Symmetric representation of elements of sporadic groups

Elena Yavorska Harris

SYMMETRIC REPRESENTATION OF ELEMENTS

OF SPORADIC GROUPS

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Approved by:

[redacted] 6/7/05

Zahid Hasan, Committee Chair, Date

[redacted]

John Sarli, Committee Member

[redacted]

Peter Williams, Committee Member

[redacted]

Peter Williams, Chair
Department of Mathematics

[redacted]

Terry Hallett,
Graduate Coordinator
Department of
Mathematics

ABSTRACT

Manipulation of elements of large sporadic groups in the usual straightforward manner, permutation representation or matrix representation, is unmanageable, inconvenient, or time consuming. Using the techniques of symmetric presentations it is possible to represent elements of groups in much shorter forms than their corresponding permutation or matrix representation. We will use these techniques to develop a nested algorithm and a computer program that will allow manipulating elements of the groups $U_3(3){:}2$, $J_2{:}2$, $G_2(4){:}2$ and $3{\cdot}Suz{:}2$ represented in nested symmetric representation form, where each element of $3{\cdot}Suz{:}2$, $G_2(4){:}2$ or $J_2{:}2$ is represented as a permutation on 14 letters of the group $PGL_2(7)$ followed by a word of length at most ten, six or four respectively.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER ONE

INTRODUCTION

Motivation

Manipulation of elements of large sporadic groups in the usual straightforward manner, permutation representation or matrix representation, is unmanageable, or inconvenient, or time consuming. Using the techniques of Symmetric presentations it is possible to represent elements of groups in much shorter forms than their corresponding permutation or matrix representation. Moreover, inversion of elements in the short form is as straightforward as permutations and multiplication can be performed manually or electronically by means of a short recursive algorithm. Computer programs have been developed in [2] to manipulate elements of J_1 in the short form, i.e. a permutation of the control group, $L_2(11)$, followed by a word in symmetric generators of length at most four. Performing these operations on the elements represented in the symmetric representation form is much more efficient and feasible.

This project undertakes to develop a nested algorithm and a computer program to manipulate elements of the groups

$U_3(3):2$, $J_2:2$, $G_2(4):2$ and $3\cdot Suz:2$, where each element of $G_2(4):2$, for example, is represented as a permutation on 14 letters of the group $PGL_2(7)$ followed by a word of length at most six, which is much more concise than permutations on 416, the lowest degree of a permutation presentation of $G_2(4):2$, letters.

To develop such an algorithm and computer program, we will need to perform double coset enumeration for these groups. In [7], the double coset enumeration of $G_2(4):2$ over $J_2:2$ and of $J_2:2$ over $U_3(3):2$ was performed. However, to make it possible to write a nested algorithm, we will slightly change the progenitors of the groups $G_2(4):2$ and $J_2:2$, use different generators for the corresponding control groups and reproduced double coset enumeration. We will also perform double coset enumeration for the groups $L_2(11)\times 3$ over A_4 and $3\cdot Suz:2$ over $G_2(4):2$.

Definitions

In this section we will list some definitions that are used in this project; these definitions can be found in [4].

G-set: Let X be a set and G be a group, then X is a G-set of G if there exists a mapping α: G x X $\rightarrow$ X, given by

$\alpha(g,X) \to gx$, such that: (i) $1x = x$ for all x in X, and (ii) $g(hx) = (gh)x$ for all g,h in G and x in X. α is called an action of G on X.

Double Coset: Let G be a group and N, K be subgroups of G not necessary distinct. Then a double coset is $NgK = \{ngk \mid n \in N \text{ and } k \in K\}$, where $g \in G$.

Orbit: Let G be a permutation group on Ω and let $\alpha \in \Omega$, then the orbit of α denoted as $O(\alpha)$ is $O(\alpha) = \{g\alpha \mid g \in G\}$.

Transitive: Let G be a permutation group on the set Ω, then G is transitive on Ω if and only if Ω is the only orbit of G, i.e. for all $x,y \in \Omega$ there exists $\sigma \in G$ such that $\sigma x=y$.

K-transitive: Let G be a permutation group on Ω, then G is said to be k-transitive if and only if for every pair of k-tuples having distinct entries in G, say $(x_1,x_2,\ldots,x_k)$ and $(y_1,y_2,\ldots,y_k)$, there exists $\sigma \in G$ such that $\sigma x_i = y_i$ for $i = 1, 2, \ldots, k$.

Word: Let X be a set and X^{-1} be a disjoint from X set, for which there is a bijection $X \to X^{-1}$. A word on X is a sequence $w = (x_1,x_2,\ldots,x_n)$, where $x_i \in X \cup X^{-1} \cup \{1\}$ for all i, such that all $x_i = 1$ for all $i > n$, where n is an integer and $n \geq 0$. The length of the word is defined to be n.

Reduced Word: A word w on X is reduced if either w is empty or $w = x_1^{\xi_1} x_2^{\xi_2} \ldots x_n^{\xi_n}$, where $x_i \in X$ and $\xi_i = \pm 1$, and x and x^{-1} are never adjusted.

Free Group: Let F be a set of all the reduced words on X, then F is a group under juxtaposition called a free group.

Free Product: Let $\{C_i \mid 1 \le i \le n\}$ be a family of groups, where $C_i = \langle t_i \rangle$. A free product $F = \prod \langle t_i \rangle$, where $1 \le i \le n$ for some integer n. The elements of F are called words.

Commutator: If $a,b \in G$, then a commutator of a and b, denoted by $[a,b]$, is $aba^{-1}b^{-1}$.

Curtis's Lemma: $\langle t_i, t_j \rangle \cap N \le C_N(N^{ij})$,

where N^{ij} is the point-wise stabilizer of $\{i,j\}$ in N (see Curtis [2]).

Notation: Here we will give the notation that we will use in this project. Let w_i and w_j be words in generators of a group G and N be a subgroup of G. We denote right cosets of N in G as Nw_i and Nw_j. If we have $Nw_i = Nw_j$, then we write $w_i \sim w_j$. We denote a double coset Nw_iN in G as $[w_i]$.

Symmetric Generation of a Group

Let G be a group and $T = \{t_1, t_2, .., t_n\} \subseteq G$, and let $T_i = \langle t_i \rangle$ for i in $\{1, 2, .., n\}$ be a cyclic subgroup of order m generated by t_i. In this project we will deal only with cases when m is equal to 2, so $T_i = \langle t_i \rangle = C_2$. Define $\Gamma = \{T_1, T_2, .., T_n\}$ and let N denote the control subgroup such that $N = N_G(\Gamma)$. T is a symmetric generating set for G if and only if (i) $G = \langle T \rangle$, i.e. $G = \langle t_1, t_2, .., t_n \rangle$ and (ii) N acts transitively on Γ.

