ZFC^-}$ so as to encode ZFC^--reductions between Graphs and HC, between Φ_α and $\mathcal{P}^\alpha_{\aleph_1}(\omega)$ for each $\alpha < \omega_1$, and between TAG_α and $\mathcal{P}^\alpha_{\aleph_1}(\omega_1)$ for each $\alpha < \omega_1$; and finally, the map sending a to the foundation rank $\mathrm{rnk}(a)$. Finding f is not hard; note, for instance, that we can find some $f_0 \in \mathbb{F}_{ZFC^-}$ such that

ZFC^--persistently, for all $\alpha < \omega_1$, $f_0 \restriction_{\{\alpha\}\times \mathrm{CSS}(\Phi_\alpha)}$ induces a ZFC^--reduction from Φ_α to $\mathcal{P}^\alpha_{\aleph_1}(\omega)$. f will be a product of several such f_i's.

Then it is straightforward to see that f works. For instance, suppose A is f-closed, and either $\mathrm{CSS}(\Phi_\alpha)_{\mathrm{ptl}} \cap A$ or else $\mathcal{P}^\alpha(\omega) \cap A$ is nonempty. Then $\alpha \in A$ since A is closed under rnk, so A will be (g, h)-closed, where g, h are the ZFC^--reductions between Φ_α and $\mathcal{P}^\alpha(\omega)$ coded by f. $\square$

The following definition is motivated by this proposition.

Definition 6.2.11. The infinite cardinal λ *admits thick sets* if for every $\alpha < \lambda^+$, and for every $f \in \mathbb{F}$, there is some f-closed $A \in \mathbb{V}_{\lambda^+}$, such that $|\mathcal{P}^\alpha(\lambda) \cap A| = \beth_\alpha(\lambda)$.

In Section 6.5 we prove the following (note that a regular strong limit is equivalently either $\aleph_0$ or inaccessible).

Theorem 6.2.12. Every regular strong limit cardinal admits thick sets. Further, it is consistent with ZFC that every regular cardinal admits thick sets; this can be achieved in a proper-class forcing extension which adds no reals.

This immediately gives the following corollaries:

Corollary 6.2.13. Suppose λ admits thick sets. Then for every $\alpha < \omega_1$, $\tau(\Phi_\alpha, \lambda) = \beth_\alpha$, and $\tau(\mathrm{TAG}_\alpha, \lambda) = \beth_\alpha(\lambda)$. Also, if Φ is Borel complete then $\tau(\Phi, \lambda) = \beth_{\lambda^+}$. In particular, this happens whenever λ is a regular strong limit, and consistently can happen for all regular λ.

Proof. Choose f as in Proposition 6.2.10.

For Φ_α, we will not actually need that λ admits thick sets: note that $\aleph_0$ is a regular strong limit, and hence admits thick sets. Then f witnesses that $\tau(\Phi_\alpha, \aleph_0) = \beth_\alpha$: suppose

$A \in \mathbb{V}_{\aleph_1}$ is f-closed. Then $|\mathrm{CSS}(\Phi_\alpha)_{\mathrm{ptl}} \cap A| = |\mathcal{P}^\alpha(\omega) \cap A|$. This is always at most $\beth_\alpha$, but since $\aleph_0$ admits thick sets, for every $g \in \mathbb{F}$ we can also arrange that A is g-closed and $|\mathcal{P}^\alpha(\omega) \cap A| = \beth_\alpha$.

The rest is similar. □

We should note that for the purposes of descriptive set theory, passing to a forcing extension without adding any reals is free: if it is convenient to assume every regular cardinal admits thick sets, then we may do so without loss of generality.

We have the following immediate consequence; the case $\alpha = 1$ was proved by Friedman and Stanley in [12], but for $\alpha > 1$, it is new that TAG_α is not Borel complete.

Theorem 6.2.14. For all $1 \leq \alpha < \omega_1$, $\Phi_{\alpha+1} \not\leq_B \mathrm{TAG}_\alpha$ (in fact $\Phi_{\alpha+1} \not\leq^*_{a\Delta^1_2} \mathrm{TAG}_\alpha$).

Proof. This is because $\tau(\Phi_{\alpha+1}, \aleph_0) = \beth_{\alpha+1} > \beth_\alpha = \tau(\mathrm{TAG}_\alpha, \aleph_0)$. □

6.3 Density and Independence Lemmas

This is the first of two technical sections, in which we prove needed facts for Theorem 6.2.12. In this section we will obtain strengthenings of Theorem 2.5.1.

We first recall some notions from Section 2.5.

Definition 6.3.1. Suppose $Y \subseteq \mathcal{P}(X)$. By a finite boolean combination from Y we mean a set of the form $a_0 \cap \ldots \cap a_{n-1} \cap (X \backslash b_0) \cap \ldots \cap (X \backslash b_{m-1})$, for some $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}$ from Y with each $ai \neq Y_j$. Y is independent over X if and only if each finite boolean combination from Y is nonempty.

The following is a special case of Theorem 2.5.4, a theorem of Engleking and Karlowicz [10].

Lemma 6.3.2. Suppose κ is an infinite cardinal. Then there is $Y \subseteq \mathcal{P}(\kappa)$ which is independent over κ with $|Y| = 2^\kappa$.

Proof. Choose $D \subseteq 2^{2^\kappa}$ of size κ such that for each $s \in [\kappa]^{<\aleph_0}$ and each $\overline{f} : 2^s \to 2$, there is some $F \in D$ such that for all $g \in 2^\kappa$, $F(g) = \overline{f}(g \restriction_s)$ (actually this determines F).

Write $D = \{F_\alpha : \alpha < \kappa\}$. For each $f \in 2^\kappa$ put $Y_f = \{\alpha < \kappa : F_\alpha(f) = 1\} \subseteq \kappa$. Let $Y = \{Y_f : f \in 2^\kappa\}$. I claim this works; clearly $|Y| = 2^\kappa$. Moreover, given $(f_i : i < i_*), (g_j : j < j_*)$ sequences of distinct elements from 2^κ with $i_*, j_* < \omega$, we can choose $s \in [\kappa]^{<\aleph_0}$ such that $f_i \restriction_s$, $g_j \restriction_s$ are all distinct. Then choose $\overline{f} : 2^s \to 2$ so that each $\overline{f}(f_i \restriction_s) = 1$, each $\overline{f}(g_j \restriction_s) = 0$. By choice of D applied to $\overline{f}$, there is some $\alpha < \kappa$ such that $F_\alpha(f_i) = 1$ and $F_\alpha(g_j) = 0$ for $i < i_*, j < j_*$; i.e. $\alpha \in Y_{f_i}$ for $i < i_*$ and $\alpha \notin Y_{g_j}$ for $j < j_*$. This suffices to show independence. □

We now wish to strengthen this, in the case where X has a topology. Some definitions will explain what we want:

Definition 6.3.3.

- Suppose X is a topological space. Then X is κ-nice if X has a basis of cardinality (at most) κ, and every nonempty open subset of X has size κ. (In particular, $|X| = \kappa$.)
- If X is a topological space and $D \subseteq X$, then say that D is κ-dense in X if whenever $\mathcal{O} \subseteq X$ is open nonempty, then $|D \cap \mathcal{O}| \geq \kappa$.
- Suppose X is a topological space and $Y \subseteq \mathcal{P}(X)$. Then Y is densely independent if every finite Boolean combination from Y is dense in X. Equivalently, for each nonempty open subset $\mathcal{O}$ of X, every finite boolean combination from Y intersects $\mathcal{O}$.

A routine diagonalizing argument shows that if X is κ-nice, then we can write X as the disjoint union of $(X_\alpha : \alpha < \kappa)$, where each X_α is dense in X and κ-nice.

Now we massage Lemma 6.3.2.

Lemma 6.3.4. Suppose X is κ-nice. Then there is $Y \subseteq \mathcal{P}(X)$ which is densely independent over X, $|Y| = 2^\kappa$.

Proof. Write X as the disjoint union of $X_\alpha : \alpha < \kappa$, where each X_α is κ-nice, and dense in X. Let $(\mathcal{O}_\alpha : \alpha < \kappa)$ be a basis of X. For each $\alpha < \kappa$ let $Y_\alpha \subseteq \mathcal{P}(X_\alpha \cap \mathcal{O}_\alpha)$ be independent over $X_\alpha \cap \mathcal{O}_\alpha$, with $|Y_\alpha| = 2^\kappa$. Write $Y_\alpha = (b^\alpha_\gamma : \gamma < 2^\kappa)$; for each $\gamma < 2^\kappa$ let $b_\gamma = \bigcup\{b^\alpha_\gamma : \alpha < \kappa\}$. Then $Y := \{b_\gamma : \gamma < 2^\kappa\}$ works, since if a is a finite Boolean combination from Y, and $\alpha < \kappa$, then $a \cap X_\alpha \cap \mathcal{O}_\alpha$ is nonempty. $\square$

And we massage again to obtain the form we will use:

Theorem 6.3.5. Suppose X is κ-nice. Then there is a sequence $(Y_\delta : \delta < 2^\kappa)$ of disjoint subsets of $\mathcal{P}(X)$ such that each $|Y_\delta| = 2^\kappa$, each Y_δ is 2^κ-dense in $\mathcal{P}(X)$ (with finite support topology), and $\bigcup_\delta Y_\delta$ is densely independent over X.

Proof. Choose an independent set $Z \subseteq \mathcal{P}(\kappa)$ with $|Z| = 2^\kappa$.

Enumerate $Z = (I_{\delta,\delta'} : \delta, \delta' < 2^\kappa)$. Also write X as the disjoint union $(X_\alpha : \alpha < \kappa)$, where each X_α is κ-nice, and dense in X. For each $\alpha < \kappa$ choose $Y'_\alpha \subseteq \mathcal{P}(X_\alpha)$ densely independent over X_α, with $|Y'_\alpha| = 2^\kappa$. Enumerate $Y'_\alpha = \{b^\gamma_\alpha : \gamma < 2^\kappa\}$. For each $\delta, \delta', \gamma < 2^\kappa$ let $c_{\delta,\delta',\gamma} \subseteq X$ be defined by $c_{\delta,\delta',\gamma} \cap X_\alpha = \emptyset$ if $\alpha \notin I_{\delta,\delta'}$, and $c_{\delta,\delta',\gamma} \cap X_\alpha = b^\gamma_\alpha$ if $\alpha \in I_{\delta,\delta'}$. Let $Y_{\delta,\delta'} \subseteq \mathcal{P}(X)$ be the set of all c such that for some $\gamma < 2^\kappa$, $\{\alpha < \kappa : c \cap X_\alpha \neq c_{\delta,\delta',\gamma} \cap X_\alpha\}$ is finite. Let $Y_\delta = \bigcup_{\delta'} Y_{\delta,\delta'}$. Then I claim $(Y_\delta : \delta < 2^\kappa)$ works.

I claim $(Y_{\delta,\delta'} : \delta, \delta' < 2^\kappa)$ is disjoint. Indeed, suppose $c \in Y_{\delta,\delta'}$ and $\overline{c} \in Y_{\overline{\delta},\overline{\delta}'}$ with $(\delta, \delta') \neq (\overline{\delta}, \overline{\delta}')$. Choose $\gamma < 2^\kappa$ such that $E = \{\alpha < \kappa : c \cap X_\alpha \neq c_{\delta,\delta',\gamma} \cap X_\alpha\}$ is finite;

similarly choose $\overline{\gamma}, \overline{E}$. Since Z is independent, there are infinitely many $\alpha \in I_{\delta,\delta'} \backslash I_{\overline{\delta},\overline{\delta'}}$; choose some such α with $\alpha \notin E \cup \overline{E}$. Then $c \cap X_\alpha = c_{\delta,\delta',\gamma} \cap X_\alpha$ and $\overline{c} \cap X_\alpha = c_{\overline{\delta},\overline{\delta}',\gamma'} \cap X_\alpha$. Now $c_{\delta,\delta',\gamma} \cap X_\alpha = b^\gamma_\alpha$ (since $\alpha \in I_{\delta,\delta'}$) and $c_{\overline{\delta},\overline{\delta}',\gamma'} \cap X_\alpha = \emptyset$ (since $\alpha \notin I_{\overline{\delta},\overline{\delta}'}$. Since $b^\gamma_\alpha \neq \emptyset$ we conclude $c \neq \overline{c}$¿

Also, each $Y_{\delta,\delta'}$ is dense in $\mathcal{P}(X)$, since it is closed under finite differences. Thus $(Y_\delta : \delta < 2^\kappa)$ is disjoint, and each Y_δ is 2^κ-dense in $\mathcal{P}(X)$. So it suffices to show $Y = \bigcup_\delta Y_\delta$ is densely independent over X.

So suppose a is a finite boolean combination from Y; say $a = \bigcap_{i<i_*} d_i^{\pm 1}$, for some $i_* < \omega$. For each $i < i_*$, choose δ_i, δ'_i with $d_i \in Y_{\delta_i,\delta'_i}$, and choose γ_i such that $\{\alpha < \kappa : d_i \cap X_\alpha \neq c_{\delta_i,\delta'_i,\gamma_i} \cap X_\alpha\}$ is finite; call this set E_i. Let $E = \bigcup_{i<i_*} E_i$. Since Z is independent, $\bigcap_{i<i_*} I_{\delta_i,\delta'_i}$ is infinite, so we can choose $\alpha \in \bigcap_{i<i_*} I_{\delta_i,\delta'_i} \backslash E$. Note that each $d_i \cap X_\alpha = c_{\delta_i \delta'_i \gamma_i}$, since $\alpha \notin E$; and $c_{\delta_i \delta'_i \gamma_i} \cap X_\alpha = b^{\gamma_i}_\alpha$ since $\alpha \in I_{\delta_i,\delta'_i}$. Let $\mathcal{O}$ be an open subset of X; since $(B^\gamma_\alpha : \gamma < \kappa)$ is densely independent, $\bigcap_{i<i_*} (b^{\gamma_i}_\alpha)^{\pm 1} \cap \mathcal{O} \neq \emptyset$.

$\square$

6.4 A Tower of $\aleph_0$-Categorical Sentences

While the tower $(\Phi_\alpha : \alpha < \omega_1)$ (or other choices which are Borel equivalent) have been the historical choice of benchmarks, it turns out that for $\alpha \geq 3$ they begin to exhibit pathologies.

Recall that F_2 is the equivalence relation on $(2^\omega)^\omega$ defined by $\overline{\eta} F_2 \overline{\tau}$ if $\{\eta_n : n < \omega\} = \{\tau_n : n < \omega\}$; so $F_2 \sim_B \Phi_2$. We like F_2 a lot; the reason for this is the following Theorem 6.24 of Kanovei, Sabok and Zapletal [31]:

Theorem 6.4.1. Let E be an analytic equivalence relation on $(2^\omega)^\omega$ with $E \supseteq F_2$. Then either E has a comeager equivalence class or else for every Borel nonmeager $C \subseteq (2^\omega)^\omega$,

$F_2 \leq_B E \restriction_C$.

In particular (taking $E = F_2$), if $C \subseteq (2^\omega)^\omega$ is nonmeager then $F_2 \leq_B F_2 \restriction_C$.

We get the following corollary:

Definition 6.4.2. If Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ and α is a countable ordinal, then let $\Phi \restriction_\alpha \in \mathcal{L}_{\omega_1\omega}$ describe the models of Φ of Scott rank less than α.

Corollary 6.4.3. Suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$. Then $\Phi_2 \leq_B \Phi$ if and only if $\Phi_2 \leq_B \Phi \restriction_\alpha$ for some $\alpha < \omega_1$. Hence whether or not $\Phi_2 \leq_B \Phi$ is absolute to forcing extensions.

Proof. For the first part, suppose $f : ((2^\omega)^\omega, F_2) \leq_B (\mathrm{Mod}(\Phi), \cong)$. For each $\alpha < \omega_1$, let $C_\alpha = f^{-1}(\mathrm{Mod}(\Phi \restriction_\alpha))$, a Borel subset of $(2^\omega)^\omega$. Since $\bigcup_\alpha C_\alpha = (2^\omega)^\omega$, we must have that C_α is nonmeager for some $\alpha < \omega_1$. (Choose a countable transitive model V of ZFC^- containing the relevant codes, and look at $x \in (2^\omega)^\omega$ Cohen over V. Then $x \in C_\alpha$ for some α less than the height of $V[x]$, but then V contains a code for C_α, hence C_α must be non-meager.) Hence, By Theorem 6.4.1 (with $E = F_2$) we get that for this choice of α, $F_2 \leq_B F_2 \restriction_{C_\alpha}$. This gives an embedding of C_α into $\Phi \restriction_\alpha$.

For the second part, note that $\Phi_2 \leq_B \Phi$ if and only if there is a countable transitive model V of ZFC^- containing Φ which believes $\Phi_2 \leq_B \Phi \restriction_\alpha$ for some countable ordinal α; so we conclude by Levy Absoluteness. $\square$

For $\alpha > 2$ this is all much more problematic. For one thing, Theorem 6.4.1 fails outright, in the following strong way:

Example 6.4.4. Let F_3 be the equivalence relation defined on $((2^\omega)^\omega)^\omega$, via $\overline{\overline{\eta}} F_3 \overline{\overline{\tau}}$ if $\{\overline{\overline{\eta}}(n) : n < \omega\} F_2 \{\overline{\overline{\tau}}(n) : n < \omega\}$. Let C be the comeager subset of $((2^\omega)^\omega)^\omega$, consisting of all $\overline{\overline{\eta}}$, such that for all $(n, m), (n', m') \in \omega \times \omega$ distinct, $\overline{\overline{\eta}}(n)(m) \neq \overline{\overline{\tau}}(n')(m')$ (these are two elements of 2^ω).

It is easy to see that $F_3 \sim_B \Phi_3$, but I claim that $F_3 \restriction_C \sim_B F_2$. The interesting direction is that $F_3 \restriction_C \leq_B F_2$, to see this, given $\overline{\overline{\eta}} \in C$, define $f(\overline{\overline{\eta}}) \in (2^\omega)^\omega$ so as to list $\{\overline{\overline{\eta}}(n)(m) : n, m < \omega\}$, as well as $\{\overline{\overline{\eta}}(n)(m) \oplus \overline{\overline{\eta}}(n)(m') : n, m, m' < \omega\}$, where $\eta \oplus \tau$ is obtained by interweaving the digits.

This is a serious pathology, and circumventing is a large difficulty in proving Theorem 6.2.12. Our main technical tool for doing this is a tower $(\Psi_\alpha : \alpha \geq 1)$ of sentences of $\mathcal{L}_{\infty\omega}$.

Definition 6.4.5. Suppose $\alpha_* \geq 1$ is an ordinal. Then let Ω_{α_*} be the set of all pairs (β, δ) where $\beta < \delta < \alpha_*$, and δ is a limit ordinal, and $\beta > 0$. Let $\Omega'_{\alpha_*} = \Omega_{\alpha_*} \cup \{(\beta, \beta+1) : \beta + 1 < \alpha_*\}$.

Let the language of Ψ_{α_*} consist of sorts $(U_\alpha : \alpha < \alpha_*)$, binary relations $R_{\beta,\alpha} \subseteq U_\beta \times U_\alpha$ for each $(\beta, \alpha) \in \Omega'_{\alpha_*}$ and binary relations $E_{\beta,\delta} \subseteq U_\delta \times U_\delta$ for each $(\beta, \delta) \in \Omega_{\alpha_*}$.

Let Ψ_{α_*} be the sentence of $\mathcal{L}_{|\alpha_*|^+\omega}$ asserting:

- $(U_\alpha : \alpha < \alpha_*)$ are disjoint and partition the universe, and each U_α is infinite:
- For all $(\beta, \delta) \in \Omega_{\alpha_*}$ and for all $a, b \in U_\delta$, $aE_{\beta,\delta}b$ if and only if for every $c \in U_\beta$: $cR_{\beta,\delta}a$ if and only if $cR_{\beta,\delta}b$.
- For all $\delta < \alpha_*$ limit and for all $a, b \in U_\delta$ distinct, there is some $(\beta, \delta) \in \Omega_{\alpha_*}$ such that $a \neg E_{\beta\delta} b$.
- (Everything that can happen, happens, part 1.) Suppose $\alpha < \alpha_*$, $\beta^i_j : i < 2, j < m$ are given with each $(\beta^i_j, \alpha) \in \Omega'_{\alpha_*}$, and $(\gamma^i_j : i < 2, j < n)$ are given with each $(\alpha, \gamma^i_j) \in \Omega'_{\alpha_*}$. Suppose $b^i_j \in U_{\beta^i_j}$ for each $i < 2, j < m$, and $c^i_j \in U_{\gamma^i_j}$ for each $i < 2, j < n$. Then there are infinitely many $a \in U_\alpha$ such that $b^0_j R_{\beta^0_j,\alpha} a$ for each

$j < m$, and $b^1_j \neg R_{\beta^1_j,\alpha} a$ for each $j < m$, and $a R_{\alpha,\gamma^0_j} c^0_j$ for each $j < n$, and $a \neg R_{\alpha,\gamma^1_j} c^1_j$ for each $j < n$.

- (Everything that can happen, happens, part 2.) Suppose $\delta < \alpha_*$ is a limit, $\beta_j : j < m$ are distinct with each $(\beta_j, \delta) \in \Omega' \alpha_*$, and $(\gamma^i_j : i < 2, j < n)$ are given with each $(\alpha, \gamma^i_j) \in \Omega'_{\alpha_*}$. Suppose $d_j \in U_\delta$ for each $j < m$, and $c^i_j \in U_{\gamma^i_j}$ for each $i < 2, j < n$. Then there are infinitely many $d \in U_\delta$ such that $d E_{\beta_j} d_j$ for each $j < m$, and $d R_{\gamma^0_j} c^0_j$ for each $j < n$, and $d \neg R_{\gamma^1_j} c^1_j$ for each $j < n$.

