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Tribonacci Convolution Triangle

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Tribonacci Convolution Triangle

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Rosa Davila

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Abstract

A lot has been said about the Fibonacci Convolution Triangle, but not much has been said about the Tribonacci Convolution Triangle. There are a few ways to generate the Fibonacci Convolution Triangle. Proven through generating functions, Koshy has discovered the Fibonacci Convolution Triangle in Pascal's Triangle, Pell numbers, and even Tribonacci numbers[KOS14]. The goal of this project is to find inspiration in the Fibonacci Convolution Triangle to prove patterns that we observe in the Tribonacci Convolution Triangle. We start this by bringing in all the information that will be useful in constructing and solving these convolution triangles and find a way to prove them in an easy way[KOS14].

Acknowledgements

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Chapter 1

Introduction

The fun I find in Pascal's Triangle is that there are many ways to construct the array. It is a very popular triangle with many applications that go along with it[KOS18]. The same is not said when you mention either the Fibonacci Convolution Triangle or the Tribonacci Convolution Triangle. The goal of this chapter is to get you to recall the information you might remember from different arrays and transition into the Convolution Triangles.

1.1 Different Arrays

We start out with one of the most popular arrays known to date. There have been many observations of patterns in Pascals Triangle and we will later see how it leads to the construction of the Fibonacci Convolution Triangle[KOS14]. We lastly follow it to both the Fibonacci Convolution Triangle, and the Tribonacci Convolution Triangle.

1.1.1 Pascal's Triangle

Pascal's Triangle is one of the most recognizable arrays because it is made up of numbers that are applicable to many areas of mathematics. This array is most recognizable for its binomial expansion[KOS11]. These numbers are seen in areas of combinatorics, statistics, and in the expansion of binomials. These numbers come from the binomial theorem:

$$(x+y)^n = \sum_{k=1}^{n} \binom{n}{k} x^{n-k} y^k.$$

An array can be constructed for each of the entries. For example, we can create a row of the entries n. The array below shows each of the entries for the first 7 rows of the binomial theorem. Notice, for each row labeled n has $n + 1$ entries since we start with zero[KOS11].

Figure 1.1: Binomial Expansion

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}
\end{array}$$

Recall that $\binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}$. Following this equation, the result of the triangle above is the triangle below.

Figure 1.2: Pascal's Triangle

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}$$

There is also another way to think of the same triangle. Start with ones on the outside, then add the two entries above. Following this pattern results in generating Pascal's Triangle.

Figure 1.3: Pascal's Triangle (constructing third row)

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1{+}1 \quad 1
\end{array}$$

In this case, the next row would have ones on the outside, and the middle number will be $1 + 1$ giving us 2, as shown below.

Figure 1.4: Pascal's Triangle

		1		
	1		1	
1		2		1

The next row will be a result of ones on the sides, and each other entry will be made up of the sum of the two numbers above it.

Figure 1.5: Pascal's Triangle (constructing fourth row)

			1			
		1		1		
	1		2		1	
1		1+2		1+2		1

In this case, the sums will be 1+2 and 2+1 with ones on the outside.

Figure 1.6: Pascal's Triangle

			1			
		1		1		
	1		2		1	
1		3		3		1

The next row is slightly more difficult since there are more entries, but you can see the pattern starting to form at this point. Again, the next row will be a result of the ones on the sides, and each other entry will be made up of the sum of the two numbers above it.

Figure 1.7: Pascal's Triangle (constructing fifth row)

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		1+3		3+3		3+1		1

In this case, the fifth row will be made up of a one on the left side, then the sums 1+3 and 3+3 and 3+1 and a one on the right side. The result of this pattern forms the array on below.

Figure 1.8: Pascal's Triangle

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

If we continue this pattern on for a few more rows, we see that it is the same triangle we previously saw. Below, we see the same triangle constructed through the sums of the two numbers directly above each entry.

Figure 1.9: Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Note that with this rule, the k^{th} term in the n^{th} row is the sum of the two entries above it. Those two terms will be on the row that is one less than the term that we speak of. Those terms will also be the $k-1^{th}$ and the k^{th} term. With this observation, we have stumbled upon Pascal's Identity[KOS11].

Theorem 1.1. *Pascal's Identity*

Let n and k be positive integers where $k \leq n$.

Then $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Since we have constructed Pascal's Triangle by adding the two entries above to create each entry, we have found Pascal's Identity[KOS11]. For the sake of this project, we are going to look at the left justified version of Pascal's Triangle. Notice in the array below, the entries of the first column are all ones, the second column is a column of consecutive numbers, the third column is a column of triangular numbers and so on and so forth[KOS11].

Table 1.1: Pascal's Triangle Left Justified

$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$
1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

1.1.2 Fibonacci Convolution Triangle

There are a few things we need to know about the Fibonacci convolution triangle before we start proving certain results about it. We need to know how to generate the array which involves the convolution of generating functions[KOS14]. We must first define generating functions[KOS18], convolutions[KOS18], and also come up with the generating function for the Fibonacci numbers themselves[KOS14]. We will learn about each of those parts separately in later chapters. For now, we should see that the following table shows the first five columns of the left-justified and offset by 1 row version of the Fibonacci convolution triangle[KOS14].

Table 1.2: Fibonacci Convolution Triangle

$F^{(0)}$	$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	$F^{(4)}$
1				
1	1			
2	2	1		
3	5	3	1	
5	10	9	4	1
8	20	22	14	5
13	38	51	40	20
21	71	111	105	65
34	130	233	256	190
55	235	474	594	511
89	420	942	1324	1295
144	744	1836	2860	3130
233	1308	3522	6020	7285

1.1.3 Tribonacci Convolution Triangle

The same goes with the Tribonacci convolution triangle. Since we are comparing this triangle to the Fibonacci convolution triangle[KOS14], we will have to learn about that triangle first. We need to understand what a generating function is in order to understand the convolution of generating functions, then find the generating function to the Tribonacci sequence of numbers. Lastly, use the convolutions of those generating functions. For now, we should see that the following table shows the first five columns of the left-justified and offset by one row of the Tribonacci convolution triangle[KOS14].

Table 1.3: Tribonacci Convolution Triangle

$T^{(0)}$	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
1	1	1	1	1
1	2	3	4	5
2	5	9	14	20
4	12	25	44	70
7	26	63	125	220
13	56	153	336	646
24	118	359	864	1800
44	244	819	2144	4810
81	499	1830	5174	12430
149	1010	4018	12200	31240
274	2027	8694	28212	76692
504	4040	18582	64168	184530
927	8004	39298	143878	436340

Chapter 2

Sequences

We see in this chapter the importance of sequences. The main point of this project is comparing the Fibonacci Convolution Triangle with the Tribonacci Convolution Triangle. These two triangles are made up of two separate sequences of numbers. It would only make sense that they follow a certain pattern which we will begin to see in this chapter.

2.1 Fibonacci Numbers

Fibonacci numbers have been seen in many different occurrences. One of the most popular of the bunch is the rabbit problem discoverd by Leonardo Fibonacci, which goes as follows[KOS18].

Suppose there are two new born rabbits, one male and the other female. Find the number of rabbits produced in a year if:

1) Each pair takes one month to become mature;

2) Each pair produces a female and male every month, from the second month on; and

3) No rabbits die during the course of the year.

For the sake of the problem, assume that the original pair of rabbits was born on January 1. assume they take a month to mature, so by the time February 1 comes along, there is still only one pair of rabbits. According to the rules, on March 1, the original pair is now two months old, and they birthed a new pair of baby rabbits, which results in a total of two pairs of rabbits. Now, on April 1, we have a total of two mature pairs of rabbits, and one new pair of baby rabbits, since the old babies were not mature enough to birth a new pair, which leaves us with a total of three pair of rabbits. The table below is a visual version of the situation we are talking about[KOS18].

Table 2.1: Rabbit Problem

Pairs of	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Adults	0	1	1	2	3	5	8	13	21	34	55	89
Babies	1	0	1	1	2	3	5	8	13	21	34	55
Total	1	1	2	3	5	8	13	21	34	55	89	144

The numbers that make up the bottom row are called the Fibonacci numbers. The sequence of numbers 1, 1, 2, 3, 5, 8, 13, ... is the Fibonacci Sequence[KOS18] that follow the recursive definition of the n^{th} Fibonacci number,

$F_n = F_{n-1} + F_{n-2}$ for n $\geq$ 3 and F_1=F_2=1.

2.2 Tribonacci Numbers

Sadly there isn't a famous story to help us come up with the Tribonacci numbers. However, the Tribonacci numbers[KOS18] are defined by the recurrence relation

$T_n = T_{n-1} + T_{n-2}\ T_{n-3}$ for n $\geq$ 4 and T_1=$T_2 = 1$ $T_3 = 2$

The first three terms of the sequence are predetermined. The sequence adds the three consecutive terms to obtain the following term. We can see what they are below. We start to see the pattern being formed after the fourth term.

$T_1 = 1$

$T_2 = 1$

$T_3 = 2$

$T_4 = T_3 + T_2 + T_1 = 1 + 1 + 2 = 4$

$T_5 = T_4 + T_3 + T_2 = 1 + 2 + 4 = 7$

$T_6 = T_5 + T_4 + T_3 = 2 + 4 + 7 = 13$

$T_7 = T_6 + T_5 + T_4 = 4 + 7 + 13 = 24$

This gives us the sequence of 1, 1, 2, 4, 7, 13, 24, ... , which are the Tribonacci numbers[KOS18].

2.3 Pell Numbers

Pell numbers have an interesting recurrence that are similar to a few different sequences[KOS11]. For instance, the Pell sequence is very similar to the Pell-Lucas sequence[KOS11]. The only difference is the first two terms. For this project, we will focus on the Pell numbers[KOS11]. The sequence follows the following relation

$P_n = 2{\cdot}P_{n-1} + P_{n-2}$ for $P_1 = 1$, $P_2 = 2$.

In order to start this sequence, we are given the first two terms. The sequence doubles the previous term, then adds the term before that. Below is a more organized version of the relation.

$P_1 = 1$

$P_2 = 2$

$P_3 = 2 \cdot P_2 + P_1 = 2 \cdot 2 + 1 = 5$

$P_4 = 2 \cdot P_3 + P_2 = 2 \cdot 5 + 2 = 12$

$P_5 = 2 \cdot P_4 + P_3 = 2 \cdot 12 + 5 = 29$

$P_6 = 2 \cdot P_5 + P_4 = 2 \cdot 29 + 12 = 70$

$P_7 = 2 \cdot P_6 + P_5 = 2 \cdot 70 + 29 = 169$

$P_8 = 2 \cdot P_7 + P_6 = 2 \cdot 169 + 70 = 408$

The numbers 1, 2, 5, 12, 29, 70, 169, 408, ... make up the first eight terms of the Pell sequence[KOS11].

Chapter 3

Generating Functions

When it gets to the proving portion of this project, we find it easier to solve the problems using generating functions[KOS14]. We will start this chapter out with defining a few terms along with a few examples. This leads to the convolution of generating functions which we will need to define and also prove the convolution triangles of different sequences. Lastly, we will find the generating functions for sequences we need to solve for. Let us first start with an introduction of generating functions[KOS18].

