ABSTRACT

Title of dissertation: USER BEHAVIOR ANALYSIS AND DATA TRADING IN MULTI-AGENT SYSTEMS

Xuanyu Cao, Doctor of Philosophy, 2017

Dissertation directed by: Professor K. J. Ray Liu
Department of Electrical and Computer Engineering

The advancement of complex social systems such as Facebook and Twitter has led to huge volume of user generated contents, which enable detailed tracking and characterization of human activities. Users of these systems interact with each other to make decisions in events such as information dissemination and to learn knowledge such as rating of online businesses. Quantitative analysis and comprehension of mechanisms of users' behaviors in these systems are both intriguing and imperative in many academic fields (e.g., economics and social/political sciences) and applications (e.g., online advertisement and management of electronic commerce). In addition, due to the high commercial and research values of these user generated data for many individuals and companies, various data trading platforms are emerging to facilitate data transactions and to extract remarkable profits from the data markets, yet few methodological trading schemes are available in the literature.

Therefore, in this dissertation, we are motivated to examine users' behaviors during several learning and decision making processes in networked systems and to design an efficient data trading mechanism systematically for markets with multiple data agents. Specifically, we first propose a graphical evolutionary game theoretic framework for information propagation over heterogeneous networks and analytically study the dynamics and stable states of the game. Theoretical results are corroborated by numerical experiments on real-world information diffusion data. Secondly, to incorporate users' long-term incentives, we propose a sequential game to model the decision-making procedures in generic popularity dynamics. Properties of the symmetric Nash equilibrium of the game are theoretically analyzed and match well with empirical observations from real world popularity dynamics such as information diffusion dynamics and paper citation dynamics. Thirdly, an evolutionary game theoretic learning algorithm is proposed for the social learning problem, where networked agents collaborate to detect some unknown system state. Theoretical analysis manifests that the stable states of the proposed distributed learning algorithm coincide with the decisions of a fictitious centralized detector. Lastly, we investigate the data trading problem in a market with multiple data owners, collectors and users. An efficient data trading mechanism based on iterative auctions is presented and we demonstrate that the mechanism converges to the socially optimal operation point and possesses appealing economic properties. Numerical studies based on data prices of real-world data transaction platforms are shown to verify the effectiveness of the proposed trading mechanisms.

USER BEHAVIOR ANALYSIS AND DATA TRADING IN MULTI-AGENT SYSTEMS

by

Xuanyu Cao

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Advisory Committee:
Professor K. J. Ray Liu, Chair/Advisor
Professor Richard J. La
Professor Behtash Babadi
Professor Min Wu
Professor Lawrence C. Washington

Dedication

To my parents.

Acknowledgments

First, I am indebted to my advisor, Professor K. J. Ray Liu, for his persistent support and guidance for both my research and my career development. He inspired me and gave me freedom to work on diverse fields of problems, which broadened my views in optimization, game theory and signal processing.

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Different from most other PhD graduates nowadays, whose academic journeys terminate after their dissertations are done, my dissertation is only an inception of my academic expedition. Will this expedition lead to a successful career? I don't know. All I can do is to keep stumbling in a seemingly correct direction.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Background and Motivation

The proliferation of ubiquitous social networks and social medias such as Facebook, Twitter and Pinterest enables convenient and intimate interactions among humans. Unlike traditional face-to-face communications, these online interactions are well recorded in digital formats and the pertinent user generated contents can be collected and trimmed by exploiting techniques such as web scraping or data crawling. These large-scale user related data provide us with unprecedented opportunities to closely investigate human users' behaviors and more importantly, the underlying complex mechanisms dominating users' learning and decision-making processes, which permeate lots of social phenomena. For instance, when a piece of information is generated in a social network, the dissemination of the information over the network is the consequence of numerous online users' interaction and decision-making (e.g., should I forward this piece of information?). As another example, in recommendation systems such as Yelp and Groupon, the ratings of businesses and products are determined by vast amount of users' learning procedures based on both internal feelings and external influences. Quantitative analysis and comprehension of the principles of social network users' behaviors in learning and decision-making processes are intriguing and crucial in many academic fields (e.g., economics and social/political sciences) and applications (e.g., online advertisement and management of electronic commerce) [24].

As most users are influenced by actions and opinions of other peers, especially their friends, the learning and decision-making procedures often involve competitions and collaborations among multiple agents. Hence, game theory is an ideal mathematical tool to examine the intricate interactions between the intelligent and strategic users, whose goals are to maximize their own benefits. Thus, in this dissertation, we are motivated to invoke various game-theoretic concepts and frameworks to study the mechanisms of users' behaviors in several learning and decision-making scenarios. Specifically, we address the following issues from game-theoretic perspectives.

- When a piece of information is propagating over a heterogeneous social network comprised of users with different hobbies and influences, every user needs to make a decision on whether to forward/mention this information or not. A natural question is how do users learn from each other to make decisions related to the information diffusion processes.

- When making decisions in the formation of popularity dynamics (e.g., information diffusion dynamics and paper citation dynamics), apart from instataneous costs and benefits, users may also have long-term incentives regarding potential prospects in the future. A key challenge is how to incorporate the notion of long-term incentives and examine its impact on users' behaviors in the system.

- In social learning, networked agents collaborate to detect some unknown system state, e.g., the quality of products and services in recommendation systems. Here, the goal is to design a distributed learning algorithm while respecting the strategic considerations of the agents.

Furthermore, the user generated contents, or more generally, big data collected by any device or system, are highly valuable resources for either commercial goals or research purposes and are desired by many individuals and companies to perform various data analytics. For example, a startup may need some data about customers' feedback on its products or services (e.g., click rates of certain websites) to enhance the quality of the businesses. However, the startup may lack the necessary capability and professions to collect and pre-process these data. Thereby, it may resort to some professional data collectors (e.g., computer scientists or engineers good at online data scraping) to help collect the data needed or directly purchase data from some data owners (e.g., social networks like Facebook or mobile service provides like Verizon who have enormous amount of useful data). Due to the pressing needs of data transactions in practice, several data trading platforms are emerging recently such as Big Data Exchange, Data Marketplace and Miscrosoft Azure Marketplace, whereas no methodological data trading scheme exists in the research literature. Therefore, in this dissertation, we endeavor to design an efficient data trading mechanism systematically, which possesses both rigorous theoretical guarantees and competitive empirical performance on data prices in real-world data trading platforms. Through both theoretical analysis and numerical experiments, we demonstrate that the proposed mechanism achieves socially optimal operation point and economically incentivizes every data agent to participate.

1.2 Outline of the Dissertation

The rest of this dissertation mainly consists of three works on game-theoretic user behavior analysis [14, 16, 17] and one work on efficient design of data trading schemes [15]. Specifically, the remaining part of the dissertation is organized as follows.

1.2.1 Evolutionary Information Diffusion over Heterogeneous Social Networks (Chapter 2)

A huge amount of information, created and forwarded by millions of people with various characteristics, is propagating through the online social networks every day. Understanding the mechanisms of the information diffusion over the social networks is critical to various applications including online advertisement and website management. Different from most of the existing works, we investigate the information diffusion from an evolutionary game-theoretic perspective and try to reveal the underlying principles dominating the complex information diffusion process over the heterogeneous social networks. Modeling the interactions among the heterogeneous users as a graphical evolutionary game, we derive the evolutionary dynamics and the evolutionarily stable states (ESSs) of the diffusion. The different payoffs of the heterogeneous users lead to different diffusion dynamics and ESSs among them, in accordance with the heterogeneity observed in real-world datasets. The theoretical results are confirmed by simulations. We also test the theory on Twitter hashtag dataset. We observe that the derived evolutionary dynamics fit the data well and can predict the future diffusion data. The results of this chapter are based on our work in [14].

1.2.2 Understanding Popularity Dynamics: Decision-Making with Long-Term Incentives (Chapter 3)

With the explosive growth of big data, human's attention has become a scarce resource to be allocated to the vast amount of data. Numerous items such as online memes, videos are generated everyday, some of which go viral, i.e., attract lots of attention, while most diminish quickly without any influence. The recorded people's interactions with these items constitute a rich amount of *popularity dynamics*, e.g., hashtags' mention count dynamics, which characterize human behaviors quantitatively. It is crucial to understand the underlying mechanisms of popularity dynamics in order to utilize the valuable attention of people efficiently. In this chapter, we propose a game-theoretic model to analyze and understand popularity dynamics. The model takes into account both the instantaneous incentives and long-term incentives during people's decision making process. We theoretically prove that the proposed game possesses a unique symmetric Nash equilibrium (SNE), which can be computed via a backward induction algorithm. We also analyze the equilibrium behavior of the proposed game, which is shown to match well with some observations from real-world popularity dynamics. Finally, by using simulations as well as experiments based on real-world popularity dynamics data, we validate the effectiveness of the theory. We find that our theory can fit the real data well and also predict the future dynamics. The results of this chapter are based on our work in [16].

1.2.3 A Graphical Evolutionary Game Approach to Social Learning (Chapter 4)

In this chapter, we study the social learning problem, in which agents of a networked system collaborate to detect the state of the nature based on their private signals. A novel distributed graphical evolutionary game theoretic learning method is proposed. In the proposed game-theoretic method, agents only need to communicate their binary decisions rather than the real-valued beliefs with their neighbors, which endows the method with low communication complexity. Under mean field approximations, we theoretically analyze the steady state equilibria of the game and show that the evolutionarily stable states (ESSs) coincide with the decisions of the benchmark centralized detector. Numerical experiments are implemented to confirm the effectiveness of the proposed game-theoretic learning method. The results of this chapter are based on our work in [17].

1.2.4 Data Trading with Multiple Owners, Collectors and Users: An Iterative Auction Mechanism (Chapter 5)

In the big data era, it is vital to allocate the vast amount of data to heterogeneous users with different interests. To clinch this goal, various agents including data owners, collectors and users should cooperate to trade data efficiently. However, the data agents (data owners, collectors and users) are selfish and seek to maximize their own utilities instead of the overall system efficiency. As such, a sophisticated mechanism is imperative to guide the agents to distribute data efficiently. In this chapter, the data trading problem of a data market with multiple data owners, collectors and users is formulated and an iterative auction mechanism is proposed to coordinate the trading. The proposed mechanism guides the selfish data agents to trade data efficiently in terms of social welfare and avoids direct access of the agents' private information. We theoretically prove that the proposed mechanism can achieve the socially optimal operation point. Moreover, we demonstrate that the mechanism satisfies appealing economic properties such as individual rationality and weakly balanced budget. Then, we expand the mechanism to non-exclusive data trading, in which the same data can be dispensed to multiple collectors and users. Simulations as well as real data experiments validate the theoretical properties of the mechanism. The results of this chapter are based on our work in [15].

Chapter 2

Evolutionary Information Diffusion over Heterogeneous Social Networks

2.1 Motivation

Online social networks such as Twitter, Facebook and Youtube are ubiquitous in daily life. Billions of people with different characteristics interact on the social networks, not only receiving a lot of information but also creating numerous amount of information. For example, about 500 millions of tweets are sent from Twitter every day [97] while around 300 thousand statuses are updated every minute on Facebook [81]. Each piece of information can either go viral, i.e., become very popular, or disappear quickly with few impact. When the user-generated information such as memes [70] and Twitter hashtags [27] propagates through the social networks, a variety of information diffusion dynamics are observed [114]. The diffusion dynamics or the popularity of the information are determined by the complicated interaction and decision-making of lots of users, which involves users' heterogeneous interests and influences. For instance, a football fan has a higher probability of retweeting a tweet about football and a user tends to post a piece of news if many of his friends have posted it. In practice, many applications are related to the information diffusion over social networks: online advertisements, political statements, rumor detection and control. All these applications call for a better understanding of the information diffusion process over the social networks composed of heterogeneous individuals. Consequently, great efforts have been devoted to studying how the information diffuses in the recent decade.

Existing works on information diffusion can be mainly classified into two categories: i) using machine learning (ML) or data mining approaches to make inference and prediction; ii) devising microscopic mechanisms to explain the information diffusion from the perspective of the individual users' interactions. Among the first category, Pinto *et al.* used early diffusion data to predict future diffusion [85] while the community structure is further exploited to improve the performance of prediction of viral memes in [111]. Yang and Leskovec proposed a clustering algorithm to identify the patterns of the diffusion dynamics of online contents [114]. Given the information diffusion data, efficient algorithms are developed to infer the underlying information diffusion network in [39, 89, 113]. Alternatively, the authors in [113] estimated the global influence of individuals in the information diffusion process. The interactions between the diffusions of multiple pieces of information are investigated in [77] while the impact of external sources on the information diffusion is considered in [78]. Cheng *et al.* tried to predict the cascades of the information diffusion [26]. Using the data from a real-world experiment, the authors in [18] studied the impact of cluster structure of the social network on the diffusion of behaviors. Similarly, taking an experimental approach, Bakshy *et al.* investigated the role of social ties on the information diffusion [7]. A common limitation of these

ML or data mining based approaches is the lack of understanding of the underlying microscopic mechanisms of the individuals' decision making that dominate the information diffusion process, which is the focus of the papers in the second category. In this category, authors in [41] and [101] developed game-theoretic mechanisms to analyze the competitive contagions in networks, such as firms' competing for users' purchase. Under a threshold model, Granovetter studied the diffusion of the collective behaviors, which are defined to be the adoption of one of two alternatives [42]. Assuming each user played the best response to the population's strategies, Morris studied the conditions for global contagion of behaviors [75]. The impact of the network structure on virus propagation was investigated in [102]. Moreover, in [58], algorithms for finding initial targets to maximize the future contagions over the networks are presented. The impact of the community structure on information diffusion was studied in a model-based approach in [79].

Recently, the authors of [54,55] proposed to use an evolutionary game-theoretic framework to model the users' interactions during the information diffusion process. Evolutionary game theory, originating from the evolutionary biology [98], was used as a promising modeling tool in various areas of signal processing such as communication networking and image processing [23,25,53,100,104]. In [54,55], it was found that the dynamics derived under the evolutionary game framework fit the real-world information diffusion dynamics well and could even make predictions on the future diffusion dynamics, suggesting a suitable and tractable paradigm for analyzing the information diffusion.

Most of the existing works treat the network users as homogeneous individuals and do not take the heterogeneity of the users into consideration. However, real-world social networks often exhibit significant heterogeneity. For example, heterogeneous aspects of the Twitter network include: (a) A variety of different topics coexist due to the heterogeneous interests of users; (b) Different users have very different follower counts, indicating different influences [9]; (c) The distribution of tweet counts is highly heterogeneous: the top 15% users account for the 85% of the tweets, suggesting that the user activity strength is heterogeneous [88]. The heterogeneity of the users' interests, influences and activities can have huge impact on information diffusion. For example, when a piece of information related to football reaches a user, whether the user is a football fan or not has huge impact on the decision-making (forwarding or not forwarding that information) of the user.

In this chapter, we study the information diffusion over the heterogeneous social networks using a graphical evolutionary game approach. Modeling users' decision making as an evolutionary game, we analyze the information diffusion dynamics. Through the study in this chapter, we provide a microeconomic framework by using a few utility parameters to describe the mechanisms of the users' decision making in the information diffusion process over the real-world heterogeneous social networks. The main contributions of this chapter can be epitomized as follows.

- We propose two mathematically tractable evolutionary game-theoretic models to characterize the impact of users' heterogeneity on the information diffusion over social networks. The two models differ in whether the user type* is a private information unknown to others or a publicly known information.

*The type of a user will be explicitly defined later in Section 2.

- For the unknown user type model, we theoretically derive the evolutionary dynamics as well as the evolutionarily stable states (ESSs). The relation between the heterogeneous payoff parameters and the heterogeneous information diffusion dynamics among different types of users is observed. In contrast, the homogeneous model in [54, 55] has to treat all types the same and can only give a mean evolutionary dynamics averaged over all types.

- For the known user type model, the evolutionary dynamics are derived and a relation between the dynamics is observed, which can be used to further simplify the dynamics. When the users manage to know the types of their neighbors through repeated interactions, the known user type model characterizes the users' decision making process more accurately than the unknown user type model.

- Using both synthetic data based simulations and real data based experiments, we validate the theoretical results. The good fitting and prediction performance on real-world datasets indicate the effectiveness of the evolutionary game modeling. In particular, our results outperform the homogeneous model in [54, 55] when characterizing the heterogeneous behaviors of different types of users.

The rest of this chapter is organized as follows. In Section 2.2, we formally state the evolutionary game-theoretic model for information diffusion. In Section 2.3, we theoretically derive the evolutionary dynamics and the ESSs for the unknown user type model. Then, the evolutionary dynamics of the known user type model are analyzed in Section 2.4. The experiments on synthetic data and real data are presented in Section 2.5. We conclude this chapter in Section 2.6.

2.2 Heterogeneous System Model

In this section, we first give a brief introduction to the preliminary concepts of evolutionary game theory. Then, we elaborate the proposed evolutionary game theoretic formulations of the information diffusion problem over heterogeneous social networks.

2.2.1 Basics of Evolutionary Game

The focus of traditional game theory is a game with static players and the solution concept is static Nash equilibrium (NE). On the contrary, evolutionary game theory [98] is concentrated on investigating the dynamics and stable states of a large population of evolving agents who interact with each other. Evolutionary game, as the name suggests, originates from the study of the evolution of species in biology, where animals or plants are modeled as players interacting with each other. Recent works [54, 55] show that it is also a very suitable model to analyze the social interactions among users of social networks.

A very important solution concept of evolutionary game theory is *evolutionarily stable state (ESS)*, which predicts the ultimate equilibrium of the evolutionary dynamics in a evolutionary game. Consider an evolutionary game with a large population of players. Suppose we have m strategies $\{1, .., m\}$ an m by m payoff matrix

U whose (i,j)-th entry u_{ij} is the payoff for strategy i verse strategy j (i.e., when a player with strategy i interacts with a player with strategy j, he will get a payoff of u_{ij}). Denote p_i the proportion of players adopting strategy i and $p = [p_1, p_2, ..., p_m]^T$ is the system state of the evolutionary game. Thus, the payoff of any sub-population with state q when interacting the whole population with state p is $q^T U p$. We call a state p^* an ESS if for any $q \neq p^*$, the following two conditions hold [98]:

1. $q^T U p^* \leq {p^*}^T U p^*$,

2. if $q^T U p^* = {p^*}^T U p^*$, then ${p^*}^T U q > q^T U q$.

The first condition is an NE condition, stating that any mutant (deviation from the ESS p^*) of any sub-population cannot make the payoff better off. The second condition guarantees that if deviation remains the payoff unchanged, then within the mutated sub-population (i.e., interacting with the sub-population state q), the ESS is strictly better than the deviated state q. This further ensures the stability of the state p^*. An important issue of evolutionary game theory is to compute the ESSs. A prevalent approach is to find the locally stable state of the evolutionary dynamics as a dynamical system $\dot{p} = f(p)$, where f is some function.

Classical evolutionary game assumes that every two players can interact with each other, implicitly making the hypothesis that the underlying interaction network is a complete graph. A useful generalization of the classical evolutionary game is the graphical evolutionary game, in which the interaction network is possibly incomplete. In graphical evolutionary game theory [80, 94], the player strategy update rule directly depends on the *fitness* of the users, which can be defined as a convex combination of the baseline fitness B and the payoff U, i.e.,

$$\pi = (1 - \alpha)B + \alpha U, \tag{2.1}$$

where π is the fitness. Here $0 < \alpha < 1$ is the selection strength, controlling the impact of the payoff on the fitness. In the literature of graphical evolutionary game theory [54, 55, 82, 83], α is generally assumed to be very small and we also make this assumption in the rest of the paper. The reason of assuming a small α is that we expect evolutions/adaptations to occur gradually and slowly. For instance, in biology, the evolution of species takes place very slowly; in adaptive signal processing (e.g., LMS algorithm), we usually adopt a small step size to inhibit abrupt intense change or instability. A small α limits the impact of payoff differences on the values of fitness, and thus reduces the gaps between the fitness of different players, which slows down the evolution. In fact, later we will see that the evolution dynamics are often proportional to α. After defining fitness, we can introduce three most prevalent strategy update rules in the literature of graphical evolutionary game theory, namely birth-death (BD), death-birth (DB) and imitation (IM).

- BD update rule: one player is chosen for reproduction with probability proportional to fitness. The chosen player's strategy replaces one of its neighbor's strategy with uniform probability.

- DB update rule: one player is chosen to abandon its strategy with uniform probability. He/she will adopt one of its neighbors' strategies with probability proportional to their fitness.

- IM update rule: one player is chosen to update its strategy with uniform probability. He/she may maintain his/her current strategy or adopt one of his/her neighors' strategies, with probability proportional to fitness.

In this chapter, we adopt the DB update rule. The other update rules can be similarly analyzed under our framework. In the following, we elaborate how to model the information diffusion over heterogeneous social networks by using evolutionary game theory.

A social network can be generally modeled as a graph, with nodes representing users and edges representing relationships. We assume there are N nodes (users) in the network and each node has some neighbors with whom it interacts. The number of neighbors k exhibits certain distributions $\lambda(k)$ (the fraction of nodes whose degree is k) in real social networks, e.g. Poisson distribution in Erdos-Renyi networks [34] and power law distribution in Barabasi-Albert scale-free networks [8]. In addition, real-world social networks usually consist of groups of users with different interests, influences and activities. To capture this heterogeneity, we categorize the users into M types, whereas the proportion of type-i users is $q(i), i = 1, 2, ..., M$. In the game-theoretic formulation, the N users are regarded as players. When a piece of information (e.g., a hashtag, a status or a meme) is generated, each user has two possible strategies: forwarding the information ($\mathcal{S}_f$) or not forwarding it ($\mathcal{S}_n$). We denote $p_f(i)$ the proportion of users adopting $\mathcal{S}_f$ among all the type-i users and p_f the proportion of users adopting $\mathcal{S}_f$ among users of all types. We shall call $p_f(i)$ and p_f *population dynamics* or *popularity dynamics* in the rest of the paper.

2.2.2 Unknown User Type Model

In real-world social networks, users often do not know the types of their neighbors/friends. For example, a user may not know whether his friend is fan of a singer or not. In this subsection, we present a model where the user type is private information that is unknown to others. Consider one social interaction where a type-i user A is interacting with one of its neighbors, a type-j user B. Because A does not know the type of B, the payoff of A should not depend on the type of B in this social interaction. Specifically, the payoff matrix of the type-i node A is:

$$\begin{array}{ccc} & \mathcal{S}_f & \mathcal{S}_n \\ \begin{array}{c}\mathcal{S}_f \\ \\ \mathcal{S}_n\end{array} & \multicolumn{2}{c}{\begin{pmatrix} u_{ff}(i) & u_{fn}(i) \\ & \\ u_{fn}(i) & u_{nn}(i) \end{pmatrix}} \end{array}.$$

When A and B both adopt $\mathcal{S}_f$, the payoff of A is $u_{ff}(i)$ regardless of the type of B. Both $u_{fn}(i)$ and $u_{nn}(i)$ are similarly defined. Here, a symmetric payoff structure is considered as in [54, 55]. In other words, when a type-i user with strategy $\mathcal{S}_f$ ($\mathcal{S}_n$) meets a user with strategy $\mathcal{S}_n$ ($\mathcal{S}_f$), its payoff is $u_{fn}(i)$. The reason of this symmetric payoff assumption is that often disagreement (one with strategy $\mathcal{S}_f$ while the other with strategy $\mathcal{S}_n$) leads to the same payoff to both sides. For instance, if a user mentions a hashtag while another user does not, then when they interact none of them can find common topic to discuss and both get the same payoff. The physical meaning of the payoff depends on the applications: if the social network nodes are social network users, then their payoffs may be their popularity; if the social network nodes are websites, then their payoffs may be their hit rates. The values of the payoff matrix depend on both the content of the information and the types of the users. For example, if the information is a recent hot topic (e.g., world cup in the summer of 2014) and forwarding it can increase users' popularity, then $u_{ff}(i)$ is big and $u_{nn}(i)$ is small. And if a group of users are very interested in that hot topic (e.g., football fans), then they may have even larger $u_{ff}(i)$ and smaller $u_{nn}(i)$ compared to other groups of users. By taking the baseline fitness to be 1 in Eq. (2.1), we can write the fitness as $\pi = 1 - \alpha + \alpha U$ (π is the fitness and U is the payoff). Here $0 < \alpha < 1$ is the selection strength, which is assumed very small conventionally. We note that different from payoff, fitness represents the level of fitting of a user in the social network. This fitting level contains not only the payoff obtained from extrinsic interactions but also a baseline fitness which encompasses intrinsic attributes of users, such as the satisfaction of the social network/website. Suppose A has k_f neighbors adopting $\mathcal{S}_f$, then the fitness of A is:

$$\pi_f(i, k_f) = 1 - \alpha + \alpha[k_f u_{ff}(i) + (k - k_f)u_{fn}(i)]. \tag{2.2}$$

One can similarly obtain $\pi_n(i, k_f)$, the fitness of A when A adopts $\mathcal{S}_n$ as follows:

$$\pi_n(i, k_f) = 1 - \alpha + \alpha[k_f u_{fn}(i) + (k - k_f)u_{nn}(i)]. \tag{2.3}$$

Furthermore, since A only knows the strategies of its neighbors but not the types of its neighbors, it regards the type of all of its neighbors the same as itself, i.e., type i. In other words, if one neighbor is adopting strategy $\mathcal{S}_f$, A consider its fitness to be $\pi_f(i, k_f)$. Otherwise, A considers its fitness to be $\pi_n(i, k_f)$.

2.2.3 Known User Type Model

Sometimes, through repeated interactions, users may somehow manage to know its neighbors' types. For instance, when a user observes that one of his friends frequently post news about football match, he may gradually know that this friend is a football fan. In this subsection, we present a model where the user types are publicly known information. Consider a social interaction where a Type-i user A is interacting with one of its neighbors, Type-j user B. Here, different from the unknown user type model, A knows the type of B. Hence the payoff of A should depend on the type of B in this social interaction. Specifically, if both A and B adopt $\mathcal{S}_f$, A gets a payoff $u_{ff}(i,j)$. If A, B adopt strategy $\mathcal{S}_f$ and $\mathcal{S}_n$ respectively, then the payoff of A is $u_{fn}(i,j)$. Similarly, we can define $u_{nf}(i,j)$ and $u_{nn}(i,j)$.

Take the baseline fitness to be 1 in Eq. (2.1) and thus the fitness of a user with strategy $\mathcal{S}_f$ or $\mathcal{S}_n$ is respectively given by:

$$\pi_f(i) = 1 - \alpha + \alpha \sum_{j=1}^{M} [k_f(j) u_{ff}(i,j) + k_n(j) u_{fn}(i,j)], \tag{2.4}$$

$$\pi_n(i) = 1 - \alpha + \alpha \sum_{j=1}^{M} [k_f(j) u_{nf}(i,j) + k_n(j) u_{nn}(i,j)], \tag{2.5}$$

where $k_f(j)$ ($k_n(j)$) denotes the number of type-j neighbors with strategy $\mathcal{S}_f$ ($\mathcal{S}_n$). The update rule is still the death-birth (DB), as described previously for the unknown type model. The difference is that now the player knows the types of his neighbors, hence can learn strategies only from those neighbors with the same type as his. The notations of this chapter are summarized in Table 2.1, in which some of the notations will be introduced in Section 2.4.

Table 2.1: Notations

N	Number of nodes in the network
k	Degree of a given node
M	Number of user types in the network
$q(i)$	The proportion of Type-i users in the network
$p_f(i)$	Proportion of users adopting $\mathcal{S}_f$ among all the type-i users
p_f	Proportion of users adopting $\mathcal{S}_f$ among users of all types
$u_{ff}(i), u_{fn}(i),$ $u_{nn}(i)$	Payoffs of Type-i users in the unknown user type model. For details, see Subsection 2.2-B.
$\pi_f(i), \pi_n(i)$	Fitness of a Type-i user with strategy $\mathcal{S}_f$ or $\mathcal{S}_n$, respectively
k_f	Number of neighbors (of a given user) adopting strategy $\mathcal{S}_f$
$\pi_f(i, k_f),$ $\pi_n(i, k_f)$	Fitness of a Type-i with k_f neighbors adopting strategy $\mathcal{S}_f$ while itself adopts strategy $\mathcal{S}_f$ or $\mathcal{S}_n$, respectively.
$p_{ff}(i, j), p_{fn}(i, j),$ $p_{nn}(i, j)$	Relationship states of Type-i users in the known user type model. For details, see Section 2.4.
$p_{f\|f}(i, j), p_{f\|n}(i, j),$ $p_{n\|f}(i, j), p_{n\|n}(i, j)$	Influence states of Type-i users in the known user type model. For details, see Section 2.4.
$u_{ff}(i, j), u_{fn}(i, j),$ $u_{nf}(i, j), u_{nn}(i, j)$	Payoffs of Type-i users in the known user type model. For details, see Subsection 2.2-C.
$k_f(j)$	Number of neighbors (of a given Type-j user) adopting strategy $\mathcal{S}_f$

2.3 Theoretical Analysis for the Unknown User Type Model

In this section, we derive the evolutionary dynamics of the network states $p_f(i), p_f$ and the corresponding evolutionarily stable states (ESSs) for the unknown user type model. The derived dynamics and ESSs connect the information diffusion process and the final steady states with the heterogeneous users' payoff matrices explicitly. We are able to give simple explanations on the ESSs of the information diffusion from the perspective of the payoff matrix.

Let's consider a type-i user with strategy $\mathcal{S}_f$ (in the following, we will call this user as the center user). Suppose among its k neighbors, there are k_f users adopting strategy $\mathcal{S}_f$ and $(k-k_f)$ users adopting strategy $\mathcal{S}_n$. The fitness $\pi_f(i,k_f)$ of the center user is given in Eq. (4.11). If the center user changes its strategy to $\mathcal{S}_n$, its fitness $\pi_n(i,k_f)$ becomes Eq. (4.12). From the perspective of the center user, a neighbor adopting strategy $\mathcal{S}_f$ (or $\mathcal{S}_n$) has fitness $\pi_f(i,k_f)$ (or $\pi_n(i,k_f)$, respectively). According to the DB update rule, the center user will adopt one of its neighbors' strategy with probability proportional to their fitness. Hence, the probability that the center user changes its strategy from $\mathcal{S}_f$ to $\mathcal{S}_n$ is given by:

$$\mathbb{P}_{f\to n}(i,k_f) = \frac{(k-k_f)\pi_n(i,k_f)}{k_f\pi_f(i,k_f)+(k-k_f)\pi_n(i,k_f)}. \tag{2.6}$$

Substituting the expressions of $\pi_f(i,k_f)$ and $\pi_n(i,k_f)$ in Eq. (4.11) and Eq. (4.12) into Eq. (2.6) yields:

$$\mathbb{P}_{f\to n}(i,k_f) \tag{2.7}$$

$$= \frac{k-k_f}{k}\frac{1+\alpha[k_f u_{fn}(i)+(k-k_f)u_{nn}(i)-1]}{1+\alpha\left[\frac{k_f}{k}(k_f u_{ff}(i)+(k-k_f)u_{fn}(i)-1)+(1-\frac{k_f}{k})(k_f u_{ff}(i)+(k-k_f)u_{fn}(i)-1)\right]} \tag{2.8}$$

$$= \frac{k-k_f}{k}+\alpha(k-k_f)\left[\frac{k_f^2}{k^2}\Delta(i)+\frac{k_f}{k}\Delta_n(i)\right]+O(\alpha^2), \tag{2.9}$$

where $\Delta(i) := 2u_{fn}(i) - u_{ff}(i) - u_{nn}(i)$, $\Delta_n(i) := u_{nn}(i) - u_{fn}(i)$ and in the last equation we invoke the fact that $\frac{1+ax}{1+bx} = 1+(a-b)x+O(x^2)$ for small x. Because α is a small quantity, we will omit the $O(\alpha^2)$ term in the following. Since the proportion of users with strategy $\mathcal{S}_f$ is p_f over the entire network, each neighbor has probability p_f of adopting strategy $\mathcal{S}_f$. Thus k_f is binomially distributed random variable with probability mass function:

$$\theta(k, k_f) = \begin{pmatrix} k \\ \\ k_f \end{pmatrix} p_f^{k_f}(1-p_f)^{k-k_f}. \tag{2.10}$$

Hence, taking expectation of Eq. (2.9) (note that k is also a r.v. and we need to take expectation of it further) gives:

$$\begin{aligned} \mathbb{E}[\mathbb{P}_{f\to n}(i,k_f)] =& 1 - p_f + \alpha\Delta(i)\left[\left(-\overline{k} + 3 - 2\overline{k^{-1}}\right)p_f^3 + \left(\overline{k} - 4 + 3\overline{k^{-1}}\right)p_f^2 + \left(1 - \overline{k^{-1}}\right)p_f\right] \\ & + \alpha\Delta_n(i)\left[-\left(\overline{k}-1\right)p_f^2 + (\overline{k}-1)p_f\right], \end{aligned} \tag{2.11}$$

where $\overline{k}$ and $\overline{k^{-1}}$ denote the expectation of k and k^{-1}, respectively. In the derivation of Eq. (2.11), we utilize the moments of binomial distribution: $\mathbb{E}[k_f|k] = kp_f$, $\mathbb{E}[k_f^2|k] = k^2p_f^2 - kp_f^2 + kp_f$, $\mathbb{E}[k_f^3|k] = k(k-1)(k-2)p_f^3 + 2(k-1)kp_f^2 + kp_f$. In each round of the DB update, one of the N users will be selected to update its strategy randomly. The proportion of type-i users with strategy $\mathcal{S}_f$ among all the users is $p_f(i)q(i)$. According to DB update rule, in order to have one Type-i user changes its strategy from $\mathcal{S}_f$ to $\mathcal{S}_n$, i.e., for $p_f(i)$ to decrease by $\frac{1}{Nq(i)}$, the chosen user in the death process should be a Type-i user with strategy $\mathcal{S}_f$, which happens with probability $q(i)p_f(i)$. After that, the user needs to change its strategy from $\mathcal{S}_f$ to $\mathcal{S}_n$, which happens with probability $\mathbb{E}[\mathbb{P}_{f\to n}(i,k_f)]$, where the expectation is with respect to the node degree k. Thus, we have:

$$\mathbb{P}\left(\delta p_f(i) = -\frac{1}{Nq(i)}\right) = p_f(i)q(i)\mathbb{E}[\mathbb{P}_{f\to n}(i,k_f)], \tag{2.12}$$

where δ denotes increment. With a similar argument as above, one can compute the probability that a type-i user changes its strategy from $\mathcal{S}_n$ to $\mathcal{S}_f$. We thus obtain:

$$\mathbb{P}\left(\delta p_f(i) = \frac{1}{Nq(i)}\right) = p_n(i)q(i)(1-\mathbb{E}[\mathbb{P}_{f\to n}(i,k_f)]). \tag{2.13}$$

Combining Eq. (2.11), Eq. (2.12) and Eq. (2.13), we deduce the expected change of $p_f(i)$:

$$\begin{aligned}\dot{p}_f(i) &= -\frac{1}{Nq(i)}\mathbb{P}\left(\delta p_f(i) = -\frac{1}{Nq(i)}\right) + \frac{1}{Nq(i)}\mathbb{P}\left(\delta p_f(i) = \frac{1}{Nq(i)}\right) \\ &= \frac{1}{N}p_f - \frac{1}{N}p_f(i) + \frac{\alpha}{N}p_f(p_f-1)\left[\Delta(i)\left(\left(\overline{k}-3+2\overline{k^{-1}}\right)p_f + 1 - \overline{k^{-1}}\right) + \Delta_n(i)(\overline{k}-1)\right],\end{aligned} \tag{2.14}$$

which is the dynamic of $p_f(i)$. Hence, from Eq. (2.14), the dynamic of p_f can be written as:

$$\begin{aligned}\dot{p}_f &= \sum_{i=1}^{M} q(i)\dot{p}_f(i) \\ &= \frac{\alpha}{N}p_f(p_f-1)\left[\overline{\Delta}\left(\left(\overline{k}-3+2\overline{k^{-1}}\right)p_f + 1 - \overline{k^{-1}}\right) + \overline{\Delta}_n(\overline{k}-1)\right],\end{aligned} \tag{2.15}$$

where $\overline{\Delta} := \sum_{i=1}^{M} q(i)\Delta(i)$ and $\overline{\Delta}_n := \sum_{i=1}^{M} q(i)\Delta_n(i)$. We summarize the theoretical evolutionary dynamics results as the following theorem, Theorem 2.1.