We will presently prove that each Ψ_α is consistent, and for $\alpha < \omega_1$, Ψ_α is $\aleph_0$-categorical. In particular, the Ψ_α's are trivial with respect to Borel complexity. For a better Friedman-Stanley tower, we could proceed as follows: let Φ'_α to be the expansion of Ψ_α, where we add infinitely many unary predicates $V_n : n < \omega$, and assert that each $V_n \subseteq U_0$, and for all $a \neq b \in U_0$, there is some n such that $a \in V_n$ if and only if $b \notin V_n$. Then $(\Phi'_\alpha : 1 \leq \alpha < \omega_1)$ turns out to be a smoother version of the tower $(\Phi_\alpha : \alpha < \omega_1)$ (we have each $\Phi'_\alpha \leq_B \Phi_{\alpha+1}$ easily, although I don't know about the reverse inequality). For what we do we will not have to explicitly deal with the Φ'_α's, and in fact they will not be mentioned again; nonetheless, we will be using similar constructions.

We are really interested in the following special models of Ψ_{α_*}:

Definition 6.4.6. Say that M is a standard model of Ψ_{α_*} if:

- $M \models \Psi_{\alpha_*}$;

- For all $a, b \in U_0^M$, $a \notin b$;

- If $\alpha > 0$ and $a \in U_\alpha$, then:

$$a = \{b \in M : \text{ there is some } \beta < \alpha \text{ with } (\beta, \alpha) \in \Omega'_{\alpha_*} \text{ and } b \in U_\beta \text{ and } b\, R_{\beta,\alpha}\, a\}.$$

Note that if M is a standard model of Ψ_{α_*}, then M is determined by its domain. Also, if $N \models \Psi_{\alpha_*}$ satisfies that for all $a, b \in U_0^N$, $a \notin b$, then there is a unique standard M with $U_0^M = U_0^N$ and $M \cong_{U_0^N} N$.

There are two important facts we need about Ψ_{α_*}:

Theorem 6.4.7. Suppose $\alpha_* \geq 1$. Then Ψ_{α_*} is consistent. In fact, suppose λ is a given cardinal cardinal with $\lambda^{<\lambda} = \lambda$ and with $\alpha_* < \lambda^+$; give $\mathcal{P}(\lambda)$ the $< \lambda$-support product topology. Suppose U_0 is 2^λ-dense in $\mathcal{P}(\lambda)$, and each $a \in U_0$ has $|a| = \lambda$. Let τ_0 be the subset topology on U_0.

Then we can find some standard standard $M = (U_\alpha, R_{\beta,\alpha}, E_{\beta,\delta} : \alpha < \alpha_*, (\beta, \alpha) \in \Omega'_{\alpha_*}, (\beta, \delta) \in \Omega_{\alpha_*}) \models \Psi_{\alpha_*}$, such that each $|U_\alpha| = \beth_{\alpha+1}(\lambda)$, and such that U_1 is densely independent over U_0.

Proof. Write $\Omega' = \Omega'_{\alpha_*}$, write $\Omega = \Omega_{\alpha_*}$.

Note that (U_0, τ_0) is 2^λ-nice, i.e. U_0 has a basis of size at most 2^λ (in fact, we can find one of size λ), and each nonempty open subset of U_0 has size 2^λ.

I claim we can find $M = (U_\beta, B_{\beta,\alpha}, \tau_\beta : \beta < \alpha_*, (\beta, \alpha) \in \Omega')$ such that:

- Each $U_\beta \subseteq \mathcal{P}^\beta(\lambda)$ is a set of size $\beth_{\beta+1}(\lambda)$, and τ_β is a topology on A_β which makes it $\beth_{\beta+1}(\lambda)$-nice;

- $(U_\beta : \beta < \alpha_*)$ are pairwise disjoint, and in fact, for every $a \in U_\beta$, $\mathrm{rnk}(a) = \lambda + \beta$;

- Each $B_{\beta,\alpha} \subseteq \mathcal{P}(U_\beta)$, and given $\beta < \alpha_*$, $(B_{\beta,\alpha} : (\beta, \alpha) \in \Omega')$ are pairwise disjoint;

- Each $U_{\beta+1} = B_{\beta,\beta+1}$, and for limit δ, U_δ is the set of sets $a \subseteq \bigcup_{(\beta,\delta)\in\Omega} U_\beta$, such that each $a_\beta \cap U_\beta \in B_{\beta,\delta}$;

- For each $\beta + 1 < \alpha_*$, $\tau_{\beta+1}$ is the topology on $U_{\beta+1}$ from considering it a subset of

$\mathcal{P}(U_\beta)$, where $\mathcal{P}(U_\beta)$ is given the finite support product topology;

- For each $(\beta, \delta) \in \Omega$ with $\delta < \alpha_*$, $E_{\beta,\delta}$ is defined as required by Ψ_{α_*}. Give U_δ the topology τ_δ consisting of finite intersections of equivalence classes of the various $E_{\beta,\delta}$'s;

- For each $\beta < \alpha_*$, $\bigcup_{(\beta,\alpha)\in\Omega'} B_{\beta,\alpha}$ is densely independent over U_β in the τ_β-topology, and each $B_{\beta,\alpha}$ $\beth_{\beta+2}(\lambda)$-dense in $\mathcal{P}(U_\beta)$, where the latter has the finite support product topology.

By induction on $\alpha < \alpha_*$ we construct $(U_\beta, B_{\beta,\gamma}, \tau_\beta : \beta < \alpha, (\beta,\gamma) \in \Omega')$. Indeed, suppose we are given $(U_\beta, B_{\beta,\gamma}, \tau_\beta : 1 \leq \beta < \alpha, (\beta,\gamma) \in \Omega')$. Let U_α, τ_α be defined as required by the clauses (three cases: for $\alpha = 0, \alpha = \beta + 1$, and α limit). In each of these cases, it is easy to check that $|U_\alpha| = \beth_{\alpha+1}(\lambda)$ and in fact U_α is $\beth_{\alpha+1}(\lambda)$-nice under τ_α (for α limit, since $(U_\beta : \beta < \alpha)$ are all disjoint, we get that $U_\alpha \cong \prod_{(\beta,\alpha)\in\Omega} B_{\beta,\alpha}$). Also, in each of the three cases it is easy to check that for all $a \in U_\alpha$, $\mathrm{rnk}(a) = \lambda + \alpha$. From this it follows that U_α is disjoint from each U_β, for $\beta < \alpha$.

Write $\kappa = \beth_{\alpha+1}$. Thus we can choose $(Y_i : i < 2^\kappa)$ as in Theorem 6.3.5, where we take $X = U_\alpha$; that is, each Y_i is 2^κ-dense in $\mathcal{P}(U_\alpha)$ under the finite support product topology, and $\bigcup_i Y_i$ is densely independent over U_α. Choose an injection $F : \{\gamma : (\alpha,\gamma) \in \Omega'\} \to 2^\kappa$. For each $(\alpha, \gamma) \in \Omega'$, define $B_{\alpha,\gamma} = Y_{F(\gamma)}$.

Let $M = \bigcup_{\beta<\alpha_*} U_\beta$. Easily, M is (the domain of) a standard model of Ψ_{α_*}, and is as required. □

We also will want the following easy observation (it is the reason we include the definable equivalence relations $E_{\beta,\delta}$ in the language):

Theorem 6.4.8. Suppose $M, N \models \Psi_{\alpha_*}$. Then $M \equiv_{\infty\omega} N$; in fact, the set of all finite partial isomorphisms from M to N is a back-and-forth system. In particular, for $\alpha_* < \omega_1$, Ψ_{α_*} is $\aleph_0$-categorical.

6.5 Constructing Thick Sets

We aim to prove Theorem 6.2.12. The idea is the following: let λ be a cardinal satisfying certain hypotheses, to be specified; let $\alpha_* < \lambda^+$, and let $f_* \in \mathbb{F}$. We want to find some f_*-closed $A \in \mathbb{V}_{\lambda^+}$ such that $|\mathcal{P}^{\alpha_*}(\lambda) \cap A| = \beth_{\alpha_*}(\lambda)$.

Choose some robust Γ such that each $f_* \in \mathbb{F}_\Gamma$. Choose $a_* \in \mathrm{HC}$ containing parameters for f_* and Γ. We will start with some transitive $V \preceq H(\lambda^+)$ with $a_*, \alpha_*, [\lambda]^{<\lambda} \in V$. Note then that $V \models \Gamma$. We will want to construct a special set $M \in \mathcal{P}^{\alpha_*+1}(\lambda)$, such that $|M| = \beth_{\alpha_*}(\lambda)$. Namely, M will be a standard model of Ψ_{α_*} with $U_0^M \subseteq \mathcal{P}(\lambda)$, as given by Theorem 6.4.7. With a careful choice of U_0^M, it will follow that there is a forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$, and a forcing notion $Q \in V$, and an V-generic filter H over Q in $\mathbb{V}[G]$, such that $M \in V[H]$. Note then that $V[H] \models \Gamma$. Then, working in either $V[H]$ or $\mathbb{V}$, we can close M off under transitive closure, pairing and and $(f_*)_{\mathrm{ptl}}$. This produces an f_*-closed set $A \in \mathbb{V}_{\lambda^+}$ with $M \in A$; it follows $|\mathcal{P}^{\alpha_*}(\lambda) \cap A| \geq \beth_{\alpha_*}(\lambda)$.

Constructing U_0^M will require some hypotheses on λ, which are met whenever λ is a regular strong limit, and can also be forced to hold for all regular λ. We describe these conditions now.

Suppose λ is a cardinal and V is a transitive model of ZFC^- with $[\lambda]^{<\lambda} \in V$ (possibly V is an inner model). If S is a set and λ is a cardinal, then recall $P_{S2\lambda}$ is the set of all partial functions from S to 2 of cardinality less than λ. We view $P_{S2\lambda}$ as adding a λ-Cohen $a \subseteq \mathcal{P}(S)$ (identifying $2^S \cong \mathcal{P}(S)$). Note that $P_{\lambda 2\lambda} \in V$, so it makes sense to say when $a \in \mathcal{P}(S)$ is λ-Cohen over V. We also view each $(P_{S2\lambda})^n = P_{S\times n\,2\lambda}$.

Definition 6.5.1. With λ, V as above, say $X \subseteq \mathcal{P}(\lambda)$ is V-symmetric if: X is 2^λ-dense in $\mathcal{P}(\lambda)$ (with the $< \lambda$-support product topology), and for each injective finite sequence $\overline{a} \in X^n$, $\overline{a}$ is λ-Cohen over V.

Lemma 6.5.2. Suppose λ is a regular strong limit. Then for every robust Γ, for every $a_* \in \mathrm{HC}$, and for every $\alpha_* < \lambda^+$, there some transitive $V \models \Gamma$ with $a_*, \alpha_*, [\lambda]^{<\lambda} \in V$ and $|V| \leq \lambda$, such that there is some V-symmetric $X \subseteq \mathcal{P}(\lambda)$. Furthermore, there is a proper-class forcing extension which does not add any reals, in which the preceding holds for all regular λ.

Proof. First suppose λ is a regular strong limit. We mimic the argument for Theorem 6.4.1 from [31]. Namely, choose some transitive $V \preceq H(\lambda^+)$ with $|V| = \lambda$ and $a \in V$. Note that $V \models \Gamma$, since $H(\lambda^+)$ does. It is easy to construct $h : 2^\lambda \to 2^{\lambda\times\lambda}$ continuous, so that for all $\overline{a} \in (2^\lambda)^{<\omega}$ injective, $h(\overline{a})$ is λ-Cohen over V. For each $\gamma < \lambda$ define $h_\gamma : 2^\lambda \to \mathcal{P}(\lambda)$ by: $\nu \in h_\gamma(f)$ if and only if $h(f)(\nu, \gamma) = 1$. Note that each $\{h_\gamma(f) : \gamma < \lambda\}$ is dense in $\mathcal{P}(\lambda)$. Define $X = \bigcup_{f \in 2^\lambda, \gamma < \lambda} h_\gamma(f)$. Clearly this works.

For the second claim, we can suppose GCH holds (since this can be arranged without adding any reals). Pass to an Easton forcing extension $\mathbb{V}[G]$ where we add $\lambda^+ = 2^\lambda$-many λ-Cohens for every regular cardinal $\lambda > \aleph_0$. Then I claim this works. The remainder of the argument takes place in $\mathbb{V}[G]$. Note that $\mathbb{R} = (\mathbb{R})^{\mathbb{V}}$ since the forcing notion is ω-closed (since we just added λ-Cohens for uncountable λ).

Suppose λ is regular, Γ is robust, $a_* \in \mathrm{HC}$, and $\alpha_* < \lambda_*$. We must find some transitive $V \models \Gamma$ with $a_*, \alpha_*, [\lambda]^{<\lambda} \in V$ and $|V| \leq \lambda$, such that there is some V-symmetric $X \subseteq \mathcal{P}(\lambda)$.

We can do $\lambda = \aleph_0$ by the first part.

Suppose $\lambda > \aleph_0$ is regular. Let $\mathbb{V}[G_\lambda]$ be the intermediate forcing extension, where we add $2^{\lambda'}$-many λ'-Cohens for every regular cardinal $\aleph_0 < \lambda' < \lambda$. Note that $[\lambda]^{<\lambda} \in \mathbb{V}[G_\lambda]$, since adding λ'-Cohens for regular $\lambda' \geq \lambda$ is $< \lambda$-closed. Also, $\mathrm{HC} \subseteq \mathbb{V}[G_\lambda]$, since $\mathrm{HC} = (\mathrm{HC})^{\mathbb{V}}$. Also there is some $Y \subseteq 2^\lambda$ which is λ-Cohen over $\mathbb{V}[G_\lambda]$. Choose a transitive $V \preceq (H(\lambda^+))^{\mathbb{V}[G_\lambda]}$ with $|V| = \lambda$ and $a_*, \alpha_*, [\lambda]^{<\lambda} \in V$. Note that $V \models \Gamma$, since $(H(\lambda^+))^{\mathbb{V}[G_\lambda]}$ does. For each $\beta < 2^\lambda$ let $x_\beta = \{\gamma < \lambda : 2^\lambda \cdot \beta + \gamma \in Y\}$ and let $X = \{x_\beta : \beta < 2^\lambda\}$. Clearly this works.

See, for instance, [45] Chapter VIII for a reference on Easton forcing. $\square$

Thus, to prove Theorem 6.2.12, it suffices to show the following.

Theorem 6.5.3. Suppose λ is a regular cardinal, such that for every robust Γ, for every $a_* \in \mathrm{HC}$, and for every $\alpha_* < \lambda^+$, there is some transitive $V \models \Gamma$ with $a_*, \alpha_*, [\lambda]^{<\lambda} \in V$ and $|V| \leq \lambda$, such that there is some V-symmetric $X \subseteq \mathcal{P}(\lambda)$. Then λ admits thick sets.

Fix some such λ for the rest of the section (note that it follows from the hypothesis that $\lambda = \lambda^{<\lambda}$). Suppose $\alpha_* < \lambda^+$ and $f_* \in \mathbb{F}$; we need to find some f_*-closed $A \in \mathbb{V}_{\lambda^+}$ with $|(\mathcal{P}^{\alpha_*}(\lambda)) \cap A| = \beth_{\alpha_*}(\lambda)$.

Choose some robust Γ such that $f_* \in \mathbb{F}_\Gamma$; choose $a_* \in \mathrm{HC}$ containing parameters for f_*, Γ. Write $\Omega = \Omega_{\alpha_*}$, $\Omega' = \Omega'_{\alpha_*}$. Choose $V \models \Gamma$ transitive with $|V| = \lambda$ and $\alpha_*, a_*, [\lambda]^{<\lambda} \in V$, such that there is some V-symmetric $X \subseteq \mathcal{P}(\lambda)$. By Theorem 6.4.7, we can find some standard $M \models \Psi_{\alpha_*}$, such that $U_0^M = X$ and each $|U_\alpha^M| = \beth_{\alpha+1}(\lambda)$.

If we can find some f_*-closed $A \in \mathbb{V}_{\lambda^+}$ with $M \in A$, then we are done: first of all, note by induction on $\beta < \alpha_*$ that each $U_\beta^M \subseteq \mathcal{P}^{\beta+1}(\lambda)$. Since A is transitive, also each $U_\beta^M \subseteq A$. If $\alpha_* = \alpha+1$ for some α, then U_α^M witnesses $|\mathcal{P}^{\alpha_*}(\lambda) \cap A| = \beth_{\alpha_*}(\lambda)$. Otherwise, $\bigcup_{\alpha<\alpha_*} U_\alpha^M$ witnesses that $|\mathcal{P}^{\alpha_*}(\lambda) \cap A| = \beth_{\alpha_*}(\lambda)$.

So we aim to find some such A.

Choose some $N \models \Psi_{\alpha_*}$ with $N \in V$ (so $|N| \leq \lambda$). Let P be the set of all finite partial isomorphisms from N to M. By Theorem 6.4.8, P adds an isomorphism $\dot{\sigma} : \check{N} \cong \check{M}$.

We identify 2^λ with $\mathcal{P}(\lambda)$, and so view $U_0^M \subseteq 2^\lambda$. Let $Q = \prod_{U_0^N} P_{\lambda 2 \lambda}$ with finite supports, so $Q \in V$. Let $\dot{g}$ be the P-name for $\dot{\sigma} \restriction_{U_0^{\check{N}}}$, a function from $U_0^{\check{N}}$ to $(2^\lambda)^{\check{\mathbb{V}}}$.

Lemma 6.5.4. *P forces that $\dot{g}$ is $\check{Q}$-generic over $\check{V}$.*

Proof. Suppose D is a dense subset of Q in V and $\sigma : N \to M$ is a finite partial isomorphism. It suffices to show that we can find some τ extending σ, so that $\tau \restriction_{U_0^N}$ extends an element of D.

Let $u = \mathrm{dom}(\sigma) \cap U_0^N$, a finite subset of U_0^N. By choice of U_0^M, we have that $s := \sigma \restriction_u$ is $P^u_{\lambda,2,\lambda}$-generic over V. We have the obvious restriction map $\pi : Q \to P_{\lambda,2,\lambda}$ (recall Q is the finite support product $\prod_{U_0^N} P_{\lambda,2,\lambda}$).

I claim that $\pi[D]$ is dense in $P^u_{\lambda 2 \lambda}$. Indeed, given $t_0 \in P^u_{\lambda 2 \lambda}$, choose $t \leq t_0$ in D, then $\pi(t) \leq t_0$ is in $\pi(D)$.

Thus we can find $t_0 \in \pi[D]$ such that $t_0 \subseteq s$. Choose $t \in D$ such that $\pi(t) = t_0$. We wish to show we can find some $\tau \in P$ extending σ and t.

Enumerate $\mathrm{dom}(t) \backslash \mathrm{dom}(t_0) = \{a_i : i < n\}$. For each $i < n$, let $\mathcal{O}_i$ be the basic open subset of 2^λ determined by $t(a_i)$ (where $t(a_i)$ is a partial function from λ to 2 of cardinality less than λ), namely $\mathcal{O}_i$ is the set of extensions of $t(a_i)$ to 2^λ. By extending t, we can suppose $(\mathcal{O}_i : i < n)$ are pairwise disjoint, and that for each $i < n$ and for each $a \in u$, $\sigma(a) \notin \mathcal{O}_i$.

For each $i < n$, since U_1^M is densely independent over U_0^M, we can find some $\tau(a_i) \in U_0^M$ such that for each $b \in U_1^N \cap \mathrm{dom}(\sigma)$, $\tau(a_i) \in \tau(b)$ if and only if $a_i R^N_{0,1} b$. Define $\tau(a) = \sigma(a)$ for all $a \in \mathrm{dom}(\sigma)$. Then $\tau \in P$ extends σ and t, so works. $\square$

Now we finish. Working in $\mathbb{V}$, let A be the least f_*-closed set with $M \in A$ (we just need A to be transitive, closed under $(f_*)_{\mathrm{ptl}}$, and closed under finite sequences). We need to show that $A \in \mathbb{V}_{\lambda^+}$.

Let $\mathbb{V}[G]$ be a P-generic forcing extension of $\mathbb{V}$, let $\sigma = \mathrm{val}(\dot{\sigma}, G)$. Write $g = \sigma \restriction_{U_0^N}$; by Lemma 6.5.4, g is Q-generic over V, hence $V[g]$ is a forcing extension of V. But then $M \in V[g]$: it can be recovered as the unique standard model of Ψ_{α_*} with $U_0^M = g[U_0^N]$, such that g extends to an isomorphism from N to M. Thus $V[g] \models \Gamma$ and $M \in V[g]$, hence $V[g]$ correctly computes A (since $(f_*)^{V[g]} \restriction_{V[g] \cap \mathbb{V}} = (f_*)_{\mathrm{ptl}} \restriction_{V[g] \cap \mathbb{V}}$). Hence $A \in V[g]$, so $\mathrm{rnk}(A) < \mathrm{rnk}(V[g]) = \mathrm{rnk}(V)$.

Back in $\mathbb{V}$, this means $A \in \mathbb{V}_{\lambda^+}$.

6.6 Schröder-Bernstein Properties

In this section, we define various Schröder-Bernstein properties of sentences $\Phi \in \mathcal{L}_{\omega_1 \omega}$. In the next section, we apply the thickness machinery to show that these properties imply a bound on the complexity of countable models of Φ, assuming large cardinals. The major example we have in mind for this is that of torsion-free abelian groups, as discussed in Chapter 7.

