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Analogues Between Leibniz's Harmonic Triangle and Pascal's Arithmetic Triangle

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A Thesis

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In Partial Fulfillment

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Master of Arts

in

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by

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Abstract

This paper will discuss the analogues between Leibniz's Harmonic Triangle and Pascal's Arithmetic Triangle by utilizing mathematical proving techniques like partial sums, committees, telescoping, mathematical induction and applying George Pólya's perspective. The topics presented in this paper will show that Pascal's triangle and Leibniz's triangle both have hockey stick type patterns, patterns of sums within shapes, and have the natural numbers, triangular numbers, tetrahedral numbers, and pentatope numbers hidden within. In addition, this paper will show how Pascal's Arithmetic Triangle can be used to construct Leibniz's Harmonic Triangle and show how both triangles relate to combinatorics and arithmetic through the coefficients of the binomial expansion. Furthermore, combinatorics plays an increasingly important role in mathematics, starting when students enter high school and continuing on into their college years. Students become familiar with the traditional arguments based on classical arithmetic and algebra, however, methods of combinatorics can vary greatly. In combinatorics, perhaps the most important concept revolves around the coefficients of the binomial expansion, studying and proving their properties, and conveying a greater insight into mathematics.

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List of Symbols

$\mathbb{N}$ = all natural numbers

$\mathbb{R}$ = all real numbers

$\mathbb{W}$ = all whole numbers

$\mathbb{Z}$ = all integer numbers

$\mathbb{Q}$ = all rational numbers

∞ = infinity

$x \in A = x$ belongs to the set A

$\forall$ = for all

$\exists$ = there exists

$\sum_{i=k}^{n}(\mathrm{a}_i) = \mathrm{a}_k + \mathrm{a}_{k+1} + \ldots + \mathrm{a}_n$

$P(n,\, r) = ({}_nP_r)$ = the number of permutations of n things taken r at a time

$C(n,\ r) = ({}_nC_r) =$ the number of combinations of n things taken r at a time

$L(n,\ r) =$ (Leibniz's numbers) $\frac{1}{(n+1)\binom{n}{r}} = \left[{n \atop r}\right] = \frac{r!(n-r)!}{n!(n+1)}$, where $0 \leq r \leq n$ and $n,\ r \in \mathbb{N}$

$! =$ (factorial) the product of all whole numbers from the number down to 1

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$\binom{-n}{r} =$ (-1)$^r \binom{n+r-1}{r}$

$t_n =$ triangular numbers $= \frac{(n+1)(n+2)}{2}$, where $n \geq 0$

$T_n =$ tetrahedral numbers $= \frac{(n+1)(n+2)(n+3)}{6}$, where $n \geq 0$

Chapter 1

Pascal's Arithmetic Triangle

1.1 Who is Blaise Pascal?

Blaise Pascal, born on June 19, 1623 in Clermont Ferrand, France, was a mathematician, physicist, and religious philosopher, who made several contributions in many of these fields. For example, in 1642 he invented the Pascaline, an early calculator that could add and subtract, which was used by his dad, a tax commissioner. In addition, with his collaboration with the Frenchman Pierre de Fermat and the Dutchman Christiaan Huygens, in 1654 the foundation of probability theory was formed and in 1657 was published as *Les Provinciales.*

Furthermore, Blaise Pascal is widely known for his collection of notes released as the *Pensées* published in 1670 after his death in Paris, France, August 19, 1662. However, in Algebra, Blaise Pascal is most recognized for the arithmetic triangle known as Pascal's Triangle, which is a triangular array of numbers that can relate to the coefficients of any binomial expansion. Even though this triangular array of numbers was well known before Pascal's time, Pascal contributed an elegant proof, plus discovered useful and interesting patterns among the rows, columns, and diagonals, publishing the work *Traité du triangle arithmétique* in 1653. Note that a binomial is a simple algebraic expression that has only two terms operated by positive whole number exponents, addition, subtraction, and multiplication. Moreover, binomial coefficients lie at the heart of combinatorial mathematics, which is why Pascal's Arithmetic Triangle has been studied by many civilizations since ancient times [Kos11, Pol09, JO19].

1.2 Pascal's Arithmetic Triangle

Let's denote $C(n,\, r)$ to represent a term in Pascal's Arithmetic Triangle, where n is the row number starting from the apex at $n \geq 0$ and r is an entry number in a row starting from the left at 0 and moving to the right, ending at n (the columns or diagonals), $0 \leq r \leq n$ and $n,\, r \in \mathbb{Z}$ (See Figure 1.1).

$$\begin{array}{ccccccc}
 & & & C(0,0) & & & \\
 & & C(1,0) & & C(1,1) & & \\
 & C(2,0) & & C(2,1) & & C(2,2) & \\
C(3,0) & & C(3,1) & & C(3,2) & & C(3,3)
\end{array}$$

Figure 1.1: Pascal's Arithemtic Triangle in $C(n,\, r)$ Form

Pascal's triangle can be constructed using the following three simple rules. First every row begins and ends with the number 1 (See Figure 1.2). Second, the triangular arrangement is symmetric along its central column through the apex. This is the column containing the central binomial coefficients, $\{1, 2, 6, 20, 70, \ldots\}$. Third, to generate the remaining terms (the inner terms of the triangle arrangement) take the sum of the two numbers directly above it (the Northwest and Northeast neighbors). For example, $1 + 2 = 3$, $4 + 1 = 5$, $5 + 10 = 15$, $21 + 7 = 28$, and $28 + 56 = 84$. However, the additional terms can be generated another way by taking the difference of the left and Southwest neighbors. For example, $3 - 1 = 2$, $5 - 4 = 1$, $15 - 5 = 10$, and $28 - 21 = 7$ (See Figure 1.3) [Sto83, Kos11].

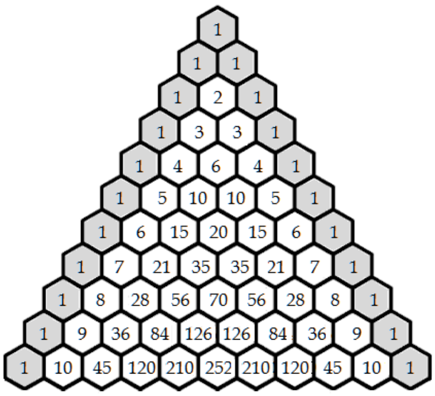

Figure 1.2: Constructing Pascal's Triangle Step 1

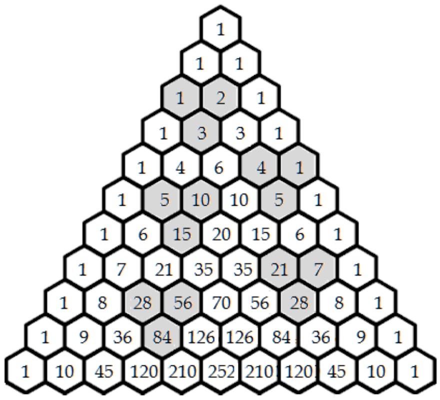

Figure 1.3: Constructing Pascal's Triangle Step 2

Above all, any entry in Pascal's Arithmetic Triangle can be found by using the following formula that has provided a fundamental link between combinatorics and arithmetic, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. For combinatorics, r represents the objects selected without replacement from n distinct objects. Lastly, let's generalize a way to construct Pascal's Arithmetic Triangle. First, the triangular array is symmetric along the vertical axis, then $C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r} = C(n, n - r)$. Next, the boundary conditions are formed using $C(n, 0) = \binom{n}{0} = \binom{n}{n} = C(n, n) = 1$, where $n \geq 0$. Lastly, the additional terms are formed using $C(n-1,\, r-1) + C(n-1,\, r) = C(n,\, r)$, where $n \geq 2$, and $1 \leq r \leq n-1$. Now let's verify that the inner numbers of Pascal's Arithmetic Triangle are formed from the sum of the two numbers immediately above it (the Northeast and Northwest neighbors). Show $C(n,\, r) = C(n-1,\, r-1) + C(n-1,\, r)$ where $n \geq 2$, and $1 \leq r \leq n-1$.

$$\begin{aligned}
C(n-1, r-1) + C(n-1, r) &= \binom{n-1}{r-1} + \binom{n-1}{r} \\
&= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\
&= \frac{r(n-1)! + (n-r)(n-1)!}{r!(n-r)!} \\
&= \frac{n(n-1)!}{r!(n-r)!} \\
&= \frac{n!}{r!(n-r)!} \\
&= C(n, r)
\end{aligned}$$

Thus $C(n,\, r) = C(n-1,\, r-1) + C(n-1,\, r)$. Verifying that the sum of the two numbers directly above (the Northeast and Northwest neighbors) can be used to generate the internal numbers of Pascal's triangular array and producing Equation 1.1, which is just Pascal's Identity [HP87, Kos11, Sto83].

$$C(n, r) = C(n-1, r-1) + C(n-1, r) \tag{1.1}$$

1.3 Some Properties and Patterns of Pascal's Triangle

Without a doubt, Pascal's Arithmetic Triangle has many interesting and useful patterns and properties that can be examined and proven. For instance, look at the diagonals of Pascal's triangular array, the first diagonal starting at the apex consists only of the number 1 and can be found using $\binom{n}{0} = C(n, 0)$ where $n \geq 0$ (See Figure 1.4). Next, look at the second diagonal to the right of the apex these are the natural numbers or known as the counting numbers $\{1, 2, 3, 4, 5, \ldots\}$ (See Figure 1.5), which can be denoted as $\mathbb{N}$. Note that the natural numbers just come from the sums of the 1 numbers, 1, $1 + 1 = 2$, $1 + 1 + 1 = 3$, $1 + 1 + 1 + 1 = 4$, and so on. Then the sum of any sequence of 1 numbers can be found using $\binom{n+1}{1} = C(n+1, 1) = (n+1)$.

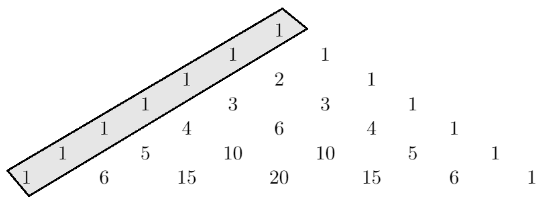

Figure 1.4: The 1 Numbers in Pascal's Triangle

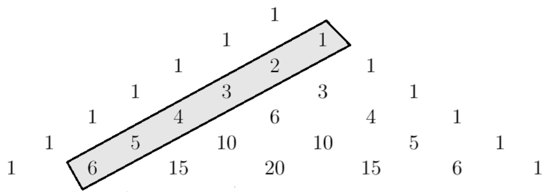

Figure 1.5: The Natural Numbers in Pascal's Triangle

Next let's examine the sum of any sequence of natural numbers by looking at the sums when $n = 0$, 1, 2, and 3 respectively.

$$
\begin{aligned}
1 &= 1 \\
1 + 2 &= 3 \\
1 + 2 + 3 &= 6 \\
1 + 2 + 3 + 4 &= 10
\end{aligned}
$$

Notice that the sum of the natural numbers yields the numbers $\{1, 3, 6, 10, \ldots\}$, which are the triangular numbers. Then the sum of any sequence of natural numbers can be founded using $\frac{1}{2}(n{+}1)(n{+}2)$, where $n \geq 0$ and $n \in \mathbb{Z}$.

