ABSTRACT

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Schottky groups are classical examples of free groups acting properly discontinuously on the complex projective line. We discuss two different applications of similar constructions. The first gives examples of 3-dimensional Lorentzian Kleinian groups which act properly discontinuously on an open dense subset of the Einstein universe. The second gives a large class of examples of free subgroups of automorphisms groups of partially cyclically ordered spaces. We show that for a certain cyclic order on the Shilov boundary of a Hermitian symmetric space, this construction corresponds exactly to representations of fundamental groups of surfaces with boundary which have maximal Toledo invariant.

GENERALIZATIONS OF SCHOTTKY GROUPS

by

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Chapter 1: Introduction

This thesis is focused around the idea of *Schottky groups* and different ways to generalize their classical construction. Such generalizations provide large classes of examples of geometrically interesting free, discrete subgroups of Lie groups. Perhaps the most commonly known version of this arises in the theory of Kleinian groups. A *classical Schottky group* is a discrete subgroup $\Gamma \subset \mathsf{PSL}(2, \mathbb{C})$ of Möbius transformations with an explicit presentation as follows. Choose $D_1^{\pm}, \ldots, D_k^{\pm}$ disjoint, closed round disks in $\mathbb{CP}^1$ and elements $g_1, \ldots, g_k \in \mathsf{PSL}(2, \mathbb{C})$ such that g_i maps the interior of D_i^- to the complement of D_i^+. The group Γ generated by the g_i is then called a classical Schottky group. It is a free group on those k generators, and it acts properly discontinuously on the complement of a Cantor set in $\mathbb{CP}^1$.

The *Ping-pong lemma* is the elementary result which proves that Schottky groups are free (Lemma 1.1.2). It was introduced in its modern generality by Tits [Tit72] in order to prove the Tits alternative. Generalizations of the construction come in two flavors. The first is more dynamical : assume that some generators in a linear group are sufficiently transverse and sufficiently attractive/repulsive, and conclude that the group they generate is free. The second is more geometric, like the construction in $\mathbb{CP}^1$. Natural hypersurfaces bounding disjoint *halfspaces* in a homogeneous space are mapped pairwise by automorphisms, and these hypersurfaces can be shown to bound a fundamental domain for the action of the resulting group. The constructions discussed here generally belong to this second type.

Previous constructions of Schottky groups using hypersurfaces include projective linear groups acting on complex projective spaces ([Nor86], [SV03]), affine Lorentzian groups acting on $\mathbb{R}^{2,1}$ ([Dru92]), and conformal Lorentzian groups acting on the Einstein universe ([CFLD14], [BCFG17]).

This last example is the one we focus on in Chapter 3. The content of this chapter is joint work with Charette, Francoeur and Goldman. In [CFLD14], Charette, Francoeur and Lareau-Dussault explain how to construct examples of Schottky subgroups of $\mathsf{O}(3,2)$ by choosing disjoint *crooked surfaces* in the Einstein universe. This was inspired by the success of *crooked planes* in the study of Schottky groups of affine Lorentzian transformations acting on Minkowski space ([Dru92], [BCDG14], [CDGM03]). The relevance of *halfspaces* in this theory, as opposed to just hypersurfaces, is motivated by the results of [BCDG14]. Namely, it is more natural to characterize disjoint halfspaces than disjoint bounding surfaces. Crooked surfaces were introduced by Frances [Fra03] in 2003 in order to conformally compactify flat affine Lorentzian manifolds. The resulting subgroups of $\mathsf{O}(3,2)$ are *Lorentzian Kleinian groups* in the sense of Frances [Fra05]. We expand upon the work of Charette-Francoeur-Lareau-Dussault by proving a complete disjointness criterion for these surfaces, a first step towards a classification of the Schottky groups that can be constructed in this way.

Theorem 1.0.1. *Two crooked surfaces $C_1, C_2 \subset \mathsf{Ein}$ are disjoint if and only if the four photons bounding the stem of C_1 are disjoint from C_2, and the four photons bounding the stem of C_2 are disjoint from C_1.*

This also builds upon the work of Danciger, Guéritaud and Kassel, who studied crooked surfaces in the context of the negatively curved anti de Sitter space. In particular, Theorem 1.0.1 specializes to the disjointness criterion for anti de Sitter crooked planes introduced in [DGK14].

The last chapter describes a different construction which applies in a much broader setting. The results in it are joint work with Nicolaus Treib. The key notion that we use is that of a *partial cyclic order* on a set, introduced by Novák in 1982 [Nov82]. They generalize cyclic orders in the same way that partial orders generalize linear orders. The circle S^1 has a cyclic ordering which is preserved by its group of orientation-preserving homeomorphisms. Schottky groups can be defined by choosing disjoint intervals and mapping them to the opposites of each other. In particular, $\mathsf{PSL}(2, \mathbb{R})$ acts on the circle by orientation-preserving diffeomorphisms, so *real* (or *Fuchsian*) Schottky groups are of this type. Section 1.1 explains this motivating example. The construction for arbitrary partial cyclic orders is an analog of this S^1 example.

Under some topological assumptions about the partially ordered set, we show that such Schottky groups admit left-continuous, equivariant limit curves. The topological hypotheses are motivated by the main class of examples that we are interested in : Shilov boundaries of Hermitian symmetric spaces. The cyclic nature of these homogeneous spaces was previously observed and used to prove strong results about discrete subgroups of Hermitian Lie groups ([Wie04], [BIW10], [BBH$^+$16]). We show the following:

Theorem 1.0.2. *Let Σ be a compact, connected, orientable surface with non-empty boundary and G a Lie group of Hermitian type. Let $\rho : \pi_1(\Sigma) \to G$ be a homomorphism. Then, ρ has maximal* Toledo invariant *if and only if there is a Schottky presentation (in this cyclic sense) for the group $\rho(\pi_1(\Sigma)) \subset G$.*

The Toledo invariant is a conjugacy invariant for representations of surface groups into Lie groups of Hermitian type that was first introduced in 1979 [Tol79]. It generalizes the Euler number for representations of fundamental groups of closed surfaces into $\mathsf{PSL}(2,\mathbb{R})$. Both invariants take values in a bounded range by the *Milnor-Wood inequality*. Goldman [Gol80] showed that the Euler number assumes its maximal value precisely for representations which correspond to holonomies of hyperbolic structures. This motivated the study of representations which have maximal Toledo invariant and they geometric properties (see [BIW10]).

The concrete description in terms of Schottky groups that we use provides a simple description of fundamental domains for the action of these representations. For maximal representations into $\mathsf{Sp}(2n,\mathbb{R})$, we give a fundamental domain and domain of discontinuity in $\mathbb{RP}^{2n-1}$. This domain of discontinuity had previously been described in the *Anosov* case by Guichard and Wienhard [GW12], but not all maximal representations are Anosov since they can contain unipotent elements.

Finally, an additional example of cyclic Schottky group is discussed, which is inspired by the work of Fock and Goncharov on positivity of flags. For simplicity we work with the space of oriented flags in $\mathbb{R}^3$, and we show that there is a partial cyclic order on this space preserved by the action of the group $\mathsf{PSL}(3, \mathbb{R})$.

1.1 Fuchsian Schottky groups

Let V be a 2-dimensional vector space over $\mathbb{R}$. Consider the projective line $\mathbb{P}(V)$. Its group of projective automorphisms is $G = \mathsf{PGL}(V)$. In this section, we describe a classical construction of free, discrete subgroups of G.

Definition 1.1.1. An *interval* $I \subset \mathbb{P}(V)$ is an open interval in any affine patch $\mathbb{P}(V)\backslash\{[v]\}$.

The complementary interval $-I$ is defined to be the interval $\mathbb{P}(V)\backslash\overline{I}$.

Let $I_1^{\pm}, \ldots, I_k^{\pm}$ be $2k$ disjoint intervals in $\mathbb{P}(V)$. Choose elements $g_1, \ldots, g_k \in G$ satisfying the condition that $g_j(-I_j^-) = I_j^+$ (see Figure 4.1). Then,

Lemma 1.1.2 (The Ping-pong lemma)**.** *The group $\Gamma \subset G$ generated by $g_1, \ldots, g_k$ is free on these generators.*

We will actually prove the more general case:

Lemma 1.1.3. *Let G be a group acting on a set X. Let $\gamma_1, \ldots, \gamma_k \in G$ $(k \geq 2)$ be elements of infinite order, and $X_1, \ldots, X_k \subset X$ be non-empty, disjoint subsets such that whenever $i \neq j$ we have*

$$\gamma_i^m(X_j) \subset X_i.$$

for any $m \neq 0$. Then, the group generated by $\gamma_1, \ldots, \gamma_k$ is free on those generators.

Proof. Consider an arbitrary reduced word $w = \gamma_{a_1}^{m_1} \dots \gamma_{a_\ell}^{m_\ell}$ in the generators γ_i (here reduced just means $a_i \neq a_{i+1}$ and $m_i \neq 0$). First, let's assume $a_1 = a_\ell$ and choose any $j \neq a_\ell$. We will look at the image of X_j under w.

$$wX_j = \gamma_{a_1}^{m_1} \dots \gamma_{a_\ell}^{m_\ell} X_j \subset \gamma_{a_1}^{m_1} \dots \gamma_{a_{\ell-1}}^{m_{\ell-1}} X_{a_\ell} \subset \cdots \subset X_{a_1}.$$

By disjointness of the X_i we get that w acts nontrivially on X. For the remaining cases ($a_1 \neq a_\ell$), consider the word $w' = \gamma_{a_1}^{m_1} w \gamma_{a_1}^{-m_1} = \gamma_{a_1}^{2m_1} \dots \gamma_{a_\ell}^{m_\ell} \gamma_{a_1}^{m_1}$. By the previous argument, w' acts nontrivially, and so w also acts nontrivially.

Lemma 1.1.2 follows from this using $\gamma_i = g_i$ and $X_i = I_i^+ \cup I_i^-$. □

If the defining intervals $I_1^\pm, \dots, I_k^\pm$ have disjoint closures, we call Γ a *Fuchsian Schottky group*. These groups are intimately related to hyperbolic structures on surfaces as follows.

Consider the upper half plane model $\{z \in \mathbb{C} \mid \Im(z) > 0\}$ of the hyperbolic plane $\mathbb{H}^2$. Its boundary is naturally the real projective line $\mathbb{RP}^1$. Any projective automorphism of this boundary extends uniquely to an isometry of $\mathbb{H}^2$, and vice versa. Any interval $I \subset \mathbb{RP}^1$ defines a unique open half plane in $\mathbb{H}^2$ by taking its convex hull, and disjoint intervals correspond to disjoint half planes. A Fuchsian Schottky group Γ thus acts on $\mathbb{H}^2$ by hyperbolic isometries. Additionally, the complement in $\mathbb{H}^2$ of the half planes defining the Schottky group is a fundamental domain for that action. The quotient is a hyperbolic surface with nonempty boundary, whose topology is determined by the combinatorics of the Schottky construction.

Chapter 2: Lorentzian geometries

In this chapter we introduce three examples of constant curvature Lorentzian 3-manifolds. Then, we define the Einstein universe Ein which is the model for conformal Lorentzian geometry. All three constant curvature examples conformally embed in a natural way into the Einstein universe, and so they help understand its geometry. In the next chapter, we study surfaces in Ein and their intersections in order to build Schottky groups acting on it. This space is also one of the main motivating examples for the general theory developed in Chapter 4.

2.1 Models for 3-dimensional Lorentzian space forms

A Lorentzian space form is a smooth manifold equipped with a constant sectional curvature Lorentzian metric. There are three cases to consider : zero curvature, negative curvature and positive curvature.

2.1.1 Minkowski space

Minkowski space is the flat model. It is analogous to Euclidean space in many ways. For instance, it is homeomorphic to $\mathbb{R}^3$ and its geodesics are straight lines.

Let $V^{2,1}$ be a 3-dimensional real vector space and $\cdot$ be a nondegenerate symmetric bilinear form of signature $(2,1)$ on $V^{2,1}$. We call such a vector space a *Lorentzian vector space*. Minkowski 3-space Min is an affine space with $V^{2,1}$ as its vector space of translations.

The form $\cdot$ on $V^{2,1}$ defines at each point of Min a *lightcone* :

Definition 2.1.1. The *lightcone* of the point $p \in \mathsf{Min}$ is the set of points $q \in \mathsf{Min}$ such that $(q-p)\cdot(q-p) = 0$.

More precisely, $\cdot$ induces a trichotomy of vectors in $V^{2,1}$ which translates to a trichotomy of points in Min whenever a base point is chosen. We call a vector $v \in V^{2,1}$

- *lightlike* whenever $v \cdot v = 0$,
- *timelike* whenever $v \cdot v < 0$, and
- *spacelike* whenever $v \cdot v > 0$

This classification of vectors into different types has consequences on every part of the geometry of Minkowski space. For instance, geodesics have a type according to their direction vector and this is preserved by any self isometries of Minkowski space.

The timelike vectors in $V^{2,1}$ are divided into two connected components. Fixing one of these components is called choosing a *time orientation* on $V^{2,1}$. We can use this structure to define a partial order on Min. The timelike vectors in a fixed component C form a proper convex cone. The partial order is then just defined by $q > p$ whenever $q - p \in C$. In this situation, we will sometimes say q is in the future of p. This partial order will be crucial for the construction described in Chapter 4.

The isometry group of Minkowski space is the *affine Lorentzian group* $V^{2,1} \rtimes \mathsf{O}(V^{2,1})$. Since $V^{2,1}$ is homogeneous space for a low dimensional Lie group, some isomorphisms between Lie algebras give different models of this space.

The Lie algebra isomorphism $\mathfrak{o}(V^{2,1}) \cong \mathfrak{sl}(2,\mathbb{R})$ has the useful application that the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of traceless 2×2 matrices itself is a model for Minkowski space. We now describe this model.

Definition 2.1.2. The *Killing form* on $\mathfrak{sl}(2,\mathbb{R})$ is the bilinear form $\mathrm{K}(X,Y) = \mathrm{Tr}(XY)/2$.

The basis

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is orthonormal for K, and this shows that this symmetric bilinear form is nondegenerate and has signature $(2,1)$. Hence, this Lie algebra inherits the geometry of Minkowski space. The classification of vectors reflects the classification of isometries of the hyperbolic plane. The timelike vectors correspond to infinitesimal elliptic isometries, the lightlike vectors to infinitesimal parabolic isometries, and the spacelike vectors to infinitesimal hyperbolic isometries.

The group of orientation preserving, time preserving isometries of Min in this model is $\mathsf{PSL}(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})$, acting on $\mathfrak{sl}(2,\mathbb{R})$ in the following way:

$$(A,Y) \cdot X := \mathrm{Ad}(A)X + Y.$$

2.1.2 Anti de Sitter space

Anti de Sitter space is the negatively curved model. It is analogous to hyperbolic space.

Let $V^{2,2}$ be a vector space of dimension 4 over $\mathbb{R}$ and $\cdot$ a symmetric bilinear form of signature $(2,2)$ on $V^{2,2}$.

Definition 2.1.3. *Anti de Sitter space* is the following submanifold of $V^{2,2}$:

$$\mathsf{AdS} := \{v \in V^{2,2} \mid v \cdot v = -1\}.$$

The ambient pseudo-Riemannian metric of signature $(2,2)$ restricts to a metric of signature $(2,1)$ on the tangent spaces of this submanifold. The isometry group of AdS is $\mathsf{O}(V^{2,2})$.

In this space, the spacelike and lightlike geodesics are all infinite, but the timelike geodesics are all closed.

Once again, low dimensional "accidental" isomorphisms between Lie algebras give different models of this space. The vector space of all 2×2 matrices over $\mathbb{R}$ has a natural quadratic form of signature $(2,2)$: the determinant. If we fix the symmetric bilinear form associated to $-\det$ instead, we get that the submanifold AdS is just the group $\mathsf{SL}(2,\mathbb{R})$.

There is an equivariant isomorphism between the actions of the identity component of $\mathsf{O}(V^{2,2})$ on AdS and the identity component of $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$ on $\mathsf{SL}(2,\mathbb{R})$ by left and right multiplication. This is due to the isomorphism $\mathfrak{o}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$.

2.1.3 de Sitter space

The third and last model for constant curvature Lorentzian 3-manifolds is *de Sitter space*. It is positively curved, and analogous to the Riemannian 3-sphere. It is defined in a similar way to anti de Sitter space : Let $V^{3,1}$ be a vector space with a bilinear form $\cdot$ of signature $(3, 1)$.

Definition 2.1.4. *de Sitter* space is the submanifold

$$\mathsf{dS} := \{v \in V^{3,1} \mid v \cdot v = 1\}.$$

Note that if we had set the condition on the right to $v \cdot v = -1$, we would have obtained a Riemannian manifold : hyperbolic 3-space. This relationship induces a duality between points in dS and totally geodesic planes in $\mathbb{H}^3$ coming from orthogonality in $V^{3,1}$. Similarly, points in $\mathbb{H}^3$ are dual to totally geodesic spacelike planes in dS.

2.2 The Einstein universe

In this section we will describe the model geometry for conformally flat Lorentzian 3-manifolds.

First, let's recall some facts about the conformal Riemannian sphere. The three constant curvature Riemannian manifolds admit conformal embeddings into the sphere. The flat Euclidean space embeds as the complement of a point using stereographic projection. The negatively curved hyperbolic space embeds as a hemisphere (or, alternatively, a disjoint union of two hyperbolic spaces embeds as the complement the equatorial sphere). Finally, the positively curved sphere embeds as the whole conformal sphere. We will see that each of these embeddings has an analog in the Lorentzian setting.

2.2.1 The projective model

The first model of the 3-dimensional Einstein universe that we will discuss is the projective model. It sits as a submanifold of projective 4-space.

Let $V^{3,2}$ be a 5-dimensional real vector space endowed with a signature $(3, 2)$ symmetric bilinear form.

Definition 2.2.1. The *Einstein universe* is the submanifold of $\mathbb{P}(V^{3,2})$ defined by

$$\mathsf{Ein} := \{[v] \in \mathbb{P}(V^{3,2}) \mid v \cdot v = 0\}$$

In order to make the conformal metric more explicit, we can choose a (positive definite) scalar product $\langle , \rangle$ on $V^{3,2}$ and look at the double cover

$$\widetilde{\mathsf{Ein}} := \{v \in V^{3,2} \mid v \cdot v = 0 \text{ and } \langle v, v \rangle = 1\}$$

Each point of Ein has exactly two representatives in this submanifold of $V^{3,2}$. Moreover, the ambient metric restricts to a Lorentzian metric on $\widetilde{\mathsf{Ein}}$ invariant under the antipodal map. Finally, choosing any other local section of the projection $\mathbb{P}$ will change this metric by a conformal map.

The group of conformal automorphisms of Ein is the orthogonal group $\mathsf{O}(V^{3,2})$.

In order to conformally embed Minkowski space into the Einstein universe, first choose any $(2, 1)$ subspace $V^{2,1} \subset V^{3,2}$. Now, $(V^{2,1})^{\perp}$ has signature $(1, 1)$, which means that there are exactly two projective equivalences of null vectors in it. Choose representatives p, q for these two null directions, normalized so that $p \cdot q = 1$.

The map $E : V^{2,1} \to V^{3,2}$ defined by $v \mapsto q - \frac{v \cdot v}{2} p + v$ embeds $V^{2,1}$ into the null cone of $V^{3,2}$.

Similarly, in order to conformally embed anti de Sitter (respectively de Sitter) space into the Einstein universe, fix a signature $(2, 2)$ (respectively $(3, 1)$) subspace U of $V^{3,2}$. Choose one of the two vectors $v \in V^{3,2}$ orthogonal to U such that $v \cdot v = 1$ (respectively $v \cdot v = -1$). Define $E : U \to V^{3,2}$ by $E(u) = u + v$. When restricted to $\mathsf{AdS} \subset U$ (respectively $\mathsf{dS} \subset U$) the image of E is in the lightcone of $V^{3,2}$.

The easiest way to show that these maps are conformal embeddings is to see that they are equivariant with respect to the transitive isometry groups of each of the spaces in question.

2.2.2 Geometric objects

The natural objects to study in Euclidean geometry are straight lines, circles and triangles. In this section, we describe classes of curves and surfaces in the Einstein universe that are natural to study.