Say that a complete first order theory T has the Schröder-Bernstein property in the class of all models if whenever $M, N \models T$ are elementarily bi-embeddable, then they are isomorphic. This notion was originally introduced by Nurmagambetov [69], [68] (without the phrase "in the class of all models"), and further studied by Goodrick in several papers, including his thesis [17], wherein he proves that if T has the Schröder-Bernstein property that T is classifiable of depth 1, i.e. $I(T, \aleph_\alpha) \leq |\alpha + \omega|^{2^{\aleph_0}}$ for all α.

We are interested in studying this phenomenon in countable model theory. To do so, we deviate from the above set-up in two ways. First, we interested in Schröder-Bernstein properties for countable structures (or generally for potential canonical Scott sentences). Second, it is convenient in applications to use the following notion of embedding.

Definition 6.6.1. Suppose $\mathcal{L}$ is a language, and M, N are $\mathcal{L}$-structures. Then say that $f : M \leq N$ is an embedding if it is a homomorphism; that is, f commutes with the function symbols, and if R is an n-ary relation, then $f[R^M] \subseteq R^N$.

This allows the most freedom. If one wanted to look at elementary embedding, then just Morleyize; generally, we can pass to $\mathcal{L}_{\omega_1\omega}$-definable expansions to get whatever notion we wanted.

Definition 6.6.2. Say that Φ has the Schröder-Bernstein property if for all $M, N \models \Psi$ countable, if $M \sim N$ then $M \cong N$.

We will eventually show that if Φ has the Schröder-Bernstein property, and if certain large cardinals hold, then this puts a bound on the thickness spectrum of Φ. In fact we show more. We will presently define the α-ary Schröder-Bernstein property for every ordinal α, and show that under large cardinals, any of these puts a bound on the thickness spectrum of Φ.

Definition 6.6.3. Suppose M, N are $\mathcal{L}$-structures and $\overline{a} \in M$, $\overline{b} \in N$ are tuples of the same length. We define what it means for $(M, \overline{a}) \sim^{SB}_\alpha (N, \overline{b})$ by induction on α. Say that $(M, \overline{a}) \sim^{SB}_0 (N, \overline{b})$ if there are $f : M \leq N$ and $g : N \leq M$ such that $f(\overline{a}) = \overline{b}$ and $g(\overline{b}) = \overline{a}$. Say that $(M, \overline{a}) \sim^{SB}_{\alpha+1} (N, \overline{b})$ if for every $a \in M$ there is $b \in M$ with $(M, \overline{a}a) \sim^{SB}_\alpha (N, \overline{b})$, and vice versa. For δ limit, say that $(M, \overline{a}) \sim^{SB}_\delta (N, \overline{b})$ if $(M, \overline{a}) \sim^{SB}_\beta (N, \overline{b})$ for every $\beta < \delta$. Finally say that $M \sim^{SB}_\alpha N$ if $(M, \emptyset) \sim^{SB}_\alpha (N, \emptyset)$.

Suppose $\Phi, \Psi \in \mathrm{CSS}(\mathcal{L})_{\mathrm{ptl}}$. Then define $\Phi \sim^{SB}_\alpha \Psi$ if for some or any forcing extension $\mathbb{V}[G]$ in which Φ, Ψ become hereditarily countable, and for some or any $M \in \operatorname{Mod}(\Phi)^{\mathbb{V}[G]}, N \in \operatorname{Mod}(\Psi)^{\mathbb{V}[G]}$, we have $(M \sim_\alpha^{SB} N)^{\mathbb{V}[G]}$. Clearly this does not depend on the choice of forcing extension or the models.

If $\Phi \in \mathcal{L}_{\omega_1\omega}$, then say that Φ has the α-ary Schröder-Bernstein property if for all $\varphi, \psi \in \mathrm{CSS}(\Phi)_{\mathrm{ptl}}$, if $\varphi \sim_\alpha^{SB} \psi$ then $\varphi = \psi$.

Note that the relation $\sim_\alpha^{SB}$ is highly nonabsolute on uncountable models; we will typically only be interested in evaluating it on countable models (possibly in forcing extensions).

We want a more explicit characterization of what it means for a pair of potential canonical Scott sentences Φ, Ψ to have $\Phi \sim_\alpha^{SB} \Psi$. For this, it is helpful to consider colored trees.

Definition 6.6.4. A colored tree is a structure $\mathcal{T} = (T, \leq_\mathcal{T}, 0_\mathcal{T}, c_\mathcal{T})$ where $(T, \leq_\mathcal{T}, 0_\mathcal{T})$ is a tree of height at most ω with root $0_\mathcal{T}$, and $c_\mathcal{T} : T \to \omega$ is a coloring. Let $\mathrm{CT} \in (\mathcal{L}_{\mathrm{ct}})_{\omega_1\omega}$ describe colored trees (formally, the language $\mathcal{L}_{\mathrm{ct}}$ includes unary predicates U_n representing $c^{-1}(n)$, for each $n < \omega$)). Note that $f : \mathcal{T} \leq \mathcal{T}'$ is an embedding if $f(0_\mathcal{T}) = 0_{\mathcal{T}'}$, and for all $s, t \in \mathcal{T}$, $s \leq_\mathcal{T} t$ implies $f(s) \leq_{\mathcal{T}'} f(t)$, and for all $s \in \mathcal{T}$, $c_\mathcal{T}(s) = c_{\mathcal{T}'}(f(s))$.

If $\mathcal{T}$ is a colored tree and $s \in \mathcal{T}$, let $\mathcal{T}_{\geq s}$ be the colored tree with root $0_{\mathcal{T}_{\geq s}} = s$, consisting of all elements of $\mathcal{T}$ extending s.

We will not be too interested in the $\sim_\alpha^{SB}$-relations on colored trees; instead we want the following very special relations.

Definition 6.6.5. Suppose $\mathcal{T}, \mathcal{T}'$ are colored trees. We define what it means for $\mathcal{T} \leq_\alpha^{\mathrm{ct}} \mathcal{T}'$ by induction on α. Put $\mathcal{T} \leq_0^{\mathrm{ct}} \mathcal{T}'$ if $c_\mathcal{T}(0_\mathcal{T}) = c_{\mathcal{T}'}(0_{\mathcal{T}'})$. For δ limit, put $\mathcal{T} \leq_\delta^{\mathrm{ct}} \mathcal{T}'$ if $\mathcal{T} \leq_\alpha^{\mathrm{ct}} \mathcal{T}'$ for all $\alpha < \delta$. Finally, put $\mathcal{T} \leq_{\alpha+1}^{\mathrm{ct}} \mathcal{T}'$ if for all $s \in \mathcal{T}$ an immediate successor of $0_{\mathcal{T}}$, there is $s' \in \mathcal{T}'$ an immediate successor of $0_{\mathcal{T}'}$, such that $\mathcal{T}_{\geq s} \sim^{\text{ct}}_{\alpha} \mathcal{T}'_{\geq s'}$. Note that $\mathcal{T} \leq \mathcal{T}'$ if $\mathcal{T} \leq^{\text{ct}}_{\alpha} \mathcal{T}'$ for all α.

Next we define what it means for $\mathcal{T} \sim^{\text{ct}}_{\alpha} \mathcal{T}'$ by induction on α. Put $\mathcal{T} \sim^{\text{ct}}_{0} \mathcal{T}'$ if $\mathcal{T} \sim \mathcal{T}'$. For δ limit, put $\mathcal{T} \sim^{\text{ct}}_{\delta} \mathcal{T}'$ if $\mathcal{T} \sim^{\text{ct}}_{\alpha} \mathcal{T}'$ for all $\alpha < \delta$. Finally, put $\mathcal{T} \sim^{\text{ct}}_{\alpha+1} \mathcal{T}'$ if for all $s \in \mathcal{T}$ an immediate successor of $0_{\mathcal{T}}$, there is $s' \in \mathcal{T}'$ an immediate successor of $0_{\mathcal{T}'}$, such that $\mathcal{T}_{\geq s} \sim^{\text{ct}}_{\alpha} \mathcal{T}'_{\geq s'}$, and vice versa.

The following lemma is very special to colored trees. For instance, the embeddability relation on uncountable dense linear orders is very complicated, even though DLO is $\aleph_0$-categorical.

Theorem 6.6.6. Suppose V is a transitive model of ZFC^-, and $\mathcal{T}, \mathcal{T}' \in V$ are colored trees. Then for $R \in \{\leq, \leq^{\text{ct}}_{\alpha}, \sim^{\text{ct}}_{\alpha} \colon \alpha \in V\}$, we have that $\mathcal{T} \; R \; \mathcal{T}'$ if and only if $(\mathcal{T} \; R \; \mathcal{T}')^V$.

Proof. First, an easy induction on $\alpha \in V$ shows that for all $\mathcal{T}, \mathcal{T}' \in V$, $\mathcal{T} \leq^{\text{ct}}_{\alpha} \mathcal{T}'$ if and only if $(\mathcal{T} \leq^{\text{ct}}_{\alpha} \mathcal{T}')^V$.

Now, if $\mathcal{T} \leq \mathcal{T}'$, then in particular $\mathcal{T} \leq^{\text{ct}}_{\alpha} \mathcal{T}'$ for all $\alpha \in V$, so $(\mathcal{T} \leq \mathcal{T}')^V$. For the converse, suppose $(\mathcal{T} \leq \mathcal{T}')^V$ and suppose towards a contradiction that $\mathcal{T} \not\leq \mathcal{T}'$. Then there is some ordinal α such that $\mathcal{T} \not\leq^{\text{ct}}_{\alpha} \mathcal{T}'$. Choose $(\mathcal{T}, \mathcal{T}')$ so as to minimize α. Note that $\alpha \notin V$, so in particular $\alpha \neq 0$; also α cannot be a limit ordinal by minimality of α. So we can write $\alpha = \beta + 1$ for some β. Choose $s \in \mathcal{T}$ an immediate successor of $0_{\mathcal{T}}$ such that for every $s' \in \mathcal{T}'$ an immediate successor of $0_{\mathcal{T}'}$, $\mathcal{T}_{\geq s} \not\leq^{\text{ct}}_{\beta} \mathcal{T}'_{\geq s'}$. Since $(\mathcal{T} \leq \mathcal{T}')^V$, we can find some $s' \in \mathcal{T}'$ an immediate successor of $0_{\mathcal{T}'}$ such that $(\mathcal{T}_{\geq s} \leq \mathcal{T}'_{\geq s'})^V$. This contradicts the minimality of α.

Hence we get the claim for $R = \leq$; another easy induction on α gets the claim for $\sim^{\text{ct}}_{\alpha}$. ∎

The following theorem, combined with Theorem 6.6.6, explains why embedding on colored trees is useful for us. As notation, if Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ then let $\mathrm{Mod}_{\mathrm{HC}}(\Phi)$ be the set of hereditarily countable models of Φ. We will only use this in the case where $\Phi = \mathrm{CT}$.

Theorem 6.6.7. Suppose $\Phi \in \mathcal{L}_{\omega_1\omega}$. Then there is a ZFC^--absolute map $f : \mathrm{CSS}(\Phi) \to \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT})$, such that ZFC^--persistently, the following holds: for all $\varphi, \psi \in \mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ and for all ordinals α, $\varphi \sim_\alpha^{SB} \psi$ if and only if $f_{\mathrm{ptl}}(\varphi) \sim_\alpha^{\mathrm{ct}} f_{\mathrm{ptl}}(\psi)$.

Proof. We can suppose the language $\mathcal{L}$ is relational. Enumerate $\mathcal{L} = (R_n : n < \omega)$, where each R_n is m_n-ary. Also, it suffices to consider the case where $\varphi, \psi \in \mathrm{CSS}(\Phi)$ (i.e., are countable), since then the same argument will run in any forcing extension.

Suppose $\varphi \in \mathrm{CSS}(\Phi)$; we describe how to construct $f(\varphi) = (S_\varphi, \leq_\varphi, c_\varphi)$. For each $n < \omega$, let $S_\infty^n(\varphi)$ be as defined in Definition 5.4.5. Then $S_\infty^{<\omega}(\varphi) = \bigcup_n S_\infty^n(\varphi)$ naturally forms a tree whose n'th level is $S_\infty^n(\varphi)$. Define a tree extension $S_\varphi \supseteq S_\infty^{<\omega}(\varphi)$ as follows: for each $\sigma(\overline{x}) \in S_\infty^n(\varphi)$, for each $n' \leq n$, and for each $s \in n^{m_{n'}}$ such that $R_{n'}(x_{s(i)} : i < m_{n'}) \in \sigma(\overline{x})$, let $t_{\sigma,n',s}$ be an immediate successor of $\sigma(\overline{x})$ (and these are the only elements we add). Define $c_\varphi \restriction_{S_\infty^{<\omega}(\varphi)}$ to be constantly 0, say, and define each $c_\varphi(t_{\sigma,n',s})$ so as to encode (n', s).

Then it is clear this works. $\square$

Since the notion of α-ary Schröder-Bernstein property highly depends on the choice of language, we cannot hope that it is a dividing line in countable model theory. The following is an abstract consequence of the α-ary Schröder-Bernstein property, which is a better candidate for this.

Definition 6.6.8. Suppose α is an ordinal. Then say that $X \subseteq \mathrm{HC}$ admits α-ary Schröder-

Bernstein invariants if, for some robust Γ: X is Γ-absolute, and there is a Γ-absolute map $f : X \to \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT})$, such that for all $\varphi, \psi \in \mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ distinct, $f_{\mathrm{ptl}}(\varphi) \not\sim^{\mathrm{ct}}_{\alpha} f_{\mathrm{ptl}}(\psi)$.

Say that Φ admits α-ary Schröder-Bernstein invariants if $\mathrm{CSS}(\Phi)$ does.

Thus, if Φ has the α-ary Schröder-Bernstein property, then Φ admits α-ary Schröder-Bernstein invariants, by Theorem 6.6.7. Actually, it is enough for Φ to have an $\mathcal{L}_{\omega_1\omega}$-definable expansion with the α-ary Schröder-Bernstein property.

We remark on the following downward Lowenheim-Skolem result.

Theorem 6.6.9. Suppose $\Phi \in \mathcal{L}_{\omega_1\omega}$, and α is an ordinal. Write $\lambda = |\alpha|$. Then:

(A) Φ has the α-ary Schröder-Bernstein property if and only if for all $\varphi, \psi \in \mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap H(\lambda^+)$, if $\varphi \sim^{SB}_{\alpha} \psi$ then $\varphi = \psi$.

(B) Φ admits α-ary Schröder-Bernstein invariants if and only if for some robust Γ, and some Γ-absolute $f : \mathrm{CSS}(\Phi) \to \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT})$, we have that for all $\varphi, \psi \in \mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap H(\lambda^+)$, if $\varphi \neq \psi$ then $f_{\mathrm{ptl}}(\varphi) \not\sim^{\mathrm{ct}}_{\alpha} f_{\mathrm{ptl}}(\psi)$.

Proof. To prove both (A) and (B), it suffices to show the following: suppose Γ is robust, and $f : \mathrm{CSS}(\Phi) \to \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT})$ is Γ-absolute. Suppose for all $\varphi, \psi \in \mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap H(\lambda^+)$, if $\varphi \neq \psi$ then $f_{\mathrm{ptl}}(\varphi) \not\sim^{\mathrm{ct}}_{\alpha} f_{\mathrm{ptl}}(\psi)$. Then for all $\varphi, \psi \in \mathrm{CSS}(\Phi)_{\mathrm{ptl}}$, if $\varphi \neq \psi$ then $f_{\mathrm{ptl}}(\varphi) \not\sim^{\mathrm{ct}}_{\alpha} f_{\mathrm{ptl}}(\psi)$.

We prove the contrapositive; so suppose some $\varphi \neq \psi \in \mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ with $f_{\mathrm{ptl}}(\varphi) \sim^{\mathrm{ct}}_{\alpha} f_{\mathrm{ptl}}(\psi)$. Choose κ regular so that $\varphi, \psi \in H(\kappa)$. Choose $V_0 \preceq H(\kappa)$ with $|V_0| \leq \lambda$ so that $\alpha \subseteq\in V_0$ and $\varphi, \psi \in V_0$ and V_0 contains parameters for f, and let V be the transitive collapse of V_0. Let φ', ψ' be the image of φ, ψ under the transitive collapse. Then $(f_{\mathrm{ptl}}(\varphi') \sim^{\mathrm{ct}}_{\alpha} f_{\mathrm{ptl}}(\psi'))^V$, but by Theorem 6.6.7, this means $f_{\mathrm{ptl}}(\varphi') \sim^{\mathrm{ct}}_{\alpha} f_{\mathrm{ptl}}(\psi')$. Since $\varphi', \psi' \in V \in H(\lambda^+)$ we conclude. $\square$

Corollary 6.6.10. Suppose $\Phi \in \mathcal{L}_{\omega_1\omega}$, and $\alpha < \omega_1$. Then Φ has the α-ary Schröder-Bernstein property if and only if for all countable $M, N \models \Phi$, if $M \sim_\alpha^{SB} N$ then $M \cong N$; moreover, this will continue to hold in every forcing extension. Φ admits α-ary Schröder-Bernstein invariants if and only if there is some robust Γ and some $f : \mathrm{CSS}(\Phi) \to \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT})$, such that for all $\varphi, \psi \in \mathrm{CSS}(\Phi)$, if $\varphi \neq \psi$ then $f_{\mathrm{ptl}}(\varphi) \not\sim_\alpha^{\mathrm{ct}} f_{\mathrm{ptl}}(\psi)$; moreover, this will hold Γ-persistently.

Proof. This follows immediately from Theorem 6.6.9, except for the two moreover clauses; these follow from Lévy's Absoluteness Principle (it suffices to check that the statement holds in every countable transitive model of ZFC^- or Γ, respectively). $\square$

6.7 Counting Colored Trees up to Biembeddability

In this section, we show that if Φ has the α-ary Schröder Bernstein property, then Φ is not Borel complete, assuming a certain large cardinal. Specifically, we will need the Erdös cardinals:

Definition 6.7.1. Suppose α is an ordinal (we will only use the case $\alpha = \omega$). Then let $\kappa(\alpha)$ be the least cardinal κ with $\kappa \to (\alpha)_2^{<\omega}$ (if it exists). (Recall that this means: whenever $F : [\kappa(\alpha)]^{<\omega} \to 2$, there is some $X \subseteq \kappa(\omega)$ of ordertype α, such that $F \restriction_{[X]^n}$ is constant for each $n < \omega$.)

$\kappa(\omega)$ is a large cardinal: it is always inaccessible and has the tree property. On the other hand, it is absolute to $\mathbb{V} = \mathbb{L}$, and well below the consistency strength of a measurable cardinal. See [28] for a description of these results.

The following is a theorem of Shelah [73]. As notation, given $\Phi \in \mathcal{L}_{\omega_1\omega}$, let $\mathrm{Mod}_{\mathbb{V}}(\Phi)$ denote the class of all models of Φ. Also, in the following theorem, the term "antichain" is used in the sense of well-quasi-ordering theory (rather than in the sense of forcing theory), so A is an antichain of for all $a, b \in A$, $a \not\leq b$ and $b \not\leq a$.

Theorem 6.7.2. Suppose $\kappa(\omega) < \infty$. Then $(\mathrm{Mod}_{\mathbb{V}}(CT), \leq)$ is a $\kappa(\omega)$-well-quasi-order (in fact a $\kappa(\omega)$-better-quasi-order). In other words, it has no descending chains nor antichains of size $\kappa(\omega)$.

This theorem is a fundamental constraint on the complexity of biembeddability relations, and will allow us to bound the complexity of sentences with the α-ary Schröder-Bernstein property.

Before proceeding, we want the following definition and technical lemma from [73] (see the proof of Theorem 5.3 there). These allow us to replace general colored trees by well-founded colored trees.

Definition 6.7.3. Suppose $\mathcal{T}$ is a colored tree and α is an ordinal. Then let $\mathcal{T} \times \alpha$ denote the colored tree of all pairs $(s, \overline{\beta})$, where $s \in \mathcal{T}$ is of height n, and $\overline{\beta} = (\beta_0, \ldots, \beta_{n-1})$ is a strictly decreasing sequence of ordinals with $\beta_0 < \alpha$. We define $c_{\mathcal{T} \times \alpha}(s, \overline{\beta}) = c_{\mathcal{T}}(s)$.

Lemma 6.7.4. Suppose $\mathcal{T}, \mathcal{T}'$ are colored trees. Then for all ordinals α, $\mathcal{T} \times \alpha \leq^{\mathrm{ct}}_{\alpha} \mathcal{T}' \times \alpha$ if and only if $\mathcal{T} \leq^{\mathrm{ct}}_{\alpha} \mathcal{T}'$ if and only if $\mathcal{T} \times \alpha \leq \mathcal{T}' \times \alpha$.

Proof. We verify by induction on α that $\mathcal{T} \times \alpha \leq^{\mathrm{ct}}_{\alpha} \mathcal{T}' \times \alpha$ implies $\mathcal{T} \leq^{\mathrm{ct}}_{\alpha} \mathcal{T}'$ implies $\mathcal{T} \times \alpha \leq \mathcal{T}' \times \alpha$ (the remaining implication is trivial). $\alpha = 0$ is immediate.

Successor stage first implication: suppose $\mathcal{T} \times (\alpha+1) \leq^{\mathrm{ct}}_{\alpha+1} \mathcal{T}' \times (\alpha+1)$, and let $s \in \mathcal{T}$ be an immediate successor of $0_{\mathcal{T}}$. Let $(s', \beta) \in \mathcal{T}' \times (\alpha+1)$ be an immediate successor of $0_{\mathcal{T}' \times (\alpha+1)}$ such that $(\mathcal{T} \times (\alpha+1))_{\geq (s, \alpha)} \leq^{\mathrm{ct}}_{\alpha} (\mathcal{T}' \times (\alpha+1))_{(s', \beta)}$. This means that $(\mathcal{T}_{\geq s}) \times \alpha \leq^{\mathrm{ct}}_{\alpha} (\mathcal{T}'_{\geq s}) \times \beta$ (since the corresponding trees are isomorphic). But easily $(\mathcal{T}'_{\geq s}) \times \beta \leq^{\mathrm{ct}} (\mathcal{T}'_{\geq s}) \times \alpha$, so we get that $(\mathcal{T}_{\geq s}) \times \alpha \leq^{\mathrm{ct}}_{\alpha} (\mathcal{T}'_{\geq s}) \times \alpha$. Thus, by the inductive hypothesis $\mathcal{T}_{\geq s} \leq^{\mathrm{ct}}_{\alpha} \mathcal{T}'_{\geq s'}$.