Likewise, examine the third diagonal to the right of the apex, these are triangular numbers $\{1, 3, 6, 10, \ldots\}$ (See Figure 1.6), which counts objects arranged in an equilateral triangle and can be denoted as t_n. Note that the number of dots in the equilateral triangle has $\frac{1}{2}(n{+}1)(n{+}2)$ numbers of dots on all sides where n is equal to the row number in Pascal's triangle (See Figure 1.7). Then any triangular number can be found using $\binom{n+2}{2} = C(n{+}2, 2) = \frac{1}{2}(n{+}1)(n{+}2)$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Figure 1.6: The Triangular Numbers in Pascal's Triangle

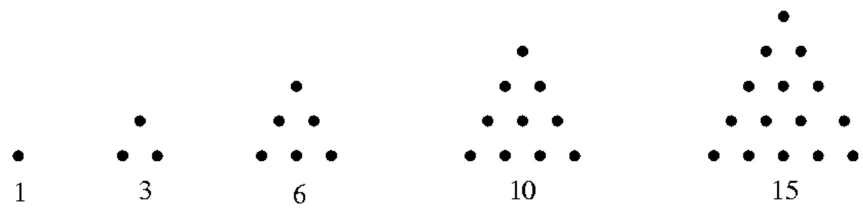

Figure 1.7: Demonstrating the Triangular Numbers

Next let's examine the sum of any sequence of triangular numbers by looking at the sums when n = 0, 1, 2, and 3 respectively.

$$1 = 1$$

$$1 + 3 = 4$$

$$1 + 3 + 6 = 10$$

$$1 + 3 + 6 + 10 = 20$$

Notice that the sum of the triangular numbers yields the numbers $\{1, 4, 10, 20, \ldots\}$, which are the tetrahedral numbers. Then the sum of any sequence of triangular numbers can be found using $\frac{1}{6}(n+1)(n+2)(n+3)$, where $n \geq 0$ and $n \in \mathbb{Z}$.

Next examine the fourth diagonal to the right of the apex, these are the tetrahedral numbers $\{1, 4, 10, 20, 35, \ldots\}$ (See Figure 1.8), which represent a pyramid with a triangular based, modeled by stacking spheres using $\frac{1}{6}(n+1)(n+2)(n+3)$ to represent the total number of spheres found in all the levels, where n is equal to the row number in Pascal's triangle. The tetrahedral numbers can be denoted as T_n (See Figure 1.9). Then any tetrahedral number can be found using $\binom{n+3}{3} = C(n+3, 3) = \frac{1}{6}(n+1)(n+2)(kn+3)$.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Figure 1.8: The Tetrahedral Numbers in Pascal's Triangle

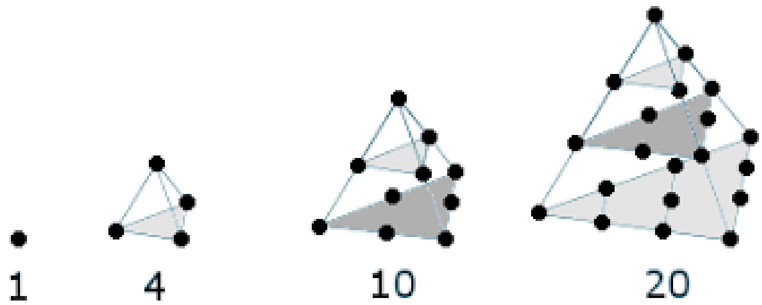

Figure 1.9: Demonstrating the Tetrahedral Numbers

Next let's examine the sum of any sequence of tetrahedral numbers by looking at the sums when $n = 0$, 1, 2, and 3 respectively.

$$1 = 1$$

$$1 + 4 = 5$$

$$1 + 4 + 10 = 15$$

$$1 + 4 + 10 + 20 = 35$$

Notice that the sum of the tetrahedral numbers yields the numbers {1,5, 15, 35, ...}, which are known as the pentatope numbers. Then the sum of any sequence of tetrahedral numbers can be found using $\frac{1}{24}(n+1)(n+2)(n+3)(n+4)$, where $n \geq 0$ and $n \in \mathbb{Z}$.

Lastly, examine the fifth diagonal to the right of the apex. These numbers

$\{1, 5, 15, 35, \ldots\}$ (See Figure 1.10), are defined as a class of figurative numbers that exist in 4D space and describe the number of vertices in a 3D configuration of tetrahedrons joined at the faces. These numbers were given a name by Pascal called the pentatope numbers. Then any pentatope number can be found using $\binom{n+4}{4} = C(n+4, 4) = \frac{1}{24}(n+1)(n+2)(n+3)(n+4)$.

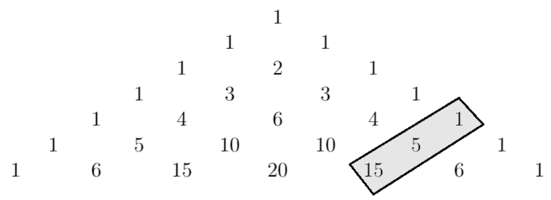

Figure 1.10: The Pentatope Numbers in Pascal's Triangle

Next let's examine the sum of any sequence of pentatope numbers by looking sums when $n = 0$, 1, 2, and 3 respectively.

$$1 = 1$$

$$1 + 5 = 6$$

$$1 + 5 + 15 = 21$$

$$1 + 5 + 15 + 21 = 42$$

Notice that the sum of pentatope numbers yields the numbers $\{1, 6, 21, 42, \ldots\}$, which are the numbers found in the sixth diagonal from the right of the apex. Then the sum of any sequence of pentatope numbers can be found using $\frac{1}{120}(n+1)(n+2)(n+3)(n+4)(n+5)$, where $n \geq 0$ and $n \in \mathbb{Z}$ [AL91, Kos11, Pol09, Tuc12].

Chapter 2

Leibniz's Harmonic Triangle

2.1 Who is Gottfried Leibniz?

Gottfried Leibniz, born July 1, 1646 in Leipzig, Germany was a philosopher, physicist, mathematician, statesman, and great polymath who knew almost everything that could be known during his time, about any subject or intellectual enterprise, remaining one of the greatest, most influential thinkers and logicians in history. In addition, he made many contributions to philosophy, engineering, physics, law, politics, and theology. One of his greatest accomplishments was the discovery of a new mathematical method of differential and integral calculus, which is a branch in mathematics that focuses on differentiation, integration, and limits of functions. He published the work in 1684. Simultaneously and independently of Leibniz, Sir Isaac Newton worked on the same mathematical method, but published his work three years later. In addition, Leibniz discovered the binary number system and invented the first calculating machine that could add, subtract, multiple, and divide.

Furthermore, from his study on harmonic series, and from taking summations of infinite series, in 1673, Leibniz's created a twist to Pascal's Arithmetic Triangle called Leibniz's Harmonic Triangle, that enabled him to work with infinite series and calculate area. Even though little is known about Leibniz's Harmonic Triangle, the terms in this triangular array of numbers that is formed from unit fractions can be denoted by $L(n,\ r)$ where n represents the row number starting from the apex where $n \geq 0$ and r represents the entry number in a row where $0 \leq r \leq n$ and $n,\ r \in \mathbb{N}$. Leibniz's Harmonic Triangle shares many interesting and similar properties with Pascal's Arithmetic Triangle and has been used heavily throughout Leibniz's many studies and different texts [BL18, Kos11].

2.2 The Harmonic Triangle

Let's denote $L(n,\ r)$ to represent a term in Leibniz's Harmonic Triangle, where n is the row number starting from the apex at $n \geq 0$ and r is an entry number in a row starting from the left at 0 and moving to the right, ending at n, where $0 \leq r \leq n$ and n, $r \in \mathbb{Z}$ (See Figure 2.1).

$$
\begin{array}{ccccccccc}
 & & & & L(0,0) & & & & \\
 & & & L(1,0) & & L(1,1) & & & \\
 & & L(2,0) & & L(2,1) & & L(2,2) & & \\
 & L(3,0) & & L(3,1) & & L(3,2) & & L(3,3) & \\
L(4,0) & & L(4,1) & & L(4,2) & & L(4,3) & & L(4,4)
\end{array}
$$

Figure 2.1: Leibniz's Harmonic Triangle in $L(n,\ r)$ Form

Leibniz's Harmonic triangle can be constructed using the following three simple rules. First, the n^{th} row begins with $L(n,\ 0) = \frac{1}{(n+1)\binom{n}{0}} = \frac{1}{n+1}$ and ends with $L(n,\ n) = \frac{1}{(n+1)\binom{n}{n}} = \frac{1}{n+1}$, where $n \geq 0$ (See Figure 2.2). Note that the two diagonals forming the boundary conditions consist of the harmonic sequence numbers $\left(\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots\right)$. Second, the triangular array is symmetric about the vertical axis through the apex, since $L(n,\ r) = \frac{1}{(n+1)\binom{n}{r}} = \frac{r!(n-r)!}{(n+1)n!} = \frac{1}{(n+1)\binom{n}{n-r}} = L(n,\ n-r)$. Third, the additional terms (the internal terms) can be generated by taking the difference of the Northwest and left neighbors. For example, $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, $\frac{1}{20} - \frac{1}{30} = \frac{1}{60}$, and $\frac{1}{105} - \frac{1}{280} = \frac{1}{168}$. Thus to construct Leibniz's Harmonic Triangle algebraically, use the formula $L(n,\ 0) = L(n,\ n)$, to find the boundary conditions and to find the additional terms use the equation $L(n,\ r) = L(n-1,\ r-1) - L(n,\ r-1)$ where $n \geq 1$, $1 \leq r \leq n-1$, and $n,\ r \in \mathbb{Z}$ (See Figure 2.3) [BJ81, Kos11, Sto83].

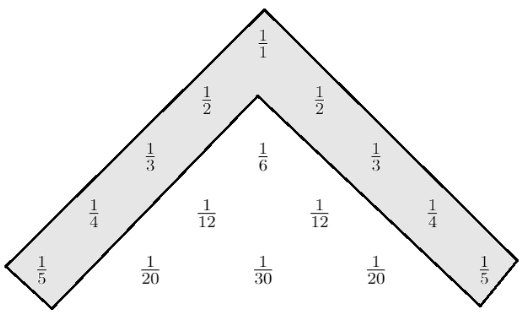

Figure 2.2: Constructing Leibniz's Triangle Step 1

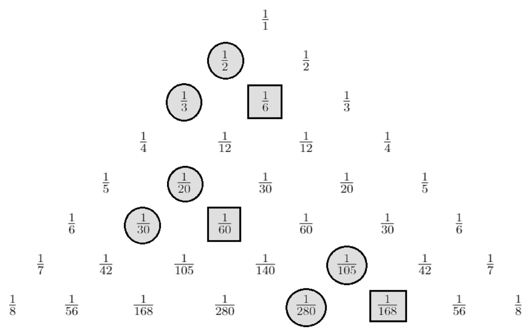

Figure 2.3: Constructing Leibniz's Triangle Step 2

Next let's verify that the remaining terms (the internal terms) of Leibniz's Harmonic Triangle can be formed from the difference of the number to the Northwest and the number to the left. That is, show $L(n,\ r) = L(n-1,\ r-1) - L(n,\ r-1)$.

$$\begin{aligned}
L(n-1,r-1)-L(n,r-1) &= \frac{1}{(n-1+1)\binom{n-1}{r-1}} - \frac{1}{(n+1)\binom{n}{r-1}} \\
&= \frac{(r-1)!(n-1-r+1)!}{n(n-1)!} - \frac{(r-1)!(n-r+1)!}{(n+1)n!} \\
&= \frac{(n+1)(r-1)!(n-r)! - (r-1)!(n-r+1)(n-r)!}{(n+1)n!} \\
&= \frac{(r-1)!(n-r)!(n+1-n+r-1)}{(n+1)n!} \\
&= \frac{r(r-1)!(n-r)!}{(n+1)n!} \\
&= \frac{r!(n-r)!}{(n+1)n!} \\
&= \frac{1}{(n+1)\binom{n}{r}} \\
&= L(n,r)
\end{aligned}$$

Thus $L(n,\, r) = L(n - 1,\, r - 1) - L(n,\, r - 1)$.