The first of these objects is the *photon*, or lightlike geodesic. This is the only type of pseudo-Riemannian geodesic which is invariant under all conformal transformations.

Definition 2.2.2. A linear subspace is called isotropic if it is contained in its orthogonal subspace. A *photon* in Ein is the projectivization of an isotropic 2-plane in $V^{3,2}$.

The conformal group $\mathsf{O}(V^{3,2})$ acts transitively on photons. We will denote the homogeneous space of photons by Pho.

Photons together with points define an incidence relation. We say that two points $p, q \in \mathsf{Ein}$ are *incident* whenever there is a photon $\varphi \in \mathsf{Pho}$ containing both : $p, q \in \varphi$. In this projective model, this is equivalent to the two isotropic lines defining p and q being orthogonal with respect to $\cdot$. Similarly, two photons are said to be incident whenever they intersect in a point. Finally, a point is incident to a photon if it is contained in that photon.

Lemma 2.2.3. *Let $p \in \mathsf{Ein}$ and $\varphi \in \mathsf{Pho}$ with p not incident to φ. Then, there is a unique photon ψ incident to both p and φ, and a unique point q also incident to both p and φ.*

Proof. Assume $p = [u]$ and $\varphi = \langle v, w\rangle$. We first show the incidence of q. Define $q = [(u \cdot v)w - (u \cdot w)v] \in \varphi$. Now,

$$((u \cdot v)w - (u \cdot w)v) \cdot u = 0,$$

so q is incident to p. If there existed another q' with these properties, we would have that p, q, q' are pairwise incident and distinct, which would imply that there is an isotropic 3-dimensional subspace of $V^{3,2}$, a contradiction.

For the rest of the lemma, notice that $\psi = \langle p, q\rangle$ is incident to both p and φ, and uniqueness of q proves uniqueness of ψ. $\square$

Definition 2.2.4. The *lightcone* of a point $[v] \in \mathsf{Ein}$ is the set of all points incident to $[v]$. Equivalently, it is the union of all photons containing $[v]$.

$$\mathcal{L}([v]) = \{[u] \in \mathsf{Ein} | u \cdot v = 0\}$$

Different types of subspaces in $V^{3,2}$ define other natural curves and surfaces.

Definition 2.2.5. A *timelike* (respectively *spacelike*) *circle* is the projectivization of the nullcone of a signature $(1, 2)$ (respectively $(2, 1)$) subspace in $V^{3,2}$.

Proposition 2.2.6. *Let $p, q \in \mathsf{Ein}$ be a pair of non-incident points. Then, $\mathcal{L}(p) \cap \mathcal{L}(q)$ is a spacelike circle. Conversely, for each spacelike circle S, there is a unique pair of points p, q with $\mathcal{L}(p) \cap \mathcal{L}(q) = S$.*

Proof. Since p, q are non-incident, they span a $(1, 1)$ subspace. The intersection of their lightcones is the projectivized lightcone of the orthogonal to that subspace, which has signature $(2, 1)$.

For the converse, consider the orthogonal complement of the $(2, 1)$ subspace defining the spacelike circle. It has signature $(1, 1)$ and so contains exactly two lightlike directions, corresponding to the points p, q. □

Proposition 2.2.7. *Three pairwise non-incident points $p, q, r \in \mathsf{Ein}$ define a unique timelike circle or spacelike circle going through them.*

Proof. By the non-incidence property, the span of p, q, r is a nondegenerate three-dimensional subspace. It cannot be positive definite since it contains null lines p, q, r. Hence, it has to be either a signature $(2, 1)$ or a signature $(1, 2)$ subspace. □

The previous proposition allows us to define a relation on triples of pairwise non-incident points in the Einstein universe.

Definition 2.2.8. We call a triple of pairwise non-incident points p, q, r in the Einstein universe a :

- timelike triple if there is a timelike circle through them, or a
- spacelike triple if there is a spacelike circle through them

Definition 2.2.9. An *Einstein torus* is the projectivization of the nullcone of a signature $(2, 2)$ subspace in $V^{3,2}$. It is an embedded copy of the 2-dimensional Einstein universe.

Definition 2.2.10. A *Riemann sphere* is the projectivization of the nullcone of a signature $(3, 1)$ subspace in $V^{3,2}$. It is an embedded copy of the conformal 2-sphere.

2.2.3 The Lagrangian Grassmannian model

In Sections 2.1.1, 2.1.2 and 2.1.3, we discussed alternate models for each of the constant curvature models using low dimensional Lie group isomorphisms. Conveniently, there is also a low dimensional isomorphism giving an alternate model for the Einstein universe. It is the isomorphism of Lie groups $\mathsf{SO}^0(3, 2) \cong \mathsf{PSp}(4, \mathbb{R})$.

Let V be a 4-dimensional vector space over $\mathbb{R}$. Equip V with a nondegenerate, skew-symmetric bilinear form ω, making it into a *symplectic vector space*. The Lie group preserving this structure is $\mathsf{Sp}(4, \mathbb{R})$. A 2-dimensional subspace of V is called *Lagrangian* if the restriction of ω to the subspace is identically zero. The Grassmannian of 2-planes in V is the space of 2-dimensional subspaces of V. The subspace consisting of only the Lagrangian subspaces is called the *Lagrangian Grassmannian*. This is the model of the Einstein universe that we will describe in this section.

In this model, points correspond to Lagrangian planes and photons correspond to lines in V. A point is incident to a photon when the line corresponding to the photon is contained in the Lagrangian plane corresponding to the point. Two points are incident when the corresponding Lagrangians intersect, and finally two photons are incident when the corresponding lines span a Lagrangian (see Section 3.2 and [BCD+08] for details).

Minkowski patches in this model are most easily described in terms of the $\mathfrak{sl}(2,\mathbb{R})$ model of Minkowski space. Let $L, L' \subset V$ be a pair of transverse 2-dimensional subspaces of V.

Definition 2.2.11. Let $f : L \to L'$ be a linear map. The *graph* of f is the linear subspace

$$\mathsf{graph}(f) := \{v + f(v) \mid v \in L\}$$

Next, assume L, L' are transverse Lagrangians, so they correspond to a pair of non-incident points in the Einstein universe.

Lemma 2.2.12. *The graph of $f : L \to L'$ is a Lagrangian subspace if and only if $\omega(u, f(v)) + \omega(f(u), v) = 0$ for all $u, v \in L$.*

Proof. Let $u, v \in L$. Then,

$$\omega(u + f(u), v + f(v)) = \omega(u, f(v)) + \omega(f(u), v).$$

□

Let $\sigma \in \mathsf{Sp}(V, \omega)$ be an involution such that $\sigma(L) = L'$. This defines a symplectic form on L as follows: $\omega^\sigma(u, v) := \omega(u, \sigma(v))$. Then, the condition of Lemma

2.2.12 is exactly the condition that $\sigma \circ f \in \mathfrak{sp}(L, \omega^\sigma)$. Therefore, we get a map

$$\mathfrak{sp}(L, \omega^\sigma) \to \mathsf{Ein}$$

$$\sigma \circ f \mapsto \mathsf{graph}(f).$$

Since $\mathfrak{sp}(L, \omega^\sigma) \cong \mathfrak{sl}(2, \mathbb{R})$, Minkowski space embeds into this model of the Einstein universe. Explicitly, let us use the standard symplectic form given by the block matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and the involution

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathsf{Sp}(4, \mathbb{R})$$

which interchanges the Lagrangians L and L' spanned respectively by the first two basis vectors and the last two basis vectors. Then, we find that ω^σ is given by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, with these choices the Lie algebra $\mathfrak{sp}(L, \omega^\sigma)$ consist of the standard traceless 2×2 matrices, and the map giving the embedding is

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -c & a \\ a & b \end{pmatrix}$$

where the 4×2 matrix on the right represents the Lagrangian spanned by its columns. We see that with these conventions, the Minkowski patch coincides with graphs of linear maps given by a symmetric matrix. Moreover, the Lorentzian quadratic form is given by minus the determinant for both the traceless and the symmetric matrix.

Let us now describe the embedding of anti de Sitter space using the $\mathsf{SL}(2,\mathbb{R})$ model in an analogous way.

Let S be a non-Lagrangian plane in V. Then, $S^\perp$ is also non-Lagrangian and $V = S \oplus S^\perp$.

Proposition 2.2.13. *The graph of a linear map $f : S \to S^\perp$ is Lagrangian if and only if $\omega(f(u), f(v)) = -\omega(u, v)$ for all $u, v \in S$.*

Proof.

$$\omega(u + f(u), v + f(v)) = \omega(u, v) + \omega(f(u), f(v))$$

□

Let i be an antisymplectic isomorphism between S and $S^\perp$. This means that $\omega(i(u), i(v)) = -\omega(u, v)$ for all $u, v \in S$. Then, the condition above translates to the condition that $i^{-1} \circ f$ preserves ω on S, that is, $i^{-1} \circ f \in \mathsf{Sp}(S, \omega)$. This defines a map

$$\begin{aligned} \mathsf{Sp}(S,\omega) &\to \mathsf{Ein} \\ i^{-1} \circ f &\mapsto \mathsf{graph}(f). \end{aligned}$$

Since $\mathsf{Sp}(S,\omega) \cong \mathsf{SL}(2,\mathbb{R})$, this defines an embedding of anti de Sitter space into this model of the Einstein universe.

In order to describe the embedding of de Sitter space into the Einstein universe, we will do something slightly different. We will use the fact that, as a homogeneous space, $\mathsf{dS} \cong \mathsf{SL}(2,\mathbb{C})/\mathsf{SL}(2,\mathbb{R})$.

Let $\omega_{\mathbb{C}}$ be a complex-valued, skew-symmetric nondegenerate bilinear form on $\mathbb{C}^2$. Then, its imaginary part defines a real valued symplectic form on $\mathbb{C}^2$. The group $\mathsf{SL}(2,\mathbb{C}) \cong \mathsf{Sp}(2,\mathbb{C})$ of symplectic linear transformations in particular preserves the imaginary part, so we get an injective homomorphism $\mathsf{SL}(2,\mathbb{C}) \to \mathsf{Sp}(4,\mathbb{R})$. The stabilizer of a real 2-plane in $\mathbb{C}$ which is *not* a complex line is isomorphic to $\mathsf{SL}(2,\mathbb{R})$, realizing dS as a subspace of the real Lagrangian Grassmannian.

We now describe some of the geometric objects from Section 2.2.2 in this model. For this purpose, we need to define the *Maslov index* of a triple of Lagrangians.

Let P, Q, R be three pairwise transverse Lagrangians in V. Denote by π_P, π_R the projections associted to the splitting $V = P \oplus R$.

Definition 2.2.14. The *Maslov index* of the triple P, Q, R, denoted $m(P,Q,R)$ is the signature of the following nondegenerate quadratic form on Q :

$$B_{P,R}(u) = \omega(\pi_P(u), \pi_R(u)).$$

Here, signature means the difference between the number of positive and negative eigenvalues of $B_{P,R}$.

Proposition 2.2.15. *The Maslov index enjoys the following properties for all P, Q, R, S*

pairwise transverse Lagrangians

- $m(P, Q, R) \in \{-2, 0, 2\}$
- $m(P, Q, R) = m(AP, AQ, AR)$ *for any* $A \in \mathsf{Sp}(4, \mathbb{R})$
- $m(P, Q, R) = m(Q, R, P)$
- $m(P, Q, R) - m(Q, R, S) + m(P, R, S) - m(Q, R, S) = 0.$

The invariant $m(P, Q, R)$ distinguishes triples on pairwise non-incident points according to their type.

Proposition 2.2.16. *Let P, Q, R be a triple of pairwise transverse Lagrangians. They form a timelike triple when $m(P, Q, R) = \pm 2$ and a spacelike triple when $m(P, Q, R) = 0$ (see Definition 2.2.8).*

Proof. Let e_1, e_2, e_3, e_4 be a symplectic basis of V. Using the action of $\mathsf{Sp}(4, \mathbb{R})$, we can assume that P is spanned by e_1, e_2 and R is spanned by e_3, e_4. By transversality, we can write Q as the graph of a unique linear map $f : P \to R$. In the bases e_1, e_2 and e_3, e_4, f will be represented by a symmetric matrix F. The quadratic form $B_{P,R}$ is also represented by F, and the Lorentzian quadratic form in this model of Minkowski space is minus the determinant. Thus, when $\det(F) < 0$, the triple is spacelike, the quadratic form is indefinite and so $m(P, Q, R) = 0$. Similarly, when $\det(F) > 0$ the triple is timelike and the quadratic form is definite so $m(P, Q, R) = \pm 2$. □

Proposition 2.2.17. *Let L, L' be transverse Lagrangians. The set of all Lagrangians intersecting both L and L' is a spacelike circle. Conversely, all spacelike circles are of this form.*

Proof. By Proposition 2.2.6, intersections of two lightcones and spacelike circles are equivalent. In the Lagrangian model, the lightcone of a Lagrangian is the set of a Lagrangians intersecting it, so the result is immediate. □

Proposition 2.2.18. *Let S be a nondegenerate 2-dimensional subspace of V (that is, the symplectic form ω does not vanish on S). The set of all Lagrangians intersecting S is an Einstein torus. The subspace S is uniquely determined by the spacelike circle, up to replacing S by its symplectic orthogonal complement $S^\perp$.*

We postpone the proof of this last statement to Chapter 3 where we investigate Einstein tori in detail (more precisely Section 3.2.1 for this proposition).

2.2.4 The Lie circles model

We show in this section that the Einstein universe is the moduli space of oriented circles in the 2-sphere. Cecil [Cec08] explains this through the usual projective model of the Einstein universe. However, as far as the author knows, there is no exposition of this using the Lagrangian Grassmannian model. We will develop the theory from this point of view.

Let $V = \mathbb{C}^2$ considered as a 4-dimensional real vector space. The determinant provides a *complex* symplectic form on $\mathbb{C}^2$, and both its real and imaginary parts are real symplectic forms. We will use the form $\omega := \Im(\det)$. A Lagrangian for this form is a *real* 2-dimensional subspace of $\mathbb{C}^2$ which is spanned by two vectors with a real determinant.

Proposition 2.2.19. *The projectivization of a real 2-plane $P \subset \mathbb{C}^2$ is a circle or a* *point in* $\mathbb{CP}^1$.

Proof. Let u, v be vectors whose $\mathbb{R}$-span is P. Denoting the projection to $\mathbb{CP}^1$ by π,

$$\pi(P) = \{\pi(ku + lv) \mid k, l \in \mathbb{R}\} = \left\{ \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \;\middle|\; k, l \in \mathbb{R} \right\}.$$

If the matrix with columns u, v is singular over $\mathbb{C}$, then the image is a single point. Otherwise, it is the image of a circle (the extended real line) by a möbius transformation, so is a circle. □

Fix a circle $C \subset \mathbb{CP}^1$ and a pair of distinct points $[p], [q] \in C$. Then, any pair of representatives for $[p], [q]$ will define a real plane in $\mathbb{C}^2$ which projects to C. Since *real* changes of basis do not affect the plane, the collection of planes which project to C is the set $\{e^{i\theta}(\mathbb{R}p + \mathbb{R}q), \theta \in [0, \pi)\}$. The *Lagrangian* planes in this set must have $\Im(\det(e^{i\theta}p, e^{i\theta}q)) = 0$. This equation has exactly two solutions which are interchanged by multiplication by i. We conclude:

Proposition 2.2.20. *The collection of real Lagrangian* 2*-planes in* $\mathbb{C}^2$ *which are not complex lines projects* 2*-to-*1 *to the collection of circles in* $\mathbb{CP}^1$.

Since multiplication by i preserves complex lines and switches the two real planes projecting to any circle, we can interpret this as a change of orientation on the set of circles. This means that Lagrangians in $\mathbb{C}^2$ correspond to oriented circles and points in $\mathbb{CP}^1$.

Proposition 2.2.21. *Assume* $L, L' \subset \mathbb{C}^2$ *are a pair of real Lagrangian* 2*-planes which are not complex lines. Assume moreover that* L *and* L' *intersect in a line* ℓ. *Then,* $\pi(L)$ *and* $\pi(L')$ *correspond to tangent circles in the plane.*

Proof. Let $u = (u_1, u_2) \in \ell$ and assume without loss of generality that $u_2 \neq 0$. Complete u to a basis of L and L' with vectors v, v', respectively. Then, in a neighborhood of $\pi(u)$, the projections can be parameterized by

$$\gamma(t) = \frac{u_1 + v_1 t}{u_2 + v_2 t}$$

and

$$\gamma'(t) = \frac{u_1 + v_1' t}{u_2 + v_2' t}.$$

The tangent vectors to these paths at $\pi(u) = \gamma(0) = \gamma'(0)$ are given respectively by

$$\dot{\gamma}(0) = \frac{u_1 v_2 - u_2 v_1}{u_2^2}$$

$$\dot{\gamma}'(0) = \frac{u_1 v_2' - u_2 v_1'}{u_2^2}.$$

To show that the circles are tangent, it remains to show that these vectors are real multiples of each other. The quotient of the two values is

$$\frac{\gamma}{\gamma'} = \frac{u_1 v_2 - u_2 v_1}{u_1 v_2' - u_2 v_1'},$$

which has a real numerator and a real denominator since we assumed that L and L' were Lagrangian. $\square$

More precisely, L and L' intersect in a line exactly when the oriented circles they correspond to are tangent with matching orientations at the tangency point.

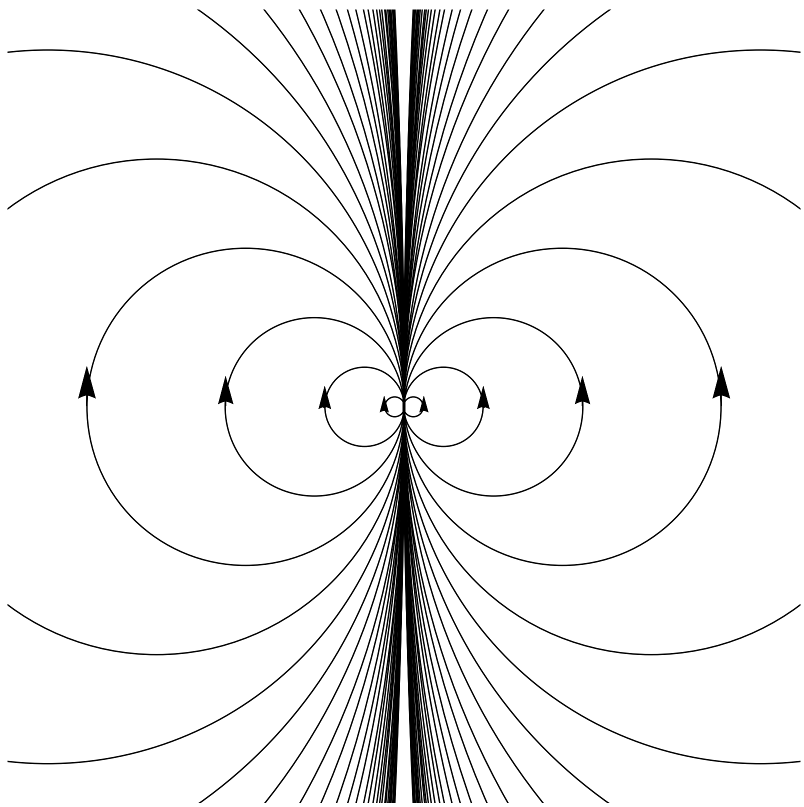


Figure 2.1: A photon in the circles model of the Einstein universe.