Successor stage, second implication: suppose $\mathcal{T} \leq^{\mathrm{ct}}_{\alpha+1} \mathcal{T}'$; given $(s, (\beta)) \in \mathcal{T} \times (\alpha+1)$ an immediate successor of $0_{\mathcal{T}\times(\alpha+1)}$, choose $s' \in \mathcal{T}'$ an immediate successor of $0_{\mathcal{T}}$ such that $\mathcal{T}_{\geq s} \leq^{\mathrm{ct}}_{\beta} \mathcal{T}'_{\geq s'}$, and note by the inductive hypothesis that $(\mathcal{T} \times (\alpha+1))_{\geq(s,(\beta))} \cong \mathcal{T}_{\geq s} \times \beta \leq \mathcal{T}'_{\geq s'} \times \beta \cong (\mathcal{T}' \times (\alpha+1))_{\geq(s',(\beta))}$.

Limit stage, first implication: suppose $\mathcal{T} \times \delta \leq^{\mathrm{ct}}_{\delta} \mathcal{T}' \times \delta$. Thus, for all $\alpha < \delta$, $\mathcal{T} \times \delta \leq^{\mathrm{ct}}_{\alpha} \mathcal{T}' \times \delta$. By the inductive hypothesis, this implies that for all $\alpha < \delta$, $(\mathcal{T} \times \delta) \times \alpha) \leq^{\mathrm{ct}}_{\alpha} (\mathcal{T}' \times \delta) \times \alpha$, but always $(\mathcal{S} \times \gamma_0) \times \gamma_1 \sim \mathcal{S} \times \min(\gamma_0, \gamma_1)$, so we get that $(\mathcal{T} \times \alpha) \leq^{\mathrm{ct}}_{\alpha} (\mathcal{T}' \times \alpha)$, hence by the inductive hypothesis again $\mathcal{T} \leq^{\mathrm{ct}}_{\alpha} \mathcal{T}'$. This holds for all $\alpha < \delta$ so $\mathcal{T} \leq^{\mathrm{ct}}_{\delta} \mathcal{T}'$.

Limit stage, second implication: suppose $\mathcal{T} \leq^{\mathrm{ct}}_{\delta} \mathcal{T}'$. Then by definition of $\leq^{\mathrm{ct}}_{\delta}$ and the inductive hypothesis, we get that $\mathcal{T} \times \alpha \leq \mathcal{T}' \times \alpha$ for all $\alpha < \delta$. Since $\mathcal{T}' \times \alpha \leq \mathcal{T} \times \delta$ and $\mathcal{T} \times \delta = \bigcup_{\alpha<\delta} \mathcal{T} \times \alpha$, we get that $\mathcal{T} \times \delta \leq \mathcal{T}' \times \delta$.

□

We can now prove the following. To fix notation, if $\mathcal{T}$ is a colored tree and $t \in T$, then inductively define $\mathrm{rnk}(\mathcal{T}, t) = \sup\{\mathrm{rnk}(\mathcal{T}, s) + 1 : s \text{ an immediate predecessor of } t\}$. Define $\mathrm{rnk}(\mathcal{T}) = \mathrm{rnk}(\mathcal{T}, 0_{\mathcal{T}})$. So $\mathcal{T}$ is well-founded if and only if $\mathrm{rnk}(\mathcal{T}) < \infty$.

Lemma 6.7.5. Suppose $\kappa(\omega)$ exists. Suppose α is a nonzero ordinal. Then there are at most $|\alpha|^{<\kappa(\omega)}$ colored trees $\mathcal{T}$ with $\mathrm{rnk}(\mathcal{T}) < \alpha$, up to biembeddability.

Proof. We proceed by induction on $\alpha \geq 1$. For $\alpha = 1$, note that colored trees of rank 1 are determined up to biembeddability by the color of $0_{\mathcal{T}}$ and, for each $n < \omega$, whether or not $0_{\mathcal{T}}$ has an (immediate) successor of color n. There are only $2^{\aleph_0} < \kappa(\omega) = 1^{<\kappa(\omega)}$-many possibilities.

The case α limit is trivial, since $\sum_{\beta<\alpha}|\beta|^{<\kappa(\omega)} \leq |\alpha|^{<\kappa(\omega)}$.

Suppose we are at stage $\alpha+1$. Write $\kappa = |\alpha|$. Let $\mathbb{S}$ be a choice of representatives for well-founded colored trees of rank $< \alpha$ up to biembeddability; so $|\mathbb{S}| \leq \kappa^{<\kappa(\omega)}$. Suppose $\mathcal{T} = (T, <, c)$ is given of rank α. Let $X_\mathcal{T}$ be the set of all $\mathcal{S} \in \mathbb{S}$ such that there is some $t \in \mathcal{T}$ of height 1 such that $\mathcal{S}$ embeds into the colored tree $\mathcal{T}_{\geq t}$. Note that $\mathcal{T}$ is biembeddable with the tree $(\mathcal{T}', <', c')$, which is defined by: $c'(0_{\mathcal{T}'}) = c(0_\mathcal{T})$, and then we put a copy of each $\mathcal{S} \in X_\mathcal{T}$ above $0_{\mathcal{T}'}$. Thus, $\mathcal{T}/\sim$ is determined by the pair $(X_\mathcal{T}, c(0_\mathcal{T}))$, where $X_\mathcal{T}$ is a downward-closed subset of $\mathbb{S}$ (ordered by embeddability $\leq$).

Thus, it suffices to show there are only $\kappa^{<\kappa(\omega)}$-many downward closed subsets of $\mathbb{S}$.

Suppose $X \subseteq \mathbb{S}$ is downward closed. Let $\mathbf{T}$ be the tree (of infinite height) $\mathbb{S}^{<\lambda}$ where λ is large (say $\lambda = |\mathbb{S}|^+$). Then inductively it is easy to find a subtree $\mathbf{S}$ of $\mathbf{T}$ such that:

- Whenever $(\mathcal{S}_\beta : \beta < \alpha) \in \mathbf{S}$, then for all $\beta < \beta' < \alpha$, $\mathcal{S}_\beta > \mathcal{S}_{\beta'}$, and each $\mathcal{S}_\beta \notin X$;
- For each $\overline{S} = (\mathcal{S}_\beta : \beta < \alpha) \in \mathbf{S}$, the set of all $\mathcal{S} \in \mathbf{S}$ such that $\overline{S}\mathcal{S} \in \mathbf{S}$ forms a maximal antichain in $\{\mathcal{S} \in \mathbb{S}\backslash X : \mathcal{S} < \mathcal{S}_\beta \text{ for all } \beta < \alpha\}$.

Since $(\mathbb{S}, \leq)$ has no descending chains of length $\kappa(\omega)$, $\mathbf{S}$ is of height at most $\kappa(\omega)$ and has no branches of length $\kappa(\omega)$. Further, since $\kappa(\omega)$ is inaccessible and $(\mathbb{S}, \leq)$ has no antichains of size $\kappa(\omega)$, each level of $\mathbf{S}$ must have size less than $\kappa(\omega)$. Thus, since $\kappa(\omega)$ has the tree property, $\mathbf{S}$ must be of height less than $\kappa(\omega)$; thus $|\mathbf{S}| < \kappa(\omega)$. Thus, it suffices to show that X is determined by $\mathbf{S}$ (since $|\mathbb{S}|^{<\kappa(\omega)} \leq \kappa^{<\kappa(\omega)}$). Define $Y = \{\mathcal{S} \in \mathbf{S} :$ there is no $(\mathcal{S}_\beta : \beta \leq \alpha) \in \mathbf{S}$ with $\mathcal{S}_\alpha \leq \mathcal{S}\}$; it suffices to show that $X = Y$.

It follows from immediately from the construction of $\mathbf{S}$ that $X \subseteq Y$; so it suffices to show that $Y \subseteq X$. So suppose $\mathcal{S} \notin X$; we show $\mathcal{S} \notin Y$. Define a chain $(\mathcal{S}_\beta : \beta < \beta_*)$ through $\mathbf{S}$ inductively, so that each $\mathcal{S} < \mathcal{S}_\beta$, for as long as possible. This process must stop, say we cannot find $\mathcal{S}\beta_*$. If $\beta_* = \beta + 1$ then this means every immediate successor of $\mathcal{S}_\beta$ is either incomparable with or below $\mathcal{S}$; by maximality of the antichain, there must be some $\mathcal{S}_{\beta_*} \leq \mathcal{S}$ with $(\mathcal{S}_\beta : \beta \leq \beta_*) \in \mathbf{S}$, as desired. The limit case is similar. $\square$

This allows us to prove the following:

Theorem 6.7.6. Suppose $\kappa(\omega)$ exists, α is an ordinal, and Φ admits α-ary Schröder-Bernstein invariants. Then for all λ, $\tau(\Phi, \lambda) \leq \beth_\alpha(\lambda^{<\kappa(\omega)})$. In particular, $\tau(\Phi, \kappa(\omega)) \leq \beth_\alpha(\kappa(\omega))$.

Proof. Let $f : \mathrm{CSS}(\Phi) \to \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT})$ witness that Φ admits α-ary Schröder-Bernstein invariants. Define $g : \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT}) \times \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT}) \to \omega_1$ via $g(\mathcal{T}, \mathcal{T}') = 0$ if $\mathcal{T} \leq \mathcal{T}'$, and otherwise $g(\mathcal{T}, \mathcal{T}') =$ the least α such that $\mathcal{T} \not\leq^{\mathrm{ct}}_\alpha \mathcal{T}'$. By Theorem 6.6.7, g is ZFC^--absolute (via the given definition). Define $h : \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT}) \times \mathrm{HC} \to \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT})$ via: $h(\mathcal{T}, s) = \mathcal{T}_{\geq s}$ if $s \in \mathcal{T}$, otherwise $h(\mathcal{T}, s)$ is some fixed $\mathcal{T}_0$.

Write $f_* = f \times g \times h$. I claim that f_* witnesses that for every λ, $\tau(\Phi, \lambda) \leq \beth_\alpha(\lambda^{<\kappa(\omega)})$. Indeed, suppose $A \in \mathbb{V}_{\lambda^+}$ is f_*-closed, i.e. f-closed, g-closed and h-closed, and let $\alpha_* = A \cap \mathrm{ON}$, so $\alpha_* < \lambda^+$.

I claim that A contains at most $\lambda^{<\kappa(\omega)}$-many colored trees up to biembeddability. Indeed, note that for all $\mathcal{T}, \mathcal{T}' \in A$ with $\mathcal{T} \not\leq \mathcal{T}'$, we have that $\mathcal{T} \not\leq^{\mathrm{ct}}_{\alpha_*} \mathcal{T}'$, since A is g-closed. Hence $\mathcal{T} \times \alpha_* \not\leq \mathcal{T}' \times \alpha_*$, by Lemma 6.7.4. Hence we conclude by Lemma 6.7.5.

I claim that for every ordinal β, A contains at most $\beth_\beta(\lambda^{<\kappa(\omega)})$-many colored trees up to $\sim^{\mathrm{ct}}_\beta$. We have just proved $\beta = 0$. Suppose we have verified β; then note that $\mathcal{T}/\sim^{\mathrm{ct}}_{\beta+1}$ is determined by $c_\mathcal{T}(0_\mathcal{T})$ along with $\{\mathcal{T}_{\geq s}/\sim^{\mathrm{ct}}_\beta$: s an immediate successor of $0_\mathcal{T}\}$. Since A is h-closed, if $\mathcal{T} \in A$ then each $\mathcal{T}_{\geq s} \in A$ so we conclude by the inductive hypothesis. Similarly, if β is limit, then $\mathcal{T}/\sim^{\mathrm{ct}}_{\beta+1}$ is determined by $c_\mathcal{T}(0_\mathcal{T})$ along with $\{\mathcal{T}_{\geq s}/\sim_{\beta'}$:

s an immediate successor of $0_{\mathcal{T}}, \beta' < \beta\}$.

Since A is f-closed, it follows that $|\mathrm{CSS}(\Phi)_{\mathrm{ptl}} \cap A| \leq \beth_\alpha(\lambda^{<\kappa(\omega)})$ as desired. $\square$

Corollary 6.7.7. Assume $\kappa(\omega)$ exists. Suppose Φ admits α-ary Schröder-Bernstein invariants for some ordinal α. Then Φ is not $\leq^*_{a\Delta^1_2}$-complete (and hence not Borel complete). If $\alpha < \omega_1$, then $\mathrm{TAG}_{\alpha+1} \not\leq^*_{a\Delta^1_2} \Phi$ (and hence $\mathrm{TAG}_{\alpha+1} \not\leq_B \Phi$).

Proof. Choose $\lambda > \alpha, \kappa(\omega)$ regular. We can choose an $|\alpha|^+$-closed forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$, such that in $\mathbb{V}[G]$, $\lambda = \lambda^{<\lambda}$ admits thick sets (first collapse $\lambda^{<\kappa}$ to λ, and then add 2^λ-many λ-Cohens, and apply Theorem 6.5.3 and the proof of Lemma 6.5.2). Since $H(|\alpha|^+)$ is unchanged, we get that Φ still admits α-ary Schröder-Bernstein invariants in $\mathbb{V}[G]$ by Theorem 6.6.9. Working in $\mathbb{V}[G]$, we see that $\tau(\Phi, \lambda) \leq \beth_\alpha(\lambda) < \beth_{\lambda^+}$, and thus Φ cannot be $\leq^*_{a\Delta^1_2}$-complete. If $\alpha < \omega_1$ then since $\tau(\mathrm{TAG}_{\alpha+1}, \lambda) = \beth_{\alpha+1}(\lambda)$, we get in fact that $\mathrm{TAG}_{\alpha+1} \not\leq^*_{a\Delta^1_2} \Phi$. $\square$

We give a concrete example.

Corollary 6.7.8. Assume $\kappa(\omega)$ exists. Then there is no Borel reduction from torsion abelian groups to colored trees, which takes nonisomorphic groups to non-biembeddable trees.

Chapter 7: Borel Complexity of Torsion-Free Abelian Groups

In this chapter, we describe the results of [82], joint with Shelah.

In [12], Friedman and Stanley leverage the Ulm analysis [86] to show that torsion abelian groups are far from Borel complete. They then pose the following question:

Question. Let TFAG be the theory of torsion-free abelian groups. Is TFAG Borel complete?

This has attracted considerable attention, but has nonetheless remained open. The following theorem of Hjorth [23] is the best known so far:

Theorem 7.0.1. $\Phi_\alpha \leq_B$ TFAG for every $\alpha < \omega_1$.

This means that if TFAG is not Borel complete, then it represents a very new phenomenon. In fact, in [12], Friedman and Stanley separately described the following question as one of the basic open problems of the general theory: if Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ and if $\Phi_\alpha \leq_B \Phi$ for each $\alpha < \omega_1$, must Φ be Borel complete?

In Sections 7.1, we give a uniform treatment of the main techniques of coding information into abelian groups. The basic idea for these codings is old, dating at least to [23] and [9]; namely, we start with a free abelian group, and then tag various subgroups by making the elements infinitely divisible by particular primes. However, to make the coding more robust we adopt an idea of [16], replacing the use of primes by an algebraically independent sequence of p-adic integers for a fixed prime p. As a first application, we show the following, where AG is the theory of abelian groups:

Theorem 7.0.2. TFAG $\sim_B$ AG. Further, if R is any countable ring, then R-mod, the theory of left R-modules, has R-mod $\leq_B$ AG.

In Section 7.2, we expand on Hjorth's proof of Theorem 7.0.1.

In Section 7.2, using the basic idea of Theorem 7.0.1, we prove the following:

Theorem 7.0.3. Suppose there is no transitive model of $ZFC^- + \kappa(\omega)$ exists. Then Graphs $\leq_{ZFC^-}$ TFAG.

Corollary 7.0.4. It is consistent with ZFC that Graphs $\leq_{ZFC^-}$ TFAG, and hence that TFAG is $\leq_{ZFC^-}$-complete.

The proofs suggest that maybe $TFAG$ has some α-ary Schröder-Bernstein property. The $\alpha = 0$ case has already been investigated: the Schröder-Bernstein property for TFAG fails, as first proved by Goodrick [17] (in fact, the failure was with elementary embedding). Recently, Calderoni and Thomas have shown in [85] that the relation of biembeddability on countable models of TFAG is Σ^1_1-complete, which is as bad as possible.

In Section 7.3, we prove the following (where we use injective group homomorphisms as our notion of embedding):

Theorem 7.0.5. For every $\alpha < \kappa(\omega)$, TFAG fails the α-ary Schröder-Bernstein property.

The construction breaks down at $\kappa(\omega)$, so the following remains open:

Question. Does TFAG have the $\kappa(\omega)$-ary Schröder-Bernstein property?

If the answer is yes, then this would imply (under the presence of $\kappa(\omega)$ that TFAG is not Borel complete.

7.1 Some Bireducibilities with TFAG

In this section, we prove in particular that AG (the theory of abelian groups) is Borel equivalent to TFAG (the theory of torsion-free abelian groups).

We set up some notation. If X is a set and G is a group we let $\oplus_X G$ denote the group of functions from X to G with finite support; so we consider $\oplus_X G \leq G^X$. For p a prime, $\mathbb{Z}[\frac{1}{p}]$ is the subring of $\mathbb{Q}$ generated by $\frac{1}{p}$; and similarly for sets of primes. $\mathbb{Z}_{(p)}$ (read: $\mathbb{Z}$ localized at the ideal (p)) is $\mathbb{Z}[\frac{1}{q} : q \neq p]$. Let $\mathbb{Z}_p$ denote the p-adic integers, i.e. the completion of $\mathbb{Z}_{(p)}$ under the p-adic metric. Let $\mathbb{Q}_p$ be the field completion of $\mathbb{Z}_p$.

Given $G \leq H$ groups, say that G is a pure subgroup of H is for every $n < \omega$, $nH \cap G = nG$. If p is a prime, say that G is a p-pure subgroup of H if for every $n < \omega$, $p^n H \cap G = p^n G$.

The following is a generalization of Hjorth's notion of "eplag."

Definition 7.1.1. Suppose $\mathcal{I}$ and $\mathcal{J}$ are countable index sets. Then let $\mathcal{L}_{\mathcal{I},\mathcal{J}}$ be the language extending the language of abelian groups, with a unary predicate symbol G_i for each $i \in I$, and a unary function symbol φ_j for each $j \in J$ (we will allow φ_j to be a partial function).

Let $\Omega_{\mathcal{I},\mathcal{J}}$ be the infinitary $\mathcal{L}_{\mathcal{F}}$-sentence such that $(G, +, G_i : i \in I, \varphi_j : j \in J) \models \Omega_{\mathcal{I},\mathcal{J}}$ if the following all hold:

- $(G, +) \equiv_{\infty\omega} \oplus_\omega \mathbb{Z}$;
- Each G_i is a subgroup of G;
- Each $\mathrm{dom}(\varphi_j)$ is either equal to all of G, or else to some G_i;
- Each $\varphi_j : \mathrm{dom}(\varphi_j) \to G$ is a homomorphism.

Let $\Omega^p_{\mathcal{I},\mathcal{J}}$ assert additionally that each G_i is a pure subgroup of G.

Some important examples: the countable models of $\Omega_{\{0\},0}$ are of the form (G, H) where G is free abelian of infinite rank and H is a subgroup of G. The countable models of $\Omega_{0,\{0\}}$ are of the form (G, φ) where G is free abelian of infinite rank and $\varphi : G \to G$ is a homomorphism. The countable models of $\Omega_{\omega,0}$ are of the form $(G, G_n : n < \omega)$, where G is free abelian of infinite rank and each G_n is a subgroup of G.

We aim to prove the following. Let AG denote the theory of abelian groups.

Theorem 7.1.2. Suppose $\mathcal{I}, \mathcal{J}$ are countable index sets, not both empty. Then $\Omega^p_{\mathcal{I},\mathcal{J}} \sim_B \Omega_{\mathcal{I},\mathcal{J}} \sim_B \mathrm{TFAG} \sim_B \mathrm{AG}$.

The proof will be via many lemmas.

Lemma 7.1.3. TFAG $\leq_B \Omega^p_{\{0\},0}$ and AG $\leq_B \Omega_{\{0\},0}$.

Proof. We describe the essential features of the construction, leaving it to the reader to check that it is Borel when formulated as an operation on Polish spaces. Suppose G is an (infinite) countable abelian group. Define $\varphi : \sum_G \mathbb{Z} \to G$ to be the augmentation map, that is given $a \in \sum_G \mathbb{Z}$, let $\varphi(a) = \sum_{n \in G'} a(b)b$. Let K be the kernel of φ. Thus $G \mapsto (\sum_G \mathbb{Z}, K)$ works, using $G \cong \sum_G \mathbb{Z}/K$. This shows AG $\leq_B \Omega_{\{0\},0}$; but note that if G is torsion-free, then K will be pure, so we also get TFAG $\leq_B \Omega^p_{\{0\},0}$. □

Lemma 7.1.4. $\Omega_{\{0\},0} \leq_B \Omega_{0,\{0\}}$. Hence, whenever $\mathcal{I}, \mathcal{J}$ are not both empty, $\Omega_{\{0\},0} \leq_B \Omega_{\mathcal{I},\mathcal{J}}$ and $\Omega^p_{\{0\},0} \leq_B \Omega^p_{\mathcal{I},\mathcal{J}}$.