Then Equation 2.1 can be used to generate the additional terms of Leibniz's Harmonic Triangle, which is just Leibniz's Identity.

$$L(n,r) = L(n-1, r-1) - L(n, r-1) \tag{2.1}$$

2.3 Some Properties and Patterns of Leibniz's Triangle

Similar to Pascal's Arithmetic Triangle, patterns and properties can be discovered about Leibniz's Harmonic Triangle. First let's examine the diagonals. The first diagonal, the diagonal along the boundary of the triangular array starting at the apex, yields the sequence $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$. These numbers are the reciprocals of the natural numbers $\frac{1}{\binom{n+1}{1}}$, which forms the harmonic sequence. Note the harmonic series, $\sum_{n=0}^{\infty} \frac{1}{n+1}$ is divergent. Then the numbers in the first diagonal of Leibniz's Harmonic Triangle can be found using $L(n, 0) = \frac{1}{(n+1)\binom{n}{0}} = \frac{1}{n+1}$, where $n \geq 0$ (See Figure 2.4).

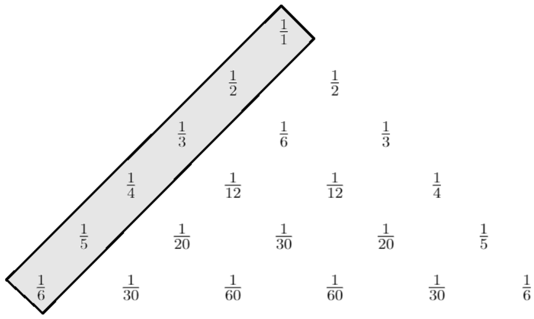

Figure 2.4: The 1^{st} Diagonal in Leibniz's Triangle

Next look at the second diagonal to the right of the apex, which yields the sequence $\{\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \ldots\}$. These numbers are twice the reciprocal of the triangular numbers, which can be denoted as $\frac{1}{2t_n}$. In addition, Christiaan Huygens proposed this following problem to Leibniz in Paris, France, 1665. Find the sum (S) of the reciprocals of the triangular numbers, so find $S = \sum_{n=0}^{\infty} \left(\frac{1}{t_n}\right)$. At first Leibniz recognized that each term of this series was actually twice the difference of the successive terms of the harmonic sequence. He found that $\frac{1}{1} - \frac{1}{2} = \frac{1}{2}$, $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, and $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$. Thus Leibniz constructed the following proof.

Proof. Find $S = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots$

Use partial sums and let $S_n = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots + \frac{1}{2t_n}$.

Next use the telescoping method, $S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n+1} - \frac{1}{(n+1)(n+2)}\right)$.

Then eliminate the additive inverses to get $S_n = \frac{1}{1}$ - $\frac{1}{(n+1)(n+2)}$.

Now take the limit as n approaches ∞ of S_n.

$S = \lim\limits_{n\to\infty} 2S_n = \lim\limits_{n\to\infty} 2\left(\frac{1}{1} - \frac{1}{(n+1)(n+2)}\right) = 2$. Thus $S = 2$. $\square$

Then Leibniz was able to find the sum of the reciprocals of the triangular number, which was 2. This also concluded that the numbers in the second diagonal of Leibniz's Harmonic Triangle are one-half the reciprocal of the triangular numbers and can be found using $\frac{1}{2\binom{n+2}{2}} = L(n + 1, 1) = \frac{1}{(n+2)\binom{n+1}{1}} = \frac{1}{(n+1)(n+2)}$, where $n \geq 0$ (See Figure 2.5).

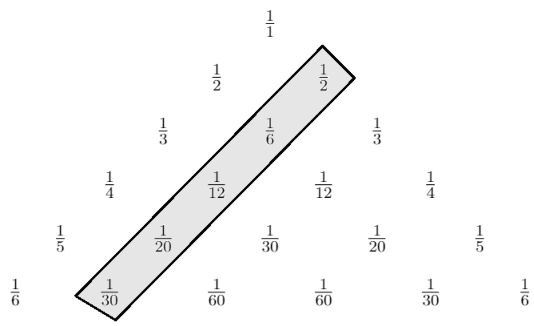

Figure 2.5: The 2^{nd} Diagonal in Leibniz's Triangle

Next look at the third diagonal from the right of the apex, which yields the sequence $\{\frac{1}{3}, \frac{1}{12}, \frac{1}{30}, \frac{1}{60}, \ldots\}$. These numbers are one-third the reciprocals of the tetrahedral numbers $\{1, 4, 10, 20, \ldots\}$, which can be denoted as $\frac{1}{3T_n}$. Notice that each term is the difference of one-half the reciprocal of two successive triangular numbers.

For example, $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$, $\frac{1}{6} - \frac{1}{12} = \frac{1}{12}$, and $\frac{1}{12} - \frac{1}{20} = \frac{1}{30}$. Then the numbers in third diagonal of Leibniz's Harmonic Triangle can be denoted as $\frac{1}{3\binom{n+3}{3}} = L(n + 2, 2) = \frac{1}{(n+3)\binom{n+2}{2}}$ $= \frac{2}{(n+1)(n+2)(n+3)}$, where $n \geq 0$ (See Figure 2.6).

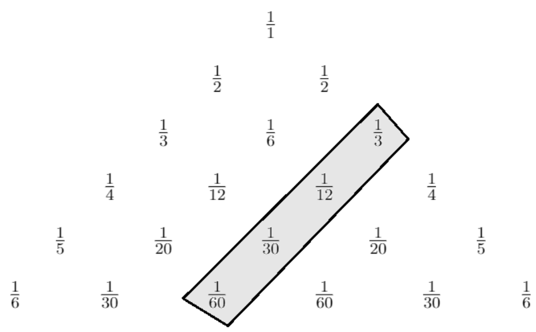

Figure 2.6: The 3^{rd} Diagonal in Leibniz's Triangle

Next look at the fourth diagonal from the right of the apex, which yields the sequence $\{\frac{1}{4}, \frac{1}{20}, \frac{1}{60}, \frac{1}{140}, \ldots\}$. These numbers are one-fourth the reciprocals of the pentatope numbers $\{1, 5, 15, 35, \ldots\}$. Notice that each term is the difference of one-third the reciprocal of two successive tetrahedral numbers.

For example, $\frac{1}{3} - \frac{1}{12} = \frac{1}{4}$, $\frac{1}{12} - \frac{1}{30} = \frac{1}{20}$, and $\frac{1}{30} - \frac{1}{60} = \frac{1}{60}$. Then the numbers in the fourth diagonal can be denoted as $\frac{1}{4\binom{n+4}{4}} = L(n+3, 3) = \frac{1}{(n+4)\binom{n+3}{3}} = \frac{6}{(n+1)(n+2)(n+3)(n+4)}$, where $n \geq 0$ (See Figure 2.7) [Kos11, Sto83, BJ81].

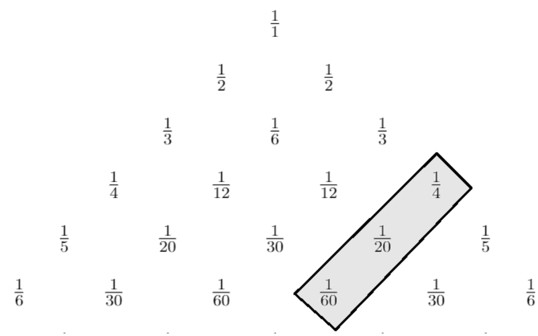

Figure 2.7: The 4^{th} Diagonal in Leibniz's Triangle

Chapter 3

Discoveries Between The Two Number Triangles

3.1 Pólya's Problems

As a result from Section 1.4 and Section 2.4, similarities and patterns can be discovered and proven between Leibniz's Harmonic Triangle and Pascal's Arithmetic Triangle. To illustrate, let's examine problems from George Pólya's *Mathematical Induction* [Pol09].

3.1.1 Who is George Pólya?

George Pólya, originally named Pólya Györg, was a Hungarian mathematician born in Budapest on December 13, 1887. He became a professor of mathematics in 1914 and later immigrated to America in 1940. Soon after arriving in America he became a mathematics professor at Brown University then at Stanford University two years later. He retired in 1953, yet continued to write and teach. Passing down his knowledge until his death in 1985, at the age of 97.

In addition, Pólya wrote *How to Solve It*, perhaps the most famous book in mathematics ever written. Interestingly, the book was not particularly a mathematics book, yet it was about how to solve problems of any kind. Pólya implied that the same method and techniques discussed in the text could fundamentally be used to solve any problems encountered in life. The method was developed for solving problems aimed towards mathematics students, to help teach them how to become better problem solvers. George Pólya's method is displayed below:

Step 1: Understanding the Problem

This may be obvious but one is not a good problem solver unless they can answer the following questions that are taken from another one of Pólya's great pieces of work, *Mathematical Discovery Volume One.*

- Do you understand all the words used in the problem?
- Can you restate the problem in your own words?
- Can you think of a picture or a diagram that might help you understand the problem?
- Is there enough information to enable you to find a solution?

Step 2: Devise a Plan

Step 3: Carry out your Plan

Step 4: Look back and Generalize

If steps two and three fail try to think of a smaller simpler problem that contains the original problem, rethink your plan, and carry out the new plan. Once the solution is found, Pólya believes a good problem solver should attempt the same problem in as many ways as possible to prove, prove, and prove again.

With this method in mind, let's look at Problem 3.1, which is example 3.52 in George Pólya's *Mathematical Discovery*, to find any associations between Pascal's Arithmetic Triangle and Leibniz's Harmonic Triangle [Pol09].

Problem 3.1 (Example 3.52 [Pol09]). Try to recognize a connection between corresponding numbers of the two triangles and, having recognized it, prove it.