3.1 Introduction to Generating Functions

Generating functions are used for both finite and infinite sequences[KOS18]. Since we are looking at sequences of numbers, we are going to concentrate more on the infinite forms of generating functions.

Definition 3.1. *Let a_0, a_1, a_2, ... be a sequence of real numbers. Then the function*

$g(x) = a_0 + a_1x^2 + a_2x^3 + ...$

is called the generating function for the sequence $\{a_n\}$[KOS18].

One of the most popular examples of generating functions follows the sequence of 1's[KOS18]. Let us start with the generating function

$$
\begin{aligned}
g(x) &= 1 + x + x^2 + x^3 + \cdots \\
&= \sum_{n=0}^{\infty} x^n \\
&= 1 + \sum_{n=1}^{\infty} x^n \\
&= 1 + x \cdot \sum_{n=1}^{\infty} x^{n-1} \\
&= 1 + x \cdot \sum_{n=0}^{\infty} x^n \\
&= 1 + x \cdot g(x).
\end{aligned}
$$

At this point, we have both sides in terms of $g(x)$. Below, we have the steps that will explicitly solve for $g(x)$.

$$
\begin{aligned}
g(x) - x \cdot g(x) &= 1 \\
g(x) \cdot (x - 1) &= 1 \\
g(x) &= \tfrac{1}{1-x}
\end{aligned}
$$

If we were to extend this out, we see that this generating function creates the same as previously stated. We can say that this is the generating function for the sequence of 1's[KOS18]. In this project, we talk about sequences rather than the generating function. Note that the coefficients of the generating function, a_0, a_1, a_2, ..., are in fact the sequences of $\{a_n\}$. This will make for a more simple way to prove the arrays.

3.2 Convolutions

Now understanding what a generating function is, and seeing how it can be applied, we can start applying the convolution to the generating function. After all, this project is on the Tribonacci Convolution Triangle. Since we are constructing a convolution triangle based on a sequence of numbers, we must have a solid understanding of the term convolution and what it brings to the array[KOS14].

Definition 3.2. *Let a(x) and b(x) be the generating functions for the sequences $\{a_n\}$ and $\{b_n\}$, then:*

$$a(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$$

$$b(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + ...$$

By multiplying the two generating functions together, we get:

$$a(x) \cdot b(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + ...$$

If we let c_n be the coefficients of x^n as such, then:

$$c_0 = a_0b_0$$
$$c_1 = a_0b_1 + a_1b_0$$
$$c_2 = a_0b_2 + a_1b_1 + a_2b_0$$
$$\vdots$$
$$c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + ... + a_{n-2}b_2 + a_{n-1}b_1 + a_nb_1$$

Then the product of the sequences $\{a_n\}$ and $\{b_n\}$, $\{c_n\}$ is called the convolution of $\{a_n\}$ and $\{b_n\}$

Note, when taking the convolution of $\{a_n\}$ and $\{b_n\}$, the coefficients of $\{c_n\}$ are the sum of the coefficients of specified products of $\{a_n\}$ and $\{b_n\}$[KOS14]. If $\{a_i\} = \{b_i\} = 1$, then we are looking at the sum of 1's as shown below, which means $\{c_n\} = n$, the natural numbers. This will become more apparent in the following section.

$$\begin{aligned} c_0 &= a_0b_0 \\ c_1 &= a_0b_1 + a_1b_0 \\ c_2 &= a_0b_2 + a_1b_1 + a_2b_0 \\ &\vdots \\ c_n &= a_0b_n + a_1b_{n-1} + a_2b_{n-2} + ... + a_{n-2}b_2 + a_{n-1}b_1 + a_nb_1 \end{aligned}$$

If $\{a_i\} = \{b_i\} = 1$, then we are looking at the sum of 1's as shown below, which means $\{c_n\} = n$, the natural numbers. This will become more apparent in the following section.

$$\begin{aligned} c_0 &= 1 \cdot 1 &&= 1 \\ c_1 &= 1 \cdot 1 + 1 \cdot 1 &&= 2 \\ c_2 &= 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 &&= 3 \\ &\vdots \\ c_n &= 1 \cdot 1 + 1 \cdot 1 + ... + 1 \cdot 1 + 1 \cdot 1 &&= \text{n} \end{aligned}$$

Convolutions have another fascinating characteristic. For generating functions who's first few constants are 0, we start to lose starting points as we take convolutions[KOS14]. For example. if we had the following functions

$$a(x) = a_1x + a_2x^2 + a_3x^3 + ...$$
$$b(x) = b_1x + b_2x^2 + b_3x^3 + ...$$

Then the convolution of the sequences $\{a_n\}$ and $\{b_n\}$, $\{c_n\}$, would be

$$c(x) = a_1b_1x^2 + (a_1b_2 + a_2b_1)x^3 + (a_1b_3 + a_2b_2 + a_3b_1)x^4 + ...$$

Notice that $a_0 = a_1 = 0$ for the terms a_0 and a_1x. Similar things will happen every time we start our generating function at a power of x. If we were to start our generating function at x^2, this would offset our convolutions by two terms. If we were to start our generating functions at x^3, this would offset our convolutions by three terms. Continuing this pattern, if we were to start our generating functions at x^n, this would offset our convolutions by n terms. This will help us out when it comes to proving our patterns later in this project.

3.3 Pascal's Generating Function

Recall Pascal's Triangle as left justified where each column starts one row lower than the previous column[KOS11]. Note, the left-most column is made up of all 1's, the second column is made up of the natural counting numbers, the third column is made up of triangular numbers, and so on and so forth. In this section, we will find the generating function that makes up each column of Pascal's Triangle. Let us start with a general generating functions $\{a_n\}$ and $\{b_n\}$.

$a(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + ...$ and

$b(x) = b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + ...$

If we let a_i=b_i=1, for i=1, 2, 3, 4, 5, ... , we have

$a(x) = x + x^2 + x^3 + x^4 + x^5 + ...$ and

$b(x) = x + x^2 + x^3 + x^4 + x^5 + ...$

It is easy to see $a(x)$ is a great representative of the first column of Pascal's Triangle as left justified, and it's respective generating function is

$$\begin{aligned} a(x) &= x + x^2 + x^3 + x^4 + x^5 + ... \\ &= \tfrac{x}{1-x} \end{aligned}$$

Now, by definition, the convolution of the first column with itself, let's say $c(x)$, is

$$\begin{aligned} a(x) \cdot b(x) &= c(x) \\ &= x^2(a_1b_1) + x^3(a_1b_2 + a_2b_1) + x^4(a_1b_3 + a_2b_2 + a_3b_1) + ... \\ &= x(1) + x^2(1+1) + x^3(1+1+1) + x^4(1+1+1+1) + ... \\ &= x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + ... \end{aligned}$$

which makes up the second column of Pascal's Triangle[KOS18], with generating function

c(x) $= \frac{x^2}{(1-x)^2}$.

The convolution of the first column, $a(x)$, with the second column, $c(x)$, let's say $d(x)$, is

$$\begin{aligned} a(x) \cdot c(x) &= d(x) \\ &= x^2(a_1c_1) + x^3(a_1c_2 + a_2c_1) + x^4(a_1c_3 + a_2c_2 + a_3c_1) + ... \\ &= x(1) + x^2(1+2) + x^3(1+2+3) + x^4(1+2+3+4) + ... \\ &= x + 3x^2 + 6x^3 + 10x^4 + 15x^5 + ... \end{aligned}$$

which makes up the third column of Pascal's Triangle[KOS18] with generating function

$d(x) = \frac{x^3}{(1-x)^3}$.

The convolution for the first column, $a(x)$, with the third column, $d(x)$, let's say $e(x)$, is

$$\begin{aligned} a(x) \cdot d(x) &= e(x) \\ &= x^2(a_1 \cdot d_1) + x^3(a_1 \cdot d_2 + a_2 \cdot d_1) + ... \\ &= x(1) + x^2(1+3) + x^3(1+3+6) + ... \\ &= x + 4x^2 + 10x^3 + 20x^4 + 35x^5 + ... \end{aligned}$$

which makes up the third column of Pascal's Triangle[KOS18] with generating function

$e(x) = \frac{x^4}{(1-x)^4}$.

At this point it is easy to see that the generating functions that make up the entire array is

$g_n(x) = \frac{x^n}{(1-x)^{n+1}}$

where the input of n makes up the n^{th} column.

3.4 Pell Generating Function

One of the other sequences of numbers that we need to discuss in this chapter are the Pell Numbers[KOS11]. Note the sequence of Pell Numbers follows the definition

$P_n = 2 \cdot P_{n-1} + P_{n-2}$ for $n \geq 3$ and $P_1 = 1, P_2 = 2$.

Let us start with a general generating function[KOS18]. Let

$$\begin{aligned} g(x) &= \sum_{n=1}^{\infty} a_n x^n \\ &= a_1 x + a_2 x^2 + \sum_{n=3}^{\infty} a_n x^n \\ &= a_1 x + a_2 x^2 + \sum_{n=3}^{\infty} (2 \cdot a_{n-1} + a_{n-2}) x^n \\ &= a_1 x + a_2 x^2 + \sum_{n=3}^{\infty} 2 \cdot a_{n-1} x^n + \sum_{n=3}^{\infty} a_{n-2} x^n \\ &= a_1 x + a_2 x^2 + 2x \cdot \sum_{n=3}^{\infty} a_{n-1} x^{n-1} + x^2 \cdot \sum_{n=3}^{\infty} a_{n-2} x^{n-2} \\ &= a_1 x + a_2 x^2 + 2x \cdot [g(x) - a_1 x] + x^2 \cdot g(x). \end{aligned}$$

Letting $a_1 = 1$ and $a_2 = 2$, we have

$$\begin{aligned} g(x) - 2x \cdot g(x) - x^2 \cdot g(x) &= a_1 x + a_2 x^2 - 2 \cdot a_1 x^2 \\ g(x) \cdot [1 - 2x - x^2] &= x + 2x^2 - 2x^2 \\ g(x) &= \frac{x}{1-2x-x^2} \end{aligned}$$

which makes up the generating function for the Pell sequence[KOS11].

3.5 Fibonacci Generating Function

As previously stated, generating functions are used a lot in this project because we can easily see them when we start proving the different patterns. In this section, we will find the generating functions that results in the sequence of Fibonacci numbers. We start with recalling the definition of Fibonacci numbers[KOS18].

Definition 3.3. *The sequence of numbers 1, 1, 2, 3, 5, 8, 13, make up the Fibonacci sequence and follow the recursive definition of the n^{th} Fibonacci number,*

$F_n = F_{n-1} + F_{n-2}$ *for* $n \geq 3$ *and* $F_1 = F_2 = 1$

Now, let us start with a general generating function[KOS18]. Let

$$\begin{aligned} g(x) &= \sum_{n=1}^{\infty} a_n x^n \\ &= a_1x + a_2x^2 + \sum_{n=3}^{\infty} a_n x^n \\ &= a_1x + a_2x^2 + \sum_{n=3}^{\infty} (a_{n-1} + a_{n-2}) x^n \\ &= a_1x + a_2x^2 + \sum_{n=3}^{\infty} a_{n-1} x^n + \sum_{n=3}^{\infty} a_{n-2} x^n \\ &= a_1x + a_2x^2 + x \cdot \sum_{n=3}^{\infty} a_{n-1} x^{n-1} + x^2 \cdot \sum_{n=3}^{\infty} a_{n-2} x^{n-2} \\ &= a_1x + a_2x^2 + x \cdot [g(x) - a_1x] + x^2 \cdot g(x) \\ &= a_1x + a_2x^2 + x \cdot g(x) - a_1x^2 + x^2 \cdot g(x) \end{aligned}$$

Letting $a_1 = a_2 = 1$, we have

$$\begin{aligned} g(x) &= x + x^2 + x \cdot g(x) - x^2 + x^2 \cdot g(x) \\ g(x) - x \cdot g(x) - x^2 \cdot g(x) &= x + x^2 - x^2 \\ g(x) \cdot [1 - x - x^2] &= x \\ g(x) &= \frac{x}{1-x-x^2} \end{aligned}$$

which is the generating function for the Fibonacci sequence[KOS18].