Let $F = 2^{*n} = \Pi\langle t_i \rangle$, where $1 \le i \le n$, be a free product of n copies of the cyclic group C_2, then $2^{*n}:N$ is an infinite progenitor, where $N = N_G(\Gamma)$ is a group of automorphisms of 2^{*n} which permutes the n cyclic subgroups by a conjugation. In this paper, we will deal with cyclic groups of order 2, so N will act by conjugation as permutations of the n involutory symmetric generators. G above is a homomorphic image of a progenitor $2^{*n}:N$ factored by relations:

$$\frac{2^{*n}:N}{\pi_1 w_1, \pi_2 w_2, \ldots, \pi_s w_s} \quad (\langle N, T \mid N_p, t_i^2 = 1, t_i\pi = t_{\pi(i)}, \pi_1 w_1 = \pi_2 w_2 = \ldots = \pi_s w_s = 1\rangle,$$

where N_p is the representation of the control group N. The index of the control group N in G is calculated as follows. Define $N^{(w)} = \{\pi \in N: Nw\pi = Nw\}$, where w is a word in symmetric generators. The number of single cosets in a double coset $[w] = N w N$ is given by

$[N : N^{(w)}] = |N|/|N^{(w)}|$,

since $Nw\pi_1 \neq Nw\pi_2 \Leftrightarrow Nw\pi_1\pi_2^{-1} \neq Nw$

$\Leftrightarrow \pi_1\pi_2^{-1} \notin N^{(w)}$

$\Leftrightarrow N^{(w)}\pi_1 \notin N^{(w)}\pi_2$.

We calculate for each double coset [w] the orbits of $N^{(w)}$ on the symmetric generators and then identify the double coset for a representative t_i from each orbit to which Nwt_i belongs. We will know that the double coset enumeration is completed when the set of single cosets is closed under right multiplication by the t_i, [2].

CHAPTER TWO

DOUBLE COSET ENUMERATION OF $L_2(11)\times 3$ OVER A_4

In this chapter we will demonstrate double coset enumeration of the group $L_2(11)\times 3$ that can be used to write elements of $L_2(11)\times 3$ as a permutation of the group A_4 followed by at most eight of the symmetric generators. A symmetric presentation of the progenitor $2^{*4}: A_4$ is given by:

$<x, y, t \mid x^3, y^3, (x*y)^2, t^2, (t, y)>$,

where the control group $N = A_4 \cong <x, y \mid x^3, y^3, (x*y)^2>$,

$x \sim (1, 2, 3)$

and $y \sim (4, 1, 2)$.

To find finite homomorphic image of the progenitor, we apply Curtis' Lemma and obtain:

$C_N(N^{41}) = <(1, 2, 3), (4, 1, 2)>$.

We factor the progenitor by the relation

$[(4, 1)(2, 3)t_4]^5 = 1$ to obtain the homomorphic image G:

$$G = \frac{2^{*4}: A_4}{[(4,1)(2,3)t_4]^5 = 1}$$

The index of N in G is 165 and $G \cong L_2(11)\times 3$.

The relation $[(4, 1)(2, 3)t_4]^5 = 1$ gives:

$[(4, 1)(2, 3)t_4]^5 = 1$

$\Rightarrow (4, 1)(2, 3)t_4(4, 1)(2, 3)t_4(4, 1)(2, 3)t_4$

$(4, 1)(2, 3)t_4(4, 1)(2, 3)t_4 = 1$

$\Rightarrow (4, 1)(2, 3)t_4t_1t_4t_1t_4 = 1$

$\Rightarrow (4, 1)(2, 3) = t_4t_1t_4t_1t_4$

$\Rightarrow N(4, 1)(2, 3) = Nt_4t_1t_4t_1t_4$

$\Rightarrow N = Nt_4t_1t_4t_1t_4$

$\Rightarrow Nt_4t_1 = Nt_4t_1t_4$ (1)

Since N is 2-transitive on 4 letters, so, [1] contains four single cosets, and [1 2] contains 12 single cosets. Order of $N^1 = N^{(1)}$ is three, and orbits of $N^{(1)}$ are {1} and {2,3,4}. $N^{(12)} = N^{(12)} = Id(N)$, and it has orbits {1}, {2}, {3}, and {4}.

Now, we will compute the double cosets [1 2 1], [1 2 2], [1 2 3] and [1 2 3 4].

Now,

$(4,1)(2,3)^{(4,1,2)} = (t_4t_1t_4t_1t_4)^{(4,1,2)}$

$\Rightarrow (1,2)(4,3) = t_1t_2t_1t_2t_1$

$\Rightarrow (1,2)(4,3)t_1t_2 = t_1t_2t_1$

$\Rightarrow [1\ 2\ 1] = [1\ 2];$

$t_1t_2t_2 = t_1 \Rightarrow [1\ 2\ 2] = [1]$;

and [1 2 3] and [1 2 4] are new double cosets (orbits).

Now $N^{123} = N^{(123)} = Id(N)$, and therefore there are twelve single cosets in the double coset [1 2 3]. The orbits of $N^{(123)}$ are {1}, {2}, {3}, and {4}. Since [1 2 3 3] = [1 2], we will consider the double cosets [1 2 3 1], [1 2 3 2] and [1 2 3 4]:

$t_1t_2t_3t_2 = t_1(2,3)(1,4)t_2t_3 = t_4t_2t_3$. Thus, $Nt_4t_2t_3 \in [1\ 2\ 4]$, since there exist $n = (1,4,3) \in N$ such that $N(t_1t_2t_4)^n = Nt_4t_2t_3$. So, [1 2 3 2] = [1 2 4].

Since,

$t_1t_2t_3t_1 = (1,2)(3,4)t_1t_2t_1t_3t_1 = (1,2)(3,4)t_1t_2(1,3)(2,4)t_1t_3$

$= (1,2)(3,4)\cdot(1,3)(2,4)t_3t_4t_1t_3$,

and there exists $n = (1,3)(2,4) \in N$ such that

$N(t_1t_2t_3t_1)^n = Nt_3t_4t_1t_3 \Rightarrow n \in N^{(1231)}$.

Then, it follows that $N^{(1231)} \geq \langle(1,3)(2,4)\rangle$, and its order is at least two, so the number of single cosets in the double coset $Nt_1t_2t_3t_1N$ is at most $|N|/|N^{(1231)}| = 12/2 = 6$.

Also, $N^{1231} = N^{(1231)} = \langle Id(N)\rangle$, so the number of single cosets in the double coset [1 2 3 4] is at most 12.

Now, we will consider [1 2 4]. $N^{124} = N^{(124)} = \langle Id(N)\rangle$, and therefore there are twelve single cosets in the double coset [1 2 4]. The orbits of $N^{(124)}$ are {1}, {2}, {3}, and {4}. Since [1 2 4 4] = [1 2], we will calculate the double cosets [1 2 4 1], [1 2 4 2] and [1 2 4 3]:

$t_1t_2t_4t_2 = t_1(2,4)(1,3)t_2t_4 = t_3t_2t_4$.