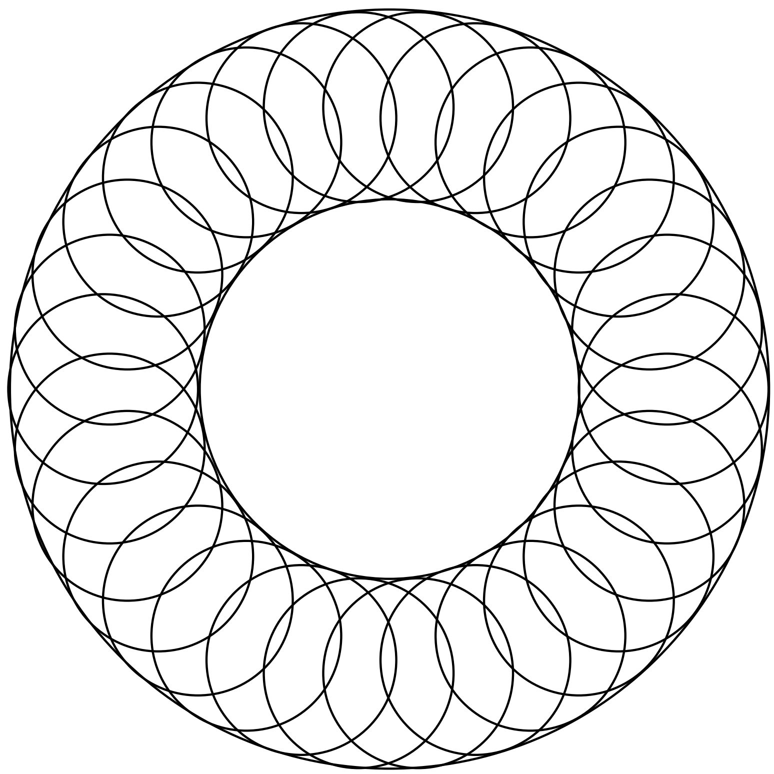


Figure 2.2: A spacelike circle in the circles model. The orientations on the outer and inner circle are opposite, and the orientations of all the other circles match that of the outer circle.

Chapter 3: Crooked Schottky groups

The content of this chapter is essentially from the preprint [BCFG17]. We want to use a class of hypersurfaces in the Einstein universe called *crooked surfaces* in order to build fundamental domains for Schottky groups. Since these surfaces are defined piecewise in a non-trivial way, the difficulty lies in finding a configuration of such surfaces which are disjoint. In order to find a disjointness criterion, we first focus on Einstein tori (Definition 2.2.9) and describe their intersections.

3.1 Einstein tori

The purpose of this section is to define an invariant $\eta \geq 0$ characterizing pairs of Einstein tori in Ein. Then, we interpret this invariant in the Lagrangian Grassmannian model. Let $V^{3,2}$ be a real vector space of dimension 5 endowed with a signature $(3,2)$ symmetric bilinear form $\cdot$.

3.1.1 Pairs of positive vectors

If $s \in V^{3,2}$ is spacelike, then $s^{\perp}$ is a subspace of signature $(2,2)$, which means that its projectivized lightcone is an Einstein torus.

A linearly independent pair of two unit-spacelike vectors s_1, s_2 spans a 2-plane

$\langle s_1, s_2 \rangle \subset V^{3,2}$ which is:

- Positive definite $\iff |s_1 \cdot s_2| < 1$;
- Degenerate $\iff |s_1 \cdot s_2| = 1$;
- Indefinite $\iff |s_1 \cdot s_2| > 1$.

The positive definite and indefinite cases respectively determine orthogonal splittings

$$V^{3,2} \cong \mathbb{R}^{3,2} = \mathbb{R}^{2,0} \oplus \mathbb{R}^{1,2}$$

$$V^{3,2} \cong \mathbb{R}^{3,2} = \mathbb{R}^{1,1} \oplus \mathbb{R}^{2,1}.$$

In the degenerate case, the null space is spanned by $s_1 \pm s_2$, where

$$s_1 \cdot s_2 = \mp 1.$$

By replacing s_2 by $-s_2$ if necessary, we may assume that $s_1 \cdot s_2 = 1$. Then $s_1 - s_2$ is null. Since $V^{3,2}$ itself is nondegenerate, there exists $v_3 \in V^{3,2}$ such that

$$(s_1 - s_2) \cdot v_3 = 1.$$

Then s_1, s_2, v_3 span a nondegenerate 3-dimensional subspace of signature $(2, 1)$.

Let H_1, H_2 be the Einstein tori respectively defined by the orthogonal subspaces $s_1^\perp, s_2^\perp$. The absolute value of the product

$$\eta(H_1, H_2) := |s_1 \cdot s_2|$$

is a nonnegative real number, depending only on the pair of Einstein hyperplanes H_1 and H_2. We have thus proved the following classification for pairs of Einstein tori:

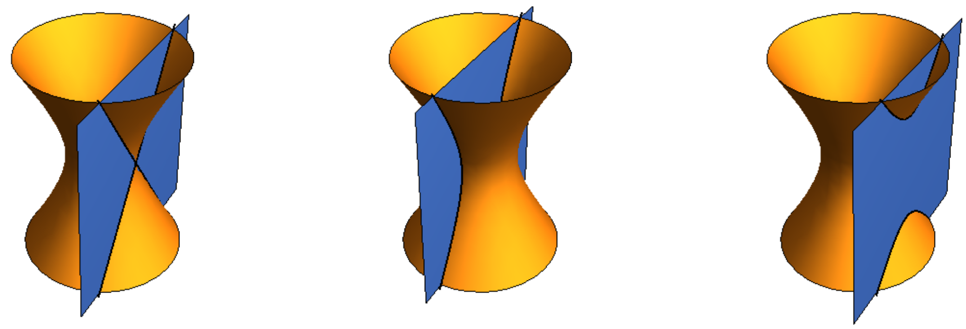
- If the span of s_1, s_2 is positive definite ($\eta(H_1, H_2) < 1$), then the intersection of the corresponding Einstein tori is the projectivised null cone of a signature $(1, 2)$ subspace, which is a timelike circle.
- If the span of s_1, s_2 is indefinite ($\eta(H_1, H_2) > 1$), then the intersection is the projectivised null cone of a signature $(2, 1)$ subspace, which is a spacelike circle.
- Finally, if the span of s_1, s_2 is degenerate ($\eta(H_1, H_2) = 1$), then the intersection is the projectivised null cone of a degenerate subspace with signature $(+, -, 0)$. This null cone is exactly the union of two isotropic planes intersecting in the degenerate direction, so when projectivising we get a pair of photons intersecting in a point.

Corollary 3.1.1. *The intersection of two Einstein tori is noncontractible in each of the two tori.*

Proof. An Einstein torus is a copy of the 2-dimensional Einstein universe. Explicitly, we can write it as $\mathbb{P}(\mathcal{N})$ where $\mathcal{N}$ is the null cone in $\mathbb{R}^{2,2}$. A computation shows that all timelike circles are homotopic, all spacelike circles are homotopic and these two homotopy classes together generate the fundamental group of the torus. Similarly, photons are homotopic to the sum of these generators and so are noncontractible. □

3.1.2 Involutions in Einstein tori

Orthogonal reflection in s defines an involution of Ein which fixes the corresponding hyperplane $H = s^{\perp}$. The orthogonal reflection in a positive vector s is



(a) Two photons (b) A timelike circle (c) A spacelike circle

Figure 3.1: The three possible types of intersection for a pair of Einstein tori, viewed in a Minkowski patch.

defined by:

$$R_s(v) = v - 2\frac{v \cdot s}{s \cdot s}s.$$

We compute the eigenvalues of the composition $R_s R_{s'}$, where s, s' are unit spacelike vectors, and relate this to the invariant η.

The orthogonal subspace to the plane spanned by s and s' is fixed pointwise by this composition. Therefore, 1 is an eigenvalue of multiplicity 3. In order to determine the remaining eigenvalues, we compute the restriction of $R_s R_{s'}$ to the subspace $\mathbb{R}s + \mathbb{R}s'$.

$$\begin{aligned} R_s R_{s'}(s) &= R_s(s - 2(s\cdot s')s') \\ &= -s - 2(s\cdot s')(s' - 2(s'\cdot s)s) \\ &= (4(s'\cdot s)^2 - 1)s - 2(s'\cdot s)s'. \end{aligned}$$

$$\begin{aligned} R_s R_{s'}(s') &= R_s(-s') \\ &= -s' + 2(s\cdot s')s. \end{aligned}$$

The matrix representation of $R_s R_{s'}$ in the basis s, s' is therefore:

$$\begin{pmatrix} 4(s'\cdot s)^2 - 1 & 2(s\cdot s') \\ -2(s'\cdot s) & -1 \end{pmatrix}.$$

The eigenvalues of this matrix are:

$$2(s\cdot s')^2 - 1 \pm 2(s\cdot s')\sqrt{(s\cdot s')^2 - 1}.$$

We observe that they only depend on the invariant $\eta = |s\cdot s'|$. The composition of involutions has real distinct eigenvalues when the intersection is spacelike, complex eigenvalues when the intersection is timelike, and a double real eigenvalue when the intersection is a pair of photons.

The case when $s_1 \cdot s_2 = 0$ is special: in that case the two involutions commute and we will say that the Einstein tori are *orthogonal*. The complement of an Einstein torus in Ein is a model for the double covering space of anti de Sitter space AdS^3. In this conformal model of AdS^3 (see [Gol15]), indefinite totally geodesic 2-planes are represented by tori which are orthogonal to the Einstein torus $\partial\mathsf{AdS}^3$.

3.2 The Lagrangian Grassmannian model

We first recall the Lagrangian model of the Einstein universe described in Section 2.2.3. Then, we make the identification between this model and the projective model more explicit. We describe Einstein tori in this context and the invariant η for a pair of tori.

Let (V, ω) be a 4-dimensional *real symplectic vector space,* that is, V is a real vector space of dimension 4 and $V \times V \xrightarrow{\omega} \mathbb{R}$ is a nondegenerate skew-symmetric bilinear form. Let $\mathsf{vol} \in \Lambda^4(V)$ be the element defined by the equation $(\omega \wedge \omega)(\mathsf{vol}) = -2$. The second exterior power $\Lambda^2(V)$ admits a nondegenerate symmetric bilinear form $\cdot$ of signature $(3, 3)$ defined by

$$(u \wedge v) \wedge (u' \wedge v') = (u \wedge v) \cdot (u' \wedge v')\mathsf{vol}.$$

The kernel

$$W := \mathsf{Ker}(\omega) \subset \Lambda^2(V)$$

inherits a symmetric bilinear form which has signature $(3, 2)$.

Define the vector $\omega^* \in \Lambda^2 V$ to be *dual* to ω by the equation

$$\omega^* \cdot (u \wedge v) = \omega(u, v),$$

for all $u, v \in V$. Because of our previous choice of vol, we have $\omega^* \cdot \omega^* = -2$. The bilinear form $\cdot$, together with the vector ω^* define a *reflection*

$$\mathsf{R}_{\omega^*} : \Lambda^2(V) \to \Lambda^2(V)$$

$$u \mapsto u + (u \cdot \omega^*)\omega^*.$$

The fixed set of this reflection is exactly the vector subspace W orthogonal to ω^*.

The *Plücker embedding* $\iota : \mathsf{Gr}(2, V) \to \mathbb{P}(\Lambda^2(V))$ maps 2-planes in V to lines in $\Lambda^2(V)$. We say that a plane in V is *Lagrangian* if the form ω vanishes identically on pairs of vectors in that plane. If we restrict ι to Lagrangian planes, then the image is exactly the set of null lines in W.

The form ω yields a relation of orthogonality on 2-planes in V. Lagrangian planes are orthogonal to themselves, and non-Lagrangian planes have a unique orthogonal complement which is also non-Lagrangian. The following proposition relates orthogonality in V with a reflection operation on $\Lambda^2(V)$.

Proposition 3.2.1. *A pair of* 2*-dimensional subspaces* $S, T \subset V$ *are orthogonal with respect to* ω *if and only if* $[\mathsf{R}_{\omega^*}(\iota(S))] = [\iota(T)]$*.*

Proof. First, assume S is Lagrangian. This means that $S = S^\perp$, and that $\iota(S) \in \omega^{*\perp}$. Hence,

$$\mathsf{R}_{\omega^*}(\iota(S)) = \iota(S) = \iota(S^\perp).$$

Next, if S is not Lagrangian, then we can find bases (u, v) of S and (u', v') of $S^\perp$ satisfying $\omega(u, v) = \omega(u', v') = 1$ and all other products between these four are zero. Then,

$$\mathsf{vol} = -u \wedge v \wedge u' \wedge v'$$

and

$$\omega^* = -u \wedge v - u' \wedge v'.$$

Consequently,

$$[\mathsf{R}_{\omega^*}(\iota(S))] = [u \wedge v + \omega(u, v)\omega^*] = [-u' \wedge v'] = [\iota(S^\perp)].$$

□

Recall that a point and a photon are called incident if the point is in the photon (Section 2.2.2). This incidence relation is reflected in the two models in the following way: A point $p \in \mathsf{Ein}$ and a photon $\phi \in \mathsf{Pho}$ are incident if and only if (p, ϕ) satisfies one of the two equivalent conditions:

- The null line in W corresponding to p lies in the isotropic 2-plane in W corresponding to ϕ.
- The Lagrangian 2-plane in V corresponding to p contains the line in W corresponding to ϕ.

The following proposition proves this equivalence :

Proposition 3.2.2. *Let $P, Q \subset V$ be two-dimensional subspaces. Then, $P \cap Q = 0$ if and only if $\iota(P) \cdot \iota(Q) \neq 0$.*

Proof. Choose bases u, v of P and u', v' of Q. Then,

$$u \wedge v \wedge u' \wedge v' \neq 0$$

if and only if u, v, u', v' span V which is equivalent to P and Q being transverse. □

The light cone of p corresponds the orthogonal hyperplane $[p]^{\perp} \subset W$ of the null line corresponding to p. In photon space $\mathbb{P}(V)$, the photons containing p form the projective space $\mathbb{P}(L)$ of the Lagrangian 2-plane L corresponding to p.

3.2.1 Nondegenerate planes and symplectic splittings

We describe the algebraic structures equivalent to an *Einstein torus* in Ein. As a reminder, these are hyperplanes of signature $(2, 2)$ inside $W \cong \mathbb{R}^{3,2}$, and describe surfaces in Ein homeomorphic to a 2-torus.

In symplectic terms, an Einstein torus corresponds to a splitting of V as a symplectic direct sum of two nondegenerate 2-planes. Let us detail this correspondence.

Define a 2-dimensional subspace $S \subset V$ to be *nondegenerate* if and only if the restriction $\omega|_S$ is nondegenerate. A nondegenerate 2-plane $S \subset V$ determines a splitting as follows. The plane

$$S^\perp := \{v \in V \mid \omega(v, S) = 0\}$$

is also nondegenerate, and defines a *symplectic complement* to S. In other words, V splits as an (internal) symplectic direct sum:

$$V = S \oplus S^\perp.$$

The corresponding Einstein torus is then the set of Lagrangians which are non-transverse to S (and therefore also to $S^\perp$).

The lines in S determine a projective line in Pho which is *not* Legendrian. Conversely, non-Legendrian projective lines in Pho correspond to nondegenerate 2-planes. This non-Legendrian line in Pho, as a set of photons, corresponds to one of the two rulings of the Einstein torus. The other ruling corresponds to the line $\mathbb{P}(S^\perp)$.

In order to make explicit the relationship between the descriptions of Einstein tori in the two models, define a map μ as follows:

$$\begin{aligned} \mu : \mathsf{Gr}(2, V) &\to W \\ S &\mapsto \iota(S) + \frac{1}{2}\omega(\iota(S))\omega^*. \end{aligned}$$

This is the composition of the Plucker embedding ι with the orthogonal projection onto W.

Lemma 3.2.3. *For S a nondegenerate plane, the image of μ is always a spacelike vector, and $\mu(S) = \mu(S^\perp)$.*

Proof. For the first part,

$$\mu(S) \cdot \mu(S) = \frac{1}{2}\omega(\iota(S))^2 > 0.$$

The second part is a consequence of the correspondence between orthogonal complements and reflection in ω^* (Proposition 3.2.1) and the fact that a vector and its reflected copy have the same orthogonal projection to the hyperplane of reflection. □

Proposition 3.2.4. *The map μ induces a bijection between spacelike lines in W and symplectic splittings of V. Under the Plücker embedding ι, the Einstein torus defined by the symplectic splitting $S \oplus S^\perp$ is sent to the Einstein torus defined by the spacelike vector $\mu(S) \in W$.*

Proof. Let $u \in W$ be a spacelike vector normalized so that $u \cdot u = 2$. Then, both vectors $u \pm \omega^*$ are null. By the fact that null vectors in $\Lambda^2(V)$ are decomposable, each $u \pm \omega^*$ corresponds to a 2-plane in V. These 2-planes are nondegenerate since

$$(u \pm \omega^*) \wedge \omega^* = -\omega(u \pm \omega^*)\mathsf{vol} = 2 \neq 0.$$

The two planes $u \pm \omega^*$ are orthogonal since they are the images of each other by the reflection R_{ω^*}, and so they are the summands for a symplectic splitting of V.

The map associating to u the splitting $u \pm \omega^*$ is inverse to the projection μ defined above.

To prove the last statement in the proposition, we apply Proposition 3.2.2. The Einstein torus defined by the splitting $S, S^\perp$ is the set of Lagrangian planes which intersect S (and $S^\perp$) in a nonzero subspace. Let P be such a plane. Then, $\iota(S) \cdot \iota(P) = 0$, which means that

$$\left(\iota(S) + \frac{1}{2}(\iota(S) \cdot \omega^*)\omega^*\right) \cdot \iota(P) = 0,$$

so $\iota(P)$ is in the Einstein torus defined by the orthogonal projection of S. Similarly, if $\iota(P)$ is orthogonal to u_S then P intersects S in a nonzero subspace. □

3.2.2 Graphs of linear maps

Now we describe pairs Einstein tori in terms of symplectic splittings of (V, ω) more explicitly.

Let A, B be vector spaces of the same dimension and $A \oplus B$ their direct sum. If $A \xrightarrow{f} B$ is a linear map, then the *graph* of f is the linear subspace $\mathsf{graph}(f) \subset A \oplus B$ consisting of all $a \oplus f(a)$, where $a \in A$. Every linear subspace $L \subset A \oplus B$ which is transverse to $B = 0 \oplus B \subset A \oplus B$ and having the same dimension as A, equals

$\mathsf{graph}(f)$ for a unique f. Furthermore, $L = \mathsf{graph}(f)$ is transverse to $A = A \oplus 0$ if and only if f is invertible, in which case $L = \mathsf{graph}(f^{-1})$ for the inverse map $B \xrightarrow{f^{-1}} A$.

Suppose that A, B are vector spaces with nondegenerate alternating bilinear forms ω_A, ω_B, respectively. Let $A \xrightarrow{f} B$ be a linear map. Its *adjugate* is the linear map

$$B \xrightarrow{\mathsf{Adj}(f)} A$$

defined as the composition

$$B \xrightarrow{\omega_B^\#} B^* \xrightarrow{f^\dagger} A^* \xrightarrow{\omega_A^\#} A \tag{3.1}$$

where $\omega_A^\#, \omega_B^\#$ are isomorphisms induced by ω_A, ω_B respectively, and $f^\dagger$ is the transpose of f. If a_1, a_2 and b_1, b_2 are bases of A and B respectively with

$$\omega_A(a_1, a_2) = 1$$

$$\omega_B(b_1, b_2) = 1,$$

then the matrices representing f and $\mathsf{Adj}(f)$ in these bases are related by:

$$\mathsf{Adj}\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{bmatrix}.$$

In particular, if f is invertible and $\dim(A) = \dim(B) = 2$, then

$$\mathsf{Adj}(f) = \mathsf{Det}(f) f^{-1}$$

where $\mathsf{Det}(f)$ is defined by $f^*(\omega_B) = \mathsf{Det}(f)\omega_A$.

Lemma 3.2.5. *Let $V = S \oplus S^\perp$. Let $S \xrightarrow{f} S^\perp$ be a linear map and let $P = \mathsf{graph}(f) \subset V$ be the corresponding 2-plane in V which is transverse to $S^\perp$.*

- *P is nondegenerate if and only if* $\mathsf{Det}(f) \neq -1$.
- *If P is nondegenerate, then its complement $P^\perp$ is transverse to S, and equals the graph*

$$P^\perp = \mathsf{graph}\big(-\mathsf{Adj}(f)\big),$$

of the negative of the adjugate map *to f*

$$S^\perp \xrightarrow{-\mathsf{Adj}(f)} S.$$

Proof. Choose a basis a, b for S. Then $a \oplus f(a)$ and $b \oplus f(b)$ define a basis for P, and

$$\begin{aligned}\omega\big(a \oplus f(a), b \oplus f(b)\big) &= \omega\big(a, b\big) + \omega\big(f(a), f(b)\big)\\ &= \big(1 + \mathsf{Det}(f)\big)\omega\big(a, b\big),\end{aligned}$$

since, by definition,

$$\omega\big(f(a), f(b)\big) = \mathsf{Det}(f)\omega\big(a, b\big).$$

Thus P is nondegenerate if and only if $1 + \mathsf{Det}(f) \neq 0$, as desired.