3.6 Fibonacci Convolutions

We are starting to see the array form in this section. We bring together the sequence of Fibonacci numbers[KOS18], the generating function of the Fibonacci sequence[KOS18], and convolution of the generating functions[KOS14]. As previously stated, we know that the function that generates the Fibonacci numbers is the function

$a(x) = \frac{x}{1-x-x^2}$.

The convolution of this function with itself, let's say $b(x)$, would be

$a(x) \cdot a(x) = \frac{x}{1-x-x^2} \cdot \frac{x}{1-x-x^2}$

$b(x) = \frac{x^2}{(1-x-x^2)^2}$

The convolution of the Fibonacci sequence with $b(x)$, let's say $c(x)$, would be

$a(x) \cdot b(x) = \frac{x}{1-x-x^2} \cdot \frac{x^2}{(1-x-x^2)^2}$

$c(x) = \frac{x^3}{(1-x-x^2)^3}$

The convolution of the Fibonacci sequence with $c(x)$, let's say $d(x)$, would be

$a(x) \cdot c(x) = \frac{x}{1-x-x^2} \cdot \frac{x^3}{(1-x-x^2)^3}$

$d(x) = \frac{x^4}{(1-x-x^2)^4}$

At this point we can see a pattern starting to form. We can now say that

$g_n(x) = \frac{x^n}{(1-x-x^2)^n}$

is the generating function for the n^{th} convolution of the Fibonacci numbers[KOS14].

3.7 Tribonacci Generating Function

The sequence of Tribonacci numbers were discussed in a previous section. Since this project is finding the similarities in patterns for the different arrays, we need to find the generating function of the Tribonacci numbers[KOS18]. Following the same steps as we have been in this chapter, we can start with the definition of Tribonacci numbers[KOS18]:

$T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$ and $T_1 = T_2 = 1, T_3 = 2$.

We can start with a general generating function. Let

$$\begin{aligned}
g(x) &= \sum_{n=1}^{\infty} a_n x^n \\
&= a_1x + a_2x^2 + a_3x^3 + \sum_{n=4}^{\infty} a_n x^n \\
&= a_1x + a_2x^2 + a_3x^3 + \sum_{n=4}^{\infty} (a_{n-1} + a_{n-2} + a_{n-3})x^n \\
&= a_1x + a_2x^2 + a_3x^3 + \sum_{n=4}^{\infty} a_{n-1}x^n + \sum_{n=4}^{\infty} a_{n-2}x^n + \sum_{n=4}^{\infty} a_{n-3}x^n
\end{aligned}$$

$$
\begin{aligned}
&= a_1x + a_2x^2 + a_3x^3 + x\sum_{n=4}^{\infty} a_{n-1}x^{n-1} \\
&\quad + x^2\sum_{n=4}^{\infty} a_{n-2}x^{n-2} + x^3\sum_{n=4}^{\infty} a_{n-3}x^{n-3} \\
&= a_1x + a_2x^2 + a_3x^3 + x[g(x) - a_1x - a_2x^2] + x^2[g(x) - a_1x] + x^3g(x)
\end{aligned}
$$

Letting $a_1 = a_2 = 1$ and $a_3 = 2$, we have

$$
\begin{aligned}
g(x) &= x + x^2 + x^3 + x[g(x) - x - x^2] \\
&\quad + x^2[g(x) - x] + x^3g(x) \\
g(x)[1 - x - x^2 - x^3] &= x \\
g(x) &= \tfrac{x}{1-x-x^2-x^3}
\end{aligned}
$$

which is the generating function for the Tribonacci sequence[KOS18].

3.8 Tribonacci Convolutions

As previously stated, we know that the function that generates the Tribonacci numbers[KOS18] is the function

$a(x) = \frac{x}{1-x-x^2-x^3}$.

The convolution of this function with itself, let's say $b(x)$, would be

$a(x) \cdot a(x) = \frac{x}{1-x-x^2-x^3} \cdot \frac{x}{1-x-x^2-x^3}$

$b(x) = \frac{x^2}{(1-x-x^2-x^3)^2}$

The convolution of the Tribonacci sequence with $b(x)$, let's say $c(x)$, would be

$a(x) \cdot b(x) = \frac{x}{1-x-x^2-x^3} \cdot \frac{x^2}{(1-x-x^2-x^3)^2}$

$c(x) = \frac{x^3}{(1-x-x^2-x^3)^3}$

The convolution of the Tribonacci sequence with $c(x)$, let's say $d(x)$, would be

$a(x) \cdot c(x) = \frac{x}{1-x-x^2-x^3} \cdot \frac{x^3}{(1-x-x^2-x^3)^3}$

$d(x) = \frac{x^4}{(1-x-x^2-x^3)^4}$

At this point we can see a pattern starting to form. We can now say that

$g_n(x) = \frac{x^n}{(1-x-x^2-x^3)^n}$

is the generating function for the n^{th} convolution of the Tribonacci numbers[KOS14].

Chapter 4

Convolution Triangles

At this point, we can start putting all the pieces together to the puzzle that is the Fibonacci Convolution Triangle, and the Tribonacci Convolution Triangle[KOS14]. We talked about the sequences of both Fibonacci numbers and Tribonacci numbers[KOS18], which lead to the generating functions of the sequences[KOS18], lastly the convolution of each[KOS14]. We can now construct each of the arrays with their respective parts. Let us start with the Fibonacci Convolution Triangle, along with some of the observations others before me have discovered[HB72][KOS14].

4.1 Fibonacci Convolution Triangles

The first, left-most column is the entries of Fibonacci Numbers, let's say $F_n^{(0)}$. The next column results in the convolution of $F_n^{(0)}$ and $F_n^{(0)}$, let's say $F_n^{(1)}$. The third column results in the convolution of $F_n^{(0)}$ and $F_n^{(1)}$, denoted $F_n^{(2)}$. The fourth column results in the convolution of $F_n^{(0)}$ and $F_n^{(2)}$, denoted $F_n^{(3)}$. At this point we start to notice a pattern. Since we start with the column labled $F_n^{(0)}$, each column is labeled one less than the actual column it is in, and is a convolution of the first column and the previous column. Here forward, we say that the m^{th} column, donoted $F_n^{(m-1)}$, is the convolution of $F_n^{(0)}$ and $F_n^{(m-2)}$ thus giving us the array in Table 1[KOS14].
Notice we have the table left justified and offset by one row. As mentioned in Chapter 3, depending on where the first generating function starts, we will have the tables offset by the numerator of the generating function. In this case, we have the generating function[KOS14] $g_n(x) = \frac{x^n}{(1-x-x^2)^n}$.

$F^{(0)}$	$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	$F^{(4)}$
1				
1	1			
2	2	1		
3	5	3	1	
5	10	9	4	1
8	20	22	14	5
13	38	51	40	20
21	71	111	105	65
34	130	233	256	190
55	235	474	594	511
89	420	942	1324	1295
144	744	1836	2860	3130
233	1308	3522	6020	7285

Table 4.1: Fibonacci Convolution Triangle

4.1.1 Sum of Pascal's Triangle Rows

One of the first observations Koshy talks about in his book is that the sum of the rows of Pascal's triangle are the Fibonacci numbers[KOS14]. In order to see this, we must first see the set up of Pascal's triangle. Recall from Chapter 1, we have Pascal's triangle left justified and offset by one.

Table 4.2: Pascal's Triangle Left Justified

$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$
1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

Now, if we were to offset the array by two positions we would have the table below. In order to see the Fibonacci numbers in Pascal's triangle, we must arrange the array in this specific way. Notice that the start of each of the columns begins two rows below the start of the previous column. The right-most column consist of the sums of each of the rows. As discussed in Chapter 3, the Fibonacci sequence is the recurrence sequence which is made up of the sum of the previous two numbers[KOS18]. We have the sequence 1, 1, 2, 3, 5, 8, ... as the first few terms of the sequence[KOS18]. Notice, if we were to continue the sequence, we would have the same numbers as seen in the right-most column.

Table 4.3: Pascal's Triangle Left Justified

$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	Sum
1						1
1						1
1	1					2
1	2					3
1	3	1				5
1	4	3				8
1	5	6	1			13
1	6	10	4			21
1	7	15	10	1		34
1	8	21	20	5		55
1	9	28	35	15	1	89

At this moment, we can see if what we see is true. Recall from Chapter 4, we said that the generating function for Pascal's triangle is $g(x) = \frac{x^{n-1}}{(1-x)^{n-1}}$. Since $g_r(x) = \frac{x^{2r}}{(1-x)^{r+1}}$ is the generating function for Pascal's triangle[KOS18] where each column starts two rows below the start of the previous column. We can say that

$$\begin{aligned}\sum_{r=0}^{\infty} g_r(x) &= \sum_{r=0}^{\infty} \frac{x^{2r}}{(1-x)^{r+1}} \\ &= \tfrac{1}{1-x} \sum_{r=0}^{\infty} (\frac{x^2}{1-x})^r.\end{aligned}$$

Recall that the sum of a geometric series is as follows

$$\sum_{n=0}^{\infty} a_0(r)^n = \tfrac{a_0}{1-r}.$$

Now, applying that same concept to this problem, we can easily see that

$$\sum_{r=0}^{\infty} \tfrac{x^{2r}}{(1-x)^{r+1}} = \tfrac{1}{1-\frac{x^2}{1-x}},$$

which forces the following

$$\begin{aligned}\sum_{r=0}^{\infty} g_r(x) &= \tfrac{1}{1-x} \cdot \tfrac{1}{1-\frac{x^2}{1-x}} \\ &= \tfrac{1}{(1-x)-x^2} \\ &= \tfrac{1}{1-x-x^2}\end{aligned}$$

Which is the generating function for the Fibonacci numbers as discussed earlier[KOS14].

4.1.2 Pascal Meets Fibonacci

One of the other proofs Koshy talks about is the convolution of each row of Pascal's triangle make up the Fibonacci convolution triangle[KOS14]. Let us start with Pascal's triangle left justified. Notice how each of the empty spaces above each column are filled with zeros.

Table 4.4: Pascal's Triangle Left Justified

$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$
1	0	0	0	0	0
1	1	0	0	0	0
1	2	1	0	0	0
1	3	3	1	0	0
1	4	6	4	1	0
1	5	10	10	5	1

Each of these columns are going to be the multipliers to Pascal's triangle offset by two rows[KOS14]. Notice each of the columns start two entries below the start of the previous column. We can see the fixed Pascal's triangle that will be used to multiply each of the entries of the columns.