Solution. First let's examine both Pascal's Arithmetic Triangle (See Figure 3.1) and Leibniz's Harmonic Triangle (See Figure 3.2). Plus let's use the information that was found from Section 1.2, Section 1.3, Section 2.2, and Section 2.3.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Figure 3.1: Pascal's Arithmetic Triangle

$$\begin{array}{ccccccccccc}
 & & & & & \frac{1}{1} & & & & & \\
 & & & & \frac{1}{2} & & \frac{1}{2} & & & & \\
 & & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & & \\
 & & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & & \\
 & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & \\
\frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6}
\end{array}$$

Figure 3.2: Leibniz's Harmonic Triangle

After analyzing both Pascal's Arithmetic Triangle and Leibniz's Harmonic Triangle some of the similarities discovered are that both triangles are symmetric about the vertical axis through the apex, both contain the natural numbers, triangular numbers, tetrahedral numbers, pentatope numbers, etc. hidden within, and both relate to binomial coefficients using combinatorics ideas and properties.

Furthermore, in Pascal's Arithmetic Triangle every term can be found using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $n,\ r \in \mathbb{Z}$ and $0 \leq r \leq n$ and Leibniz's Harmonic Triangle can be constructed using Pascal's Arithmetic Triangle. First take the reciprocal of each term in Pascal's triangle to get $\frac{1}{\binom{n}{r}}$. Next multiple the denominator by $(n + 1)$, to get $\frac{1}{(n+1)\binom{n}{r}}$. Then $\frac{1}{(n+1)\binom{n}{r}}$ is the formula that can be used to find every term in Leibniz's Harmonic Triangle. Thus $L(n,\ r) = \frac{1}{(n+1)\binom{n}{r}}$ can denote a term in the Leibniz's Harmonic Triangle where $n \geq 0$ and $0 \leq r \leq n$. Now let's verify this.

Notice that $L(n,\ 0) = \frac{1}{(n+1)\binom{n}{0}} = \frac{1}{n+1}$ and that $L(n,\ n) = \frac{1}{(n+1)\binom{n}{n}} = \frac{1}{n+1}$. Thus $L(n,\ n) = L(n,\ 0)$. Then for $r = 0$, the boundary condition of Leibniz's Harmonic Triangle holds for all $n \geq 0$ and n,$r \in \mathbb{Z}$. Next the internal numbers of Leibniz's Harmonic Triangle can be found using $L(n,\ r) = L(n - 1,\ r - 1) - L(n,\ r - 1)$, which is Equation 2.1 that had been proven previously. However, the additional terms of Leibniz's Harmonic Triangle can be generated another way. Similar to how the additional terms of Pascal's Arithmetic Triangle are generated by taking the sum of the two terms directly above it, the Northwest and Northeast neighbors (See figure 3.3). Instead take the sum of the two terms directly below, the Southwest and Southeast neighbors to generate the additional terms of Leibniz's Harmonic Triangle (See Figure 3.4). To create the equation, first use the recursion formula, then the explicit form of the binomial coefficients to get $L(n,\ r - 1) + L(n,\ r) = L(n - 1,\ r - 1)$. Now let's verify this.

$$\begin{aligned}
L(n, r-1) + L(n, r) &= \frac{1}{(n+1)\binom{n}{r-1}} + \frac{1}{(n+1)\binom{n}{r}} \\
&= \frac{(r-1)!(n-r+1)!}{(n+1)n!} + \frac{r!(n-r)!}{(n+1)n!} \\
&= \frac{(r-1)!(n-r)!(n+1)}{(n+1)n!} \\
&= \frac{(r-1)!(n-r)!}{n(n-1)!} \\
&= \frac{1}{n\binom{n-1}{r-1}} \\
&= L(n-1, r-1)
\end{aligned}$$

Then $L(n,\ r-1) + L(n,\ r) = L(n$-1, r-1) can be used to generate the additional terms of Leibniz's Harmonic Triangle and Equation 3.1 is formed.

$$L(n, r-1) + L(n, r) = L(n-1, r-1) \tag{3.1}$$

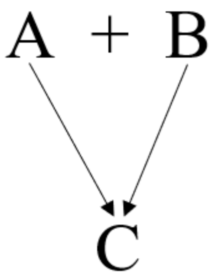

Figure 3.3: Generating Terms in Pascal's Triangle

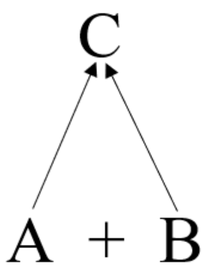

Figure 3.4: Generating Terms in Leibniz's Triangle

Thus Leibniz's Harmonic Triangle can be constructed using Pascal's Arithmetic Triangle, showing a relationship between these two triangles [Sto83, Kos11].

3.1.2 The Hockey Stick Pattern

First, recall Equation 1.1 that was discussed and proven previously, it will be helpful in Problem 3.3. Next, restate Equation 1.1 as a theorem and prove it again using committee forming.

Theorem 3.2. *(Pascal's Identity)* $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ *for any positive integer* r *and* n.

Proof. By committee forming

Consider choosing a committee of r members from n people. One way to do this is to just $\binom{n}{r}$. Another way, is to first decide whether or not to choose a specific person to join the committee or not. If a person is chosen to be on the committee then there are still $r - 1$ members left to select from now $n - 1$ people, which can be done $\binom{n-1}{r-1}$ ways. If that person is not chosen then there are still r members to be selected from now $n - 1$ people, which can be done $\binom{n-1}{r}$ ways. Then the total choices are $\binom{n-1}{r-1} + \binom{n-1}{r}$.
Thus $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ □

Now, let's look at Problem 3.3, which is example 3.34 in George Pólya *Mathematical Discovery* [Pol09], to find a specific pattern within Pascal's Arithmetic Triangle

Problem 3.3 (Example 3.34 [Pol09])**.** Consider the sum of the first six numbers along the third avenue of the Pascal's triangle, $1 + 4 + 10 + 20 + 35 + 56 = 126$. Locate this sum in the Pascal triangle, try to observe analogous facts, generalize, and prove.

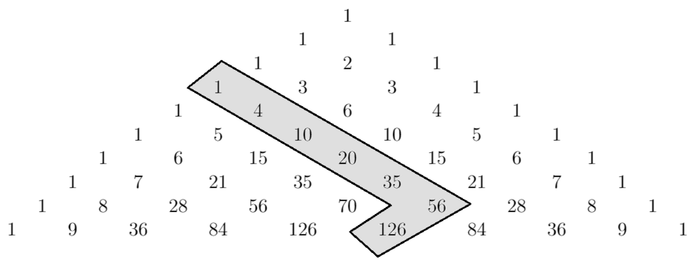

Figure 3.5: Hockey Stick Pattern in Pascal's Triangle Problem 3.3

Solution. Notice that 1 + 4 + 10 + 20 + 35 + 56 = 126 in Figure 3.5 forms a hockey stick shape pattern, where the handle of the hockey stick is the series of numbers (1 + 4 + 10 + 20 + 35 + 56) and the head of the hockey stick is the sum of the series (126). Next let's examine similar examples (Figure 3.6 and Figure 3.7) to see if the pattern continues.

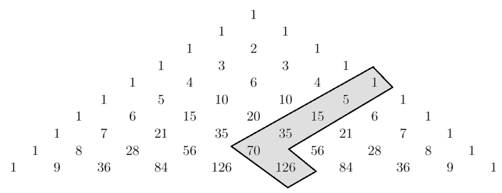

Figure 3.6: Hockey Stick Pattern in Pascal's Triangle Example 1

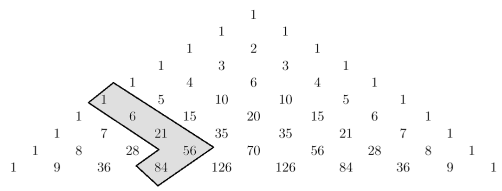

Figure 3.7: Hockey Stick Pattern in Pascal's Triangle Example 2

First examine Figure 3.6 where the series of numbers 1 + 5 + 15 + 35 + 70 has a sum of 126, which forms the same hockey stick shape pattern with the handle of the hockey stick as the series of numbers and the head of the hockey stick as the sum. Now examine Figure 3.7 where the series of numbers 1 + 6 + 21 + 56 has a sum of 84, which forms the same hockey stick shape pattern with the handle of the hockey stick as the series of numbers and the head of the hockey stick as the sum.

After observing Figure 3.5, Figure 3.6, and Figure 3.7, a pattern is formed in a hockey stick shape pattern, where the handle of the hockey stick must begin at an entry on the edge of the triangle, then continue down diagonally away from the edge forming the stick, which can be of any length. Then ending at the head of the hockey stick, which is the entry either Southeast or Southwest of the last entry of the stick. Thus a generalized formula for the hockey stick pattern in Pascal's Arithmetic Triangle can be formed.

Theorem 3.4. *The Hockey Stick Pattern in Pascal's Arithmetic Triangle:*
Let $C(n, r) = \binom{n}{r}$, denote the entry r in row n of Pascal's Arithmetic Triangle, then $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-2}{r-1} + \ldots + \binom{r-1}{r-1}$, for any integer $n \geq 1$ and $1 \leq r \leq n$. Where the left hand side represents the head of the hockey stick and the right hand side represents the stick of the hockey stick.

Proof. Use mathematical induction.

Let $P_n : \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-2}{r-1} + \ldots + \binom{r-1}{r-1}$, where $n \geq 1$ and $0 \leq r \leq n$.

1. Verify P_1 is true when $r = n$.

$$\begin{aligned}\binom{n}{r} &= \binom{n-1}{r-1}\\ \binom{n}{n} &= \binom{n-1}{n-1}\\ \binom{1}{1} &= \binom{1-1}{1-1}\\ 1 &= \binom{0}{0}\\ 1 &= 1\end{aligned}$$

2. Assume P_k is true when $k \geq r$.

   Then $P_k : \binom{k}{r} = \binom{k-1}{r-1} + \binom{k-2}{r-1} + \ldots + \binom{r-1}{r-1}$.

3. Show P_{k+1} is true.

   That is prove $P_{k+1} : \binom{k+1}{r} = \binom{k}{r-1} + \binom{k-1}{r-1} + \binom{k-2}{r-1} + \ldots + \binom{r-1}{r-1}$.

   First take the assumption P_k and add $\binom{k}{r-1}$ to both sides to get

   $\binom{k}{r} + \binom{k}{r-1} = \binom{k}{r-1} + \binom{k-1}{r-1} + \binom{k-2}{r-1} + \ldots + \binom{r-1}{r-1}$.

   Next apply Theorem 3.2 (Pascal's Identity) to the left hand side of the equation to get $\binom{k+1}{r} = \binom{k}{r-1} + \binom{k-1}{r-1} + \binom{k-2}{r-1} + \ldots + \binom{r-1}{r-1}$.

   Then P_{k+1} is true.

Thus Theorem 3.4, the Hockey Stick Theorem has been proven for Pascal's Arithmetic Triangle using mathematical induction [HP87, Kos11, Sto83]. □

Now let's looks at Problem 3.5, which is example 3.53 in Pólya's *Mathematical Discovery* [Pol09], to explore a pattern within Leibniz's Harmonic Triangle.

Problem 3.5 (Example 3.53 [Pol09]). Prove:

1. $\frac{1}{1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots$

2. $\frac{1}{2} = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \dots$

3. $\frac{1}{3} = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \dots$

Solution. (Problem 3.5 Question 1) Use partial sums and the telescoping method to prove the infinite series, $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots = \frac{1}{1}$.

Proof. The series $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots$ has the following partial sums.