For the second assertion, suppose that P is nondegenerate. Since $P, P^\perp, S, S^\perp \subset V$ are each 2-dimensional, the following conditions are equivalent:

- P is transverse to $S^\perp$;
- $P \cap S^\perp = 0$;
- $P^\perp + S = V$;
- $P^\perp$ is transverse to S.

Thus $P^\perp = \mathsf{graph}(g)$ for a linear map $S^\perp \xrightarrow{g} S$.

We express the condition that $\omega(P, P^{\perp}) = 0$ in terms of f and g: For $s \in S$ and $t \in S^{\perp}$, the symplectic product is zero if anly only if

$$\omega\big(s + f(s), t + g(t)\big) = \omega\big(s, g(t)\big) + \omega\big(f(s), t\big) \tag{3.2}$$

vanishes. This condition easily implies that $g = -\mathsf{Adj}(f)$ as claimed. $\square$

The following proposition relates the invariant η defined for a pair of spacelike vectors with the invariant Det associated to a pair of symplectic splittings.

Proposition 3.2.6. *Let $S \oplus S^{\perp}$ be a symplectic splitting and $f : S \to S^{\perp}$ be a linear map with $\mathsf{Det}(f) \neq -1$. Let $T = \mathsf{graph}(f)$ be the symplectic plane defined by f. Then,*

$$\eta(\mu(S), \mu(T)) = \frac{|\mu(S) \cdot \mu(T)|}{\sqrt{(\mu(S) \cdot \mu(S))(\mu(T) \cdot \mu(T))}} = \frac{|1 - \mathsf{Det}(f)|}{|1 + \mathsf{Det}(f)|}.$$

Proof. Let u, v be a basis for S such that $\omega(u, v) = 1$. Then, $u + f(u), v + f(v)$ is a basis for T. Moreover,

$$\iota(S) \cdot \iota(T)\mathsf{vol} = u \wedge v \wedge (u + f(u)) \wedge (v + f(v)) = u \wedge v \wedge f(u) \wedge f(v).$$

We can compute which multiple of vol this last expression represents by using the normalization $(\omega \wedge \omega)(\mathsf{vol}) = -2$ and the computation

$$(\omega \wedge \omega)(u \wedge v \wedge f(u) \wedge f(v)) = 2\mathsf{Det}(f).$$

We deduce that

$$\iota(S) \cdot \iota(T) = -\mathsf{Det}(f).$$

Now we compute $\mu(S) \cdot \mu(T)$:

$$\begin{aligned}\mu(S) \cdot \mu(T) &= \left(\iota(S) + \frac{1}{2}\omega(\iota(S))\omega^*\right) \cdot \left(\iota(T) + \frac{1}{2}\omega(\iota(T))\omega^*\right) \\ &= -\mathsf{Det}(f) + (1 + \mathsf{Det}(f)) - \frac{1}{2}(1 + \mathsf{Det}(f)) \\ &= 1/2(1 - \mathsf{Det}(f)).\end{aligned}$$

Finally, by the proof of Lemma 3.2.3, $\mu(S) \cdot \mu(S) = \frac{1}{2}$ and $\mu(T) \cdot \mu(T) = \frac{1}{2}(1 + \mathsf{Det}(f))^2$. Combining these computations finishes the proof of the statement. □

3.3 Disjoint crooked surfaces

In this section we apply the techniques developed above in order to prove a full disjointness criterion for pairs of crooked surfaces in Ein.

We work in the Lagrangian framework of Section 2.2.3 with the symplectic vector space (V, ω).

Let u_+, u_-, v_+, v_- be four vectors in V such that

$$\omega(u_+, v_-) = \omega(u_-, v_+) = 1$$

and all other products between these four vanish. This means that we have Lagrangians

$$P_0 := \mathbb{R}v_+ + \mathbb{R}v_-,$$

$$P_\infty := \mathbb{R}u_+ + \mathbb{R}u_-, and$$

$$P_\pm := \mathbb{R}v_\pm + \mathbb{R}u_\pm$$

representing the points of intersection of the photons associated to $[u_+], [u_-], [v_+], [v_-]$. We call this configuration of four points and four photons a *lightlike quadrilateral.*

The *crooked surface* C determined by this configuration is a subset of Ein consisting of three pieces : two *wings* and a *stem*. The two wings are foliated by photons, and we will denote by $\mathcal{W}_+, \mathcal{W}_-$ the sets of photons covering the wings. Each wing is a subset of the light cone of P_+ and P_-, respectively. Identifying points in $\mathbb{P}(V)$ with the photons they represent, the foliations are as follows:

$$\mathcal{W}_+ = \{[tu_+ + sv_+] \mid ts \geq 0\},$$

$$\mathcal{W}_- = \{[tu_- + sv_-] \mid ts \leq 0\}.$$

We will sometimes abuse notation and use the symbol $\mathcal{W}_\pm$ to denote the collection of points in the Einstein universe which is the union of these collections of photons.

The stem $\mathcal{S}$ is the subset of the Einstein torus determined by the splitting $S_1 \oplus S_2 := (\mathbb{R}u_+ + \mathbb{R}v_-) \oplus (\mathbb{R}u_- + \mathbb{R}v_+)$ consisting of timelike points with respect to P_0, P_∞ :

$$\mathcal{S} = \{\mathbb{R}w + \mathbb{R}w' \mid w \in S_1, w' \in S_2, |m(P_0, L, P_\infty)| = 2\}.$$

Note that this definition gives only the *interior* of the stem as defined in [CFLD14]. This crooked surface is the closure in Ein of a *crooked plane* in the Minkowski patch defined by the complement of the light cone of P_∞.

Theorem 3.3.1. *Let C_1, C_2 be two crooked surfaces with intersecting stems. Then, the stem of C_1 intersects a wing of C_2 or vice versa. That is, crooked surfaces cannot intersect in their stems only.*

Proof. The stem consists of two disjoint, contractible pieces. To see this, note that this set is contained in the Minkowski patch defined by P_∞. There, the Einstein torus containing the stem is a timelike plane through the origin, and the timelike points in this plane form two disjoint quadrants. Let K be the intersection of the two Einstein tori containing the stems of C_1 and C_2. Then, K is noncontractible in either tori (Corollary 3.1.1), so it can't be contained in the interior of the stem. Therefore, ℓ must intersect the boundary of the stem which is part of the wings. $\square$

Lemma 3.3.2. *Let $p_0, p_\infty, p \in \mathsf{Ein}$ be three points in the Einstein universe. The point p is timelike with respect to p_0, p_∞ if and only if the intersection of the three light cones of p, p_0, p_∞ is empty.*

Proof. We work in the model of Ein given by lightlike lines in a vector space of signature $(3, 2)$. If p is timelike with respect to p_0, p_∞, then it lies on a timelike curve which means that the subspace generated by p, p_0, p_∞ has signature $(1, 2)$. Therefore, its orthogonal complement is positive-definite and contains no lightlike vectors, so the intersection of the light cones is empty. The converse is similar. $\square$

Lemma 3.3.3. *A photon represented by a vector $p \in V$ is disjoint from the crooked surface C if and only if the following two inequalities are satisfied:*

$$\omega(p, v_+)\omega(p, u_+) > 0$$

$$\omega(p, v_-)\omega(p, u_-) < 0.$$

Proof. Write p in the basis u_+, u_-, v_+, v_- :

$$p = au_+ + bu_- + cv_+ + dv_-.$$

Then,

$$a = \omega(p, v_-) \quad b = \omega(p, v_+)$$

$$c = -\omega(p, u_-) \quad d = -\omega(p, u_+).$$

The photon p is disjoint from $\mathcal{W}_+$ if and only if the following equation has no solutions:

$$\omega(p, tu_+ + sv_+) = 0.$$

This happens exactly when $bd < 0$. Similarly, p is disjoint from $\mathcal{W}_-$ if and only if $ac > 0$. These two equations are equivalent to the ones in the statement of the Lemma, therefore it remains only to show that under these conditions, p is disjoint from the stem.

The Lagrangian plane P representing the intersection of p with the Einstein torus containing the stem is generated by p and $au_+ + dv_-$. We want to show that P cannot intersect the stem in a point which is timelike with respect to P_0, P_∞.

The intersection of the light cones of P_0 and P_∞ consists of planes of the form: $\mathbb{R}(su_+ + tu_-) + \mathbb{R}(s'v_+ + t'v_-)$ where $st' + ts' = 0$. We want to show that no point represented by such a plane is incident to P. Two Lagrangian planes are incident when their intersection is a non-zero subspace. Equivalently, they are incident if they do not span V. We have :

$$\begin{aligned} &\det(p, au_+ + dv_-, su_+ + tu_-, s'v_+ + t'v_-) \\ &= (-bdss' + catt') \det(u_+, u_-, v_+, v_-) \\ &= k(bds^2 + act^2) \det(u_+, u_-, v_+, v_-), \end{aligned}$$

where $t' = kt, s' = -ks,\ k \neq 0$. There exist t, s making this determinant vanish because bd, ac have different signs. This means that the point where p intersects the Einstein torus containing the stem is not timelike and therefore outside the stem. □

Theorem 3.3.4. *Two crooked surfaces C, C' given respectively by the configurations u_+, u_-, v_+, v_- and u'_+, u'_-, v'_+, v'_- are disjoint if and only if the four photons u'_+, u'_-, v'_+, v'_- do not intersect C and the four photons u_+, u_-, v_+, v_- do not intersect C'.*

Proof. Let us first show that the wing $\mathcal{W}_+$ of C does not intersect C'. By Lemma 3.3.3, it suffices to show that

$$\omega(tu_+ + sv_+, v'_+)\omega(tu_+ + sv_+, u'_+) > 0$$

and

$$\omega(tu_+ + sv_+, v'_-)\omega(tu_+ + sv_+, u'_-) < 0$$

for all $s, t \in \mathbb{R}$ such that $st \geq 0$ (with s and t not both zero).

We have

$$\begin{aligned} &\omega(tu_+ + sv_+, v'_+)\omega(tu_+ + sv_+, u'_+) \\ &= t^2\omega(u_+, v'_+)\omega(u_+, u'_+) + st\omega(u_+, v'_+)\omega(v_+, u'_+) \\ &\quad + st\omega(v_+, v'_+)\omega(u_+, u'_+) + s^2\omega(v_+, v'_+)\omega(v_+, u'_+). \end{aligned}$$

By hypothesis, neither u_+, v_+ intersect C', and neither u'_+, v'_+ intersect C. Therefore, using again Lemma 3.3.3 and $st \geq 0$, we see that each term in this sum is nonnegative and that at least one of them must be strictly positive. Therefore,

$$\omega(tu_+ + sv_+, v'_+)\omega(tu_+ + sv_+, u'_+) > 0.$$

The proof that

$$\omega(tu_+ + sv_+, v'_-)\omega(tu_+ + sv_+, u'_-) < 0$$

is similar. Therefore, $\mathcal{W}_+$ does not intersect C'.

In an analogous way, one can show that $\mathcal{W}_-$ does not intersect C'. Therefore, the wings of the crooked surface C do not intersect C'. Hence, to show that C and C' are disjoint, it only remains to show that the stem of C does not intersect C'.

By symmetry, the wings of C' do not intersect C, which means in particular that they do not intersect the stem of C. Consequently, the stem of C can only intersect the stem of C'. However, according to Theorem 3.3.1, if the stem of C intersects the stem of C', it must necessarily intersect its wings as well, which is not the case here. Therefore, we conclude that C and C' must be disjoint. □

By Lemma 3.3.3, this disjointness criterion can be expressed explicitly as 16 inequalities (two for each of the 8 photons defining the two crooked surfaces). There is some redundancy in these inequalities, but there does not seem to be a natural way to reduce the system.

3.4 Anti de Sitter crooked planes

In this section, we show that the criterion for disjointness of *anti de Sitter* crooked planes described in [DGK14] is a special case of Theorem 3.3.4, when embedding the double cover of anti de Sitter space in the Einstein universe.

Theorem 3.4.1 ([DGK14], Theorem 3.2)**.** *Let ℓ, ℓ' be geodesic lines of $\mathbb{H}^2$ and $g \in \mathsf{PSL}(2,\mathbb{R})$. Then, the* AdS *crooked planes defined by (I,ℓ) and (g,ℓ') are disjoint if and only if for any endpoints ξ of ℓ and ξ' of ℓ', we have $\xi \neq \xi'$ and $d(\xi, g\xi') - d(\xi,\xi') < 0$.*

In this criterion, the difference $d(p, gq) - d(p, q)$ for $p, q \in \partial\mathbb{H}^2$ is defined as follows : choose sufficiently small horocycles C, D through p, q respectively. Then, $d(p, gq) - d(p, q) := d(C, GD) - d(C, D)$ and this quantity is independent of the choice of horocycles.

3.4.1 AdS as a subspace of Ein

Let V_0 be a real two dimensional symplectic vector space with symplectic form ω_0. Denote by V the four dimensional symplectic vector space $V = V_0 \oplus V_0$ equipped with the symplectic form $\omega = \omega_0 \oplus -\omega_0$. This vector space V will have the same role as in Section 3.2.

The Lie group $\mathsf{Sp}(V_0) = \mathsf{SL}(V_0)$ is a model for the double cover of anti de Sitter 3-space. We will show how to embed this naturally inside the Lagrangian Grassmannian model of the Einstein Universe in three dimensions.

Define

$$i : \mathsf{SL}(V_0) \to \mathrm{Gr}(2, V)$$

$$f \mapsto \mathsf{graph}(f)$$

The graph of $f \in \mathsf{Sp}(V_0)$ is a Lagrangian subspace of $V = V_0 \oplus V_0$. This means that $i(\mathsf{SL}(V_0)) \subset \mathsf{Lag}(V) \cong \mathsf{Ein}$. This map is equivariant with respect to the homomorphism:

$$\mathsf{SL}(V_0) \times \mathsf{SL}(V_0) \to \mathsf{Sp}(V)$$

$$(A, B) \mapsto B \oplus A.$$

The involution of Ein induced by the linear map

$$I \oplus -I : V_0 \oplus V_0 \mapsto V_0 \oplus V_0,$$

where I denotes the identity map on V_0, preserves the image of i. It corresponds to the two-fold covering $\mathsf{SL}(V_0) \to \mathsf{PSL}(V_0)$. The fixed points of this involution are exactly the complement of the image of i, corresponding to the conformal boundary of AdS.

3.4.2 Crooked surfaces and AdS crooked planes

As in [Gol15], we say that a crooked surface is *adapted to an AdS patch* if it is invariant under the involution $I \oplus -I$. More precisely, two of the opposite vertices are fixed (they lie on the boundary of AdS) and the two others are swapped. If we denote the four photons by u_-, u_+, v_-, v_+, this means $v_- = (I \oplus -I)u_-$ and $v_+ = (I \oplus -I)u_+$.

3.4.2.1 AdS crooked planes based at the identity

For concreteness, choose a basis of V to identify it with $\mathbb{R}^4$. We will represent a plane in $\mathbb{R}^4$ by a 4×2 matrix whose columns generate the plane, up to multiplication on the right by an invertible 2×2 matrix. For example, $\mathsf{graph}(f)$ corresponds to the matrix:

$$\begin{pmatrix} I \\ \\ f \end{pmatrix}.$$

The identity element of $\mathsf{SL}(V_0)$ maps to the plane

$$\begin{pmatrix} I \\ \\ I \end{pmatrix}$$

and its image under the involution $I \oplus -I$ is

$$\begin{pmatrix} I \\ \\ -I \end{pmatrix}.$$

In order to complete this to a lightlike quadrilateral, we choose a pair of vectors $a, b \in V_0$ (2×1 column vectors). Then, the four vertices of the lightlike quadrilateral are:

$$\begin{pmatrix} I \\ \\ I \end{pmatrix}, \begin{pmatrix} a & a \\ \\ a & -a \end{pmatrix}, \begin{pmatrix} b & b \\ \\ b & -b \end{pmatrix}, \begin{pmatrix} I \\ \\ -I \end{pmatrix}.$$

We will say that such a lightlike quadrilateral is based at I and defined by the vectors a, b. Its lightlike edges are the photons represented by vectors:

$$u_+ = \begin{pmatrix} a \\ \\ a \end{pmatrix}, u_- = \begin{pmatrix} -a \\ \\ a \end{pmatrix}$$

$$v_+ = \begin{pmatrix} b \\ \\ b \end{pmatrix}, v_- = \begin{pmatrix} b \\ \\ -b \end{pmatrix}.$$

3.4.2.2 AdS crooked planes based at f

In order to get an AdS crooked plane based at a different point $f \in \mathsf{SL}(V_0)$, we map the crooked plane by an element of the isometry group $\mathsf{SL}(V_0) \times \mathsf{SL}(V_0) \subset \mathsf{Sp}(V)$.

The easiest way is to use an element of the form :

$$\begin{pmatrix} I & 0 \\ 0 & f \end{pmatrix}.$$

This corresponds to left multiplication by f in $\mathsf{SL}(V)$.

Applying f to a lightlike quadrilateral, we get a lightlike quadrilateral with vertices of the form:

$$\begin{pmatrix} I \\ f \end{pmatrix}, \begin{pmatrix} I \\ -f \end{pmatrix}, \begin{pmatrix} a & -a \\ fa & fa \end{pmatrix}, \begin{pmatrix} b & b \\ fb & -fb \end{pmatrix}$$

and edges of the form:

$$\begin{pmatrix} a \\ fa \end{pmatrix}, \begin{pmatrix} -a \\ fa \end{pmatrix}$$

$$\begin{pmatrix} b \\ fb \end{pmatrix}, \begin{pmatrix} -b \\ fb \end{pmatrix}.$$

3.4.3 Disjointness

The disjointness criterion for crooked surfaces in the Einstein Universe is given by 16 inequalities. Using the symmetries imposed by an AdS patch, we can reduce them to 4 inequalities.

Using the involution defining the AdS patch, we can immediately reduce the number of inequalities by half. This is because both surfaces are preserved by the involution, and their defining photons are swapped in pairs. (So for example, we only have to check that u_+ and u_- are disjoint from the other surface, for each surface.)

The second reduction comes from the fact that for AdS crooked planes, we only need to check that the four photons from the first crooked surface are disjoint from the second, and then the four from the second are automatically disjoint from the first.

For a crooked surface based at the identity with lightlike quadrilateral defined by the vectors $a, b \in V_0$ and another based at f with quadrilateral defined by $a', b' \in V_0$, the inequalities reduce to:

$$\omega_0(a', b)^2 > \omega_0(fa', b)^2$$

$$\omega_0(a', a)^2 > \omega_0(fa', a)^2$$

$$\omega_0(b', b)^2 > \omega_0(fb', b)^2$$

$$\omega_0(b', a)^2 > \omega_0(fb', a)^2.$$

What remains is to interpret these four inequalities in terms of hyperbolic geometry. We first define an equivariant map from $\mathbb{P}(V_0)$ to $\partial\mathbb{H}^2$. As a model of the boundary of $\mathbb{H}^2$, we use the projectivized null cone for the Killing form in $\mathfrak{sl}(2,\mathbb{R})$. Define

$$\eta : V_0 \to N(\mathfrak{sl}(2,\mathbb{R}))$$

$$a \mapsto -aa^T J,$$

where a is a column vector representing a point in $\mathbb{P}(V_0)$. This map associates to the vector a the tangent vector to the identity of the photon between I and the boundary point $\begin{pmatrix} a & a \\ a & -a \end{pmatrix}$. Note that the image of η is contained in the upper part of the null cone.

Lemma 3.4.2. *η is equivariant with respect to the action of* $\mathsf{SL}(V_0)$.