Proof. Suppose $(G, H) \models \Omega_{\{0\},0}$ is a given countable model; so G is free abelian of infinite rank and H is a subgroup of G. Write $G' = G \times H'$; where $H' \cong H$; note that H' and hence G' is free abelian, since subgroups of free abelian groups are free. Define $\varphi : G' \to G'$

via $\varphi \restriction_G = 0$ and $\varphi \restriction_{H'}: H' \cong H$. Then $(G, H) \mapsto (G', \varphi)$ works, using $G = \ker(\varphi)$ and $H = \operatorname{im}(\varphi)$.

The second claim follows trivially (note $\Omega^p_{0,\{0\}} = \Omega_{0,\{0\}}$). $\square$

Lemma 7.1.5. For any countable index sets $\mathcal{I}, \mathcal{J}$, $\Omega_{\mathcal{I},\mathcal{J}} \leq_B \Omega_{\omega,0}$ and $\Omega^p_{\mathcal{I},\mathcal{J}} \leq_B \Omega^p_{\omega,0}$.

Proof. Write $\mathcal{I}' = \mathcal{I} \cup \mathcal{J} \cup \{*_0, *_1\}$ (we suppose this is a disjoint union). We show that $\Omega_{\mathcal{I},\mathcal{J}} \leq_B \Omega_{\mathcal{I}',0}$ and $\Omega^p_{\mathcal{I},\mathcal{J}} \leq_B \Omega^p_{\mathcal{I}',0}$.

Suppose $(G, G_i : i \in \mathcal{I}, \varphi_j : j \in \mathcal{J}) \models \Omega_{\mathcal{I},\mathcal{J}}$. Define $G' = G \times G$; for each $i \in \mathcal{I}$, define G'_i to be the copy of G_i in the first factor of G'; for each $j \in \mathcal{J}$, define G'_j to be the graph of φ_j; define $G'_{*_0} = G \times 0$; and finally let G'_{*_1} be the graph of the identify function $id_G : G \to G$. Then $(G', G'_i : i \in \mathcal{I}') \models \Omega_{\mathcal{I}',0}$ works. Also note that if each G_i is pure, then so is each $G'_{i'}$; this is because the graph of a partial homomorphism is pure if and only if its domain is pure. $\square$

Lemma 7.1.6. $\Omega_{\omega,0} \leq_B \Omega^p_{\omega,0}$.

Proof. By the preceding lemma, it suffices to find index sets $\mathcal{I}, \mathcal{J}$ such that $\Omega_{\omega,0} \leq_B \Omega^p_{\mathcal{I},\mathcal{J}}$. Write $\mathcal{I} = \omega \cup \{*\}$, write $\mathcal{J} = \omega$.

Suppose $(G, G_n : n < \omega) \models \Omega_{\omega,0}$. We define $G' = G \times \oplus_{n<\omega} \sum_{G_n} \mathbb{Z}$. For each $n < \omega$ let $G'_n = \oplus_{G_n} \mathbb{Z}$; let $G'_* = G$. Finally, define $\varphi_n : G'_n \to G'$ to be the augmentation map $\oplus_{G_n} \mathbb{Z} \mapsto G_n$. Then clearly $(G', G'_i : i \in \mathcal{I}, \varphi_j : j \in \mathcal{J})$ works ($G = G'_*$ and each $G_n = \operatorname{Im}(\varphi_n)$). $\square$

Note that to finish the proof of Theorem 7.1.2, it suffices to show that $\Omega^p_{\omega,0} \leq_B$ TFAG. Indeed, we would then have that for any countable index sets $\mathcal{I}, \mathcal{J}$ not both empty, TFAG $\leq_B \Omega^p_{\{0\},0} \leq_B \Omega^p_{\mathcal{I},\mathcal{J}} \leq_B \Omega^p_{\omega,0} \leq_B$ TFAG, and thus these are all equivalent; and similarly, AG $\leq_B \Omega_{\mathcal{I},\mathcal{J}} \leq_B \Omega_{\omega,0} \leq_B \Omega^p_{\omega,0} \leq_B$ AG, and so these are also all equivalent.

This remaining reduction is more involved than the others; the basic idea for it is due to Goodrick [16]. To begin, we need the following lemma. The point is that if G is a p-pure subgroup of $\oplus_\omega \mathbb{Z}_p$, then the isomorphism type of $(\mathbb{Z}_p(G), G)$ depends only on the isomorphism type of G.

Lemma 7.1.7. Suppose G is a p-pure subgroup of $\oplus_\omega \mathbb{Z}_p$. Then there is a $\mathbb{Z}_p$-module isomorphism $\varphi : (\mathbb{Z}_p \otimes G)/(p^\infty(\mathbb{Z}_p \otimes G)) \to \mathbb{Z}_p G$ which is the identity on G, where $\mathbb{Z}_p \otimes G$ is the tensor product (over $\mathbb{Z}$), and where $\mathbb{Z}_p G$ is the $\mathbb{Z}_p$-submodule of $\oplus_\omega \mathbb{Z}_p$ generated by G.

Proof. Define $\psi(\gamma, a) = \gamma a$, going from $\mathbb{Z}_p \times G$ to $\mathbb{Z}_p G$. ψ is clearly a $\mathbb{Z}$-bilinear map, so it induces a group homomorphism $\varphi_0 : \mathbb{Z}_p \otimes G \to \mathbb{Z}_p G$. Clearly φ_0 is 0 on $p^\infty(\mathbb{Z}_p \otimes G)$ so this induces a map $\varphi : (\mathbb{Z}_p \otimes G)/(p^\infty(\mathbb{Z}_p \otimes G)) \to \mathbb{Z}_p G$. We check this works. Clearly φ is surjective and the identity on G, and preserves the $\mathbb{Z}_p$-action. So it suffices to check the kernel of φ_0 is $p^\infty(\mathbb{Z}_p \otimes G)$.

Given $\gamma \in \mathbb{Z}_p$ and $n < \omega$, let $\gamma \restriction_n \in \{0, \ldots, p^n - 1\}$ be the unique element with $\gamma - \gamma \restriction_n \in p^n \mathbb{Z}_p$ (recall that $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ in the p-adic metric; so choose $(k_m : m < \omega)$ a sequence from $\mathbb{Z}$ converging to γ and note that k_m mod p^n must eventually be constant).

Suppose $\sum_{i<n} \gamma_i a_i = 0$; we want to show $\sum_{i<n} \gamma_i \otimes a_i \in p^\infty(\mathbb{Z}_p \otimes G)$. Note that for each m, $\sum_{i<n} \gamma_i a_i \in p^m(\oplus_\omega \mathbb{Z}_p)$. Hence, for each m, $b_m := \sum_{i<n} \gamma_i \restriction_m a_i \in p^m G$, using that G is p-pure. Note that in $\mathbb{Z}_p \otimes G$, $\sum_{i<n} \gamma_i \restriction_m \otimes a_i = 1 \otimes b_m$, since we can move all the $\gamma_i \restriction_m$'s to the right-hand side. Thus $1 \otimes b_m - \sum_{i<n} \gamma_i \otimes a_i \in (p^m \mathbb{Z}_p) \otimes G$, as it is equal to $\sum_{i<n} (\gamma_i \restriction_m - \gamma_i) \otimes a_i$. Thus $\sum_{i<n} \gamma_i \otimes a_i \in p^m(\mathbb{Z}_p \otimes G)$ for all m, as desired. □

Finally:

Lemma 7.1.8. $\Omega^p_{\omega,0} \leq_B \mathrm{TFAG}$.

Proof. Let p be a prime.

Let $(\gamma_n : 1 \leq n < \omega)$ be a sequence of algebraically-independent elements of $\mathbb{Z}_p$ over $\mathbb{Q}$, such that each γ_n is a unit of $\mathbb{Z}_p$ (in particular is not divisible by p). Write $\gamma_0 = 1$. Note then that $(\gamma_n : n < \omega)$ is linearly independent over $\mathbb{Q}$.

Let $(\oplus_\omega \mathbb{Z}, G_n : n < \omega) \models \Omega^p_{\omega,0}$; we can suppose $G_0 = G_1 = \oplus_\omega \mathbb{Z}$. Let G be the p-pure subgroup of $\oplus_\omega \mathbb{Z}_p$ generated by $\bigcup_{n<\omega} \gamma_n G_n$ (that is, close off under addition, inverses, and division by p within $\oplus_\omega \mathbb{Z}_p$). We want to check that the map $\overline{G} \mapsto G$ works.

First, suppose $(\oplus_\omega \mathbb{Z}, G_n : n < \omega) \cong (\oplus_\omega \mathbb{Z}, G'_n : n < \omega)$; we want to verify that the corresponding groups G, G' are isomorphic. Let φ be the isomorphism. Then φ lifts canonically to an isomorphism $\varphi^* : \oplus_\omega \mathbb{Z}_p \cong \oplus_\omega \mathbb{Z}_p$ (let $(e_i : i < \omega)$ be the standard basis of $\oplus_\omega \mathbb{Z}$, define $\varphi^*(\sum_i \gamma_i e_i) = \sum_i \gamma_i \varphi(e_i)$, where $(e_i : i < \omega)$ is the standard basis of $\oplus_\omega \mathbb{Z}$; more abstractly, $\varphi^* = 1 \otimes \varphi$ where we view $\oplus_\omega \mathbb{Z}_p = \mathbb{Z}_p \otimes \oplus_\omega \mathbb{Z}$). Then clearly $\varphi^* \restriction_G$ is an isomorphism onto G'.

For the reverse it suffices, by Lemma 7.1.7, to show we can canonically recover each G_n from $(\mathbb{Z}_p G, G)$.

Note that every $a \in G$ can be written as $\sum_{n<\omega} \gamma_n p^{k(n)} b_n$, for some $k(n) \in \mathbb{Z}$, $b_n \in G_n$ with all but finitely many $b_n = 0$, and $k(n) = 0$ whenever $b_n = 0$. (Not all such sums are in G; G contains such sums which are additionally in $\oplus_\omega \mathbb{Z}_p$.) We call this a representation of a if each $p \not| b_n$. Then representations are unique: for suppose $\sum_{n<\omega} \gamma_n p^{k(n)} b_n = \sum_{n<\omega} \gamma_n p^{k'(n)} b'_n$. Let $i \in \omega$; then we have $\sum_{n<\omega} \left(p^{k(n)} b_n(i) - p^{k'(n)} b'_n(i)\right) \gamma_n = 0$. By linear independence of $(\gamma_n : n < \omega)$ this implies each $p^{k(n)} b_n(i) = p^{k'(n)} b'_n(i)$. Since this holds for each i we have each $p^{k(n)} b_n = p^{k'(n)} b'_n$. Then by divisibility assumptions we have that each $b_n = b'_n$ and so each $k(n) = k'(n)$.

Suppose $f \in \mathbb{Z}_p G$ and let $1 \leq m < \omega$. It suffices to show that $a \in G_m$ if and only if $a \in G$ and $\gamma_m a \in G$: left to right follows from our assumption that $\gamma_0 = 1$. For right to left: let $\sum_{n<\omega} \gamma_n p^{k(n)} b_n$ be the representation of a, and let $\sum_{n<\omega} \gamma_n p^{k'(n)} b'_n$ be the representation of $\gamma_m a$. Let $i \in \omega$. Then $\sum_{n<\omega} \gamma_m \gamma_n p^{k(n)} b_n(i) = \sum_{n<\omega} \gamma_n p^{k'(n)} b'_n(i)$. Note that the only time $\gamma_m \gamma_n = \gamma_k$ is when $n = 0$, $k = m$. Thus by linear independence of $(\gamma_n : n < \omega)^\frown (\gamma_m \gamma_n : 1 \leq n < \omega)$ we have that $b_n = 0$ for all $n \neq 0$, and $b'_n = 0$ for all $n \neq m$. In particular, $a = p^k b$ for some $b \in G_m$. Since $\oplus_\omega \mathbb{Z}$ is p-pure in $\oplus_\omega \mathbb{Z}_p$ and since G_m is p-pure in $\oplus_\omega \mathbb{Z}$, we have that $a \in G_m$. $\square$

Remark 7.1.9. It is easy to add to the list in Theorem 7.1.2. For instance, we can additionally insist that each φ_j is a pure embedding, i.e. preserves the divisibility relations.

A much stronger condition is the following: let $\Omega^*_{\mathcal{I},\mathcal{J}}$ be $\Omega_{\mathcal{I},\mathcal{J}}$ together with the second-order assertion saying, given $(G, G_i : i \in I, \varphi_j : j \in J)$, that there is a basis $\mathcal{B}$ of G (as a $\mathbb{Z}$-module) such that each G_i is spanned by basis elements of $\mathcal{B}$ and each φ_j takes basis elements to basis elements. All of the known complexity of TFAG is also present in $\Omega^*_{\omega,\{0\}}$; see the next section.

Finally, we aim towards showing that whenever R is a countable ring, then R-mod (the theory of left R-modules) is Borel reducible to AG. This will not be used in the remainder of the chapter.

Definition 7.1.10. Suppose $\mathcal{I}, \mathcal{J}$ are countable index sets. Let $\Omega^-_{\mathcal{I},\mathcal{J}}$ be the $\mathcal{L}_{\mathcal{I},\mathcal{J}}$-theory such that $(G, +, G_i, \varphi_j : i \in \mathcal{I}, j \in \mathcal{J}) \models \Omega^-_{\mathcal{I},\mathcal{J}}$ if:

- $(G, +)$ is an abelian group;
- Each G_i is a subgroup of G;
- Each $\mathrm{dom}(\varphi_j)$ is either all of G or else some G_i;

- Each $\varphi_j : \mathrm{dom}(\varphi_j) \to G$ is a homomorphism.

So the only difference with $\Omega_{\mathcal{I},\mathcal{J}}$ is that we are no longer requiring $G \equiv_{\infty\omega} \oplus_\omega \mathbb{Z}$.

Theorem 7.1.11. For all countable index sets $\mathcal{I}, \mathcal{J}$, we have $\Omega^-_{\mathcal{I},\mathcal{J}} \sim_B \mathrm{AG}$.

Proof. Clearly $\mathrm{AG} \leq_B \Omega^-_{\mathcal{I},\mathcal{J}}$. (Given $G \models \mathrm{AG}$, let each $G_i = G$ and let each φ_j be the identity of G.) Also, we have by exactly the same argument as before that each $\Omega^-_{\mathcal{I},\mathcal{J}} \leq_B \Omega^-_{\omega,0}$. So it suffices to show that $\Omega^-_{\omega,0} \leq_B \Omega_{\omega \cup \{*\},0}$.

Given $(G, G_n : n < \omega) \models \Omega^-_{\omega,0}$ (that is, G is an abelian group and each G_n is a subgroup of G), write $G' = \oplus_G \mathbb{Z}$; let G'_* be the kernel of the augmentation map $G' \to G$; and let $G'_n = G'_* + \oplus_{G_n} \mathbb{Z}$. Then $(G', G'_n : n < \omega, G'_*)$ works, using $G \cong G'/G_*$ via an isomorphism that takes each G_n to G'_n/G_*. □

Corollary 7.1.12. Suppose R is a countable ring. Then R-mod $\leq_B \mathrm{AG}$.

Proof. An R-module $(M, +, \cdot_r : r \in R)$ can be viewed as a model of $\Omega^-_{0,R}$, and this gives a reduction R-mod $\leq_B \Omega^-_{0,R}$. □

7.2 Embedding Graphs into TFAG

In this section, we prove Theorem 7.0.3: if there is no transitive model of $ZFC^- + \kappa(\omega)$ exists, then Graphs $\leq_{ZFC^-}$ TFAG.

We will split the proof of Theorem 7.0.3 into two main subtheorems.

Theorem 7.2.1. There is a Borel map $f : \mathrm{Mod}(\mathrm{CT}) \to \mathrm{Mod}(\mathrm{TFAG})$ such that for all $\mathcal{T}, \mathcal{T}' \in \mathrm{Mod}(\mathrm{CT})$: if $\mathcal{T} \cong \mathcal{T}'$ then $f(\mathcal{T}) \cong f(\mathcal{T}')$, and if $f(\mathcal{T}) \cong f(\mathcal{T}')$ then $\mathcal{T} \sim \mathcal{T}'$. (In fact, we will get that for every $t \in T$, there is $t' \in T'$ of the same height with $\mathcal{T}_{\geq t} \sim \mathcal{T}'_{\geq t'}$, and conversely.)

Theorem 7.2.2. Suppose there is no transitive model of $ZFC^- + \kappa(\omega)$ exists. Then graphs admit 0-ary Schröder-Bernstein invariants; in fact, we can choose the witnessing reduction to be ZFC^--absolute.

We are essentially following Hjorth's proof of Theorem 7.0.1 in [23], although Theorem 7.1.2 will make our life easier. Recall from Corollary 6.7.8 that if $\kappa(\omega)$ exists, then the conclusion of Theorem 7.2.2 fails.

Also, note that to prove Theorem 7.0.3, it suffices to establish Theorem 7.2.1 and Theorem 7.2.2.

Proof of Theorem 7.2.1.

Suppose $\mathcal{T} = (T, <_T, c_T) \models \mathrm{CT}$. We define a model $\mathcal{T} \otimes \mathbb{Z}$ of $\Omega_{\omega\times\omega,\{0\}}$. ($f$ will be the function $\mathcal{T} \mapsto \mathcal{T} \otimes \mathbb{Z}$.) Let the underlying group of $\mathcal{T} \otimes \mathbb{Z}$ be $\oplus_T \mathbb{Z}$ (which we suppose is infinite); define the group homomorphism $\pi_{\mathcal{T}} : \oplus_T \mathbb{Z} \to \oplus_T \mathbb{Z}$ by $\pi_{\mathcal{T}}(a)(t) = \sum_{s \in \mathrm{succ}_{\mathcal{T}}(t)} a(s)$. Viewing $T \subseteq \oplus_T \mathbb{Z}$ in the obvious way, note that $\pi_{\mathcal{T}}(0_{\mathcal{T}}) = 0$, and for all $s \neq 0_{\mathcal{T}}$, $\pi_{\mathcal{T}}(s)$ is the immediate predecessor of s. For each $n, i < \omega$ write $G_{\mathcal{T},n,i} = \oplus_t \mathbb{Z}$, where the sum is over all $t \in T$ with $\mathrm{ht}(t) = n$ and $c_{\mathcal{T}}(t) = i$. Let $\mathcal{T} \otimes \mathbb{Z}$ be the structure $(\oplus_{\mathcal{T}} \mathbb{Z}, G_{\mathcal{T},n,i}, \pi_{\mathcal{T}} : n, i < \omega)$.

Let $\mathrm{CT} \otimes \mathbb{Z}$ be the Σ^1_1-sentence describing the closure under isomorphism of $\{\mathcal{T} \otimes \mathbb{Z} : \mathcal{T} \models \mathrm{CT}\}$. Clearly, if $\mathcal{T}_1 \cong \mathcal{T}_2$ then $\mathcal{T}_1 \otimes \mathbb{Z} \cong \mathcal{T}_2 \otimes \mathbb{Z}$.

Fix some countable $\mathcal{T} \models \mathrm{CT}$. We perform some analysis on $\mathcal{T} \otimes \mathbb{Z}$; write $G = \oplus_T \mathbb{Z}$.

For each $\bar{i} = (i_m : m < n+1) \in \omega^{n+1}$, let $G_{\mathcal{T},\bar{i}}$ be the subgroup of all $a \in G$ such that for each $m \leq n$, $\pi^m(a) \in G_{\mathcal{T},i_{n-m}}$. Also let $G_{\mathcal{T},\emptyset} = 0$. Note that π takes $G_{\mathcal{T},\bar{i}}$ to $G_{\mathcal{T},\bar{i}\restriction_n}$; also, G is the direct sum of the various $G_{\mathcal{T},\bar{i}}$'s. Further, $G_{\mathcal{T},\bar{i}}$ is spanned by $\{t \in T : \mathrm{ht}(t) = n, \bar{c}_{\mathcal{T}}(t) = \bar{i}\}$, where $\bar{c}_{\mathcal{T}}(t) = (c_{\mathcal{T}}(t\restriction_0), c_{\mathcal{T}}(t\restriction_1), \ldots, c_{\mathcal{T}}(t))$.

For each $a \in G_{\mathcal{T},\bar{i}}$ nonzero, let T^*_a denote the set of all b such that for some $\bar{i} \subseteq \bar{j}$,

$b \in G_{\mathcal{T},\bar{j}}$ and $\pi_{\mathcal{T}}^{\lg(\bar{j})-\lg(\bar{i})}(b) = a$. If we define $c^*_a(b) = \bar{j}(\lg(\bar{j})-1)$, and if we let $b \leq_f b'$ if and only if some $\pi^m(b') = b$, then $(T^*_a, \leq_a, c^*_a) = \mathcal{T}^*_a$ is a colored tree. We wish to understand $\mathcal{T}^*_a/\sim$ in terms of $\mathcal{T}$. The following definition will be relevant:

Definition 7.2.3. If $(\mathcal{S}_k : k < k_*)$ are colored trees, then the product $\prod_{k<k_*} \mathcal{S}_k$ is the colored tree whose elements are all sequences $(s_k : k < k_*)$, where for some $n < \omega$, each s_k has height n, and for some $(i_m : m \leq n) \in \omega^{n+1}$, we have for all $m \leq n$, $c_{\mathcal{S}_k}(s_k \restriction_m) = i_m$. Then we define the color of $(s_k : k < k_*)$ to be i_n. Clearly, $\prod_{k<k_*} \mathcal{S}_k \leq \mathcal{S}_{k'}$ for each $k' < k_*$, via projection onto the k'-factor. In fact, $\mathcal{T} \leq \prod_{k<k_*} \mathcal{S}_k$ if and only if $\mathcal{T} \leq \mathcal{S}_k$ for each $k < k_*$. This is because if $\mathcal{T} \leq \prod_{k<k_*} \mathcal{S}_k$, then we can compose with the projection maps to get $\mathcal{T} \leq^{\mathrm{ct}} \mathcal{S}_k$ for each k; and if $f_k : \mathcal{T} \leq^{\mathrm{ct}} \mathcal{S}_k$ for each $k < k_*$, we can define $f : \mathcal{T} \leq^{\mathrm{ct}} \prod_{k<k_*} \mathcal{S}_k$ via $f(t) = (f_k(t) : k < k_*)$.