- $S_0 = \frac{1}{2}$
- $S_1 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$
- $S_2 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$
- $S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$
- $S_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}$
- $S_5 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} = \frac{6}{7}$
- $S_6 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} = \frac{7}{8}$

Notice by partial sums, as the number of terms increase the infinite series converges to $\frac{1}{1}$, leading to the assumption that $\frac{1}{1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots$.
Now let $S_n = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots + \frac{1}{(n+1)(n+2)}$, where $n \geq 0$ represents the n^{th} partial sum of the series.

Next apply the telescoping method to S_n to get

$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right).$

Then cancel the additive inverses to get $S_n = \frac{1}{1} - \frac{1}{n+2}$.

Next take the limit as n approaches infinity to S_n.

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{n+2}\right) \\
S_\infty &= \lim_{n\to\infty} \left(\frac{1}{1}\right) - \lim_{n\to\infty} \left(\frac{1}{n+2}\right) \\
S_\infty &= \left(\frac{1}{1} - \frac{1}{\infty}\right) \\
S_\infty &= \left(\frac{1}{1} - 0\right) \\
S_\infty &= \frac{1}{1}
\end{aligned}$$

Thus $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = \frac{1}{1}$ is proven by partial sums and the telescoping method. □

Solution. (Problem 3.5 Question 1) Use mathematical induction to prove the infinite series, $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = \frac{1}{1}$.

Proof. Let $S_n = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots + \frac{1}{(n+1)(n+2)}$, where $n \geq 0$ represents the n^{th} partial sum of the series.

From the telescoping method $S_n = \frac{1}{1} - \frac{1}{n+2} = \frac{n+1}{n+2}$.

Then prove $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$.

1. Verify S_0 holds.

$$S_n = \frac{n+1}{n+2}$$
$$S_0 = \frac{0+1}{0+2}$$
$$\frac{1}{2} = \frac{1}{2}$$

2. Assume S_k holds.

   Then $S_k = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$.

3. Show S_{k+1} holds.

   That is prove $S_{k+1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} = \frac{k+2}{k+3}$.

   First take the assumption S_k and add $\frac{1}{(k+2)(k+3)}$ to both sides to get

   $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} = \frac{k+1}{k+2} + \frac{1}{(k+2)(k+3)}$.

   Next simplify the right hand side of the equation, $\frac{k+1}{k+2} + \frac{1}{(k+2)(k+3)}$

$$= \frac{(k+1)(k+3)+1}{(k+2)(k+3)}$$
$$= \frac{k^2+4k+4}{(k+2)(k+3)}$$
$$= \frac{(k+2)(k+2)}{(k+2)(k+3)}$$
$$= \frac{k+2}{k+3}$$

   Then S_{k+1} holds.

Thus $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = \frac{1}{1}$ is proven by mathematical induction. $\square$

Solution. (Problem 3.5 Question 2) Use partial sums and the telescoping method to prove the infinite series, $\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots = \frac{1}{2}$.

Proof. The series $\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots$ has the following partial sums.

- $S_0 = \frac{1}{3}$
- $S_1 = \frac{1}{3} + \frac{1}{12} = \frac{5}{12}$
- $S_2 = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} = \frac{9}{20}$
- $S_3 = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} = \frac{7}{15}$
- $S_4 = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} = \frac{10}{21}$
- $S_5 = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \frac{1}{168} = \frac{27}{56}$

Notice by partial sums, as the number of terms increase the infinite series converges to $\frac{1}{2}$, leading to the assumption that $\frac{1}{2} = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots$.

Now let $S_n = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots + \frac{2}{(n+1)(n+2)(n+3)}$, where $n \geq 0$ represents the n^{th} partial sum of the series.

Next apply the telescoping method to S_n to get

$S_n = \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{12}\right) + \left(\frac{1}{12} - \frac{1}{20}\right) + \left(\frac{1}{20} - \frac{1}{30}\right) + \ldots + \left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+2)(n+3)}\right)$.

Then cancel the additive inverses to get $S_n = \frac{1}{2} - \frac{1}{(n+2)(n+3)}$.

Next take the limit as n approaches infinity to S_n.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{(n+2)(n+3)}\right)$$
$$S_\infty = \lim_{n\to\infty}\left(\frac{1}{2}\right) - \lim_{n\to\infty}\left(\frac{1}{n^2+5n+6}\right)$$
$$S_\infty = \left(\frac{1}{2} - \frac{1}{\infty}\right)$$
$$S_\infty = \left(\frac{1}{2} - 0\right)$$
$$S_\infty = \frac{1}{2}$$

Thus $\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots = \frac{1}{2}$ is proven by partial sums and the telescoping method. $\square$

Solution. (Problem 3.5 Question 2) Use mathematical induction to prove the infinite series, $\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \ldots = \frac{1}{2}$.

Proof. Let $S_n = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \ldots + \frac{2}{(n+1)(n+2)(n+3)}$, where $n \geq 0$ represents the n^{th} partial sum of the series.

From the telescoping method $S_n = \frac{1}{2} - \frac{1}{(n+2)(n+3)} = \frac{(n+4)(n+1)}{2(n+2)(n+3)}$.

Then prove $\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \ldots + \frac{2}{(n+1)(n+2)(n+3)} = \frac{(n+4)(n+1)}{2(n+2)(n+3)}$.

1. Verify S_0 holds.

$$S_n = \frac{(n+4)(n+1)}{2(n+2)(n+3)}$$
$$S_0 = \frac{(0+4)(0+1)}{2(0+2)(0+3)}$$
$$\frac{1}{3} = \frac{4}{12}$$
$$\frac{1}{3} = \frac{1}{3}$$

2. Assume S_k holds.

   Then $S_k = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \ldots + \frac{2}{(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{2(k+2)(k+3)}$.

3. Show S_{k+1} holds.

   That is prove $S_{k+1} = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \ldots + \frac{2}{(k+1)(k+2)(k+3)} + \frac{2}{(k+2)(k+3)(k+4)}$ $= \frac{(k+2)(k+5)}{2(k+3)(k+4)}$.

   First take the assumptions S_k and add $\frac{2}{(k+2)(k+3)(k+4)}$ to both sides to get

   $\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \ldots + \frac{2}{(k+1)(k+2)(k+3)} + \frac{2}{(k+2)(k+3)(k+4)} = \frac{(k+1)(k+4)}{2(k+2)(k+3)} + \frac{2}{(k+2)(k+3)(k+4)}$.

   Next simplify the right hand side of the equation, $\frac{(k+1)(k+4)}{2(k+2)(k+3)} + \frac{2}{(k+2)(k+3)(k+4)}$

$$= \frac{(k+1)(k+4)^2 + 4}{2(k+2)(k+3)(k+4)}$$
$$= \frac{k^3 + 9k^2 + 24k + 20}{2(k+2)(k+3)(k+4)}$$
$$= \frac{(k+2)(k+2)(k+3)}{2(k+2)(k+3)(k+4)}$$
$$= \frac{(k+2)(k+5)}{2(k+3)(k+4)}$$

Then S_{k+1} holds.

Thus $\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots = \frac{1}{2}$ is proven by mathematical induction. $\square$

Solution. (Problem 3.5 Question 3) Use partial sums and the telescoping method to prove the infinite series, $\frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots = \frac{1}{3}$.

Proof. The series $\frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots$ has the following partial sums.

- $S_0 = \frac{1}{4}$
- $S_1 = \frac{1}{4} + \frac{1}{20} = \frac{3}{10}$
- $S_2 = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} = \frac{19}{60}$
- $S_3 = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} = \frac{34}{105}$
- $S_4 = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} = \frac{55}{168}$

Notice by partial sums, as the number of terms increase the infinite series converges to $\frac{1}{3}$, leading to the assumption that $\frac{1}{3} = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots$.

Now let $S_n = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots + \frac{6}{(n+1)(n+2)(n+3)(n+4)}$, where $n \geq 0$ represents the n^{th} partial sum of the series.

Next apply the telescoping method to S_n to get

$S_n = \left(\frac{1}{3} - \frac{1}{12}\right) + \left(\frac{1}{12} - \frac{1}{30}\right) + \left(\frac{1}{30} - \frac{1}{60}\right) + \ldots + \left(\frac{2}{(n+1)(n+2)(n+3)} - \frac{2}{(n+2)(n+3)(n+4)}\right)$.

Then cancel the additive inverses to get $S_n = \frac{1}{3} - \frac{2}{(n+2)(n+3)(n+4)}$.

Next take the limit as n approaches infinity to S_n.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{2}{(n+2)(n+3)(n+4)}\right)$$
$$S_\infty = \lim_{n\to\infty}\left(\frac{1}{3}\right) - \lim_{n\to\infty}\left(\frac{2}{n^3+9n^2+26n+24}\right)$$
$$S_\infty = \left(\frac{1}{3} - \frac{2}{\infty}\right)$$
$$S_\infty = \left(\frac{1}{3} - 0\right)$$
$$S_\infty = \frac{1}{3}$$

Thus $\frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots = \frac{1}{3}$ is proven by partial sums and the telescoping method. $\square$

Solution. (Problem 3.5 Question 3) Use mathematical induction to prove the infinite series, $\frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots = \frac{1}{3}$.

Proof. Let $S_n = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots + \frac{6}{(n+1)(n+2)(n+3)(n+4)}$, where $n \geq 0$ represents the n^{th} partial sum of the series.

From the telescoping method $S_n = \frac{1}{3} + \frac{1}{(n+2)(n+3)(n+4)} = \frac{(n+1)(n^2+8n+18)}{3(n+2)(n+3)(n+4)}$.

Then prove $\frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \ldots + \frac{6}{(n+1)(n+2)(n+3)(n+4)} = \frac{(n+1)(n^2+8n+18)}{3(n+2)(n+3)(n+4)}$.

1. Verify S_0 holds.

$$S_n = \frac{(n+1)(n^2+8n+18)}{3(n+2)(n+3)(n+4)}$$
$$S_0 = \frac{(0+1)((0)^2+8(0)+18)}{3(0+2)(0+3)(0+4)}$$
$$\frac{1}{4} = \frac{18}{72}$$
$$\frac{1}{4} = \frac{1}{4}$$

2. Assume S_k holds.

   Then $S_k = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \ldots + \frac{6}{(k+1)(k+2)(k+3)(k+4)} = \frac{(k+1)(k^2+8k+18)}{3(k+2)(k+3)(k+4)}$.

3. Show S_{k+1} holds.

   That is prove $S_{k+1} = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \ldots + \frac{6}{(k+1)(k+2)(k+3)(k+4)} + \frac{6}{(k+2)(k+3)(k+4)(k+5)}$ $= \frac{(k+2)((k+1)^2+8(k+1)+18)}{3(k+3)(k+4)(k+5)} = \frac{(k+2)(k^2+10k+27)}{3(k+3)(k+4)(k+5)}$.

   First take the assumption S_k and add $\frac{6}{(k+2)(k+3)(k+4)(k+5)}$ to both side to get

   $\frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \ldots + \frac{6}{(k+1)(k+2)(k+3)(k+4)} + \frac{6}{(k+2)(k+3)(k+4)(k+5)} = \frac{(k+1)(k^2+8k+18)}{3(k+2)(k+3)(k+4)}$ $+ \frac{6}{(k+2)(k+3)(k+4)(k+5)}$.