Proof.

$$\eta(Aa) = -Aa(Aa)^T J = -Aaa^T A^T J = -Aaa^T JA^{-1} = A\eta(a)A^{-1}.$$

□

Lemma 3.4.3. *Let $a, b \in V_0$. Then, $\omega_0(a,b)^2 = -K(\eta(a), \eta(b))$.*

Proof.

$$\begin{aligned}\omega_0(a,b)^2 &= -a^T Jbb^T Ja \\ &= a^T J\eta(b)a \\ &= \mathsf{Tr}(a^T J\eta(b)a \\ &= \mathsf{Tr}(aa^T J\eta(b)) \\ &= -\mathsf{Tr}(\eta(a)\eta(b)) \\ &= -K(\eta(a), \eta(b)).\end{aligned}$$

□

Note that the expression $\omega_0(a,b)$ is not projectively invariant, but the sign of $\omega_0(a,b)^2 - \omega_0(a,fb)^2$ is.

Corollary 3.4.4. *The following inequalities are equivalent*

$$\omega_0(a,b)^2 - \omega_0(a,fb)^2 > 0,$$

$$K(\eta(a), f\eta(b)f^{-1}) > K(\eta(a), \eta(b)).$$

Finally, we want to show that the four inequalities above imply the DGK criterion. Let A, B, A', B' denote respectively $\eta(a), \eta(b), \eta(a'), \eta(b')$. Then, A, B, A', B' represent endpoints of two geodesics g, g' in the hyperbolic plane. We want to show

$$d(\xi, f\xi' f^{-1}) - d(\xi, \xi') < 0$$

for $\xi \in \{A, B\}$ and $\xi' \in \{A', B'\}$.

We use the hyperboloid model of $\mathbb{H}^2$, $\{X \in \mathfrak{sl}(2, \mathbb{R}) \mid K(X, X) = -1\}$. Consider horocycles $C_\xi(r) = \{X \in \mathbb{H}^2 \mid K(X, \xi) = -r\}$ and $C_{\xi'}(r') = \{X \in \mathbb{H}^2 \mid K(X, \xi') = -r'\}$ at ξ and ξ' respectively. The distance between these two horocycles is given by the formula

$$d(C_\xi(r), C_{\xi'}(r')) = \operatorname{arccosh}\left(-\frac{1}{2}\left(\frac{K(\xi, \xi')}{2rr'} + \frac{2rr'}{K(\xi, \xi')}\right)\right).$$

Similarly,

$$d(C_\xi(r), fC_{\xi'}(r')f^{-1}) = \operatorname{arccosh}\left(-\frac{1}{2}\left(\frac{K(\xi, f\xi' f^{-1})}{2rr'} + \frac{2rr'}{K(\xi, f\xi' f^{-1})}\right)\right).$$

We know that $K(\xi, f\xi' f^{-1}) > K(\xi, \xi')$. If r, r' are sufficiently small, by increasingness of the function $x \mapsto x + \frac{1}{x}$ for $x > 1$ and increasingness of arccosh we conclude $d(C_\xi(r), C_{\xi'}(r')) > d(C_\xi(r), fC_{\xi'}(r'))$, which is what we wanted.

Chapter 4: Partial cyclic orders

The projective line $\mathbb{RP}^1$ admits a cyclic order invariant under the action of projective automorphisms. The simplest Schottky groups are defined by their action on $\mathbb{RP}^1$ by such automorphisms. Indeed, the application of the Ping-pong Lemma only relies on this cyclic order, and it is possible to define Schottky subgroups of the orientation-preserving homeomorphism group $\mathrm{Homeo}^+(\mathbb{RP}^1)$. This motivates the idea of finding analogs of this cyclic order in other spaces in order to define Schottky groups in a broader context. The contents of this chapter are mostly from the joint preprint with N. Treib [BT16].

4.1 Definitions

A partial cyclic order is a relation on triples which is analogous to a partial order, but generalizing a cyclic order instead of a linear order. The definition we use was introduced in 1982 by Novák [Nov82].

Definition 4.1.1. A *partial cyclic order* (PCO) on a set C is a relation $\rightarrow$ on triples in C satisfying, for any $a, b, c, d \in C$:

- if $\overrightarrow{abc}$, then $\overrightarrow{bca}$ (*cyclicity*).
- if $\overrightarrow{abc}$, then not $\overrightarrow{cba}$ (*asymmetry*).

- if $\overrightarrow{abc}$ and $\overrightarrow{acd}$, then $\overrightarrow{abd}$ (*transitivity*).

If in addition the relation satisfies:

- If a, b, c are distinct, then either $\overrightarrow{abc}$ or $\overrightarrow{cba}$ (*totality*),

then it is called a *total cyclic order*.

Let C, D be partially cyclically ordered sets.

Definition 4.1.2. A map $f : C \to D$ is called *increasing* if $\overrightarrow{abc}$ implies $\overrightarrow{f(a)f(b)f(c)}$. An automorphism of a partial cyclic order is an increasing map $f : C \to C$ with an increasing inverse. We will denote by G the group of all automorphisms of C.

Any subset $X \subset C$ such that the restriction of the partial cyclic order is a total cyclic order on X will be called a *cycle*. We will also use the term cycle for (ordered) tuples $(x_1, \ldots, x_n) \in C^n$ if the cyclic order relations between the points in C agree with the cyclic order given by the ordering of the tuple.

Definition 4.1.3. Let $a, b \in C$. The *interval* between a and b is the set $(a, b) := \{x \in C \mid \overrightarrow{axb}\}$. The set of all intervals generates a natural topology on C under which automorphisms of the partial cyclic order are homeomorphisms. We call this topology the *interval topology* on C. We call C *first countable* when its interval topology is first countable. We need this last condition to justify the use of the sequential definition of continuity, for instance in the proof of Theorem 4.2.7.

The *opposite* of an interval $I = (a, b)$ is the interval (b, a), also denoted by $-I$.

Example 4.1.4. The circle S^1 admits a total cyclic order. The relation on triples is : $\overrightarrow{abc}$ whenever (a, b, c) are in counterclockwise order around the circle. The automorphism group of this cyclic order is the group of orientation preserving homeomorphisms of the circle.

Example 4.1.5. We can define a product cyclic order on the torus $S^1 \times S^1$. Define the relation to be $\overrightarrow{xyz}$ whenever $\overrightarrow{x_1y_1z_1}$ and $\overrightarrow{x_2y_2z_2}$. This is not a total cyclic order. Some intervals in this cyclically ordered space are shown in Figure 4.2.

Example 4.1.6. Every strict partial order $<$ on a set X induces a partial cyclic order in the following way: define $\overrightarrow{abc}$ if and only if either $a < b < c$, $b < c < a$, or $c < a < b$. The cyclic permutation axiom is automatic and the two other axioms follow from the antisymmetry and transitivity axioms of a partial order.

The key topological property that we will need in the next section is a notion of completeness that we can associate to a space carrying a PCO.

Definition 4.1.7. A sequence $a_1, a_2, \dots \in C$ is *increasing* if and only if $\overrightarrow{a_ia_ja_k}$ whenever $i < j < k$.

Equivalently, the map $a : \mathbb{N} \to C$ defined by $a(i) = a_i$ is increasing, where the cyclic order on $\mathbb{N}$ is given by $\overrightarrow{ijk}$ whenever $i < j < k$, $j < k < i$ or $k < i < j$ (as in Example 4.1.6).

Definition 4.1.8. A partially cyclically ordered set C is *increasing-complete* if every increasing sequence converges to a unique limit in the interval topology.

The following is a natural equivalence relation for increasing sequences.

Definition 4.1.9. Two increasing sequences a_n and b_m are called *compatible* if they admit subsequences a_{n_k} and b_{m_l} making the combined sequence $a_{n_1}, b_{m_1}, a_{n_2}, b_{m_2}, \dots$ increasing.

Lemma 4.1.10. *Let C be an increasing-complete partially cyclically ordered set, and let a_n and b_m be compatible increasing sequences. Then their limits agree.*

Proof. Any increasing sequence has a unique limit, and any subsequence of an increasing sequence therefore has the same unique limit.
The combined sequence (see the previous definition) is increasing, hence its unique limit must agree with the unique limits of both subsequences a_{n_k} and b_{m_l}. □

To complete this list of definitions related to PCOs, we finish with two further restrictions on a set with a PCO which will be useful in Section 4.2.2.

Definition 4.1.11. A partially cyclically ordered set C is *proper* if for any increasing quadruple $(a, b, c, d) \in C^4$, we have $\overline{(b, c)} \subset (a, d)$. Here, "bar" denotes the closure in the interval topology.

Definition 4.1.12. Two points $a, b \in C$ in a partially cyclically ordered set C are called *comparable* if there exists a point $c \in C$ with either $\overrightarrow{abc}$ or $\overrightarrow{acb}$.

Definition 4.1.13. A PCO set C is *full* if whenever (a, b) is non-empty for some pair a, b, then (b, a) is also non-empty. Equivalently, whenever a, b are comparable then both intervals they bound are non-empty.

Remark 4.1.14. *The motivation for the term "full" stems from the following construction. Assume we have a non-empty interval (a, b). Then we can find a point $c \in (a, b)$ and another point $d \in (b, a)$. But then the point d also lies in the interval (c, a), so by fullness, there is a point inside (a, c) as well. Continuing in this fashion, we can subdivide all resulting intervals further and further, and thereby construct a* *countably infinite subset $X \subset (a,b)$ with the following two properties: Firstly, X is a cycle. Secondly, for any pair x_1, x_2 of distinct elements of X, the intersection $(x_1, x_2) \cap X$ is nonempty.*

4.2 Generalized Schottky groups

Throughout this section, C denotes a partially cyclically ordered set and $G = \mathsf{Aut}(C)$.

4.2.1 Definition of generalized Schottky group

Let Σ be the interior of a compact, connected, oriented surface with boundary of Euler characteristic $\chi < 0$. Then, the fundamental group $\pi_1(\Sigma)$ is free on $g = 1-\chi$ generators. Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be the holonomy of a finite area hyperbolization of Σ. In this section, we construct free subgroups of G using Γ as a combinatorial model.

It is well known that there is a presentation for Γ of the following form : Γ is freely generated by $A_1, \ldots, A_g \in \mathrm{PSL}(2,\mathbb{R})$ and there are $2g$ disjoint open intervals $I_1^+, \ldots, I_g^+, I_1^-, \ldots, I_g^- \subset \mathbb{RP}^1 \cong S^1$ such that $A_j(-I_j^-) = I_j^+$. Moreover, we have that $\bigcup\limits_i \overline{I_i^+} \cup \bigcup\limits_i \overline{I_i^-} = \mathbb{RP}^1$ (Figure 4.1).

The cyclic ordering on S^1 gives a cyclic ordering to the intervals in the definition.

We call a k-th order interval the image of any I_j^+(respectively I_j^-) by a reduced word $W = \gamma_1\gamma_2 \ldots \gamma_{k-1}$ of length $k-1$ with $\gamma_{k-1} \neq A_j^{-1}$(respectively $\gamma_{k-1} \neq A_j$).

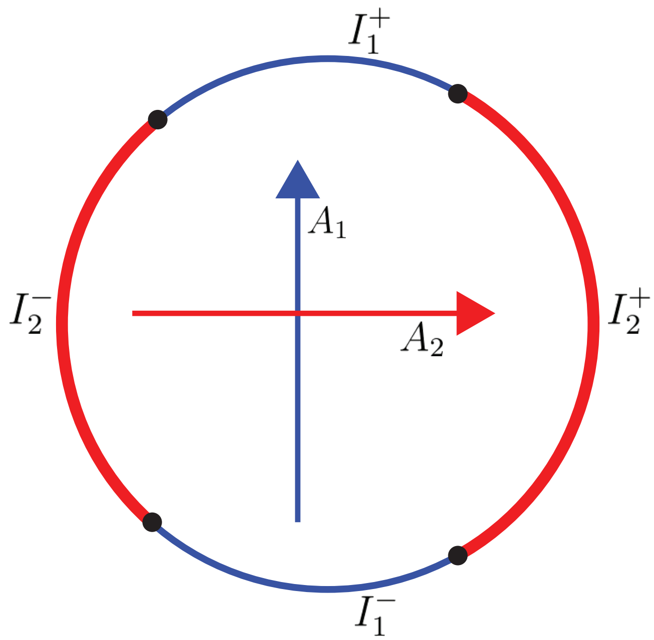


Figure 4.1: A combinatorial model for the once punctured torus.

There are exactly $(2g)(2g-1)^{k-1}$ k-th order intervals. There is a natural bijection between words of length k and k-th order intervals. We use this bijection to index k-th order intervals : I_W is the interval corresponding to the word W. For any fixed k, the k-th order intervals are all pairwise disjoint, and so they are cyclically ordered. This induces a cyclic ordering on words of length k in Γ. The union of all closures of k-th order intervals is all of $\mathbb{RP}^1$.

The following easy lemma, which is a reformulation of transitivity, motivates our definition of generalized Schottky groups in G.

Lemma 4.2.1. *Let $(a, b, c) \in C^3$ be a cycle. Then we have $(b, c) \subset (b, a)$. In particular, the intervals (a, b) and (b, c) are disjoint.*

Proof. Let $x \in (b, c)$, so we have $\overrightarrow{bxc}$. By transitivity, together with $\overrightarrow{bca}$, this implies

$\overrightarrow{bxa}$. $\square$

We now define generalized Schottky subgroups of G by asking for a setup of intervals similar to the $\mathrm{PSL}(2,\mathbb{R})$ case and requiring generators to pair them the same way.

Definition 4.2.2. Let ξ_0 be an increasing map from the set of endpoints of the intervals $I_1^+,\dots,I_g^+$, $I_1^-,\dots,I_g^-$ into C. For $I_i^\pm = (a_i^\pm, b_i^\pm)$, define the corresponding interval $J_i^\pm = (\xi_0(a_i^\pm), \xi_0(b_i^\pm)) \subset C$. Assume there exist $h_1,\dots,h_g \in G$ which pair the endpoints of $J_i^\pm$ in the same way that the g_i pair the $I_i^\pm$, so that $h_i(-J_i^-) = J_i^+$. We call the image of the induced homomorphism $\Gamma \to G$ sending A_i to h_i a *generalized Schottky group*, and the intervals $J_i^\pm$ used to define it a set of *Schottky intervals* for this group.

Remark 4.2.3.

1. *A generalized Schottky group will in general have many possible choices of a set of Schottky intervals. We will only use this term when a specific choice of both generators and intervals is fixed.*
2. *Since the cyclic ordering is a property of $\mathbb{RP}^1$ which is not shared by $\mathbb{CP}^1$, the Schottky groups defined here do* not *generalize the more well known $\mathbb{CP}^1$ Kleinian case.*
3. *The requirement that the combinatorial model be a finite-area hyperbolization is artificial. It is helpful in order to avoid having to separate our analysis into several cases. We could use a model where the intervals have disjoint closures and the construction would work in the same way. Such models always admit*

a choice of Schottky generators with contiguous Schottky intervals as well, so they form a strict subset. In Section 4.4.2 we will use intervals with disjoint closures to describe domains of discontinuity in $\mathbb{RP}^{2n-1}$.

4. *Our use of the term "Schottky" differs slightly from most references in that we allow for the closures of the Ping-pong subsets to intersect. This is sometimes called "Schottky-type".*

With this setup, we can define k-th order intervals in C in the same way as above but starting with the intervals $J_i^{\pm}$ and their images under words in the h_i (see Figure 4.2). As above, denote by J_W the interval corresponding to W. Note that since ξ_0 is increasing, the k-th order intervals in C are also cyclically ordered, where the ordering is the same as the ordering of the corresponding intervals in $\mathbb{RP}^1$.

Proposition 4.2.4. *The group generated by $h_1 \dots h_g$ is free on those generators.*

Proof. Define $J_i = J_i^+ \cup J_i^-$. Note that $J_i \cap J_j = \emptyset$ whenever $i \neq j$. Moreover, for any $n \neq 0$, $h_i^n(J_j) \subset J_i$ and so the proposition follows from the Ping-pong lemma. □

The endpoints of k-th order intervals in C satisfy the same cyclic order relations as the corresponding endpoints in S^1, and we can extend ξ_0 to an increasing equivariant map defined on the countable dense set of all endpoints of k-th order intervals in S^1. We denote this set by S^1_Γ.

4.2.2 Limit curves

Lemma 4.2.5. *Let C be a partially cyclically ordered set which is full and proper, and $(x_1, \dots, x_6) \in C^6$ a cycle. Let $I_1 = (x_1, x_2), I_2 = (x_3, x_4), I_3 = (x_5, x_6)$ and*

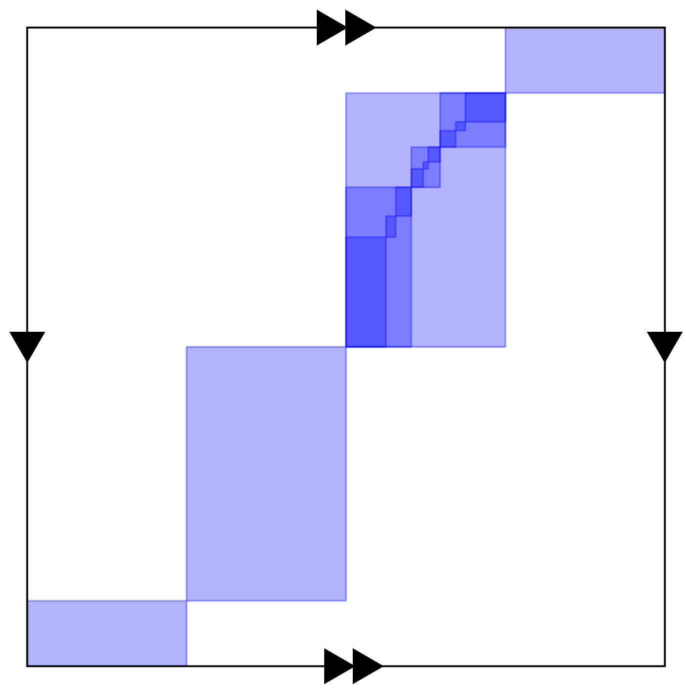


Figure 4.2: Some first, second and third order intervals for a generalized Schottky group acting on $S^1 \times S^1$.

$a_i \in \overline{I_i}$ be arbitrary points in the closures of the intervals. Then $\overrightarrow{a_1a_2a_3}$.

Proof. Since C is full, we can choose auxiliary points y_i such that the 12-tuple $(y_1, x_1, x_2, y_2, y_3, x_3, x_4, y_4, y_5, x_5, x_6, y_6)$ is a cycle. This allows us to conclude that $\overline{(x_i, x_{i+1})} \subset (y_i, y_{i+1})$ for odd i as C is proper. Since $(y_1, \ldots, y_6)$ is a cycle, transitivity implies the lemma. □

Lemma 4.2.6. *Let $P_n \to P$ be an increasing sequence in a proper, increasing complete, PCO set C. Assume Q_n is another sequence with $Q_n \in \overline{(P_n, P_{n+1})}$ for all n. Then Q_n converges to P and is 3-increasing in the following sense: whenever $i + 2 < j < k - 2$, we have $\overrightarrow{Q_iQ_jQ_k}$.*

Proof. For every $n \geq 2$, $Q_n \in (P_{n-1}, P_{n+2})$ by properness, which already implies that Q_n is 3-increasing. Now, consider the following sequence:

$$P_1, Q_2, P_4, Q_5, \ldots, P_{3n+1}, Q_{3n+2}, \ldots$$

It is increasing, and admits a subsequence which is also a subsequence of P_n. Since increasing sequences have unique limits, this sequence must converge to P. The increasing subsequence Q_{3n+2} therefore converges to P. Using the same argument, we see that Q_{3n+1} and Q_{3n} also converge to P, so in fact the sequence Q_n converges to P. □

We now come to the main theorem of this section, which explains how to construct a boundary map for generalized Schottky groups, under some topological assumptions.

The boundary map we construct will be left-continuous as a map from S^1 to some first countable topological space C. To avoid confusion, let us fix the definition here: In a small neighborhood U of a point $a \in S^1$, the cyclic order induces a linear order. A sequence $a_n \in U$ converges to a from the left if $a_n < a$ and $a_n \xrightarrow{n\to\infty} a$. The function f is left-continuous at a if $f(a_n) \to f(a)$ for all sequences a_n converging to a from the left.