Theorem 2.1 (Evolutionary Dynamics) *In the unknown user type model, the evolutionary dynamics for the network states $p_f(i)$ and p_f are given in Eq.* (2.14) *and Eq.* (2.15), *respectively.*

From Theorem 2.1, we observe that the population dynamics $p_f(i)$ in Eq. (2.14) depend on both the global population dynamics p_f and the type-specific utility-related parameters $\Delta(i), \Delta_n(i)$. Consequently, a connection between the heterogeneous type-specific payoff matrix and the heterogeneous information diffusion dynamics of each time is established explicitly. Additionally, comparing Eq. (2.15) with the evolutionary population dynamics of a homogeneous social network given in [55] and [54], we note that the global population dynamics p_f evolve as if the network is homogeneous with corresponding payoff matrix being the weighted average (with weights $q(i)$) of those among all the types.

Given the dynamical system described in Theorem 2.1, we want to identify its ESSs. This is accomplished by the following theorem, Theorem 2.2.

Theorem 2.2 (ESSs) *In the unknown user type model, the ESSs of the network are as follows:*

$$p_f^* = \begin{cases} 0, & \text{if} \quad \overline{u}_{nn} > \overline{u}_{fn}, \\ 1, & \text{if} \quad \overline{u}_{ff} > \overline{u}_{fn}, \\ \dfrac{\overline{\Delta}_n(1-\overline{k}) + \overline{\Delta}(\overline{k^{-1}}-1)}{\overline{\Delta}(\overline{k}-3+2\overline{k^{-1}})}, & \text{if} \ \max\{\overline{u}_{ff}, \overline{u}_{nn}\} < \overline{u}_{fn}, \end{cases} \tag{2.16}$$

$$p_f^*(i) = p_f^* + \alpha p_f^*(p_f^*-1)\left[\Delta(i)\left(\left(\overline{k}-3+2\overline{k^{-1}}\right)p_f^* + 1 - \overline{k^{-1}}\right) + \Delta_n(i)(\overline{k}-1)\right], \tag{2.17}$$

where $\overline{u}_{ff} = \sum_{i=1}^{M} q(i)u_{ff}(i)$ *and* $\overline{u}_{fn}, \overline{u}_{nn}$ *are similarly defined. Recall that* $\Delta(i) = 2u_{fn}(i) - u_{ff}(i) - u_{nn}(i), \Delta_n(i) = u_{nn}(i) - u_{fn}(i)$ *and* $\overline{\Delta} = \sum_{i=1}^{M} q(i)\Delta(i)$, $\overline{\Delta}_n = \sum_{i=1}^{M} q(i)\Delta_n(i)$. *Note that it is possible that the system has more than one ESS.*

Proof: Letting the R.H.S. of Eq. (2.14) be zero, we obtain the three equilibrium points for the dynamic of p_f:

$$p_f^* = 0,\ 1,\ \frac{\overline{\Delta}_n(1-\overline{k}) + \overline{\Delta}(\overline{k^{-1}}-1)}{\overline{\Delta}(\overline{k}-3+2\overline{k^{-1}})}. \tag{2.18}$$

Given p_f^*, the equilibrium state of $p_f(i)$ can be derived from Eq. (2.14) as stated in Eq. (2.17).

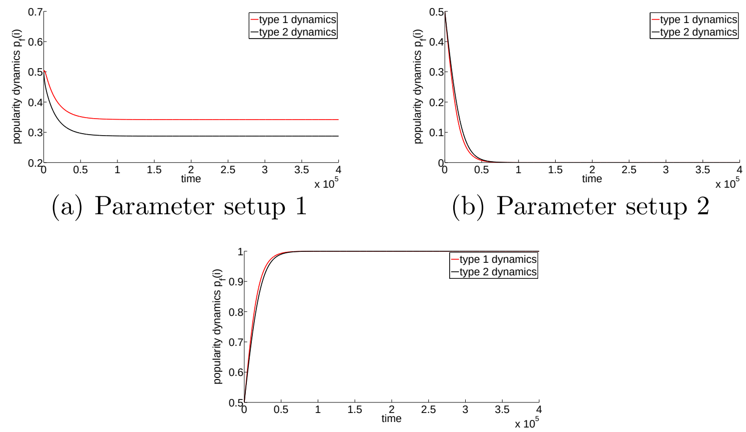
For an equilibrium point to be an ESS, it needs to be locally asymptotically stable for the underlying dynamical system. Note that for each i, $p_f(i)$ and p_f can be regarded as a dynamical system consisting of two states as indicated by Eq. (2.14) and Eq. (2.15). The Jacobian matrix of the system is given by:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \dot{p}_f(i)}{\partial p_f(i)} & \dfrac{\partial \dot{p}_f(i)}{\partial p_f} \\ \dfrac{\partial \dot{p}_f}{\partial p_f(i)} & \dfrac{\partial \dot{p}_f}{\partial p_f} \end{bmatrix}, \tag{2.19}$$

where

$$\begin{aligned}
\frac{\partial \dot{p}_f(i)}{\partial p_f(i)} &= -\frac{1}{N}, \\
\frac{\partial \dot{p}_f(i)}{\partial p_f} &= \frac{1}{N} + \frac{\alpha}{N}(2p_f-1)\left[\Delta(i)\left(\overline{k}-3+2\overline{k^{-1}}\right)p_f + \Delta(i)(1-\overline{k^{-1}}) + \Delta_n(i)(\overline{k}-1)\right] \\
&\quad + \frac{\alpha\Delta(i)}{N}(p_f^2-p_f)(\overline{k}-3+2\overline{k^{-1}}), \\
\frac{\partial \dot{p}_f}{\partial p_f(i)} &= 0, \\
\frac{\partial \dot{p}_f}{\partial p_f} &= \frac{\alpha}{N}(2p_f-1)\left[\overline{\Delta}\left(\overline{k}-3+2\overline{k^{-1}}\right)p_f + \overline{\Delta}(1-\overline{k^{-1}}) + \overline{\Delta}_n(\overline{k}-1)\right] \\
&\quad + \frac{\alpha\Delta}{N}\left(p_f^2-p_f\right)\left(\overline{k}-3+2\overline{k^{-1}}\right).
\end{aligned} \tag{2.20}$$

Since $\mathbf{J}$ is an upper triangular matrix and $\frac{\partial \dot{p}_f(i)}{\partial p_f(i)}$ is always negative, the condition for stability is simply $\frac{\partial \dot{p}_f}{\partial p_f} < 0$. Substituting the three equilibrium points in Eq. (2.18)



(a) Parameter setup 1 (b) Parameter setup 2

(c) Parameter setup 3

Fig. 2.1: Evolutionary dynamics under different parameter setups. Parameter setup 1: $u_{ff}(1) = 0.4, u_{ff}(2) = 0.2, u_{fn} = 0.6, u_{fn}(2) = 0.4, u_{nn}(1) = 0.3, u_{nn}(2) = 0.5$; Parameter setup 2: $u_{ff}(1) = 0.4, u_{ff}(2) = 0.2, u_{fn} = 0.3, u_{fn}(2) = 0.5, u_{nn}(1) = 0.6, u_{nn}(2) = 0.4$; $u_{ff}(1) = 0.6, u_{ff}(2) = 0.4, u_{fn} = 0.3, u_{fn}(2) = 0.5, u_{nn}(1) = 0.4, u_{nn}(2) = 0.2$. In every setup, we have $q(1) = q(2) = 0.5, N = 1000, k = 20$. The ESSs match the assertions in Theorem 2.2: some dynamics decrease to 0 (subfigure b) or increase to 1 (subfigure c) while some will stay at some stable state between 0 and 1 (subfigure a).

into it yields the conditions for the three possible ESSs given in Eq. (2.16), where we make use of the fact that the node degree k is generally much larger than 1 in practice. $\square$

The ESS results Eq. (2.16) in Theorem 2.2 can be interpreted easily as follows. If $\overline{u}_{ff}$ is large enough (larger than $\overline{u}_{fn}$), i.e., on average the players favor forwarding the information, then $p_f^* = 1$ is an ESS of the network. The ESS $p_f^* = 0$ can be similarly interpreted. On the contrary, if neither $\overline{u}_{ff}$ nor $\overline{u}_{nn}$ is not large enough (both smaller than $\overline{u}_{fn}$), an ESS between 0 and 1 is in presence. As shown in Fig. 2.1, for different parameter setups, we have different evolutionary dynamics. Some dynamics decrease to 0 (Fig. 2.1-b) or increase to 1 (Fig. 2.1-c) while some will stay at some stable state between 0 and 1 (Fig. 2.1-a). The corresponding ESSs are correctly predicted by Theorem 2.2. We observe that the population dynamics $p_f(i)$ always vary quickly at first and gradually slow down the varying speed until finally converge to a stable state. This can be explained by Eq. (2.15). As p_f gets closer and closer to the ESS (be it 0, 1, or some number between 0 and 1), the absolute value of R.H.S. of Eq. (2.15) gets smaller and hence the varying speed of p_f slows down until it finally equals to the ESS. Meanwhile, when p_f is stable, according to Eq. (2.14), all the type specific population dynamics $p_f(i)$ will also converge to their respective ESSs.

2.4 Theoretical Analysis for Known User Type Model

In this section, the evolutionary dynamics for the known user type model are derived. It is observed that the influence states (which we will define later) always keep track of the corresponding population states, which can be exploited to further simplify the dynamics.

Since a user's type and strategy affect its neighbors' payoffs, they may also influence the neighbors' strategies. Thus, the edge information is also required to fully characterize the network state. Specifically, we define network edge states as $p_{ff}(i,j), p_{fn}(i,j), p_{nn}(i,j)$, where $p_{ff}(i,j)$ ($p_{nn}(i,j)$) denotes the proportion of edges connecting a type-i user with strategy $\mathcal{S}_f$ ($\mathcal{S}_n$) and a type-j user with strategy $\mathcal{S}_f$ ($\mathcal{S}_n$), and $p_{fn}(i,j)$ denotes the proportion of edges connecting a type-i user with strategy $\mathcal{S}_f$ and a type-j user with strategy $\mathcal{S}_n$. Moreover, we denote $p_{f|f}(i,j)$ the percentage of type-i neighbors adopting strategy $\mathcal{S}_f$, given a center type-j user using strategy $\mathcal{S}_f$. Similarly, we can define $p_{f|n}(i,j), p_{n|f}(i,j), p_{n|n}(i,j)$. In summary, we have *population states* (e.g. $p_f(i)$), *relationship states* (e.g. $p_{ff}(i,j)$) and *influence states* (e.g. $p_{f|f}(i,j)$) as the network states. Because these states are related to each other, we only need a subset of them to characterize the entire network state. For example, we can use $p_f(i), 1 \leq i \leq M$ and $p_{ff}(i,j), 1 \leq i \leq j \leq M$ to compute all the other states.

Consider a type-i user using strategy $\mathcal{S}_f$. Rigorously speaking, $k_f(j)$ and $k_n(j)$ are random variables with expectation $kq(j)p_{f|f}(j,i)$ and $kq(j)p_{n|f}(j,i)$ respectively. Since in real world social networks, k is relatively large (more than 100 for typical online social networks such as Facebook) and a small number of types (i.e., M) is enough to capture the user behaviors, we approximate $k_f(j), k_n(j)$ with their expectations for ease of analysis in the following. This approximation can be justified as follows. Recall the Chernoff bound: Suppose $X_1, X_2, ..., X_n$ are independent random variables taking values in $[0,1]$, $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}(X)$. Then, for any $0 < \delta < 1$, we have: (i) $\mathbb{P}(X \geq (1+\delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{3}\right)$; (ii) $\mathbb{P}(X \leq (1-\delta)\mu) \leq \exp\left(-\frac{\delta^2\mu}{2}\right)$. In our case, for a Type-i user with strategy $\mathcal{S}_f$ and k neighbors, each one of its neighbors is a Type-j user with strategy $\mathcal{S}_f$ with probability $q(j)p_{f|f}(j,i)$ independently. Let the random variable X_l ($l = 1, ..., k$) be 1 if the l-th neighbor is a Type-j with strategy $\mathcal{S}_f$ and be 0 otherwise. Thus, X_l's are i.i.d. random variables. Denote $X = \sum_{l=1}^{k} X_i$ the total number of Type-j neighbors with strategy $\mathcal{S}_f$, which is $k_f(j)$ in our context. Because M is small, usually each $q(j), j =$

$1, 2..., M$ (altogether sum to 1) is not too small. Furthermore k is large and $p_{f|f}(j,i)$ is generally not too small. Hence, $\mu = \mathbb{E}(X) = kq(j)p_{f|f}(j,i)$ is large. Applying the multiplicative form of Chernoff bound, we can assert that X is close to its expectation with high probability. Thus, it is reasonable to replace $k_f(j)$ with its expectation. Similar arguments hold for $k_n(j)$. With this approximation, Eq. (2.4) becomes

$$\pi_f(i) = 1 - \alpha + \alpha k \sum_{j=1}^{M} q(j)[p_{f|f}(j,i)u_{ff}(i,j) + p_{n|f}(j,i)u_{fn}(i,j)]. \tag{2.21}$$

Similarly, if a type-i user is adopting strategy $\mathcal{S}_n$, its fitness Eq. (2.5) can be approximated as:

$$\pi_n(i) = 1 - \alpha + \alpha k \sum_{j=1}^{M} q(j)[p_{f|n}(j,i)u_{nf}(i,j) + p_{n|n}(j,i)u_{nn}(i,j)]. \tag{2.22}$$

Now, consider a type-i center user using strategy $\mathcal{S}_f$, who is selected to update its strategy. On average, there are $kp_{f|f}(i,i)$ type-i neighbors using strategy $\mathcal{S}_f$ and $kp_{n|f}(i,i)$ type-i neighbors using strategy $\mathcal{S}_n$. Thereby, according to the DB update rule, the probability that the center user will update its strategy to be $\mathcal{S}_n$ is:

$$\mathbb{P}_{f\to n}(i) = \frac{\pi_n(i)p_{n|f}(i,i)}{\pi_f(i)p_{f|f}(i,i) + \pi_n(i)p_{n|f}(i,i)}. \tag{2.23}$$

The probability that a type-i user with strategy $\mathcal{S}_f$ is chosen to update its strategy is $q(i)p_f(i)$. Hence, we have:

$$\mathbb{P}\left(\delta p_f(i) = -\frac{1}{Nq(i)}\right) = q(i)p_f(i)\mathbb{E}[\mathbb{P}_{f\to n}(i)]. \tag{2.24}$$

Similarly, we can analyze the situation where a type-i user with strategy $\mathcal{S}_n$ is selected to update its strategy. And we obtain:

$$\mathbb{P}_{n\to f}(i) = \frac{\pi_f(i)p_{f|n}(i,i)}{p_{f|n}(i,i)\pi_f(i) + p_{n|n}(i,i)\pi_n(i)}. \tag{2.25}$$

$$\mathbb{P}\left(\delta p_f(i) = \frac{1}{Nq(i)}\right) = q(i)p_n(i)\mathbb{E}[\mathbb{P}_{n\to f}(i)]. \tag{2.26}$$

We know that:

$$\dot{p}_f(i) = -\frac{1}{Nq(i)}\mathbb{P}\left(\delta p_f(i) = -\frac{1}{Nq(i)}\right) + \frac{1}{Nq(i)}\mathbb{P}\left(\delta p_f(i) = \frac{1}{Nq(i)}\right). \tag{2.27}$$

For ease of notation, we temporarily denote that $a = k\sum_{j=1}^{M} q(j)[p_{f|n}(j,i)u_{nf}(i,j) + p_{n|n}(j,i)u_{nn}(i,j)]$ and $b = k\sum_{j=1}^{M} q(j)[p_{f|f}(j,i)u_{ff}(i,j) + p_{n|f}(j,i)u_{fn}(i,j)]$. Thus, the first term in Eq. (2.27) can be rewritten as:

$$-\frac{1}{Nq(i)}\mathbb{P}\left(\delta p_f(i) = -\frac{1}{Nq(i)}\right) \tag{2.28}$$

$$= -\frac{p_f(i)p_{n|f}(i,i)}{N}\mathbb{E}\left\{\frac{1+\alpha(a-1)}{1+\alpha[(b-1)p_{f|f}(i,i)+(a-1)p_{n|f}(i,i)]}\right\} \tag{2.29}$$

$$= -\frac{p_f(i)p_{n|f}(i,i)}{N}\mathbb{E}[1 + p_{f|f}(i,i)(a-b)\alpha] + O(\alpha^2), \tag{2.30}$$

where we make use of the fact that $p_{f|f}(i,i) + p_{n|f}(i,i) = 1$, which can be easily seen from the definition. The expectation is taken over k. Similarly, we can derive the second term in Eq. (2.27) as:

$$\frac{1}{Nq(i)}\mathbb{P}\left(\delta p_f(i) = \frac{1}{Nq(i)}\right) = \frac{p_n(i)p_{f|n}(i,i)}{N}\mathbb{E}[1 + \alpha p_{n|n}(i,i)(b-a)] + O(\alpha^2). \tag{2.31}$$

Noticing the fact that $p_f(i)p_{n|f}(i,i) = p_n(i)p_{f|n}(i,i)$, we obtain:

$$\begin{aligned}\dot{p}_f(i) \approx\ & \frac{\alpha\overline{k}}{N}p_f(i)p_{n|f}(i,i)(p_{n|n}(i,i) + p_{f|f}(i,i)) \\ & \times\sum_{j=1}^{M} q(j)[p_{f|f}(j,i)u_{ff}(i,j) + p_{n|f}(j,i)u_{fn}(i,j) - p_{f|n}(j,i)u_{nf}(i,j) - p_{n|n}(j,i)u_{nn}(i,j)],\end{aligned} \tag{2.32}$$

where $\overline{k}$ denotes the average degree of the network and we omit the $O(\alpha^2)$ terms. Next, we compute the dynamics of $p_{ff}(i,l)$ (or equivalently, $p_{f|f}(i,l)$). To change the value of $p_{ff}(i,l)$, either a type-i user or a type-l user changes its strategy. If $i \neq l$, there are totally four situations: i) a type-i user changes its strategy from $\mathcal{S}_f$

to $\mathcal{S}_n$; ii) a type-i user changes its strategy from $\mathcal{S}_n$ to $\mathcal{S}_f$; iii) a type-l user changes its strategy from $\mathcal{S}_f$ to $\mathcal{S}_n$; iv) a type-l user changes its strategy from $\mathcal{S}_n$ to $\mathcal{S}_f$. They correspond to the following four equations:

$$\begin{aligned}
&\mathbb{P}\left(\delta p_{ff}(i,l) = -\frac{2}{N}q(l)p_{f|f}(l,i)\right) = q(i)p_f(i)\mathbb{P}_{f\to n}(i) \approx q(i)p_f(i)p_{n|f}(i,i),\\
&\mathbb{P}\left(\delta p_{ff}(i,l) = -\frac{2}{N}q(i)p_{f|f}(i,l)\right) = q(l)p_f(l)\mathbb{P}_{f\to n}(l) \approx q(l)p_f(l)p_{n|f}(l,l),\\
&\mathbb{P}\left(\delta p_{ff}(i,l) = \frac{2}{N}q(l)p_{f|n}(l,i)\right) = q(i)p_n(i)\mathbb{P}_{n\to f}(i) \approx q(i)p_n(i)p_{f|n}(i,i),\\
&\mathbb{P}\left(\delta p_{ff}(i,l) = \frac{2}{N}q(i)p_{f|n}(i,l)\right) = q(l)p_n(l)\mathbb{P}_{n\to f}(l) \approx q(l)p_n(l)p_{f|n}(l,l),
\end{aligned} \tag{2.33}$$

where in the last step we omit $O(\alpha)$ terms, i.e., treating α as 0. The reason that we omit $O(\alpha)$ terms instead of $O(\alpha^2)$ terms as before is that we have nonzero $O(1)$ terms here. Combining the four equations in Eq. (2.33), we get (for $i \neq l$):

$$\begin{aligned}
\dot{p}_{ff}(i,l) = &-\frac{2}{N}q(l)p_{f|f}(l,i)\mathbb{P}\left(\delta p_{ff}(i,l) = -\frac{2}{N}q(l)p_{f|f}(l,i)\right)\\
&-\frac{2}{N}q(i)p_{f|f}(i,l)\mathbb{P}\left(\delta p_{ff}(i,l) = -\frac{2}{N}q(i)p_{f|f}(i,l)\right)\\
&+\frac{2}{N}q(l)p_{f|n}(l,i)\mathbb{P}\left(\delta p_{ff}(i,l) = \frac{2}{N}q(l)p_{f|n}(l,i)\right)\\
&+\frac{2}{N}q(i)p_{f|n}(i,l)\mathbb{P}\left(\delta p_{ff}(i,l) = \frac{2}{N}q(i)p_{f|n}(i,l)\right)\\
= &\frac{2}{N}q(i)q(l)p_f(i)p_{n|f}(i,i)(p_{f|n}(l,i) - p_{f|f}(l,i))\\
&+\frac{2}{N}q(i)q(l)p_f(l)p_{n|f}(l,l)(p_{f|n}(i,l) - p_{f|f}(i,l))\\
= &\frac{2}{N}q(i)q(l)p_f(i)(1-p_{f|f}(i,i))\left[\frac{p_f(l)}{p_n(i)}(1-p_{f|f}(i,l)) - p_{f|f}(l,i)\right]\\
&+\frac{2}{N}q(i)q(l)p_f(l)(1-p_{f|f}(l,l))\left[\frac{p_f(i)}{p_n(l)}(1-p_{f|f}(l,i)) - p_{f|f}(i,l)\right],
\end{aligned} \tag{2.34}$$

where we have used the equalities $p_{n|f}(i,i) = 1 - p_{f|f}(i,i)$ and $p_{f|n}(l,i) = \frac{p_f(l)}{p_n(i)}(1 - p_{f|f}(i,l))$ in the last step so as to substitute all the influence states by $p_{f|f}(\cdot,\cdot)$.

Similarly we can derive the dynamics of $p_{ff}(i,i)$ as follows:

$$\dot{p}_{ff}(i,i) = \frac{2}{Np_n(i)} q^2(i) p_f(i)(1 - p_{f|f}(i,i))(p_f(i) - p_{f|f}(i,i)). \tag{2.35}$$

Recall Eq. (2.32), where we note that the population dynamics $p_f(\cdot)$ evolves at the speed of $O(\alpha)$. From Eq. (2.34) and Eq. (2.35), we observe that the relationship dynamics $p_{ff}(\cdot,\cdot)$ (hence the influence dynamics $p_{f|f}(\cdot,\cdot)$) evolve at the speed of $O(1)$. Due to the assumption that α is very small, the relationship dynamics and influence dynamics change at a much faster speed than population dynamics do. This implies that we can select a time window with an appropriate length such that the population dynamics $p_f(\cdot)$ basically remain unchanged while the relationship dynamics $p_{ff}(\cdot,\cdot)$ and influence dynamics $p_{f|f}(\cdot,\cdot)$ vary a lot. In the following, we focus on such a time period in which the population dynamics $p_f(\cdot)$ remains a constant and only relationship dynamics and influence dynamics vary with time. Taking derivative w.r.t time on both sides of the equation $p_{ff}(i,l) = 2q(i)q(l)p_f(i)p_{f|f}(l,i)$, $i \neq l$, we obtain:

$$\dot{p}_{ff}(i,l) = 2q(i)q(l)p_f(i)\dot{p}_{f|f}(l,i). \tag{2.36}$$

Combining Eq. (2.34) and Eq. (2.36) yields the dynamics of $p_{f|f}(l,i)$, $l \neq i$:

$$\begin{aligned} \dot{p}_{f|f}(l,i) = &\frac{1}{N}(1 - p_{f|f}(i,i)) \left[\frac{p_f(l)}{p_n(i)}(1 - p_{f|f}(i,l)) - p_{f|f}(l,i) \right] \\ &+ \frac{1}{N}(1 - p_{f|f}(l,l)) \left[\frac{p_f(l)}{p_n(l)}(1 - p_{f|f}(l,i)) - \frac{p_f(l)}{p_f(i)} p_{f|f}(i,l) \right]. \end{aligned} \tag{2.37}$$

Leveraging the equation $p_f(i)p_{f|f}(l,i) = p_f(l)p_{f|f}(i,l)$, we can further simplify Eq. (2.37) as follows:

$$\dot{p}_{f|f}(l,i) = \frac{1}{N}(p_f(l) - p_{f|f}(l,i)) \left[\frac{1 - p_{f|f}(i,i)}{p_n(i)} + \frac{1 - p_{f|f}(l,l)}{p_n(l)} \right], \forall l \neq i. \tag{2.38}$$

On the other hand, if $l = i$, then $\dot{p}_{ff}(i,i) = q^2(i)p_f(i)\dot{p}_{f|f}(i,i)$. Thus, from Eq. (2.35), we obtain:

$$\dot{p}_{f|f}(i,i) = \frac{2}{Np_n(i)}(1 - p_{f|f}(i,i))(p_f(i) - p_{f|f}(i,i)), \forall i. \tag{2.39}$$

Since Eq. (2.39) is equivalent to letting $i = l$ in Eq. (2.38), we know that Eq. (2.38) applies to any i, l (not necessarily unequal). Recall that in Eq. (2.38), we treat the population dynamics $p_f(i), p_n(i)$ as constants. In other words, we are considering a small time period where the population dynamics do not vary with time while the influence dynamics $p_{f|f}(\cdot,\cdot)$ vary according to the deduced dynamics Eq. (2.38). Next, we show that in this small time period, the influence dynamics $p_{f|f}(\cdot,\cdot)$ will converge to the corresponding population dynamics $p_f(\cdot)$.

We first solve the ODE Eq. (2.39) with single variable $p_{f|f}(i,i)$. Without loss of generality, we assume the initial value of $p_{f|f}(i,i)$ is less than $p_f(i)$. Thus, by solving Eq. (2.39), we have:

$$p_{f|f}(i,i) = p_f(i) - \frac{p_n(i)}{e^{\frac{4t}{N}+C_i} - 1}, \tag{2.40}$$

where $C_i := \ln\left(1 - p_{f|f}(i,i)\big|_{t=0}\right) - \ln\left(p_f(i) - p_{f|f}(i,i)\big|_{t=0}\right)$ is a constant. From Eq. (2.40), we see that $\lim_{t\to+\infty} p_{f|f}(i,i) = p_f(i)$. Substituting Eq. (2.40) into Eq. (2.38), we obtain:

$$\dot{p}_{f|f}(l,i) = \frac{1}{N}(p_f(l) - p_{f|f}(l,i))\left[\frac{e^{\frac{4t}{N}+C_i}}{e^{\frac{4t}{N}+C_i} - 1} + \frac{e^{\frac{4t}{N}+C_l}}{e^{\frac{4t}{N}+C_l} - 1}\right]. \tag{2.41}$$

Hence, by solving for $p_{f|f}(l,i)$, we have:

$$\ln\left|p_f(l) - p_{f|f}(l,i)\right| - \ln\left|p_f(l) - p_{f|f}(l,i)\big|_{t=0}\right| + \frac{2t}{N} = -\frac{1}{N}\int_0^t \left(\frac{1}{e^{\frac{4\sigma}{N}+C_i} - 1} + \frac{1}{e^{\frac{4\sigma}{N}+C_l} - 1}\right) d\sigma. \tag{2.42}$$

The R.H.S. of Eq. (2.42) is clearly a bounded quantity as t goes to infinity. Hence, from the L.H.S., we observe that $\ln \left|p_f(l) - p_{f|f}(l,i)\right| \to -\infty$ as $t \to +\infty$. In other words, $\lim_{t\to+\infty} p_{f|f}(l,i) = p_f(l), \forall l \neq i$. We summarize the results obtained for the evolutionary dynamics in the known user type model as the following theorem, Theorem 2.3.

Theorem 2.3 *In the known user type model, the population dynamics $p_f(i)$ are given in Eq. (2.32) while the relationship dynamics $p_{ff}(i,l)$ are given in Eq. (2.34) (for $i \neq l$) and Eq. (2.35) (for $i = l$).*

The population dynamics evolve at a much slower speed than the influence dynamics and the relationship dynamics. In a small time period such that the population states $p_f(\cdot)$ remain constants, the influence dynamics $p_{f|f}(l,i)$ are given by Eq. (2.38) (for any l, i). In such a small time period, each influence state $p_{f|f}(l,i)$ will converge to the corresponding fixed population state $p_f(l)$.

According to Theorem 2.3, since the influence state will keep track of the corresponding population state, we can make the approximation that $p_{f|f}(l,i) = p_f(l), \forall l, i$. Thus, the population dynamics can be further simplified into the following form.

Corollary 2.1 *In the known user type model, the population dynamics $p_f(i)$ for each type $i = 1, 2, ...M$ are (approximately) given by:*

$$\dot{p}_f(i) = \frac{\alpha\overline{k}}{N} p_f(i) p_n(i) \sum_{j=1}^{M} q(j)[p_f(j)(u_{ff}(i,j) - u_{nf}(i,j)) + p_n(j)(u_{fn}(i,j) - u_{nn}(i,j))]. \tag{2.43}$$

2.5 Experiments

In this section, we implement synthetic data as well as real data experiments to verify the theoretical results on information diffusion dynamics and ESSs. First, using synthetic data, we show that the simulations match the theoretical findings well. Then, using real data, we find that the theoretical dynamics also fit the real-world information diffusion dynamics well and can even make predictions for the future diffusion dynamics.

2.5.1 Synthetic Data Experiments

In this subsection, we conduct simulations to validate the theoretical evolutionary dynamics and ESSs. We set $M = 2$, i.e., the network consists of two types of users. We synthesize a constant degree network, i.e., all the nodes have the same degree (k is a deterministic constant). We first consider the unknown user type model. The payoff parameters of the two types of players are set as following: $u_{ff}(1) = 0.4, u_{ff}(2) = 0.2, u_{fn}(1) = 0.6, u_{fn}(2) = 0.4, u_{nn}(1) = 0.3, u_{nn}(2) = 0.5$. Other parameters are $N = 1000, k = 20, q(1) = q(2) = 0.5, \alpha = 0.05$. The result is reported in Fig 4.1. The theoretical dynamics match the simulation dynamics well and the theoretical ESSs are near the simulated ESSs with average relative ESS error[†] 3.54%. If we model the heterogeneous network as a homogeneous one

[†]The average relative ESS error is calculated as follows. We denote these two simulated ESSs (for two different types, respectively) as x_1 and x_2. We denote the two theoretical ESSs as y_1 and y_2. Then the average relative ESS error is $\frac{1}{2}(|y_1 - x_1|/x_1 + |y_2 - x_2|/x_2)$. If we use homogeneous network to model, we only have one global theoretical ESS z. In such a case, the average relative

like in [54, 55], i.e., all the payoffs are set to be the average over all types, then the average relative ESS error is 6.83%, indicating the advantage of the proposed heterogeneous model. In addition, we simulate the evolutionary dynamics under another utility parameter setup in Fig. 2.3 and observe that the simulated dynamics still match well with the theoretical ones. Furthermore, to manifest the extreme ESSs highlighted in Theorem 2.2, i.e., ESSs of 0 and 1, we alter the utility parameters to simulate and the results are shown in Fig. 2.4, where population dynamics with ESSs of 0 and 1 are exhibited, respectively. We observe that the theoretical dynamics again match well with the simulated ones. Simulation results for Erdos-Renyi network [34] and Barabasi-Albert network [8] with the same parameter setup are shown in Fig. 2.5-(a),(b) respectively. The population dynamics is very similar to that of the constant degree network, and the theoretical dynamics still fit the simulated one well. In Fig. 2.6, we simulate the information diffusion of a heterogeneous network with three types of users. We observe that the theoretical dynamics still match well with the simulated ones. All of the above results demonstrate the effectiveness and accuracy of the proposed heterogeneous network theory.

Next, we implement a simulation for the known user type model with payoff parameters randomly chosen as follows:

ESS error is computed as $\frac{1}{2}(|z-x_1|/x_1 + |z-x_2|/x_2)$.

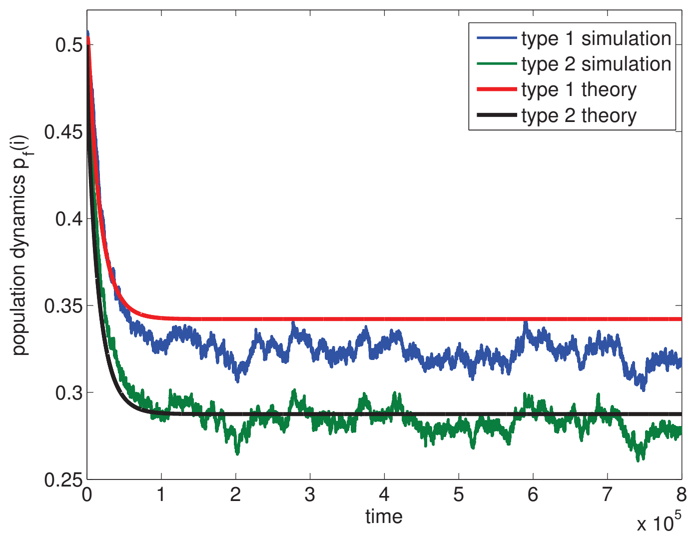


Figure 2.2: Simulation results of the evolution dynamics for the unknown user type model. The theoretical dynamics fit the simulation dynamics well and the ESSs are predicted accurately. The average relative ESS error of the heterogeneous model is 3.54%. If we model the entire network as a homogeneous one as in [54, 55], the average relative ESS error becomes 6.83%, indicating the advantage of the heterogeneous model in this chapter.

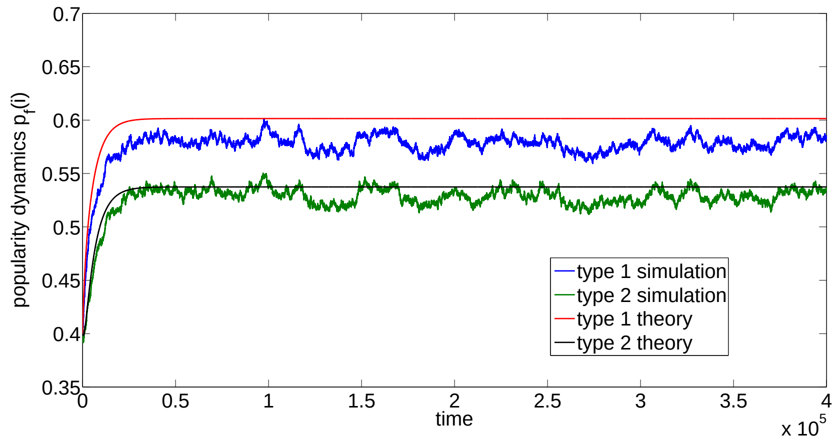
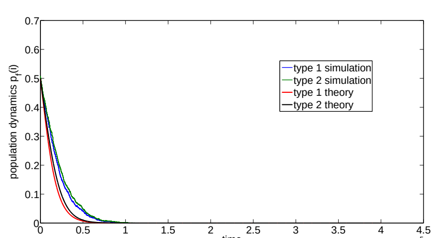
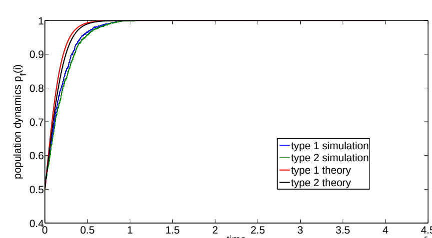


Figure 2.3: Simulation results of evolution dynamics for the unknown user type model with another utility parameter setup: $u_{ff}(1) = 0.5$, $u_{ff}(2) = 0.1$, $u_{fn}(1) = 0.8$, $u_{fn}(2) = 0.5$, $u_{nn}(1) = 0.1$, $u_{nn}(2) = 0.3$. We observe that the simulated dynamics still match well with the theoretical ones.