$Nt_3t_2t_4 \in [1\ 2\ 3]$, since there exist $n = (1,3,4) \in N$ such that $N(t_1t_2t_3)^n = Nt_3t_2t_4$, so [1 2 4 2] = [1 2 3].

Now,

$t_1t_2t_4t_1 = (1,2)(3,4)t_1t_2t_1t_4t_1 = (1,2)(3,4)t_1t_2(1,4)(2,3)t_1t_4$

$= (1,2)(3,4)\cdot(1,4)(2,3)t_4t_3t_1t_4$.

There exists $n = (1,4)(2,3) \in N$ such that

$N(t_1t_2t_4t_1)^n = Nt_4t_3t_1t_4$.

Then, it follows that $n \in N^{(1241)}$ and

$N^{1241} = N^{(1241)} \geq \langle(1,4)(2,3)\rangle$. So the order of $N^{(1241)}$ is at least two, and therefore the number of single cosets in the double coset [1 2 4 1] is at most $|N|/|N^{(1241)}| = 12/2 = 6$.

Also, $N^{1243} = N^{(1243)} = \langle Id(N)\rangle$, so the number of single cosets in the double coset [1 2 4 3] is at most 12.

By previous calculations, we need to consider the following double cosets: [1 2 3 4], [1 2 3 1], [1 2 4 3] and [1 2 4 1].

$N^{1234} = N^{(1234)} = \langle Id(N)\rangle$; therefore, the double coset

[1 2 3 4] has at most twelve single cosets. The orbits of $N^{(1234)}$ are {1}, {2}, {3} and {4}.

Since [1 2 3 4 4] = [1 2 3], we will calculate the double cosets [1 2 3 4 1], [1 2 3 4 2] and [1 2 3 4 3]:

$N^{12341} = N^{12341} = \langle Id(N) \rangle$, so the double coset [1 2 3 4 1] has at most 12 single cosets;

$N^{12342} = N^{12342} = \langle Id(N) \rangle$, so the double coset [1 2 3 4 2] has at most 12 single cosets;

and finally:

$t_1t_2t_3t_4t_3 = t_1t_2(1,2)(3,4)t_3t_4 = t_2t_1t_3t_4.$

There exists $n = (1,2)(3,4) \in N$ such that

$N(t_1t_2t_4t_3)^n = Nt_2t_1t_3t_4$, so [1 2 3 4 3] = [1 2 4 3].

$N^{1231} = N^{(1231)} \geq \langle (1,3)(2,4) \rangle$; therefore, the double coset [1 2 3 1] has at most $|N|/|N^{(1231)}| = 12/2 = 6$ single cosets. The orbits of $N^{(1231)}$ are {1,3} and {2,4}. Since [1 2 3 1 1] = [1 2 3], we will calculate [1 2 3 1 2]:

there exists $n = (1,3,2) \in N$ such that

$N(t_2t_3t_1t_2t_3)^n = Nt_1t_2t_3t_1t_2$, and

$t_2t_3t_1t_2t_3 = t_2t_3t_1(1,4)(2,3)t_2t_3t_2 = (1,4)(2,3)t_3t_2t_4t_2t_3t_2$

$= (1,4)(2,3)t_3(1,3)(2,4)t_2t_4t_3t_2$

$= (1,4)(2,3)\cdot(1,3)(2,4)t_1t_2t_4t_3t_2.$

It follows that

$t_2t_3t_1t_2t_3 \sim t_1t_2t_4t_3t_2$

$\Rightarrow N(t_2t_3t_1t_2t_3)^n \sim N(t_1t_2t_4t_3t_2)^n$

$\Rightarrow Nt_1t_2t_3t_1t_2 \sim N(t_1t_2t_4t_3t_2)^n$

$\Rightarrow Nt_1t_2t_3t_1t_2 \in [1\ 2\ 4\ 3\ 2]$

$\Rightarrow [1\ 2\ 3\ 1\ 2] = [1\ 2\ 4\ 3\ 2]$.

$N^{1243} = N^{(1243)} = Id(N)$; therefore, the double coset [1 2 4 3] has at most twelve single cosets. The orbits of $N^{(1243)}$ are {1}, {2}, {3} and {4}.

Since [1 2 4 3 3] = [1 2 4], we will calculate the double cosets [1 2 4 3 4], [1 2 4 3 1] and [1 2 4 3 2].

$t_1t_2t_4t_3t_4 = t_1t_2(1,2)(3,4)t_4t_3 = t_2t_1t_4t_3.$

There exists $n = (1,2)(3,4) \in N$ such that

$N(t_1t_2t_3t_4)^n = Nt_2t_1t_4t_3$, so $Nt_2t_1t_4t_3 \in [1\ 2\ 3\ 4]$, and therefore [1 2 4 3 4] = [1 2 3 4].

Now, $N^{12431} = N^{(12431)} = \langle Id(N) \rangle$, so there are at most 12 single cosets in the double coset [1 2 4 3 1].

Finally, $N^{12432} = N^{(12432)} = \langle Id(N) \rangle$, so there are at most 12 single cosets in the double coset [1 2 4 3 2].

$N^{1241} = N^{(1241)} \geq \langle (1,4)(2,3) \rangle$; therefore, the double coset [1 2 4 1] has at most $|N|/|N^{(1241)}| = 12/2 = 6$ single cosets. The orbits of $N^{(1241)}$ are {1,4} and {2,3}. Since [1 2 4 1 1] = [1 2 4], we will calculate [1 2 4 1 2]:

There exists $n = (1,4,2) \in N$ such that

$N(t_2t_4t_1t_2t_4)^n = Nt_1t_2t_4t_1t_2$, and

$t_2t_3t_1t_2t_3 = t_2t_4t_1(1,3)(2,4)t_2t_4t_2 = (1,3)(2,4)t_4t_2t_3t_2t_4t_2$

$= (1,3)(2,4)t_4(1,4)(2,3)t_2t_3t_4t_2$

$=(1,4)(2,3)\cdot(1,3)(2,4)t_1t_2t_3t_4t_2.$

It follows that

$t_2t_4t_1t_2t_4 \sim t_1t_2t_3t_4t_2$

$\Rightarrow N(t_2t_4t_1t_2t_4)^n = N(t_1t_2t_3t_4t_2)^n$

$\Rightarrow Nt_1t_2t_4t_1t_2 = N(t_1t_2t_3t_4t_2)^n$

$\Rightarrow Nt_1t_2t_4t_1t_2 \in [1\ 2\ 3\ 4\ 2]$, i.e. $[1\ 2\ 4\ 1\ 2] = [1\ 2\ 3\ 4\ 2]$.