Table 4.5: Pascal's Triangle Left Justified and Offset by Two Entries

$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$
1					
1					
1	1				
1	2				
1	3	1			
1	4	3			
1	5	6	1		
1	6	10	4		
1	7	15	10	1	
1	8	21	20	5	
1	9	28	35	15	1

Using this fixed array, let us multiply each entry by the first column of Pascals triangle then add the rows to a single entry. This will make up the first column of the Fibonacci convolution triangle[KOS14].

Table 4.6: Pascal's Triangel Offset by Two Entries

1	1	1	1	1	1
1					
1					
1	1				
1	2				
1	3	1			
1	4	3			
1	5	6	1		
1	6	10	4		
1	7	15	10	1	
1	8	21	20	5	
1	9	28	35	15	1

As you see here the first entry in the first column will be multiplied with all the entries in the first column, the second entry in the first column will be multiplied to all the entries in the second column, the third entry in the first column will be multiplied to every entry in the third column, ..., the n^{th} entry in the first column will be multiplied to every entry in the n^{th} column, ... , then all entries in each row will be added together. This will make up the first column of the Fibonacci convolution triangle[KOS14].

Table 4.7: First Column of Fibonacci Convolution Triangle

1		1		1		1		1	sum
1·1									1
1·1									1
1·1	+	1·1							2
1·1	+	1·2							3
1·1	+	1·3	+	1·1					5
1·1	+	1·4	+	1·3					8
1·1	+	1·5	+	1·6	+	1·1			13
1·1	+	1·6	+	1·10	+	1·4			21
1·1	+	1·7	+	1·15	+	1·10	+	1·1	34

Notice the right-most column above. Those are the numbers of the Fibonacci sequence[KOS18]. Which is what we were looking for. This is very similar to the first observation Koshy mentioned in his book[KOS14]. Since we were using a column of all 1's, we were looking at the sum of Pascal's triangle rows. Fascinating, but we are looking for the Fibonacci convolution triangle. Let us see if we are on the right track with the next column. We are going to apply the same steps as the previous table to the next table, but this time we will use the second column of Pascal's triangle. We are going to use the first entry in the second column to multiply with all the entries in the first column, the second entry in the second column will be multiplied to all the entries in the second column, the third entry in the second column will be multiplied to every entry in the third column, ..., the n^{th} entry in the second column will be multiplied to every entry in the n^{th} column, ... , then all entries in each row will be added together. This will make up the second column of the Fibonacci convolution triangle[KOS14].

Table 4.8: Second Column of Fibonacci Convolution Triangle

0		1		2		3		4	sum
0·1									0
0·1									0
0·1	+	1·1							1
0·1	+	1·2							2
0·1	+	1·3	+	2·1					5
0·1	+	1·4	+	2·3					10
0·1	+	1·5	+	2·6	+	3·1			20
0·1	+	1·6	+	2·10	+	3·4			38
0·1	+	1·7	+	2·15	+	3·10	+	4·1	71

Notice the first two entries of the second column are zeros. This tells us we are offsetting the second column of the Fibonacci convolution triangle[KOS14] by two rows, which means that we are going to start the second column two rows below the start of the first column. If this pattern continues, we will have the start of each of the columns starting two rows below the start of the previous column. I also see that these are indeed the same entries of the second row of the Fibonacci convolution triangle[KOS14]. If we continue the pattern we should get the third column of the Fibonacci convolution triangle[KOS14]. We start with using the first entry in the third column to multiply with all the entries in the first column, the second entry in the third column will be multiplied to all the entries in the second column, the third entry in the third column will be multiplied to every entry in the third column, ..., the n^{th} entry in the third column will be multiplied to every entry in the n^{th} column, ... , then all entries in each row will be added together. This will make up the third column of the Fibonacci convolution triangle[KOS14].

Table 4.9: Third Column of Fibonacci Convolution Triangle

0		0		1		3		6	sum
0·1									0
0·1									0
0·1	+	0·1							0
0·1	+	0·2							0
0·1	+	0·3	+	1·1					1
0·1	+	0·4	+	1·3					3
0·1	+	0·5	+	1·6	+	3·1			9
0·1	+	0·6	+	1·10	+	3·4			22
0·1	+	0·7	+	1·15	+	3·10	+	6·1	51

The first four entries of this column are made up of zeros. This gives us more evidence that each of the columns generated in this way are offset by two rows. The other entries are the same entries of the third column of the Fibonacci convolution triangle[KOS14]. Let us try this one last time to truly see the pattern. Let us use the first entry in the fourth column to multiply with all the entries in the first column, the second entry in the fourth column will be multiplied to all the entries in the second column, the third entry in the fourth column will be multiplied to every entry in the third column, ..., the n^{th} entry in the fourth column will be multiplied to every entry in the n^{th} column, ... , then all entries in each row will be added together. This will make up the fourth column of the Fibonacci convolution triangle[KOS14].

Table 4.10: Fourth Column of Fibonacci Convolution Triangle

0		0		0		1		4	sum
0·1									0
0·1									0
0·1	+	0·1							0
0·1	+	0·2							0
0·1	+	0·3	+	0·1					0
0·1	+	0·4	+	0·3					0
0·1	+	0·5	+	0·6	+	1·1			1
0·1	+	0·6	+	0·10	+	1·4			4
0·1	+	0·7	+	0·15	+	1·10	+	4·1	14

At this point, we can gather the information we have to see if what we have is indeed the Fibonacci convolution triangle[KOS14]. The table below is made up of the columns we just created by the multipliers and sums of the rows. This table is the Fibonacci convolution triangle offset by two rows[KOS14].

Table 4.11: Fibonacci Convolution Triangle

$F^{(0)}$	$F^{(1)}$	$F^{(2)}$	$F^{(3)}$
1	0	0	0
1	0	0	0
2	1	0	0
3	2	0	0
5	5	1	0
8	10	3	0
13	20	9	1
21	38	22	4
34	71	51	14

It seems as if we have constructed the Fibonacci convolution triangle[KOS14] in this way, however, we need to prove this to be true. Since $g_r(x) = \frac{x^r}{(1-x)^{r+1}}$ is the generating function for Pascal's triangle where each column starts two rows below the start of the previous column, and we input $\frac{x^2}{1-x}$ we get the following

$$\begin{aligned} g_r(\tfrac{x^2}{1-x}) &= \frac{(\frac{x^2}{1-x})^r}{(1-\frac{x^2}{1-x})^{r+1}} \\ &= \frac{(\frac{x^2}{1-x})^r}{(1-\frac{x^2}{1-x})^{r+1}} \cdot \frac{(1-x)^{r+1}}{(1-x)^{r+1}} \\ &= \frac{(1-x)x^{2r}}{((1-x)-x^2)^{r+1}} \\ &= \frac{(1-x)x^{2r}}{(1-x-x^2)^{r+1}}. \end{aligned}$$

Lastly, by dividing both sides by $(1-x)$, we have

$$\frac{1}{1-x} \cdot g_r(\tfrac{x^2}{1-x}) = \frac{x^{2r}}{(1-x-x^2)^{r+1}}$$

Which is the generating function for the rth column of the Fibonacci convolution triangle[KOS14].

4.1.3 Fibonacci Meets Pell

In his book, Koshy talks about a discovery where the sum of the rows in the Fibonacci convolution triangle make up the Pell sequence[KOS14]. Recall the sequence of Pell numbers[KOS11] as the sequence of 1, 2, 5, 12, 29, 70, ... The table below gives a visual of the situation.

This pattern seems to work with the Fibonacci convolution triangle being left justified and offset by one row[KOS14]. This will be helpful when proving the conjecture. Since the

Table 4.12: Sums of Fibonacci Convolution Triangle Result in Pell Numbers

$F^{(0)}$	$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	$F^{(4)}$	$F^{(5)}$	$F^{(6)}$	$F^{(7)}$	Row Sums
1								1
1	1							2
2	2	1						5
3	5	3	1					12
5	10	9	4	1				29
8	20	22	14	5	1			70
13	38	51	40	20	6	1		169
21	71	111	105	65	27	7	1	408

r^{th} column of the Fibonacci convolution triangle is modeled by the generating function $g_r(x) = \frac{x^r}{(1-x-x^2)^{r+1}}$ then the sum of the rows of the Fibonacci convolution triangle is modeled by[KOS14]

$$\begin{aligned}\sum_{r=0}^{\infty} g_r(x) &= \sum_{r=0}^{\infty} \frac{x^r}{(1-x-x^2)^{r+1}} \\ &= \frac{1}{1-x-x^2} \sum_{r=0}^{\infty} \frac{x^r}{(1-x-x^2)^r} \\ &= \frac{1}{1-x-x^2} \sum_{r=0}^{\infty} (\frac{x}{1-x-x^2})^r.\end{aligned}$$

Recall that the sum of a geometric series is as follows

$$\sum_{n=0}^{\infty} a_0(r)^n = \frac{a_0}{1-r}.$$

Which means we have the following equivalence

$$\sum_{r=0}^{\infty} (\frac{x}{1-x-x^2})^r = \frac{1}{1-\frac{x}{1-x-x^2}}$$

and thus we have the following

$$\begin{aligned}\sum_{r=0}^{\infty} g_r(x) &= \frac{1}{1-x-x^2} \cdot \frac{1}{1-\frac{x}{1-x-x^2}} \\ &= \frac{1}{(1-x-x^2)-x} \\ &= \frac{1}{1-2x-x^2}\end{aligned}$$

which is the generating function of the Pell numbers[KOS14].

4.1.4 Fibonacci meets Tribonacci

Another proven pattern Koshy talks about is the sum of the rows adding to the Tribonacci sequence of numbers[KOS14]. In this case, we must first get the Fibonacci convolution triangle as left justified and offset by three rows as seen in the table below

Table 4.13: Sum of Rows in Fibonacci Convolution Triangle are Tribonacci Numbers

$F^{(0)}$	$F^{(1)}$	$F^{(2)}$	Row Sums
1			1
1			1
2			2
3	1		4
5	2		7
8	5		13
13	10	1	24
21	20	3	44

Notice that the columns are offset by 3 rows. Even though the pattern looks to be this exact pattern, we must prove the conjecture. Since the r^{th} column of the Fibonacci convolution triangle is generated by the function $g_r(x) = \frac{x^r}{(1-x-x^2)^{r+1}}$, then we can easily use the function $g_r(x) = \frac{x^{3r}}{(1-x-x^2)^{r+1}}$ to generate the Fibonacci convolution triangle where each column starts three rows below the the start of the previous column[KOS14]. Now, the sum of each of the rows.

$$\begin{aligned}
\sum_{r=0}^{\infty} g_r(x) &= \sum_{r=0}^{\infty} \frac{x^{3r}}{(1-x-x^2)^{r+1}} \\
&= \frac{1}{1-x-x^2} \sum_{r=0}^{\infty} \frac{x^{3r}}{(1-x-x^2)^r} \\
&= \frac{1}{1-x-x^2} \sum_{r=0}^{\infty} (\frac{x^3}{1-x-x^2})^r \\
&= \frac{1}{1-\frac{x^3}{1-x-x^2}} \\
&= \frac{1}{1-x-x^2} \cdot \frac{1}{1-\frac{x^3}{1-x-x^2}} \\
&= \frac{1}{(1-x-x^2)-x^3} \\
&= \frac{1}{1-x-x^2-x^3}
\end{aligned}$$

which is the generating function of the Tribonacci numbers[KOS14].