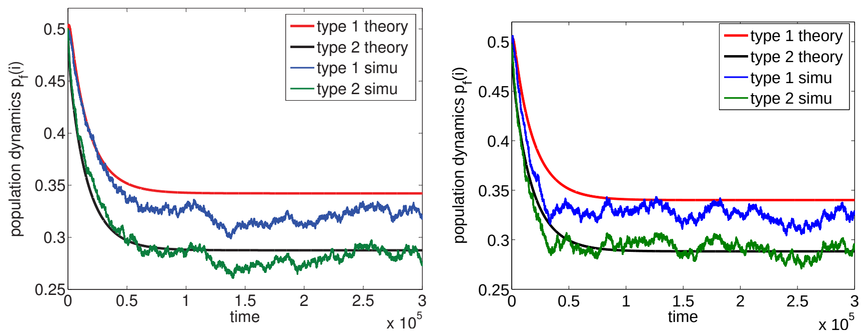


(a) Population dynamics with ESS of 0



(b) Population dynamics with ESS of 1

Fig. 2.4: Simulations for unknown user type model: population dynamics wit ESSs of 0 and 1, respectively. In (a), the utility parameters are: $u_{ff}(1) = 0.4$, $u_{ff}(2) = 0.2$, $u_{fn}(1) = 0.3$, $u_{fn}(2) = 0.5$, $u_{nn}(1) = 0.6$, $u_{nn}(2) = 0.4$. In (b), the utility parameters are: $u_{ff}(1) = 0.6$, $u_{ff}(2) = 0.4$, $u_{fn}(1) = 0.3$, $u_{fn}(2) = 0.5$, $u_{nn}(1) = 0.4$, $u_{nn}(2) = 0.2$.



(a) Erdos-Renyi network (b) Barabasi-Albert network

Fig. 2.5: More simulations of the evolutionary dynamics for the unknown user type model with different networks.

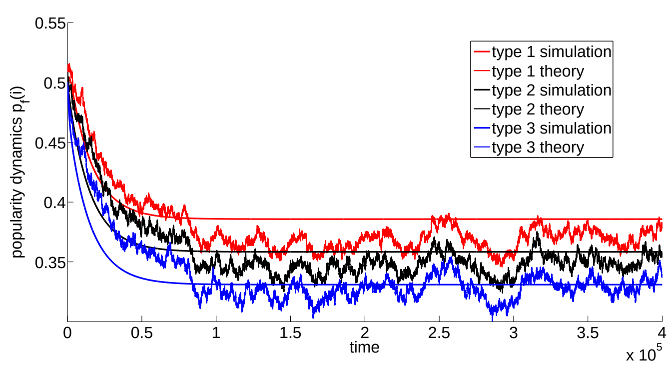


Figure 2.6: Simulation results for unknown user type model with three types of users. We observe that the theoretical dynamics still match well with the simulated ones.

$$u_{ff} = \begin{bmatrix} 0.5882 & 0.0116 \\ 0.8688 & 0.1590 \end{bmatrix}, u_{fn} = \begin{bmatrix} 0.9619 & 0.7370 \\ 0.5595 & 0.7180 \end{bmatrix},$$
$$u_{nf} = \begin{bmatrix} 0.9339 & 0.9864 \\ 0.3288 & 0.4593 \end{bmatrix}, u_{nn} = \begin{bmatrix} 0.2479 & 0.3385 \\ 0.6570 & 0.2437 \end{bmatrix}. \tag{2.44}$$

The other parameters are $N = 1000, k = 20, q(1) = 0.5518, q(2) = 0.4482, \alpha = 0.05$. The simulated and theoretical population dynamics are shown in Fig. 2.7, where the known user type model based theoretical dynamics and the simulated dynamics match well. In Fig. 2.7, we also plot the evolutionary dynamics given by the theory of the unknown user type model. This does not match the simulated evolutionary dynamics under the known user type model, indicating the necessity of the theory of the known user type model. Simulations under two different parameter setups are shown in Fig. 2.8, where the theoretical dynamics and the simulated dynamics match. In Fig. 2.8-(a), the utility parameters are set as follows:

$$u_{ff} = \begin{bmatrix} 0.4228 & 0.1052 \\ 0.9184 & 0.5182 \end{bmatrix}, u_{fn} = \begin{bmatrix} 0.9641 & 0.9865 \\ 0.3008 & 0.7058 \end{bmatrix},$$
$$u_{nf} = \begin{bmatrix} 0.7453 & 0.7104 \\ 0.8943 & 0.9505 \end{bmatrix}, u_{nn} = \begin{bmatrix} 0.3199 & 0.6119 \\ 0.3162 & 0.4556 \end{bmatrix}. \tag{2.45}$$

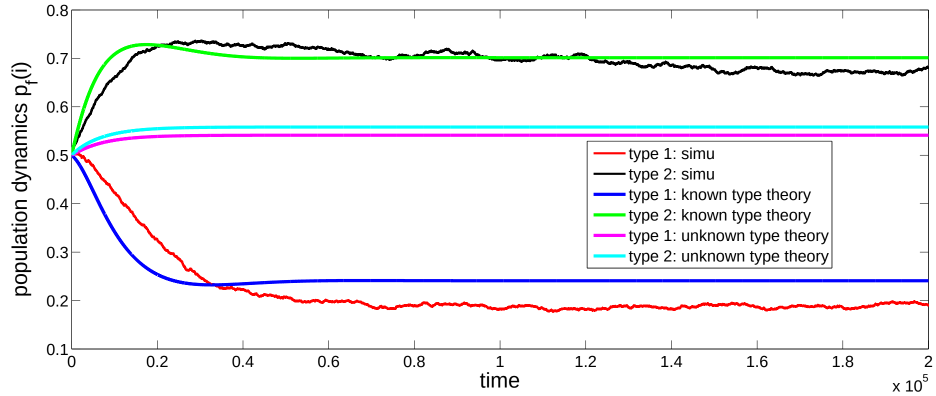


Figure 2.7: Simulation of evolutionary dynamics: the known user type model.

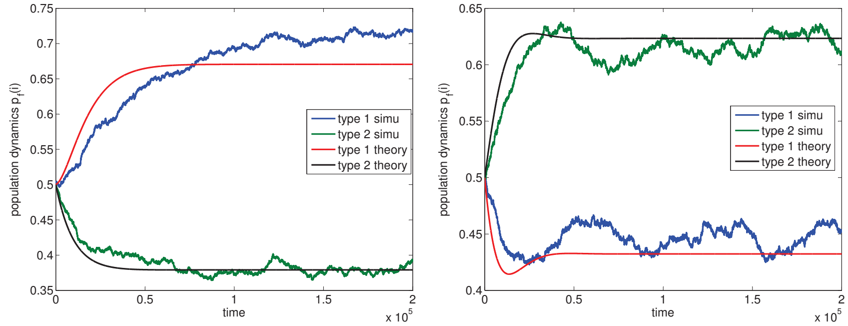
And in Fig. 2.8-(b), the utility parameters are set as follows:

$$
\begin{aligned}
u_{ff} &= \begin{bmatrix} 0.6673 & 0.1855 \\ 0.0703 & 0.2549 \end{bmatrix}, u_{fn} = \begin{bmatrix} 0.7964 & 0.1144 \\ 0.9288 & 0.9262 \end{bmatrix}, \\
u_{nf} &= \begin{bmatrix} 0.7979 & 0.1071 \\ 0.8047 & 0.4854 \end{bmatrix}, u_{nn} = \begin{bmatrix} 0.2721 & 0.7794 \\ 0.7564 & 0.0574 \end{bmatrix}.
\end{aligned}
\tag{2.46}
$$

In Fig. 2.8-(b), we observe some oscillations of the simulated dynamics. The reason may be that the number of parameters in the known user type model is relatively large and the strategy update rule is more complicated than the unknown user type model, which may lead to unstable behaviors of the users.

2.5.2 Real Data Experiments

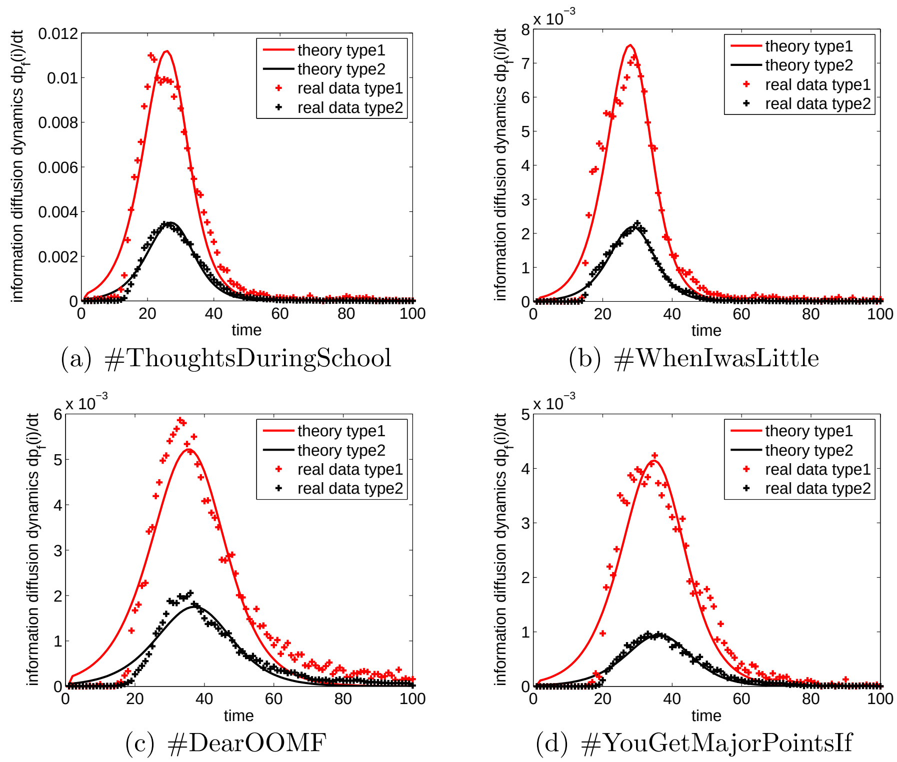
In this subsection, we use the Twitter hashtag dataset in [111] to validate the theory. The dataset, comprising sequences of adopters and timestamps for the observed hashtags, is based on sampled public tweets from March 24, 2012 to April 25, 2012. To characterize the heterogeneity of the users, we classify the users into two



(a) Parameter setup 1 (b) Parameter setup 2

Fig. 2.8: Known user type model: more simulations of the evolutionary dynamics with different parameter setups.

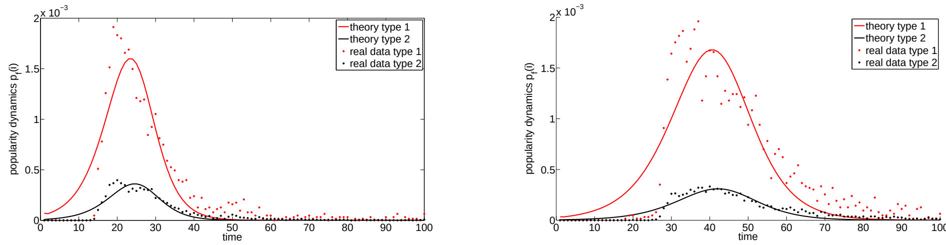
types. The classification is based on the users' activity. Specifically, we compute the number of hashtags each user has mentioned. Then, the top 10% users with highest number of hashtag mentioning are categorized as Type-1 users while the remaining users are categorized as Type-2 ones. After classification, the number of type-1 users is 62757 while that of type-2 users is 533262. We set k to be 100, a typical number of neighbors/friends in social networks. Since the dataset does not contain the network structure of the users, we postulate the network to be a constant degree network where each user has the same degree $k = 100$. The selection strength α is not important in the curve fitting/prediction process, since it can be absorbed into the payoff parameters as it always multiplies with all the payoff parameters. In our dataset, the physical unit of time indices is not specified. In the following experiments, we choose appropriate time slot length so that (i) the data dynamics are smooth (so the time slot length cannot be too small), (ii) the data dynamics vary continuously and can correctly reflect the variation of the diffusion dynamics of real data (so the time slot length cannot be too large).



(a) #ThoughtsDuringSchool (b) #WhenIwasLittle

(c) #DearOOMF (d) #YouGetMajorPointsIf

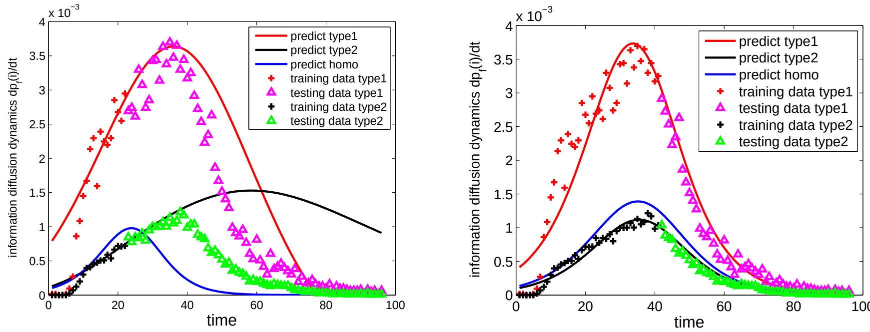
Fig. 2.9: Fitting results for the unknown user type model. Type-1 users are always more active than type-2 users because $p_f(1)$ is always larger than $p_f(2)$. The proposed theoretical dynamics fit the information diffusion dynamics of the real-world heterogeneous social networks well, which validates the effectiveness of considering the individuals' interactions. The theory suggests that the heterogeneous behavior dynamics of online users are consequences of their heterogeneous payoff structures.



(a) #ididnttextback (b) #imhappywhen

Fig. 2.10: Fitting results for the unknown user type model. Two less popular hashtags, #ididnttextback and #imhappywhen, are fitted. The fitting is still accurate though the data become more noisy as these two hashtags are less popular.

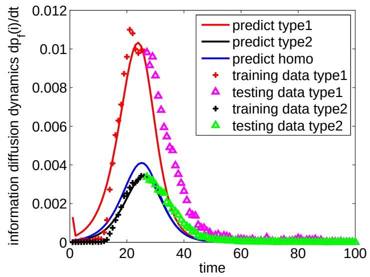
We first fit the theoretical dynamics for the unknown user type model in Eq. (2.14) and Eq. (2.15) with the real data. We use the real data to estimate the parameters (i.e., $\Delta(i)$ and $\Delta_n(i)$) in Eq. (2.14) and Eq. (2.15), and then calculate the theoretical dynamics based on the estimated parameters. We invoke the Matlab function `lsqcurvefit` to implement the curve fitting, or in other words, to estimate the payoff parameters. The parameter estimation process is built inside this Matlab function. Given data and a function to be fit, `lsqcurvefit` selects the optimal parameters in order to minimize the squared fitting error. The fitting results for four popular hashtags are reported in Fig. 2.9. Type-1 users are more active than type-2 users since the population state $p_f(1)$ is always larger than $p_f(2)$. We observe that the proposed theoretical dynamics fit the real-world information diffusion dynamics well, indicating the effectiveness of taking the heterogeneous users' interactions and decision making into account. In the curve fitting of the dynamics of the hashtag #ThoughtsDuringSchool, the utility parameters are estimated to satisfy: $u_{ff}(1) - u_{fn}(1) = -3.32$, $u_{nn}(1) - u_{fn}(1) = -0.578$, $u_{ff}(2) - u_{fn}(2) = -0.64$,



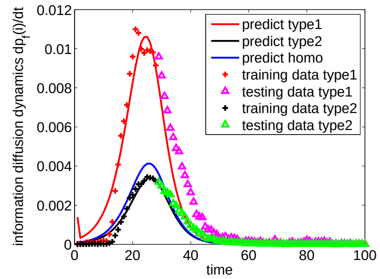
(a) Using data up to time 22 (b) Using data up to time 41

Fig. 2.11: Predictions. The heterogeneous game-theoretic model can predict future diffusion dynamics. The predictions made by the heterogeneous model outperforms that of the homogeneous one in [54].

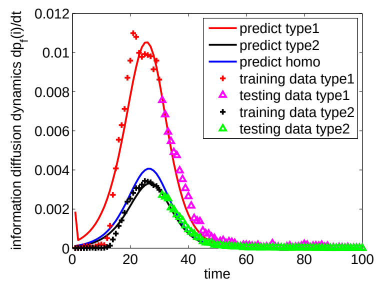
$u_{nn}(2) - u_{fn}(2) = -0.004$. From these relationships, we see that for real-world information diffusion data, the estimated utility parameters satisfy the condition $\bar{u}_{fn} > \max\{\bar{u}_{ff}, \bar{u}_{nn}\}$. From Theorem 2.2, we see that this condition leads to an ESS between 0 and 1, which is clearly the case in most real-world applications. In the previous subsection on simulations, the utility parameters are also chosen in compliance with this condition (e.g., Fig.3 and Fig. 4) and are hence justified by the real data. Furthermore, we see that $u_{nn}(1)$ is much smaller than $u_{fn}(1)$ while $u_{nn}(2)$ is basically the same as $u_{fn}(2)$. To some extent, this explains why Type-1 users are more active than Type-2 users. Furthermore, we fit two less popular hashtags #ididnttextback and #imhappywhen (with peak mention counts about 1/6 of that of the hashtag #ThougtsDuringSchool). The results are reported in Fig. 2.10 from which we observe that the fitting is still accurate though the data become more noisy as these two hashtags are less popular, indicating the robustness of our approach.



(a) Using data up to time 26

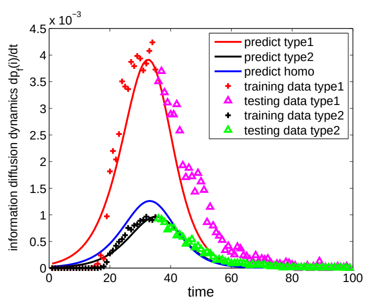


(b) Using data up to time 28

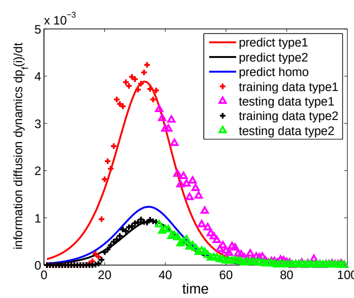


(c) Using data up to time 30

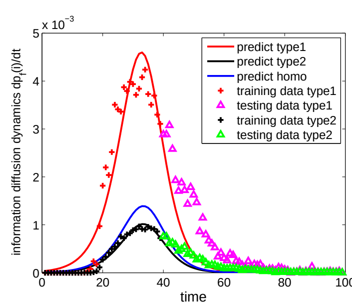
Fig. 2.12: Predictions for Twitter hashtag #ThoughtsDuringSchool.



(a) Using data up to time 36

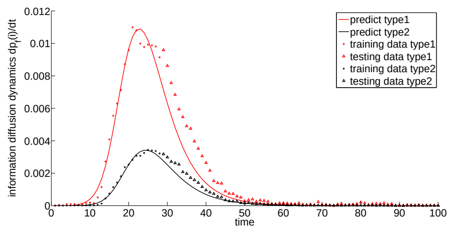


(b) Using data up to time 38

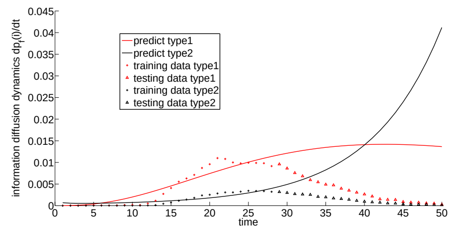


(c) Using data up to time 40

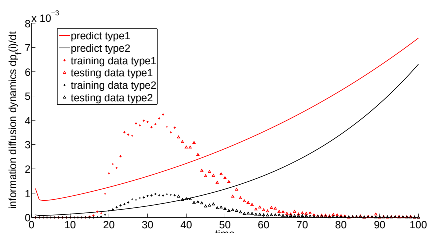
Fig. 2.13: Predictions for Twitter hashtag #YouGetMajorPointsIf.



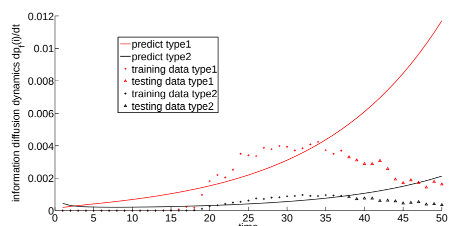
(a) [106], #ThoughtsDuringSchool, using data up to time 28



(b) [70], #ThoughtsDuringSchool, using data up to time 28



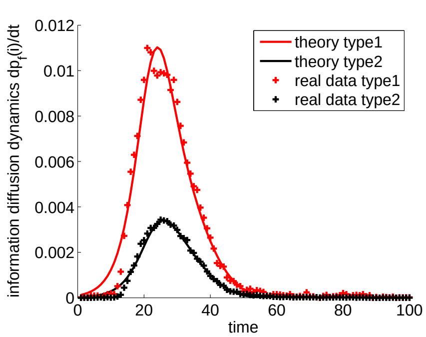
(c) [106], #YouGetMajorPointsIf, using data up to time 38



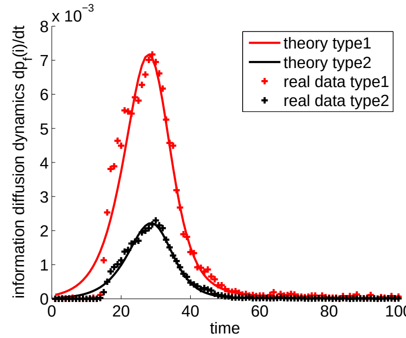
(d) [70], #YouGetMajorPointsIf, using data up to time 38

Fig. 2.14: Prediction results of [106] and [70]. Comparisons subfigures (a)(b) with Fig. 2.12-(b) and subfigures (c)(d) with Fig. 2.13-(b) highlight the advantage of the proposed game-theoretic approach. In particular, the results in subfigures (b)(c)(d) fail to give meaningful predictions.

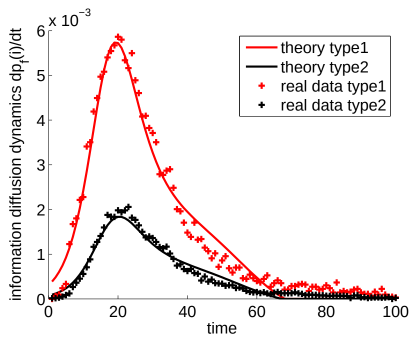
In addition, we conduct experiments on the prediction of future diffusion dynamics. Specifically, we only use part of the data to train the payoff parameters in Eq. (2.14), Eq. (2.15), and use the trained parameters to predict future diffusion dynamics. To compare with the homogeneous model in [54, 55], we also model the heterogeneous network as a homogeneous one and use the homogeneous network theory in [54] to make predictions, which serve as benchmarks. The prediction results for one popular hashtag #WhenIwasLittle are shown in Fig. 2.11. Two different training data lengths are investigated. The heterogeneous game-theoretic model can predict the future diffusion dynamics well. In contrast, by modeling the network as a homogeneous one, the prediction does not match the real data well, especially for type-1 users. The reason is that the prediction made by the homogeneous model can be thought of as a prediction of the overall diffusion dynamics averaged over the two types. But, type-1 users are active minority (10% of all the users). So, its diffusion dynamic is far from the average one and is poorly predicted. The prediction results of two other Twitter hashtags #ThoughtsDuringSchool and #YouGetMajorPointsIf are shown in Fig. 2.12 and Fig. 2.13, respectively. For both hashtags, the prediction performance of our heterogeneous model is good. In addition, we perform predictions for future 10 time slots immediately after the peak of the diffusion dynamics is observed for the 8 most popular hashtags in the dataset. The average relative error of the heterogeneous game model is 23% while that of the homogeneous game model in [54] is 47%. Furthermore, prediction results of the existing methods in [106] and [70] are reported in Fig. 2.14. Comparison with the corresponding prediction results of the proposed approach in Fig. 2.12-(b) and Fig.



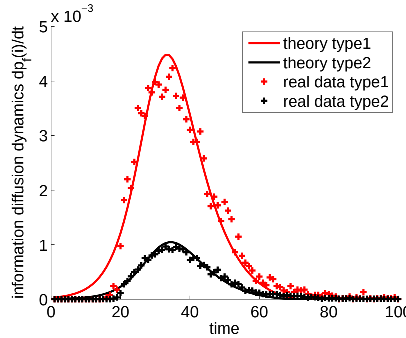
(a) #ThoughtsDuringSchool



(b) #WhenIwasLittle



(c) #DearOOMF

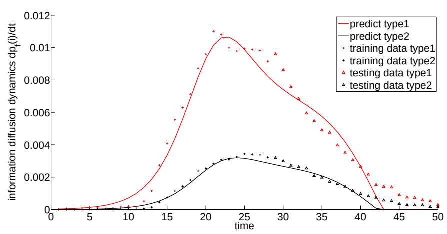


(d) #YouGetMajorPointsIf

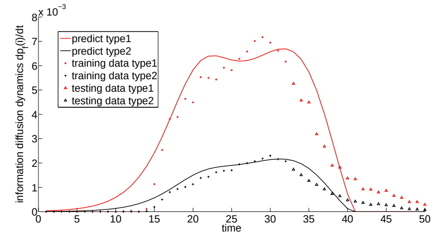
Fig. 2.15: Fitting results of the known user type model for the four popular Twitter hashtags.

2.13-(b) demonstrate the advantage of the proposed game-theoretic approach.

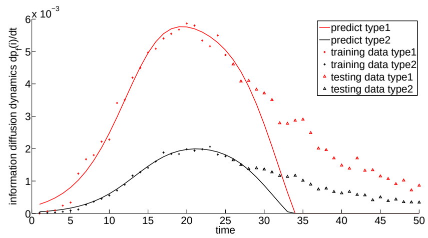
Lastly, we fit the theoretical dynamics of the known user type model with the real data of the four popular Twitter hashtags. As shown in Fig. 2.15, the theoretical dynamics fit the real data well. However, the prediction performance of the known user type model is not stable, as shown in Fig. 2.16. The reason may be that the known user type model involves more parameters and the observed data quality is not high enough to estimate them accurately.



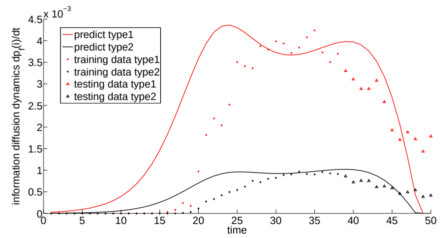
(a) #ThoughtsDuringSchool



(b) #WhenIwasLittle



(c) #DearOOMF



(d) #YouGetMajorPointsIf

Fig. 2.16: Known user type model: prediction results for various Twitter hashtags. The prediction performance of the known user type model is not stable. Sometimes, it is accurate (subfigures (a) and (b)) while sometimes not (subfigures (c) and (d)).

2.6 Summary

From the real data experiments, we see that sometimes the known user type model cannot predict the future dynamics of information diffusion well. We ascribe this to the quality of the data, i.e., the time resolution of the data is not good enough or equivalently the data is not smooth enough when we narrow the time window, since the known user type model involves more parameters than the unknown user type model and needs better data to estimate all the parameters accurately. Another reason is that unlike Facebook, in Twitter network (from which the data are collected), users often follow celebrities rather than acquaintances, which implies that Twitter users may not know their friends' types very well. Hence, the known user type model may not fit the Twitter network well. But, in the corresponding simulations, since the setup is just the known user type model, the theoretical dynamics still match the simulated ones well, demonstrating the theory itself is accurate.

Overall, we present an evolutionary game-theoretic framework to analyze the information diffusion over the heterogeneous social networks. The theoretical results fit and predict the information diffusion data generated by real-world social networks well, confirming the effectiveness of the heterogeneous game-theoretic modeling approach. The derived evolutionary dynamics can be absorbed to improve the state-of-art machine learning based method in the literature of information diffusion. More importantly, with a few parameters, our model gives a game-theoretic interpretation to the mechanism of the individuals' decision-making in the information diffusion process over the heterogeneous social networks.

Chapter 3

Understanding Popularity Dynamics: Decision-Making with Long-Term Incentives

3.1 Motivation

In the big data era nowadays, people not only read lots of data but also create vast amount of data everyday through interactions. For instance, Twitter users may mention or retweet a hashtag; Youtube users can like or dislike a video; researchers may quote keywords in papers. All of these interactions lead to a notion of *popularity dynamics* such as: Twitter hashtags' mention count dynamics and research keywords' quotation dynamics. The popularity dynamics can describe and track people's interactions with different types of items. In general, people can only pay limited attention to a limited number of items. When the number of items are growing drastically, they can only focus on certain items of their great interest. Meanwhile, in the real world, some items go viral, i.e., appealing to lots of interactions and attentions from people, while most items diminish quickly without any impact. To manage and utilize people's valuable interactions and attention better, it is crucial to understand the underlying mechanisms of the popularity dynamics and thus explain the reason why some items are so successful while others aren't.

The process of the generation of popularity dynamics is complicated and involves the decision-making of many individuals. Individual's decision is influenced by many factors including the quality and timeliness of the item, the personal preference of the individual and others' decisions. An example of Twitter hashtag is illustrated in Fig. 3.1. Thus, to model the generation process of the popularity dynamics accurately, we need to take many factors into consideration: the intrinsic attribute of an item, the decaying of the attractiveness of an item as time passes, the heterogeneity of individuals' interests, and the influence of others' decisions, i.e., network externality [105]. Since this involves the interactions between multiple decision makers, game theory [99] can be a very suitable mathematical modeling tool here. By appealing to game theory, we can incorporate all the aforementioned factors into the model of popularity dynamics and the equilibrium of the formulated game can facilitate the understanding or even prediction of users' behaviors in popularity dynamics.

In the literature, game theory has been utilized to model popularity dynamics [54, 55]. However, most of existing game-theoretic models only consider the *instantaneous incentives* of players, i.e., the decision-makers are myopic in the sense that they only decide based on the current state of the system or the current decisions of his neighbors in the network. All the myopic players in the network iteratively update their decisions, leading to a popularity dynamics of an item. On the contrary, in real-world, individuals are usually more farsighted: they may predict the subsequent behaviors of other individuals and then maximize their future

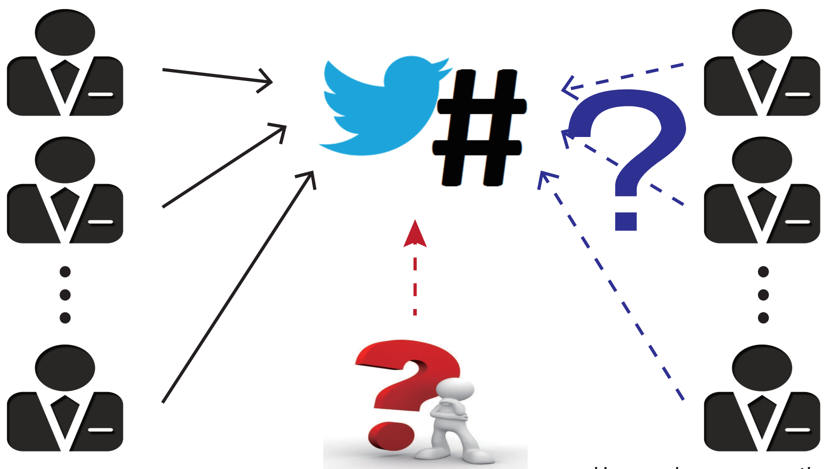


Figure 3.1: An illustration of the decision making problem of popularity dynamics. We use the mentioning of a Twitter hashtag as an example here. Consider an arbitrary Twiter user who observes a Twitter hashtag. He needs to decide whether to mention this hashtag or not based on many factors including the intrinsic quality and timeliness of the hashtag, his own interest, current popularity of the hashtag and future actions of other users.

benefits based on the predictions. In other words, individuals can have *long-term incentives* which depend on other individuals' future actions. For instance, when a Twitter user is deciding whether to forward a tweet or not, he may take the future influence of the tweet and the future actions of other users into account: will this tweet become popular in the future or will many other users also forward this tweet? This is illustrated in Fig. 3.1. Different from the previous game-theoretic frameworks for information diffusion dynamics [54,55], our model incorporates both instantaneous incentives and long-term incentives of individuals. The latter depends on subsequential individuals' actions in the future. Our main contributions in this chapter can be epitomized as follows.

- We propose a game-theoretic framework to model the sequential decision making process of general popularity dynamics. The model incorporates both instantaneous incentives and long-term incentives so that the decision-makers are farsighted enough to take others' future actions into account when making decisions.

- We theoretically show that the formulated game has a unique symmetric Nash equilibrium (SNE). We observe that the SNE is in pure strategy form and possesses a threshold structure. Furthermore, we design a backward induction algorithm to compute the SNE.

- From real data, we observe that: (i) most popularity dynamics first increase and then decrease and (ii) for some dynamics, when they are increasing, the increasing speed gradually slows down until they reach the peak and when they are decreasing, the decreasing speed also gradually slows down. We theoretically analyze these properties at the SNE of the proposed game-theoretic model. We find that the equilibrium behavior of the proposed game confirms with real data.

- The proposed theory is validated by both simulations and experiments based on real data. It is shown that the proposed game-theoretic model can even predict future dynamics of real data.

The roadmap of the rest of the paper is as follows. In Section 3.2, we review the existing literature on popularity dynamics. In Section 3.3, we describe the model in detail and formulate the problem formally. In Section 3.4, equilibrium analysis is conducted. In Section 3.5, a property of the equilibrium is shown. Simulations and real data experiments are carried out in Section 3.6. In Section 3.7, we conclude this chapter.

3.2 Related Works

Recently, intensive research efforts have been devoted to network users' behavior dynamics due to its importance [71]. In [87], Ratkiewicz *et al.* studied the popularity dynamics of online webpages and online topics. They proposed a model to combine classic preferential attachment [8] with the random popularity shifts incurred by exogenous factors. Shen *et al.* [95] proposed to use reinforced Poisson processes to model the popularity dynamics and presented a statistical inference approach to predict the future dynamics accordingly.

An important special case of popularity dynamics is the information diffusion dynamics over social networks, which have attracted tremendous research efforts in the recent decade. The abundant literature on information diffusion dynamics can be divided into two categories. In the first category, researchers combine data mining/machine learning techniques with empirical observations from real-world datasets and propose simple models to explain the observed phenomena. Yang and Leskovec [114] studied the temporal shapes of online information dynamics and clustered these temporal dynamics into several patterns, which suggest several types of information dynamics. In [70], the authors empirically studied the temporal dynamics of online memes and discovered interesting phenomena such as an average 2.5 hours lag between the peaks of a phrase in news media and in blogs respectively. The role of social networks, i.e., the influence between network users, in information diffusion is studied in [7] through an experimental approach. In [40, 89, 113], with machine learning approaches, the underlying implicit diffusion networks are inferred from the observed information cascades to better understand the diffusion processes. Guille and Hacid [43] proposed a predictive model for information diffusion process, which could predict the future information dynamics accurately. In the second category, game-theoretic analyses were conducted to understand the information diffusion processes from the perspective of individual user's decision making. This category has closer relationship with this chapter. Jiang *et al.* [54, 55] exploited evolutionary game theory to model and analyze the information diffusion dynamics, where the information diffusion is treated as the consequence of the games played by the network users. In [54, 55], the users were assumed to be myopic and didn't take other individuals' future actions into account when making decisions. Furthermore, in [37], the authors proposed a sequential game model to analyze the voting and answering behaviors in social computing systems such as Stack Overflow, which inspired our model in this chapter. The differences between the model in [37] and the model here are: (i) we focus on characterizing the temporal dynamics of the interactions while the main goal of [37] was to describe users' behaviors (voting and answering) when facing with certain system states; (ii) we include preferential attachment [8] (a universal phenomenon in network science that items with large popularity are more visible and hence can gain new popularity more easily) into our model while [37] didn't.