   Next simplify the right hand side of the equation, $\frac{(k+1)(k^2+8k+18)}{3(k+2)(k+3)(k+4)} + \frac{6}{(k+2)(k+3)(k+4)(k+5)}$

$$= \frac{(k+1)(k^2+8k+18)(k+5)+18}{3(k+2)(k+3)(k+4)(k+5)}$$
$$= \frac{k^4+14k^3+71k^2+148k+108}{3(k+2)(k+3)(k+4)(k+5)}$$
$$= \frac{(k+2)(k+2)(k^2+10k+27)}{3(k+2)(k+3)(k+4)(k+5)}$$
$$= \frac{(k+2)(k^2+10k+27)}{3(k+3)(k+4)(k+5)}$$

Then S_{k+1} holds.

Thus $\frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots = \frac{1}{3}$ is proven by mathematical induction. $\square$

After proving the three infinite series in Problem 3.5, look at the problems within Leibniz's Harmonic triangle to see if a pattern can be recognized (See Figure 3.8, Figure 3.9, and Figure 3.10) [Kos11, Tuc12].

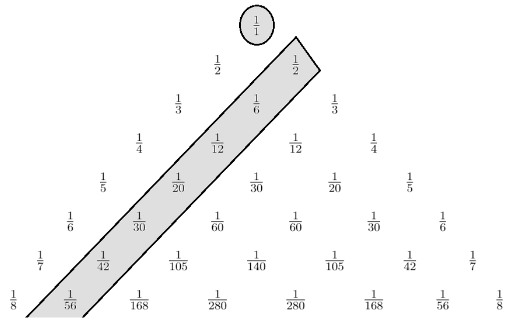

Figure 3.8: Problem 3.5 Question 1

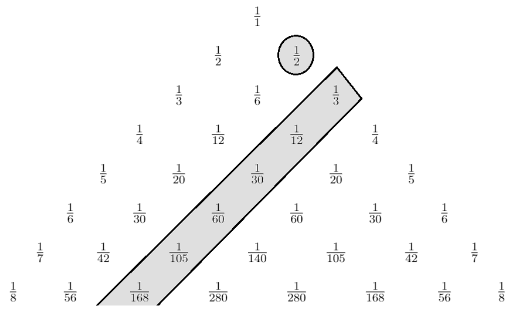

Figure 3.9: Problem 3.5 Question 2

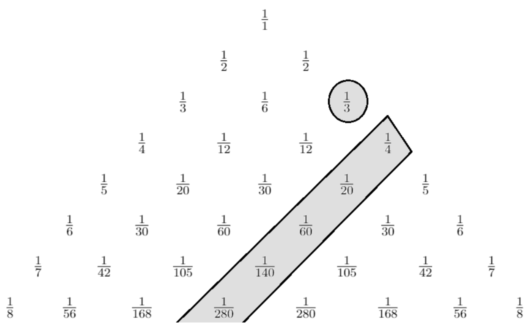

Figure 3.10: Problem 3.5 Question 3

Notice that a hockey stick type pattern is formed. However, let's see if this pattern continues. Now examine Problem 3.6, which is example 3.54 and the last example that will be used from George Pólya's *Mathematical Discovery* [Pol09].

Problem 3.6 (Example 3.54 [Pol09])**.** Find the sum $\frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots$ and generalize.

First by partial sums and telescoping let's find the sum of Problem 3.6, then try to generalize a formula and prove it.

Solution. Let's find the sum of $\frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots$ by examining the partial sums.

- $S_0 = \frac{1}{12}$
- $S_1 = \frac{1}{12} + \frac{1}{30} = \frac{7}{60}$
- $S_2 = \frac{1}{12} + \frac{1}{30} + \frac{1}{60} = \frac{2}{15}$
- $S_3 = \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} = \frac{1}{7}$
- $S_4 = \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \frac{1}{168} = \frac{25}{168}$
- $S_5 = \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \frac{1}{168} + \frac{1}{252} = \frac{11}{72}$
- $S_6 = \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \frac{1}{168} + \frac{1}{252} + \frac{1}{360} = \frac{7}{45}$

Notice as the number of terms increase the infinite series converges to $\frac{1}{6}$, leading to the assumption that $\frac{1}{6} = \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \frac{1}{168} + \ldots$.

Now let $S_n = \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \frac{1}{168} + \ldots + \frac{2}{(n+2)(n+3)(n+4)}$, where $n \geq 0$ represent the n^{th} partial sum of the series.

Next apply the telescoping method to S_n to get

$S_n = \left(\frac{1}{6} - \frac{1}{12}\right) + \left(\frac{1}{12} - \frac{1}{20}\right) + \left(\frac{1}{20} - \frac{1}{30}\right) + \left(\frac{1}{30} - \frac{1}{42}\right) + \ldots + \left(\frac{1}{(n+2)(n+3)} - \frac{1}{(n+3)(n+4)}\right)$.

Then cancel the additive inverses to get $S_n = \frac{1}{6}$ - $\frac{1}{(n+3)(n+4)}$.

Next take the limit as n approaches infinity to S_n.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{6}-\frac{1}{(n+3)(n+4)}\right)$$
$$S_\infty = \lim_{n\to\infty}\left(\frac{1}{6}\right)-\lim_{n\to\infty}\left(\frac{1}{n^2+7n+12}\right)$$
$$S_\infty = \left(\frac{1}{6}-\frac{1}{\infty}\right)$$
$$S_\infty = \left(\frac{1}{6}-0\right)$$
$$S_\infty = \frac{1}{6}$$

Thus by partial sums and the telescoping method $\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\frac{1}{105}+\frac{1}{168}+\ldots=\frac{1}{6}$.

Now look at the sum found in Problem 3.6 within Leibniz's Harmonic Triangle to see if the hockey stick type pattern continues (See Figure 3.11).

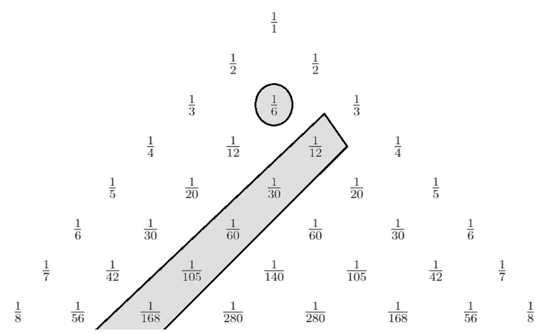

Figure 3.11: Problem 3.6

After observing Figure 3.8, Figure 3.9, Figure 3.10, and Figure 3.11 the hockey stick type pattern continues. Now let's break down what was found to construct a generalized equation and prove it.

- From Problem 3.5 Question 1, it was proven that $\frac{1}{1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$, which can be rewritten as $L(0,\,0) = L(1,\,1) + L(2,\,1) + L(3,\,1) + L(4,\,1) + \ldots$.

- From Problem 3.5 Question 2, it was proven that $\frac{1}{2} = \frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \ldots$, which can be rewritten as $L(1,\,1) = L(2,\,2) + L(3,\,2) + L(4,\,2) + L(5,\,2) + \ldots$.

- From Problem 3.5 Question 3, it was proven that $\frac{1}{3} = \frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \ldots$, which can be rewritten as $L(2,\,2) = L(3,\,3) + L(4,\,3) + L(5,\,3) + L(6,\,3) + \ldots$.

- From Problem 3.6 it was proven that $\frac{1}{6} = \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots$, which can be rewritten as $L(2,\,1) = L(3,\,2) + L(4,\,2) + L(5,\,2) + L(6,\,2) + \ldots$.

Notice that the shape formed in the figures above is similar to the hockey stick pattern in Pascal's Arithmetic Triangle (See Problem 3.3). However, the hockey stick design in Figure 3.8, Figure 3.9, Figure 3.10 and Figure 3.11 has the head of the stick located at the top of the triangle either on a boundary entry or an internal entry of the triangle with the handle of the stick headed towards the bottom of the triangle infinitely. Thus using the information from Problem 3.5 and Problem 3.6, a generalized formula for the hockey stick type pattern in Leibniz's Harmonic Triangle can be developed.

Theorem 3.7. *The Hockey Stick Pattern in Leibniz's Harmonic Triangle:*

$L(n,\,r) = L(n+1,\,r+1) + L(n+2,\,r+1) + L(n+3,\,r+1) + \ldots$, where $n \geq 0$, $0 \leq r \leq n$, and $n,\,r \in \mathbb{Z}$. In other words $\sum_{n=k}^{\infty} L(n+1,\,r+1) = L(k,\,r)$, where $k \geq 0$, $0 \leq r \leq k$, and $k,\,r \in \mathbb{Z}$.

Proof. Use mathematical induction.

Let P_n: $L(n,\,r) = L(n+1,\,r+1) + L(n+2,\,r+1) + L(n+3,\,r+1) + L(n+4,\,r+1) + \ldots$, where $n \geq 0$, $0 \leq$ r $\leq$ n, and n, $r \in \mathbb{Z}$.

1. Verify P_0 is true when $r = 0$.

$$L(n,r) = L(n+1,r+1) + L(n+2,r+1) + L(n+3,r+1) + \dots$$

$$L(0,0) = L(0+1,0+1) + L(0+2,0+1) + L(0+3,0+1) + \dots$$

$$\frac{1}{1} = L(1,1) + L(2,1) + L(3,1) + \dots$$

$$\frac{1}{1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots$$

   From Problem 3.3.1 P_0 is true.

2. Assume P_k is true.

   Then P_k: $L(k,\ r) = L(k{+}1,\ r{+}1) + L(k{+}2,\ r{+}1) + L(k{+}3,\ r{+}1) + \dots$.

3. Show P_{k+1} is true.

   That is prove P_{k+1}: $L(k{+}1,\ r) = L(k{+}2,\ r{+}1) + L(k{+}3,\ r{+}1) + \dots$.

   First take the assumption P_k and subtract $L(k{+}1,\ r{+}1)$ from both sides to get $L(k,\ r) - L(k{+}1,\ r{+}1) = L(k{+}1,\ r{+}1) - L(k{+}1,\ r{+}1) + L(k{+}2,\ r{+}1) + \dots$.

   Simplify to get $L(k,\ r) - L(k{+}1,\ r{+}1) = L(k{+}2,\ r{+}1) + L(k{+}3,\ r{+}1) + \dots$.

   Next apply Equation 2.1 (Leibniz's Identity) to the left hand side of the equation to get $L(k{+}1,\ r) = L(k{+}2,\ r{+}1) + L(k{+}3,\ r{+}1) + \dots$.

   Then P_{k+1} is true.

Thus Theorem 3.7 is proven by mathematical induction [Kos11, Tuc12]. $\square$

3.2 Other Patterns

Lastly, let's explore other patterns within Pascal's Arithmetic Triangle and Leibniz's Harmonic Triangle dealing with sums and shapes.

Problem 3.8. Examine the results of taking the sum of the terms within a rhombus shape of any size in Pascal's Arithmetic Triangle. Look for a pattern, generalize it, then prove it.