4.2 Tribonacci Convolution Triangles

Tribonacci convolution triangle is first set as left justified, and aligned at the top to where the first row of numbers all result in 1. The first, left-most column is the sequence of Tribonacci numbers[KOS18], we say $T_n^{(0)}$. The next column results in the convolution of $T_n^{(0)}$ and $T_n^{(0)}$, let's say $T_n^{(1)}$. The third column results in the convolution of $T_n^{(0)}$ and $T_n^{(1)}$, we say $T_n^{(2)}$. The fourth column results in the convolution of $T_n^{(0)}$ and $T_n^{(2)}$, we say $T_n^{(3)}$. Noticing the same symmetry from the Fibonacci convolution triangle, we see that each column is labeled one less than the actual column it is in, and is a convolution of the first column and its previous column. Here forward, we say that the m^{th} column, denoted $T_n^{(m-1)}$, is the convolution of $T_n^{(0)} \cdot T_n^{(m-2)}$, thus forming the array in the Table below[KOS14].

Table 4.14: Tribonacci Convolution Triangle

$T^{(0)}$	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
1	1	1	1	1
1	2	3	4	5
2	5	9	14	20
4	12	25	44	70
7	26	63	125	220
13	56	153	336	646
24	118	359	864	1800
44	244	819	2144	4810
81	499	1830	5174	12430
149	1010	4018	12200	31240

4.2.1 Tribonacci Meets Fibonacci

In his book, Koshy got the Tribonacci sequence from the Fibonacci convolution triangle fairly easily[KOS14]. He wanted to know if there was a way to get to the Fibonacci sequence out of the Tribonacci convolution triangle. Recall the Pascal Meets Fibonacci proof from section 4.1.2. In that section, we used columns in Pascal's triangle as multipliers to get us the Fibonacci convolution triangle[KOS14]. We can use a similar process to get the Fibonacci convolution triangle from the Tribonacci convolution triangle[KOS14]. The main difference is in the columns we use as multipliers and the array we use. Rather than using all positive multipliers, as we did in the other example, we will alternate from positive to negative multipliers. This will help us get the numbers we wish for the Fibonacci convolution triangle[KOS14]. Below we see the convolution of the first column that makes up Pascal's triangle and the Tribonacci convolution triangle to make up the first row of the Fibonacci convolution triangle[KOS14].

Table 4.15: First Column of Fibonacci Convolution Triangle

1		-1		1	Sums
$1 \cdot 1$					1
$1 \cdot 1$					1
$1 \cdot 2$					2
$1 \cdot 4$	-	$1 \cdot 1$			3
$1 \cdot 7$	-	$1 \cdot 2$			5
$1 \cdot 13$	-	$1 \cdot 5$			8
$1 \cdot 24$	-	$1 \cdot 12$	+	$1 \cdot 1$	13
$1 \cdot 44$	-	$1 \cdot 26$	+	$1 \cdot 3$	21

This makes up the first column in the Fibonacci convolution triangle[KOS14]. Doing the same action using the second column of Pascal's triangle, we have the table below.

Table 4.16: Second Column of Fibonacci Convolution Triangle

0		1		-2	Sums
$0 \cdot 1$					0
$0 \cdot 1$					0
$0 \cdot 2$					0
$0 \cdot 4$	+	$1 \cdot 1$			1
$0 \cdot 7$	+	$1 \cdot 2$			2
$0 \cdot 13$	+	$1 \cdot 5$			5
$0 \cdot 24$	+	$1 \cdot 12$	-	$2 \cdot 1$	10
$0 \cdot 44$	+	$1 \cdot 26$	-	$2 \cdot 3$	20

This makes up the second column of the Fibonacci convolution triangle[KOS14]. Again, we construct the third column in the same way.

Table 4.17: Third Column of Fibonacci Convolution Triangle

0		0		1	Sums
$0 \cdot 1$					0
$0 \cdot 1$					0
$0 \cdot 2$					0
$0 \cdot 4$	+	$0 \cdot 1$			0
$0 \cdot 7$	+	$0 \cdot 2$			0
$0 \cdot 13$	+	$0 \cdot 5$			0
$0 \cdot 24$	+	$0 \cdot 12$	+	$1 \cdot 1$	1

At this point we can say we have enough evidence to gather the columns and call it the Fibonacci convolution triangle[KOS14]. The table below is the columns we constructed in the previous steps and gathered in their respective order.

Table 4.18: Fibonacci Convolution Triangle

$F^{(0)}$	$F^{(1)}$	$F^{(2)}$
1	0	0
1	0	0
2	0	0
3	1	0
5	2	0
8	5	0
13	10	1
21	20	3

Here, we can visually see the pattern forming, but we have to rule out the fact that it was just coincidence. We start with Pascal's generating function[KOS18]. Since $g_r(x) = \frac{x^r}{(1-x)^{r+1}}$ is the generating function for Pascal's triangle where each column starts one row below the start of the previous column. Since we are alternating between positive and negative numbers, we must look at the negative input of x.

$$\begin{aligned} g_r(-x) &= \frac{(-x)^r}{(1-(-x))^{r+1}} \\ &= \frac{(-1)^r x^r}{(1+x)^{r+1}} \end{aligned}$$

Recall that the generating function for the Tribonacci sequence[KOS18] in this problem is $g(x) = \frac{x^3}{(1-x-x^2-x^3)}$. Let us input the Tribonacci sequence into Pascal's triangle generating function[KOS14].

$$\begin{aligned} g_n\left(\frac{x^3}{(1-x-x^2-x^3)}\right) &= \frac{\left(\frac{x^3}{(1-x-x^2-x^3)}\right)^n}{\left(1+\frac{x^3}{(1-x-x^2-x^3)}\right)^{n+1}} \\ &= \frac{\left(\frac{x^3}{1-x-x^2-x^3}\right)^n}{\left(1+\frac{x^3}{(1-x-x^2-x^3)}\right)^{n+1}} \cdot \frac{(1-x-x^2-x^3)^{n+1}}{(1-x-x^2-x^3)^{n+1}} \\ &= \frac{(1-x-x^2-x^3)x^{3n}}{(1-x-x^2-x^3+x^3)^{n+1}} \\ &= \frac{(1-x-x^2-x^3)(-1)^n x^{3n}}{(1-x-x^2)^{n+1}} \end{aligned}$$

By dividing both sides by $(-1)^n(1-x-x^2-x^3)$ we get

$$g_n(x) = \frac{x^{3n}}{(1-x-x^2)^{n+1}}$$

Which represents the generating function where n is the column of the Fibonacci convolution triangle[KOS14].

Chapter 5

Our Findings

One of the most difficult parts after finding a pattern was finding a way to prove what I had found. The book I had been looking at worked with the column generating functions[KOS14]. Then I came across a journal article from Hoggat and Bicknell that found the convolution triangles in row generating functions[HB72]. Hoggat and Bicknell proved their finding in a different way, however, I believe I have made the process more simple. Let us start with a brief explanation of what we need in order to find the terms themselves. We need a way to write the rows so that we can find the individual terms. The way Hoggat and Bicknel[HB72] found it was through the function $g(x) = \frac{1}{(1-x)^n}$. Writing the function in this way will open the opportunity to find the m^{th} term in that row. Later in this chapter, we will prove the relation $R_n = \frac{N_n}{(1-x)^n}$, where N_n is the numerator of the row generating function[HB72]. This table comes from the generating function expansion since the m^{th} term of a generating function comes from

$$\begin{aligned} g(x) &= \frac{1}{(1-x)^n} \\ &= (1 + x + x^2 + \cdots + x^m + \cdots)^n \\ &= (1 + \binom{1+n-1}{1}x + \binom{2+n-1}{2}x^2 + \cdots + \binom{m+n-1}{m}x^m + \cdots) \end{aligned}$$

Therefore, the term itself is the coefficent $C_n^{(m)} = \binom{m+n-1}{m}N_n$

Notice that in this way, we are able to find each individual coefficient. Table 5.1 is a visual of what each of those terms would look like[MER03].

Table 5.1: Term Coefficients for Row Generating Functions

	$C^{(0)}$	$C^{(1)}$	$C^{(2)}$	$\cdots$	$C^{(m)}$
R_1	$1 \cdot N_1$	$\binom{1+1-1}{1} \cdot N_1$	$\binom{2+1-1}{2} \cdot N_1$	$\cdots$	$\binom{m+1-1}{m} \cdot N_1$
R_2	$1 \cdot N_2$	$\binom{1+2-1}{1} \cdot N_2$	$\binom{2+2-1}{2} \cdot N_2$	$\cdots$	$\binom{m+2-1}{m} \cdot N_2$
R_3	$1 \cdot N_3$	$\binom{1+3-1}{1} \cdot N_3$	$\binom{2+3-1}{2} \cdot N_3$	$\cdots$	$\binom{m+3-1}{m} \cdot N_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
R_n	$1 \cdot N_n$	$\binom{1+n-1}{1} \cdot N_n$	$\binom{2+n-1}{2} \cdot N_n$	$\cdots$	$\binom{m+n-1}{m} \cdot N_n$

5.1 Fibonacci Convolution Triangle

The first thing we need to show is that there exists row generating functions rather than column generating functions. This is easily proved through st induction[POL81].

5.1.1 Row Generators

We can start the process by seeing the difference and similarities in rows. We can start with the very first row. Notice it is a row of ones. Since we have proven this relation in a previous chapter[KOS18], we can say $R_1 = \frac{1}{1-x}$. The second row is a row of the natural numbers, which is similar to the second row of Pascal's triangle[KOS18], which would make $R_2 = \frac{1}{(1-x)^2}$. We start to notice a pattern in the third row[HB72]. If we were to offset the third row by one column, and add it to the previous two, we would get the third row as shown below[HB72].

$$\begin{array}{cccccccc} & R_1 & 1 & 1 & 1 & 1 & 1 \\ & R_2 & 1 & 2 & 3 & 4 & 5 \\ + & R_3 & & 2 & 5 & 9 & 14 \\ \hline & R_3 & 2 & 5 & 9 & 14 & 20 \end{array}$$

This pattern worked for the third row, let us see if it works out for the fourth row[HB72].

$$\begin{array}{cccccccc} & R_2 & 1 & 2 & 3 & 4 & 5 \\ & R_3 & 2 & 5 & 9 & 14 & 20 \\ + & R_4 & & 3 & 10 & 22 & 40 \\ \hline & R_4 & 3 & 10 & 22 & 40 & 65 \end{array}$$

Continuing this for the following row would yield

	R_3	2	5	9	14	20	
	R_4	3	10	22	40	65	
+	R_5		5	20	51	105	.
	R_5	5	20	51	105	190	

At this point, we can say that each row, starting with the third row comes from the pattern $R_n = x \cdot R_n + R_{n-1} + R_{n-2}$ which would give us $R_n = \frac{R_{n-1}+R_{n-2}}{1-x}$. Let us find the first few rows[HB72].