There are also many domain specific research literature on popularity dynamics. The citation dynamics were studied in [106] and a universal formula was proposed to characterize the temporal citation dynamics of individual papers. The channel popularity dynamics of Internet Protocol TV were investigated in [86] while the authors in [68] proposed a model predict the dynamics of news. Furthermore, a model for Twitter dynamics was presented in [60] while the dynamics of viral marketing were studied in [69].

3.3 Model

In the generation process of popularity dynamics, multiple intelligent decision makers decide whether to interact with an item or not with the goal of maximizing their own utilities. The system has network externality [105], i.e., the utility of an individual is affected by other individuals' actions, as explained in Section 3.1. Game theory is a mathematical tool to study the decision-making of multiple strategic agents where one's utility is influenced by others' actions, and thus very suitable for modeling the popularity dynamics. Additionally, there are various equilibrium concepts in the game theory literature which can serve as predictions of individuals' behaviors in popularity dynamics and thus promote the understanding of the mechanisms of popularity dynamics. In this section, we will introduce a game-theoretic model of the popularity dynamics in detail.

Suppose an item, item $\mathcal{A}$, is generated. The item can be an online meme, an online video or a keyword in scientific research. People decide whether to interact with item $\mathcal{A}$ or not sequentially. For instance, Twitter users decide whether to mention a hashtag or not; Youtube users decide whether to like a video or not; researchers decide whether to quote a keyword in their papers or not. We view the cumulative interaction dynamics x_t, i.e., the total number of interactions up to time t, as a stochastic dynamical system. We assume people, i.e., players of the game, arrive at the system at discrete time instants $t \in \mathbb{N}$ (one player arrives at each time instant) and decide whether to interact with item $\mathcal{A}$ or not. Players are heterogeneous and have different *types*, which indicate the relevances of the item to the different players. For example, for a Twitter hashtag related to football, football fans have higher types than normal users; for a research keyword related to signal processing, researchers specializing in signal processing have higher types than other researchers. We suppose that each player's type θ is a random variable distributed in $[0, 1]$ with probability density function (PDF) $h(\theta)$. The above concepts are

Table 3.1: Game-theoretic model for popularity dynamics

Game-theoretic model	Popularity dynamics concepts
System state at time t	Cumulative interactions x_t
Players	People arriving at the system
Player type	Relevance of the item to the player $\theta \in [0, 1]$
Action set of each player	$\{$`interacting`, `not interacting`$\}$

summarized in Table 5.1.

To complete the game-theoretic formulation, we need to define the utilities of the players. As stated in Section 3.1, the utility should encompass both the immediate effect and the future effect of the interactions. Furthermore, due to the preferential attachment property of networks [8], items which already get many interactions should be more visible, i.e., easier to be found by players arriving at the system. Combining all these factors, the proposed model can be illustrated in Fig. 3.2. When a player arrive at the system with state x, the item will be *visible* to him with some probability related to the current state of the system. If the item is visible to the player and he chooses to interact with the item, then he will get an *instantaneous reward* which depends on both the type (e.g., hobbies) of the player and the state of the system. Afterwards, whenever there is a new interaction with the item (occurs at say, state y), the aforementioned player will obtain a *future reward* because the current interacting player may pay attention to him. The overall utility of the player will be a discounted sum of the instantaneous reward and all the future rewards. In the following, we specify these components of the model in

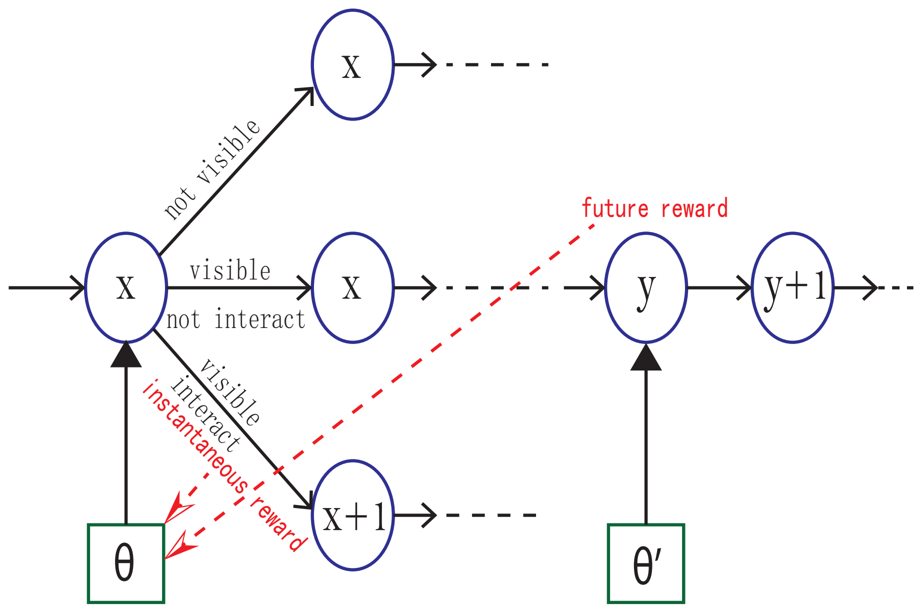


Figure 3.2: Illustration of the state transition in the system model. The numbers inside the blue circle are the current states. The numbers inside the green square are the types of the arriving players.

more detail.

3.3.1 Instantaneous Reward

Each player choosing to interact with item $\mathcal{A}$ gets an instantaneous reward $R(x, \theta)$, where x is the state of the system when the interaction occurs and θ is the type of the player. For instance, if a Twitter user is interested in a hashtag, then by mentioning this hashtag, the user will gain some immediate utility. The instantaneous reward depends on the system state since the immediate utilities of an item at different stages (e.g., incipient stage, blooming stage and ending stage) are different. The instantaneous reward also depends on the type of the interacting player because the same item is of different relevances to players of different types: a football fan can get much more utility by mentioning a football related hashtag than a normal Twitter user. Note that $R(x, \theta)$ can also be negative since the interaction implicitly incurs a cost for the player, e.g., by mentioning a hashtag, a Twitter user needs to spend some time and efforts during the manipulation. Now, we impose five assumptions on the function $R(x, \theta)$ as follows.

1. $R(x, \theta)$ decreases with respect to x.

2. $R(x, \theta)$ strictly increases with respect to θ.

3. $R(x, \theta)$ is continuous with respect to θ, for each given x.

4. $R(0, 0) < 0$ and $R(0, 1) > 0$.

5. $\lim_{x \to \infty} R(x, 1) < 0$.

The five assumptions can be justified as follows respectively. (1) Taking timeliness of the item into account, players who interact with the item early (when x is small) should get higher utility than those who interact later (when x is large). For example, a Twitter user mentioning up to date hashtags should gain higher instantaneous reward than a user mentioning outdated hashtags. (2) Those players who find the item more relevant gain higher instantaneous reward by interacting with it. (3) A technical assumption. (4) Initially (i.e., $x = 0$), some players' instantaneous rewards are positive while some are negative. (5) Even for those who find the item very relevant ($\theta = 1$), if the item is very outdated ($x \to \infty$), then it is no longer attractive.

3.3.2 Future Reward

For a player `B` choosing to interact with item $\mathcal{A}$, whenever there is a subsequent interaction with item $\mathcal{A}$, this subsequent interacting player will pay attention to player `B` with probability $\frac{1}{x}$, where x is the system state when this subsequent interaction occurs. Thus player `B` will receive an expected reward of $\frac{1}{x}$ due to the possible attention he gets. This reward is called the *future reward* since it is obtained after the interaction occurs. For instance, if a Twitter user `B` mentions a hashtag $\mathcal{A}$, and later, when hashtag $\mathcal{A}$ has already been mentioned x times, another user `C` also mentions hashtag $\mathcal{A}$. In such a case, user `C` may visit those users who have mentioned hashtag $\mathcal{A}$, and with probability $\frac{1}{x}$, user `B` will be visited by user `C`. We further assume players discount future reward with factor $0 < \lambda < 1$, which is a common assumption in dynamic games and sequential decision making. The instantaneous reward and the future reward together constitute the utility of an interacting player.

3.3.3 Visibility Probability

We assume one player arrives at the system at each time instant. Item $\mathcal{A}$ is *visible* to a player with probability $f(x) \in [0, 1]$, where x is the system state when the player arrives. In other words, after a player arrives, he/she will notice item $\mathcal{A}$ with probability $f(x)$. We also impose several assumptions on the visibility probability function $f(x)$ as follows.

- $f(x)$ increases with x. *Justification: Popular items are more visible. This is* *also refereed as the 'rich gets richer' or preferential attachment phenomenon in network science [8].*

- $f(0) > 0$. *Justification: Even the most unpopular item is visible with positive probability.*

3.3.4 Action Rule and Utility Function

When a player arrives at the system and sees item $\mathcal{A}$, he/she needs to decide whether to interact with the item or not based on the current system state x and his/her type θ. For sake of generality, we allow the players to use mixed action rule $\pi : \mathbb{N} \times [0,1] :\rightarrow [0,1]$, where $\pi(x,\theta)$ is the probability of choosing the action `interacting` when the state is x and the type of the player is θ. Denote the set of all possible mixed action rules as Π. We denote $g_\pi(x)$ the long-term utility of an interacting player starting from state x while the subsequent players use action rule π.

Denote $p^\pi(x) = \mathbb{E}_\theta[\pi(x,\theta)]$, i.e., the expected probability of a new interaction when the system state is x and users adopt action rule π. Thus, the long term utility $g_\pi(x)$ can be computed recursively as follows. $\forall x \geq 1$:

$$g_\pi(x) = \overbrace{\frac{f(x)p^\pi(x)}{x}}^{\text{instantaneous reward at the current time slot}} + \underbrace{\lambda\left\{f(x)p^\pi(x)g_\pi(x+1) + \left[1 - f(x)p^\pi(x)\right]g_\pi(x)\right\}}_{\text{reward in future time slots}}. \tag{3.1}$$

Denote $u(x,\theta,a,\pi)$ the utility of a type-θ player who enters the system in state x

and takes action a while other players adopt action rule π. Thus,

$$u(x,\theta,a,\pi) = \begin{cases} R(x,\theta) + \lambda g_\pi(x+1), & \text{if } a = \texttt{interacting}, \\ \\ 0, & \text{if } a = \texttt{not interacting}. \end{cases} \tag{3.2}$$

If a player chooses mixed action, i.e., `interacting` with probability q, then his/her utility is:

$$\mathcal{U}(x,\theta,q,\pi) = q[R(x,\theta) + \lambda g_\pi(x+1)]. \tag{3.3}$$

3.3.5 Solution Concept

In this chapter, the solution concept is chosen to be the *symmetric Nash equilibrium (SNE)*, which is defined in the following.

Definition 3.1 *An action rule π^* is said to be a symmetric Nash equilibrium (SNE) if:*

$$\pi^*(x,\theta) \in \arg\max_{q\in[0,1]} \mathcal{U}(x,\theta,q,\pi^*), \quad \forall x \in \mathbb{N}, \theta \in [0,1]. \tag{3.4}$$

In an SNE, no player wants to deviate unilaterally, hence the action rule is self enforcing.

3.4 Equilibrium Analysis

In this section, we show that there is a unique SNE of the formulated game. A backward induction algorithm for computing this unique SNE is also presented.

The infinite-horizon sequential game is effectively of finite length, given the following lemma, which says that no one will interact with item $\mathcal{A}$ after a certain number of interactions is reached.

Lemma 3.1 *There exists an $\tilde{x} \in \mathbb{N}$ such that $\forall x \geq \tilde{x}, \theta \in [0,1], \pi \in \Pi$:*

$$u(x, \theta, \texttt{interacting}, \pi) < u(x, \theta, \texttt{not interacting}, \pi), \tag{3.5}$$

i.e., the action $\texttt{interacting}$ *is strictly dominated by the action* $\texttt{not interacting}$ *regardless of the player type and other players' action rule.*

Proof: The long-term utility of an interacting player starting from state x while subsequent players use action rule π can be upper bounded as follows:

$$g_\pi(x) = \mathbb{E}_{\{x_t\}_{t=2}^{\infty}}\left[\sum_{t=1}^{\infty} \lambda^{t-1} \frac{f(x_t)p^\pi(x_t)}{x_t} \middle| \pi, x_1 = x\right] \leq \sum_{t=1}^{\infty} \lambda^{t-1}\frac{1}{x} = \frac{1}{(1-\lambda)x} \to 0, \text{ as } x \to \infty. \tag{3.6}$$

Note that $\lim_{x \to \infty} R(x, 1) < 0$. So, there exists an $\tilde{x} \in \mathbb{N}$, such that $\forall x \geq \tilde{x}$:

$$R(x, 1) + \frac{\lambda}{(1-\lambda)x} < 0. \tag{3.7}$$

Hence, $\forall x \geq \tilde{x}, \theta \in [0,1], \pi \in \Pi$:

$$R(x, \theta) + \lambda g_\pi(x+1) \leq R(x, 1) + \frac{\lambda}{(1-\lambda)x} < 0, \tag{3.8}$$

i.e., $u(x, \theta, \texttt{interacting}, \pi) < u(x, \theta, \texttt{not interacting}, \pi)$ due to the utility expression in (3.2). □

Denote $\hat{x} = \max\{x \in \mathbb{N} | R(x, 1) > 0\}$. We design a backward induction algorithm, Algorithm 3.1, to compute the SNE. We first show that the action rule obtained from Algorithm 3.1 is indeed an SNE.

Theorem 3.1 *(Existence of the SNE) The action rule π^* computed by Algorithm 3.1 is an SNE.*

Proof: According to Lemma 3.1, $\forall x \geq \tilde{x}, \theta \in [0,1] : \arg\max_{q \in [0,1]} \mathcal{U}(x, \theta, q, \pi^*) = \{0\}$. Thus, $\pi^*(x, \theta) \in \arg\max_{q \in [0,1]} \mathcal{U}(x, \theta, q, \pi^*), \forall x \geq \tilde{x}, \theta \in [0,1]$.

Algorithm 3.1: Computation of the unique equilibrium

Inputs:

Instantaneous reward function $R(x, \theta)$.

Player type PDF $h(\theta)$.

Visibility probability function $f(x)$.

Discount factor λ.

Outputs:

Unique equilibrium action rule $\pi^*(x, \theta)$.

When $x \geq \hat{x} + 1$: $\pi^*(x, \theta) = 0, \theta \in [0, 1]$; $g_{\pi^*}(x) = 0$.

Let $x = \hat{x}$.

while $x \geq 0$ **do**

  if $R(x, 0) + \lambda g_{\pi^*}(x + 1) > 0$ **then**

    $\theta_x^* = 0$,

  else

    θ_x^* is the unique solution of $R(x, \theta) + \lambda g_{\pi^*}(x + 1) = 0$.

  end if

  Compute:

$$\pi^*(x, \theta) = \begin{cases} 1, & \text{if } \theta \geq \theta_x^*, \\ 0, & \text{if } \theta < \theta_x^*, \end{cases} \tag{3.9}$$

$$p^{\pi^*}(x) = \int_{\theta_x^*}^{1} h(\theta) d\theta, \tag{3.10}$$

$$g_{\pi^*}(x) = \frac{1}{1 - \lambda[1 - f(x) p^{\pi^*}(x)]} \left[\frac{f(x) p^{\pi^*}(x)}{x} + \lambda f(x) p^{\pi^*}(x) g_{\pi^*}(x + 1) \right]. \tag{3.11}$$

  $x \leftarrow x - 1$.

end while

When $\hat{x}+1 \le x \le \tilde{x}-1$, we have the following. $\forall \theta \in [0,1]$:

$$u(x,\theta,\texttt{interacting},\pi^*) = R(x,\theta) + \lambda g_{\pi^*}(x+1) = R(x,\theta) \le R(x,1) \le 0. \quad (3.12)$$

So, $\pi^*(x,\theta) = 0 \in \arg\max_{q\in[0,1]} \mathcal{U}(x,\theta,q,\pi^*)$.

When $x = \hat{x}$, we still have $u(\hat{x},\theta,\texttt{interacting},\pi^*) = R(x,\theta)$. If $\theta \ge \theta_x^*$, we have $u(\hat{x},\theta,\texttt{interacting},\pi^*) \ge 0$ and thus $\pi^*(x,\theta) = 1 \in \arg\max_{q\in[0,1]} \mathcal{U}(x,\theta,q,\pi^*)$. If $\theta < \theta_x^*$, we have $u(\hat{x},\theta,\texttt{interacting},\pi^*) < 0$ and hence $\pi^*(x,\theta) = 0 \in \arg\max_{q\in[0,1]}$ $\mathcal{U}(x,\theta,q,\pi^*)$.

When $x \le \hat{x}-1$, we discuss two cases:

- Case 1: $R(x,0) + \lambda g_{\pi^*}(x+1) > 0$. In this case, $\theta_x^* = 0$ and $\pi^*(x,\theta) = 1, \forall\theta \in [0,1]$. Thus, $u(x,\theta,\texttt{interacting},\pi^*) \ge R(x,0) + \lambda g_{\pi^*}(x+1) > 0, \forall\theta$. Hence, $\pi^*(x,\theta) = 1 \in \arg\max_{q\in[0,1]} \mathcal{U}(x,\theta,q,\pi^*), \forall\theta$.

- Case 2: $R(x,0) + \lambda g_{\pi^*}(x+1) \le 0$. In such a case, if $\theta \ge \theta_x^*$, then $u(x,\theta,\texttt{interacting},\pi^*) \ge 0$ and thus $\pi^*(x,\theta) = 1 \in \arg\max_{q\in[0,1]} \mathcal{U}(x,\theta,q,\pi^*)$. Otherwise, if $\theta < \theta_x^*$, then $u(x,\theta,\texttt{interacting},\pi^*) < 0$ and $\pi^*(x,\theta) = 0 \in \arg\max_{q\in[0,1]} \mathcal{U}(x,\theta,q,\pi^*)$.

Overall, we always have $\pi^*(x,\theta) \in \arg\max_{q\in[0,1]} \mathcal{U}(x,\theta,q,\pi^*), \forall x, \theta \in [0,1]$. $\square$

We further prove that the π^* computed in Algorithm 3.1 is indeed the unique SNE.

Theorem 3.2 *(Uniqueness of the SNE) Suppose the distribution of player type θ is atomless, i.e., $h(\theta)$ is finite for every $\theta \in [0,1]$. Then, any SNE $\tilde{\pi}$ differs with π^* for zero mass players, i.e., $\mathbb{P}\{\tilde{\pi}(x,\theta) \ne \pi^*(x,\theta)\} = 0$ for every $x \in \mathbb{N}$.*

Proof: Suppose $\tilde{\pi}$ is an SNE, i.e., $\tilde{\pi}(x,\theta) \in \arg\max_{q\in[0,1]} \mathcal{U}(x,\theta,q,\tilde{\pi}), \forall x,\theta \in [0,1]$. In the following, we show that $\tilde{\pi}$ differs from π^* for zero-mass players. We discuss for different values of x as follows.

- Case 1: $x \geq \tilde{x}$. From Lemma 3.1, we know that $\forall \theta \in [0,1], u(x,\theta,\texttt{interacting},\tilde{\pi}) < 0$. So, $\tilde{\pi}(x,\theta) = 0 = \pi^*(x,\theta)$.

- Case 2: $\hat{x}+1 \leq x \leq \tilde{x}-1$. First, we note that $u(\tilde{x}-1,\theta,\texttt{interacting},\tilde{\pi}) = R(\tilde{x}-1,\theta)$. When $\theta < 1$, $u(\tilde{x}-1,\theta,\texttt{interacting},\tilde{\pi}) < 0$, and thus $\tilde{\pi}(\tilde{x}-1,\theta) = 0$. Hence, $p^{\tilde{\pi}}(\tilde{x}-1) = 0$ and $g_{\tilde{\pi}}(\tilde{x}-1) = 0$. Suppose we have $\tilde{\pi}(x+1,\theta) = g_{\tilde{\pi}}(x+1) = 0$, where $\hat{x}+1 \leq x \leq \tilde{x}-2$. Then, for $\theta < 1$, we have $u(x,\theta,\texttt{interacting},\tilde{\pi}) = R(x,\theta) < 0$. Hence, $\tilde{\pi}(x,\theta) = 0, \forall\theta$. Thus, $p^{\tilde{x}}(x) = g_{\tilde{x}}(x) = 0$. By induction, we know that $\tilde{\pi}(x,\theta) = g_{\tilde{\pi}}(x) = 0, \forall \hat{x}+1 \leq x \leq \tilde{x}-1$ and $\theta < 1$. In particular, we still have $\tilde{\pi}(x,\theta) = \pi^*(x,\theta)$, $\forall \hat{x}+1 \leq x \leq \tilde{x}-1$ and $\theta < 1$.

- Case 3: $x = \hat{x}$. $u(\hat{x},\theta,\texttt{interacting},\tilde{\pi}) = R(\hat{x},\theta)$. If $\theta > \theta^*_{\hat{x}}$, we have $u(\hat{x},\theta,\texttt{interacting},\tilde{\pi}) > 0$ and thus $\tilde{\pi}(\hat{x},\theta) = 1$. Similarly, if $\theta < \theta^*_{\hat{x}}$, we have $\tilde{\pi}(\hat{x},\theta) = 0$. So, $\tilde{\pi}(\hat{x},\theta) = \pi^*(\hat{x},\theta), \forall\theta \neq \theta^*_{\hat{x}}$. Hence, $p^{\tilde{\pi}}(\hat{x}) = p^{\pi^*}(\hat{x}), g_{\tilde{\pi}}(\hat{x}) = g_{\pi^*}(\hat{x}), \tilde{\pi}(\hat{x},\theta) = \pi^*(\hat{x},\theta), \forall\theta \neq \theta^*_{\hat{x}}$.

- Case 4: $x \leq \hat{x}-1$. Suppose for some $1 \leq x \leq \hat{x}-1$, we have $g_{\tilde{x}}(x+1) = g_{\pi^*}(x+1)$ and $\tilde{\pi}(x+1,\theta) = \pi^*(x+1,\theta), \forall\theta \neq \theta^*_{x+1}$. We note that these already hold for $x = \hat{x}-1$ according to Case 3. Thus, we have $u(x,\theta,\texttt{interacting},\tilde{\pi}) = R(x,\theta) + \lambda g_{\tilde{\pi}}(x+1) = R(x,\theta) + \lambda g_{\pi^*}(x+1)$. If $\theta > \theta^*_x$, we have $u(x,\theta,\texttt{interacting},\tilde{\pi}) > 0$ and thus $\tilde{\pi}(x,\theta) = 1$. Otherwise, if $\theta < \theta^*_x$, we have $u(x, \theta, \texttt{interacting}, \tilde{\pi}) < 0$ and $\tilde{\pi}(x, \theta) = 0$. Consequently, we have $\tilde{\pi}(x, \theta) = \pi^*(x, \theta), \forall \theta \neq \theta_x^*$ and hence $g_{\tilde{\pi}}(x) = g_{\pi^*}(x)$. So, by induction, we have $\tilde{\pi}(x, \theta) = \pi^*(x, \theta), \forall \theta \neq \theta_x^*, \forall x \leq \hat{x} - 1, \theta \neq \theta_x^*$.

In all, we have $\tilde{\pi} = \pi^*$ except for a zero mass amount of players. □

Remark 3.1 *The unique SNE of the game is in pure strategy form and possesses a threshold structure. For every state x, there exists a threshold θ_x^* such that a player of type θ will interact with item $\mathcal{A}$ if and only if $\theta \geq \theta_x^*$.*

3.5 Popularity Dynamics at the Equilibrium

In this section, we first observe some properties of popularity dynamics from real data. Then, we analyze the corresponding properties at the equilibrium of the proposed game. We find that the equilibrium behavior of the proposed game-theoretic model confirms with the real data.

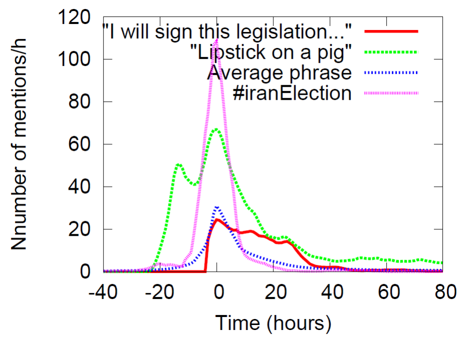
3.5.1 Observations from real data

In Fig. 3.3, we plot mention dynamics of popular memes and sum citation dynamics of all the papers published in *Nature* in 1990. Here, we use the dynamics of memes and the citation dynamics of papers as examples of popularity dynamics.

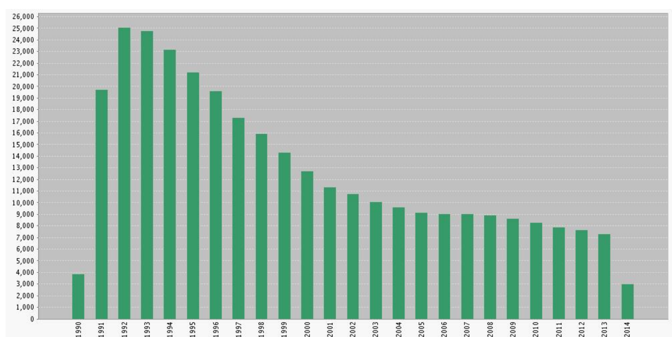
We observe that, typically, the popularity dynamics of an item will first increase and then decrease, leading to a peak in the dynamics. This is a general *first order* property of popularity dynamics. Thus, a natural question is: does the equilibrium behavior of the proposed game-theoretic model possess this property?

Intuitively, it should. The reason is as follows. At first, according to our model, the visibility probability is low since the item has few interactions. As time goes, the item accumulates more interactions so that the visibility probability increases and the interaction rate, i.e., the dynamics observed in Fig. 3.3, also increases. When time is sufficiently large, the visibility probability basically saturates. With the augment of the cumulative interactions, the instantaneous reward and long-term reward decreases so that few players will further interact with the item, leading to a decrease in interaction rate. In next subsection, we will formally state and prove this first order property.

Furthermore, we observe that some popularity dynamics, especially the citation dynamics of papers as in Fig. 3.3-(b), Fig. 3.6 and Fig. 3.7-(c)(d), have the following *second order* property: when it is increasing, its increasing speed gradually slows down and when it is decreasing, its decreasing speed also gradually slows down. This behavior is reasonable. Many items' interaction rates increase drastically when they first come out and keep increasing (but at a lower speed) until they reach the peak. Later, after the items are no longer that popular, their interaction rates decrease quickly and will keep decreasing for some time (but at a lower speed). In next subsection, we will formally state and prove this second order property under certain assumptions.



(a) Mention dynamics of popular memes.



(b) Sum citation dynamics of all the papers published in Nature in 1990.

Fig. 3.3: Real-world popularity dynamics [46, 114].

3.5.2 Properties at the equilibrium

Generally, the unique SNE should be computed using the backward induction as specified in Algorithm 3.1, which is hard to analyze. To facilitate analysis, we further restrict attention to models satisfying the following three assumptions:

(1) (Linear reward) $R(x, \theta) = -x + a\theta - b$, where $a > b > 0$.

(2) (Uniform player type distribution) $h(\theta) = 1,\ \forall \theta \in [0, 1]$.

(3) (Saturated visibility probability) There exists a $\check{x} \in \mathbb{N}$ less than $\hat{x} = \lfloor a - b \rfloor$ such that: $\forall x \geq \check{x}:\ f(x) = 1$, and $\forall 1 \leq x \leq \check{x}$:

$$\frac{f(x)}{f(x-1)} \geq 1 + \frac{1 + \frac{\lambda}{x}}{a - b - \check{x}}. \tag{3.13}$$

These three assumptions can be justified as follows. (1) Linear reward is used to simplify the analysis and it indeed increases with θ and decreases with x, which coincides with the assumptions in Section 3.3. (2) Uniform player type distribution is also to simplify calculations though our analysis is applicable to more complicated distributions in principle. (3) When the number of interactions is large enough, the item becomes 'famous' enough so that it is visible to everyone arriving at the system. Before this saturation occurs, however, it increases at a speed not too slow. Note that the R.H.S. of (3.13) is very close to 1 since the numerator of the ratio is close to 1 while the denominator is some integer much larger than 1 generally. So, the assumption is indeed very weak.

Denote $r(x) = f(x)p^{\pi^*}(x)$, i.e., the probability that there is a new interaction at state x in the SNE. We first show the first order property of the SNE.

Theorem 3.3 ***(First order characterization of the SNE)*** *Suppose the assumptions (1)(2)(3) hold, the SNE π^* satisfies the following:*

- *For $1 \leq x \leq \check{x}$: $r(x) \geq r(x-1)$;*
- *For $x \geq \check{x}$: $r(x) \geq r(x+1)$.*

In other words, the interaction rate $r(x)$ first increases and then decreases.

Proof: According to the assumptions of linear reward and uniform player type distribution, we can obtain closed form expression of the iterative update of θ_x^* and $p^{\pi^*}(x)$ in Algorithm 3.1 as follows: $\forall x \leq \hat{x} = \lfloor a - b \rfloor$:

$$p^{\pi^*}(x) = 1 - \frac{1}{a}(x + b - \lambda g_{\pi^*}(x+1))^+, \tag{3.14}$$

$$g_{\pi^*}(x) = \frac{1}{1 - \lambda(1 - f(x)p^{\pi^*}(x))}\left[\frac{f(x)p^{\pi^*}(x)}{x} + \lambda f(x)p^{\pi^*}(x)g_{\pi^*}(x+1)\right], \tag{3.15}$$

where $x^+ \triangleq \max\{x, 0\}$.

We first consider the case $x \geq \check{x}$. In the following, we show that g_{π^*} is decreasing for $\check{x} \leq x \leq \hat{x} + 1$. When $\check{x} \leq x \leq \hat{x}$, noticing that $f(x) = 1$, we rewrite (3.15) as:

$$g_{\pi^*}(x) - g_{\pi^*}(x+1) = \frac{p^{\pi^*}(x)}{1 - \lambda(1 - p^{\pi^*}(x))}\left[\frac{1}{x} - \frac{1-\lambda}{p^{\pi^*}(x)}g_{\pi^*}(x+1)\right]. \tag{3.16}$$

Since $g_{\pi^*}(\hat{x}+1) = 0$, we have:

$$0 = g_{\pi^*}(\hat{x}+1) \leq g_{\pi^*}(\hat{x}) = \frac{p^{\pi^*}(\hat{x})}{[1 - \lambda(1 - p^{\pi^*}(\hat{x}))]\hat{x}} \leq \frac{p^{\pi^*}(x)}{(1-\lambda)\hat{x}}. \tag{3.17}$$

Suppose $g_{\pi^*}(x) \geq g_{\pi^*}(x+1)$ and $g_{\pi^*}(x) \leq \frac{p^{\pi^*}(x)}{(1-\lambda)x}, \forall m \leq x \leq \hat{x}$ for some $\check{x} + 1 \leq m \leq \hat{x}$ (note that we already show that these hold for $m = \hat{x}$). We next show $g_{\pi^*}(m-1) \geq g_{\pi^*}(m)$ and $g_{\pi^*}(m-1) \leq \frac{p^{\pi^*}(m-1)}{(1-\lambda)(m-1)}$.

According to (3.14), for $m \leq x \leq \hat{x}$:

$$p^{\pi^*}(x-1) = 1 - \frac{1}{a}(x - 1 + b - \lambda g_{\pi^*}(x))^+, \tag{3.18}$$

$$p^{\pi^*}(x) = 1 - \frac{1}{a}(x + b - \lambda g_{\pi^*}(x+1))^+. \tag{3.19}$$

Since $g_{\pi^*}(x) \geq g_{\pi^*}(x+1), \forall m \leq x \leq \hat{x}$, comparing the above two expressions, we have $p^{\pi^*}(x-1) \geq p^{\pi^*}(x), \forall m \leq x \leq \hat{x}$. In particular, $p^{\pi^*}(m-1) \geq p^{\pi^*}(m)$, thus,

$$g_{\pi^*}(m) \leq \frac{p^{\pi^*}(m)}{(1-\lambda)m} \leq \frac{p^{\pi^*}(m-1)}{(1-\lambda)(m-1)}. \tag{3.20}$$

Hence, by (3.16) and (3.20), we obtain:

$$g_{\pi^*}(m-1) - g_{\pi^*}(m) = \frac{p^{\pi^*}(m-1)}{1 - \lambda(1 - p^{\pi^*}(m-1))} \left[\frac{1}{m-1} - \frac{1-\lambda}{p^{\pi^*}(m-1)} g_{\pi^*}(m) \right] \geq 0. \tag{3.21}$$

So, $g_{\pi^*}(m-1) \geq g_{\pi^*}(m)$. In addition:

$$g_{\pi^*}(m-1) = \mathbb{E}\left[\sum_{t=1}^{\infty} \lambda^{t-1} \frac{p^{\pi^*}(x_t)}{x_t} \middle| \pi^*, m-1 \right] \leq \sum_{t=1}^{\infty} \lambda^{t-1} \frac{p^{\pi^*}(m-1)}{m-1} = \frac{p^{\pi^*}(m-1)}{(1-\lambda)(m-1)}. \tag{3.22}$$

Hence, by induction, we have $g_{\pi^*}(x) \geq g_{\pi^*}(x+1)$ and $g_{\pi^*}(x) \leq \frac{p^{\pi^*}(x)}{(1-\lambda)x}, \forall \check{x} \leq x \leq \hat{x}$. Thus, by (3.14), we have: $p^{\pi^*}(x-1) \geq p^{\pi^*}(x), \forall \check{x} \leq x \leq \hat{x}$. Note that $p^{\pi^*}(x) = 0, \forall x \geq \hat{x}+1$ since $\pi^*(x, \theta) = 0, \forall x \geq \hat{x}+1, \theta \in [0,1]$. Thus, for $x \geq \check{x}$: $p^{\pi^*}(x) \geq p^{\pi^*}(x+1)$. So, for $x \geq \check{x}$: $r(x) \geq r(x+1)$.

Next, we consider the case $x \leq \check{x} (\leq \hat{x})$. In such a case, we rewrite the update equations (3.14) and (3.15) in terms of g_{π^*} and r as follows:

$$g_{\pi^*}(x) = \frac{r(x)}{1 - \lambda(1 - r(x))} \left[\frac{1}{x} + \lambda g_{\pi^*}(x+1) \right], \forall 1 \leq x \leq \check{x}, \tag{3.23}$$

$$r(x) = \left[1 - \frac{1}{a}(x + b - \lambda g_{\pi^*}(x+1))^+ \right] f(x), \forall 0 \leq x \leq \check{x}. \tag{3.24}$$

Rewriting (3.23) yields: $\forall 1 \le x \le \check{x}$:

$$g_{\pi^*}(x) - g_{\pi^*}(x+1) = \frac{r(x)}{1-\lambda(1-r(x))}\left[\frac{1}{x} - \frac{1-\lambda}{r(x)} g_{\pi^*}(x+1)\right] \le \frac{r(x)}{1-\lambda(1-r(x))}\frac{1}{x} \le \frac{1}{x}. \tag{3.25}$$

For $1 \le x \le \check{x}$:

$$\frac{r(x)}{r(x-1)} = \frac{1-\frac{1}{a}(x+b-\lambda g_{\pi^*}(x+1))^+}{1-\frac{1}{a}(x-1+b-\lambda g_{\pi^*}(x))^+}\frac{f(x)}{f(x-1)}. \tag{3.26}$$

For $1 \le x \le \check{x}$, from (3.25) we have:

$$(x+b-\lambda g_{\pi^*}(x+1)) - (x-1+b-\lambda g_{\pi^*}(x)) = 1+\lambda[g_{\pi^*}(x) - g_{\pi^*}(x+1)] \le 1+\frac{\lambda}{x}. \tag{3.27}$$

So,

$$(x+b-\lambda g_{\pi^*}(x+1))^+ - (x-1+b-\lambda g_{\pi^*}(x))^+ \le 1+\frac{\lambda}{x}. \tag{3.28}$$

Hence,

$$\left[1-\frac{1}{a}(x+b-\lambda g_{\pi^*}(x+1))^+\right] - \left[1-\frac{1}{a}(x-1+b-\lambda g_{\pi^*}(x))^+\right] \le \frac{1}{a}\left(1+\frac{\lambda}{x}\right). \tag{3.29}$$

We further know that:

$$1-\frac{1}{a}(x+b-\lambda g_{\pi^*}(x+1))^+ \ge 1-\frac{1}{a}(x+b) \ge 1-\frac{1}{a}(\check{x}+b). \tag{3.30}$$

Thus,

$$\frac{1-\frac{1}{a}(x-1+b-\lambda g_{\pi^*}(x))^+}{1-\frac{1}{a}(x+b-\lambda g_{\pi^*}(x+1))^+} - 1 \le \frac{\frac{1}{a}\left(1+\frac{\lambda}{x}\right)}{1-(x+b-\lambda g_{\pi^*}(x+1))^+} \tag{3.31}$$

$$\le \frac{\frac{1}{a}\left(1+\frac{\lambda}{x}\right)}{1-\frac{1}{a}(\check{x}+b)} \tag{3.32}$$

$$= \frac{1+\frac{\lambda}{x}}{a-b-\check{x}}. \tag{3.33}$$

Note that for $1 \leq x \leq \check{x}$:

$$1 + \frac{1 + \frac{\lambda}{x}}{a - b - \check{x}} \leq \frac{f(x)}{f(x-1)}. \tag{3.34}$$

Combining (3.33) and (3.34) yields:

$$\frac{1 - \frac{1}{a}(x - 1 + b - \lambda g_{\pi^*}(x))^+}{1 - \frac{1}{a}(x + b - \lambda g_{\pi^*}(x+1))^+} \leq \frac{f(x)}{f(x-1)}, \tag{3.35}$$

which, according to (3.26), is equivalent to:

$$r(x) \geq r(x-1), \tag{3.36}$$

where $1 \leq x \leq \check{x}$.