Both $N^{(12341)}$ and $N^{(12342)}$ are equal to Id(N); thus, each of the double cosets [1 2 3 4 1] and [1 2 3 4 2] has at most twelve single cosets. The orbits of $N^{(12341)}$ and $N^{(12342)}$ are {1}, {2}, {3} and {4}. So, for [1 2 3 4 1] we will have:

$[1\ 2\ 3\ 4\ 1\ 1] = [1\ 2\ 3\ 4]$;

$t_1t_2t_3t_4t_1t_4 = t_1t_2t_3(1,4)(2,3)t_4t_1 = (1,4)(2,3)t_4t_3t_2t_4t_1$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_4t_3t_4t_2t_4t_1$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_4t_3(1,3)(2,4)t_4t_2t_1$

$= (1,4)(2,3)\cdot(1,2)(3,4)\cdot(1,3)(2,4)t_2t_1t_4t_2t_1.$

There exists $n = (1,2)(3,4) \in N$ such that

$N(t_1t_2t_3t_1t_2)^n = Nt_2t_1t_4t_2t_1$, but as we have seen before, $Nt_1t_2t_3t_1t_2 \in [1\ 2\ 4\ 3\ 2]$; thus, $Nt_2t_1t_4t_2t_1 \in [1\ 2\ 4\ 3\ 2]$, and $[1\ 2\ 3\ 4\ 1\ 4] = [1\ 2\ 4\ 3\ 2]$.

Now, we will consider [1 2 3 4 1 2].

$t_1t_2t_3t_4t_1t_2 = (1,4)(2,3)t_4t_2t_3t_2t_4t_1t_2$

$= (1,4)(2,3)\cdot(1,3)(2,4)t_2t_4t_1t_2t_4t_2t_1t_2$

$= (1,4)(2,3)\cdot(1,3)(2,4)\cdot(1,2)(3,4)t_1t_3t_2t_1t_3t_2t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)\cdot(1,2)(3,4)\cdot(1,2)(3,4)t_2t_4t_2t_1t_2t_3t_2t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)\cdot(1,4)(2,3)t_3t_1t_3t_4t_2t_3t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)\cdot(1,4)(2,3)\cdot(1,3)(2,4)t_3t_1t_4t_2t_3t_1$

$= (1,4)(2,3)t_2t_1t_4t_1t_2t_3t_1$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_1t_2t_3t_1t_2t_1t_3t_1$

$= (1,4)(2,3)\cdot(1,2)(3,4)\cdot(1,3)(2,4)t_3t_4t_1t_3t_4t_1t_3$

$= (1,4)(2,3)\cdot(1,2)(3,4)\cdot(1,3)(2,4)\cdot(1,3)(2,4)t_1t_2t_1t_3t_1t_4t_1t_3$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_4t_3t_4t_2t_1t_4t_3$

$= (1,4)(2,3)\cdot(1,2)(3,4)\cdot(1,2)(3,4)t_4t_3t_2t_1t_4t_3$

$= (1,4)(2,3)t_4t_3t_2t_1t_4t_3$.

There exists $n = (1,4)(2,3) \in N$ such that $N(t_1t_2t_3t_4t_1t_2)^n = Nt_4t_3t_2t_1t_4t_3$. Therefore,

$N^{123412} = N^{(123412)} \geq \langle(1,4)(2,3)\rangle$ of order at least two, so there are at most six single cosets in [1 2 3 4 1 2].

Finally, $N^{123413} = N^{(123413)} = \langle Id(N)\rangle$, so the double coset [1 2 3 4 1 3] has at most 12 single cosets.

For [1 2 3 4 2] we will have:

[1 2 3 4 2 2] = [1 2 3 4] and the double cosets [1 2 3 4 2 1], [1 2 3 4 2 3] and [1 2 3 4 2 4].

$t_1t_2t_3t_4t_2t_4 = t_1t_2t_3(1,3)(2,4)t_4t_2 = (1,3)(2,4)t_3t_4t_1t_4t_2$

$= (1,3)(2,4)t_3(1,4)(2,3)t_4t_1t_2$

$= (1,3)(2,4)\cdot(1,4)(2,3)t_2t_4t_1t_2$.

There exists $n = (1,2,4) \in N$ such that

$N(t_1t_2t_4t_1)^n = Nt_2t_4t_1t_2$, so $Nt_2t_4t_1t_2 \in$ [1 2 4 1], and therefore [1 2 3 4 2 4] = [1 2 4 1].

$t_1t_2t_3t_4t_2t_1 = t_1(1,4)(2,3)t_2t_3t_2t_4t_2t_1$

$= (1,4)(2,3)t_4t_2t_3t_2t_4t_2t_1 = (1,4)(2,3)t_4t_2t_3(1,3)(2,4)t_2t_4t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)t_2t_4t_1t_2t_4t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)t_2t_4t_1t_2t_4t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)\cdot(1,3)(2,4)t_2t_4t_2t_1t_2t_4t_1$

$= (1,4)(2,3)t_2t_4(1,2)(3,4)t_2t_1t_4t_1$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_1t_3t_2t_1t_4t_1$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_1t_3t_2(1,4)(2,3)t_1t_4$

$= (1,4)(2,3)\cdot(1,2)(3,4)\cdot(1,4)(2,3)t_4t_2t_3t_1t_4$

$= (1,2)(3,4)t_4t_2t_3t_1t_4$.

There exists $n = (1,3,4) \in N$ such that

$N(t_4t_2t_3t_1t_4)^n = Nt_1t_2t_4t_3t_1$, so $Nt_4t_2t_3t_1t_4 \in [1\ 2\ 4\ 3\ 1]$, and therefore [1 2 3 4 2 1] = [1 2 4 3 1].

Now, $N^{123423} = N^{(123423)} = \langle Id(N) \rangle$, so the double coset [1 2 3 4 2 3] has at most 12 single cosets.

Both $N^{(12431)}$ and $N^{(12432)}$ are equal to Id(N); thus, each of the double cosets [1 2 4 3 1] and [1 2 4 3 2] has at most twelve single cosets. The orbits of $N^{(12431)}$ and $N^{(12432)}$ are {1}, {2}, {3} and {4}.

For [1 2 4 3 1] we will calculate the double cosets [1 2 4 3 1 1] = [1 2 4 3], [1 2 4 3 1 2], [1 2 4 3 1 3] and [1 2 4 3 1 4].