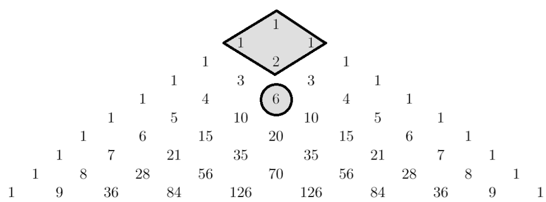

Figure 3.12: Problem 3.8 Example 1

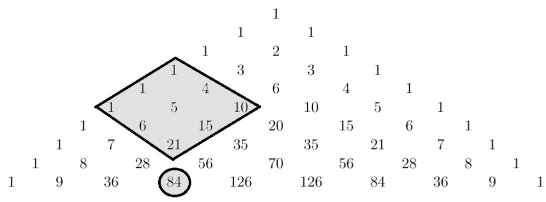

Figure 3.13: Problem 3.8 Example 2

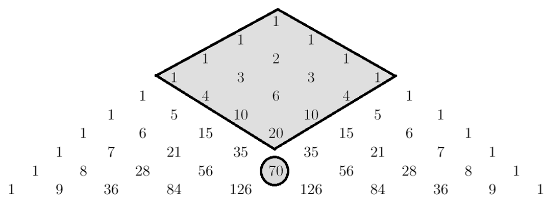

Figure 3.14: Problem 3.8 Example 3

First, let's find a pattern

- Look at Figure 3.12 and take the sum of the terms within the rhombus shape $1 + 1 + 1 + 2 = 5$, which is the number directly below the bottom vertex of the rhombus shape $6 - 1$.

  Notice that the sums within the rhombus shape are just hockey stick patterns discussed in Problem 3.3.

  $1 + 1 = 2 \implies \binom{0}{0} + \binom{1}{0} = \binom{2}{1}$.

  $1 + 2 = 3 \implies \binom{1}{1} + \binom{2}{1} = \binom{3}{2}$.

  Next, observe that the sum $2 + 3 = 5$ is the same as $2 + 3 = 6 - 1$, which is just part of another hockey stick, $1 + 2 + 3 = 6$.

  $\implies \binom{1}{1} + \binom{2}{1} + \binom{3}{1} = \binom{4}{2}$.

  However, the term $1 = \binom{1}{1}$ is part of the handle of the hockey stick not needed and can be subtracted from both sides to get $\binom{2}{1} + \binom{3}{2} = \binom{4}{2} - \binom{1}{1}$.

  $\implies 2 + 3 = 6 - 1$.

  $\implies 2 + 3 = 5$.

  Then the sum of the rhombus shape is just the sum of the hockey sticks within the rhombus shape ($1 + 1 = 2$ and $1 + 2 = 3$) equal to the difference of two other hockey sticks ($1 + 2 + 3 = 6$ and $1 = 1$).

- Next, look at Figure 3.13 and take the sum of the terms within the rhombus shape $1 + 1 + 1 + 4 + 5 + 6 + 10 + 15 + 21 = 64$, which is the number directly below the bottom vertex of the rhombus shape $84 - 20$.

  Notice that the sums within the rhombus shape are just hockey stick patterns discussed in Problem 3.3.

$1 + 4 + 10 = 15 \implies \binom{3}{0} + \binom{4}{1} + \binom{5}{2} = \binom{6}{2}$.

$1 + 5 + 15 = 21 \implies \binom{4}{0} + \binom{5}{1} + \binom{6}{2} = \binom{7}{2}$.

$1 + 6 + 21 = 28 \implies \binom{5}{0} + \binom{6}{1} + \binom{7}{2} = \binom{8}{2}$.

Next, observe that the sum $15 + 21 + 28 = 64$ is the same as $15 + 21 + 28 = 84 - 20$, which is just part of another hockey stick, $1 + 3 + 6 + 10 + 15 + 21 + 28 = 84$.

$\implies \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} + \binom{8}{2} = \binom{9}{3}$.

However, the terms $1 = \binom{2}{2}$, $3 = \binom{3}{2}$, $6 = \binom{4}{2}$, and $10 = \binom{5}{2}$ are part of the handle of the hockey stick not needed and can be subtracted from both sides to get

$\binom{6}{2} + \binom{7}{2} + \binom{8}{2} = \binom{9}{3} - \left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2}\right]$.

$\implies 15 + 21 + 28 = 84 - (1 + 3 + 6 + 10)$.

$\implies 15 + 21 + 28 = 84 - 20$.

$\implies \binom{6}{2} + \binom{7}{2} + \binom{8}{2} = \binom{9}{3} - \binom{6}{3}$.

Then the sum of the rhombus shape is just the sum of the hockey sticks within the rhombus shape ($1 + 4 + 10 = 15$, $1 + 5 + 15 = 21$, and $1 + 6 + 21 = 28$) equal to the difference of two other hockey sticks ($1 + 3 + 6 + 10 + 15 + 21 + 28 = 84$ and $1 + 3 + 6 + 10 = 20$).

- Lastly, look at Figure 3.14 and take the sum of the terms within the rhombus shape $1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 + 3 + 3 + 4 + 6 + 4 + 10 + 10 + 20 = 69$, which is the number directly below the bottom vertex of the rhombus shape $70 - 1$. Notice that the sums within the rhombus shape are just hockey stick patterns discussed in Problem 3.3.

$1 + 1 + 1 + 1 = 4 \implies \binom{0}{0} + \binom{1}{0} + \binom{2}{0} + \binom{3}{0} = \binom{4}{1}.$

$1 + 2 + 3 + 4 = 10 \implies \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} = \binom{5}{2}.$

$1 + 3 + 6 + 10 = 20 \implies \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{6}{3}.$

$1 + 4 + 10 + 20 = 35 \implies \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} = \binom{7}{4}.$

Next observe that the sum $4 + 10 + 20 + 35 = 69$ is the same as $4 + 10 + 20 + 35 = 70 - 1$, which is just part of another hockey stick $1 + 4 + 10 + 20 + 35 = 70$.

$\implies \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} + \binom{7}{3} = \binom{8}{4}.$

However, the term $1 = \binom{3}{3}$ is part of the handle of the hockey stick not needed and can be subtracted from both sides to get $\binom{4}{3} + \binom{5}{3} + \binom{6}{3} + \binom{7}{3} = \binom{8}{4} - \binom{3}{3}.$

$\implies 10 + 20 + 35 = 70 - 1.$

$\implies 10 + 20 + 35 = 69.$

Then the sum of the rhombus shape is just the sum of the hockey sticks within the rhombus shape ($1 + 4 + 10 = 15$, $1 + 5 + 15 = 21$, and $1 + 6 + 21 = 28$) equal to the difference of two other hockey sticks ($1 + 3 + 6 + 10 + 15 + 21 + 28 = 84$ and $1 + 3 + 6 + 10 = 20$).

After observing Figure 3.12, Figure 3.13, and Figure 3.14 notice that these rhombus shape patterns have similar properties and restrictions that are found in the hockey stick pattern within Pascal's Arithmetic Triangle. The rhombus shape has to include the boundary terms of the triangular array, it can not contain only the internal terms. Then the rhombus shape must continue downwards to any size. Thus a generalized equation can be discovered to represent a rhombus shape pattern within Pascal's Arithmetic Triangle with a side length of any number of terms.

First, to construct the general equation to the sum of terms inside a rhombus shape within Pascal's Arithmetic Triangle reference Figure 3.15, which shows Pascal's Arithmetic Triangle in the form of binomial coefficients using combinations. Then, let $\sum_{n=k-m+1}^{k}\left[\binom{n-2}{r-1} + \binom{n-3}{r-2} + \ldots + \binom{n-m-1}{0}\right]$ represent the sum of the terms within the rhombus shape, where m is the number of terms in one side length of the rhombus shape and k is the n^{th} position in Pascal's Arithmetic Triangle representing the term below the bottom vertex of the rhombus shape. Notice that the sum of the terms inside the rhombus shape creates a hockey stick pattern, where the handle of the hockey stick is as long as m. Then the sum of the terms inside a rhombus shape with m number of terms can be represented by another hockey stick. Thus $\sum_{n=k-m+1}^{k}\left[\binom{n-2}{r-1} + \binom{n-3}{r-2} + \ldots + \binom{n-m-1}{0}\right] = \binom{k-1}{r-1} + \binom{k-2}{r-1} + \ldots + \binom{k-m}{r-1}$.

$$
\begin{array}{lccccccccccccc}
n=0: & & & & & & & \binom{0}{0} & & & & & & \\
n=1: & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
n=2: & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
n=3: & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
n=4: & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
n=5: & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
n=6: & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

Figure 3.15: Pascal's Triangle in Combination Form

Theorem 3.9. *The Rhombus Shape Pattern in Pascal's Arithmetic Triangle:*

0 Let $\sum_{n=k-m+1}^{k}\left[\binom{n-2}{r-1} + \binom{n-3}{r-2} + \ldots + \binom{n-m-1}{0}\right] = \binom{k-1}{r-1} + \binom{k-2}{r-1} + \ldots + \binom{k-m}{r-1}$ *represent the sum of terms within the rhombus shape, where* m *represents the length of terms for one side of the rhombus, and* k *represents the* n^{th} *position in Pascal's triangle. Then the* *sum of terms inside a rhombus shape within Pascal's Arithmetic Triangle can be found using* $\binom{k-1}{r-1} + \binom{k-2}{r-1} + \ldots + \binom{k-m}{r-1} = \binom{k}{r} - \binom{k-m}{r}$*, for all* $k \geq 4$*,* $r \geq 1$*,* $1 \leq m \leq k$*, and* k*,* m*,* $r \in \mathbb{Z}$*.*

Proof. Use mathematical induction.

Let P_n: $\binom{n-1}{r-1} + \binom{n-2}{r-1} + \ldots + \binom{n-m}{r-1} = \binom{n}{r} - \binom{n-m}{r}$, where $n \geq 4$, $r \geq 2$, and $2 \leq m \leq n$.

- Verify P_4 is true when $r = 2$ and $m = 2$.

$$\binom{n-1}{r-1} + \binom{n-2}{r-1} + \ldots + \binom{n-m}{r-1} = \binom{n}{r} - \binom{n-m}{r}$$

$$\binom{3}{1} + \binom{2}{1} = \binom{4}{2} - \binom{2}{2}$$

$$3 + 2 = 6 - 1$$

$$3 + 2 = 5$$

- Assume P_k is true.

  Then $\binom{k-1}{r-1} + \binom{k-2}{r-1} + \ldots + \binom{k-m}{r-1} = \binom{k}{r} - \binom{k-m}{r}$.

- Show P_{k+1} holds.

That is prove $\binom{k}{r-1} + \binom{k-1}{r-1} + \binom{k-2}{r-1} + \ldots + \binom{k-m+1}{r-1} = \binom{k+1}{r} - \binom{k-m+1}{r}$.

First take the assumption P_k and add $\binom{k}{r-1}$ and subtract $\binom{k-m}{r-1}$ to both sides of the equation to get $\binom{k}{r-1} + \binom{k-1}{r-1} + \binom{k-2}{r-1} + \ldots + \binom{k-m+1}{r-1} = \binom{k}{r} - \binom{k-m}{r} + \binom{k}{r-1} - \binom{k-m}{r-1}$.

Next rearrange the right hand side of the equation, $\binom{k}{r} - \binom{k-m}{r} + \binom{k}{r-1} - \binom{k-m}{r-1}$ to get $\left[\binom{k}{r} + \binom{k}{r-1}\right] - \left[\binom{k-m}{r} + \binom{k-m}{r-1}\right]$.