It is worth noting that to check for left-continuity at a point a, it is in fact sufficient to check the convergence of $f(a_n)$ for increasing sequences a_n converging to a from the left. The reason is the following: Assume a_n is a sequence converging to a from the left such that $f(a_n)$ does not converge to $f(a)$. Then it has a subsequence such that $f(a_{n_k})$ stays bounded away from $f(a)$. But since $a_{n_k} \to a$ from the left, we can pick a further subsequence which is increasing and still a counterexample to left-continuity.

Theorem 4.2.7. *Let $\rho : \Gamma \to G$ be the map defining a generalized Schottky group. Assume that C is first countable, increasing-complete, full and proper. Then there is a left-continuous, equivariant, increasing boundary map $\xi : S^1 \to C$.*

Proof. We construct the map ξ as follows: Recall that $S^1_\Gamma \subset S^1$ denotes the domain of ξ_0 and is a dense subset. For $x \in S^1$, pick any increasing sequence $x_n \in S^1_\Gamma$ converging to x and set

$$\xi(x) = \lim_{n\to\infty} \xi_0(x_n).$$

First of all, let us show that this value is well-defined. Since x_n is an increasing sequence, the increasing map ξ_0 maps it to an increasing sequence in C which therefore has a unique limit. Furthermore, this limit does not depend on the choice of

x_n: Let y_m be another increasing sequence converging to x. Then the two sequences $\xi_0(x_n)$ and $\xi_0(y_m)$ are compatible, so they have the same limit by Lemma 4.1.10.

We then verify that ξ is equivariant. Let $x \in S^1$, $\gamma \in \Gamma$, and $x_n \to x$ an increasing sequence, so we have $\xi(x) = \lim_{n\to\infty} \xi_0(x_n)$. Then $\gamma(x_n)$ is an increasing sequence converging to $\gamma(x)$, so by continuity of $\rho(\gamma)$ and equivariance of ξ_0, we have the following equalities:

$$\rho(\gamma)(\xi(x)) = \lim_{n\to\infty} \rho(\gamma)(\xi_0(x_n)) = \lim_{n\to\infty} \xi_0(\gamma(x_n)) = \xi(\gamma(x)).$$

Next, we show that it is left-continuous. Assume $x_n \in S^1$ is a sequence converging to x from the left. As explained above, without loss of generality we can take x_n to be an increasing sequence. We pick points $y_n \in S^1_\Gamma$ such that $y_n \in (x_{n-1}, x_n)$. Then y_n is increasing and $x_n \in (y_n, y_{n+1})$. Furthermore, y_n also converges to x, hence

$$\xi(x) = \lim \xi_0(y_n). \tag{4.1}$$

Now, for each n, let $\{a_k(n)\}_{k\in\mathbb{N}} \subset S^1_\Gamma$ be an increasing sequence converging to x_n, so

$$\xi(x_n) = \lim_{k\to\infty} \xi_0(a_k(n)). \tag{4.2}$$

Then $a_k(n) \in (y_n, y_{n+1})$ for large k, so

$$\lim_{k\to\infty} \xi_0(a_k(n)) \in \overline{(\xi_0(y_n), \xi_0(y_{n+1}))} \tag{4.3}$$

because ξ_0 is increasing. Now Lemma 4.2.6 applies and, combined with (4.1), (4.2) and (4.3), tells us that $\xi(x_n)$ converges to $\xi(x)$.

The final property we need to check is that ξ is increasing. Assume that we have $\overrightarrow{xyz}$ for points $x, y, z \in S^1$. By density of S^1_Γ, we can find a cycle $(a_1, a_2, b_1, b_2, c_1, c_2) \in$

$(S^1_\Gamma)^6$ such that $x \in (a_1, a_2)$, $y \in (b_1, b_2)$, $z \in (c_1, c_2)$. As in the proof of left-continuity, this implies that $\xi(x) \in \overline{(\xi_0(a_1), \xi_0(a_2))}$, and similar for the other two points. Using Lemma 4.2.5, we conclude $\overrightarrow{\xi(x)\xi(y)\xi(z)}$. □

The very general construction described in this section applies to many examples. For instance, various notions of positivity in homogeneous spaces give rise to partial cyclic orders. More specifically, the Shilov boundary of Hermitian symmetric spaces admits a PCO satisfying all the above properties, and the next section dedicated to this example. It is also possible, using techniques similar to Fock-Goncharov total positivity [FG06], to construct a PCO on spaces of complete oriented flags. We will explain this partial cyclic order in Section 4.5 and how it can be used to describe convex projective structures on surfaces with boundary (see also [BT17]).

4.3 Hermitian symmetric spaces of tube type

In this section, we show that the Shilov boundary of a Hermitian symmetric space of tube type X admits a partial cyclic order invariant under the biholomorphism group of X

A motivating example is the case where X is the Siegel upper half space of 2×2 complex matrices with positive-definite imaginary part. The Shilov boundary, in this case, identifies with the Einstein universe. The partial cyclic order arises from the causal structure on Ein.

We prove that Shilov boundaries satisfy the topological assumptions from Theorem 4.2.7, so we have a boundary map for every generalized Schottky subgroup.

Then, using the machinery of Section 4.2, we show that Schottky subgroups in this case correspond to maximal representations.

4.3.1 The partial cyclic order on the Shilov boundary

Let V be a real Euclidean vector space. That is, V is equipped with a scalar product $\langle , \rangle$.

Definition 4.3.1. A *symmetric cone* $\Omega \subset V$ is an open convex cone which is self-dual and homogeneous. More precisely, the dual cone

$$\Omega^* := \{v \in V \mid \langle u, v\rangle > 0, \forall u \in \overline{\Omega} \backslash \{0\}\}$$

equals Ω itself, and the subgroup of $\mathsf{GL}(V)$ preserving Ω acts transitively on Ω.

A *tube type domain* is a domain of the form $X = V + i\Omega \subset V_{\mathbb{C}}$ in the complexification of V, where Ω is a symmetric cone. Let G be the group of biholomorphisms of X.

The vector space V admits a *Euclidean Jordan algebra* structure associated to the symmetric cone Ω. The two structures (symmetric cone and Jordan algebra) determine each other [FK94].

Definition 4.3.2. A Jordan algebra is a vector space V over $\mathbb{R}$ together with a bilinear product $(u, v) \mapsto uv \in V$ satifsying:

$$uv = vu$$

and

$$u(u^2v) = u^2(uv)$$

for all $u, v \in V$.

Definition 4.3.3. A Jordan algebra V is *Euclidean* if it admits an identity element e, and there exists a positive definite inner product $\langle , \rangle$ on V such that

$$\langle uv, w \rangle = \langle v, uw \rangle$$

for all $u, v, w \in V$. The *cone of squares* of V is

$$C = \{v^2 \mid v \in V\}.$$

The interior C° of C is a symmetric cone, and coincides with Ω for the Jordan algebra structure induced by Ω.

Example 4.3.4. Consider $V = \mathbb{R}^{2,1}$ a 3-dimensional real vector space with Lorentzian inner product $u \cdot v = u_1v_1 + u_2v_2 - u_3v_3$. The set $\Omega = \{v \in V \mid v \cdot v < 0, v_3 > 0\}$ of future-pointing timelike vectors is a symmetric cone. The Jordan algebra structure associated to this cone is given by the product:

$$(u_1, u_2, u_3)(v_1, v_2, v_3) = (u_1v_3 - u_3v_1, u_2v_3 - u_3v_2, u_1v_1 + u_2v_2 + u_3v_3).$$

Example 4.3.5. The set of $n \times n$ real symmetric matrices is a Jordan algebra with product $A \star B = (AB + BA)/2$. The corresponding symmetric cone is the cone of positive-definite matrices.

There is a spectral theorem for Euclidean Jordan algebras :

Proposition 4.3.6 ([FK94] Theorem III.1.2)**.** *Let $v \in V$ with $\dim(V) = k$. Then, there exist unique real numbers $\lambda_1, \ldots, \lambda_k$, and a* Jordan frame *of primitive orthogonal idempotents $c_1, \ldots, c_k$ (that is, $c_i^2 = c_i$, $c_ic_j = 0$ for $i \neq j$, and $\sum c_i = e$) such* *that*

$$v = \lambda_1 c_1 + \ldots \lambda_k c_k.$$

The λ_i *are called the* eigenvalues *of* v.

Definition 4.3.7. The partial order $<_\Omega$ on a Jordan algebra V is defined by $x <_\Omega y$ if and only if $y - x \in \Omega$.

The Cayley transform is the classical biholomorphic map which sends the upper half plane to the unit disk in $\mathbb{C}$. We will use the following generalization to Jordan algebras in order to define a bounded realization of tube type domains.

Definition 4.3.8. Let $\mathsf{D} = \{z \in V_\mathbb{C} \mid z + ie \text{ is invertible}\}$, where e is the identity of the Jordan algebra and we extend the multiplication linearly to the complexification of V.

The *Cayley transform* is the map $p : \mathsf{D} \to V_\mathbb{C}$ defined by

$$p(v) = (v - ie)(v + ie)^{-1}.$$

Proposition 4.3.9 ([FK94], Theorem X.4.3)**.** *The Cayley transform* p *maps the tube type domain* $X = V \oplus i\Omega$ *biholomorphically onto a bounded domain* $B \subset V_\mathbb{C}$*, which we call the* bounded domain realization *of* X *(also known as the* Harish-Chandra realization*).*

Definition 4.3.10. If B is a bounded domain in $\mathbb{C}^n$, denote by $C(B)$ the set of continuous functions on $\overline{B}$ which are holomorphic on B. The *Shilov boundary* $\mathcal{S}$ of B is the smallest closed subset of ∂B such that, for all $f \in C(B)$ we have

$$\max_{z \in \bar{B}} |f(z)| = \max_{z \in \mathcal{S}} |f(z)|.$$

By extension, the Shilov boundary of a tube type domain X is the Shilov boundary of its bounded domain realization. The action of the group G of biholomorphisms of B extends smoothly to its Shilov boundary.

Proposition 4.3.11 ([FK94], Proposition X.2.3)**.** *The Cayley transform $p : V \to V_{\mathbb{C}}$ maps the vector space V into the Shilov boundary $\mathcal{S}$ and $\overline{p(V)} = \mathcal{S}$.*

Using the following notion of transversality, we can make the previous proposition more precise and say explicitly which points are in the image of p.

Definition 4.3.12. Two points $x, y \in \mathcal{S}$ are called *transverse* if the pair $(x, y) \in \mathcal{S} \times \mathcal{S}$ belongs to the unique open G-orbit for the diagonal action.

The image of the Cayley transform is exactly the set of points $x \in \mathcal{S}$ which are transverse to a fixed point which we denote by ∞. [Wie04, Section 6.6.1]

The next object we need to define is the generalized Maslov index (generalizing the case of the Lagrangian Grassmannian in Definition 2.2.14). This index is a function on ordered triples of points in $\mathcal{S}$, invariant under G. It will be used in order to define a partial cyclic order on $\mathcal{S}$, extending the partial cyclic order induced by $<_\Omega$ on $p(V) \subset \mathcal{S}$.

The generalized Maslov index is defined in [Cle04] using the notion of Γ-radial convergence. For our purposes we will use the following equivalent definition, given in the same paper.

Definition 4.3.13. Let $x, y, z \in \mathcal{S}$. Applying an element of G, we may assume $x, y, z \in p(V)$. Let $v_x, v_y, v_z \in V$ be the vectors which map respectively to x, y, z

under the Cayley transform p. Then, the *generalized Maslov index* of x, y, z is the integer

$$\mathsf{M}(x,y,z) := \mathsf{k}(v_y - v_x) + \mathsf{k}(v_z - v_y) + \mathsf{k}(v_x - v_z)$$

where $\mathsf{k}(v)$ is the difference between the number of positive eigenvalues of v and the number of negative eigenvalues of v in its spectral decomposition.

When x, y are transverse to z, equivalently, we can map z to ∞ using an element of G and define

$$\mathsf{M}(x,y,\infty) = \mathsf{k}(v_y - v_x)$$

Proposition 4.3.14. *The Maslov index enjoys the following properties :*

- *G-invariance :* $\mathsf{M}(gx, gy, gz) = \mathsf{M}(x,y,z)$.
- *Skew-symmetry :* $\mathsf{M}(x_1,x_2,x_3) = \mathsf{sgn}(\sigma)\mathsf{M}(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$. *(for any permutation $\sigma \in S_3$)*
- *Cocycle identity :* $\mathsf{M}(y,z,w) - \mathsf{M}(x,z,w) + \mathsf{M}(x,y,w) - \mathsf{M}(x,y,z) = 0$.
- *Boundedness :* $|\mathsf{M}(x,y,z)| \leq \mathrm{rk}(X)$

These properties allow us to define a partial cyclic order on the Shilov boundary.

Proposition 4.3.15. *The relation $\overrightarrow{xyz}$ if and only if $\mathsf{M}(x,y,z) = \mathrm{rk}(X)$ defines a G-invariant partial cyclic order on $\mathcal{S}$.*

Proof. Since M is skew-symmetric, the relation automatically satisfies the first two axioms of a partial cyclic order. To prove the third axiom, assume $\mathsf{M}(x,y,z) =$

$\mathsf{M}(x, z, w) = \mathrm{rk}(X)$. By the cocycle identity,

$$\mathsf{M}(y, z, w) - \mathsf{M}(x, z, w) + \mathsf{M}(x, y, w) - \mathsf{M}(x, y, z) = 0$$

and so

$$\mathsf{M}(y, z, w) + \mathsf{M}(x, y, w) = 2\,\mathrm{rk}(X)$$

which is only possible if $\mathsf{M}(y, z, w) = \mathsf{M}(x, y, w) = \mathrm{rk}(X)$. $\square$

The Shilov partial cyclic order $\to$ is closely related with the causal structure on $\mathcal{S}$ introduced by Kaneyuki [Kan91]. Namely, whenever $\overrightarrow{xyz}$, there is a future-oriented closed timelike curve going through x, y, z in that order. Informally, y is in the intersection of the future of x and the past of z. The following two lemmas describe some immediate properties of cyclically ordered triples.

Lemma 4.3.16 ([Wie04], Lemma 5.5.4)**.** *Let $x, y, z \in \mathcal{S}$ with $\overrightarrow{xyz}$. Then x, y, z are pairwise transverse.*

Lemma 4.3.17. *Assume $x, y \in V$. Then, $\overrightarrow{xy\infty}$ if and only if $x <_\Omega y$.*

Proof. The cone Ω coincides with the region where $\mathsf{k}(v) = \mathrm{rk}(X)$. $\square$

Remark 4.3.18. *The interval topology on $\mathcal{S}$ is the same as the usual manifold topology.*

Proposition 4.3.19. *The PCO defined by $\to$ on $\mathcal{S}$ is increasing-complete, full and proper.*

Proof. We first show that it is increasing-complete. Let $x_1, x_2, \ldots$ be an increasing sequence in $\mathcal{S}$. Let $g \in G$ be such that $gx_2 = \infty$. Then, since we have $\overrightarrow{x_k x_{k+1} x_2}$

for all $k \geq 3$, the sequence $gx_3, gx_4, \dots$ is an increasing sequence transverse to ∞. Hence, there exist $v_3, v_4, \dots \in V$ with $p(v_k) = gx_k$.

This new sequence is increasing with respect to $<_\Omega$. Moreover, it is bounded since we have $\overrightarrow{gx_k gx_1 gx_2}$ for all $k > 2$, so $v_k <_\Omega v_1$ where $p(v_1) = gx_1$. The tail of the sequence is contained in $\overline{(v_3, v_1)}$ which is compact, so it has an accumulation point. If w, w' are two accumulations points of the sequence, let w_k, w'_k be subsequences converging respectively to each of them. Passing to subsequences if necessary, we can arrange so that $w_k <_\Omega w'_k$ for all k, and so $w'_k - w_k \in \Omega$. This implies $w' - w \in \overline{\Omega}$, and by the same argument we can also show $w - w' \in \overline{\Omega}$. Since Ω is a proper convex cone (in the sense of [FK94]), its closure does not contain any opposite pairs, so $w = w'$.

Now we turn to fullness of the PCO. Whenever an interval (x, y) is nonempty, its endpoints have to be transverse by Lemma 4.3.16. We can therefore apply an element of G to map x to ∞ and y inside $p(V)$. Then Lemma 4.3.17 shows that the interval (y, x) is also nonempty.

Finally, we show that the PCO is proper. Let $(x_1, x_2, x_3, x_4) \in \mathcal{S}^4$ be a cycle. Using an element of G, we can assume that x_4 is ∞, so that $x_1, x_2, x_3 \in p(V)$. Let $v_i \in V$ be the vector such that $p(v_i) = x_i$ for $i = 1, 2, 3$. Now the cyclic relations $\overrightarrow{x_1 x_2 \infty}$ and $\overrightarrow{x_2 x_3 \infty}$ imply that both $v_2 - v_1$ and $v_3 - v_2$ lie in the cone Ω. The interval (x_2, x_3) is therefore given by $p\left((v_2 + \Omega) \cap (v_3 - \Omega)\right)$. This implies the claim since $(v_2 + \Omega) \cap (v_3 - \Omega)$ is a relatively compact set in V whose closure is contained in $v_1 + \Omega$, which is mapped onto (x_1, ∞) by p. $\square$

4.3.2 Maximal representations

In the previous section we defined a PCO on the Shilov boundary $\mathcal{S}$ of a Hermitian symmetric space of tube type on which the group of holomorphic isometries G acts by order-preserving diffeomorphisms. We recall that this action is transitive on transverse pairs. The Schottky construction described in Section 4.2 therefore gives maps $\rho : \Gamma \to G$ where Γ is the fundamental group of a surface with boundary. Maximal representations are a class of geometrically interesting representations and we will show in this section that they correspond to Schottky subgroups. They are defined by associating a natural invariant to the representation and requiring it to attain its maximal possible value. While the study of this invariant was originally restricted to closed surfaces ([Tol79], [DT87], [Tol89]), the definition was extended to surfaces with boundary in [BIW10].

Let X be a Hermitian symmetric space and ω be the Kähler form on X. Then, ω defines a continuous, bounded cohomology class $\kappa_G^b \in H^2_{cb}(G, \mathbb{R})$ called the Kähler class. If $\rho : \pi_1(\Sigma) \to G$ is a representation, the pullback $\rho^*\kappa_G^b$ is a bounded cohomology class in $H^2_b(\pi_1(\Sigma), \mathbb{R}) \cong H^2_b(\Sigma, \mathbb{R})$. In order to get an invariant out of this class, we use the isomorphism $j : H^2_b(\Sigma, \partial\Sigma, \mathbb{R}) \to H^2_b(\Sigma, \mathbb{R})$ (see [BIW10] for details).

Definition 4.3.20. The *Toledo invariant* is the real number

$$\mathsf{T}(\rho) = \langle j^{-1}\rho^*\kappa_G^b, [\Sigma, \partial\Sigma]\rangle$$

where $[\Sigma, \partial\Sigma]$ is the relative fundamental class.

The Toledo invariant satisfies a sharp bound generalizing the *Milnor-Wood-inequality*: $|\mathsf{T}(\rho)| \leq |\chi(\Sigma)| \operatorname{rk}(X)$. A representation ρ is called *maximal* whenever equality is attained. The key to our analysis is the following characterization from [BIW10] (Theorem 8) :

Theorem 4.3.21. *Let $h : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ be a complete finite area hyperbolization of the interior of Σ and $\rho : \Gamma \to G$ a representation into a group of Hermitian type. Then ρ is maximal if and only if there exists a left continuous, equivariant, increasing map*

$$\xi : S^1 \to \mathcal{S}$$

where $\mathcal{S}$ is the Shilov boundary of the bounded symmetric domain associated to G.

Using this characterization and our earlier construction of a boundary map for generalized Schottky representations, we see that the two notions agree:

Theorem 4.3.22. *The representation $\rho : \Gamma \to G$ is maximal if and only if it admits a Schottky presentation.*

Proof. Assume ρ is Schottky. Proposition 4.3.19 states that all the prerequisites of Theorem 4.2.7 are fulfilled. Therefore, there exists a boundary map ξ satisfying the conditions of the characterization above, so ρ is maximal.