$\square$

Next we turn to the second order property of the SNE.

Theorem 3.4 ***(Second order characterization of the SNE)*** *Suppose that the assumptions (1)(2)(3) hold. Further assume that (i)* $\lambda \leq \frac{1}{a-b}$ *and (ii)* $\forall\ 2 \leq x \leq \check{x}$: $f(x) + \left(1 + \frac{1+\frac{\lambda}{x}}{a-b-\check{x}}\right) f(x-2) \leq 2f(x-1)$. *Then the SNE* π^* *satisfies the following:*

- *For* $2 \leq x \leq \check{x}$: $0 \leq r(x) - r(x-1) \leq r(x-1) - r(x-2)$;
- *For* $x \geq \check{x} + 2$: $0 \leq r(x-1) - r(x) \leq r(x-2) - r(x-1)$.

In other words, we have: (a) when $r(x)$ *is increasing, its increasing speed gradually slows down; (b) when* $r(x)$ *is decreasing, its decreasing speed also gradually slows down.*

Proof: We first consider the case $\check{x} + 2 \leq x \leq \hat{x} - 1$. From $\lambda \leq \frac{1}{a-b}$, we get:

$$\lambda \left(a - b - \check{x} + \frac{\lambda}{(1-\lambda)(\check{x}+1)} \right) \leq 1. \tag{3.37}$$

Since $p^{\pi^*}(x)$ is decreasing for $x \geq \check{x}$, we have:

$$p^{\pi^*}(x) \leq p^{\pi^*}(\check{x}) = 1 - \frac{1}{a}(\check{x} + b - \lambda g_{\pi^*}(\check{x}+1)) \leq 1 - \frac{1}{a}(\check{x}+b) + \frac{\lambda}{a(1-\lambda)(\check{x}+1)} \leq \frac{1}{\lambda a}, \tag{3.38}$$

where in the second last inequality and last inequality we make use of (3.6) and (3.37) respectively. Furthermore,

$$\frac{p^{\pi^*}(x)}{1 - \lambda(1 - p^{\pi^*}(x))} = \frac{1}{\lambda + \frac{1-\lambda}{p^{\pi^*}(x)}} \leq 1. \tag{3.39}$$

So, together with (4.11), we have:

$$\frac{\lambda p^{\pi^*}(x)}{1 - \lambda(1 - p^{\pi^*}(x))} \leq \frac{1}{a p^{\pi^*}(x)}. \tag{3.40}$$

For any $\check{x} \leq x \leq \hat{x}$:

$$x + b - \lambda g_{\pi^*}(x+1) \geq \check{x} + b - \lambda g_{\pi^*}(\check{x}+1) \geq \check{x} + b - \frac{\lambda}{(1-\lambda)(\check{x}+1)} \geq 0. \tag{3.41}$$

So, from (3.14), for any $\check{x} + 2 \leq x \leq \hat{x} - 1$, we obtain:

$$p^{\pi^*}(x-1) - p^{\pi^*}(x) = \frac{1}{a} + \frac{\lambda}{a} g_{\pi^*}(x) - \frac{\lambda}{a} g_{\pi^*}(x+1) \geq \frac{1}{a}, \tag{3.42}$$

where the last inequality is due to the monotonicity of $g_{\pi^*}(x)$ for $x \geq \check{x}$. Combining (4.12) and (4.10) yields $\forall \check{x} + 2 \leq x \leq \hat{x} - 1$:

$$\frac{\lambda p^{\pi^*}(x)}{1 - \lambda(1 - p^{\pi^*}(x))} \leq \frac{p^{\pi^*}(x-1) - p^{\pi^*}(x)}{p^{\pi^*}(x)}. \tag{3.43}$$

From (3.15), noticing that $f(x) = 1$, we have $\forall \check{x} + 2 \leq x \leq \hat{x} - 1$:

$$g_{\pi^*}(x+1) \geq \frac{1}{1 - \lambda(1 - p^{\pi^*}(x))} \frac{p^{\pi^*}(x)}{x}, \tag{3.44}$$

and thus

$$
\begin{aligned}
\frac{g_{\pi^*}(x)}{g_{\pi^*}(x+1)} &\leq \frac{p^{\pi^*}(x)}{1-\lambda(1-p^{\pi^*(x)})}\left\{\frac{1}{x}\frac{x[1-\lambda(1-p^{\pi^*}(x))]}{p^{\pi^*}(x)}+\lambda\right\} \\
&= 1+\frac{\lambda p^{\pi^*}(x)}{1-\lambda(1-p^{\pi^*}(x))} \\
&\leq \frac{p^{\pi^*}(x-1)}{p^{\pi^*}(x)},
\end{aligned}
\tag{3.45}
$$

where the last inequality is due to (3.43). Hence, $\forall \check{x}+2 \leq x \leq \hat{x}-1$:

$$
\frac{g_{\pi^*}(x+1)}{p^{\pi^*}(x)} \geq \frac{g_{\pi^*}(x)}{p^{\pi^*}(x-1)}. \tag{3.46}
$$

From (3.15), we have:

$$
g_{\pi^*}(x)-g_{\pi^*}(x+1) = \frac{p^{\pi^*}(x)}{1-\lambda(1-p^{\pi^*}(x))}\left[\frac{1}{x}-\frac{1-\lambda}{p^{\pi^*}(x)}g_{\pi^*}(x+1)\right], \tag{3.47}
$$

$$
g_{\pi^*}(x-1)-g_{\pi^*}(x) = \frac{p^{\pi^*}(x-1)}{1-\lambda(1-p^{\pi^*}(x-1))}\left[\frac{1}{x-1}-\frac{1-\lambda}{p^{\pi^*}(x-1)}g_{\pi^*}(x)\right]. \tag{3.48}
$$

From monotonicity of $p^{\pi^*}(x)$ on $x \geq \check{x}$, we have:

$$
\frac{p^{\pi^*}(x)}{1-\lambda(1-p^{\pi^*}(x))} \leq \frac{p^{\pi^*}(x-1)}{1-\lambda(1-p^{\pi^*}(x-1))}. \tag{3.49}
$$

Combining (3.46) and (3.49) yields:

$$
g_{\pi^*}(x)-g_{\pi^*}(x+1) \leq g_{\pi^*}(x-1)-g_{\pi^*}(x). \tag{3.50}
$$

From (3.14) and (3.41), we get:

$$
2p^{\pi^*}(x-1)-p^{\pi^*}(x)-p^{\pi^*}(x-2) = \frac{\lambda}{a}\left[2g_{\pi^*}(x)-g_{\pi^*}(x-1)-g_{\pi^*}(x+1)\right] \leq 0. \tag{3.51}
$$

So, $r(x-1)-r(x) \leq r(x-2)-r(x-1), \forall \check{x}+2 \leq x \leq \hat{x}-1$. Furthermore, since $0 = 2g_{\pi^*}(\hat{x}+1) \leq g_{\pi^*}(\hat{x})+g_{\pi^*}(\hat{x}+2)$, we have:

$$
\begin{aligned}
&2\left[1-\frac{1}{a}(\hat{x}+b-\lambda g_{\pi^*}(\hat{x}+1))\right] \\
&\leq 1-\frac{1}{a}(\hat{x}-1+b-\lambda g_{\pi^*}(\hat{x}))+1-\frac{1}{a}(x+1+b-\lambda g_{\pi^*}(\hat{x}+2)) \\
&< 1-\frac{1}{a}(\hat{x}-1+b-\lambda g_{\pi^*}(\hat{x})).
\end{aligned}
\tag{3.52}
$$

Hence,

$$p^{\pi^*}(\hat{x}) \leq \frac{1}{2}p^{\pi^*}(\hat{x}-1) = \frac{1}{2}(p^{\pi^*}(\hat{x}-1) + p^{\pi^*}(\hat{x}+1)) \tag{3.53}$$

Thus, we have $r(\hat{x}) - r(\hat{x}+1) \leq r(\hat{x}-1) - r(\hat{x})$. Since $\lambda \leq \frac{1}{a-b}$, we have $\hat{x} - 1 \leq (1-\lambda(1-p^{\pi^*}(\hat{x})))\hat{x}$. Together with $p^{\pi^*}(\hat{x}) \leq \frac{1}{2}p^{\pi^*}(\hat{x}-1)$, we have:

$$\frac{p^{\pi^*}(\hat{x}-1)}{\hat{x}-1} \geq \frac{2}{1-\lambda(1-p^{\pi^*}(\hat{x}))}\frac{p^{\pi^*}(\hat{x})}{\hat{x}} = 2g_{\pi^*}(\hat{x}) \geq (2 - 2\lambda + \lambda p^{\pi^*}(\hat{x}-1))g_{\pi^*}(\hat{x}). \tag{3.54}$$

So,

$$g_{\pi^*}(\hat{x}-1) = \frac{1}{1-\lambda(1-p^{\pi^*}(\hat{x}-1))}\left[\frac{p^{\pi^*}(\hat{x}-1)}{\hat{x}-1} + \lambda p^{\pi^*}(\hat{x}-1)g_{\pi^*}(\hat{x})\right] \geq 2g_{\pi^*}(\hat{x}). \tag{3.55}$$

Thereby,

$$2p^{\pi^*}(\hat{x}-1) - p^{\pi^*}(\hat{x}) - p^{\pi^*}(\hat{x}-2) = \frac{\lambda}{a}[2g_{\pi^*}(\hat{x}) - g_{\pi^*}(\hat{x}-1)] \leq 0. \tag{3.56}$$

So, $r(\hat{x}-1) - r(\hat{x}) \leq r(\hat{x}-2) - r(\hat{x}-1)$. Hence, overall, $r(x-1) - r(x) \leq r(x-2) - r(x-1), \forall x \geq \check{x} + 2$.

Now, consider $2 \leq x \leq \check{x}$. Because $g_{\pi^*}(x+1) \leq \frac{1}{(1-\lambda)(x+1)} \leq \frac{1}{(1-\lambda)x}$, we have $\frac{1}{x} - \frac{1-\lambda}{r(x)}g_{\pi^*}(x+1) \geq \frac{1}{x}\left(1 - \frac{1}{r(x)}\right)$. Hence, from (3.25), we get:

$$g_{\pi^*}(x) - g_{\pi^*}(x+1) \geq \frac{r(x)}{1-\lambda(1-r(x))}\frac{1}{x}\left(1 - \frac{1}{r(x)}\right) = -\frac{1-r(x)}{1-\lambda(1-r(x))}\frac{1}{x} \geq -\frac{1}{\lambda}. \tag{3.57}$$

Thus,

$$(x + b - \lambda g_{\pi^*}(x+1))^+ \geq (x - 1 + b - \lambda g_{\pi^*}(x))^+. \tag{3.58}$$

So,

$$\frac{1 - \frac{1}{a}(x + b - \lambda g_{\pi^*}(x+1))^+}{1 - \frac{1}{a}(x - 1 + b - \lambda g_{\pi^*}(x))^+} \leq 1. \tag{3.59}$$

Together with (3.33), we know that:

$$
\begin{aligned}
&\frac{1-\frac{1}{a}(x+b-\lambda g_{\pi^*}(x+1))^+}{1-\frac{1}{a}(x-1+b-\lambda g_{\pi^*}(x))^+} f(x) + \frac{1-\frac{1}{a}(x-2+b-\lambda g_{\pi^*}(x-1))^+}{1-\frac{1}{a}(x-1+b-\lambda g_{\pi^*}(x))^+} f(x-2) \\
&\le f(x) + \left(1 + \frac{1+\frac{\lambda}{x}}{a-b-\check{x}}\right) f(x-2) \qquad (3.60) \\
&\le 2f(x-1).
\end{aligned}
$$

Thus, $r(x)+r(x-2) \le 2r(x-1)$, i.e., $r(x)-r(x-1) \le r(x-1)-r(x-2), \forall\, 2 \le x \le \check{x}$.

□

Remark 3.2 *Assumption (i) of Theorem 3.4 requires the discount factor* λ *to be sufficiently small, or in other words, players of the popularity dynamics game are myopic and don't care about future rewards very much. Assumption (ii) is basically equivalent to* $f(x) - f(x-1) \le f(x-1) - f(x-2), \forall 2 \le x \le \check{x}$ *since the ratio in the parenthesis of (ii) is usually very small. This requires* $f(x)$*'s increasing speed is slowing down as* x *approaches* $\check{x}$*, which is a reasonable assumption. Moreover, we notice that Theorem 3.4 does not cover all the situations of popularity dynamics. There are real-world popularity dynamics, such as those in Fig. 3.3-(a), which have more complicated second order patterns. For example, during increasing phase of the dynamics, the increasing speed can first increase and then decrease. Due to the intricacy of these second order patterns, we don't give theoretical discussions about them here.*

3.6 Simulations and Real Data Experiments

In this section, we conduct simulations and real data experiments to validate the theoretical results obtained. We choose the form of instantaneous reward function to be linear, i.e., $R(x, \theta) = -x + a\theta - b$, where $a > b > 0$.

3.6.1 Simulations

In our simulation, we define the visibility probability function $f(x)$ in the following form:

$$f(x) = \alpha \left(1 - e^{-\beta x}\right) + 1 - \alpha, \tag{3.61}$$

where α, β are parameters controlling the initial visibility probability and the increasing speed of the visibility probability. We set the discount factor to be $\lambda = 0.5$. For different parameter setups of a, b, α, β, we stochastically simulate the equilibrium popularity dynamics calculated by Algorithm 3.1 many times and then take average of them. Here, the equilibrium behaviors are stochastic because (i) the user types are random variables; (ii) whether the item is visible to the arriving player is random. We also theoretically compute the expected equilibrium popularity dynamics by Algorithm 3.1, which serve as the theoretical dynamics. Specifically, for theoretical dynamics, at each time instant, we replace the actual stochastic equilibrium behavior with the expected equilibrium behavior. This deviation may affect the system state at the next time instant, which in turn influence the equilibrium behaviors at the next time instant since players' strategies depend on the system state. In other words, the deviation caused by using the expected equilibrium behaviors to approximate the actual stochastic equilibrium behaviors may propagate and accumulate. The simulations are aimed at verifying that this approximation does not hurt much, i.e., the theoretical dynamics can still match well with the

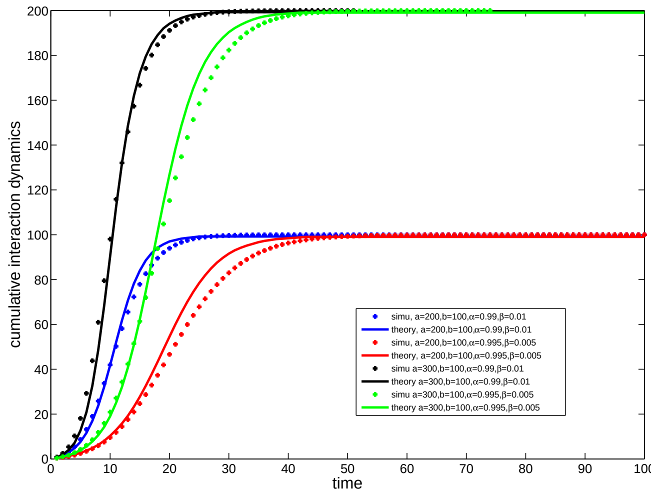
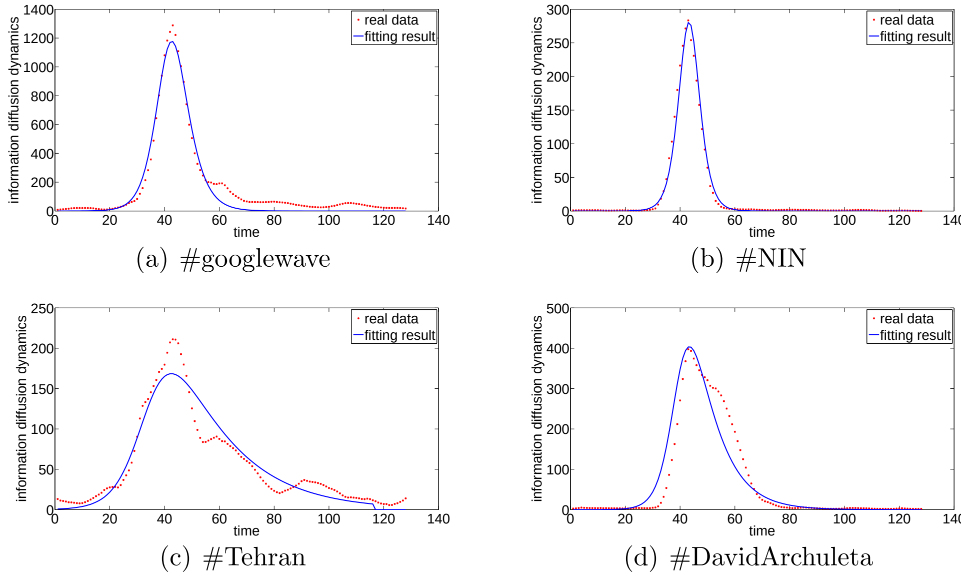


Figure 3.4: Simulation results under different parameter setups.

simulated ones. The theoretical cumulative dynamics as well as the corresponding simulated cumulative dynamics are shown in Fig. 4.1, from which we observe that (i) the theoretical dynamics indeed match well with the simulated dynamics; (ii) the proposed game-theoretic model can flexibly generate popularity dynamics of different shapes by tuning the parameters.

3.6.2 Real data experiments

In this subsection, real data experiments are carried out to verify that the proposed theory matches well with the real-world popularity dynamics. The datasets we use here are Twitter hashtag dataset [114] and the citation data of papers from the Web of Science [46]. The Twitter hashtag data are the mentioning counts of popular hashtags from June to December 2009. We first use the equilibrium computed by Algorithm 3.1 to fit the mention dynamics of four popular Twitter hashtags in



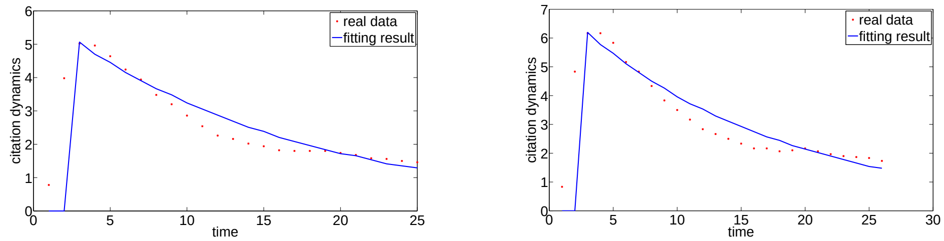
(a) #googlewave (b) #NIN

(c) #Tehran (d) #DavidArchuleta

Fig. 3.5: Fitting Twitter hashtag dynamics.

Fig. 3.5. To fit a popularity dynamics, we use the dynamics data to estimate the parameters of the proposed model and then use the estimated parameters to generate a theoretical dynamics, which is the fitting result. We observe that the theoretical fitting dynamics match well with the real-world dynamics. We further fit the average citation dynamics of the papers published in Nature 1990 and Science 1990, respectively, in Fig. 3.6. We remark that the fitting is still very accurate, though the temporal shape of the citation dynamics are very different from that of the Twitter hashtag dynamics, confirming the universality of our theory for popularity dynamics.

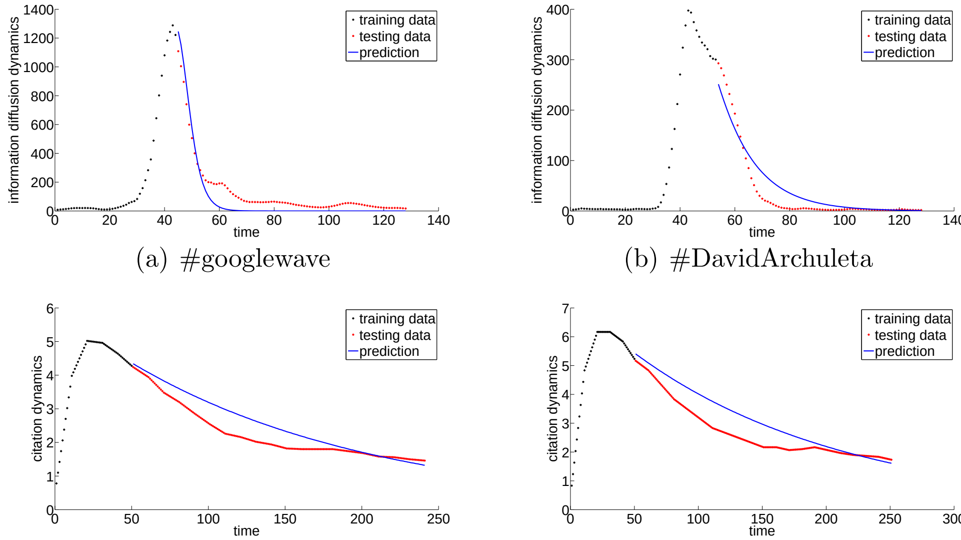
Additionally, we can even exploit the equilibrium of the proposed game to predict future dynamics for real data. To this end, we use part of the dynamics data to train the proposed game-theoretic model, i.e., estimate the parameters in the model, and then predict future dynamics by using the trained model. The



(a) Nature (b) Science

Fig. 3.6: Fitting paper citation dynamics.

prediction results are reported in Fig. 3.7, from which we see that the prediction is quite accurate. To highlight the advantage of the proposed approach, we compare with the prediction results of two existing methods, namely the methods in [106] and [54], which are reported in Figures 3.8 and 3.9, respectively. The four dynamics to be predicted are the same as those in Fig. 3.7. First, we note that the approach in [106] is proposed for citation dynamics. From Fig. 3.8, we observe that the method of [106] fails in predicting the dynamics of the two hashtags (subfigures (a) and (b)). Even for prediction of citation dynamics (subfigures (c) and (d)), our approach outperforms the method in [106]. Second, noting that the method in [54] is designed for information diffusion dynamics, we observe that our approach still outperforms it when predicting the dynamics of two hashtags (Fig. 3.9-(a) and Fig. 3.9-(b)). When it comes to the prediction of citation dynamics, our approach is much better than the method in [54] (Fig. 3.9-(c) and Fig. 3.9-(d)). These comparisons demonstrate that our proposed approach is universally good for general popularity dynamics. Even compared with methods specifically designed for a certain kind of popularity dynamics (e.g., [54] for information diffusion and [106] for citations), our method is still better. In addition, the performance enhancement over the



(a) #googlewave (b) #DavidArchuleta

(c) Nature (d) Science

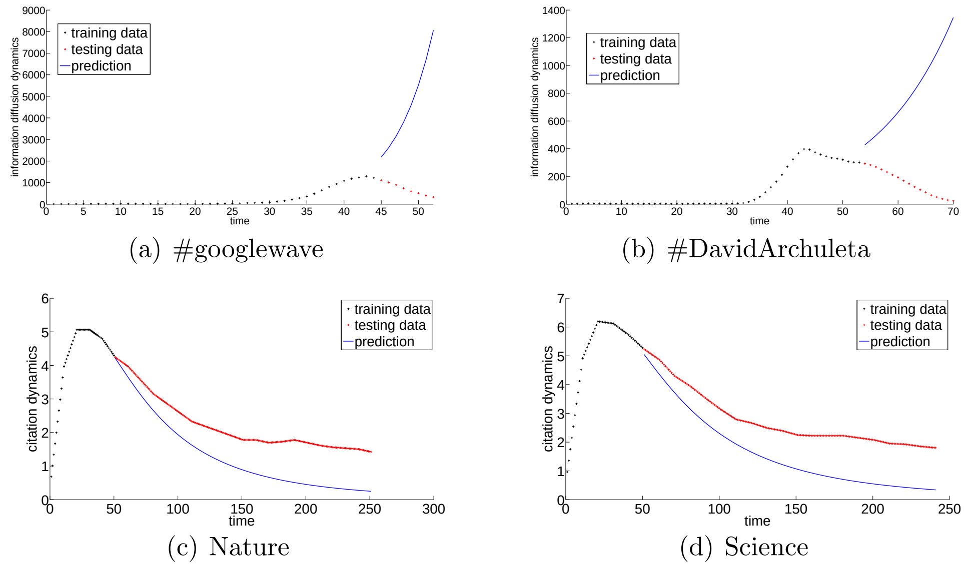
Fig. 3.7: Predicting future dynamics.

method in [54] can be ascribed to the fact that the model in [54] is merely based on instantaneous incentives while our model incorporates long-term incentives as well, which suggests the importance and necessity of taking long-term incentives of individuals into account.

Generally, the prediction is accurate when the training period includes the peak of the dynamics. However, sometimes, we may even predict future dynamics accurately without knowing the peak, which is illustrated by a Twitter hashtag #Tehran in Fig. 3.10.

3.7 Summary

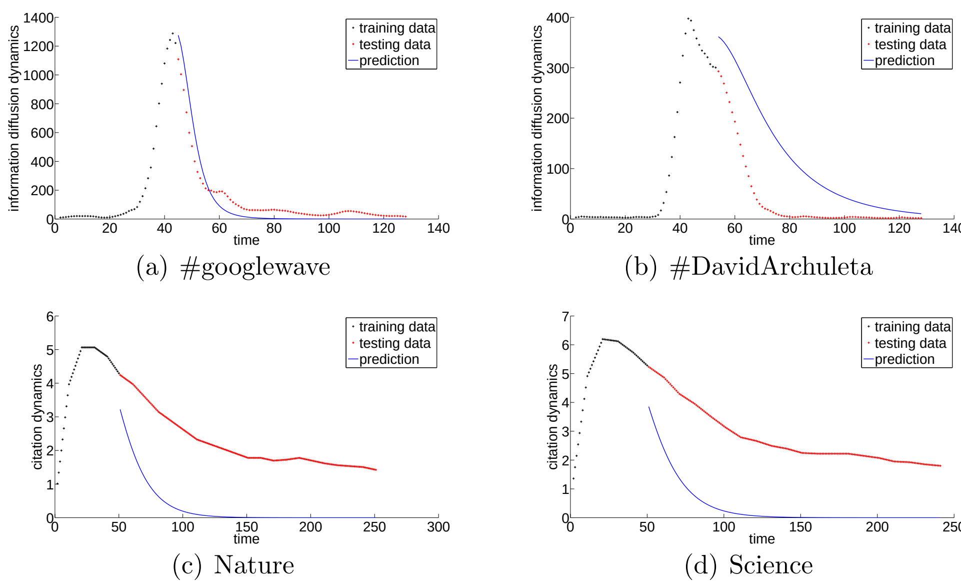
In this chapter, a sequential game is proposed to characterize the mechanisms of popularity dynamics. We prove that the proposed game has a unique SNE, which is a pure strategy action rule with a threshold structure and can be computed using



(a) #googlewave (b) #DavidArchuleta

(c) Nature (d) Science

Fig. 3.8: Prediction results of the method in [106]



(a) #googlewave (b) #DavidArchuleta

(c) Nature (d) Science

Fig. 3.9: Prediction results of the method in [54]

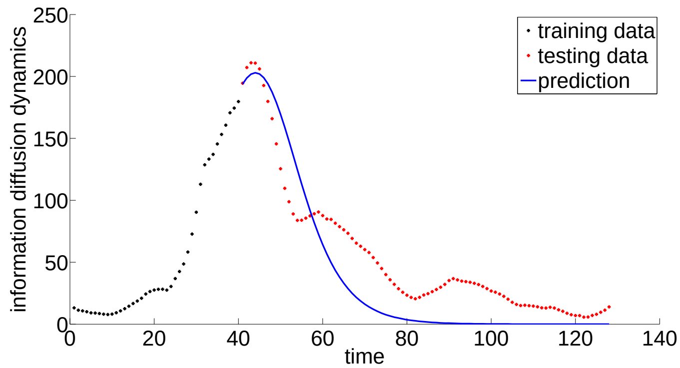


Figure 3.10: Prediction before reaching the peak of the dynamics: Twitter hashtag #Tehran.

a backward induction algorithm. Moreover, at the equilibrium of the proposed game, we analyze some properties observed from the real data, demonstrating that the equilibrium behavior of the proposed game confirms with real-world popularity dynamics. The theory is validated by both simulations and experiments based on real data. The proposed model can even predict future dynamics.

Chapter 4

A Graphical Evolutionary Game Approach to Social Learning

4.1 Motivation

In the recent decade, tremendous research efforts have been devoted to the social learning problems, in which agents of networked systems learn from not only their own private signals but also other agents. Applications of such social learning problems are ubiquitous in fields such as state estimation in power systems, distributed detection over sensor networks and behavior analysis of social networks.

The setups of the social learning problems can be sorted into two categories. In the first category, agents arrive at the system sequentially and make one-shot decisions consecutively based on their own private observations and the actions of their predecessors. In [63,64], the sequential detection problem was studied by using partially observable Markov decision processes (POMDP). The impact of the memory size of the agents was investigated in [30,61]. The effect of noisy communications in sequential detection was considered in [116] while the benefits of randomness of decision making were studied in [109]. Furthermore, a Chinese restaurant game-theoretic analysis of agents' sequential decision making processes was presented in [105]. The problem formulation of this letter is closer to the second category of social learning setups, in which agents are fixed and networked and update their actions iteratively based on their own private signals and neighbor agents' actions or beliefs. In this line of models, the consensus hypothesis testing over networks was investigated in [12, 108], while its applications in wireless communications were considered in [76]. Moreover, a non-Bayesian social learning method with the property of asymptotic learning was proposed in [51]. Overviews on topics in social learning, distributed detection/estimation over networks were presented in [28, 29, 84].

As the social learning problems involve interactions (learning and decision making) between multiple agents, game theory emerges as an appropriate mathematical tool to study them [99]. In [32,33], a Bayesian quadratic network game filter was proposed for rational agents to learn the state of the world in a cooperative manner. A Bayesian dynamic game model of social learning was investigated and the conditions of asymptotic learning were presented in [2]. Additionally, the network users' decision making problems were studied with a Bayesian game formulation in [65, 66].

In this chapter, inspired by the recent success of evolutionary game theory in diverse fields [14, 53–55, 80, 83], we propose a graphical evolutionary game-theoretic method for social learning. In the proposed method, based on a death-birth decision update rule, agents only need to communicate their binary decisions instead of the real-valued beliefs with their neighbors, which endows the method with low communication complexity. By invoking mean field approximations, we analyze the steady state equilibria of the game and show that the evolutionarily stable states (ESSs) [110] coincide with the decisions of the benchmark centralized detector.

Lastly, we present numerical results to confirm the effectiveness of the proposed game-theoretic learning method.

4.2 Problem Formulation

Consider a network of N agents or nodes (the two terms are used interchangeably in the following). Assume for simplicity that the network is k-regular, i.e., the degree (number of neighbors) of each agent is k. In practice, many networks are k-regular graphs. For example, many sensor networks are grid networks over 2-dimensional plane and are thus 4-regular graphs [56]; many cellular communication networks are comprised of hexagon cells (each hexagon cell corresponds to the service area of one base station) and are hence 6-regular graphs. In the social learning problem, there is an unknown state of the nature $\theta \in \{0, 1\}$ to be detected by all the nodes in a collaborative manner based on their individual private signals or measurements. Suppose the prior distribution of θ is $\Pr(\theta = 0) = \Pr(\theta = 1) = 0.5$. Agents are sorted into I categories depending on the qualities of their private signals, i.e., the usefulness of the private signals in detecting the unknown state θ. Suppose agent n has some private signal s_n and its type is i. Then, its private belief is $p_i = \Pr\{\theta = 0|s_n\}$. Clearly, if p_i is close to 0 or 1, then the signals of type-i agents are useful for detecting θ. Oppositely, if p_i is close to 0.5, then the signals of type-i agents are not very useful.

4.2.1 The Centralized Detector

In this subsection, a centralized detector, i.e., a detector utilizing the signals of all agents in a centralized manner, is derived as a performance benchmark. Assume that, given the true state θ, the signals $s_1, ..., s_N$ (henceforth $s_{1:N}$ for shorthand) are conditionally independent, i.e., $p(s_{1:N}|\theta) = \Pi_{n=1}^N p(s_n|\theta)$. With the signals $s_{1:N}$ of all N nodes, a centralized processor can form the posterior distribution $\Pr(\theta = 0|s_{1:N})$ according to the Bayesian rule as follows:

$$\Pr(\theta = 0|s_{1:N}) \tag{4.1}$$

$$= \frac{p(s_{1:N}|\theta = 0)\Pr(\theta = 0)}{p(s_{1:N}|\theta = 0)\Pr(\theta = 0) + p(s_{1:N}|\theta = 1)\Pr(\theta = 1)} \tag{4.2}$$

$$= \frac{\Pi_{n=1}^N p(s_n|\theta = 0)}{\Pi_{n=1}^N p(s_n|\theta = 0) + \Pi_{n=1}^N p(s_n|\theta = 1)} \tag{4.3}$$

$$= \frac{1}{1 + \Pi_{i=1}^I \left(\frac{1-p_i}{p_i}\right)^{Nq_i}}, \tag{4.4}$$

where we denote the proportion of type-i agents as q_i. With a threshold of 0.5 for the posterior distribution $\Pr(\theta = 0|s_{1:N})$, the decision rule of the centralized detector $\hat{\theta}_c$ is given by:

$$\sum_{i=1}^{I} q_i \log\left(\frac{1-p_i}{p_i}\right) \underset{\hat{\theta}_c=0}{\overset{\hat{\theta}_c=1}{\gtrless}} 0. \tag{4.5}$$

4.2.2 A Graphical Evolutionary Game Framework

The centralized detector has several drawbacks such as large communication overhead and vulnerability to link failures which make it infeasible in many applications. Therefore, we are motivated to find another detection algorithm with the following favorable properties.

- `P1` The detection algorithm is distributed, i.e., each agent only communicates with its neighbors and no centralized entity is needed.

- `P2` Agents only interchange their current binary decisions on the state θ instead of their real-valued beliefs (posterior distributions) on θ. This reduces the communication complexity significantly.

- `P3` The detection algorithm produces the same result as the centralized detector (4.5) does, possibly asymptotically if the algorithm is iterative.