Then apply Theorem 3.2 (Pascal's Identity) to get $\left[\binom{k}{r} + \binom{k}{r-1}\right] - \left[\binom{k-m}{r} + \binom{k-m}{r-1}\right] = \binom{k+1}{r} - \binom{k-m+1}{r}$.

Then P_{k+1} is true.

Thus Theorem 3.9 is proven by mathematical induction. $\square$

Through similarities between Pascal's Arithmetic Triangle and Leibniz's Harmonic Triangle, the rhombus shape pattern paves the way to explore similar sum patterns within Leibniz's Harmonic Triangle. Finally, let's look at Problem 3.10, which will explore the sum of the terms in a triangular shape found within Leibniz's Harmonic Triangle.

Problem 3.10. Examine what happens when the sum of an infinite triangular design is taken within Leibniz's Harmonic Triangle. Look for a pattern, generalize it, then prove it.

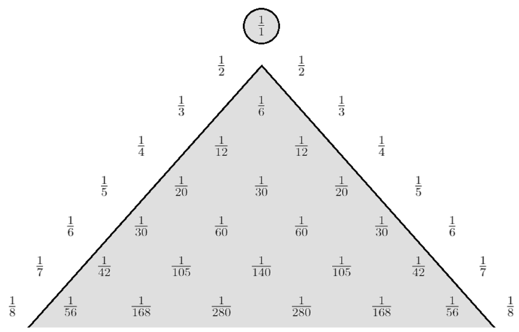

Figure 3.16: Problem 3.10 Example 1

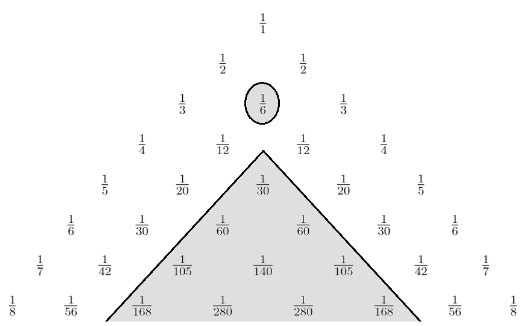

Figure 3.17: Problem 3.10 Example 2

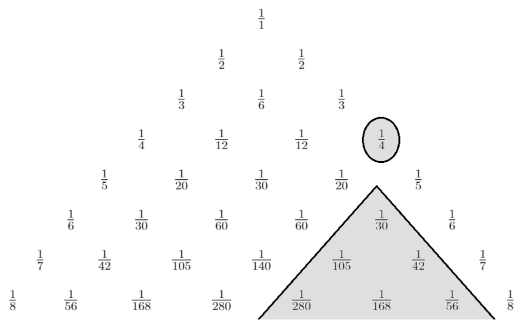

Figure 3.18: Problem 3.10 Example 3

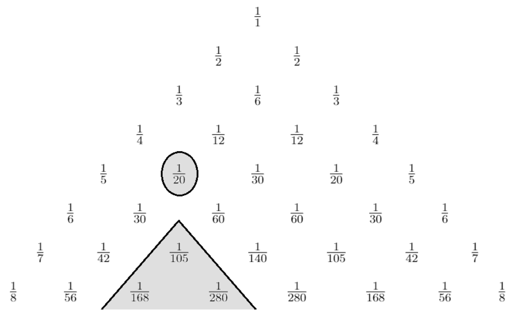

Figure 3.19: Problem 3.10 Example 4

First let's find a pattern.

- Look at Figure 3.16 and take the infinite sums

  $\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \dots$

  $+ \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \frac{1}{168} + \dots$

  $+ \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \dots$

  $+ \frac{1}{30} + \frac{1}{105} + \frac{1}{280} + \dots$

  $+ \frac{1}{42} + \frac{1}{168} + \dots$

  $+ \frac{1}{56} + \dots$

  First take the sum of the diagonals to get

$$\begin{aligned}
\frac{1}{2} &= \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \dots \\
\frac{1}{6} &= \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \frac{1}{168} + \dots \\
\frac{1}{12} &= \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \dots \\
\frac{1}{20} &= \frac{1}{30} + \frac{1}{105} + \frac{1}{280} + \dots \\
\frac{1}{30} &= \frac{1}{42} + \frac{1}{168} + \dots
\end{aligned}$$

The sums form hockey stick patterns discussed in Problem 3.5 and Problem 3.6. Next take the sum of the numbers found for each diagonal to get

$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = \frac{1}{1}$ (shown in Problem 3.5 question 1). Notice that taking the sum of hockey stick patterns is just another hockey stick pattern.

- Look at Figure 3.17 and take the infinite sums

  $\frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \frac{1}{168} + \ldots$

  $+ \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots$

  $+ \frac{1}{105} + \frac{1}{280} + \ldots$

  $+ \frac{1}{168} + \ldots$

  First take the sum of the diagonals to get

$$\begin{aligned}
\frac{1}{12} &= \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \frac{1}{168} + \ldots \\
\frac{1}{30} &= \frac{1}{60} + \frac{1}{140} + \frac{1}{280} + \ldots \\
\frac{1}{60} &= \frac{1}{105} + \frac{1}{280} + \ldots
\end{aligned}$$

  The sums form hockey stick patterns discussed in Problem 3.5 and Problem 3.6. Next take the sum of the numbers found for each diagonal to get

  $\frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \frac{1}{105} + \ldots = \frac{1}{6}$ (shown in Problem 3.6). Notice that taking the sum of hockey stick patterns is just another hockey stick pattern.

- Look at Figure 3.18 and take the infinite sums

  $\frac{1}{30} + \frac{1}{105} + \frac{1}{280} + \ldots$

  $+ \frac{1}{42} + \frac{1}{168} + \ldots$

$+ \frac{1}{56} + \ldots$

First take the sum of the diagonals to get

$$\frac{1}{20} = \frac{1}{30} + \frac{1}{105} + \frac{1}{280} + \ldots$$
$$\frac{1}{30} = \frac{1}{42} + \frac{1}{168} + \ldots$$
$$\frac{1}{42} = \frac{1}{56} + \ldots$$

The sums form hockey stick patterns discussed in Problem 3.5 and Problem 3.6.

Next take the sum of the numbers found for each diagonal to get

$\frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \ldots = \frac{1}{4}$. Notice that taking the sum of hockey stick patterns is just another hockey stick pattern.

- Look at Figure 3.19 and take the infinite sums

  $\frac{1}{105} + \frac{1}{168} + \ldots$

  $+ \frac{1}{280} + \ldots$

  First take the sum of the diagonals to get

$$\frac{1}{30} = \frac{1}{105} + \frac{1}{168} + \ldots$$
$$\frac{1}{105} = \frac{1}{280} + \ldots$$

  The sums form hockey stick patterns discussed in Problem 3.5 and Problem 3.6.

  Next take the sum of the numbers found for each diagonal to get

  $\frac{1}{30} + \frac{1}{105} + \frac{1}{280} + \ldots = \frac{1}{20}$. Notice that taking the sum of hockey stick patterns is just another hockey stick pattern.

Now based on what has been discovered, let's construct a generalized formula for the infinite triangular design in Leibniz's Harmonic Triangle. First, it was found that the sum of the terms in an infinite triangular design is just the sum of hockey sticks. However, the sum of hockey sticks is just another hockey stick.

Then $\sum\limits_{n=m}^{\infty}\left[L(n,\, r) + L(n+1,\, r+1) + \dots\right] = \sum\limits_{n=m}^{\infty} L(n - 1,\, r)$ can simplify to $\sum\limits_{n=m}^{\infty} L(n - 1,\, r) = L(m - 2,\, r - 1)$, where $m \geq 2$, $1 \leq r \leq m$, and $m, r \in \mathbb{N}$.

Theorem 3.11. *An infinite triangular design in Leibniz's Harmonic Triangle:* $L(m-1,\, r) + L(m,\, r) + L(m+1,\, r) + L(m+2,\, r) + \dots = L(m-2,\, r-1)$. *In other words,* $\sum\limits_{n=m}^{\infty} L(n-1,\, r) = L(m-2,\, r-1)$, *where* $m \geq 2$, $1 \leq r \leq m$, *and* $m, r \in \mathbb{N}$.

Proof. Use mathematical induction

Let P_m: $L(m-2,\, r-1) = L(m-1,\, r) + L(m,\, r) + L(m+1,\, r) + L(m+2,\, r) + \dots$

1. Show P_2 is true when $r = 1$.

$$L(m-2, r-1) = L(m-1, r) + L(m, r) + L(m+1, r) + \dots$$

$$L(2-2, 1-1) = L(2-1, 1) + L(2, 1) + L(2+1, 1) + \dots$$

$$L(0, 0) = L(1, 1) + L(2, 1) + L(3, 1) + \dots$$

$$\frac{1}{1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots \checkmark$$

From Problem 3.3.1 P_2 is true.

2. Assume P_k is true.

Then P_k: $L(k - 2, r - 1) = L(k - 1, r) + L(k, r) + L(k + 1, r) + \ldots$

3. Show P_{k+1} is true.

   That is prove P_{k+1}: $L(k - 1, r - 1) = L(k, r) + L(k + 1, r) + L(k + 2, r) + \ldots$

   First take the assumption P_k and subtract $L(k - 1, r)$ from both sides to get

   $L(k - 2, r - 1) - L(k - 1, r) = L(k - 1, r) - L(k - 1, r) + L(k, r) + \ldots$

   Then simplify to get $L(k - 2, r - 1) - L(k - 1, r) = L(k, r) + L(k + 1, r) + \ldots$.

   Next apply Equation 2.1 (Leibniz's Identity) to the left hand side of the equation

   to get $L(k - 1, r - 1) = L(n, r) + L(k + 1, r) + L(k + 2, r) + \ldots$

   Then P_{k+1} holds

Thus Theorem 3.11 is proven by mathematical induction. $\square$

Bibliography

[AL91] S.C. Althoen and C.B. Lacampagne. Tetrahedral numbers as sums of sqaure numbers. *Mathematics Magazine*, pages 104–108, 1991.

[BJ81] Marjorie BickNell-Johnson. Diagonal sums in the harmonic triangle. *Fibonacci Quarterly*, pages 196–199, 1981.

[BL18] Yvon Belaval and Brandon C. Look. Gottfried wilhelm leibniz. *Encyclopedia Britannica*, 2018.

[HP87] Peter Hilton and Jean Pedersen. Looking into pascal's triangle: Combinatorics, arithmetic, and geometry. *Mathematics Magazine*, pages 305–316, 1987.

[JO19] Lucien Jerphagnon and Jean Orcibal. Blaise pascal. *Encyclopedia Britannica*, 2019.

[Kos11] Thomas Koshy. *Triangular Arrays with Applications.* Oxford University Press, New York, 2011.

[Pol09] George Polya. *Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving.* Ishi Press, New York, 2009.

[Sto83] Ivan D. Stones. The harmonic triangle: Opportunities for pattern identification and generalization. *The Mathematics Teacher*, pages 350–354, 1983.

[Tuc12] Alan Tucker. *Applied Combinatorics*. John Wiley and Sons, USA, 2012.