Conversely, if ρ is maximal, then we have such a map ξ. Choosing a Schottky presentation for the hyperbolisation h, we get a Schottky presentation for ρ by using the intervals $(\xi(a), \xi(b))$ where (a, b) is some Schottky interval in the presentation for h. Equivariance and positivity of ξ ensure that these intervals fit our definition of generalized Schottky groups. □

Theorem 4.3.22, as stated, assumes that G is of tube type. However, this assumption is not necessary. This is because of the following observations. Let X be a Hermitian symmetric space, and $\mathcal{S}$ its Shilov boundary. Then, in the same way as for tube type, the generalized Maslov index defines a partial cyclic order on $\mathcal{S}$. Let $x, y \in \mathcal{S}$ be transverse. Then, x, y are contained in the Shilov boundary of a unique maximal tube type subdomain of X [Wie04, Lemma 4.4.2]. Moreover, this is also true of any increasing triple in $\mathcal{S}$ [Wie04, Proposition 5.1.4]. This means that any increasing subset of $\mathcal{S}$ is contained in the Shilov boundary of a tube type subdomain, and so the proofs of this section generalize to arbitrary Hermitian symmetric spaces.

4.4 Schottky groups in $\mathrm{Sp}(2n, \mathbb{R})$

In this section, we consider the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, acting on $\mathbb{R}^{2n}$ equipped with a symplectic form ω, and describe the construction of Schottky groups in detail.

4.4.1 The Maslov index in $\mathrm{Sp}(2n, \mathbb{R})$

Definition 4.4.1. Let P, Q be transverse Lagrangians in $\mathbb{R}^{2n}$. We associate to them an antisymplectic involution σ_{PQ} defined using the splitting $\mathbb{R}^{2n} = P \oplus Q$:

$$\sigma_{PQ} : P \oplus Q \to P \oplus Q$$
$$(v, w) \mapsto (-v, w)$$

We call this antisymplectic involution the *reflection* in the pair P, Q. This generalizes the projective reflection in $\mathbb{RP}^1$.

We will sometimes abuse notation and use σ_{PQ} to denote the induced transformation on Grassmannians.

Using this involution, we associate a symmetric bilinear form to the pair P, Q :

Definition 4.4.2.

$$\mathcal{B}_{PQ}(v, w) := \omega(v, \sigma_{PQ}(w))$$

This bilinear form is nondegenerate and has signature (n, n).

Definition 4.4.3. Let P, Q, R be pairwise transverse Lagrangians in $\mathbb{R}^{2n}$. The *Maslov index* of the triple (P, Q, R) is the index of the restriction of $\mathcal{B}_{PR}$ to Q. We denote it by $\mathsf{M}(P, Q, R)$.

Remark 4.4.4. *This is a special case of Definition 4.3.13 which covered all Shilov boundaries, and it specializes to Definition 2.2.14 in the Einstein universe when* $n = 2$.

Since $\mathrm{Lag}(\mathbb{R}^{2n})$ is the Shilov boundary for the bounded domain realization of the symmetric space of $\mathrm{Sp}(2n, \mathbb{R})$, it is an example of the general construction in Section 4.3. In fact, the Maslov index we just defined agrees with the more general version that we introduced before. Hence, the relation defined by $\overrightarrow{PQR}$ whenever $\mathsf{M}(P, Q, R) = n$ is a partial cyclic order on $\mathrm{Lag}(\mathbb{R}^{2n})$, enabling us to apply the constructions and results from Section 4.2.

We also remark that the definition makes sense for any isotropic subspace Q, not only the maximal isotropic ones.

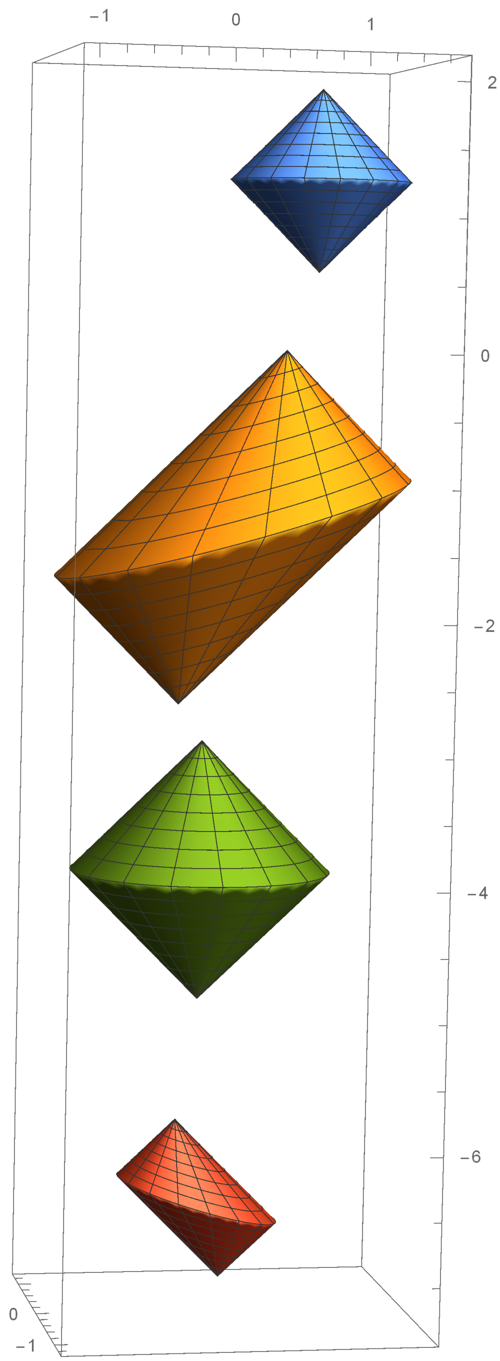


Figure 4.3: Four intervals between Lagrangians in increasing order.

The following property of the Maslov index is well-known.

Proposition 4.4.5. *The Maslov index classifies orbits of triples of pairwise transverse Lagrangians, i.e. the map*

$$(P, Q, R) \mapsto \mathsf{M}(P, Q, R)$$

induces a bijection from orbits of pairwise transverse Lagrangians under $\mathrm{Sp}(2n, \mathbb{R})$ *to the set* $\{-n, -n+2, \ldots, n\}$.

The Maslov index and the reflection in a pair of Lagrangians are related in the following way:

Proposition 4.4.6.

$$\mathsf{M}(P, \sigma_{PQ}(V), Q) = -\mathsf{M}(P, V, Q).$$

Proof.

$$\mathcal{B}_{PQ}\big(\sigma_{PQ}(u), \sigma_{PQ}(v)\big) = \omega(\sigma_{PQ}(u), v) = -\omega(u, \sigma_{PQ}(v)) = -\mathcal{B}_{PQ}(u, v). \qquad \square$$

The proposition above means that reflections reverse the partial cyclic order.

4.4.2 Fundamental domains

In the special case of $\mathrm{Sp}(2n, \mathbb{R})$, the Schottky groups we obtain admit nice fundamental domains for their action on $\mathbb{RP}^{2n-1}$. The domain of discontinuity which is the orbit of this fundamental domain is in general hard to describe, but it simplifies in some cases.

We will proceed as follows: First, we associate a "halfspace" in $\mathbb{RP}^{2n-1}$ to each interval in $\mathrm{Lag}(\mathbb{R}^{2n})$ and explain how to construct the fundamental domain. Then we cover some preliminaries which will allow us to explicitly identify the domain of discontinuity for generalized Schottky groups modeled on an infinite area hyperbolization without cusps. More specifically, we explain how to identify an interval with the symmetric space associated with $\mathrm{GL}(n, \mathbb{R})$ and how to use a contraction property from [Bou93] for maps sending one interval into another.

4.4.2.1 Positive halfspaces and fundamental domains

Definition 4.4.7. Let P, Q be an ordered pair of transverse Lagrangians. We define the *positive halfspace* $\mathcal{P}(P, Q)$ as the subset

$$\mathcal{P}(P, Q) := \{\ell \in \mathbb{RP}^{2n-1} \mid \mathcal{B}_{PQ}|_{\ell \times \ell} > 0\}.$$

It is the set of positive lines for the form $\mathcal{B}_{PQ}$.

The positive halfspace $\mathcal{P}(P, Q)$ is bounded by the conic defined by $\mathcal{B}_{PQ} = 0$. This type of bounding hypersurface was introduced by Guichard and Wienhard in order to describe Anosov representations of closed surfaces into $\mathrm{Sp}(2n, \mathbb{R})$. They are also the boundaries of $\mathbb{R}$-tubes defined in [BP15]. A symplectic linear transformation $T \in \mathrm{Sp}(2n, \mathbb{R})$ acts on positive halfspaces in the following way : $T\mathcal{P}(P, Q) = \mathcal{P}(TP, TQ)$.

Proposition 4.4.8. *Let P, Q be an ordered pair of Lagrangians. Then,*

$$\mathcal{P}(Q, P) = \overline{\mathcal{P}(P, Q)}^{c} = \sigma_{PQ}(\mathcal{P}(P, Q))$$

Proof. For the first equality,

$$\mathcal{B}_{QP}(v,w) = \omega(v,\sigma_{QP}(w)) = \omega(v,-\sigma_{PQ}(w)) = -\mathcal{B}_{PQ}(v,w).$$

For the second equality, notice that $\mathcal{B}_{PQ}(\sigma_{PQ}(v),\sigma_{PQ}(w)) = -\mathcal{B}_{PQ}(v,w)$. □

Proposition 4.4.9. *A positive halfspace is the projectivisation of an interval, that is,*

$$\mathcal{P}(P,Q) = \bigcup_{L\in(P,Q)} \mathbb{P}(L)$$

Proof. If $\ell \subset L$ for some $L \in (P,Q)$, then

$$\mathcal{B}_{PQ}|_{\ell\times\ell} > 0$$

and so $\ell \in \mathcal{P}(P,Q)$.

Conversely, if $\ell \in \mathcal{P}(P,Q)$, then we wish to find a Lagrangian $L \supset \ell$ with $\mathsf{M}(P,L,Q) = n$. Consider the subspace $V = \langle \ell, \sigma_{PQ}(\ell)\rangle$. The form $\mathcal{B}_{PQ}$ has signature $(1,1)$ on that subspace, and so its orthogonal has signature $(n-1,n-1)$. Moreover, the form ω is nondegenerate on V so $V^{\perp_\omega}$ is a symplectic subspace. Notice that

$$V^{\perp_\mathcal{B}} = \langle \ell, \sigma_{PQ}(\ell)\rangle^{\perp_\mathcal{B}} = \ell^{\perp_\mathcal{B}} \cap (\sigma_{PQ}(\ell))^{\perp_\mathcal{B}} = \ell^{\perp_\mathcal{B}} \cap \ell^{\perp_\omega} = V^{\perp_\omega}.$$

So we can pick a positive definite Lagrangian $L' \subset V^\perp$, which will be orthogonal to ℓ for both ω and $\mathcal{B}_{PQ}$, so $L = \langle L', \ell\rangle$ is a positive definite Lagrangian containing ℓ. □

Lemma 4.4.10. *If (P,Q,R,S) is a cycle in $\mathrm{Lag}(\mathbb{R}^{2n})$ and $V \in \mathrm{Lag}(\mathbb{R}^{2n})$ such that $\mathsf{M}(P,V,Q) = n$, then $\mathsf{M}(R,V,S) = -n$.*

Proof. Using the cocycle relation,

$$\mathsf{M}(V,Q,R) - \mathsf{M}(P,Q,R) + \mathsf{M}(P,V,R) - \mathsf{M}(P,V,Q) = 0$$

so

$$\mathsf{M}(V,Q,R) + \mathsf{M}(P,V,R) = 2n$$

which implies that $\mathsf{M}(V,Q,R) = \mathsf{M}(P,V,R) = n$.

Similarly,

$$\mathsf{M}(V,R,S) - \mathsf{M}(P,R,S) + \mathsf{M}(P,V,S) - \mathsf{M}(P,V,R) = 0$$

so

$$\mathsf{M}(V,R,S) + \mathsf{M}(P,V,S) = 2n$$

which means that $\mathsf{M}(V,R,S) = n$ and so $\mathsf{M}(R,V,S) = -n$. □

Now we can prove the disjointness criterion for positive halfspaces.

Proposition 4.4.11. *If (P,Q,R,S) is a cycle in $\mathrm{Lag}(\mathbb{R}^{2n})$, then $\mathcal{P}(P,Q)$ is disjoint from $\mathcal{P}(R,S)$.*

Proof. Let $\ell \in \mathcal{P}(P,Q)$. By Proposition 4.4.9, $\ell \subset L$ for some Lagrangian L with $\mathsf{M}(P,L,Q) = n$. By Lemma 4.4.10, $\mathsf{M}(R,L,S) = -n$ which means that $\mathcal{B}_{RS}|_\ell < 0$ and so $\ell \notin \mathcal{P}(R,S)$. □

For any generalized Schottky group, we can use this previous proposition to construct a fundamental domain. If the defining intervals for the Schottky group are $(a_1^\pm, b_1^\pm), \ldots, (a_g^\pm, b_g^\pm) \subset \mathrm{Lag}(\mathbb{R}^{2n})$, let

$$D = \bigcap_{j=1}^{g} \left(\mathcal{P}(a_j^+, b_j^+) \cup \mathcal{P}(a_j^-, b_j^-)\right)^C.$$

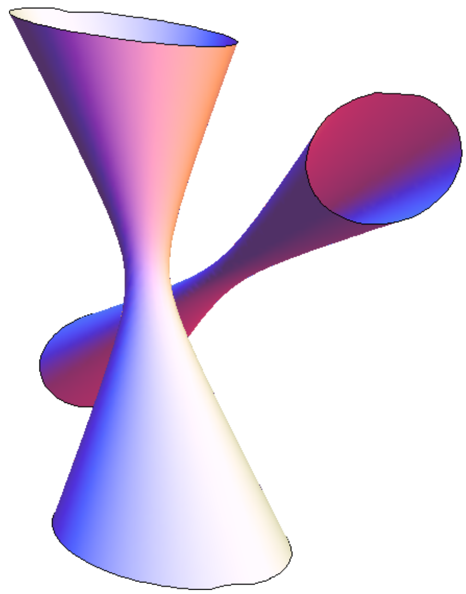


Figure 4.4: A pair of disjoint positive halfspaces in $\mathbb{RP}^3$

That is, D is the subset of $\mathbb{RP}^{2n-1}$ which is the complement of the positive halfspaces defined by each interval. It is a closed subset since each positive halfspace is open. The interiors of the translates of D are all disjoint by the two previous propositions and the boundary components are identified pairwise, so D is a fundamental domain for its orbit (Fig. 4.5). This orbit is in general hard to describe, but in some cases we can identify it precisely.

In the definition of generalized Schottky subgroups, we required that the model be a finite area hyperbolization. This is an artificial requirement which made the analysis of maximal representations simpler. In what follows, we will assume that the model Schottky group acting on $\mathbb{RP}^1$ is defined by intervals with disjoint closures, so it corresponds to an infinite area hyperbolization. The advantage of using intervals with disjoint closures lies in the contraction property proven in [Bou93] which we will exploit later.

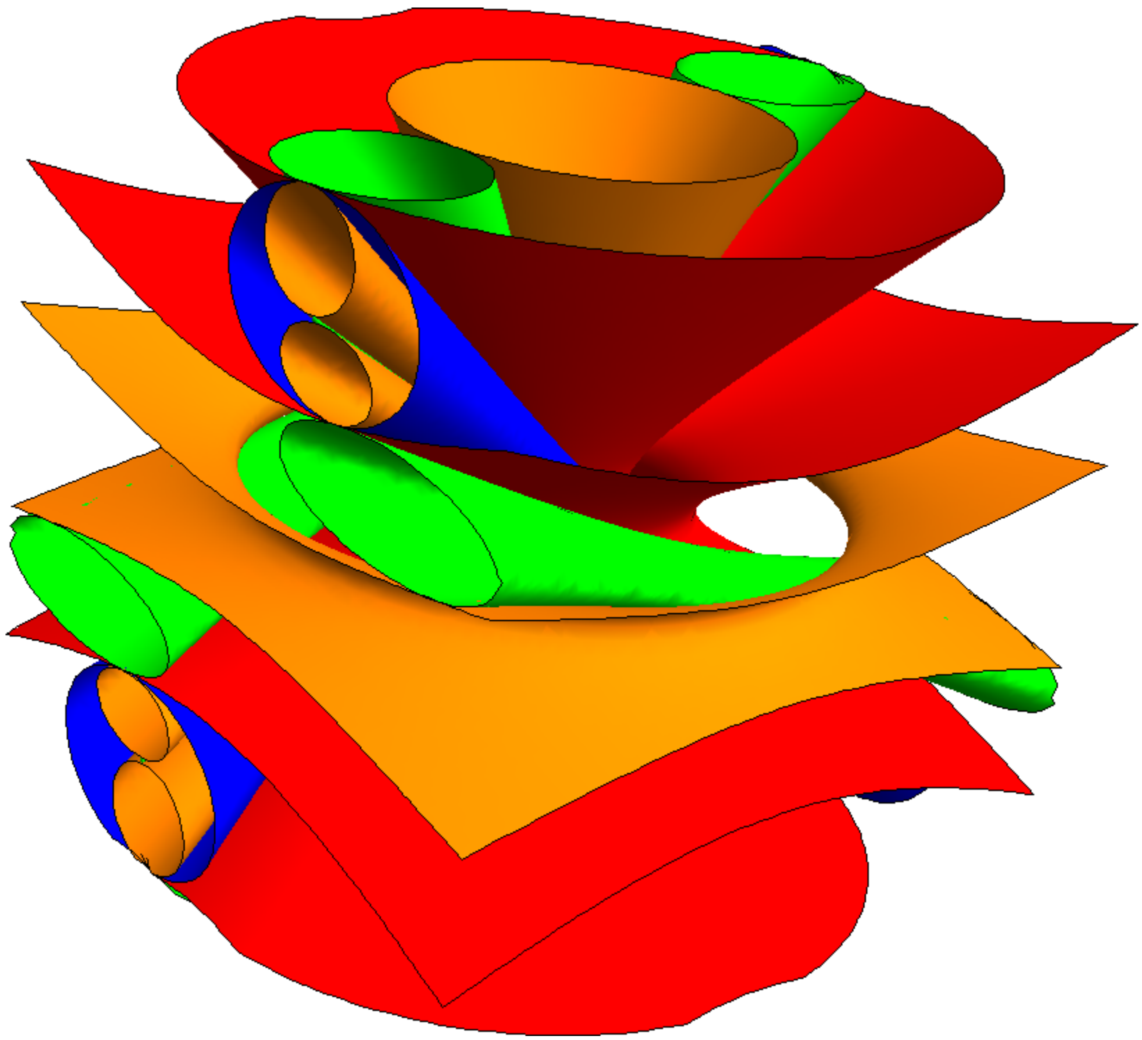


Figure 4.5: The first two generations of positive halfspaces for a two-generator Schottky group in $\mathrm{Sp}(4, \mathbb{R})$.

4.4.2.2 Intervals as symmetric spaces

We will now describe how to identify an interval in $\mathrm{Lag}(\mathbb{R}^{2n})$ with the symmetric space associated with $\mathrm{GL}(n,\mathbb{R})$, endowing any interval with a Riemannian metric.

Let $P, Q \in \mathrm{Lag}(\mathbb{R}^{2n})$ be two transverse Lagrangians. As we saw earlier in Corollary 4.3.16, all Lagrangians in the interval (P, Q) have to be transverse to Q, so they are graphs of linear maps $f : P \to Q$. The isotropy condition on f is given by

$$\omega(v + f(v), v' + f(v')) = \omega(v, f(v')) + \omega(f(v), v') = 0 \quad \forall v, v' \in P.$$

Now we recall from our discussion of the Maslov index that we can associate the bilinear form

$$\mathcal{B}_{PQ} : P \oplus Q \to \mathbb{R}$$
$$(v, w) \mapsto \omega(v, \sigma_{PQ}(w))$$

to this splitting, and the index of its restriction to $\mathsf{graph}(f)$ is the Maslov index $\mathsf{M}(P, \mathsf{graph}(f), Q)$. We observe that this restriction is given by

$$\mathcal{B}_{PQ}(v + f(v), v' + f(v')) = \omega(v, f(v')) - \omega(f(v), v') = 2\omega(v, f(v')),$$

where the last equation follows from the isotropy condition on f. This bilinear form on $\mathsf{graph}(f)$ can also be seen as a symmetric bilinear form on P. Maximality of the Maslov index then translates to this form being positive definite.