In this subsection, we present a graphical evolutionary game social learning approach, which satisfies the aforementioned three properties. Suppose each agent n has a decision $d_n \in \{0, 1\}$ on the state θ and the decision d_n shall be updated iteratively in the game. When a type-i agent n interacts with one of its neighbors, agent m, the utility of agent n is summarized in the following table for different combinations of actions of the two interacting parties n and m:

	$d_m = 0$	$d_m = 1$
$d_n = 0$	$\log(1 - p_i) + u$	$-\log(1 - p_i)$
$d_n = 1$	$-\log p_i$	$-\log p_i + u$

Here $u \geq 0$ is some non-negative constant used to capture the fact that agents tend to imitate their neighbors (or friends in social networks) and reach consensus. Additionally, agent n also tends to adhere to its own private belief p_i. As such, we reward or penalize the utility of agent n for actions conforming to or deviating from its belief p_i, respectively. The usage of logarithmic terms in the utility is inspired by the centralized detector (4.5). For an agent with total utility U through interactions with her neighbors, we further define her fitness π as a convex combination of U and 1: $\pi = 1 - \alpha + \alpha U$, where $\alpha > 0$ is some small positive constant called the selection strength in evolutionary game theory. The bigger the selection strength α, the more heavily the fitness π depends on the utility U and the bigger the advantage of agents with large utility. For a type-i agent with k_0 neighbors making decision 0, if he makes decision $d = 0$, then his fitness is:

$$\pi_0(i, k_0) = 1 - \alpha + \alpha[k_0(-\log(1 - p_i) + u) - (k - k_0)\log(1 - p_i)]. \tag{4.6}$$

If he makes decision $d = 1$, then his fitness is:

$$\pi_1(i, k_0) = 1 - \alpha + \alpha[-k_0 \log p_i + (k - k_0)(-\log p_i + u)]. \tag{4.7}$$

Based on fitness, agents can update their decisions according to some strategy update rule. In the literature of graphical evolutionary game theory [14, 54, 55, 82, 83], there are mainly three strategy update rules: the death-birth process, the birth-death process and the imitation process. In this letter, we will focus on the death-birth update rule and other rules can be similarly analyzed. In the death-birth update rule, at each time slot, one agent is selected to abandon her decision uniformly randomly (death process) and the chosen agent update her decision to be one of her neighbors' decisions with probability proportional to their fitness (birth process). This decision update process continues repeatedly across time. In this chapter, our goal is to study the agents' steady state behaviors in this update process.

The proportion of adoption of decision 0 among type-i agents is denoted as x_i while the proportion of adoption of decision 0 among all agents is denoted as x.

We call x_i the population dynamics of type-i agents and x the population dynamics of the entire network (or simply population dynamics). Obviously, we have $x = \sum_{i=1}^{I} q_i x_i$. Our goal in this letter is to study the steady state equilibrium of the population dynamics x and show that this equilibrium coincides with the centralized detector (4.5).

We note that the gossip method proposed in [108] tackles a similar social learning problem and also possesses properties `P1`, `P2` and `P3`. In this chapter, we take an alternative approach based on evolutionary game theory as opposed to the gossip based method in [108]. The proposed game-theoretic social learning method takes agents' rational learning and decision-making behaviors (such as learning from neighbors with high fitness) into consideration and is thus more amenable to practical implementations in systems with intellegient or strategic agents, e.g., social networks.

4.3 Algorithm Development and Equilibrium Analysis

In this section, we develop the detailed algorithm of the game-theoretic social learning method and analyze the corresponding steady state equilibrium, i.e., the evolutionarily stable state (ESS) [110], of the population dynamics x. Suppose, at a time instant, a type-i agent with decision 0 is chosen to abandon her decision. According to the death-birth update rule, this agent should update her decision to be one of her neighbors' decisions with probability proportional to fitness. However, as we only allow the agents to communicate their decisions d rather than their private beliefs p (property `P2`), the chosen agent is unaware of her neighbors' fitness, which depend on their private beliefs. As such, the chosen agent will update her decision as if all of her neighbors' types are i, i.e., their beliefs are p_i, and only take the neighbors' decisions into consideration. Thus, the probability that the chosen agent will change her decision from 0 to 1 is given by:

$$\Pr_{0\to 1}(i,k_0) = \frac{(k-k_0)\pi_1(i,k_0)}{k_0\pi_0(i,k_0)+(k-k_0)\pi_1(i,k_0)}. \tag{4.8}$$

Exploiting the expressions of fitness in (4.6) and (4.7) and making use of the first order approximation $\frac{1+a\alpha}{1+b\alpha} \approx 1+(a-b)\alpha$ for small α, we compute the transition probability $\Pr_{0\to 1}(i,k_0)$ in (4.9).

$$\begin{aligned}
&\Pr_{0\to 1}(i,k_0)\\
&= \frac{k-k_0}{k}\frac{1+\alpha[-k_0\log p_i+(k-k_0)(-\log p_i+u)-1]}{1+\alpha\left\{\frac{k_0}{k}[k_0(-\log(1-p_i)+u)-(k-k_0)\log(1-p_i)-1]+\left(1-\frac{k_0}{k}\right)[-k_0\log p_i+(k-k_0)(-\log p_i+u)-1]\right\}}\\
&\approx \frac{k-k_0}{k}+\alpha(k-k_0)\left[\left(\log\frac{1-p_i}{p_i}+u\right)\frac{k_0}{k}-2u\frac{k_0^2}{k^2}\right]
\end{aligned} \tag{4.9}$$

Note that k_0 is a binomially distributed random variable with probability mass function (PMF) $\beta(k,k_0) = \begin{pmatrix} k \\ k_0 \end{pmatrix} x^{k_0}(1-x)^{k-k_0}$. Using the moments of binomial distribution, we obtain $\mathbb{E}[k_0]=kx$, $\mathbb{E}[k_0^2]=(k^2-k)x^2+kx$, $\mathbb{E}[k_0^3]=k(k-1)(k-2)x^3+3k(k-1)x^2+kx$. Thus, we can compute the expected transition probability averaged over k_0:

$$\begin{aligned}
&\mathbb{E}_{k_0}\left[\Pr_{0\to 1}(i,k_0)\right]\\
&= 1-x+\alpha\left(\log\frac{1-p_i}{p_i}+u\right)[-(k-1)x^2+(k-1)x]\\
&\quad -2u\alpha[(-k+3-2k^{-1})x^3+(k-4+3k^{-1})x^2\\
&\quad +(1-k^{-1})x]
\end{aligned} \tag{4.10}$$

Noticing that the probability of choosing a type-i agent with decision 0 to abandon her decision is $q_i x_i$, we write the PMF of the increment of x_i, denoted as δx_i, in the following:

$$\Pr\left(\delta x_i = -\frac{1}{Nq_i}\right) = q_i x_i \mathbb{E}\left[\Pr_{0\to 1}(i, k_0)\right]. \tag{4.11}$$

Similarly, by considering the scenario where a type-i agent with decision 1 is selected to abandon her decision, we get:

$$\Pr\left(\delta x_i = \frac{1}{Nq_i}\right) = q_i(1 - x_i)\mathbb{E}\left[\Pr_{1\to 0}(i, k_0)\right] \tag{4.12}$$

$$= q_i(1 - x_i)\left(1 - \mathbb{E}\left[\Pr_{0\to 1}(i, k_0)\right]\right). \tag{4.13}$$

We approximate the discrete time decision update system with a continuous time version, as per convention in the analysis of graphical evolutionary game [14, 54, 55, 82, 83]. Thus, utilizing (4.10), (4.11) and (4.12), we derive the evolutionary dynamics of x_i as follows:

$$\begin{aligned}
\dot{x}_i &= \frac{1}{Nq_i}\Pr\left(\delta x_i = \frac{1}{Nq_i}\right) - \frac{1}{Nq_i}\Pr\left(\delta x_i = -\frac{1}{Nq_i}\right) \\
&= \frac{1}{N}\left(1 - x_i - \mathbb{E}\left[\Pr_{0\to 1}(i, k_0)\right]\right) \\
&= \frac{x}{N} - \frac{x_i}{N} + \frac{\alpha}{N}x(x-1)\Big\{2u\big[\left(-k + 3 - 2k^{-1}\right)x \\
&\quad - 1 + k^{-1}\big] + \left(\log\frac{1-p_i}{p_i} + u\right)(k-1)\Big\}
\end{aligned} \tag{4.14}$$

Taking a weighted average over all types, we get the evolutionary dynamics of the population dynamics x as:

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{I} q_i \dot{x}_i \\ &= \frac{\alpha}{N} x(x-1)\big\{2u\left[\left(-k+3-2k^{-1}\right)x - 1 + k^{-1}\right] \\ &\quad + (\lambda+u)(k-1)\big\},\end{aligned} \tag{4.15}$$

where $\lambda \triangleq \sum_{i=1}^{I} q_i \log \frac{1-p_i}{p_i}$. Note that λ is just the discriminant used in the centralized detector (4.5). Now, we are ready to present the main theorem of this letter regarding the ESS of the social learning game.

Theorem 4.1 *(i) Suppose the degree $k \geq 2$. Then, the set of evolutionarily stable states (ESSs) $\mathcal{X}^*$ of the social learning game is:*

$$\mathcal{X}^* = \begin{cases} \{0\}, & \text{if} \quad \lambda > u - 2k^{-1}u, \\ \{1\}, & \text{if} \quad \lambda < -u + 2k^{-1}u, \\ \{0,1\}, & \text{if} \quad -u + 2k^{-1}u < \lambda < u - 2k^{-1}u. \end{cases}$$

(ii) If we further assume that the initial value of the population dynamics x is $x(0) = 0.5$, which can be achieved by a random guess by all agents, then the ESS x^* that the population dynamics x converges to is:

$$x^* = \begin{cases} 0, & \text{if} \quad \lambda > 0, \\ 1, & \text{if} \quad \lambda < 0. \end{cases} \tag{4.16}$$

Proof: (i) Letting $\dot{x} = 0$ in the population dynamics (4.15) yields three equilibria 0, 1, and $\tilde{x}$, where $\tilde{x} \triangleq \frac{\lambda}{2u(1-2k^{-1})} + \frac{1}{2}$. For an equilibrium point to be an ESS, it needs to be a locally asymptotically stable for the underlying dynamical system.

To test the stability of the three equilibria, we form the Jacobian matrix $\mathbf{J} \in \mathbb{R}^{2\times 2}$ of the dynamical system (x_i, x) specified in equations (4.14) and (4.15):

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \dot{x}_i}{\partial x_i} & \frac{\partial \dot{x}_i}{\partial x} \\ \\ \frac{\partial \dot{x}}{\partial x_i} & \frac{\partial \dot{x}}{\partial x} \end{bmatrix}. \tag{4.17}$$

The entries of $\mathbf{J}$ are computed as follows:

$$\frac{\partial \dot{x}_i}{\partial x_i} = -\frac{1}{N}, \quad \frac{\partial \dot{x}}{\partial x_i} = 0,$$

$$\begin{aligned}\frac{\partial \dot{x}_i}{\partial x} &= \frac{1}{N} + \frac{\alpha}{N}(2x-1)\Big[-2u\big(\big(k-3+2k^{-1}\big)x + 1 - k^{-1}\big) \\ &+ \left(\log\frac{1-p_i}{p_i} + u\right)(k-1)\Big] + \frac{2u\alpha}{N}x(x-1)\left(-k+3-2k^{-1}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial \dot{x}}{\partial x} &= \frac{\alpha}{N}(2x-1)\big[-2u\big(\left(k-3+2k^{-1}\right)x + 1 - k^{-1}\big) \\ &\quad + (u+\lambda)(k-1)\big] + \frac{2u\alpha}{N}x(x-1)(-k+3-2k^{-1})\end{aligned}$$

As $\mathbf{J}$ is upper triangular and $\frac{\partial \dot{x}_i}{\partial x_i}$ is negative, the locally asymptotically stability is equivalent to $\frac{\partial \dot{x}}{\partial x} < 0$. Therefore, $x = 0$ is an ESS iff $\frac{\partial \dot{x}}{\partial x}|_{x=0} < 0$, i.e., $\lambda > -u + 2k^{-1}u$. Similarly, $x = 1$ is an ESS iff $\frac{\partial \dot{x}}{\partial x}|_{x=1} < 0$, i.e., $\lambda < u - 2k^{-1}u$. $x = \tilde{x}$ is an ESS iff $\frac{\partial \dot{x}}{\partial x}|_{x=\tilde{x}} < 0$, i.e., $\tilde{x} < 0$ or $\tilde{x} > 1$, which contradict to the fact that the population dynamics is within $[0, 1]$. So, $\tilde{x}$ can never be an ESS. We thus conclude the first part of the theorem.

(ii) If $\lambda > u - 2k^{-1}u$, then the unique ESS is 0 and the population dynamics x will converge to it. Similarly, if $\lambda < -u + 2k^{-1}u$, then the unique ESS is 1 and the population dynamics x will converge to it. In these two circumstances, (4.16) evidently holds. If $-u + 2k^{-1}u < \lambda < u - 2k^{-1}u$, then the set of ESSs $\mathcal{X}^*$ contains both 0 and 1 and we need to ascertain which ESS will the population dynamics x converge to. Recall the evolutionary dynamics of x in (4.15) and we note that if $x > \tilde{x}$, then $\dot{x} > 0$ and x is increasing; if $x < \tilde{x}$, then $\dot{x} < 0$ and x is decreasing. Recall that the initial value of x is $x(0) = 0.5$. If $\lambda > 0$, then $\tilde{x} > 0.5 = x(0)$. So, x is decreasing initially, which means x will become even smaller and $x < \tilde{x}$ still hods. Therefore, x is always decreasing and the ESS it converges to is 0. Analogously, if $\lambda < 0$, then the ESS x converges to is 1. □

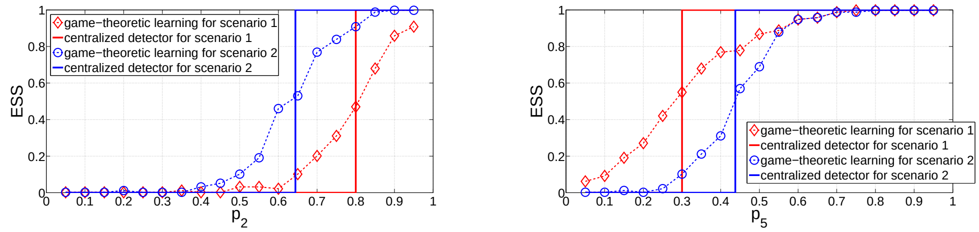
Remark 4.1 *Part (ii) of Theorem 4.1 establishes that the steady state of the game-theoretic social learning method coincides with the decision of the centralized detector, i.e., the game-theoretic social learning method possesses property* **P3**.

4.4 Numerical Results

In this section, numerical results are presented to corroborate the proposed game-theoretic social learning approach. We simulate a random regular network with $N = 1000$ nodes (agents) and the degree of each node is $k = 20$. The game parameters are chosen to be $\alpha = 0.05$ and $u = 0.5$. All experimental results are averages over 100 independent trials.

We first consider a network of $I = 2$ types of agents. The belief of the first type is fixed to be $p_1 = 0.2$. We consider two scenarios (i) $q_1 = q_2 = 0.5$; (ii) $q_1 = 0.3$, $q_2 = 0.7$. The relation between the ESS and p_2 is reported in Fig. 4.1-(a) for the two scenarios, respectively. , The ESSs are computed as the average proportion of agents with decision 0 over the 100 trials. The decisions of the centralized detector are also plotted as a benchmark. We observe that the ESSs of the game-theoretic learning are close to the decisions of the centralized detector in both scenarios. The gaps between the ESSs of the game-theoretic learning method and the decisions of the centralized detector are consequences of the randomness of the graphical evolutionary game formulation (e.g., the birth process in the death-birth decision update rule is subject to randomness). Note that the theoretical result (e.g., Theorem 4.1) is based on mean-field approximations, i.e., replacing random variables with their expectations to simplify analysis. Therefore, though Theorem 4.1 asserts that the steady states of the game-theoretic learning method coincide with the decisions of the centralized detector, there exist some gaps between the two in numerical experiments. Since the game-theoretic learning method is fully distributed and only requires communications of agents' binary decisions instead of their real-valued beliefs, it is still more desirable in many applications, especially those in need of low communication overhead and robustness.

We further conduct experiments for networks with $I = 5$ types of agents. The beliefs of the first four types are set to be $p_1 = 0.6$, $p_2 = 0.7$, $p_3 = 0.5$, $p_4 = 0.4$. We consider two scenarios: (i) $q_1 = q_2 = q_3 = q_4 = q_5 = 0.2$; (ii) $q_1 = 0.2$, $q_2 = 0.1$, $q_3 = 0.1$, $q_4 = 0.1$, $q_5 = 0.5$. The relation between the ESSs and p_5 is illustrated in Fig. 4.1-(b). The decisions of the centralized detector are also shown as a comparison. Similar to the experiments with 2 types of agents, the ESSs of the game-theoretic learning method can still match the decisions of the centralized detector approximately, which confirms the effectiveness of the proposed game-theoretic social learning method for different numbers of types.



(a) Three types of agents (b) Five types of agents

Fig. 4.1: Performance of the proposed social learning method

The typical number of iterations (or equivalently, time slots) needed to converge to the ESS is between 5×10^4 to 10^5. Though the iteration number seems huge, the actual convergence time in real networks is not large given the fact that the length of each time slot is very small (the length of time slot is approximately inversely proportional to the number of agents N since one agent is chosen to update her decision in each time slot).

4.5 Summary

In this letter, a graphical evolutionary game based social learning method is proposed. The method is fully distributed and only requires communications of agents' binary decisions instead of their real-valued beliefs, which endows the proposed method with low communication complexity. Theoretical analysis under mean field approximations indicates that the evolutionarily stable states of the game coincide with the decisions of the centralized detector. Numerical experiments are implemented to validate the performance of the game-theoretic learning method.

Chapter 5

Data Trading with Multiple Owners, Collectors and Users: An Iterative Auction Mechanism

5.1 Motivation

In the big data era, vast amount of data are generated and exploited by various agents. For example, numerous memes such as Twitter hashtags are produced in online social networks and millions of videos are uploaded to Youtube. Many software/APP developers may need certain online data (such as the click-through rate of some advertisements or mention count dynamics of some memes) to enhance the quality of their products. As another example, with the development of data procurement and storage capability, many organizations own databases of the statistics of their fields, e.g., hospitals may have data about the clinical performances of medicines. In order to conduct research, researchers need to access these data owned by organizations. In all these circumstances, we face the problem of allotting/trading data from the data owners (e.g., social networks/websites or organizations) to the data users (e.g., software companies or researchers). In fact, several data trading markets or companies have already emerged recently, such as the Data Marketplace, Big Data Exchange and Microsoft Azure Marketplace. However, these data markets are still at the incipient stage and lack appropriate regulations. Economically, the data agents are selfish and seek to maximize their own utilities instead of the overall system efficiency. As such, a sophisticated mechanism is imperative to guide the agents to distribute or trade data efficiently.

The problem of coordinating data trading in a data market falls into the general topic of resource trading/allocation in networks, for which abundant works have been done in the past decades. For communication networks, by using optimization and game theoretic techniques, researchers propose various algorithms to allocate power [47, 48] or channels [44, 91] to communication nodes or access points. For cognitive radio networks, spectrum resources are allotted among primary users and secondary users [52, 103]. For power networks or smart grids, power or voltage resources are distributed to devices and apparatuses in order to maintain high-performance and stable power systems [36, 67, 72]. The most relevant resource allocation/trading problem to this chapter is the privacy trading problem [90]. In most privacy trading problems investigated in the current literature, a single data collector is aimed at collecting binary data from multiple data owners in order to estimate some statistics. From example, each data owner may have a binary answer (yes/no) to some problem and the data collector wants to estimate the proportion of data owners with the answer yes. The involved data are private and leakage of them to the data collector compromises the security of data owners. The loss from this compromising of privacy can be quantified by the differential privacy [31]. As such, data owners should be somehow compensated by the data collector. Additionally, data owners are selfish and may not report their true data to the data collector. Therefore, from the perspective of the data collector, a mechanism is needed to collect accurate data at a low cost from the data owners. To this end, Ghosh and Roth proposed an auction mechanism for a single data collector to collect data from multiple data owners [38]. Along this line, Fleischer and Lyu extended the auction mechanism to the scenario where individual data owner's valuation of the data privacy was correlated with the data themselves [35]. Furthermore, Xu *et al.* proposed a contract-theoretic mechanism to collect general private data which are not necessarily binary [112].

However, there are two limitations of existing models of data trading in the aforementioned works [35, 38, 112]. First, in the existing models, there is only one single data collector. This is not the case in most real-world data market, where multiple data collectors (such as many companies or groups like Big Data Exchange) often coexist and compete with each other. Second, in most data markets, the data collectors usually do not exploit the data by themselves. Instead, they often sell the data to data users, who are not capable of collecting and storing massive datasets but need data to develop projects or conduct research. For example, many APP developers are small companies who cannot afford collecting necessary data to develop APPs and thus need to purchase data from professional data collecting companies. In other words, in data markets, besides data owners and collectors, there are data users who can make use of the data but are not able to collect data by themselves. In this chapter, we take the above mentioned two limitations of existing works into consideration and investigate the data trading problem in a market with multiple data owners, collectors and users (in the following, we use the term *data agents* to refer to data owners, collectors and users).

Due to the existence of multiple collectors and users, the problem in this chapter is significantly different from the data trading in [35, 38, 112]. Instead of maximizing the profit of a single collector as in previous works, we consider from a system designer's perspective and are aimed at maximizing the overall social welfare, which quantifies the operation efficiency of the data market. However, in practice, the data agents are usually selfish and seek to maximize their own utilities instead of the overall system performance. In order to coordinate the data trading among multiple selfish agents, we resort to the *iterative auction* mechanism, which is initially proposed in [57]. In iterative auction, the auctioneer announces the resource allocation and payment rules to the bidders. Then, the selfish bidders submit appropriate bids to the auctioneer with the goal of maximizing their own utilities. Based on the submitted bids, the auctioneer adjusts the resource allocation and payment rules and another round of auction starts. Through careful design of the mechanism, the iterative auction may converge to an operation point with satisfactory properties. The iterative auction has already been successfully applied to resource allocation in communication networks [13, 49, 50, 73].

The contribution of this chapter is epitomized in the following.

- We present a data market model with multiple data owners, collectors and users who have heterogeneous utility functions. Considering from the perspective of the system designer, we formulate corresponding social welfare maximization problem.

- An iterative auction mechanism is proposed to coordinate the data trading among the data agents. The mechanism avoids direct access to the data agents' utility functions, which are private information unknown to the system designer. The selfish nature of individual data agents is also respected in the mechanism.

- We theoretically show that the proposed mechanism converges to the socially optimal operation point. We also analytically substantiate that the mechanism possesses appealing economic properties including individual rationality and weakly balanced budget.

- We also extend the mechanism to the non-exclusive data trading scenario, where the same data can be used by multiple data users repeatedly.

- Simulations as well as real data experiments are implemented to validate the theoretical results of the mechanism.

The roadmap of this chapter is as follows. In Section 5.2, our model of the data market is presented and the social welfare maximization problem is formulated. In Section 5.3, we design an iterative auction mechanism to coordinate the data trading. The convergence analysis and economic properties of the proposed mechanism are presented in Section 5.4. Then, we extend the mechanism to the non-exclusive data trading scenario in Section 5.5. In Section 5.6, simulation results and real data experiments are shown. Lastly, we conclude the paper in Section 5.7.

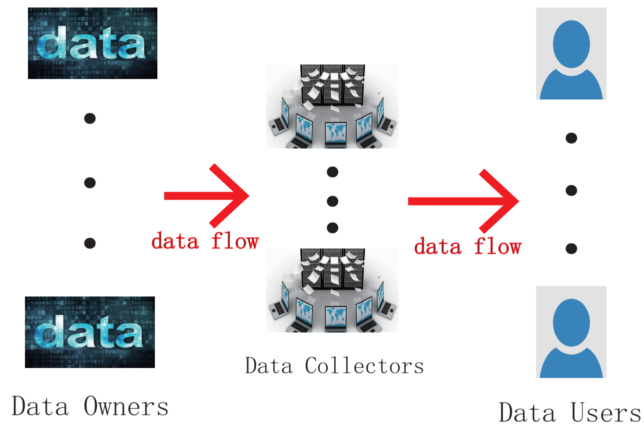


Figure 5.1: A data market with multiple data owners, collectors and users

5.2 Model

In this section, we describe the model of a data market with multiple data owners, collectors and users in detail. Then, we formulate the associated social welfare maximization problem and motivate the iterative auction mechanism.

Consider a data market with M data owners, N data collectors and L data users as shown in Fig. 5.1. In real world, the data owners correspond to those sources or producers of the data such as websites with online user data or organizations with certain statistics. The data users can be any companies or individuals who either consume the data or exploit data to develop projects and to make profits. For example, a software company may need certain user record data to develop an APP. Often, in a data market, data users do not interact with the data owners directly due to the limited data collection, storage and processing capability of many data users. Instead, between data owners and users, there may exist data collectors who are able to collect, store and process massive datasets.

The collectors collect data from the owners through various methods such as web scraping for websites or direct inquiries to organizations with certain statistics. The specific data collection manner depends on the form of the data. After obtaining the (massive) data, the collectors store them and further process them to be more sanitary and user-friendly. Lastly, the collectors sell the data to users according to the different demands of users.

Different from prior works [35, 38, 112], we assume the existence of multiple data owners, collectors and users competing with each other, which is the case in reality as explained in Section 5.1. This makes the problem more challenging because of the conflicting interests and selfishness of the data agents, which necessitates a framework different from the traditional auction theoretic approach in [35, 38] and contract theoretic approach in [112]. Next, we describe the data trading among the data agents and their utility functions in detail.

5.2.1 Data Owners

Suppose owner m (there are M data owners in total) entitles collector n to collect x_{mn} amount of data, which is the maximum amount of data that collector n can get from owner m. For instance, a website may give a data collector (e.g., a web scraper) access to a certain part of data in that website; an organization may allow a data collector to access certain records or statistics of the organization. Due to the exposure of its data, the owner m suffers a loss of $U_m(\mathbf{x}_m)$, where $\mathbf{x}_m = [x_{m1}, ..., x_{mN}]$. This loss may stem from compromise of privacy or leakage of lucrative information/technologies. For example, if a social network allows some of its users' data to be accessed by companies or researchers, its users' privacy will be compromised and the social network may lose popularity among online users.

We assume that the data here are exclusive, i.e., the same data can only be assigned to one collector and one user. For example, software companies (data users) may need tailored data (e.g., click-through rate of specific web pages or advertisements in order to monitor the users' feedback) to develop their own softwares or APPs. These data are useful only to this user and are useless for others, i.e., these data are exclusive. In Section 5.5, we extend the proposed mechanism to non-exclusive data trading scenario, where the same data can be used by multiple users.

We assume that owner m has C_m amount of data in total. In real world, when the data exposure or leakage is tiny, the data owner may hardly suffer any loss. However, if the data exposure is severe, e.g., larger than a certain threshold, the privacy loss will increase faster and faster with the amount of data exposure. In order to capture this second order property of loss function of data owners, we assume that the loss function U_m is a convex function.

5.2.2 Data Collectors

Suppose collector n (there are N data collectors in total) collects y_{mn} data from owner m. Clearly, y_{mn} is no larger than x_{mn}. When it is strictly smaller than x_{mn}, the collector n does not collect all the authorized data from owner m due to the loss from collection efforts. We assume that the collecting procedure incurs a loss of $V_n(\mathbf{y}_n)$ for collector n, where $\mathbf{y}_n = [y_{1n}, ..., y_{Mn}]^T$. In real world, the collecting procedure can be data scraping from websites or direct inquiry to organizations etc., depending on the form and availability of the data. The collection and basic trimming/processing of the massive data need significant efforts of the collectors. In addition, the storage of the massive datasets also necessitate lots of apparatuses and devices. All of these contribute to the loss of the data collectors. Often, with the increase of the data to be collected, the difficulty (and hence efforts) of data collection increases faster and faster due to reasons such as the limitations on the internet connections and computers' processing speed (if the data amount is huge, collectors need to greatly enhance their internet connections or computer devices, which is costly). Therefore, we assume V_n is a convex function.

5.2.3 Data Users

Lastly, data user l (there are L data users in total) buys z_{nl} amount of data from collector n. The gain of user l is $W_l(\mathbf{z}_l)$, where $\mathbf{z}_l = [z_{1l}, ..., z_{Nl}]^T$. For instance, by exploiting the user feedback data such as click-through rate, a software/APP developer can enhance its product and makes more profits. As per conventions of the resource allocation literature, the gain function W_l is assumed to be a concave function.

5.2.4 Social Welfare Maximization

As the interests of the data agents conflict with each other (e.g., the data owners want to sell the data with high price while the data collector wants to gain the data at low cost) and the data agents are selfish, a system designer is needed to coordinate the agents' behaviors to maximize overall system efficiency or *social welfare*, which is defined as the difference between the total gain of users and total loss of owners and collectors. The corresponding social welfare maximization problem **SWM** can be formulated as follows.

$$\textbf{SWM}: \quad \texttt{Maximize}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \quad -\sum_{m=1}^{M} U_m(\mathbf{x}_m) - \sum_{n=1}^{N} V_n(\mathbf{y}_n) + \sum_{l=1}^{L} W_l(\mathbf{z}_l) \tag{5.1}$$

$$\texttt{s.t.} \quad \sum_{n=1}^{N} x_{mn} \leq C_m, \quad \forall m, \tag{5.2}$$

$$\sum_{l=1}^{L} z_{nl} \leq \sum_{m=1}^{M} y_{mn}, \quad \forall n, \tag{5.3}$$

$$y_{mn} \leq x_{mn}, \quad \forall m, n. \tag{5.4}$$

The first constraint is the total data constraint at each data owner. The second constraint is the data constraint at each collector where the total amount of sold data is no larger than the amount of total collected data. The third constraint means that the data collected by a collector n from an owner m is no bigger than the data that owner m entitles collector n to collect.

SWM is a convex optimization problem and can be solved in a centralized manner by using state-of-the-art optimization toolbox such as `CVX` [11]. However, in real-world applications, we cannot directly solve the **SWM** to coordinate the data trading due to the following reasons.

- First, data agents (data owners, collectors and users) are selfish and seek to maximize their own utilities instead of the social welfare. As a result, even if the system designer computes the socially optimal point by solving **SWM**, the optimal solution cannot be enforced given the selfishness of the data agents.

- Second, the utility functions U, V, W are private information of the agents which is unknown to the system designer. Thereby, **SWM** cannot be solved at the system designer's side in a centralized fashion.

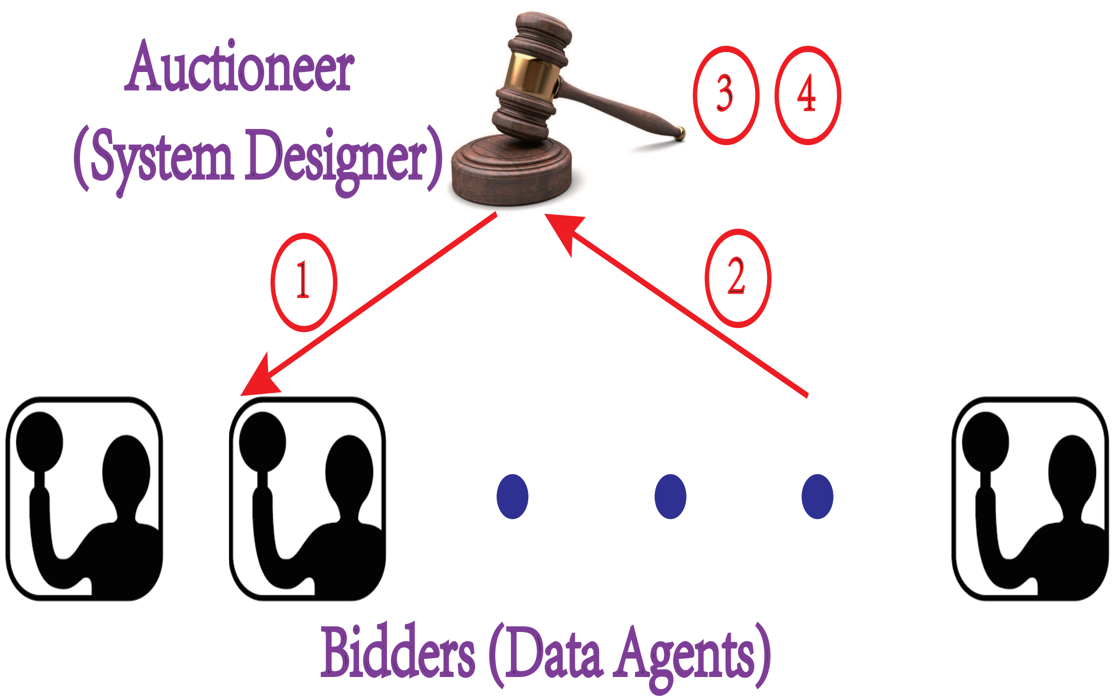
In order to elicit the private information of the agents and guide the selfish agents to cooperate to achieve social optimum, we resort to iterative auction mechanism [57]. The presumption of this mechanism is that the agents are price-takers, meaning that the each agent takes the announced prices as fixed and does not expect any impact of its action on the prices. This hypothesis holds when either (1) the agents have limited computational capability and thus limited rationality so that they do not consider the effects of their actions on pricing; or (2) the number of agents is large so that each agent has little influence on the prices.

5.3 Mechanism Design

In this section, we design an iterative auction mechanism for the data trading problem formulated in Section 5.2. Our design goal is to guide the selfish agents to trade data at a socially optimal point while respecting each agent's private information, i.e., avoiding direct inquiry of the agents' utility functions. The proposed iterative auction mechanism is illustrated in Fig. 5.2. The system designer serves as the auctioneer and the data agents are the bidders. Analogous to many auction mechanisms in the literature [62], the agents submit *bids* to signal their valuations of the resources, or data in this context. The first step of the mechanism is that the system designer announces the data allocation and pricing/reimbursement rules to the agents. In the second step, based on these rules, each agent calculates and submits an appropriate bid in order to maximize her own utility in accordance with her selfishness. In the third step, the system designer computes the data allocation result according to the submitted bids and the data allocation rule. The aforementioned three steps are common in auction theory. The unique feature of iterative auction lies in the fourth step, in which the system designer adjusts the data allocation and pricing/reimbursement rules based on the data allocation results. Then, the system designer announces these new rules and another auction begins. This iterative process continues until the system designer observes convergence. In the following subsections, we describe each step of the mechanism in more detail.

5.3.1 The System Designer's Problem

As explained in Section 5.2, a difficulty for the system designer to solve the **SWM** is that the she is unaware of the loss and gain functions U, V, W, which are private information of the agents. Thus, the system designer has to replace these unknown functions with some known functions. In addition, denote the bid that owner m submits to the system designer by $\mathbf{s}_m = [s_{m1}, ..., s_{mN}] \succeq \mathbf{0}$, where $\succeq$ means componentwise inequality. Similarly, denote the bid of collector n by



1) The system designer announces the current data allocation and pricing/reimbursement rules (as functions of bids).
2) According to the announced data allocation and pricing/reimbursement rules, the agents compute their bids so as to maximize their own utilities
3) The system designer updates the data allocation based on the submitted bids and current allocation rule.
4) The system designer updates the data allocation and pricing/reimbursement rules based on the current data allocation. Another iteration starts.

Figure 5.2: An illustration of the proposed iterative auction mechanism, which iterates the four steps depicted in the figure.