$$R_1 = \frac{1}{1-x}$$

$$R_2 = \frac{1}{(1-x)^2}$$

$$\begin{aligned} R_3 &= \frac{R_2+R_1}{1-x} \\ &= \frac{\frac{1}{(1-x)^2}+\frac{1}{1-x}}{1-x} \\ &= \frac{1+(1-x)}{(1-x)(1-x)^2} \\ &= \frac{2-x}{(1-x)^3} \\ R_4 &= \frac{R_3+R_2}{1-x} \\ &= \frac{\frac{2-x}{(1-x)^3}+\frac{1}{(1-x)^2}}{1-x} \\ &= \frac{(2-x)+(1-x)}{(1-x)(1-x)^3} \\ &= \frac{3-2x}{(1-x)^4} \\ R_5 &= \frac{R_4+R_3}{1-x} \\ &= \frac{\frac{3-2x}{(1-x)^4}+\frac{2-x}{(1-x)^3}}{1-x} \\ &= \frac{(3-2x)+(2-x)(1-x)}{(1-x)(1-x)^4} \\ &= \frac{3-2x+2-3x+x^2}{(1-x)^5} \\ &= \frac{5-5x+x^2}{(1-x)^5} \end{aligned}$$

Notice that the degree of the denominator of the n^{th} row is n. If we were to separate the numerator from the denominator, we would have $R_n = \frac{N_n}{(1-x)^n}$ for some numerator N_n. If we were to take the numerator as the sum of the two numerators of the rows, we notice that we have the sum of two quotients with different degrees in their denominator[HB72]. This seems to follow the relation of $R_n = \frac{N_n}{(1-x)^n} = \frac{N_{n-1}+(1-x)N_{n-2}}{(1-x)^n}$ where the numerator of the n^{th} row is $N_n = N_{n-1} + (1-x)N_{n-2}$. We will use this relation to help prove our findings.

5.1.2 Sum of Rows for the m^{th} Term

In this section, we talked about the findings Hoggat and Bicknell talked about in their journal article[HB72]. They originally found this relation, however, I found a way to make it a little easier to prove. Notice the highlighted entries in the table below.

.

Table 5.2: Sum of Rows

	$F^{(0)}$	$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	$F^{(4)}$
R_1	1	1	1	1	1
R_2	**1**	**2**	**3**	**4**	5
R_3	**2**	**5**	**9**	**14**	20
R_4	3	10	22	**40**	65
R_5	5	20	51	105	190
R_6	8	38	111	256	511

For each entry in the m^{th} column, we have the sum of the two rows above it, up the the m^{th} entry of their respective columns resulting in the m^{th} term that we speak of. For example, notice in table 5.2, we have the sum of the two rows above the entry 40 resulting in 40. We can prove this with induction. Recall table 5.1 which shows the m^{th} term in the n^{th} column through the coefficients of the generating functions. Using those coefficients, we can solve this relation by induction. In order to prove this by induction, we must prove the two different shifts. A horizontal shift in terms and a vertical shift in terms. Before we start, let's note the following equivalence:

$$\begin{aligned} C_n^{(m)} &= \binom{m+n-1}{m} N_n \\ &= \binom{m+n-1}{m} \cdot [N_{n-1} + (1-x)N_{n-2}] \\ &= \binom{m+n-1}{m} N_{n-1} + \binom{m-1+n-1}{m} N_{n-2} \\ &= \binom{m+n-1}{m} N_{n-1} + \binom{m+n-2}{m} N_{n-2} \end{aligned}$$

We know by the table that $C_3^{(4)} = 20$. The following is an example of the relation above:

$$\begin{aligned} C_3^{(4)} &= \binom{4+3-1}{4} N_3 \\ &= \binom{4+3-1}{4} \cdot [N_{3-1} + (1-x)N_{3-2}] \\ &= \binom{4+3-1}{4} N_2 + \binom{4-1+3-1}{4} N_1 \\ &= \binom{6}{4} N_2 + \binom{5}{4} N_1 \\ &= 15N_2 + 5N_1 \\ &= 15+5 \\ &= 20 \end{aligned}$$

We want to prove the m^{th} term in the n^{th} row is the sum of the two rows above it. We will first prove the horizontal shift in terms.

Step 1: Prove the first term of the series. In this case the first term we can apply the rule to is $C_3^{(0)}$.

$$
\begin{aligned}
C_3^{(0)} &= \binom{0+3-1}{0} \cdot N_3 \\
&= \binom{2}{0} \cdot N_3 \\
&= 1 \cdot N_3
\end{aligned}
$$

Now that we have the LHS, let us see what the result of the RHS is.

$$
\begin{aligned}
\binom{0+1-1}{0} \cdot N_1 + \binom{0+2-1}{0} \cdot N_2 &= \binom{0+2-1}{0} \cdot N_3 \\
&= 1 \cdot N_3
\end{aligned}
$$

Since we got the same results for the two sides, we have proven the first term to be true.

Now, we have to check the entire rule.

Step 2: Assume $C_n^{(k)}$ is true, prove $C_n^{(k+1)}$

We take the following to be true.

$$
\begin{array}{rcl}
(1 + \binom{1+n-2-1}{1} + \binom{2+n-2-1}{2} + \cdots + \binom{k+n-2-1}{k}) \cdot N_{n-2} & & \\
+ & = & \binom{k+n-1}{k} \cdot N_n \\
(1 + \binom{1+n-1-1}{1} + \binom{2+n-1-1}{2} + \cdots + \binom{k+n-1-1}{k}) \cdot N_{n-1} & &
\end{array}
$$

If we simplify, we get

$$
\begin{array}{rcl}
(1 + \binom{n-2}{1} + \binom{n-1}{2} + \cdots + \binom{k+n-3}{k}) \cdot N_{n-2} & & \\
+ & = & \binom{k+n-1}{k} \cdot N_n \\
(1 + \binom{n-1}{1} + \binom{n}{2} + \cdots + \binom{k+n-2}{k}) \cdot N_{n-1} & &
\end{array}
$$

We must now prove for the term $C_n^{(k+1)}$. The RHS is fairly easy to prove.

$$
\begin{aligned}
C_n^{(k+1)} &= \binom{k+1+n-1}{k+1} \cdot N_n \\
&= \binom{k+n}{k+1} \cdot N_n
\end{aligned}
$$

Now we prove that the sum of the two rows above $C_n^{(k+1)}$ results in $C_n^{(k+1)}$.

$$
\begin{array}{c}
(1 + \binom{1+n-2-1}{1} + \binom{2+n-2-1}{2} + \cdots + \binom{k+n-2-1}{k} + \binom{k+1+n-2-1}{k+1}) \cdot N_{n-2} + \\
(1 + \binom{1+n-1-1}{1} + \binom{2+n-1-1}{2} + \cdots + \binom{k+n-1-1}{k} + \binom{k+1+n-1-1}{k+1}) \cdot N_{n-1}
\end{array}
$$

When we simplify we get

$$
\begin{array}{c}
(1 + \binom{n-2}{1} + \binom{n-1}{2} + \cdots + \binom{k+n-3}{k} + \binom{k+n-2}{k+1}) \cdot N_{n-2} + \\
(1 + \binom{n-1}{1} + \binom{n}{2} + \cdots + \binom{k+n-2}{k} + \binom{k+n-1}{k+1}) \cdot N_{n-1}
\end{array}
$$

By the rule of induction, we substitute and get

$$
\binom{k+n}{k} \cdot N_n + \binom{k+n-1}{k+1} \cdot N_{n-1} + \binom{k+n-2}{k+1} \cdot N_{n-2}
$$

By the rule of the n^{th} term, we have

$\binom{k+n-1}{k} \cdot N_n + \binom{k+n-1}{k+1} \cdot N_n$

Lastly, by Pascal's identity, we have

$\binom{k+n}{k+1} \cdot N_n$

Since $\binom{k+n}{k+1} \cdot N_n = \binom{k+n}{k+1} \cdot N_n$ we can say that the sum of the two rows above the $C_n^{(m)}$ term results in $C_n^{(m)}$ for a horizontal shift in terms.

Now that we proved this relation to be true in the rows, let's see the proof by induction for the shifts in columns. In this case, we will prove the first column to be true. Note that the first column is the Fibonacci sequence. Knowing that every entry is the sum of the two entries above it in the first column, we can shift horizontally to prove this conjecture works for every entry in the Fibonacci convolution triangle.

Step 1: Prove the first term of the series. In this case the first term we can apply the rule to is $C_3^{(0)}$. We know this to be true since we have proved this for the horizontal shifts in the array. Let's move on to step 2.

Step 2: Assume $C_k^{(0)}$ is true, prove $C_{k+1}^{(0)}$

We take the following to be true.

$$\binom{0+k-1}{0} N_n = \binom{0+k-1-1}{0} N_{k-1} + \binom{0+k-2-1}{0} N_{k-2}$$

When we simplify we get

$$\binom{k-1}{0} N_k = \binom{0+k-2}{0} N_{k-1} + \binom{0+k-3}{0} N_{k-2}$$

Now we prove $C_{k+1}^{(0)}$ to be the sum of the two numbers above it. Let's start with the LHS.

$$C_{k+1}^{(0)} = \binom{0+k+1-1}{0} N_{k+1}$$

When we simplify we get

$$\binom{k}{0} N_{k+1}$$

By the relation of the N_n, we get the following

$$\begin{aligned} & \binom{k}{0} \cdot [N_k + (1-x)N_{k-1}] \\ = & \binom{k}{0} N_k + \binom{k-1}{0} N_{k-1} \end{aligned}$$

When we simplify we get

$$\begin{aligned} & 1 \cdot N_k + 1 \cdot N_{k-1} \\ = & N_k + N_{k-1} \end{aligned}$$

Now that we have the result for the LHS, lets see the result of the RHS.

$$\begin{aligned} C_{k+1}^{(0)} &= \binom{0+k-1}{0} N_k + \binom{0+k-1-1}{0} N_{k-1} \\ &= \binom{k-1}{0} N_k + \binom{k-2}{0} N_{k-1} \\ &= 1 \cdot N_k + 1 \cdot N_{k-1} \\ &= N_k + N_{k-1} \end{aligned}$$

Since we proved the horizontal shift to be true, and since the LHS=RHS, we know that we can use this conjecture to generate the entire Fibonacci convolution triangle. Therefore, the sum of the two rows up to the m^{th} entry above $C_n^{(m)}$ is $C_n^{(m)}$.

5.1.3 The Sum of Distinct Term for the m^{th} Term

If you were to offset the columns by two rows, we would have the following table.

Table 5.3: Sum of Terms Offset by Two Rows

$F^{(0)}$	$F^{(1)}$	$F^{(2)}$	$F^{(3)}$
1			
1			
2	1		
3	2		
5	5	1	
8	10	3	
13	20	9	1
21	38	22	4
34	71	**51**	**14**
55	130	111	**40**
89	235	233	**105**

Notice that in this table, we have a few cells highlighted. The sum of 105 = 40+14+51. More importantly, each term is the sum of the two entries above it, and the entry to the left of the highest one. It might be a little hard to see this one, so we have the table below to show us where those numbers came from.