Claim 1. Suppose $a \in G_{\mathcal{T},\bar{i}}$ is nonzero; enumerate $\mathrm{supp}(a) = \{t_k : k < k_*\}$. Then $\mathcal{T}^*_a \sim \prod_{k<k_*} \mathcal{T}_{\geq t_k}$.

Proof. First we will define an embedding $f : \mathcal{T}^*_a \leq^{\mathrm{ct}} \prod_{k<k_*} \mathcal{T}_{\geq t_k}$. We will define $f(b)$ inductively on the height of $b \in \mathcal{T}^*_a$; our inductive hypothesis will be that $f(b) = (t_k : k < k_*)$ is a sequence from $\mathrm{supp}(b)$, and if we let $\bar{i}$ be such that $b \in G_{\mathcal{T},\bar{i}}$, then each $\bar{c}_{\mathcal{T}}(t_k) = \bar{i}$.

So we are given b and $f(b) = (t_k : k < k_*)$. Suppose $i < \omega$ and $c \in G_{\mathcal{T},\bar{i}i}$ satisfies that $\pi_{\mathcal{T}}(c) = b$. Then $\pi_{\mathcal{T}}[\mathrm{supp}(c)] \supseteq \mathrm{supp}(b)$, so for each $k < k_*$ we can find $s_k \in \mathrm{supp}(c)$ with $\pi_{\mathcal{T}}(s_k) = t_k$. Clearly then we can define $f(c) = (s_k : k < k_*)$, and continue.

For the reverse embedding $\prod_{k<k_*} \mathcal{T}_{\geq t_k} \leq^{\mathrm{ct}} \mathcal{T}^*_a$, write $a = \sum_{k<k_*} \lambda_k t_k$, and send $(s_k : k < k_*) \in \prod_{k<k_*} \mathcal{T}_{\geq t_k}$ to $\sum_{k<k_*} \lambda_k s_k \in \mathcal{T}^*_a$. □

Given an ω-labeled tree $\mathcal{S}$, let $G_{\mathcal{T},\bar{i},\mathcal{S}}$ be the set of all $a \in G_{\mathcal{T},\bar{i}}$ such that $\mathcal{S} \leq^{\mathrm{ct}} T^*_a$, along with $a = 0$. From the preceding claim it is clear that $G_{\mathcal{T},\bar{i},\mathcal{S}}$ is a subgroup of $G_{\mathcal{T},\bar{i}}$.

Also, let $G_{\mathcal{T},\bar{i},>\mathcal{S}} = \sum_{\mathcal{S}<\mathcal{S}'} G_{\mathcal{T},\bar{i},\mathcal{S}'}$, where the sum is over all colored trees $\mathcal{S}'$ such that $\mathcal{S} \leq \mathcal{S}'$ but not reversely.

Note that if $a \in G_{\mathcal{T},\bar{i}}$, then always $a \in G_{\mathcal{T},\bar{i},\mathcal{T}_a^*}$, but sometimes also $a \in G_{\mathcal{T},\bar{i},>\mathcal{T}_a^*}$. Say that a is good if this is not the case, i.e. $a \in G_{\mathcal{T},\bar{i},\mathcal{T}_a^*} \backslash G_{\mathcal{T},\bar{i},>\mathcal{T}_a^*}$.

Claim 2. Suppose $a \in G_{\mathcal{T},\bar{i}}$. Then a is good if and only if a is nonzero, and there is some $t \in \mathrm{supp}(a)$ such that $\mathcal{T}_a^* \sim \mathcal{T}_{\geq t}$.

Proof. Enumerate $\mathrm{supp}(a) = \{t_k : k < k_*\}$, and write $a = \sum_{k<k_*} \lambda_k t_k$. Then by Claim 1, $\mathcal{T}_a^* \leq \prod_{k<k_*} \mathcal{T}_{\geq t_k}$, so $\mathcal{T}_a^* \leq \mathcal{T}_{\geq t_k}$ for each $k < k_*$.

If a is good, then we cannot have each $\mathcal{T}_a^* < \mathcal{T}_{\geq t_k}$, so some $\mathcal{T}_{\geq t_k} \sim \mathcal{T}_a^*$ as desired. For the converse, suppose $t \in \mathrm{supp}(a)$ satisfies that $\mathcal{T}_a^* \sim \mathcal{T}_{\geq t}$. Suppose we write $a = \sum_{i<i_*} b_i$. Then $t \in \mathrm{supp}(b_i)$ for some $i < i_*$. By Claim 1, $\mathcal{T}_{b_i}^* \leq \mathcal{T}_{\geq t}$, and thus $\mathcal{T}_{b_i}^* \not> \mathcal{T}_{\geq t} \sim \mathcal{T}_a^*$. $\square$

In particular, if $a \in G_{\mathcal{T},\bar{i}}$ is good, then $\mathcal{T}_a^* \sim \mathcal{T}_{\geq t}$ for some $t \in \bar{c}_{\mathcal{T}}^{-1}(\bar{i})$, and so we can recover $\{\mathcal{T}_{\geq t}/\sim : t \in T, \mathrm{ht}(t) = n\}$ from the isomorphism class of $\mathcal{T} \otimes \mathbb{Z}$, for each n. This concludes the proof of Theorem 7.2.1.

Before continuing on to the proof of Theorem 7.2.2, we need some set-theoretic observations.

First, we note that various familiar facts about $\kappa(\omega)$ continue to hold when the ambient set theory is just ZFC^- (less suffices as well).

Say that a cardinal κ (in a model of ZFC) is totally indescribable if for every n, for every sentence φ in the language of set theory with an extra relation symbol, and for every $R \subseteq \mathbb{V}_\kappa$ with $(\mathbb{V}_{\kappa+n}, \in, R) \models \varphi$, there is an $\alpha < \kappa$ such that $(\mathbb{V}_{\alpha+n}, \in, R \cap \mathbb{V}_\alpha) \models \varphi$. This is a large cardinal notion; it implies that κ is weakly compact. In fact, weak compactness is equivalent to this condition when restricted to $n = 1$. This is due to Hanf and Scott;

see Theorem 6.4 of Kanamori [28].

Lemma 7.2.4. Work in ZFC^-.

(A) Suppose $\kappa \to (\omega)^{<\omega}_2$ and N is a transitive model of ZFC^- containing κ (possible a proper class). Then $(\kappa \to (\omega)^{<\omega}_2)^N$.

(B) If $\mathbb{V} = \mathbb{L}$ (we really just need global choice), and if $\kappa(\omega)$ exists, then $\kappa(\omega)$ is inaccessible (i.e., $\kappa(\omega)$ is a regular cardinal, and for all $\alpha < \kappa(\omega)$, $2^{|\alpha|}$ exists and is less than $\kappa(\omega)$). In particular $\mathbb{L}_{\kappa(\omega)} = \mathbb{V}_{\kappa(\omega)} \models ZFC$.

(C) If $\mathbb{V} = \mathbb{L}$ and if $\kappa(\omega)$ exists, then $\mathbb{V}_{\kappa(\omega)} \models$ "There exist totally indescribable cardinals."

(D) If $\mathbb{V} = \mathbb{L}$, then $\kappa(\omega)$ is the least cardinal κ such that whenever $f : [\kappa]^{<\omega} \to 2$, there is an increasing sequence $(\alpha_n : n < \omega)$ from κ such that for all n, $f(\alpha_0, \ldots, \alpha_{n-1}) = f(\alpha_1, \ldots, \alpha_n)$.

(E) If $\mathbb{V} = \mathbb{L}$, then $\kappa(\omega)$ is the least cardinal κ such that there is no antichain $(\mathcal{T}_\alpha : \alpha < \kappa(\omega))$ of ω-colored trees; by an antichain I mean that for all $\alpha < \beta < \kappa(\omega)$, $\mathcal{T}_\alpha \not\leq^{\mathrm{ct}} \mathcal{T}_\beta$ and $\mathcal{T}_\beta \not\leq^{\mathrm{ct}} \mathcal{T}_\alpha$. (If $\kappa(\omega)$ does not exist then we just mean that for every cardinal κ, there is an antichain of length κ.)

Note that Corollary 7.0.4 follows from Theorem 7.0.3 and (B). (C) provides a strengthening: it is consistent with $ZFC+$"There is a totally indescribable cardinal" that Graphs $\leq_{ZFC^-}$ TFAG.

Proof. All of these are routine modifications of the case where the ambient set theory is ZFC; (A) and (D) are due to Silver [83]. (B) is also due to Silver [84], or see Corollary

7.6 of Kanamori [28]. (C) is due to Silver and Reinhardt, see Exercise 9.18 of [28]. (E) is due to Shelah [73]; for the reader's convenience we provide a proof.

First suppose $\kappa < \kappa(\omega)$. Choose some $f : [\kappa]^{<\omega} \to 2$ failing (D). For each $\alpha < \kappa$, we define a colored tree $\mathcal{T}_\alpha$ as follows. Namely, let T_α be all finite increasing sequences of ordinals from κ whose first term is α; let $<_{\mathcal{T}_\alpha}$ be initial segment. Let $c_{\mathcal{T}_\alpha}(s) = f(s)$. Let $\mathcal{S}_\alpha$ be $\mathcal{T}_\alpha$ together with the tree of descending sequences from α, with the new elements all colored 2.

Note that for all $\alpha_0 < \alpha_1 < \kappa$, $\mathcal{T}_{\alpha_0} \not\leq \mathcal{T}_{\alpha_1}$, as given an embedding $\rho : \mathcal{T}_\alpha \leq^{\mathrm{ct}} \mathcal{T}_\beta$, we can inductively find $\alpha_n : n < \omega$ such that for all n, $\rho(\alpha_i : i < n) = (\alpha_i : 1 \leq i \leq n+1)$; but this clearly contradicts the hypothesized property of f. From this it follows that $(\mathcal{S}_\alpha : \alpha < \kappa)$ is the desired antichain.

In the other direction, suppose $(\mathcal{T}_\alpha : \alpha < \kappa(\omega))$ is a sequence of colored trees. Write $\kappa = \kappa(\omega)$; choose an elementary substructure $H \leq (\mathbb{V}_\kappa, \ldots)$ (using $<_{\mathbb{L}}$) such that H is the Skolem hull of an infinite set of indiscernible ordinals $\{\alpha_n : n < \omega\}$. Then it is easy to check that $\mathcal{T}_{\alpha_0} \leq \mathcal{T}_{\alpha_1}$. □

We can now finish.

Proof of Theorem 7.2.2.

Suppose A is a hereditarily countable set. We describe a colored tree $\mathcal{T}_A = (T_A, <_A, c_A)$, and then show that whenever $A \neq A'$, we have that $\mathcal{T}_A \not\sim \mathcal{T}_{A'}$. Moreover, the operation $A \mapsto \mathcal{T}_A$ will be ZFC^--absolute.

Having done this, $\varphi \mapsto \mathcal{T}_\varphi : \mathrm{CSS}(\mathrm{Graphs}) \to \mathrm{Mod}_{\mathrm{HC}}(\mathrm{CT})$ will witness that Graphs admits 0-ary Schröder-Bernstein invariants via a ZFC^--absolute reduction.

So it is enough to define $A \mapsto \mathcal{T}_A$. Let A be given, and let $\alpha = \mathrm{rnk}(A)$, where rnk is foundation rank. Let $(\mathcal{S}_\beta : \beta \leq \alpha)$ be the $<_{\mathbb{L}}$-least antichain of colored trees indexed by $\alpha+1$. This is computed correctly in any transitive model of ZFC^-, since if M is any transitive model of ZFC^- with $\alpha \in M$, then $\mathbb{L}^M$ does not believe that $\kappa(\omega)$ exists, and so $\mathbb{L}^M$ can find a $<_{\mathbb{L}^M}$-least sequence $(\mathcal{S}_\beta : \beta \leq \alpha)$ such that $\mathbb{L}^M \models (\mathcal{S}_\beta : \beta \leq \alpha)$ is an antichain. But the property of being an antichain of colored trees of length $\alpha+1$ is absolute to models of ZFC^-; thus $(\mathcal{S}_\beta : \beta \leq \alpha)$ is the $<_{\mathbb{L}}$-least antichain of colored trees indexed by $\alpha+1$.

We define a preliminary colored tree $\mathcal{T}_{0,A} = (T_{0,A}, <_{0,A}, c_{0,A})$. Let $(T_{0,A}, <_{0,A})$ be the tree of all nonempty finite sequences $(a_0, \ldots, a_n)$ from $\mathrm{tcl}(A \cup \{A\})$ such that $a_0 = A$ and $\mathrm{rnk}(a_0) > \mathrm{rnk}(a_1) > \ldots > \mathrm{rnk}(a_n)$. Given $(a_0, \ldots, a_n) \in T_{0,A}$, let $c_{0,A}(a_0, \ldots, a_n) = 0$ if $a_{n-1} \in a_n$, and $c_{0,A}(a_0, \ldots, a_n) = 1$ otherwise. Let $\mathcal{T}_A$ be obtained from $\mathcal{T}_{0,A}$ as follows: above each $(a_0, \ldots, a_n) \in T_{0,A}$, put a copy of $(S_\beta, <_{S_\beta})$, where β is the foundation rank of a_n; given $t \in S_\beta$, let the color of the copy of t above $(a_0, \ldots, a_n)$ be $c_{\mathcal{S}_\beta}(t) + 2$.

Suppose $\mathcal{T}_A \sim \mathcal{T}_{A'}$. Let $\alpha = \mathrm{rnk}(A)$ and let $\alpha' = \mathrm{rnk}(A')$. Choose $f : \mathcal{T}_A \leq \mathcal{T}_{A'}$ and $f' : \mathcal{T}_{A'} \leq \mathcal{T}_A$ witnessing that $\mathcal{T}_A \sim \mathcal{T}_{A'}$. Note that $f \restriction_{T_{A,0}}$ and $f' \restriction_{T_{A',0}}$ witness that $\mathcal{T}_{A,0}$ and $\mathcal{T}_{A',0}$ are biembeddable; since $\mathcal{T}_{A,0}$ is well-founded of rank α, and $\mathcal{T}_{A',0}$ is well-founded of rank α', this implies $\alpha = \alpha'$. Let $(\mathcal{S}_\beta : \beta \leq \alpha)$ be as above.

Now, consider the embedding $h := f' \circ f : \mathcal{T}_A \leq \mathcal{T}_A$. I claim that $h \restriction_{\mathcal{T}_{0,A}}$ must be the identity. This suffices, since it implies $\mathcal{T}_{0,A} \cong \mathcal{T}_{0,A'}$ and hence $A = A'$.

Suppose $(a_0, \ldots, a_n) \in T_{0,A}$; write $\beta = \mathrm{rnk}(a_n)$ and write $h(a_0, \ldots, a_n) = (b_0, \ldots, b_n)$. We show by induction on β that $a_n = b_n$; this suffices. Note that $\mathcal{S}_\beta \leq \mathcal{S}_{\mathrm{rnk}(b_n)}$, and hence $\mathrm{rnk}(b_n) = \beta$ also (this is the key point!).

If $\beta = 0$, then $a_n = b_n = \emptyset$. Suppose we have verified the claim for all $\gamma < \beta$. We show that for every $a \in \mathrm{tcl}(A \cup \{A\})$ with $\mathrm{rnk}(a) < \beta$, we have that $a \in a_n$ if and only if $a \in b_n$. Indeed, suppose a is given. Write $h(a_0, \ldots, a_n, a) = (b_0, \ldots, b_n, b)$. By construction of the coloring, we have that $a \in a_n$ if and only if $b \in b_n$; but by the inductive hypothesis, we have that $a = b$.

7.3 Schröder-Bernstein Properties for TFAG

In this section, we prove Theorem 7.3, namely: for every $\alpha < \kappa(\omega)$, TFAG fails the α-ary Schröder-Bernstein property. The construction breaks down at $\kappa(\omega)$, so the following remains open:

Question. Does TFAG have the $\kappa(\omega)$-ary Schröder-Bernstein property?

The remainder of this section is a proof of Theorem 7.3. Throughout, we abbreviate $\sim_\alpha^{SB}$ to $\sim_\alpha$. Also, for our notion of embedding $\leq$ on TFAG, we want to take injective group homorphisms; to align with the terminology of Chapter 6, we should add a unary predicate for $\{(a, b) : a \neq b\}$, although we suppress this.

We remark on the following easy lemma.

Lemma 7.3.1. Suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$, and α is an ordinal. Suppose there are $M, N \models \Phi$ such that $M \sim_\alpha N$ but $M \not\equiv_{\infty\omega} N$. Then Φ fails the α-ary Schröder-Bernstein property.

Proof. Clearly $\text{css}(M) \sim_\alpha \text{css}(N)$ but $\text{css}(M) \neq \text{css}(N)$. $\square$

In the remainder of this section, we prove the following:

Theorem 7.3.2. Suppose $\kappa(\omega)$ does not exist. Then for every ordinal α, TFAG fails the α-ary Schröder-Bernstein property.

Note that Theorem 7.0.5 follows: for every $\alpha < \kappa(\omega)$, TFAG fails the α-ary Schröder-Bernstein property. This is because we can always apply Theorem 7.3.2 in $\mathbb{V}_{\kappa(\omega)}$.

So, in the remainder of this section, suppose $\kappa(\omega)$ does not exist; equivalently, for every cardinal λ, there is an antichain of colored trees of length λ.

First of all, we note the following lemma:

Lemma 7.3.3. Suppose $\mathcal{I}, \mathcal{J}$ are countable index sets, not both empty; let $F : \Omega^p_{\mathcal{I},\mathcal{J}} \leq_B$ TFAG be the Borel reduction from the proof of Theorem 7.1.2 (that is, the composition of the reductions from Lemma 7.1.5 and Lemma 7.1.8). Suppose $\overline{G}^0, \overline{G}^1 \in \mathrm{Mod}(\Omega_{\mathcal{I},\mathcal{J}})$ and $\alpha < \omega_1$. If $\overline{G}^0 \sim_{2\cdot(\omega\cdot\alpha)} \overline{G}^1$, then $F(\overline{G}^0) \sim_\alpha F(\overline{G}^1)$.

Hence, if $\Omega^p_{\mathcal{I},\mathcal{J}}$ fails the α-ary Schröder-Bernstein property for every ordinal $\alpha < \kappa(\omega)$, then so does TFAG.

Proof. The final claim follows, since the first part continues to hold in forcing extensions.

Write $\mathcal{I}' = \mathcal{I} \cup \mathcal{J} \cup \{*_0, *_1\}$ (we suppose this is a disjoint union).

Let $F_0 : \Omega^p_{\mathcal{I},\mathcal{J}} \leq_B \Omega^p_{\mathcal{I}',0}$ be as in Lemma 7.1.5 and let $F_1 : \Omega^p_{\omega,0} \leq_B$ TFAG be as in Lemma 7.1.8.

First we look at F_0. We recap the definition of F_0, for the reader's convenience. Suppose $\overline{G} = (G, G_i : i \in \mathcal{I}, \varphi_j : j \in \mathcal{J}) \models \Omega^p_{\mathcal{I},\mathcal{J}}$ is countable. Define $G' = G \times G$; for each $i \in \mathcal{I}$, define G'_i to be the copy of G_i in the first factor of G'; for each $j \in \mathcal{J}$, define G'_j to be the graph of φ_j; define $G'_{*_0} = G \times 0$; and finally let G'_{*_1} be the graph of the identify function $id_G : G \to G$. Then $F(G, G_i : i \in I, \varphi_j : j \in J)$ is $\overline{G}' = (G', G'_{i'} : i' \in \mathcal{I}')$ (suppressing the coding that arranges everything to have universe ω).

Suppose $\overline{G}_0, \overline{G}_1 \models \Omega^p_{\mathcal{I},\mathcal{J}}$ are countable, and define $\overline{G}'_0, \overline{G}'_1$ as above. Then it is easy to check that for all $((a^0_i, a^1_i) : i < i_*)$ from $\overline{G}'_0$ and all $(b^0_i, b^1_i) : i < i_*)$ from $\overline{G}'_1$, if $f : (\overline{G}_0, (a^j_i : i < i_*, j < 2)) \leq (\overline{G}_1, (b^j_i : i < i_*, j < 2)$, then $f \times f : (\overline{G}'_0, ((a^0_i, a^1_i) : i < i_*)) \leq (\overline{G}'_1, ((b^0_i, b^1_i) : i < i_*))$. From this it follows by an easy inductive argument that for all $\beta < \omega_1$, if $(\overline{G}_0, (a^j_i : i < i_*, j < 2)) \sim_{2\cdot\beta} (\overline{G}_1, (b^j_i : i < i_*, j < 2)$, then

$(\overline{G}'_0, ((a^0_i, a^1_i) : i < i_*)) \sim_\beta (\overline{G}'_1, ((b^0_i, b^1_i) : i < i_*))$.