$\mathbf{t}_n = [t_{1n}, ..., t_{Mn}]^T \succeq \mathbf{0}$ and the bid of user l by $\mathbf{r}_l = [r_{1l}, ..., r_{Nl}]^T \succeq \mathbf{0}$. The bids signal the agents' valuations of the data and should be incorporated into the loss and gain functions in the system designer's perspective. In the iterative auction mechanism, the system designer makes the following utility function replacements to avoid direct access of the private information of the agents:

$$U_m(\mathbf{x}_m) \leftarrow \sum_{n=1}^{N} \frac{s_{mn}}{2} x_{mn}^2, \tag{5.5}$$

$$V_n(\mathbf{y}_n) \leftarrow \sum_{m=1}^{M} \frac{t_{mn}}{2} y_{mn}^2, \tag{5.6}$$

$$W_l(\mathbf{z}_l) \leftarrow \sum_{n=1}^{N} r_{nl} \log z_{nl}. \tag{5.7}$$

Note that through these replacements, the convexity/concavity of the functions U, V, W are preserved. Then, the **SWM** is transformed into the following designer's allocation problem **DAP**.

$$\textbf{DAP}: \quad \texttt{Maximize}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \quad \sum_{l=1}^{L}\sum_{n=1}^{N} r_{nl} \log z_{nl} - \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{s_{mn}}{2} x_{mn}^2 - \sum_{n=1}^{N}\sum_{m=1}^{M} \frac{t_{mn}}{2} y_{mn}^2 \tag{5.8}$$

$$\texttt{s.t.} \quad \text{the constraints (5.2), (5.3) and (5.4)} \tag{5.9}$$

Denote the dual variables associated with constraints (5.2), (5.3) and (5.4) by $\boldsymbol{\lambda} \in \mathbb{R}^M, \boldsymbol{\mu} \in \mathbb{R}^N, \boldsymbol{\eta} \in \mathbb{R}^{M \times N}$, respectively. The Lagrangian of **DAP** is:

$$\begin{aligned} L(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta}) &= \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{s_{mn}}{2} x_{mn}^2 + \sum_{n=1}^{N}\sum_{m=1}^{M} \frac{t_{mn}}{2} y_{mn}^2 - \sum_{l=1}^{L}\sum_{n=1}^{N} r_{nl} \log z_{nl} \\ &+ \sum_{m=1}^{M} \lambda_m \left(\sum_{n=1}^{N} x_{mn} - C_m \right) + \sum_{n=1}^{N} \mu_n \left(\sum_{l=1}^{L} z_{nl} - \sum_{m=1}^{M} y_{mn} \right) + \sum_{m=1}^{M}\sum_{n=1}^{N} \eta_{mn}(y_{mn} - x_{mn}). \end{aligned} \tag{5.10}$$

Thus, the Karush-Kuhn-Tucker (KKT) conditions of **DAP** can be written as follows.

$$\textit{Primal Feasibility}: \quad \sum_{n=1}^{N} x_{mn} \le C_m, \forall m, \tag{5.11}$$

$$\sum_{l=1}^{L} z_{nl} \le \sum_{m=1}^{M} y_{mn}, \quad \forall n, \tag{5.12}$$

$$y_{mn} \le x_{mn}, \quad \forall m, n, \tag{5.13}$$

$$\textit{Dual Feasibility}: \quad \boldsymbol{\lambda} \succeq \mathbf{0}, \quad \boldsymbol{\mu} \succeq \mathbf{0}, \quad \boldsymbol{\eta} \succeq \mathbf{0}, \tag{5.14}$$

$$\textit{Complementary Slackness}: \quad \lambda_m \left(\sum_{n=1}^{N} x_{mn} - C_m \right) = 0, \forall m, \tag{5.15}$$

$$\mu_n \left(\sum_{l=1}^{L} z_{nl} - \sum_{m=1}^{M} y_{mn} \right) = 0, \forall n, \tag{5.16}$$

$$\eta_{mn}(y_{mn} - x_{mn}) = 0, \tag{5.17}$$

$$\textit{Stationarity}: \quad s_{mn} x_{mn} + \lambda_m - \eta_{mn} = 0, \forall m, n, \tag{5.18}$$

$$t_{mn} y_{mn} - \mu_n + \eta_{mn} = 0, \forall m, n, \tag{5.19}$$

$$-\frac{r_{nl}}{z_{nl}} + \mu_n = 0, \forall n, l. \tag{5.20}$$

From equations (5.18), (5.19) and (5.20), we obtain the *data allocation rule*:

$$x_{mn} = \frac{\eta_{mn} - \lambda_m}{s_{mn}}, \; y_{mn} = \frac{\mu_n - \eta_{mn}}{t_{mn}}, \; z_{nl} = \frac{r_{nl}}{\mu_n}, \; \forall m, n, l. \tag{5.21}$$

The data allocation rule prescribes how the data are allocated given the submitted bids $\mathbf{S} = [s_{mn}]_{M\times N}, \mathbf{T} = [t_{mn}]_{M\times N}, \mathbf{R} = [r_{nl}]_{N\times L}$. The allocation rule is parameterized by the Lagrangian multipliers $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta}$. Given a set of $\{\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta}\}$, an allocation rule is defined according to Eq. (5.21), i.e., a relationship between the data allocation and the bids is specified. As stated in the first step of the mechanism in Fig.

5.2, besides data allocation rule, the system designer also needs to specify the data pricing/reimbursement rule, i.e., the price and reimbursement of data as functions of the bids of the agents. In other words, for owner m, given its bid $\mathbf{s}_m$, the system designer needs to reimburse $f_m(\mathbf{s}_m)$ amount of money to compensate her loss due to privacy compromise. Similarly, the system designer will reimburse $g_n(\mathbf{t}_n)$ amount of money to collector n given her bid $\mathbf{t}_n$. Furthermore, the system designer will charge user l $h_l(\mathbf{r}_l)$ amount of money given her bid $\mathbf{r}_l$. As a mechanism designer, we need to appropriately design the pricing/reimbursement functions f_m, g_n, h_l so that the data allocation will gradually converge to the socially optimal point, i.e., the optimal point of **SWM**. In the following subsections, we specify how to design these pricing/reimbursement functions in detail.

5.3.2 Owners' Problems

For owner m, if she bids $\mathbf{s}_m$, she will get an reimbursement of $f_m(\mathbf{s}_m)$ as well as a loss of $U_m\left(\frac{\eta_{m1}-\lambda_m}{s_{m1}}, ..., \frac{\eta_{mN}-\lambda_m}{s_{mN}}\right)$, according to the data allocation rule in Eq. (5.21). Hence, the utility maximization problem of owner m can be written as:

$$\texttt{Maximize}_{\mathbf{s}_m \succeq \mathbf{0}} \quad f_m(\mathbf{s}_m) - U_m\left(\frac{\eta_{m1}-\lambda_m}{s_{m1}}, ..., \frac{\eta_{mN}-\lambda_m}{s_{mN}}\right). \tag{5.22}$$

The first order optimality condition of owner m's problem is:

$$\frac{\partial f_m(\mathbf{s}_m)}{\partial s_{mn}} + \frac{\partial U_m}{\partial x_{mn}} \frac{\eta_{mn}-\lambda_m}{s_{mn}^2} = 0, \forall n. \tag{5.23}$$

In order to design a suitable f_m such that the data allocation will converge to the socially optimal point, we need to compare Eq. (5.23) with the optimality condition of **SWM**. To this end, we write the Lagrangian of **SWM** as follows:

$$\begin{aligned}
&\tilde{L}(\mathbf{X},\mathbf{Y},\mathbf{Z},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\eta}) = \sum_{m=1}^{M} U_m(\mathbf{x}_m) + \sum_{n=1}^{N} V_n(\mathbf{y}_n) - \sum_{l=1}^{L} W_l(\mathbf{z}_l) \\
&+ \sum_{m=1}^{M} \lambda_m \left(\sum_{n=1}^{N} x_{mn} - C_m \right) + \sum_{n=1}^{N} \mu_n \left(\sum_{l=1}^{L} z_{nl} - \sum_{m=1}^{M} y_{mn} \right) + \sum_{m=1}^{M}\sum_{n=1}^{N} \eta_{mn}(y_{mn} - x_{mn}).
\end{aligned} \tag{5.24}$$

The constraints of **SWM** and **DAP** are the same and the only difference is the objective function. Thus, in the KKT conditions of **SWM**, the primal feasibility, dual feasibility and complementary slackness conditions are the same as those of **DAP**, i.e., equations (5.11)-(5.17), while stationarity condition of **SWM** is:

$$\frac{\partial U_m(\mathbf{x}_m)}{\partial x_{mn}} + \lambda_m - \eta_{mn} = 0, \quad \forall m, n, \tag{5.25}$$

$$\frac{V_n(\mathbf{y}_n)}{\partial y_{mn}} - \mu_n + \eta_{mn} = 0, \quad \forall m, n, \tag{5.26}$$

$$-\frac{W_l(\mathbf{z}_l)}{z_{nl}} + \mu_n = 0, \quad \forall n, l. \tag{5.27}$$

Combining equations (5.23) and (5.25), we derive:

$$\frac{\partial f_m(\mathbf{s}_m)}{\partial s_{mn}} = \frac{\lambda_m - \eta_{mn}}{s_{mn}^2}\frac{\partial U_m}{\partial x_{mn}} = \frac{\lambda_m - \eta_{mn}}{s_{mn}^2}(\eta_{mn} - \lambda_m) = -\frac{(\lambda_m - \eta_{mn})^2}{s_{mn}^2}. \tag{5.28}$$

Therefore, we set the reimbursement rule of owner m to be $f_m(\mathbf{s}_m) = \sum_{n=1}^{N} \frac{(\lambda_m - \eta_{mn})^2}{s_{mn}}$.

5.3.3 Collectors' Problems

For collector n, if she bids $\mathbf{t}_n$, she will get a reimbursement of $g_n(\mathbf{t}_n)$ and a loss of $V_n\left(\frac{\mu_n - \eta_{1n}}{t_{1n}}, ..., \frac{\mu_n - \eta_{Mn}}{t_{Mn}}\right)$. Thereby, the utility maximization problem of collector n is:

$$\texttt{Maximize}_{\mathbf{t}_n \succeq \mathbf{0}} \quad g_n(\mathbf{t}_n) - V_n\left(\frac{\mu_n - \eta_{1n}}{t_{1n}}, ..., \frac{\mu_n - \eta_{Mn}}{t_{Mn}}\right). \tag{5.29}$$

The optimality condition of collector n's problem is:

$$\frac{\partial g_n(\mathbf{t}_n)}{\partial t_{mn}} + \frac{\partial V_n}{\partial y_{mn}}\frac{\mu_n - \eta_{mn}}{t_{mn}^2} = 0, \quad \forall m. \tag{5.30}$$

Combining equations (5.26) and (5.30) yields:

$$\frac{\partial g_n(\mathbf{t}_n)}{\partial t_{mn}} = \frac{\eta_{mn} - mu_n}{t_{mn}^2} \frac{\partial V_n}{\partial y_{mn}} = \frac{\eta_{mn} - mu_n}{t_{mn}^2}(\mu_n - \eta_{mn}) = -\frac{(\mu_n - \eta_{mn})^2}{t_{mn}^2}. \quad (5.31)$$

So, the reimbursement function of collector n should be $g_n(\mathbf{t}_n) = \sum_{m=1}^{M} \frac{(\mu_n - \eta_{mn})^2}{t_{mn}}$.

5.3.4 Users' Problems

For user l, if she bids $\mathbf{r}_l$, she will be charged $h_l(\mathbf{r}_l)$ and has a gain of $W_l\left(\frac{r_{1l}}{\mu_1}, ..., \frac{r_{Nl}}{\mu_N}\right)$. Thus, the utility maximization problem of user l is:

$$\texttt{Maximize}_{\mathbf{r}_l \succeq \mathbf{0}} \quad -h_l(\mathbf{r}_l) + W_l\left(\frac{r_{1l}}{\mu_1}, ..., \frac{r_{Nl}}{\mu_N}\right). \quad (5.32)$$

The optimality condition of user l's problem is:

$$-\frac{\partial h_l(\mathbf{r}_l)}{\partial r_{nl}} + \frac{\partial W_l}{\partial z_{nl}} \frac{1}{\mu_n} = 0, \forall n. \quad (5.33)$$

Combining equations (5.27) and (5.33) yields:

$$\frac{\partial h(\mathbf{r}_l)}{\partial r_{nl}} = \frac{1}{\mu_n} \frac{\partial W_l}{\partial z_{nl}} = \frac{1}{\mu_n} \cdot \mu_n = 1. \quad (5.34)$$

Thus, we design the price function of user l to be $h_l(\mathbf{r}_l) = \sum_{n=1}^{N} r_{nl}$.

5.3.5 Summary of Algorithm

The owners' problem (5.22), the collectors' problem (5.29) and the users' problem (5.32) together specify how the bids are chosen in the second stage of the mechanism in Fig. 5.2. Then, in the third stage, the system designer computes the new data allocation result based on these submitted bids and the data allocation rule in Eq. (5.21). In the fourth stage, we update the dual variables $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta}$ (or equivalently, update the data allocation rule and data pricing/reimbursement rule) by invoking the subgradient method:

$$\lambda_m \leftarrow \left(\lambda_m + \alpha\left(\sum_{n=1}^{N} x_{mn} - C_m\right)\right)^+, \quad \forall m \tag{5.35}$$

$$\mu_n \leftarrow \left(\mu_n + \alpha\left(\sum_{l=1}^{L} z_{nl} - \sum_{m=1}^{M} y_{mn}\right)\right)^+, \quad \forall n \tag{5.36}$$

$$\eta_{mn} \leftarrow (\mu_{mn} + \alpha(y_{mn} - x_{mn}))^+, \forall m, n, \tag{5.37}$$

where $\alpha > 0$ is the step length and $x^+ = \max\{x, 0\}$. The proposed iterative auction mechanism is summarized in Algorithm 5.1. We remark that Algorithm 5.1 is a distributed algorithm: each data agent solves its own utility maximization problem in a parallel manner and the interactions between the agents. Algorithm 5.1 clearly resolves the two difficulties for directly solving **SWM** in Subsection 5.2-D: (i) each agent maximizes her own utility in accordance with her selfishness; (ii) the system designer does not direct access the private information of the agents, i.e., the loss/gain functions U, V, W. Instead the system designer gradually and implicitly elicits this information through iterative auctions.

5.4 Convergence and Economic Properties of the Mechanism

In this section, we theoretically show that the proposed iterative auction mechanism for data trading can indeed converge to the socially optimal operating point, i.e., the optimal point of **SWM**. Moreover, we prove that the mechanism has two appealing economic properties, i.e., individual rationality and weakly balanced budget, which makes the mechanism economically viable.

Algorithm 5.1: The proposed iterative auction mechanism

1: Initialize $\mathbf{X}^{(0)}, \mathbf{Y}^{(0)}, \mathbf{Z}^{(0)}, \boldsymbol{\lambda}^{(0)}, \boldsymbol{\mu}^{(0)}, \boldsymbol{\eta}^{(0)}$ to be non-negative. Set the time index τ to be 0.

2: Repeat the following until convergence:

3: The system designer announces $\boldsymbol{\lambda}^{(\tau)}, \boldsymbol{\mu}^{(\tau)}, \boldsymbol{\eta}^{(\tau)}$.

4: $\tau \leftarrow \tau + 1$.

5: Each owner m solves its problem (5.22) to get $\mathbf{s}_m^{(\tau)}$.

6: Each collector n solves its problem (5.29) to get $\mathbf{t}_n^{(\tau)}$.

7: Each user l solves its problem (5.32) to get $\mathbf{r}_l^{(\tau)}$.

8: The system designer computes the new $\mathbf{X}^{(\tau)}, \mathbf{Y}^{(\tau)}, \mathbf{Z}^{(\tau)}$ according to the current allocation rule (5.21) and the submitted bids $\mathbf{S}^{(\tau)}$, $\mathbf{T}^{(\tau)}$ and $\mathbf{R}^{(\tau)}$.

9: The system designer updates the dual variables:

$$\lambda_m^{(\tau)} = \left(\lambda_m^{(\tau-1)} + \alpha \left(\sum_{n=1}^{N} x_{mn}^{(\tau)} - C_m\right)\right)^+, \quad \forall m \tag{5.38}$$

$$\mu_n^{(\tau)} = \left(\mu_n^{(\tau-1)} + \alpha \left(\sum_{l=1}^{L} z_{nl}^{(\tau)} - \sum_{m=1}^{M} y_{mn}^{(\tau)}\right)\right)^+, \quad \forall n \tag{5.39}$$

$$\eta_{mn}^{(\tau)} = \left(\eta_{mn}^{(\tau-1)} + \alpha \left(y_{mn}^{(\tau)} - x_{mn}^{(\tau)}\right)\right)^+, \forall m, n. \tag{5.40}$$

5.4.1 Convergence Analysis

When designing the mechanism in Section 5.3, we make a connection between the data allocation rule, the optimality condition of each agent's utility maximization problem and the KKT conditions of **SWM**. Intuitively, the mechanism should guide the data allocation towards the solution of **SWM**. In this subsection, we rigorously demonstrate this convergence result. To make the analysis tractable, we assume that the step size α in the update of dual variables (5.38), (5.39) and (5.40) is very small, which is a reasonable assumption in the literature of subgradient method in optimization theory [10] and LMS algorithm in adaptive signal processing [45]. Thus, we can approximate Algorithm 5.1 with a continuous-time version by taking the time slot to be α. From Eq. (5.38), we know that λ_m is always non-negative. If $\lambda_m^{(\tau-1)} > 0$, since α is very small, the quantity inside the parenthesis of Eq. (5.38) is still positive. Thus, $\lambda_m^{(\tau)} = \lambda_m^{(\tau-1)} + \alpha\left(\sum_{n=1}^{N} x_{mn}^{(\tau)} - C_m\right)$. Noting that the time slot length is α, a small positive number, we have $\frac{d\lambda_m}{d\tau} = \sum_{n=1}^{N} x_{mn} - C_m$. If $\lambda_m^{(\tau-1)} = 0$, we can similarly derive that $\frac{d\lambda_m}{d\tau} = \left(\sum_{n=1}^{N} x_{mn} - C_m\right)^+$. Define the notation (for $x, y \in \mathbb{R}$ and $y \geq 0$):

$$(x)_y^+ = \begin{cases} x, & \text{if } \; y > 0, \\ x^+, & \text{if } \; y = 0. \end{cases} \tag{5.41}$$

Then, we have:

$$\frac{d\lambda_m}{d\tau} = \left(\sum_{n=1}^{N} x_{mn} - C_m\right)_{\lambda_m}^+. \tag{5.42}$$

Similarly, we have:

$$\frac{d\mu_n}{d\tau} = \left(\sum_{l=1}^{L} z_{nl} - \sum_{m=1}^{M} y_{mn} \right)^{+}_{\mu_n}, \tag{5.43}$$

$$\frac{d\eta_{mn}}{d\tau} = (y_{mn} - x_{mn})^{+}_{\eta_{mn}}. \tag{5.44}$$

Now, we are ready to state the convergence result.

Theorem 5.1 *Suppose the step size α in Algorithm 5.1 is small enough. Then, the data allocation $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ of Algorithm 5.1 converges to the optimal point of **SWM**. Moreover, the dual variables $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta})$ of Algorithm 5.1 converge to the dual optimal point of **SWM**.*

Proof: Denote the dual optimal point of **SWM** by $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{\eta}^*)$. Define the Lyapunov function:

$$H(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta}) = \frac{1}{2} \sum_{m=1}^{M} (\lambda_m - \lambda_m^*)^2 + \frac{1}{2} \sum_{n=1}^{N} (\mu_n - \mu_n^*)^2 + \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{N} (\eta_{mn} - \eta_{mn}^*)^2. \tag{5.45}$$

Taking derivative of Z with respect to the (continuous) time τ yields:

$$\frac{dH}{d\tau} = \sum_{m=1}^{M} (\lambda_m - \lambda_m^*) \frac{d\lambda_m}{d\tau} + \sum_{n=1}^{N} (\mu_n - \mu_n^*) \frac{d\mu_n}{d\tau} + \sum_{m=1}^{M} \sum_{n=1}^{N} (\eta_{mn} - \eta_{mn}^*) \frac{d\eta_{mn}}{d\tau} \tag{5.46}$$

$$= \sum_{m=1}^{M} (\lambda_m - \lambda_m^*) \left(\sum_{n=1}^{N} x_{mn} - C_m \right)^{+}_{\lambda_m} + \sum_{n=1}^{N} (\mu_n - \mu_n^*) \left(\sum_{l=1}^{L} z_{nl} - \sum_{m=1}^{M} y_{mn} \right)^{+}_{\mu_n}$$
$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} (\eta_{mn} - \eta_{mn}^*) (y_{mn} - x_{mn})^{+}_{\eta_{mn}} \tag{5.47}$$

$$\leq \sum_{m=1}^{M} (\lambda_m - \lambda_m^*) \left(\sum_{n=1}^{N} x_{mn} - C_m \right) + \sum_{n=1}^{N} (\mu_n - \mu_n^*) \left(\sum_{l=1}^{L} z_{nl} - \sum_{m=1}^{M} y_{mn} \right)$$
$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} (\eta_{mn} - \eta_{mn}^*) (y_{mn} - x_{mn}), \tag{5.48}$$

where we use equations (5.42), (5.43) and (5.44) to get Eq. (5.47). The reason of inequality (5.48) is as follows. If $\lambda_m = 0$, then $\left(\sum_{n=1}^{N} x_{mn} - C_m\right)^+_{\lambda_m} = \left(\sum_{n=1}^{N} x_{mn} - C_m\right)^+ \geq \sum_{n=1}^{N} x_{mn} - C_m$. Since $\lambda_m - \lambda_m^* = -\lambda_m^* \leq 0$, we have $(\lambda_m - \lambda_m^*)\left(\sum_{n=1}^{N} x_{mn} - C_m\right)^+_{\lambda_m} \leq (\lambda_m - \lambda_m^*)\left(\sum_{n=1}^{N} x_{mn} - C_m\right)$. If If $\lambda_m > 0$, we evidently have $(\lambda_m - \lambda_m^*)\left(\sum_{n=1}^{N} x_{mn} - C_m\right)^+_{\lambda_m} = (\lambda_m - \lambda_m^*)\left(\sum_{n=1}^{N} x_{mn} - C_m\right)$. In all, we always have $(\lambda_m - \lambda_m^*)\left(\sum_{n=1}^{N} x_{mn} - C_m\right)^+_{\lambda_m} \leq (\lambda_m - \lambda_m^*)\left(\sum_{n=1}^{N} x_{mn} - C_m\right)$ and similar inequalities hold for the other two terms in (5.47), leading to inequality (5.48). In Step 5, the optimal point of the problem (5.22) should satisfy the optimality condition (5.23). Noting the form of the reimbursement function f we design in Subsection 5.3-B, we have:

$$-\frac{(\lambda_m - \eta_{mn})^2}{s_{mn}^2} + \frac{\partial U_m(\mathbf{x}_m)}{\partial x_{mn}} \frac{\eta_{mn} - \lambda_m}{s_{mn}^2} = 0, \tag{5.49}$$

which leads to:

$$\lambda_m = \eta_{mn} - \frac{\partial U_m(\mathbf{x}_m)}{\partial x_{mn}}. \tag{5.50}$$

Similarly, from the optimality condition (5.30), we get

$$\mu_n = \eta_{mn} + \frac{\partial V_n(\mathbf{y}_n)}{\partial y_{mn}}. \tag{5.51}$$

And from the optimality condition (5.33), we obtain:

$$\mu_n = \frac{\partial W_l(\mathbf{z}_l)}{\partial z_{nl}}. \tag{5.52}$$

Denote the optimal point of **SWM** by $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{Z}^*)$. Since SWM is a convex optimization problem, KKT condition is necessary and sufficient for optimality. Hence, the primal optimal point $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{Z}^*)$ together with dual optimal point $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{\eta}^*)$

should satisfy the stationarity condition (5.25), (5.26) and (5.27), which can be further rewritten as:

$$\lambda_m^* = \eta_{mn}^* - \frac{\partial U_m(\mathbf{x}_m^*)}{\partial x_{mn}}, \tag{5.53}$$

$$\mu_n^* = \eta_{mn}^* + \frac{\partial V_n(\mathbf{y}_n^*)}{\partial y_{mn}}, \tag{5.54}$$

$$\mu_n^* = \frac{\partial W_l(\mathbf{z}_l^*)}{\partial z_{nl}}. \tag{5.55}$$

Hence, according to equations (5.50) and (5.53), we have:

$$\begin{aligned} &\sum_{m=1}^{M}(\lambda_m - \lambda_m^*)\left(\sum_{n=1}^{N} x_{mn} - \sum_{n=1}^{N} x_{mn}^*\right) \\ &= \sum_{m=1}^{M}\sum_{n=1}^{N}\left(\eta_{mn} - \frac{\partial U_m(\mathbf{x}_m)}{\partial x_{mn}} - \eta_{mn}^* + \frac{\partial U_m(\mathbf{x}_m^*)}{\partial x_{mn}}\right)(x_{mn} - x_{mn}^*), \end{aligned} \tag{5.56}$$

which can be further rewritten as:

$$\begin{aligned} &\sum_{m=1}^{M}(\lambda_m - \lambda_m^*)\left(\sum_{n=1}^{N} x_{mn} - \sum_{n=1}^{N} x_{mn}^*\right) + \sum_{m=1}^{M}\sum_{n=1}^{N}(\eta_{mn} - \eta_{mn}^*)(x_{mn}^* - x_{mn}) \\ &= \sum_{m=1}^{M}\sum_{n=1}^{N}\left(\frac{\partial U_m(\mathbf{x}_m^*)}{\partial x_{mn}} - \frac{\partial U_m(\mathbf{x}_m)}{\partial x_{mn}}\right)(x_{mn} - x_{mn}^*). \end{aligned} \tag{5.57}$$

Similarly, from equations (5.52) and (5.55), we obtain:

$$\sum_{n=1}^{N}(\mu_n - \mu_n^*)\left(\sum_{l=1}^{L} z_{nl} - \sum_{l=1}^{L} z_{nl}^*\right) = \sum_{n=1}^{N}\sum_{l=1}^{L}\left(\frac{\partial W_l(\mathbf{z}_l)}{\partial z_{nl}} - \frac{\partial W_l(\mathbf{z}_l^*)}{\partial z_{nl}}\right)(z_{nl} - z_{nl}^*). \tag{5.58}$$

And combining equations (5.51) and (5.54) yields:

$$\begin{aligned} &\sum_{n=1}^{N}(\mu_n - \mu_n^*)\left(\sum_{m=1}^{M} y_{mn}^* - \sum_{m=1}^{M} y_{mn}\right) \\ &= \sum_{m=1}^{M}\sum_{n=1}^{N}\left(\eta_{mn} + \frac{\partial V_n(\mathbf{y}_n)}{\partial y_{mn}} - \eta_{mn}^* - \frac{\partial V_n(\mathbf{y}_n^*)}{\partial y_{mn}}\right)(y_{mn}^* - y_{mn}), \end{aligned} \tag{5.59}$$

which can be rewritten as:

$$\begin{aligned}&\sum_{n=1}^{N}(\mu_n-\mu_n^*)\left(\sum_{m=1}^{M}y_{mn}^*-\sum_{m=1}^{M}y_{mn}\right)+\sum_{m=1}^{M}\sum_{n=1}^{N}(\eta_{mn}-\eta_{mn}^*)(y_{mn}-y_{mn}^*)\\&=\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\frac{\partial V_n(\mathbf{y}_n)}{\partial y_{mn}}-\frac{\partial V_n(\mathbf{y}_n^*)}{\partial y_{mn}}\right)(y_{mn}^*-y_{mn}).\end{aligned} \tag{5.60}$$

Moreover, since the primal optimal point $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{Z}^*)$ together with dual optimal point $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{\eta}^*)$ should satisfy the KKT conditions of **SWM**, including conditions (5.11)-(5.17) (this part of KKT conditions coincides with that of **DAP**), from the complimentary slackness conditions, we have:

$$\lambda_m^*\left(\sum_{n=1}^{N}x_{mn}^*-C_m\right)=0, \tag{5.61}$$

$$\mu_n^*\left(\sum_{l=1}^{L}z_{nl}^*-\sum_{m=1}^{M}y_{mn}^*\right)=0, \tag{5.62}$$

$$\eta_{mn}^*(y_{mn}^*-x_{mn}^*)=0. \tag{5.63}$$

Further notice that $\lambda_m, \mu_n, \eta_{mn} \geq 0$ and $\sum_{n=1}^{N} x_{mn}^* \leq C_m, \sum_{l=1}^{L} z_{nl}^* \leq \sum_{m=1}^{M} y_{mn}^*$. Thus, we get:

$$(\lambda_m-\lambda_m^*)\left(\sum_{n=1}^{N}x_{mn}^*-C_m\right)\leq 0, \tag{5.64}$$

$$(\mu_n-\mu_n^*)\left(\sum_{l=1}^{L}z_{nl}^*-\sum_{m=1}^{M}y_{mn}^*\right)\leq 0, \tag{5.65}$$

$$(\eta_{mn}-\eta_{mn}^*)(y_{mn}^*-x_{mn}^*)\leq 0. \tag{5.66}$$

Adding the six equations and inequalities (5.57), (5.58), (5.60), (5.64), (5.65) and (5.66) gives:

$$\begin{aligned}
&\sum_{m=1}^{M}(\lambda_m - \lambda_m^*)\left(\sum_{n=1}^{N} x_{mn} - C_m\right) + \sum_{n=1}^{N}(\mu_n - \mu_n^*)\left(\sum_{l=1}^{L} z_{nl} - \sum_{m=1}^{M} y_{mn}\right) \\
&\quad + \sum_{m=1}^{M}\sum_{n=1}^{N}(\eta_{mn} - \eta_{mn}^*)(y_{mn} - x_{mn}), \\
&\leq \sum_{m=1}^{M}\sum_{n=1}^{N}\left(\frac{\partial U_m(\mathbf{x}_m^*)}{\partial x_{mn}} - \frac{\partial U_m(\mathbf{x}_m)}{\partial x_{mn}}\right)(x_{mn} - x_{mn}^*) \\
&\quad + \sum_{n=1}^{N}\sum_{l=1}^{L}\left(\frac{\partial W_l(\mathbf{z}_l)}{\partial z_{nl}} - \frac{\partial W_l(\mathbf{z}_l^*)}{\partial z_{nl}}\right)(z_{nl} - z_{nl}^*) \\
&\quad + \sum_{m=1}^{M}\sum_{n=1}^{N}\left(\frac{\partial V_n(\mathbf{y}_n)}{\partial y_{mn}} - \frac{\partial V_n(\mathbf{y}_n^*)}{\partial y_{mn}}\right)(y_{mn}^* - y_{mn}) \\
&\leq 0
\end{aligned} \tag{5.67}$$

The last inequality of (5.67) is due to the convexity/concavity of the functions U, V, W. Specifically, since U_m, V_n are convex functions and W_l is concave function, we have:

$$\left(\nabla U_m(\mathbf{x}_m^*) - \nabla U_m(\mathbf{x}_m)\right)^T(\mathbf{x}_m^* - \mathbf{x}_m) \geq 0, \quad \forall m, \tag{5.68}$$

$$\left(\nabla W_l(\mathbf{z}_l) - \nabla W_l(\mathbf{z}_l^*)\right)^T(\mathbf{z}_l - \mathbf{z}_l^*) \leq 0, \quad \forall l, \tag{5.69}$$

$$\left(\nabla V_n(\mathbf{y}_n) - \nabla V_n(\mathbf{y}_n^*)\right)^T(\mathbf{y}_n - \mathbf{y}_n^*) \geq 0, \forall n. \tag{5.70}$$

Adding inequalities (5.68), (5.69) and (5.70) together over all m, n, l yields the last inequality of (5.67). Combining the inequalities (5.48) and (5.67), we obtain $\frac{dH}{d\tau} \leq 0$. Thus, according to LaSalle's invariance principle [59], $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta})$ converges to $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{\eta}^*)$. Comparing equations (5.50), (5.51) and (5.52) with equations (5.53), (5.54) and (5.55), we conclude that $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ converges to $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{Z}^*)$. $\square$

5.4.2 Economic Properties

Implementation of the proposed iterative auction mechanism in real-world data trading market necessitates brilliant economic properties of the mechanism. In this subsection, we show that the proposed mechanism has appealing economic properties. First, the proposed mechanism is clearly *efficient* since it converges to the socially optimal point. Second, the proposed mechanism possesses the *incentive compatibility* property because in each auction iteration, each agent is maximizing her own utility selfishly. To ensure that each agent complies to the mechanism voluntarily, the mechanism needs to guarantee that every agent has non-negative utility, i.e., the mechanism should be *individually rational*. This is shown in the following proposition.

Proposition 5.1 *Assume that* $U_m(\boldsymbol{0}) = 0, V_n(\boldsymbol{0}) = 0, W_l(\boldsymbol{0}) = 0, \forall m, n, l$. *Then, when Algorithm 5.1 converges, every data agent has non-negative utility, i.e., the proposed mechanism is individually rational.*

Proof: As shown in Theorem 5.1, when Algorithm 5.1 converges, $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ becomes $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{Z}^*)$ and $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta})$ becomes $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{\eta}^*)$. Thus, according to the allocation rule (5.21), the bids $(\mathbf{S}, \mathbf{T}, \mathbf{R})$ become $(\mathbf{S}^*, \mathbf{T}^*, \mathbf{R}^*)$ defined as follows:

$$s^*_{mn} = \frac{\eta^*_{mn} - \lambda^*_m}{x^*_{mn}}, \tag{5.71}$$

$$t^*_{mn} = \frac{\mu^*_n - \eta^*_{mn}}{y^*_{mn}}, \tag{5.72}$$

$$r^*_{nl} = z^*_{nl}\mu^*_n. \tag{5.73}$$

Since U_m is convex, we have:

$$0 = U_m(\mathbf{0}) \geq U_m(\mathbf{x}_m^*) + \nabla U_m(\mathbf{x}_m^*)^T(\mathbf{0} - \mathbf{x}_m^*), \tag{5.74}$$

which can be rewritten as:

$$\sum_{n=1}^{N} \frac{\partial U_m(\mathbf{x}_m^*)}{\partial x_{mn}} x_{mn}^* - U_m(\mathbf{x}_m^*) \geq 0. \tag{5.75}$$

By Eq. (5.53), we further derive:

$$\sum_{n=1}^{N} (\eta_{mn}^* - \lambda_m^*) x_{mn}^* - U_m(\mathbf{x}_m^*) \geq 0, \tag{5.76}$$

which by Eq. (5.71) can be written as:

$$\sum_{n=1}^{N} \frac{(\lambda_m^* - \eta_{mn}^*)^2}{s_{mn}^*} - U_m(\mathbf{x}_m^*) \geq 0. \tag{5.77}$$

Note that the left hand side is exactly the utility of owner m when Algorithm 5.1 converges. So, owner m has non-negative utility. Similarly, from the convexity of V_n, we have:

$$V_n(\mathbf{y}_n^*) \leq \sum_{m=1}^{M} y_{mn}^* \frac{\partial V_n(\mathbf{y}_n^*)}{\partial y_{mn}}, \tag{5.78}$$

which by equations (5.54) and (5.72) can be rewritten as:

$$\sum_{m=1}^{M} \frac{(\mu_n^* - \eta_{mn}^*)}{t_{mn}^*} - V_n(\mathbf{y}_n^*) \geq 0. \tag{5.79}$$

Notice that the left hand side is just the utility of collector n when Algorithm 5.1 converges. We thus assert that each collector has non-negative utility. From the concavity of W_l, we obtain:

$$W_l(\mathbf{z}_l^*) \geq \sum_{n=1}^{N} z_{nl}^* \frac{\partial W_l(\mathbf{z}_l^*)}{\partial z_{nl}}, \tag{5.80}$$

which by equations (5.55) and (5.73) is be written as:

$$-\sum_{n=1}^{N} r_{nl}^* + W_l(\mathbf{z}_l^*) \geq 0. \tag{5.81}$$

Hence, each user has non-negative utility. Overall, we conclude that the mechanism is individually rational. □

We can further show that the system designer has *weakly balanced budget*, i.e., the income (through the data reimbursement/pricing) of the system designer in the mechanism is non-negative when Algorithm 5.1 converges. In other words, the system designer does not need to inject any money into the data market in order to implement the mechanism.