Conversely, given a symmetric bilinear form b on P, we obtain, for any $v' \in P$, a linear functional

$$\left(v \mapsto \frac{1}{2}b(v,v')\right) \in P^*.$$

Using the isomorphism

$$Q \to P^*$$

$$w \mapsto \omega(\cdot, w),$$

we see that there is a unique vector $f(v') \in Q$ such that $b(v,v') = 2\omega(v, f(v'))$ $\forall v \in P$. This uniquely defines a linear map $f : P \to Q$, and

$$2\left(\omega(v, f(v')) + \omega(f(v), v')\right) = b(v,v') - b(v',v) = 0,$$

so $\mathsf{graph}(f)$ is a Lagrangian. The Maslov index $\mathsf{M}(P, \mathsf{graph}(f), Q)$ is maximal if and only if b is positive definite. This identifies (P,Q) with the space of positive definite symmetric bilinear forms on P, which is the symmetric space of $\mathrm{GL}(P)$.

The stabilizer in $\mathrm{Sp}(2n,\mathbb{R})$ of the pair (P,Q) can be identified with $\mathrm{GL}(P)$ since any element $A \in \mathrm{GL}(P)$ uniquely extends to a linear symplectomorphism of $\mathbb{R}^{2n}$ fixing Q: The linear forms $v \mapsto \omega(A(v), w)$ on P, for $w \in Q$, give rise to a unique automorphism $A^* : Q \to Q$ such that

$$\omega(A(v), w) = \omega(v, A^*(w)).$$

Then $A \oplus (A^*)^{-1}$ is the unique symplectic extension of A fixing Q; we abuse notation slightly and denote it by A as well. It acts on graphs $f : P \to Q$ by

$$f \mapsto AfA^{-1},$$

and on bilinear forms on P by

$$(A \cdot b)(v, v') = b(A^{-1}v, A^{-1}v').$$

The identification of graphs and bilinear forms is equivariant with respect to these actions. In particular, $\mathrm{Stab}_{\mathrm{Sp}(2n,\mathbb{R})}(P, Q)$ identifies with the isometry group of the symmetric space (P, Q).

4.4.2.3 The Riemannian distance on intervals

Here is a simple formula for the Riemannian distance between two points in the interval (P, Q):

Definition 4.4.12. Let f, g be linear maps from P to Q whose graphs are elements of (P, Q). Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the automorphism fg^{-1}. Then, define

$$d_{PQ}(f, g) = \sqrt{\sum_{i=1}^{n} \log(\lambda_i)^2}.$$

The following useful proposition is proved in [Bou93].

Proposition 4.4.13. *Let $T \in \mathrm{Sp}(2n, \mathbb{R})$ such that $\overline{T(P, Q)} \subset (P, Q)$. Then, T is a Lipschitz contraction for the distance d_{PQ}.*

Corollary 4.4.14. *Let $T \in \mathrm{Sp}(2n, \mathbb{R})$ such that $\overline{T(P, Q)} \subset (R, S)$. Then, for any $X, Y \in (P, Q)$,*

$$d_{RS}(TX, TY) \leq C d_{PQ}(X, Y)$$

for some constant $0 < C < 1$.

Let us now prove the main lemma for the description of domains of discontinuity. Let $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$ define a generalized Schottky group in $\mathrm{Sp}(2n, \mathbb{R})$. Assume that the model Γ is defined by intervals with distinct endpoints, so that the intervals in $\mathrm{Lag}(\mathbb{R}^{2n})$ have disjoint closures.

Lemma 4.4.15. *Let $\gamma \in \rho(\Gamma)$ be a word of reduced length ℓ in the generators T_i and their inverses, with first letter T and last letter S. We denote their attracting and repelling intervals by $I^\pm$ and $J^\pm$. Then, for any Schottky interval $K \neq J^-$ and $X, Y \in K$,*

$$d_{I^+}(\gamma(X), \gamma(Y)) < C^\ell d_K(X, Y)$$

for some $0 < C < 1$ depending only on the set of generators.

Proof. Since a generator T_k maps the interval $-I_k^-$ into I_k^+, we can consider it as a map from any Schottky interval $L \neq I_k^-$ into I_k^+. All of these maps are Lipschitz contractions by Corollary 4.4.14.

Now let C be the maximum Lipschitz constant of all such maps, for $1 \leq k \leq 2g$. We have

$$d_{J^+}(SX, SY) < C d_K(X, Y).$$

Composing contractions, we obtain

$$d_{I^+}(\gamma(X), \gamma(Y)) < C^\ell d_K(X, Y).$$

□

4.4.2.4 The domain of discontinuity

Now we analyze the orbit $\rho(\Gamma) \cdot D$ of the fundamental domain $D \subset \mathbb{RP}^{2n-1}$ which was defined in Section 4.4.2.1. Using Lemma 4.4.15, we first define a map from the boundary of Γ, which is a Cantor set, into $\mathrm{Lag}(\mathbb{R}^{2n})$.

Proposition 4.4.16. *Let $L \in \bigcap_{j=1}^{g}(-I_j^+ \cap -I_j^-)$ where $I_j^\pm$ are the defining intervals of the generalized Schottky group. The evaluation map $\eta_0(\gamma) = \rho(\gamma)(L)$ induces an injective map $\eta : \partial\Gamma \to \mathrm{Lag}(\mathbb{R}^{2n})$ independent of the choice of L. Moreover, η is continuous and increasing.*

Proof. Let $x \in \partial\Gamma$ be a boundary point. Then x corresponds to a unique infinite sequence in the generators T_i and their inverses, where this sequence is reduced in the sense that no letter is followed by its inverse. We denote by $x^{(k)} \in \Gamma$ the word consisting of the first k letters of x. Then the map η will be defined by taking the limit

$$\eta(x) = \lim_{k\to\infty} \rho(x^{(k)})(L).$$

Let us first check that this limit does in fact exist. Recall that we introduced k-th order intervals and a bijection between words of length k and k-th order intervals in Section 4.2. By the specific choice of L, its image $\rho(x^{(k)})(L)$ has to lie in the interval $I_{x^{(k)}}$ corresponding to the word $x^{(k)}$. Since the first k letters of any word $x^{(m)}$, $m > k$ agree with $x^{(k)}$, the intervals $I_{x^{(k)}}$ form a nested sequence. Now we want to make use of the contraction property from the previous subsection. We first observe that since our model uses Schottky intervals with disjoint closures, second order intervals are relatively compact subsets of first order intervals. Let $I^{(2)} \subset I^{(1)}$ be such a configuration. Since the number of second order intervals is finite, there is a uniform bound M such that

$$\mathrm{diam}_{I^{(1)}}(I^{(2)}) < M,$$

where we used the metric on the symmetric space $I^{(1)}$. Then, denoting the first letter of x by T, Lemma 4.4.15 tells us that

$$\mathrm{diam}_{I_T}(I_{x_k}) < MC^{k-2}.$$

This contracting sequence of nested subsets of the symmetric space I_T thus has a unique limit, and η is well-defined. By the same argument, we see that this limit does not depend on the choice of L.

We now show continuity of η. Let $y_n \to x$ be a sequence in $\partial\Gamma$ converging to x. This implies that for any $N \in \mathbb{N}$, we can find n_0 such that for all $n \geq n_0$, the first N letters of y_n and x agree. In this situation, $\eta(y_n)$ and $\eta(x)$ lie in the same interval $I_{x^{(N)}}$ and so we conclude, if the first letter of x is T, that

$$d_{I_T}(\eta(x), \eta(y_n)) < MC^{N-2}.$$

Finally, we prove positivity in a similar way to Theorem 4.2.7. For any $x, y, z \in \partial\Gamma$ such that $\overrightarrow{xyz}$ (where we use the natural embedding of $\partial\Gamma$ in S^1 to get the cyclic order) we can find a large enough K so that I_{x_K},I_{y_K} and I_{z_K} have disjoint closures. But since the cyclic relations on k-th order intervals are the same in S^1 as in $\mathrm{Lag}(\mathbb{R}^{2n})$, for any $P \in I_{x_K}, Q \in I_{y_K}, R \in I_{z_K}$ we have $\overrightarrow{PQR}$. In particular, $\overrightarrow{\eta(x)\eta(y)\eta(z)}$. □

Remark 4.4.17. *The map η that we define is related to the map ξ of Theorem 4.2.7. In this case, the endpoints of k-th order intervals are not dense in S^1, so we cannot get a map on the whole circle. However, because the intervals have disjoint closures, we get continuity on both sides rather than just left-continuity.*

The next lemma relates the construction of the limit map η with the positive halfspaces that intervals define in $\mathbb{RP}^{2n-1}$.

Lemma 4.4.18. *Let L_1^k, L_2^k be sequences of Lagrangians such that $L_1^k \to L$ and $L_2^k \to L$ with $\overrightarrow{L_1^k L L_2^k}$ for all k. Then,*

$$\bigcap_{k=1}^{\infty} \overline{\mathcal{P}(L_1^k, L_2^k)} = \bigcap_{k=1}^{\infty} \mathcal{P}(L_1^k, L_2^k) = \mathbb{P}(L).$$

Proof. Assume $\mathcal{B}_{L_1^k L_2^k}(v,v) \geq 0$ for all k. Then we can find $v_k \xrightarrow{k\to\infty} v$ such that $\mathcal{B}_{L_1^k L_2^k}(v_k, v_k) > 0$ for all k. Now, by Proposition 4.4.9, v_k can be completed to a Lagrangian L^k with $\mathsf{M}(L_1^k, L^k, L_2^k) = n$, so $L^k \subset (L_1^k, L_2^k)$ for all k, which implies $L^k \to L$, and so $v \in L$. □

Now we are ready to describe the orbit $\rho(\Gamma)D$.

The union of D with the positive halfspaces defining the Schottky group is all of $\mathbb{RP}^{2n-1}$, by definition of D. Denote by Γ_ℓ the set of words in Γ of length up to ℓ. Then, the union of $\rho(\Gamma_\ell)D$ with the projectivizations (positive halfspaces) of all ℓ-th order intervals again covers all of $\mathbb{RP}^{2n-1}$. Thus, when taking words of arbitrary length in Γ, these two pieces become respectively the full orbit $\rho(\Gamma)D$ and limits of nested positive halfspaces, which by Lemma 4.4.18 collapse to the projectivization of a single Lagrangian. We conclude:

Theorem 4.4.19. *The orbit $\rho(\Gamma)D$ is the complement of a Cantor set of projectivized Lagrangian n-planes in $\mathbb{RP}^{2n-1}$. This Cantor set is exactly the projectivization of the increasing set of Lagrangians defined by the boundary map η.*

Remark 4.4.20. *The symplectic structure on $\mathbb{R}^{2n}$ induces a contact structure on*

$\mathbb{RP}^{2n-1}$ *preserved by the symplectic group. The projectivizations of Lagrangian subspaces correspond to* Legendrian $(n-1)$*-dimensional planes in* $\mathbb{RP}^{2n-1}$.

4.5 Oriented flags in three dimensions

The theory of *positive configurations* of flags of Fock and Goncharov [FG06] hints at the existence of a partial cyclic order on the space of flags in $\mathbb{R}^n$.

Since positivity of triples of flags is preserved under all permutations, we have to look at oriented flags. The space of *oriented flags* in $\mathbb{R}^n$ admits a partial cyclic order, with some care needed when n is even. For ease of exposition, we describe this ordering for $n = 3$ and the resulting Schottky groups. We treat the general case in the work in progress [BT17].

4.5.1 Hyperconvex configurations

Definition 4.5.1. An *oriented flag* in $\mathbb{R}^3$ is a sequence of subspaces $\ell \subset P \subset \mathbb{R}^3$ together with a choice of orientation on each subspace. We will sometimes denote and oriented flag by the pair (ℓ, P).

Definition 4.5.2. A *basis* for an oriented flag (ℓ, P) is an oriented basis e_1, e_2, e_3 of $\mathbb{R}^3$ such that e_1 is an oriented basis for ℓ and e_1, e_2 is an oriented basis for P.

If we fix an oriented basis B of $\mathbb{R}^3$, we can also denote an oriented flag F by a 3×3 matrix whose columns are the coordinates in the basis B for a flag basis of F.

We denote by $\mathcal{F}$ the space of oriented flags.

Fix a choice of orientation on $\mathbb{R}^3$. Throughout this section, we will be considering *oriented* vector spaces. The direct sum operation can be applied to oriented vector spaces, with the orientation on the sum being given by the concatenation of two oriented bases.

Definition 4.5.3. A pair of oriented flags $(\ell, P), (\ell', P')$ is called *transverse* if the following condition holds :

$$\ell \oplus P' = P \oplus \ell' = \mathbb{R}^3,$$

where equality is understood in terms of *oriented* vector spaces.

Proposition 4.5.4. *Transversality of oriented flags is symmetric. That is, if F_1, F_2 are transverse, then F_2, F_1 are transverse.*

Proof. Let e_1, e_2, e_3 and e_1', e_2', e_3' be oriented bases for F_1, F_2, respectively. Assume F_1, F_2 are transverse. Then, F_2, F_1 are transverse if the following two bases are oriented :

$$e_1', e_2, e_3,$$

$$e_2', e_3', e_1.$$

But these bases are cyclic permutations of e_2, e_3, e_1' and e_1, e_2', e_3' which are oriented bases by transversality of F_1, F_2. □

Example 4.5.5. Let e_1, e_2, e_3 be the canonical oriented basis of $\mathbb{R}^3$. The oriented flags $([e_1], [e_1] \oplus [e_2])$ and $([e_3], [e_2] \oplus [e_3])$ are transverse. If we switch the orientation on the 2-plane of the second flag, they are not considered transverse anymore since the orientation given by (e_1, e_3, e_2) does not coincide with the fixed orientation.

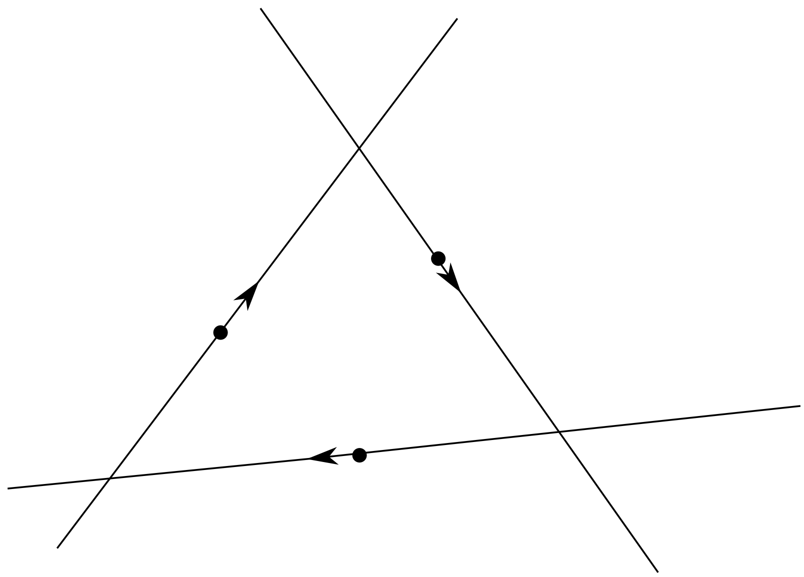


Figure 4.6: A hyperconvex configuration of three oriented flags, projected to the 2-sphere of directions.

Definition 4.5.6. A triple of oriented flags (F_1, F_2, F_3) is called *hyperconvex* if F_1, F_2, F_3 are pairwise transverse and the following equality of oriented vector spaces holds :

$$\ell_1 \oplus \ell_2 \oplus \ell_3 = \mathbb{R}^3.$$

The group $\mathrm{SL}(3, \mathbb{R})$ acts on $\mathcal{F}$ preserving transversality and hyperconvexity of triples, since it consists of orientation-preserving linear transformations.

Proposition 4.5.7. $\mathrm{SL}(3, \mathbb{R})$ *acts transitively on pairs of transverse flags in* $\mathcal{F}$.

Proof. Let F_1, F_2 be a pair of transverse flags. Let A_1, A_2 be 3×3 matrices representing F_1, F_2 in the standard basis. Since $\det(A_1), \det(A_2) > 0$ we can multiply by a positive scalar and assume $A_1, A_2 \in \mathrm{SL}(3, \mathbb{R})$. Multiply the pair by A_1^{-1} to get $I, A_1^{-1}A_2$.

Denote $B = A_1^{-1}A_2$. The stabilizer of the flag I is the set of upper triangular matrices with positive entries on the diagonal. Transversality implies that there is a choice of basis for the flag F_2 which makes B a matrix of the form

$$\begin{pmatrix} a & b & 1 \\ c & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which we can send using a unipotent upper triangular matrix to the standard

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We will denote this standard matrix by $\bar{I}$. □

We now show that hyperconvexity gives a partial cyclic order on the set of oriented flags in $\mathbb{R}^3$. Cyclicity and asymmetry follow from the fact that permuting a basis changes its orientation according to the sign of the permutation. It remains to show that this order is transitive.

Proposition 4.5.8. *Let*

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

The triple of flags given by matrices $I, A\bar{I}, \bar{I}$ is hyperconvex if and only if $a, b, c > 0$ and $ac - b > 0$. Such a matrix A is called totally positive.

Proof. The transversality of the second and third flags is equivalent to the positivity of the determinants

$$\det \begin{pmatrix} 0 & b & -a \\ 0 & c & -1 \\ 1 & 1 & 0 \end{pmatrix} = ac - b$$

$$\det \begin{pmatrix} 0 & 0 & b \\ 0 & -1 & c \\ 1 & 0 & 1 \end{pmatrix} = b.$$

The hyperconvexity of the triple is then equivalent to the positivity of the determinant

$$\det \begin{pmatrix} 1 & b & 0 \\ 0 & c & 0 \\ 0 & 1 & 1 \end{pmatrix} = c.$$

The condition $a > 0$ is redundant and included for symmetry.

□

A simple calculation shows the following proposition.

Proposition 4.5.9. *The set of totally positive matrices forms a subsemigroup of the group of unipotent upper triangular matrices.*

Proposition 4.5.10 (Transitivity)**.** *If* (F_1, F_2, F_3) *is hyperconvex and* (F_1, F_3, F_4) *is hyperconvex, then* (F_1, F_2, F_4) *is hyperconvex.*

Proof. Without loss of generality, F_1 is given by the identity matrix and F_3 is given by $\bar{I}$. By transversality of F_1, F_2 and F_1, F_4 we can write $F_2 = A\bar{I}$ and $F_4 = B\bar{I}$ where A, B are unipotent upper triangular matrices.

Since (F_1, F_2, F_3) is hyperconvex, Proposition 4.5.8 shows that A is totally positive. Similarly, since (F_1, F_3, F_4) is hyperconvex, we know that $(I, B^{-1}\bar{I}, \bar{I})$ is hyperconvex and so B^{-1} is totally positive. Since totally positive matrices form a semigroup, $B^{-1}A$ is also totally positive, which means that $(I, B^{-1}A\bar{I}, \bar{I})$ is hyperconvex. Left multiplying by B throughout, we conclude that $(I, A\bar{I}, B\bar{I}) = (F_1, F_2, F_4)$ is hyperconvex. □

As so we have proven

Theorem 4.5.11. *The hyperconvexity relation on triples of oriented flags is a partial cyclic order. We will denote it by* $\overrightarrow{F_1F_2F_3}$ *as in the previous sections.*

We proved in Proposition 4.5.8 that the interval between two oriented flags (F_1, F_2) can be identified with the open set of totally positive, unipotent, upper triangular 3×3 matrices. These sets are homeomorphic to balls in the space of oriented flags.

By similar arguments to those of Theorem 4.3.19, we obtain the following:

Theorem 4.5.12. *The PCO on the space of oriented flags* $\mathcal{F}$ *is increasing-complete, full and proper (it satisfies the hypotheses of Theorem 4.2.7).*

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