$t_1t_2t_4t_3t_1t_3 = t_1t_2t_4(1,3)(2,4)t_3t_1 = (1,3)(2,4)t_3t_4t_2t_3t_1$

$= (1,3)(2,4)\cdot(1,2)(3,4)t_3t_4t_3t_2t_3t_1$

$= (1,3)(2,4)\cdot(1,2)(3,4)t_3t_4(1,4)(2,3)t_3t_2t_1$

$= (1,3)(2,4)\cdot(1,2)(3,4)\cdot(1,4)(2,3)t_2t_1t_3t_2t_1$

$= (1,3)(2,4)\cdot(1,2)(3,4)\cdot(1,4)(2,3)t_2(1,3)(2,4)t_1t_3t_1t_2t_1$

$= (1,3)(2,4)\cdot(1,2)(3,4)\cdot(1,4)(2,3)\cdot(1,3)(2,4)t_4t_1t_3t_1t_2t_1$

$= (1,3)(2,4)t_4t_1t_3(1,2)(3,4)t_1t_2$

$= (1,3)(2,4)\cdot(1,2)(3,4)t_3t_2t_4t_1t_2.$

There exists $n = (1,3,4) \in N$ such that

$N(t_1t_2t_3t_4t_2)^n = Nt_3t_2t_4t_1t_2$, so $Nt_3t_2t_4t_1t_2 \in [1\ 2\ 3\ 4\ 2]$, and therefore $[1\ 2\ 4\ 3\ 1\ 3] = [1\ 2\ 3\ 4\ 2]$.

$t_1t_2t_4t_3t_1t_4 = t_1t_2(1,2)(3,4)t_4t_3t_4t_1t_4 = (1,2)(3,4)t_2t_1t_4t_3t_4t_1t_4$

$= (1,2)(3,4)t_2t_1t_4t_3(1,4)(2,3)t_4t_1$

$= (1,2)(3,4)\cdot(1,4)(2,3)t_3t_4t_1t_2t_4t_1.$

There exists $n = (1,3)(2,4) \in N$ such that

$N(t_1t_2t_3t_4t_2t_3)^n = Nt_3t_4t_1t_2t_4t_1$, so

$Nt_3t_4t_1t_2t_4t_1 \in [1\ 2\ 3\ 4\ 2\ 3]$, and therefore

$[1\ 2\ 4\ 3\ 1\ 4] = [1\ 2\ 3\ 4\ 2\ 3]$.

$t_1t_2t_4t_3t_1t_2 = (1,2)(3,4)t_1t_2t_1t_4t_3t_1t_2$

$= (1,2)(3,4)t_1t_2(1,4)(2,3)t_1t_4t_1t_3t_1t_2$

$= (1,2)(3,4)\cdot(1,4)(2,3)t_4t_3t_1t_4t_1t_3t_1t_2$

$= (1,2)(3,4)\cdot(1,4)(2,3)t_4t_3t_1t_4(1,3)(2,4)t_1t_3t_2$

$= (1,2)(3,4)\cdot(1,4)(2,3)\cdot(1,3)(2,4)t_2t_1t_3t_2t_1t_3t_2$

$= t_2t_1t_3t_2t_1t_3t_2$

$= t_2(1,3)(2,4)t_1t_3t_1t_2t_1t_3t_2 = (1,3)(2,4)t_4t_1t_3t_1t_2t_1t_3t_2$

$= (1,3)(2,4)t_4t_1t_3(1,2)(3,4)t_1t_2t_3t_2$

$= (1,3)(2,4)\cdot(1,2)(3,4)t_3t_2t_4t_1t_2t_3t_2$

$= (1,3)(2,4)\cdot(1,2)(3,4)t_3t_2t_4t_1(1,4)(2,3)t_2t_3$

$= (1,3)(2,4)\cdot(1,2)(3,4)\cdot(1,4)(2,3)t_2t_3t_1t_4t_2t_3$

$= t_2t_3t_1t_4t_2t_3$

There exists $n = (1,2,3) \in N$ such that

$N(t_1t_2t_3t_4t_1t_2)^n = Nt_2t_3t_1t_4t_2t_3$, so

$Nt_2t_3t_1t_4t_2t_3 \in$ [1 2 3 4 1 2] and

[1 2 4 3 1 2] = [1 2 3 4 1 2].

Now, we will consider the double coset [1 2 4 3 2]. Since [1 2 4 3 2 2] = [1 2 4 3], we will calculate the double cosets [1 2 4 3 2 1], [1 2 4 3 2 3] and [1 2 4 3 2 4].

$t_1t_2t_4t_3t_2t_3 = t_1t_2t_4(1,4)(2,3)t_3t_2 = (1,4)(2,3)t_4t_3t_1t_3t_2$

$= (1,4)(2,3)t_4(1,3)(2,4)t_3t_1t_2$

$= (1,4)(2,3)\cdot(1,3)(2,4)t_2t_3t_1t_2$.

There exists $n = (1,2,3) \in N$ such that

$N(t_1t_2t_3t_1)^n = Nt_2t_3t_1t_2$, so

$Nt_2t_3t_1t_2 \in$ [1 2 3 1], and

[1 2 4 3 2 3] = [1 2 3 1].

$t_1t_2t_4t_3t_2t_4 = t_1t_2(1,2)(3,4)t_4t_3t_4t_2t_4 = (1,2)(3,4)t_2t_1t_4t_3t_4t_2t_4$

$= (1,2)(3,4)t_2t_1t_4t_3(1,3)(2,4)t_4t_2$

$= (1,2)(3,4)\cdot(1,3)(2,4)t_4t_3t_2t_1t_4t_2$.

There exists $n = (1,4)(2,3) \in N$ such that

$N(t_1t_2t_3t_4t_1t_3)^n = Nt_4t_3t_2t_1t_4t_2$, so

$Nt_4t_3t_2t_1t_4t_2 \in$ [1 2 3 4 1 3] and

[1 2 4 3 2 4] = [1 2 3 4 1 3].

$t_1t_2t_4t_3t_2t_1 = t_1(1,3)(2,4)t_2t_4t_2t_3t_2t_1 = (1,3)(2,4)t_3t_2t_4t_2t_3t_2t_1$

$= (1,3)(2,4)t_3t_2t_4(1,4)(2,3)t_2t_3t_1$

$= (1,3)(2,4)\cdot(1,4)(2,3)t_2t_3t_1t_2t_3t_1$

$= (1,3)(2,4)\cdot(1,4)(2,3)\cdot(1,4)(2,3)t_2t_3t_2t_1t_2t_3t_1$

$= (1,3)(2,4)t_2t_3(1,2)(3,4)t_2t_1t_3t_1$

$= (1,3)(2,4)\cdot(1,2)(3,4)t_1t_4t_2t_1t_3t_1$

$= (1,3)(2,4)\cdot(1,2)(3,4)t_1t_4t_2(1,3)(2,4)t_1t_3$

$= (1,3)(2,4)\cdot(1,2)(3,4)\cdot(1,3)(2,4)t_3t_2t_4t_1t_3.$

There exists $n = (1,3,4) \in N$ such that

$N(t_1t_2t_3t_4t_1)^n = Nt_3t_2t_4t_1t_3$, so

$Nt_1t_2t_4t_3t_2t_1 \sim Nt_3t_2t_4t_1t_3 \in [1\ 2\ 3\ 4\ 1]$, and

$[1\ 2\ 4\ 3\ 2\ 1] = [1\ 2\ 3\ 4\ 1]$.

$N^{(123412)}$ has two orbits: $\{1,4\}$ and $\{2,3\}$.