Proposition 5.2 *When Algorithm 5.1 converges, the income of the system designer through data reimbursement/pricing in the mechanism is non-negative. In other words, the mechanism has weakly balanced budget.*

Proof: The income of the system designer through data reimbursement/pricing is:

$$\sum_{l=1}^{L} h_l(\mathbf{r}_l^*) - \sum_{m=1}^{M} f_m(\mathbf{s_m}^*) - \sum_{n=1}^{N} g_n(\mathbf{t}_n^*) \tag{5.82}$$

$$= \sum_{l=1}^{L}\sum_{n=1}^{N} r_{nl}^* - \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{(\lambda_m^* - \eta_{mn}^*)}{s_{mn}^*} - \sum_{n=1}^{N}\sum_{m=1}^{M} \frac{(\mu_n^* - \eta_{mn}^*)^2}{t_{mn}^*} \tag{5.83}$$

$$= \sum_{l=1}^{L}\sum_{n=1}^{N} z_{nl}^*\mu_n^* - \sum_{m=1}^{M}\sum_{n=1}^{N} x_{mn}^*(\eta_{mn}^* - \lambda_m^*) - \sum_{n=1}^{N}\sum_{m=1}^{M} y_{mn}^*(\mu_n^* - \eta_{mn}^*) \tag{5.84}$$

$$= \sum_{m=1}^{M}\sum_{n=1}^{N} \eta_{mn}^*(y_{mn}^* - x_{mn}^*) + \sum_{n=1}^{N} \mu_n^* \left(\sum_{l=1}^{L} z_{nl}^* - \sum_{m=1}^{M} y_{mn}^* \right) + \sum_{m=1}^{M}\sum_{n=1}^{N} x_{mn}^*\lambda_m^* \tag{5.85}$$

$$\geq 0 \tag{5.86}$$

where Eq. (5.84) comes from equations (5.71), (5.72) and (5.73). The reason of the last step is: $\eta^*_{mn}(y^*_{mn} - x^*_{mn}) = 0, \mu^*_n \left(\sum_{l=1}^{L} z^*_{nl} - \sum_{m=1}^{M} y^*_{mn}\right) = 0$ due to complimentary slackness (5.62) and (5.63) and $x^*_{mn} \geq 0, \lambda^*_m \geq 0$. $\square$

5.5 Extension to Non-Exclusive Data Trading

In previous sections, we assume that the data are exclusive, i.e., the same data can be dispensed to only one user and one collector. However, in many real-world data markets, the data can be *non-exclusive*, i.e., the same data can be allotted to multiple collectors and users. For example, many software/APP developers (data users) may want to access the same online data of some social network (data owner); or many researchers (data users) may want to use the same data from an organization (data owner) to conduct research. In this section, we formulate the data trading problem with non-exclusive data and extend the proposed mechanism in Section 5.3 to this scenario.

Since the same data can be distributed to multiple collectors, different collectors' data can overlap each other. To avoid purchasing the same data from different collectors, we assume that each user buys data from only one single collector. Equivalently, from the collectors' perspective, each collector n serves a set of users $\mathcal{L}_n$ and users in $\mathcal{L}_n$ only purchase data from collector n. For example, in real world, a data collection company may occupy the most of the share of the local market in some region and becomes the monopoly in the local region. Basically all data users in this region will purchase data only from this data collector. Note that the sets

$\mathcal{L}_n, n = 1, ..., N$ are mutually exclusive and $\bigcup_{n=1}^{N} \mathcal{L}_n = \{1, ..., L\}$. Each user l purchases from its designated collector z_{ml} amount owner m's data. Other notations are the same as the exclusive data trading model in Section 5.2. The social welfare maximization problem for non-exclusive data trading can be formulated as follows.

$$\begin{aligned}
\texttt{Maximize}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \quad & -\sum_{m=1}^{M} U_m(\mathbf{x}_m) - \sum_{n=1}^{N} V_n(\mathbf{y}_n) + \sum_{l=1}^{L} W_l(\mathbf{z}_l) && (5.87)\\
\texttt{s.t.} \quad & y_{mn} \leq x_{mn}, \quad \forall m, n, && (5.88)\\
& x_{mn} \leq C_m, \quad \forall m, n, && (5.89)\\
& z_{ml} \leq y_{mn}, \quad \forall m, n, l \in \mathcal{L}_n. && (5.90)
\end{aligned}$$

The first constraint means that the data collected by collectors should be no more than the data authorized by the owners. The second constraint is the data constraint at each owner. Instead of total data constraint in the exclusive data trading scenario, the data constraint becomes individual data constraint in the non-exclusive data trading scenario. The third constraint indicates that the data purchased by users are no greater than the data collected by collectors. Similar to the exclusive data trading scenario, it is inviable to directly solve this social welfare maximization problem and enforce the solution for the data agents. Hence, we go through similar procedures as in Section 5.3 to obtain an iterative auction mechanism which can achieve the social optimum while respecting agents' private information (their loss/gain functions) and selfishness. The mechanism is summarized in Algorithm 5.2 and the design details are omitted. In Algorithm 5.2, we denote the Lagrangian multipliers corresponding to constraints (5.88), (5.89) and

(5.90) by $\boldsymbol{\mu} \in \mathbb{R}^{M\times N}, \boldsymbol{\lambda} \in \mathbb{R}^{M\times N}$ and $\boldsymbol{\eta} \in \mathbb{R}^{M\times L}$, respectively.

5.6 Simulations and Real Data Experiments

In this section, we present simulations as well as real data experiments to validate the theoretical results for the proposed iterative auction mechanism. We consider both exclusive data trading and non-exclusive data trading.

5.6.1 Simulations

Consider a data market with $M = 2$ data owners, $N = 2$ data collectors and $L = 4$ data users. The total data amount of owners 1 and 2 are set to be 2 and 4, respectively. The owners' convex loss functions are defined as follows:

$$U_m(\mathbf{x}_m) = a_m \left(\sum_{n=1}^{2} e^{x_{mn}} - 2 \right), \quad m = 1, 2, \tag{5.100}$$

where $a_1 = 0.1, a_2 = 0.3$. The collectors' convex loss functions are defined as:

$$V_n(\mathbf{y}_n) = b_n \sum_{m=1}^{2} y_{mn}^2, \quad n = 1, 2, \tag{5.101}$$

where $b_1 = 0.5, b_2 = 1$. The users' concave gain functions are:

$$W_l(\mathbf{z}_l) = c_l \sum_{n=1}^{2} \log(1 + z_{nl}), \quad l = 1, 2, 3, 4, \tag{5.102}$$

where $c_1 = \frac{3}{2}, c_2 = \frac{7}{6}, c_3 = \frac{5}{6}, c_4 = \frac{1}{2}$.

We first consider the exclusive data trading scenario. We simulate the proposed iterative auction mechanism in Algorithm 5.1. In Fig. 5.3, we validate the convergence behavior of the mechanism. The relative error used in Fig. 5.3 is

Algorithm 5.2: The iterative auction mechanism for non-exclusive data trading

1: Initialize $\mathbf{X}^{(0)}, \mathbf{Y}^{(0)}, \mathbf{Z}^{(0)}, \boldsymbol{\lambda}^{(0)}, \boldsymbol{\mu}^{(0)}, \boldsymbol{\eta}^{(0)}$ to be non-negative. Set the time index τ to be 0.

2: Repeat the following until convergence:

3: The system designer announces $\boldsymbol{\lambda}^{(\tau)}, \boldsymbol{\mu}^{(\tau)}, \boldsymbol{\eta}^{(\tau)}$.

4: $\tau \leftarrow \tau + 1$.

5: Each owner m solves the following problem to get $\mathbf{r}_m^{(\tau)}$:

$$\texttt{Maximize}_{\mathbf{r}_m \succeq \mathbf{0}} \quad \sum_{n=1}^{N} \frac{\left(\lambda_{mn}^{(\tau)} - \mu_{mn}^{(\tau)}\right)^2}{r_{mn}} - U_m \left(\frac{\mu_{m1}^{(\tau)} - \lambda_{m1}^{(\tau)}}{r_{m1}}, ..., \frac{\mu_{mN}^{(\tau)} - \lambda_{mN}^{(\tau)}}{r_{mN}} \right). \tag{5.91}$$

6: Each collector n solves the following problem to get $\mathbf{s}_n^{(\tau)}$:

$$\texttt{Maximize}_{\mathbf{s}_n \succeq \mathbf{0}} \quad \sum_{m=1}^{M} \frac{\left(\sum_{l \in \mathcal{L}_n} \eta_{ml}^{(\tau)} - \mu_{mn}\right)^2}{s_{mn}} - V_n \left(\frac{\sum_{l \in \mathcal{L}_n} \eta_{1l}^{(\tau)} - \mu_{1n}^{(\tau)}}{s_{1n}}, ..., \frac{\sum_{l \in \mathcal{L}_n} \eta_{Ml}^{(\tau)} - \mu_{Mn}^{(\tau)}}{s_{Mn}} \right). \tag{5.92}$$

7: Each user l solves the following problem to get $\mathbf{t}_l^{(\tau)}$:

$$\texttt{Maximize}_{\mathbf{t}_l \succeq \mathbf{0}} \quad -\sum_{m=1}^{M} t_{ml} + W_l \left(\frac{t_{1l}}{\eta_{1l}^{(\tau)}}, ..., \frac{t_{Ml}}{\eta_{Ml}^{(\tau)}} \right). \tag{5.93}$$

8: The system designer computes the new $\mathbf{X}^{(\tau)}, \mathbf{Y}^{(\tau)}, \mathbf{Z}^{(\tau)}$ according to:

$$x_{mn}^{(\tau)} = \frac{\mu_{mn}^{(\tau)} - \lambda_{mn}^{(\tau)}}{r_{mn}^{(\tau)}}, \tag{5.94}$$

$$y_{mn}^{(\tau)} = \frac{\sum_{l \in \mathcal{L}_n} \eta_{ml}^{(\tau)} - \mu_{mn}^{(\tau)}}{s_{mn}^{(\tau)}}, \tag{5.95}$$

$$z_{ml}^{(\tau)} = \frac{t_{ml}^{(\tau)}}{\eta_{ml}^{(\tau)}}. \tag{5.96}$$

9: The system designer updates the dual variables:

$$\lambda_{mn}^{(\tau)} = \left(\lambda_{mn}^{(\tau-1)} + \alpha \left(x_{mn}^{(\tau)} - C_m\right)\right)^+, \quad \forall m, n, \tag{5.97}$$

$$\mu_{mn}^{(\tau)} = \left(\mu_{mn}^{(\tau-1)} + \alpha \left(y_{mn}^{(\tau)} - x_{mn}^{(\tau)}\right)\right)^+, \quad \forall m, n, \tag{5.98}$$

$$\eta_{ml}^{(\tau)} = \left(\eta_{ml}^{(\tau-1)} + \alpha \left(z_{ml}^{(\tau)} - y_{mn}^{(\tau)}\right)\right)^+, \forall m, n, l \in \mathcal{L}_n. \tag{5.99}$$

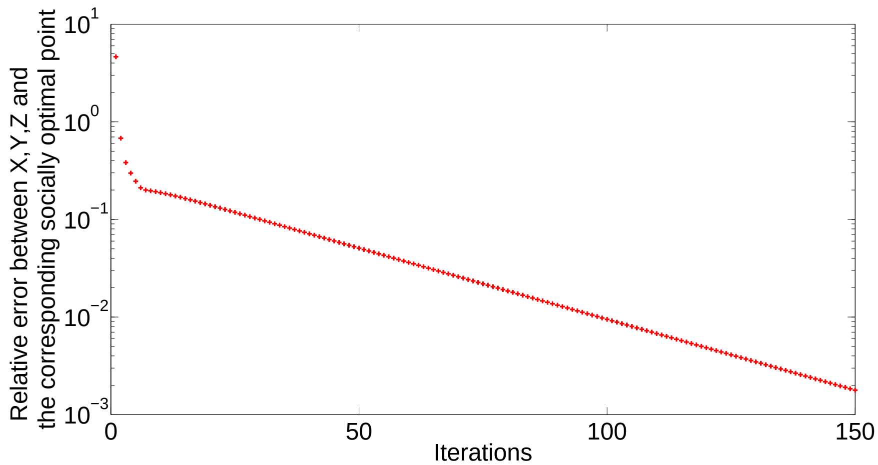


Figure 5.3: Convergence of the iterative auction mechanism to the socially optimal point, i.e., the optimal point of **SWM**.

$\max\left\{\frac{||X-X^*||_F}{||X^*||_F}, \frac{||Y-Y^*||_F}{||Y^*||_F}, \frac{||Z-Z^*||_F}{||Z^*||_F}\right\}$, where $||\cdot||_F$ means the Frobenius norm. As guaranteed by Theorem 5.1, the mechanism converges to the socially optimal point, i.e., the mechanism is efficient. We further investigate the economic properties of the mechanism through simulations in Fig. 5.4. We report the utilities of the owner 1, collector 1 and user 1 as the algorithm gradually converges. As asserted in Proposition 5.1, the mechanism is individually rational: the three data agents in Fig. 5.4 have non-negative utilities when the algorithm converges. Furthermore, we show the budget balance (income) of the system designer and find that as assured by Proposition 5.2, the budget balance is non-negative when the algorithm converges. Next, we turn to the non-exclusive data trading scenario. We set $\mathcal{L}_1 = \{1, 2\}, \mathcal{L}_2 = \{3, 4\}$. Other simulation setup remains unchanged and we simulate the iterative auction mechanism in Algorithm 5.2. As exhibited in Fig. 5.5, the mechanism still converges to the socially optimal point.

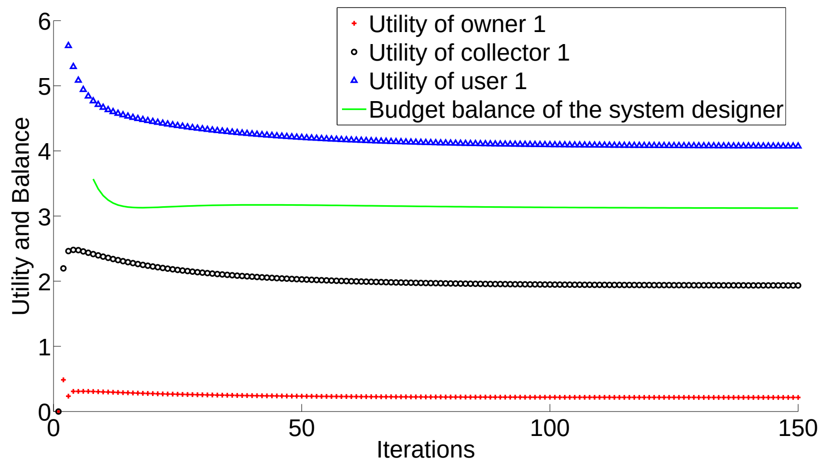


Figure 5.4: The utilities of owner 1, collector 1 and user 1 and the budget balance (income) of the system designer.

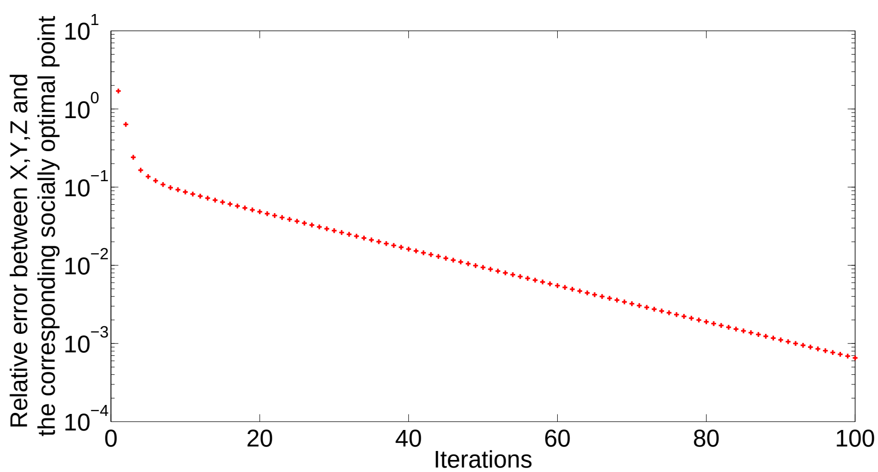
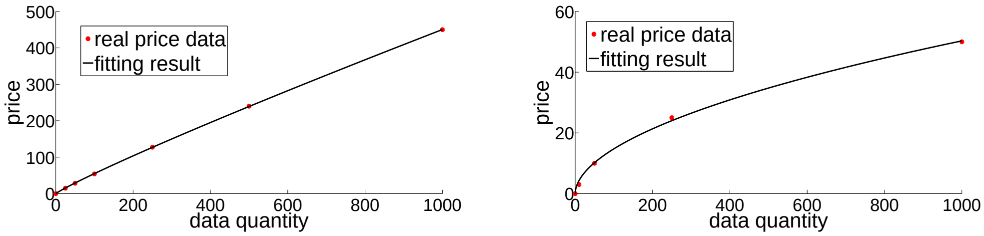


Figure 5.5: Convergence of the iterative auction mechanism to the socially optimal point: non-exclusive data trading.



(a) Wealth score data price: $y = 0.821x^{0.9131}$ (b) Text analytics data price: $y = 1.267x^{0.5329}$

Fig. 5.6: Fitting the real-world data price

5.6.2 Real Data Experiments

In this subsection, we use real data to get the loss/gain functions of the data agents and investigate the performance of the proposed mechanism on them. We still consider a data market with $M = 2$ owners, $N = 2$ collectors and $L = 4$ users. We first use real data prices to estimate the users' gain functions. To this end, we fit the prices of the two datasets, namely the wealth score dataset and the text analytics dataset, in the Microsoft Azure Marketplace [1] (a data trading platform) with the function $y = ax^b$. The fitting results are shown in Fig. 5.6, which are very accurate. The sum of these two price functions can be regarded as the mean user gain function. To introduce heterogeneity into users' gain functions, we multiple a coefficient onto this mean user gain to get individual users' gains as follows:

$$W_l(\mathbf{z}_l) = c'_l \sum_{n=1}^{2} \alpha_n z_{nl}^{\beta_n}, \quad l = 1, 2, 3, 4 \tag{5.103}$$

where $\alpha_1 = 0.821, \alpha_2 = 1.267, \beta_1 = 0.9131, \beta_2 = 0.5329, c'_1 = 1/2, c'_2 = 5/6, c'_3 = 7/6, c'_4 = 3/2$.

Next, we estimate the owners' loss functions. In [112], a relationship between the information loss and the privacy breach level in anonymization is obtained from real data [3]. Specifically, the privacy leakage is quantified by the k-anonymity, which means that the probability that an individual item being re-identified by an attacker is no higher than $1/k$. Thus, $1/k$ can be regarded as the loss of the data owner. (total data amount $-$ $\mathtt{IL}$) can be regarded as the effective amount of data obtained by a collector, where $\mathtt{IL}$ means the information loss. The relationship between k and $\mathtt{IL}$ is estimated to be $\mathtt{IL} = -0.4804k^{-0.2789} + 0.7883$, which can be rewritten as $1/k = (2.0816(0.7883 - \mathtt{IL}))^{3.5855}$. We set 0.7883 to be the total amount of data and thus $y = (2.0816x)^{3.5855}$ can be regarded as the average owners' loss function. By varying the coefficients, we introduce heterogeneity to the loss function and finally set:

$$U_m(\mathbf{x}_m) = a'_m \sum_{n=1}^{2} (\theta_n x_{mn})^{3.5855}, \quad m = 1, 2, \tag{5.104}$$

where $\theta_1 = 1.5816, \theta_2 = 2.5816, a'_1 = 5, a'_2 = 15$. As for the collectors' loss functions V_n, it is hard to find corresponding real data and we directly use quadratic functions in simulation setups for them. Other experiment setups are the same as those of simulations.

With the loss/gain functions estimated from real data, we test the performance of the proposed iterative auction mechanism. We first consider the exclusive data trading. The total data amounts of owner 1 and owner 2 are 0.25 and 0.5, respectively. As shown in Fig. 5.7, the mechanism still converges to the socially optimal point. In Fig. 5.8, we further observe that the individual rationality and weakly balanced budget still hold as the utilities of owner 1, collector 1 and user 1 as well as the budget balance of the system designer are all non-negative. Then, we

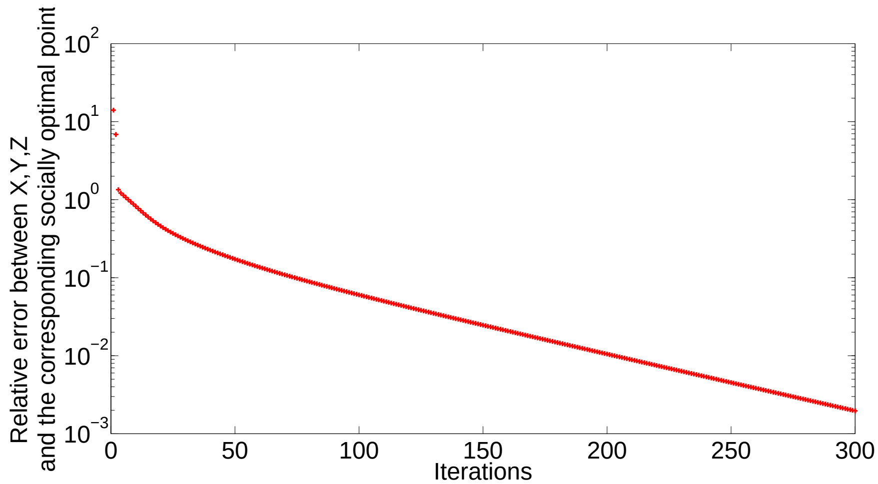


Figure 5.7: Convergence of the iterative auction mechanism to the socially optimal point in real data experiment.

change to the non-exclusive data trading and alter the total data amounts of owner 1 and owner 2 to be 0.2 and 0.4, respectively. We remark that the mechanism still converges to the socially optimal point, as illustrated in Fig. 5.9.

Lastly, we endeavor to compare the proposed iterative auction mechanism with the contract-theoretic approach in [112]. The model of [112] consists of multiple data owners and one single data collector without the notion of data users. To accommodate to this, we consider $M = 4$ owners, $N = 1$ collector and $L = 1$ user in our model. As per setups of real data experiments, we set the loss function of owners to be:

$$U_m(x_m) = a''_m(2.0816x_m)^{3.5855}, \quad m = 1, 2, 3, 4, \tag{5.105}$$

where $a''_1 = 5, a''_2 = \frac{25}{3}, a''_3 = \frac{35}{3}, a''_4 = 15$. The total data amount of each owner is

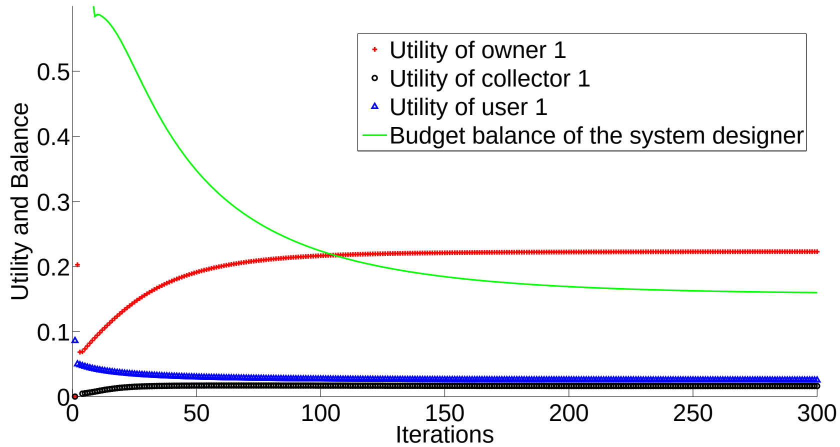


Figure 5.8: The utilities of owner 1, collector 1 and user 1 and the budget balance (income) of the system designer in real data experiment.

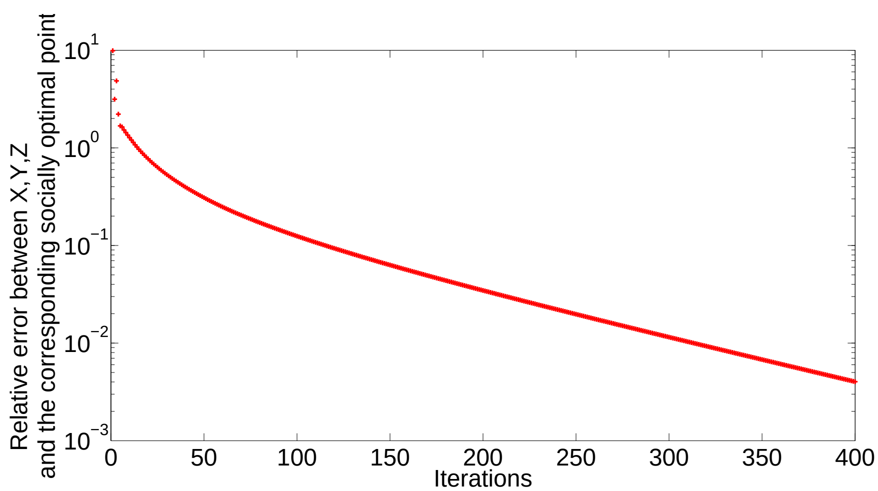


Figure 5.9: Convergence of the iterative auction mechanism to the socially optimal point in real data experiment: non-exclusive data trading.

0.08. Moreover, we set the gain function of the single user to be:

$$W_1(z_1) = 0.82105 z_1^{0.5329}. \tag{5.106}$$

The loss function of the single data collector still takes the quadratic form previously used, i.e., $V_1(\mathbf{y}_1) = \frac{1}{2}\mathbf{y_1}^T\mathbf{y_1}$. Since the model in [112] only considers linear owner loss, we use $\tilde{U}_m(x_m) = 2.0816 a''_m x_m$ for [112]. Besides, the model in [112] sets the collector's gain to be a square root function. Hence, we use $\tilde{W}_1(z_1) = 0.82105 z_1^{0.5}$ for [112]. Note that the collector in [112] plays the role of end user and we translates that into the user in our model. In the model of [112], we need to specify a required total amount of data, i.e., $q_{\text{req}} = \sum_{m=1}^{M} x_m$, which we set to be 0.16, i.e., the half of the sum of total data amounts of all the owners. We first simulate the proposed iterative auction mechanism, which still converges to the socially optimal point, as illustrated in Fig. 5.10. The socially optimal point is $X = Y = [0.08, 0.08, 0.074, 0.074]^T, Z = 0.3015$ and the optimal social welfare (which is obtained by the proposed mechanism) is 0.373. Then, we simulate the contract-theoretic approach of [112], which gives the data allocation $X = Y = [0.080.0800]^T, Z = 0.16$ and a social welfare of 0.2812. Thus, we observe that the proposed mechanism can achieve a higher social welfare than [112].

According to the experiments and simulations, a practical issue of the proposed iterative auction mechanism is that it may need hundreds of iterations to converge. This requires the bidders (agents) to bid for hundreds of times. A common solution to this issue is to equip each bidder with some bidding software, which can automatically bid for the agent according to some preset bidding rule such as

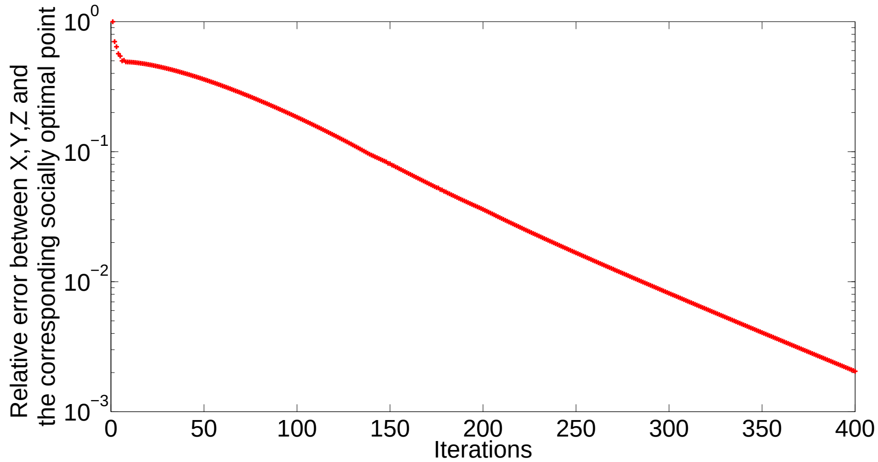


Figure 5.10: Convergence of the iterative auction mechanism to the socially optimal point in the comparison experiment.

the one specified in the proposed iterative auction mechanism. With the help of such bidding softwares, the bidding processes can be very fast and accomplish hundreds of iterations quickly, making the proposed mechanism practical. In fact, fast iterative bidding with the assist of bidding softwares is already used in practice such as the eBay auction.

5.7 Summary

In this chapter, we study the data trading problem with multiple data owners, collectors and users. We present an iterative auction mechanism to guide the selfish agents to behave in a socially optimal way without direct access of their private information. We theoretically prove the convergence as well as economic properties (individual rationality and weakly balanced budget) of the mechanism. Simulations and real data experiments are carried out to confirm the theoretical properties of the proposed mechanism.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

In this dissertation, we have presented several game-theoretic analyses of user behaviors in social systems. We have also designed an efficient data trading mechanism for data markets with multiple data agents. These works shed some light on the fundamentals of the comprehension, design and optimization of modern social networks, data markets or more generally, multi-agent systems.

First, we study the information propagation problem in heterogeneous social networks composed of users with different hobbies and influences or more abstractly, types. Two distinct network scenarios are considered, namely the unknown user type model and the known user type model, depending on whether users know the types of their neighbors. Modeling users' learning and decision-making processes as a graphical evolutionary game, we theoretically derive the evolutionary dynamics and the evolutionarily stable states (ESSs) of the information diffusion game. Numerical experiments based on both synthetic data and real-world information dessemination data are presented to confirm the validity of the proposed game-theoretic models for information diffusion.

Second, incorporating the notion of long-term incentives of users, we propose a novel sequential game theoretic model for users' decision making procedure in the formation of generic popularity dynamics, e.g., information diffusion dynamics and paper citation dynamics. The existence and uniqueness of the symmetric Nash equilibrium (SNE) of the game are proved. First and second order properties of the SNE are demonstrated, confirming our empirical observations from real-world popularity dynamics. Furthermore, the game-theoretic model and analysis are corroborated by fitting and prediction experiments based on real-world information propagation data and paper citation data.

Third, we examine the social learning problem, in which networked agents collaborate to detect some unknown state of the nature, which can be the quality of some products or services in recommendation systems in practice. A distributed evolutionary game theoretic learning algorithm is proposed and each agent only needs to communicate its binary action with its neighbors, making the algorithm computationally efficient. Theoretical analysis manifests that the ESS of the game coincides with the decision of a fictitious centralized detector, highlighting the optimality of the proposed learning algorithm.

Finally, we investigate the data trading problem in data markets with multiple data owners, collectors and users. An iterative auction based data trading mechanism is proposed to guide the selfish agents to trade data efficiently without direct access their private information. The proposed mechanism is shown to converge to the socially optimal operation point and has appealing economic properties including individual rationality and weakly balanced budget. Additionally, the mechanism is validated through numerical experiments based on real-world data prices from Microsoft Azure Marketplace, an emerging data trading market.

This dissertation has broad applications in multi-agent systems with intelligent users and modern data trading platforms. Firstly, the proposed game-theoretic frameworks deepen our comprehension of social network users' behaviors. With the presented game-theoretic analysis, we can go one step further to design smart mechanisms to guide users to behave in a certain way. For instance, we may give online users appropriate virtual badges or virtual coins to alter their utility functions and thus influence the outcomes (or equilibria of the games) of users' interactions/decision-making. This can be used to incentivize users to adopt certain behaviors that we desire. In fact, some simple incentive mechanisms have already been implemented in several successful real-world social websites, e.g., Stack Overflow (a popular Q&A website), where virtual badges and sophisticated rating mechanisms are used to incentivize users to raise meaningful questions and to provide high quality answers. Secondly, the iterative auction based data trading mechanism proposed in this dissertation can be applied to real-world data trading systems as a novel trading paradigm in contrast to the traditional fixed price trading scheme used in most existing data trading platforms such as Microsoft Azure Marketplace. While respecting the selfishness and privacy concerns of data agents, the proposed mechanism guarantees overal system efficiency, which cannot be achieved by existing data trading methods. Actually, in practice, some simple auction mechanisms have already been used in electronic trading platforms (e.g., eBay Auction) for goods other than data, indicating the promising prospect of auction based trading schemes.

6.2 Future Work

In this section, I point out two potential future research directions, namely signal processing for social networks and data trading beyond our existing work in Chapter 5.

6.2.1 Signal Processing for Social Networks

The first three works in this dissertation, i.e., Chapters 2, 3, 4, are focused on game-theoretic analysis for social network users' behaviors. One commmon implicit assumption made in most game-theoretic works is that the players should be rational so that the various solution concepts of the game, e.g., Nash equilibrium, subgame perfect equilibrium, evolutionarily stable states, can hold in practice. Sometimes, this rationality hypothesis is overly restrictive as many practical agents are not fully rational due to factors such as limited computational capability. For instance, a social network user may not make the best response with respect to her neighbors' actions because calculation of such best response consumes too much efforts for the user. Consequently, though game theory is a powerful tool to give insightful explanations of the underlying mechanisms of many social phenomena, it is not suitable for those data driven tasks such as statistical inference, estimation and detection (or at least, one cannot solely rely on game theory for those tasks). This limitation motivates me to pursue a more data-centric approach for problems in social networks by invoking tools from signal processing, pattern recognition and optimization theory in the future.

In fact, there are plenty of emerging well-posed signal processing problems in social networks or more generally, network science. For instance, researchers often encounter the so called *topology inference* problem in network science [5, 96], e.g., inferring an influence graph of social entities from the observed social events and inferring a brain connection network from the available brain signals. Generally speaking, in topology inference, we are given some data or signals over a network and our goal is to estimate the underlying network topology. To some extent, the topology inference problem can be regarded as the opposite of classical distributed signal processing problems, in which we are given the network topology and our goal is to process the signals over the network.

Another example of signal processing in social networks is *cascade tracking* [5, 6]. When a piece of information propagates over social networks, it generates a path of information dissemination, which is called information cascade. For instance, an information cascade of $\mathtt{A} \to \mathtt{B} \to \{\mathtt{C},\mathtt{D}\}$ means that the information propagates from `A` to `B` and later from `B` to `C` and `D`. Information cascades are important for many applications such as identification of critical/influential users in social networks. Unfortunately, in practice, information cascades are often not directly observable since users usually do not identify who influences whom in social networks. Therefore, we have to apply statistical signal processing techniques to infer the hidden information cascades from the available data such as timestamps of infections, e.g., mentioning a certain phrase or purchasing a certain product. Other network signal processing issues encompass community detection, signal recovery and signal sampling over graphs, etc [4, 19–22, 74, 92, 93].

6.2.2 Data Trading: Quantity versus Quality

In Chapter 5, we design an efficient data trading scheme for data markets with multiple data agents. As more and more data are desired by more and more individuals and companies to perform various data analytics for either research or businesses, it is foreseeable that data trading will become more important in the future and deserves more research efforts. Actually, besides our work in [15], i.e., Chapter 5 of this dissertation, there have been several research efforts on data trading in recent years [107, 112, 115, 117, 118]. Hitherto, most works on data trading are either solely focused on the quantity aspect of the data [15, 115, 117] or merely concentrated on the quality aspect of the data (e.g., privacy concerns of data providers) [107, 112], yet no joint consideration of both quantity and quality of the data exists in the literature.

Therefore, in the future, I am motivated to investigate the quantity and quality aspects of the data jointly in the data trading problems. The goal of this research is twofold. The first goal is to steer the data markets to trade an *appropriate* quantity of data among the data agents subject to their selfishness and privacy constraints. The appropriateness here is measured from the perspective of either social welfare or the profits of certain parties of the data market. The second goal is to incentivize the data providers to offer high quality data by designing an intellegient incentive mechanism. This goal will promote the reliability of data and is particularly crucial for modern data acquisition methods such as crowdsourcing, which, though being efficient and convenient, has poor reliability.

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