Next we look at F_1. Let $(\gamma_n : 1 \leq n < \omega)$ be as in Lemma 7.1.8, i.e. a sequence of algebraically independent units of $\mathbb{Q}_p$; and let $\gamma_0 = 1$. Let $\overline{G} = (\oplus_\omega \mathbb{Z}, G_n : n < \omega)$ be a countable model of $\Omega^p_{\omega,0}$; we only consider the case where $G_0 = G_1 = \oplus_\omega \mathbb{Z}$, without loss of generality. Then recall $F_1(\overline{G})$ is (isomorphic to) G, where G is the p-pure subgroup of $\oplus_\omega \mathbb{Z}_p$ generated by $\bigcup_n \gamma_n G_n$. Recall that every $a \in G$ can be written as a sum $a = \sum_{n<\omega} \gamma_n p^{k(n)} b_n$, where each $k(n) \in \mathbb{Z}$, $b_n \in G_n$ and all but finitely many $k(n), b_n$ are 0. Say that this is a weak representation of a (it may not be a full representation; we don't require that $p \not| b_n$ in G_n.)

Suppose $\overline{G}^j = (\oplus_\omega \mathbb{Z}, G^j_n : n < \omega)$ are countable models of $\Omega^p_{\omega,0}$ for $j < 2$; let G^0, G^1 be defined from $\overline{G}^0, \overline{G}^1$ as above. Suppose $f : \overline{G}^0 \leq \overline{G}^1$. Define $f_* : \oplus_\omega \mathbb{Z}_p \to \oplus_\omega \mathbb{Z}_p$ via $f_*(\sum_n \gamma_n e_n) = \sum_n \gamma_n f(e_n)$, where $(e_n : n < \omega)$ is the standard basis. Moreover, $f_* \restriction_{G^0} : G^0 \leq G^1$, since f_* preserves the action of $\mathbb{Z}_p$.

Suppose $(a_i : i < i_*)$ is a sequence from $\oplus_\omega \mathbb{Z}$, and suppose $(a'_i : i < i_*)$ is a sequence from $\oplus_\omega \mathbb{Z}$. Suppose for each $i < i_*$, $a_i = \sum_{n \in \Gamma_i} \gamma_n p^{k_i(n)} b_{i,n}$ is a weak representation with respect to $\overline{G}^0$, and $a'_i = \sum_{n \in \Gamma_i} \gamma_n p^{k_i(n)} b'_{i,n}$ is a weak representation with respect to $\overline{G}^1$, for finite sets $\Gamma_i \subset \omega$. Suppose finally that $f : (\overline{G}^0, (b_{i,n} : n \in \Gamma_i, i < i_*)) \leq (\overline{G}^1, (b'_{i,n} : n \in \Gamma_i, i < i_*))$. Then note that each $f_*(p^{k_i(n)} b_{i,n}) = p^{k_i(n)} b'_{i,n}$, hence each $f_*(a_i) = a'_i$, hence $f_* : (G^0, (a_i : i < i_*)) \leq (G^1, (a'_i : i < i_*))$.

From this, an easy inductive argument shows that if $(\overline{G}^0, (b_{i,n} : n \in \Gamma_i, i < i_*)) \sim_{\omega \cdot \alpha} (\overline{G}^1, (b'_{i,n} : n \in \Gamma_i, i < i_*))$, then $(G^0, (a_i : i < i_*)) \sim_\alpha (G^1, (a'_i : i < i_*))$.

□

Thus it suffices to show that some $\Omega^p_{\mathcal{I},\mathcal{J}}$ fails the α-ary Schröder-Bernstein property for all α.

For the next lemma, we make the obvious definitions for $\Omega^p_{\mathcal{I},\mathcal{J}}$ in the case where the index sets are possibly uncountable.

Lemma 7.3.4. Suppose $\kappa(\omega)$ does not exist. Suppose $\mathcal{I}, \mathcal{J}$ are index sets, and suppose $\overline{G}^0, \overline{G}^1 \models \Omega_{\mathcal{I},\mathcal{J}}$. Then we can find $\mathbf{F}(\overline{G}^0), \mathbf{F}(\overline{G}^1) \models \Omega^p_{\omega\times\omega\cup\{0,1\},\{0,1\}}$, such that $\overline{G}^0 \equiv_{\infty\omega} \overline{G}^1$ if and only if $\mathbf{F}(\overline{G}^0) \equiv_{\infty\omega} \mathbf{F}(\overline{G}^1)$, and for every ordinal β, if $\overline{G}^0 \sim_\beta \overline{G}^1$ then $\mathbf{F}(\overline{G}^0) \sim_\beta \mathbf{F}(\overline{G}^1)$.

Proof. We can suppose $\mathcal{J} = \emptyset$, by applying the construction from Lemma 7.1.5.

Choose λ large enough so that $\mathcal{I}, \overline{G}^0, \overline{G}^1$ all are of size at most λ. We can suppose $\mathcal{I} = \lambda$.

Let $(\mathcal{T}_\gamma : \gamma < \lambda)$ be a family of pairwise-non-biembeddable colored trees. Let $\mathcal{T}$ be the colored tree such that $c_{\mathcal{T}}(0) = 0$ (say), and for each $\gamma < \lambda$, there are λ-many $t \in T$ of height 1 such that $\mathcal{T}_{\geq t} \cong \mathcal{T}_\gamma$, and for each $t \in T$ of height 1, $\mathcal{T}_{\geq t}$ is isomorphic to some such $\mathcal{T}_\gamma$.

Recall the definition of $\mathcal{T} \otimes \mathbb{Z} = (G_{\mathcal{T}}, G_{\mathcal{T},n,i}, \pi : n, i < \omega) \models \Omega^p_{\omega\times\omega,\{0\}}$ from Theorem 7.2.1. For each $\gamma < \lambda$, let $\mathcal{E}_\gamma$ be the set of all $t \in T$ of height 1 such that $\mathcal{T}_{\geq t} \cong \mathcal{T}_\gamma$. Let $\hat{G}_{\mathcal{T},\gamma}$ denote the subgroup of $G_{\mathcal{T}}$ spanned by $\mathcal{E}_\gamma$. Note that each $\hat{G}_{\mathcal{T},\gamma}$ is $\mathcal{L}_{\infty\omega}$-definable, since $(\mathcal{T}_\gamma : \gamma < \lambda)$ is an antichain, and so $g \in \hat{G}_{\mathcal{T},\gamma}$ if and only if $g = 0$ or else $\mathcal{T}_\gamma$ embeds into $\mathcal{T}^*_g$.

Let $\mathbf{F}(\overline{G}^\ell) = (G_{\mathcal{T}} \oplus G^\ell, G_{\mathcal{T},n,i}, H^0, H^1, \pi, \psi^\ell : n, i < \omega) \models \Omega^p_{\omega\times\omega\cup\{0,1\},\{0,1\}}$, where $H^0 = \mathcal{T} \otimes \mathbb{Z}$, $H^1 = G^\ell$, and where $\psi^\ell : G_{\mathcal{T}} \to G^\ell$ satisfies:

- $\psi^\ell(t) = 0$ for all $t \in T$ not of height 1,
- For every $\gamma < \lambda$, $\psi \restriction_{\mathcal{E}_\gamma} : \mathcal{E}_\gamma \to G^\ell_\gamma$ is λ-to-one.

It is easy to check that this works.

$\square$

Thus, to finish it suffices to verify the following:

Lemma 7.3.5. Suppose $\kappa(\omega)$ does not exist. Suppose $\alpha_* < \kappa(\omega)$. Then for some index set $\mathcal{I}$, there are $\overline{G}^0_*, \overline{G}^1_* \models \Omega^p_{\mathcal{I},\{0\}}$, with $\overline{G}^0_* \sim_{\alpha_*} \overline{G}^1_*$ yet $\overline{G}^0_* \not\equiv_{\infty\omega} \overline{G}^1_*$.

Our idea is the following: given $\overline{G} = (G, G_i : i \in \mathcal{I}, \varphi) \models \Omega^p_{\mathcal{I},\{0\}}$, define $X^{\overline{G}} := G \setminus \bigcup_i G_i$ and define $\leq^{\overline{G}}$ to be the partial order of $X^{\overline{G}}$ given by: $a \leq^{\overline{G}} b$ if and only if $\varphi^n(a) = b$ for some $n < \omega$, satisfying further that for all $m < n$, $\varphi^m(a) \in X^{\overline{G}}$. Then we will arrange that $(X^{\overline{G}^0_*}, \leq^{\overline{G}^0_*})$ is ill-founded, but $(X^{\overline{G}^1_*}, \leq^{\overline{G}^1_*})$ is well-founded. It turns out we can make $\overline{G}^0_* \sim_{\alpha_*} \overline{G}^1_*$ without upsetting this.

We will be approximating $\overline{G}^0_*$ and $\overline{G}^1_*$ as a union of chains. To control the eventual behavior of $(X^{\overline{G}^i_*}, \leq^{\overline{G}^i_*})$, we will be defining upper bounds to the rank function at each stage. The following are the approximations we will be using:

Definition 7.3.6. Given an index set $\mathcal{I}$, let $\Gamma_{\mathcal{I}}$ denote all tuples $(\overline{G}, \mathcal{B}, \rho)$ where:

- $\overline{G} = (G, G_i, \varphi : i \in \mathcal{I}) \models \Omega^p_{\mathcal{I},\{0\}}$;
- G is free abelian (this is not redundant, since $\Omega^p_{\mathcal{I},\{0\}}$ only asserts that $G \equiv_{\infty\omega} \oplus_\omega \mathbb{Z}$) and $\mathcal{B}$ is a basis of G;
- $\varphi : G \to G$;
- $\rho : X^{\overline{G}} \to \mathrm{ON} \cup \{\infty\}$ satisfies: for all $a, b \in X^{\overline{G}}$, if $\varphi(b) = a$ and $\rho(b) < \infty$ then $\rho(a) < \rho(b)$. Hence $\rho(a) \geq \mathrm{rnk}(a)$ where rnk is the rank function for $(X^{\overline{G}}, \leq^{\overline{G}})$.
- For all $a \in X$ and for all $n \in \mathbb{Z}$ nonzero, $\rho(a) = \rho(na)$.

When we write $\overline{G}, \overline{G}', \overline{G}^\ell$, etc., then we will always have $\overline{G} = (G, G_i, \varphi : i \in I)$, $\overline{G}' = (G', G'_i, \varphi' : i \in I), \overline{G}^\ell = (G^\ell, G^\ell_i, \varphi^\ell : i \in I)$, etc.

Definition 7.3.7. Suppose $\mathcal{I}, \mathcal{I}'$ are index sets with $\mathcal{I} \subseteq \mathcal{I}'$. Suppose $(\overline{G}, \mathcal{B}, \rho) \in \Gamma_{\mathcal{I}}$ and $(\overline{G}', \mathcal{B}', \rho') \in \Gamma_{\mathcal{I}'}$. Then say that $(\overline{G}', \mathcal{B}', \rho')$ extends $(\overline{G}, \mathcal{B}, \rho)$ if:

- $G \subseteq G'$ and $\mathcal{B} \subseteq \mathcal{B}'$;
- For each $i \in \mathcal{I}$, $G'_i \cap G = G_i$;
- For each $i \in \mathcal{I}' \backslash \mathcal{I}$, $G'_i \cap G = 0$;
- $\varphi' \restriction_{G_i} = \varphi$;
- $\rho' \restriction_{X^{\overline{G}}} = \rho$.

The following lemma is immediate.

Lemma 7.3.8. Suppose $\delta < \lambda^+$ is a limit ordinal, $(\mathcal{I}_\gamma : \gamma < \delta)$ is an increasing chain of index sets, and $((\overline{G}^\gamma, \mathcal{B}^\gamma, \rho^\gamma) : \gamma < \delta)$ is a sequence satisfying each $(\overline{G}^\gamma, \mathcal{B}^\gamma, \rho^\gamma) \in \Gamma_{\mathcal{I}_\gamma}$ and for $\gamma < \gamma'$, $(\overline{G}^{\gamma'}, \mathcal{B}^{\gamma'}, \rho^{\gamma'})$ extends $(\overline{G}^\gamma, \mathcal{B}^\gamma, \rho^\gamma)$. Then the natural union of the chain $(\overline{G}, \mathcal{B}, \rho)$ extends each $(\overline{G}^\gamma, \mathcal{B}^\gamma, \rho^\gamma)$.

The final set of definitions describe the embeddings we will use to arrange $\overline{G}^0_* \sim_{\alpha_*} \overline{G}^1_*$.

Definition 7.3.9. If $(\overline{G}, \mathcal{B}, \rho) \in \Gamma_{\mathcal{I}}$, then say that H is a basic subgroup of G if H is spanned by $H \cap \mathcal{B}$. By $\overline{G} \restriction_H$ we mean $(H, G_i \cap H, \varphi \restriction_H : i \in I) \models \Omega^p_{\mathcal{I},\{0\}}$. By $(\overline{G}, \mathcal{B}, \rho) \restriction_H$ we mean $(\overline{G} \restriction_H, \mathcal{B} \cap H, \rho \restriction_{X^{\overline{G} \restriction_H}})$.

Suppose $(\overline{G}, \mathcal{B}, \rho), (\overline{G}', \mathcal{B}', \rho') \in \Gamma_{\mathcal{I}}$. Then by a -1-embedding from $(\overline{G}, \mathcal{B}, \rho)$ into $(\overline{G}', \mathcal{B}', \rho')$, we mean a map f where $f : \overline{G} \leq \overline{G}'$ is an embedding and $f[\mathcal{B}] \subseteq \mathcal{B}'$. For an ordinal $\alpha \geq 0$, say that f is an α-embedding if additionally: $f[X^{\overline{G}}] \subseteq X^{\overline{G}'}$, and for all $a \in X^{\overline{G}}$, if $\rho(a) < \omega \cdot \alpha$, then $\rho(a) = \rho'(f(a))$.

For all $\alpha \geq -1$, say that f is a partial α-embeddding from $(\overline{G}, \mathcal{B}, \rho)$ into $(\overline{G}', \mathcal{B}', \rho')$ if for some basic subgroup D of $\overline{G}$, f is an an α-embedding from $(\overline{G}, \mathcal{B}, \rho) \restriction_D$ to $(\overline{G}', \mathcal{B}', \rho')$.

Finally, we describe the construction of $\overline{G}^0_*, \overline{G}^1_*$. We will build them as a union of chains. In the outer layer, we will construct, by induction on $n < \omega$, index sets $\mathcal{I}^n$, and, for each $\ell < 2$, $(\overline{G}^{\ell,n}, \mathcal{B}^{\ell,n}, \rho^{\ell,n}) \in \Gamma_{\mathcal{I}^n}$ with a privileged element $e^n \in G^{0,n}$ for $n > 0$, and for each $\ell < 2$ a set $\mathcal{F}^{\ell,n}$, satisfying various constraints. The goal is that $(e^n : n < \omega)$ will witness that $X^{\overline{G}^{0,n}}$ is ill-founded, and $\mathcal{F}^{\ell,n}$ will be a set of partial embeddings from $\overline{G}^{\ell,n}$ to $\overline{G}^{1-\ell,n}$, which will be used to arrange that $\overline{G}^0_* \sim_{\alpha_*} \overline{G}^1_*$. Formally, we need the following requirements:

1. For $n < m < \omega$, $(\overline{G}^{\ell,m}, \mathcal{B}^{\ell,m}, \rho^{\ell,m})$ extends $(\overline{G}^{\ell,n}, \mathcal{B}^{\ell,n}, \rho^{\ell,n})$;

2. For each $n > 0$, $e^n \in X^{\overline{G}^{0,n}}$, and $\varphi^{0,n+1}(e_{n+1}) = e_n$ (so necessarily each $\rho^{0,n}(e_n) = \infty$).

3. For all $a \in X^{\overline{G}^{1,n}}$, $\rho^{1,n}(a) < \infty$.

4. For all n, ℓ, $(\varphi^{\ell,n})^n = 0$ (i.e. $\varphi^{\ell,n}$ iterated n-many times is 0);

5. Each $\mathcal{F}^{\ell,n}$ is a set of tuples (α, D, R, f), where $-1 \leq \alpha \leq \alpha_*$, and f is a partial α-embedding from $(\overline{G}^{\ell,n}, \mathcal{B}^{\ell,n}, \rho^{\ell,n})$ to $(\overline{G}^{1-\ell,n}, \mathcal{B}^{1-\ell,n}, \rho^{1-\ell,n})$ with domain D and range R;

6. For each $n < m$, and for each $\ell < 2$, $\mathcal{F}^{\ell,n} \subseteq \mathcal{F}^{\ell,m}$;

7. If $(\alpha, D, R, f) \in \mathcal{F}^{\ell,n}$ and $\alpha \geq 0$, then $(\alpha, R, D, f^{-1}) \in \mathcal{F}^{1-\ell,n}$ (in particular f^{-1} is a partial α-embedding);

8. Suppose $(\alpha, D, R, f) \in \mathcal{F}^{\ell,n}$, and suppose either $\beta < \alpha$ or else $\beta = -1$. Then for

every $a \in G^{\ell,n+1}$, there is some $D' \supseteq D \cup \{a\}$, $R' \supseteq R$, and $f' \supseteq f$ such that $(\beta, D', R', f') \in \mathcal{F}^{\ell,n+1}$;

9. $G^{0,0} = G^{1,0} = 0$ (this determines each $(\overline{G}^{\ell,0}, \mathcal{B}^{\ell,0}, \rho^{\ell,0})$), and $(\alpha_*, 0, 0, 0) \in \mathcal{F}^{0,0}$.

Having done this, let $(\overline{G}^{\ell}_*, \mathcal{B}^{\ell}_*, \rho^{\ell}_*)$ be the union of the chain $((\overline{G}^{\ell,m}, \mathcal{B}^{\ell,m}, \rho^{\ell,m}) : m < \omega)$, as promised by Lemma 7.3.8. Then $\overline{G}^0_* \not\equiv_{\infty\omega} \overline{G}^1_*$, since $(X^{\overline{G}^0_*}, \leq^{\overline{G}^0_*})$ is ill-founded (by condition (2)) while $(X^{\overline{G}^1_*}, \leq^{\overline{G}^1_*})$ is well-founded (by condition (3). On the other hand, it is clear that for all $n < \omega$, for all $(\alpha, D, R, f) \in \mathcal{F}^{\ell,n}$ with $\alpha \geq 0$, and for all finite tuples $\overline{a} \in D$, we have $(\overline{G}^{\ell}_*, \overline{a}) \sim_\alpha (\overline{G}^{1-\ell}_*, f(\overline{a}))$ (by condition (8)). Thus $\overline{G}^0_* \sim_{\alpha_*} \overline{G}^1_*$.

So it remains to show this construction is possible. This will mostly be achieved by the following two lemmas, which will allow us to handle the key condition (8) without disturbing any of the other hypotheses:

Lemma 7.3.10. Suppose $(\overline{G}^{\ell}, \mathcal{B}^{\ell}, \rho^{\ell}) \in \Gamma_{\mathcal{I}}$ for each $\ell < 2$. Suppose f is a partial -1-embedding from $(\overline{G}^0, \mathcal{B}^0, \rho^0)$ to $(\overline{G}^1, \mathcal{B}^1, \rho^1)$. Finally, suppose each $(\varphi^i)^{n+1} = 0$. Then we can find an index set $\mathcal{I}'$, and an extension $(\overline{G}^2, \mathcal{B}^2, \rho^2)$ of $(\overline{G}^1, \mathcal{B}^1, \rho^1)$ in $\Gamma_{\mathcal{I}'}$, such that $X^{\overline{G}_2} = X^{\overline{G}_1}$, and f extends to a -1-embedding h from $(\overline{G}^0, \mathcal{B}^0, \rho^0)$ to $(\overline{G}^2, \mathcal{B}^2, \rho^2)$, and finally $(\varphi^2)^{n+1} = 0$.

Proof. Let D be the domain of f and let R be its range. Recall that we require D and R to be basic subgroup of G, that is, $\mathcal{B} \cap D$ spans D. Let $\mathcal{I}' \supseteq \mathcal{I}$ be large enough.

Write $\mathcal{A} = \mathcal{B}^0 \backslash (\mathcal{B} \cap D)$. Let $G^2 = G^1 \times \oplus_{\mathcal{A}} \mathbb{Z}$. Write $H = 0 \times \oplus_{\mathcal{A}} \mathbb{Z}$, and let $g : \mathrm{span}_{G^0}(\mathcal{A}) \cong H$ be the natural isomorphism. Let $\mathcal{B}^2$ be $\mathcal{B}^1 \cup g[\mathcal{A}]$. Define $h : G^0 \to G^2$ via $h \restriction_D = f$ and $h \restriction_{\mathrm{span}(\mathcal{A})} = g$.

Define $\varphi^2 : G^2 \to G^2$ via: $\varphi^2 \restriction_{G^1} = \varphi^1$, and $\varphi^2 \restriction_H = g \circ \varphi^0 \circ g^{-1}$. For each $i \in \mathcal{I}$, let $G^2_i = G^1_i$.

Let $G_i^2 : i \in \mathcal{I}' \backslash \mathcal{I}$ enumerate all singly generated pure subgroups of G^2 which are not contained in G^1. Note then that $X^{\overline{G}^2} = X^{\overline{G}^1}$ so we must let $\rho^2 = \rho^1$ and then clearly we are done.