$Nt_1t_2t_3t_4t_1t_2t_2 = Nt_1t_2t_3t_4t_1$, so $[1\ 2\ 3\ 4\ 1\ 2\ 2]=[1\ 2\ 3\ 4\ 1]$.

$t_1t_2t_3t_4t_1t_2t_1 = (1,2)(3,4)t_2t_1t_4t_3t_1t_2$

$= (1,2)(3,4)\cdot(1,4)(2,3)t_3t_1t_4t_1t_3t_1t_2$

$= (1,2)(3,4)\cdot(1,4)(2,3)\cdot(1,3)(2,4)t_1t_3t_2t_1t_3t_2$

$= (1,2)(3,4)t_2t_4t_2t_1t_2t_3t_2 = (1,2)(3,4)\cdot(1,4)(2,3)t_3t_1t_3t_4t_2t_3$

$= (1,2)(3,4)\cdot(1,4)(2,3)\cdot(1,3)(2,4)t_3t_1t_4t_2t_3$

$= t_3t_1t_4t_2t_3.$

There exists $n = (1,3,2) \in N$ such that

$N(t_1t_2t_4t_3t_1)^n = Nt_3t_1t_4t_2t_3$. So, $Nt_1t_2t_3t_4t_1t_2t_1 \sim Nt_3t_1t_4t_2t_3 \in$ [1 2 4 3 1]. Thus, [1 2 3 4 1 2 1] = [1 2 4 3 1].

We will consider [1 2 3 4 1 3]:

$N^{123413} = N^{(123413)} = \langle Id(N) \rangle$, so there are twelve single cosets in the double coset [1 2 3 4 1 3], and there are four orbits of $N^{(123413)}$: {1}, {2}, {3} and {4}.

$Nt_1t_2t_3t_4t_1t_3t_4 \in$ [1 2 3 4 1 3 4].

$t_1t_2t_3t_4t_1t_3t_3 = t_1t_2t_3t_4t_1$, so $Nt_1t_2t_3t_4t_1t_3t_3 \in$ [1 2 3 4 1] or [1 2 3 4 1 3 3] = [1 2 3 4 1].

$t_1t_2t_3t_4t_1t_3t_1 = (1,3)(2,4)t_3t_4t_1t_2t_1t_3$

$= (1,3)(2,4)\cdot(1,2)(3,4)t_4t_3t_1t_2t_3$.

There exists $n = (1,4)(2,3) \in N$ such that

$N(t_1t_2t_4t_3t_2)^n = Nt_4t_3t_1t_2t_3$, so we will have

$Nt_1t_2t_3t_4t_1t_3t_1 \sim Nt_4t_3t_1t_2t_3 \in$ [1 2 4 3 2], and

[1 2 3 4 1 3 1] = [1 2 4 3 2].

$t_1t_2t_3t_4t_1t_3t_2 = (1,2)(3,4)t_2t_1t_3t_4t_3t_1t_3t_2$

$= (1,2)(3,4)\cdot(1,3)(2,4)t_4t_3t_1t_2t_3t_1t_2$

$= (1,2)(3,4)\cdot(1,3)(2,4)\cdot(1,4)(2,3)t_1t_2t_4t_2t_3t_2t_1t_2$

$= (1,2)(3,4)t_2t_1t_3t_1t_4t_2t_1 = (1,2)(3,4)\cdot(1,3)(2,4)t_4t_1t_3t_4t_2t_1$

$= (1,2)(3,4)\cdot(1,3)(2,4)\cdot(1,4)(2,3)t_4t_1t_4t_3t_4t_2t_1$

$= (1,2)(3,4)t_3t_2t_4t_3t_2t_1 = (1,2)(3,4)\cdot(1,3)(2,4)t_1t_2t_4t_2t_3t_2t_1$

$= (1,2)(3,4)\cdot(1,3)(2,4)\cdot(1,4)(2,3)t_4t_3t_1t_2t_3t_1$

$= (1,2)(3,4)t_3t_4t_1t_2t_1t_3t_1 = (1,2)(3,4)\cdot(1,3)(2,4)t_1t_2t_3t_4t_1t_3$.

So, $Nt_1t_2t_3t_4t_1t_3t_2 \sim Nt_1t_2t_3t_4t_1t_3 \in [1\ 2\ 3\ 4\ 1\ 3]$ and $[1\ 2\ 3\ 4\ 1\ 3\ 2] = [1\ 2\ 3\ 4\ 1\ 3]$.

Now, we will study [1 2 3 4 2 3]:

$N^{123423} = N^{(123423)} = <Id(N)>$, so there are twelve single cosets in the double coset [1 2 3 4 2 3], and there are four orbits of $N^{(123423)}$: {1}, {2}, {3} and {4}.

$Nt_1t_2t_3t_4t_2t_3t_1 \in [1\ 2\ 3\ 4\ 2\ 3\ 1]$.

$Nt_1t_2t_3t_4t_2t_3t_3 \sim Nt_1t_2t_3t_4t_2 \in [1\ 2\ 3\ 4\ 2]$ or $[1\ 2\ 3\ 4\ 2\ 3\ 3] = [1\ 2\ 3\ 4\ 2]$.

$t_1t_2t_3t_4t_2t_3t_2 = (1,4)(2,3)t_4t_3t_2t_1t_2t_3$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_3t_4t_2t_1t_3$.

There exists $n = (1,3)(2,4) \in N$ such that $N(t_1t_2t_4t_3t_1)^n = Nt_3t_4t_2t_1t_3$, so we will have $Nt_1t_2t_3t_4t_2t_3t_2 \sim Nt_3t_4t_2t_1t_3 \in [1\ 2\ 4\ 3\ 1]$ or $[1\ 2\ 3\ 4\ 2\ 3\ 2] = [1\ 2\ 4\ 3\ 1]$.

$t_1t_2t_3t_4t_2t_3t_4 = (1,3)(2,4)t_3t_4t_1t_4t_2t_4t_3t_4$

$= (1,3)(2,4)\cdot(1,2)(3,4)t_4t_3t_2t_3t_1t_4t_3$

$= (1,3)(2,4)\cdot(1,2)(3,4)\cdot(1,4)(2,3)t_1t_3t_2t_1t_4t_3$

$= (1,3)(2,4)t_1t_3t_1t_2t_1t_4t_3 = (1,3)(2,4)\cdot(1,2)(3,4)t_2t_4t_1t_2t_4t_3$

$= (1,3)(2,4)\cdot(1,2)(3,4)\cdot(1,4)(2,3)t_3t_4t_1t_4t_2t_4t_3$

$= (1,3)(2,4)t_1t_2t_3t_4t_2t_3$.

Thus, $Nt_1t_2t_3t_4t_2t_3t_4 \sim Nt_1t_2t_3t_4t_2t_3 \in [1\ 2\ 3\ 4\ 2\ 3]$ or $[1\ 2\ 3\ 4\ 2\ 3\ 4] = [1\ 2\ 3\ 4\ 2\ 3]$.