Table 5.4: Sum of Terms

$F^{(0)}$	$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	$F^{(4)}$
1	1	1	1	1
1	2	3	4	5
2	5	9	**14**	20
3	10	22	**40**	65
5	20	**51**	**105**	190
8	38	111	256	511

Here we can see that it generates the first few entries, but does it generate the entire array? We can prove this through induction in two cases. First for the horizontal shift, then for the vertical shift.

We want to prove that $C_n^{(m)} = C_n^{(m-1)} + C_{n-1}^{(m)} + C_{n-2}^{(m)}$. Let us first see the case of the horizontal shift.

Step 1: Prove the first term which, in this case is $C_3^{(1)}$

We can start by seeing the result of the LHS.

$$\begin{aligned} C_3^{(1)} &= \binom{1+3-1}{1} \cdot N_3 \\ &= \binom{3}{1} \cdot N_3 \end{aligned}$$

Now, let us see the result for the RHS.

$$\begin{aligned} C_3^{(0)} + C_2^{(1)} + C_1^{(2)} &= \binom{0+3-1}{0} \cdot N_3 + \binom{1+2-1}{1} \cdot N_2 + \binom{2+1-1}{2} \cdot N_1 \\ &= \binom{2}{0} \cdot N_3 + \binom{1+2-1}{1} \cdot N_3 \\ &= \binom{2}{0} \cdot N_3 + \binom{2}{1} \cdot N_3 \\ &= \binom{3}{1} \cdot N_3 \end{aligned}$$

Since $\binom{3}{1} \cdot N_3 = \binom{3}{1} \cdot N_3$, we can say that the relation works for the first term. Now that we have finished the first step, we can now move on to step two.

Step 2: Assume $C_n^{(k)}$ is true, prove $C_n^{(k+1)}$

We take the following to be true.

$$\begin{aligned} C_n^{(k)} &= C_n^{(k-1)} + C_{n-1}^{(k)} + C_{n-2}^{(k)} \\ \binom{k+n-1}{k} \cdot N_n &= \binom{k-1+n-1}{k-1} \cdot N_n + \binom{k+n-1-1}{k} \cdot N_{n-1} + \binom{k+n-2-1}{k} \cdot N_{n-2} \\ \binom{k+n-1}{k} \cdot N_n &= \binom{k+n-2}{k-1} \cdot N_n + \binom{k+n-2}{k} \cdot N_{n-1} + \binom{k+n-3}{k} \cdot N_{n-2} \end{aligned}$$

Next we prove the relation for the $C_n^{(k+1)}$ term. Let us first start with the LHS.

$$
\begin{aligned}
C_n^{(k+1)} &= \tbinom{k+1+n-1}{k+1} \cdot N_n \\
&= \tbinom{k+n}{k+1} \cdot N_n
\end{aligned}
$$

Now that we have the LHS, the next thing we do is see if we get the same result for the RHS.

$$
\begin{aligned}
& C_n^{(k-1)} + C_{n-1}^{(k)} + C_{n-2}^{(k)} \\
= \; & \tbinom{k+1+n-2}{k} \cdot N_n + \tbinom{k+1+n-2}{k+1} \cdot N_{n-1} + \tbinom{k+1+n-3}{k+1} \cdot N_{n-2}
\end{aligned}
$$

When we simplify we get

$$= \tbinom{k+n-1}{k} \cdot N_n + \tbinom{k+n-1}{k+1} \cdot N_{n-1} + \tbinom{k+n-2}{k+1} \cdot N_{n-2}$$

By substituting the n^{th} numerator we get

$$= \tbinom{k+n-1}{k} \cdot N_n + \tbinom{k+n-1}{k+1} \cdot N_n$$

When we simplify we get

$$= \tbinom{k+n}{k+1} \cdot N_n$$

Since $\binom{k+n}{k+1} \cdot N_n = \binom{k+n}{k+1} \cdot N_n$, we can say the relation holds true for the horizontal shift. If we were to do the proof for the vertical shift, we can see that if we prove this relation for the first column, we can prove it for the entire array. Notice, if we were to take the two terms above the first term in the n^{th} row, we have the exact proof we proved in the previous section. Note, that the entry to the left of any term in the first column is zero. Therefore, we have already proven the relation to be true for the vertical shift. Therefore, each term in the array can be made up of the two numbers above it and the one to the left of it.

5.2 Tribonacci Convolution Triangle

The Tribonacci convolution triangle is very similar to the Fibonacci convolution triangle in that it is made up of column generating functions of Tribonacci numbers[KOS14]. Again, Hoggat and Bicknell proved the following proof, but they did it in such a way that made it more difficult to understand[HB72]. I used the row generating function they found and used induction to prove both their findings, and my own. We can start with the row generating function[HB72].

5.2.1 Row Generators

Picking up parts from the Fibonacci row generating function[HB72], we can see that the first three rows of the Tribonacci convolution triangle are the same as the ones we found in the Fibonacci convolution triangle. We start seeing a difference in the 4^{th} row. Similar to the Fibonacci convolution triangle, if we were to offset the fourth row by one column, and add it to the previous three, we would get the third row[HB72]. Below is a visual of what I am stating.

	R_1	1	1	1	1	1
	R_2	1	2	3	4	5
	R_3	2	5	9	14	20
+	R_4		4	12	25	44
	R_4	4	12	25	44	70

This pattern gives us the fourth row. Let's see if this works for the fifth row.

	R_2	1	2	3	4	5
	R_3	2	5	9	14	20
	R_4	4	12	25	44	70
+	R_5		7	26	63	125
	R_5	7	26	63	125	220

This gives us the fifth row of the Tribonacci convolution triangle[HB72]. Let's give it one more try before we see the pattern.

	R_3	2	5	9	14	20
	R_4	4	12	25	44	70
	R_5	7	26	63	125	220
+	R_6		13	56	153	336
	R_6	13	56	153	336	646

At this point, it is safe to see the pattern[HB72] we have found to being $R_n = x \cdot R_n + R_{n-1} + R_{n-2} + R_{n-3}$ which would give us $R_n = \frac{R_{n-1}+R_{n-2}+R_{n-3}}{(1-x)}$. Similar to the Fibonacci convolution triangle we can say that $R_n = \frac{N_n}{(1-x)^n}$ for some numerator N_n. This will allow us to use the coefficient chart in the beginning of the chapter. It also seems to follow the relation $R_n = \frac{N_n}{(1-x)^n} = \frac{N_{n-1}+(1-x)N_{n-2}+(1-x)^2N_{n-3}}{(1-x)^n}$ where the numerator of the n^{th} row is $N_n = N_{n-1} + (1-x)N_{n-2} + (1-x)^2N_{n-3}$.

5.2.2 Sum of the Rows for the m^{th} Term

Each of the terms in the Tribonacci convolution triangle are the result of the sum of the three rows above it up to the that term[HB72]. The table below is a good representation of the situation.

Table 5.5: Sum of Rows

$T^{(0)}$	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
1	1	1	1	1
1	**2**	**3**	**4**	5
2	**5**	**9**	**14**	20
4	**12**	**25**	**44**	70
7	26	63	**125**	220
13	56	153	336	646

Notice that 125 = (1+2+3+4)+(2+5+9+14)+(4+12+25+44), which is the sum of the three rows above 125. This generates the entire Tribonacci convolution triangle. This can be easily proven by induction as shown below[HB72]. Before we start, let's note the following equivialence:

$$\begin{aligned} C_n^{(m)} &= \binom{m+n-1}{m} N_n \\ &= \binom{m+n-1}{m} \cdot [N_{n-1} + (1-x)N_{n-2} + (1-x)^2 N_{n-3}] \\ &= \binom{m+n-1}{m} N_{n-1} + \binom{m-1+n-1}{m} N_{n-2} + \binom{m-2+n-1}{m} N_{n-3} \\ &= \binom{m+n-1}{m} N_{n-1} + \binom{m+n-2}{m} N_{n-2} + \binom{m+n-3}{m} N_{n-3} \end{aligned}$$

We want to prove the m^{th} term in the n^{th} row is the sum of the three rows above it.

Step 1: Prove the first term of the series. In this case, the first term we can apply will be $C_4^{(0)}$. Let's start with finding the LHS.

$$\begin{aligned} C_4^{(0)} &= \binom{0+4-1}{0} \cdot N_4 \\ &= \binom{3}{0} \cdot N_4 \\ &= 1 \cdot N_4 \end{aligned}$$

Now that we have the result for the LHS, let's see what we get for the RHS.

$$\begin{aligned} C_1^{(0)} + C_2^{(0)} + C_3^{(0)} &= \binom{0+1-1}{0} N_1 + \binom{0+2-1}{0} N_2 + \binom{0+3-1}{0} N_3 \\ &= \binom{0+1-1}{0} N_4 \\ &= 1 \cdot N_4 \end{aligned}$$

Since $N_4 = N_4$, we can say that the conjecture is true for the first term. The next thing we have to do is prove the general case.

Step 2: Assume $C_n^{(k)}$ is true, prove $C_n^{(k+1)}$

We take the following to be true.

$$\binom{k+n-1}{k}N_n = \begin{array}{l}(1+\binom{1+n-3-1}{1}+\binom{2+n-3-1}{2}+\cdots+\binom{k+n-3-1}{k})\cdot N_{n-3}+\\(1+\binom{1+n-2-1}{1}+\binom{2+n-2-1}{2}+\cdots+\binom{k+n-2-1}{k})\cdot N_{n-2}+\\(1+\binom{1+n-1-1}{1}+\binom{2+n-1-1}{2}+\cdots+\binom{k+n-1-1}{k})\cdot N_{n-1}\end{array}$$

By simplifying, we get

$$\binom{k+n-1}{k}N_n = \begin{array}{l}(1+\binom{n-3}{1}+\binom{n-2}{2}+\cdots+\binom{k+n-4}{k})\cdot N_{n-3}+\\(1+\binom{n-2}{1}+\binom{n-1}{2}+\cdots+\binom{k+n-3}{k})\cdot N_{n-2}+\\(1+\binom{n-1}{1}+\binom{n}{2}+\cdots+\binom{k+n-2}{k})\cdot N_{n-1}\end{array}$$

The next thing we do is prove the following term. Let us start with the LHS.

$$\begin{aligned}C_n^{(k+1)} &= \binom{k+1+n-1}{k+1}\cdot N_n\\ &= \binom{k+n}{k+1}\cdot N_n\end{aligned}$$

Now that we have the LHS, let's see what the result of the RHS is.

$$\begin{array}{l}(1+\binom{1+n-3-1}{1}+\binom{2+n-3-1}{2}+\cdots+\binom{k+n-3-1}{k}+\binom{k+1+n-3-1}{k+1})\cdot N_{n-3}+\\(1+\binom{1+n-2-1}{1}+\binom{2+n-2-1}{2}+\cdots+\binom{k+n-2-1}{k}+\binom{k+1+n-2-1}{k+1})\cdot N_{n-2}+\\(1+\binom{1+n-1-1}{1}+\binom{2+n-1-1}{2}+\cdots+\binom{k+n-1-1}{k}+\binom{k+1+n-1-1}{k+1})\cdot N_{n-1}\end{array}$$

After simplifying, we get

$$\begin{array}{l}(1+\binom{n-3}{1}+\binom{n-2}{2}+\cdots+\binom{k+n-4}{k}+\binom{k+n-3}{k+1})\cdot N_{n-3}+\\(1+\binom{n-2}{1}+\binom{n-1}{2}+\cdots+\binom{k+n-3}{k}+\binom{k+n-2}{k+1})\cdot N_{n-2}+\\(1+\binom{n-1}{1}+\binom{n}{2}+\cdots+\binom{k+n-2}{k}+\binom{k+n-1}{k+1})\cdot N_{n-1}\end{array}$$

By the induction hypothesis, we substitute and get

$$\binom{k+n-1}{k}N_n+\binom{k+n-1}{k+1}N_{n-1}+\binom{k+n-2}{k+1}N_{n-2}+\binom{k+n-3}{k+1}N_{n-3}$$

By the definition of the n^{th} numerator, we substitute and get

$$\binom{k+n-1}{k}N_n+\binom{k+n-1}{k+1}N_n$$

Lastly, by Pascal's identity, we have

$$\binom{k+n}{k+1}N_n$$

Since $\binom{k+n}{k+1}N_n=\binom{k+n}{k+1}N_n$, we have proven the sum of three rows above any given term makes up the entry in the case of a horizontal shift.