□

Lemma 7.3.11. Suppose $(\overline{G}^\ell, \mathcal{B}^\ell, \rho^\ell) \in \Gamma_\mathcal{I}$ for each $\ell < 2$. Suppose $0 \leq \beta < \alpha$, and f is a partial α-embedding from $(\overline{G}^0, \mathcal{B}^0, \rho^0)$ to $(\overline{G}^1, \mathcal{B}^1, \rho^1)$ such that f^{-1} is also a partial α-embedding. Finally, suppose each $(\varphi^i)^{n+1} = 0$. Then we can find an index set $\mathcal{I}'$, and an extension $(\overline{G}^2, \mathcal{B}^2, \rho^2)$ of $(\overline{G}^1, \mathcal{B}^1, \rho^1)$ in $\Gamma_{\mathcal{I}'}$, such that:

- f extends to an β-embedding h from $(\overline{G}^0, \mathcal{B}^0, \rho^0)$ to $(\overline{G}^2, \mathcal{B}^2, \rho^2)$;
- h^{-1} is a partial β-embedding from $(\overline{G}^2, \mathcal{B}^2, \rho^2)$ to $(\overline{G}^0, \mathcal{B}^0, \rho^0)$;
- For all $a \in X^{\overline{G}^2} \backslash X^{\overline{G}^1}$, $\rho^2(a) < \omega \cdot \alpha$;
- $(\varphi^2)^{n+1} = 0$.

Proof. Let D be the domain of f and let R be its range. Let $\mathcal{I}' \supseteq \mathcal{I}$ be large enough.

Write $\mathcal{B}^0 = (\mathcal{B} \cap D) \cup \mathcal{A}$. Let $G^2 = G^1 \times \oplus_\mathcal{A} \mathbb{Z}$. Write $H = 0 \times \oplus_\mathcal{A} \mathbb{Z}$, and let $g : \mathrm{span}_{G^0}(\mathcal{A}) \cong H$ be the natural isomorphism. Let $\mathcal{B}^2$ be $\mathcal{B}^1 \cup g[\mathcal{A}]$. Define $h : G^0 \to G^2$ via $h \restriction_D = f$ and $h \restriction_{\mathrm{span}(\mathcal{A})} = g$.

Define $\varphi^2 : G^2 \to G^2$ via: $\varphi^2 \restriction_{G^1} = \varphi^1$, and $\varphi^2 \restriction_H = g \circ \varphi^0 \circ g^{-1}$. For each $i \in \mathcal{I}$, let $G_i^2 = G_i^1$. It remains to define G_i^2 for $i \in \mathcal{I}' \backslash \mathcal{I}$, and then to define ρ^2.

Let $G_i^2 : i \in \mathcal{I}' \backslash \mathcal{I}$ enumerate all singly generated pure subgroups of G^2 which are not contained in G^1 and which are not contained in $R + H$. Note then that $X^{\overline{G}^2} = X^{\overline{G}^1} \cup h[X^{\overline{G}^0}]$. We define ρ^2 as follows: suppose $a \in X^{\overline{G}^2}$. If $a \in X^{\overline{G}^1}$ then we must let $\rho^2(a) = \rho^1(a)$. Suppose instead $a \in h[X^{\overline{G}^0}] \backslash X^{\overline{G}^1}$; write $a = h(a')$. If $\rho^0(a) < \omega \cdot \beta$

then let $\rho^2(a) = \rho^0(a)$. Otherwise, let k be largest such that there is $c' \in X^{\overline{G}^0}$ such that $(\varphi^0)^k(c') = a'$, and for all $k' < k$, $(\varphi^0)^{k'}(c') \in X^{\overline{G}^0}$, and finally $\rho^0(c') \geq \omega \cdot \beta$; let $\rho^2(a) = \omega \cdot \beta + k$. Note that $k \leq n$ since $(\varphi_0)^{n+1} = 0$.

Now I claim this works. First of all:

Claim. Suppose $a \in h[X^{\overline{G}^0}]\backslash X^{\overline{G}^1}$; write $h(a') = a$. Then $\rho^0(a') \geq \rho^2(a)$.

Proof. This is immediate if $\rho^0(a') < \omega \cdot \beta$, so suppose instead $\rho^0(a') \geq \omega \cdot \beta$; let c', k be as in the definition of $\rho^2(a)$. Then $\rho^0(a') = \rho^0((\varphi_0)^k(c')) \geq \rho^0(c') + k \geq \omega \cdot \beta + k = \rho^2(a)$. $\square$

We show $(\overline{G}^2, \mathcal{B}^2, \rho^2) \in \Gamma_{\mathcal{I}'}$. We must check that for all $a, b \in X^{\overline{G}^2}$ with $\varphi^2(b) = a$ and with $\rho^2(a) < \infty$, we have that $\rho^2(b) < \rho^2(a)$. If $b \in X^{\overline{G}^1}$, then $a \in X^{\overline{G}^1}$ and this is clear. Suppose $b \in h[X^{\overline{G}^0}]\backslash X^{\overline{G}^1}$, and $a \in X^{\overline{G}^1}$; note that $a \in f[X^{\overline{G}^0}] \subseteq h[X^{\overline{G}^0}]$; write $a = f(a')$ and write $b = h(b')$. We consider two further subcases. If $\rho^0(a') = \rho^1(a)$, then $\rho^2(a) = \rho^0(a') > \rho^0(b') \geq \rho^2(b)$, using the claim. If $\rho^0(a') \neq \rho^1(a)$, then since f, f^{-1} are both α-embeddings we must have $\rho^0(a'), \rho^1(a) \geq \omega \cdot \alpha$. Hence $\rho^2(a) = \rho^1(a) \geq \omega \cdot \alpha > \omega \cdot \beta + n \geq \rho^2(b)$. Finally, suppose both $a, b \in h[X^{\overline{G}^0}]\backslash X^{\overline{G}^1}$. Write $a = h(a')$, $b = h(b')$. If $\rho^0(a') < \omega \cdot \beta$ then $\rho^2(a) = \rho^0(a') > \rho^0(b') \geq \rho^2(b)$. If $\rho^0(a') \geq \omega \cdot \beta$ and $\rho^0(b') < \omega \cdot \beta$, then $\rho^2(a) \geq \omega \cdot \beta > \rho^0(b') = \rho^2(b)$. Finally, if $\rho^0(a')$ and $\rho^0(b')$ are both $\geq \omega \cdot \beta$, then let k be as in the definition of $\rho^2(b)$, i.e. so that $\rho^2(b) = \omega \cdot \beta + k$; clearly then $\rho^2(a) \geq \omega \cdot \beta + (k+1)$.

To finish, it is clear that for all $a' \in \overline{G}^0$, if either $\rho^0(a') < \omega \cdot \beta$ or else $\rho^2(h(a')) < \omega \cdot \beta$, then $\rho^0(a') = \rho^2(h(a'))$; hence h is a β-embedding and h^{-1} is a partial β-embedding. $\square$

Now, suppose we are given $(\overline{G}^{\ell,n}, \mathcal{B}^{\ell,n}, \rho^{\ell,n})$, $\mathcal{F}^{\ell,n}$, and e^n satisfying (1) through (9). We explain how to get $(\overline{G}^{\ell,n+1}, \mathcal{B}^{\ell,n+1}, \rho^{\ell,n+1})$, $\mathcal{F}^{\ell,n+1}$, and e^{n+1}.

Define $G^0 = G^{0,n} \times \mathbb{Z}$, let $e^{n+1} = (0,1) \in G^0$. Let $\mathcal{I} \supseteq \mathcal{I}^n$ be sufficiently large. For each $i \in \mathcal{I}^n$ let $G^0_i = G^{0,n}_i$. Choose $(G^0_i : i \in \mathcal{I}\backslash\mathcal{I}^n)$ so as to enumerate the singly-generated pure subgroups of G^0 which are not contained in $G^{0,n}$ and which do not contain e^{n+1}. Define φ^0 via $\varphi^0 \restriction_{G^{0,n}} = \varphi^{0,n}$ and $\varphi^0(e^{n+1}) = e^n$ (or, if $n = 0$ then let $\varphi^0(e_1) = 0$). We have defined $\overline{G}^0 \models \Omega^p_{\mathcal{I},\{0\}}$, an extension of $\overline{G}^{0,n}$. Note that $X^{\overline{G}^0} = X^{\overline{G}^{n,0}} \cup \{me^{n+1} : m \in \mathbb{Z}, m \neq 0\}$. Let $\mathcal{B}^0 = \mathcal{B}^{0,n} \cup \{e^{n+1}\}$, and define each $\rho^0(me^{n+1}) = \infty$.

Define $G^1 = G^{1,n}$; for each $i \in \mathcal{I}^n$, let $G^1_i = G^1$, and for each $i \in \mathcal{I} \backslash \mathcal{I}^n$, and let $G^1_i = 0$; let $\varphi^1 = \varphi^{1,n}$. Finally, let $\mathcal{F}^\ell = \mathcal{F}^{\ell,n}$ for each $\ell < 2$.

The only thing left to do is arrange (8) to hold. For this, apply Lemmas 7.3.10 and 7.3.11 repeatedly, using Lemma 7.3.8 at limit stages.

This concludes the proof of Theorem 7.3.2, and hence of Theorem 7.0.5.

Bibliography

[1] J. Baldwin. *Categoricity*, University Lecture Series, http://homepages.math.uic.edu/ jbaldwin/pub/AEClec.pdf.

[2] J. Baldwin, A. H. Lachlan. On strongly minimal sets. *J Symb Loic* 26, 1 (1971) 79-96.

[3] J. Baker and K. Kunen. Limits in the Uniform Ultrafilters. *Trans. Amer. Math. Soc.* 353, 10 (2001) 4083-4093.

[4] J. Barwise. *Admissible sets and structures.* Perspectives in mathematical logic. Springer Berlin Heidelberg, 1975.

[5] S. Buechler. Lascar Strong Types in Some Simple Theories. *J Symb Logic* 64,2 (1999) 817-824.

[6] E. Casanovas. The number of types in simple theories. *Ann. Pure Appl. Logic* 98 (1999), 69-86.

[7] A. Chernikov, I. Kaplan. Forking and Dividing in NTP_2 theories. *J Symb Logic* 77, 1 (2012) 1-20.

[8] M. Džamonja and S. Shelah. On $\trianglelefteq^*$-maximality. *Ann. Pure App. Logic* 125, 1-3 (2004) 119-158.

[9] P. C. Ekloff and S. Shelah, *Absolutely rigid systems and absolutely indecomposable groups,* in Abelian groups and modules (Dublin, 1998), Trends Math., Birkh́auser, Basel, 1999, 257-268.

[10] R. Engleking and M. Karlowicz. Some theorems of set theory and their topological consequences. *Fund Math* 57 (1965), 275-285.

[11] P. Erdös, A. Hajnal, Máté, and Rado. *Combinatorial Set Theory: Partition Relations for Cardinals.* Studies in Logic and the Foundations of Mathematics (1984).

[12] H. Friedman and L Stanley. A Borel reducibility theory for classes of countable structures. *J. Symbolic Logic* 54 (1989), no. 3, 894-914.

[13] S. Fuchino, H. Mildenberger, S. Shelah and P. Vojtas. On absolutely divergent series. *Fund. Math.* 160 (1999) 255-268.

[14] S. Gao. *Invariant Descriptive Set Theory.* Chapman & Hall/CRC Pure and Applied Mathematics. CRC Press, 2008.

[15] V. Gitman, J. Hamkins, and T. Johnstone, *What is the theory ZFC without power set?*, arXiv:1110.2430, August 2015.

[16] J. Goodrick, *The Schroder-Bernstein property for theories of abelian groups*, arXiv:0705.1850, May 2007.

[17] J. Goodrick, *When are elementarily bi-embeddable structures isomorphic?*, Ph.D. thesis under Thomas Scanlon at UC Berkeley, 2007.

[18] J. Goodrick, B. Kim, A. Kolesnikov. Type-amalgamation properties and polygroupoids in stable theories. *J. Math. Log.* 15 (2015).

[19] L. Harrington, A. S. Kechris, and A. Louveau. A Glimm-Effros dichotomy for Borel equivalence relations. *J. Amer. Math. Soc.* 3 (1990), 903-928.

[20] M. Harrison-Trainor. A first-order theory of Ulm type. Preprint: https://arxiv.org/abs/1702.06586.

[21] W. Hodges. *Model Theory.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1993.

[22] G. Hjorth. *Classification and Orbit Equivalence Relations,* volume 75 of *Mathematical Surveys and Monographs.* American Mathematical Society, 1999.

[23] G. Hjorth, *The isomorphism relation on countable torsion free abelian groups*, Fund. Math. 175 (2002), 241-257.

[24] P. Holy, R. Krapf, P. Lücke, A. Nijegomir, and P. Schlicht. Class forcing, the forcing theorem and boolean completions. *Journal of Symbolic Logic* 81, 4 (2016) 1500-1530.

[25] G. Horth, A. Kechris and A. Louveau. Borel equivalence relations induced by actions of the symmetric group. *Ann. Pure App. Logic* 92 (1998) 63-112.

[26] E. Hrushovski, Pseudo-finite fields and related structures in *Model theory and applications* (ed. L. Belair et al), 151-212, Quaderni di Mathematica, Volume 11 (Seconda Universita di Napoli, 2002).

[27] T. Jech. *Set theory.* The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[28] A. Kanamori. *The Higher Infinite,* second edition, Springer-Verlag, Berlin, 2009.

[29] A. Kanamori. Levy and Set theory. *Ann. Pure App. Logic* 140 (2006) 233-252.

[30] V. Kanovei. *Borel equivalence relations.* Structure and classification. University Lecture Series, 44. *American Mathematical Society,* Providence, RI, 2008.

[31] V. Kanovei, M. Sabox, and Jindřich Zapletal, *Canonical Ramsey Theory on Polish Spaces*, volume 202 of *Cambridge Tracts in Mathematics.* Cambridge University Press, 2013.

[32] I. Kaplan and S. Shelah. Forcing a countable structure to belong to the ground model. *Mathematical Logic Quarterly, 2015* 62, 6 (2016) 530-546.

[33] A. Kechris. *Classical Descriptive Set Theory.* Graduate Texts in Mathematics. Springer New York, First Edition, 1995.

[34] H. J. Keisler. Ultraproducts which are not saturated. *J Symb Logic* 32 (1967) 23-46.

[35] H. J. Keisler. *Model Theory for Infinitary Logic.* North Holland, 1971.

[36] B. Kim. Forking in Simple Unstable Theories, *J of London Math. Soc.* 57 (1998), 25767.

[37] B. Kim. *Simplicity Theory*, Oxford University Press, Oxford, 2014.

[38] B. Kim and K. Kim. Notions around tree property 1. *Ann. Pure App. Logic* 162, 9 (2011) 698-709.

[39] B. Kim, A. Kolesnikov, and A. Tsuboi. Generalized amalgamation and n-simplicity. *Ann. Pure Appl. Logic* 155 (2008), no. 2, 97-114.

[40] A. Kolesnikov. n-simple theories. *Ann. of Pure and Appl. Logic* 131 (3005) 227261.

[41] M. Koerwien. Comparing Borel Reducibility and Depth of an ω-stable Theory. *Notre Dame Journal of Formal Logic.* 50, 4 (2009) 365-380.

[42] M. Koerwien. A Complicated ω-stable depth 2 theory. *J of Symb Logic.* 76, 1 (2011) 47-65.

[43] A. Kruckman, N. Ramsey. Generic expansion and Skolemization in $NSOP_1$ theories. Preprint, https://arxiv.org/abs/1706.06616.

[44] K. Kunen. Set theory. Studies in Logic (London), 34. *College Publications,* London, 2011.

[45] K. Kunen. Ultrafilters and independent sets. *Trans. Amer. Math. Soc.* 172 (1972) 299-306.

[46] K. Kunen. Weak P-points in BN - N. *Proc. Bolyai Janos Soc. Colloq. on Topology,* 1978, pp. 741-749.

[47] M. Laskowski and S. Shelah. Borel completeness of some $\aleph_0$-stable theories. *Fund Math*, 2013.

[48] Łoś. Quelques remarques, theoremes et problemes sur les classes definissables dalgebres, *Mathematical Interpretations of Formal Systems*, North-Holland (1995) 98-113.

[49] M. Malliaris. The characteristic sequence of a first-order formula. *J Symb Logic* 75, 4 (2010) 1415-1440.

[50] M. Malliaris. Hypergraph sequences as a tool for saturation of ultrapowers. *J Symb Logic* 77, 1 (2012) 195-223.

[51] M. Malliaris. Independence, order, and the interaction of ultrafilters and theories. *Ann. Pure App. Logic*, 163, 11 (2012) 1580-1595.

[52] M. Malliaris. *Persistence and Regularity in Unstable Model Theory.* PhD thesis. University of California, Berkeley, 2009.

[53] M. Malliaris. Realization of φ-types and Keisler's order. *Ann. Pure Appl. Logic* 157 (2009) 220-224.

[54] M. Malliaris and S. Shelah. Cofinality spectrum theorems in model theory, set theory, and general topology. *J. Amer. Math. Soc.* 29 (2016), 237–297.

[55] M. Malliaris and S. Shelah. Constructing regular ultrafilters from a model-theoretic point of view. *Trans. Amer. Math. Soc.* 367 (2015), 8139–8173

[56] M. Malliaris and S. Shelah. A dividing line within simple unstable theories. *Advances in Math* 249 (2013) 250–288.

[57] M. Malliaris and S. Shelah. Existence of optimal ultrafilters and the fundamental complexity of simple theories. *Advances in Math* 290 (2016) 614-681.

[58] M. Malliaris and S. Shelah. Keisler's order has infinitely many classes. *Israel J. of Math*, accepted.

[59] M. Malliaris and S. Shelah. Model-theoretic applications of cofinality spectrum problems. Preprint, https://arxiv.org/pdf/1503.08338.pdf.

[60] M. Malliaris and S. Shelah. Model-theoretic properties of ultrafilters built by independent families of functions. *J Symb Logic* 79, 1 (2014) 103-134.

[61] M. Malliaris and S. Shelah. A new look at interpretability and saturation. Preprint, https://arxiv.org/pdf/1709.04899.pdf.

[62] R. Mansfield. Theory of Boolean Ultrapowers. *Ann. Pure Appl. Logic* 2, 3 (1971) 297-323.

[63] D. Marker. The Borel Complexity of Isomorphism for Theories with Many Types. *Notre Dame Journal of Formal Logic* 48, 1 (2007) 93-97.

[64] D. Marker. *Model theory : an introduction,* volume 217 of *Graduate texts in mathematics.* Springer, New York, 2002.

[65] M. Morley. Categoricity in power. *Trans. Am. Math. Soc.* 114 (1965), 514-538.

[66] A. Mostowski. Proofs of Non-Deducibility in Intuitionistic Functional Calculus. *J. Symbolic Logic* 13, 4 (1948), 204-207.

[67] J. Nešitřil, V. Rödl. Ramsey Classes of Set Systems. *Journal of Combinatorial Set Theory* 34 (1983) 183-201.

[68] T. Nurmagambetov, *Characterization of ω-stable theories with a bounded number of dimensions*, Algebra i Logika 28, 5 (1989), 388396.

[69] T. Nurmagambetov, *The mutual embeddability of models,* Theory of Algebraic Structures (in Russian), Karagand. Gos. Univ., 1985, 109115.

[70] H. Rasiowa, R. Sikorski. Algebraic treatment of the notion of satisfiability. *Fund. Math.* 40, 1 (1953) 62-95.

[71] H. Rasiowa, R. Sikorski. A proof of the completeness theorem of Gödel. *Fund. Math.* 37, 1 (1950) 193-200.

[72] D. Scott. A proof of the independence of the continuum hypothesis. *Math. Systems Theory* 1 (1967), 89-111.

[73] S. Shelah, *Better quasi-orders for uncountable cardinals,* Israel J. of Math., 42, 3 (1982), 177-226.

[74] S. Shelah. Borel sets with large squares. *Fund. Math.* 159 (1999).

[75] S. Shelah. *Classification Theory*, North-Holland, Amsterdam, 1978.

[76] S. Shelah. A comment on '$\mathfrak{p} < \mathfrak{t}$'. Canadian Math Bulletin 52 (2009) 303-314.

[77] S. Shelah. Every two elementarily equivalent models have isomorphic ultrapowers. *Israel J. Math.,* 10 (1971), 224-233.

[78] S. Shelah. Toward classifying unstable theories. *Ann. Pure Appl. Logic* 80 (1996) 229-255.

[79] S. Shelah. Simple unstable theories. *Ann. Math. Logic* 19 (1980), no. 3, 177-203.

[80] S. Shelah and A. Usvyatsov. More on SOP_1 and SOP_2. *Ann. Pure App. Logic* 155, 1 (2008) 16-31.

[81] S. Shelah and D. Ulrich. $\leq_{SP}$ can have infinitely many classes. Preprint.

[82] S. Shelah and D. Ulrich. Torsion-Free Abelian Groups are Consistently $a\Delta^1_2$-complete. Preprint.

[83] Silver, Jack H. *A large cardinal in the constructible universe.*Fund. Math. 69 (1970), 93-100.

[84] Silver, Jack H. *Some applications of model theory in set theory.* Thesis (Ph.D.)- University of California, Berkeley. 1966.

[85] Calderoni, Filippo; Thomas, Simon. The bi-embeddability relation for countable abelian groups. Preprint.

[86] H. Ulm, Zur Theorie der abzählbar-unendlichen Abelschen Gruppen. (German) *Math. Ann.* 107 (1933), no. 1, 774-803

[87] D. Ulrich. Low is a dividing Line in Keisler's order. *Proc. Amer. Math. Soc.*, submitted April 2017.

[88] D. Ulrich. Keisler's order is not linear, assuming a supercompact. *J Symb Logic*, accepted. arXiv:1702.01237.

[89] D. Ulrich, R. Rast, C. Laskowski, *Borel Complexity and Potential Canonical Scott Sentences*, Fund. Math. 239, 2 (2017) 101-147.

[90] Vaughan, Jerry E. Small uncountable cardinals and topology. With an appendix by S. Shelah. *Open problems in topology,* 195-218, North-Holland, Amsterdam, 1990.

[91] P. Vopěnka. The limits of sheaves over extremally disconnected compact Hausdorff spaces. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 15 (1967), 1-4.