Now, we will show that $N^{1234134} = N^{(1234134)} \geq <(1,4,3)>$.

$t_1t_2t_3t_4t_1t_3t_4 = (1,4)(2,3)t_4t_3t_2t_4t_1t_4t_3t_4$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_3t_4t_1t_3t_2t_4t_3$

$= (1,4)(2,3)\cdot(1,2)(3,4)\cdot(1,2)(3,4)t_3t_4t_3t_1t_3t_2t_4t_3$

$= (1,4)(2,3)\cdot(1,3)(2,4)t_1t_2t_3t_1t_2t_4t_3$

$= (1,4)(2,3)\cdot(1,3)(2,4)\cdot(1,4)(2,3)t_4t_2t_3t_2t_1t_2t_4t_3$

$= (1,4)(2,3)\cdot(1,3)(2,4)\cdot(1,4)(2,3)\cdot(1,2)(3,4)t_3t_1t_4t_2t_1t_4t_3$

$= (1,3)(2,4)\cdot(1,2)(3,4)\cdot(1,3)(2,4)t_1t_3t_4t_2t_4t_1t_4t_3$

$= (1,2)(3,4)\cdot(1,4)(2,3)t_4t_2t_1t_3t_4t_1t_3$.

There exists $n = (1,4,3) \in N$ such that $N(t_1t_2t_3t_4t_1t_3t_4)^n = Nt_4t_2t_1t_3t_4t_1t_3$. Therefore, $N^{1234134} = N^{(1234134)} \geq <(1,4,3)>$, and $[1\ 2\ 3\ 4\ 1\ 3\ 4]$ has at most $|N|/|N^{(1234134)}| = 12/3 = 4$ single cosets. $N^{(1234134)}$ has two orbits: $\{1,4,3\}$ and $\{2\}$.

$Nt_1t_2t_3t_4t_1t_3t_4t_4 = Nt_1t_2t_3t_4t_1t_3 \in [1\ 2\ 3\ 4\ 1\ 3]$, i.e. $[1\ 2\ 3\ 4\ 1\ 3\ 4\ 4] = [1\ 2\ 3\ 4\ 1\ 3]$ and

$Nt_1t_2t_3t_4t_1t_3t_4t_2 \in [1\ 2\ 3\ 4\ 1\ 3\ 4\ 2]$.

Now, we will show that $N^{12341342} = N^{(12341342)} = N$, and hence there is one single coset in the double coset [1 2 3 4 1 3 4 2].

$t_1t_2t_3t_4t_1t_3t_4t_2 = (1,4)(2,3)t_4t_3t_2t_4t_1t_4t_3t_4t_2$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_3t_4t_1t_3t_2t_4t_3t_2$

$= (1,4)(2,3)\cdot(1,2)(3,4)\cdot(1,3)(2,4)t_1t_2t_3t_1t_2t_4t_2t_3t_2$

$= (1,4)(2,3)t_4t_3t_2t_4t_3t_1t_2t_3$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_4t_3t_4t_2t_4t_3t_1t_2t_3$

$= (1,4)(2,3)\cdot(1,2)(3,4)\cdot(1,3)(2,4)t_2t_1t_4t_2t_3t_1t_2t_3$

$= (1,4)(2,3)t_3t_4t_1t_2t_3t_2t_1t_2t_3$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_4t_3t_2t_1t_4t_2t_1t_3$.

There exists $n = (1,4)(2,3) \in N$ such that $N(t_1t_2t_3t_4t_1t_3t_4t_2)^n = Nt_4t_3t_2t_1t_4t_2t_1t_3$. Therefore, $N^{(12341342)}$ can be obtained by fixing the point 2 in $N^{(1234134)} \geq \langle(1,4,3)\rangle$ and adding to it n:

$N^{12341342} = N^{(12341342)} = \langle(1,4,3),\ (1,4)(2,3)\rangle = N$.

$N^{(12341342)}$ has a single orbit {1, 2, 3, 4}, so

$Nt_1t_2t_3t_4t_1t_3t_4t_2t_2 = Nt_1t_2t_3t_4t_1t_3t_4 \in$ [1 2 3 4 1 3 4] or [1 2 3 4 1 3 4 2 2] = [1 2 3 4 1 3 4].

Now, we will show that the double coset [1 2 3 4 2 3 1] has at most four single cosets. We need to show that the order of $N^{(1234231)}$ is at least three.

$t_1t_2t_3t_4t_2t_3t_1 = (1,4)(2,3)t_4t_2t_3t_2t_4t_2t_3t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)t_2t_4t_1t_2t_4t_3t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)\cdot(1,3)(2,4)t_2t_4t_2t_1t_2t_4t_3t_1$

$= (1,4)(2,3)\cdot(1,2)(3,4)t_1t_3t_2t_1t_4t_3t_1$

$= (1,4)(2,3)\cdot(1,2)(3,4)\cdot(1,4)(2,3)t_4t_2t_3t_1t_4t_1t_3t_1$

$= (1,2)(3,4)\cdot(1,3)(2,4)\cdot(1,3)(2,4)t_4t_2t_1t_3t_1t_2t_1t_3$

$= (1,2)(3,4)\cdot(1,2)(3,4)t_3t_1t_2t_4t_1t_2t_3 = t_3t_1t_2t_4t_1t_2t_3.$

There exists $n = (1,3,2) \in N$ such that

$N(t_1t_2t_3t_4t_2t_3t_1)^n = Nt_3t_1t_2t_4t_1t_2t_3$, so

$N^{1234231} = N^{(1234231)} \geq <(1,3,2)>$; therefore, there are at most

$|N|/|N^{(1234231)}| = 12/3 = 4$ single cosets in the double coset

[1 2 3 4 2 3 1]. The orbits of $N^{(1234231)}$ are {1,2,3} and {4}.

$Nt_1t_2t_3t_4t_2t_3t_1t_1 = Nt_1t_2t_3t_4t_2t_3 \in$ [1 2 3 4 2 3], and

$Nt_1t_2t_3t_4t_2t_3t_1t_4 \in$ [1 2 3 4 2 3 1 4].

Now, we will show that $N^{(12342314)} = N$, and therefore the double coset [1 2 3 4 2 3 1 4] has one single coset.