Now let's prove the relation for a vertical shift.

Step 1: Prove the first term of the series. In this case the first term we can apply the rule to is $C_4^{(0)}$. We know this to be true since we have proved this for the horizontal shifts in array. Let's move on to step 2.

Step 2: Assume $C_k^{(0)}$ is true, prove $C_{k+1}^{(0)}$

We take the following to be true.

$$\begin{aligned}\binom{0+k-1}{0}N_k &= \binom{0+k-1-1}{0}N_{k-1}+\binom{0+k-2-1}{0}N_{k-2}+\binom{0+k-3-1}{0}N_{k-3}\\ \binom{k-1}{0}N_k &= \binom{k-2}{0}N_{k-1}+\binom{k-3}{0}N_{k-2}+\binom{k-4}{0}N_{k-3}\end{aligned}$$

Now we prove $C_{k+1}^{(0)}$ to be the sum of the two numbers above it. Let's start with the LHS.

$$C_{k+1}^{(0)} = \binom{0+k+1-1}{0}N_{k+1}$$

By simplifying, we get

$$\binom{k}{0}N_{k+1}$$

By substituting the n^{th} numerator, we get

$$\binom{k}{0}\cdot[N_k+(1-x)N_{k-1}+(1-x)^2N_{k-2}]$$

By distributing, we get

$$\binom{k}{0}N_k+\binom{k-1}{0}N_{k-1}+\binom{k-2}{0}N_{k-2}$$

By simplifying, we get

$$\begin{aligned}&1\cdot N_k+1\cdot N_{k-1}+1\cdot N_{k-2}\\ =\ &N_k+N_{k-1}+N_{k-2}\end{aligned}$$

Now that we have the result for the LHS, lets see the result of the RHS.

$$C_{k+1}^{(0)} = \binom{0+k-1}{0}N_k+\binom{0+k-1-1}{0}N_{k-1}+\binom{0+k-2-1}{0}N_{k-2}$$

By simplifying, we get

$$\begin{aligned} & \binom{k-1}{0}N_k + \binom{k-2}{0}N_{k-1} + \binom{k-3}{0}N_{k-2} \\ = \quad & 1 \cdot N_k + 1 \cdot N_{k-1} + 1 \cdot N_{k-2} \\ = \quad & N_k + N_{k-1} + N_{k-2} \end{aligned}$$

Since the LHS=RHS, and since we proved the horizontal shift, we know the sum of the three rows up to the m^{th} entry above $C_n^{(m)}$ is $C_n^{(m)}$.

5.2.3 The Sum Distinct Terms for the m^{th} Term

In order to see the actual pattern, we must first see the Tribonacci convolution triangle as left justified, and offset by three rows as in the table below. This allows us to see the distinct terms that make up the m^{th} term in the n^{th} row.

Table 5.6: Sum of Terms Offset by Three Rows

$T^{(0)}$	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
1				
1				
2				
4	1			
7	2			
13	5			
24	12	1		
44	26	3		
81	56	9		
149	118	25	1	
274	244	**63**	**4**	
504	499	153	**14**	
927	1010	359	**44**	1
1705	2027	819	**125**	5

In this case, we see 125 = 44 + 14 + 4 + 63. It is easy to see the pattern forming, but we should see these numbers on the simply left justified Tribonacci convolution triangle as show below.

At this point, we want to prove $C_n^{(m)} = C_n^{(m-1)} + C_{n-1}^{(m)} + C_{n-2}^{(m)} + C_{n-3}^{(m)}$. Let's prove this by form of induction. We want to prove the m^{th} term in the n^{th} row is the sum of the three entries above it, and the one to the left.

Step 1: Prove the first term of the series. In this case, the first term we can apply will

Table 5.7: Sum of Distinct Terms

$T^{(0)}$	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
1	1	1	1	1
1	2	3	**4**	5
2	5	9	**14**	20
4	12	25	**44**	70
7	26	**63**	**125**	220

be $C_4^{(0)}$. Let's start by finding the LHS of the equation.

$$\begin{aligned} C_4^{(0)} &= \binom{0+4-1}{0} N_4 \\ &= \binom{5}{0} N_4 \\ &= 1 \cdot N_4 \end{aligned}$$

Now that we have the LHS, let's see if the result of the RHS is the same. Note that the term to the left of $C_4^{(0)}$=0.

$$\begin{aligned} & 0 + C_3^{(1)} + C_2^{(1)} + C_1^{(1)} \\ =\ & \binom{1+3-1}{1} N_3 + \binom{1+2-1}{1} N_2 + \binom{1+1-1}{1} N_1 \\ =\ & \binom{3}{1} N_3 + \binom{2}{1} N_2 + \binom{1}{1} N_1 \\ =\ & \binom{4}{0} N_4 \\ =\ & 1 \cdot N_4 \\ =\ & N_4 \end{aligned}$$

Since N_4=N_4, then we have proven the first term to follow the conjecture. Next, we have to prove the pattern for the general term.

Step 2: Assume $C_n^{(k)}$ is true, prove $C_n^{(k+1)}$

Take the following to be true.

$$\begin{aligned} C_n^{(k)} &= C_n^{(k-1)} + C_{n-1}^{(k)} + C_{n-2}^{(k)} + C_{n-3}^{(k)} \\ \binom{k+n-1}{k} N_n &= \binom{k+n-2}{k-1} N_n + \binom{k+n-2}{k} N_{n-1} + \binom{k+n-3}{k} N_{n-2} + \binom{k+n-4}{k} N_{n-3} \end{aligned}$$

Using the above statement, we need to prove $C_n^{(k+1)}$. Let's start with the LHS.

$$\begin{aligned} C_n^{(k+1)} &= \binom{k+1+n-1}{k+1} N_n \\ &= \binom{k+n}{k+1} N_n \end{aligned}$$

Now that we have the result for the LHS, let's see if the RHS matches.

$$\begin{aligned} & C_n^{(k)} + C_{n-1}^{(k+1)} + C_{n-2}^{(k+1)} + C_{n-3}^{(k+1)} \\ =\ & \binom{k+n-1}{k} N_n + \binom{k+1+n-2}{k+1} N_{n-1} + \binom{k+1+n-3}{k+1} N_{n-2} + \binom{k+1+n-4}{k+1} N_{n-3} \\ =\ & \binom{k+n-1}{k} N_n + \binom{k+n-1}{k+1} N_{n-1} + \binom{k+n-2}{k+1} N_{n-2} + \binom{k+n-3}{k+1} N_{n-3} \\ =\ & \binom{k+n-1}{k} N_n + \binom{k+n-1}{k+1} N_n \\ =\ & \binom{k+n}{k+1} N_n \end{aligned}$$

Since the LHS=RHS, we have proven the relation to be true for the case of the horizontal shift. If we were to do the proof for the vertical shift, we can see that if we prove this relation for the first column, we can prove it for the entire array. Notice, if we were to take the three terms above the first term in the n^{th} row, we have the exact proof we proved in the previous section. Note, that the entry to the left of any term in the first column is zero. Therefore, we have already proven the relation to be true for the vertical shift. Therefore, each term in the array can be made up of the two numbers above it and the one to the left of it.

Chapter 6

Conclusion

Finding the patterns in Chapter 5 about the Fibonacci and Tribonacci convolution triangles was one of the most thrilling parts of this project. I found them early on and was really excited to prove them. I had been using the column generating functions[KOS14] which helped prove the conjectures for the early observations. I realized that there was a way to find row generating functions[HB72] I could use to prove our findings. While proving our findings I noticed something worth stating.

6.1 Fibonacci Convolution Triangle

As I was proving my findings for the Fibonacci convolution triangle[HB72], I noticed that they looked very similar to the second step of induction of the sum of the rows. Notice that each of the entries is the sum of the previous term in the row and the two new terms of that pattern as shown below.

Table 6.1: Sum of Two Rows

$F^{(0)}$	$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	$F^{(4)}$
1	1	1	1	1
1	2	3	4	5
2	5	9	14	20
3	10	22	40	65
5	**20**	**51**	105	190
8	**38**	**111**	256	511
13	71	**233**	594	1295

Notice that 233 is already the sum of the entries of the two rows above it. When we look for the entries of 594, rather than taking the sum of the entire two rows above it, we simply take the sum of the 233, which is the term next to it, and add it to the other two missing entries of that column. We can see that in the table below.

Table 6.2: Sum of Terms

$F^{(0)}$	$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	$F^{(4)}$
1	1	1	1	1
1	2	3	4	5
2	5	9	14	20
3	10	22	40	65
5	20	51	**105**	190
8	38	111	**256**	511
13	71	**233**	594	1295

6.2 Tribonacci Convolution Triangle

Similarly to the Tribonacci convolution triangle, I noticed a similar pattern in the Tribonacci convolution triangle[HB72]. The sum of the four distinct terms was similar to the second step of induction of the sum of the rows. Noticing that the same relation was found I wanted to see a visual of what was going on. Below, we have the Tribonacci convolution triangle.

Table 6.3: Sum of Three Rows

$T^{(0)}$	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
1	1	1	1	1
1	2	3	4	5
2	5	9	14	20
4	**12**	**25**	44	70
7	**26**	**63**	125	220
13	**56**	**153**	336	646
24	118	**359**	864	1800
44	244	819	2144	4810

Notice that 359 is already the sum of the entries of the two rows above it. Then we look for the entries of 864, rather than taking the sum of the entire rows above it, we can add the entry to the left of it, and the two entries above it. Offsetting the array by 3 rows make it easier to see the sum of the entries. This pattern, as we proved, generates the entire Tribonacci convolution triangle.

Table 6.4: Sum of Terms

$T^{(0)}$	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
1	1	1	1	1
1	2	3	4	5
2	5	9	14	20
4	12	25	**44**	70
7	26	63	**125**	220
13	56	153	**336**	646
24	118	**359**	864	1800

Notice that 359 is already the sum of the entries of the two rows above it. Then we look for the entries of 864, rather than taking the sum of the entire rows above it, we can add the entry to the left of it, and the two entries above it. Offsetting the array by 3 rows make it easier to see the sum of the entries. This pattern, as we proved, generates the entire Tribonacci convolution triangle.

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