$t_1t_2t_3t_4t_2t_3t_1t_4 = (1,3)(2,4)t_3t_4t_1t_4t_2t_4t_3t_1t_4$

$= (1,3)(2,4)\cdot(1,2)(3,4)t_4t_3t_2t_3t_1t_4t_3t_4t_1t_4$

$= (1,3)(2,4)\cdot(1,2)(3,4)\cdot(1,4)(2,3)t_1t_2t_3t_2t_4t_1t_2t_4t_1$

$= (1,4)(2,3)t_4t_3t_2t_3t_4t_1t_4t_2t_4t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)t_2t_1t_4t_1t_2t_3t_4t_2t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)\cdot(1,4)(2,3)t_3t_1t_4t_2t_3t_4t_2t_1$

$= (1,3)(2,4)\cdot(1,4)(2,3)t_2t_4t_1t_2t_3t_2t_4t_2t_1$

$= (1,3)(2,4)\cdot(1,4)(2,3)\cdot(1,3)(2,4)t_4t_2t_3t_4t_1t_2t_4t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)t_4t_2t_4t_3t_4t_1t_2t_4t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)\cdot(1,2)(3,4)t_3t_1t_4t_3t_1t_2t_4t_1$

$= (1,4)(2,3)t_2t_1t_4t_1t_3t_1t_2t_4t_1$

$= (1,4)(2,3)\cdot(1,3)(2,4)t_4t_3t_2t_1t_3t_2t_4t_1$.

There exists $n = (1,4)(2,3) \in N$ such that

$N(t_1t_2t_3t_4t_2t_3t_1t_4)^n = Nt_4t_3t_2t_1t_3t_2t_4t_1$, so

$N^{12342314} = N^{(12342314)} \geq \langle(1,3,2),\ (1,4)(2,3)\rangle = N$. Thus, there is one single coset in the double coset [1 2 3 4 2 3 1 4].

The orbit of $N^{(12342314)}$ is $\{1,2,3,4\}$.

$Nt_1t_2t_3t_4t_2t_3t_1t_4t_4 = Nt_1t_2t_3t_4t_2t_3t_1 \in$ [1 2 3 4 2 3 1].

We have completed the manual double coset enumeration of $L_2(11)\times 3$ over A_4. The Cayley graph is shown in Figure 1.

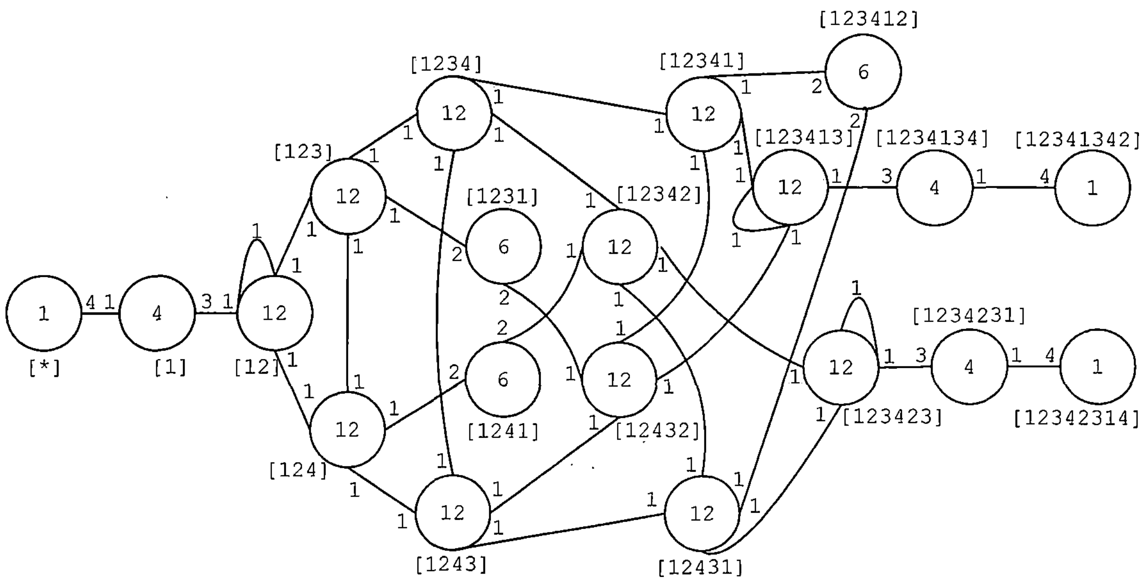

Figure 1. Cayley Graph of $L_2(11)\times 3$ Over A_4

From the Cayley graph above, it can be seen that each element of $L_2(11)$x3 can be written as a permutation of A_4, on 4 letters followed by a word in symmetric generators of length at most 8.

For the action of the four symmetric generators on the cosets of $L_2(11)$x3 over A_4, we label the 165 cosets as shown in Figure 2.

1	[]	43	[4,2,3,1]	84	[1,3,4,2]	125	[3,1,2,4,3]
2	[1]	44	[3,2,1,3]	85	[2,3,1,4]	126	[4,3,2,1,4]
3	[2]	45	[3,4,2,3]	86	[1,2,3,4]	127	[4,2,1,3,4]
4	[3]	46	[1,2,4,1]	87	[1,4,2,3,1]	128	[3,4,1,2,3,1]
5	[2,1]	47	[3,4,2,1]	88	[4,2,3,1,2]	129	[1,3,4,2,1,4]
6	[4]	48	[4,3,2,1]	89	[3,4,2,1,4]	130	[4,1,2,3,4]
7	[3,1]	49	[3,2,4]	90	[3,1,4,2,1]	131	[1,2,3,4,1,2]
8	[3,2]	50	[2,3,1,2]	91	[1,2,4,3,2]	132	[1,4,2,3,1,2]
9	[4,1]	51	[2,4,3,1]	92	[1,2,4,3,1]	133	[2,3,1,4,2,1]
10	[1,3]	52	[1,3,4,1]	93	[2,1,3,4,1]	134	[3,1,2,4,3,2]
11	[1,2]	53	[2,4,3,2]	94	[4,3,1,2,3]	135	[2,3,1,4,3,1]
12	[2,3]	54	[2,3,4,1]	95	[4,2,1,3,2]	136	[4,3,2,1,3,2]
13	[4,3]	55	[4,3,1,2]	96	[1,4,2,3,4]	137	[4,1,3,2,1,3]
14	[3,2,1]	56	[4,1,2,3]	97	[2,3,4,1,2]	138	[1,4,2,3,4,2]
15	[2,4]	57	[3,1,4,3]	98	[1,2,3,4,2]	139	[2,4,3,1,4,3]
16	[4,2]	58	[3,2,4,1,2]	99	[3,4,1,2,4]	140	[1,2,3,4,2,3]
17	[1,4]	59	[2,3,4,2]	100	[3,1,2,4,1]	141	[2,1,4,3,1,4]
18	[3,4]	60	[1,4,3,2]	101	[2,4,1,3,2]	142	[3,2,4,1,3,2]
19	[2,3,1]	61	[2,4,1,3]	102	[1,3,2,4,1]	143	[3,4,1,2,3,4]
20	[4,3,1]	62	[4,1,3,2]	103	[3,4,2,1,3]	144	[4,1,3,2,4]
21	[1,3,2]	63	[3,2,4,1]	104	[1,3,4,2,1]	145	[3,1,2,4,3,1]
22	[2,4,1]	64	[4,2,1,3]	105	[1,2,3,4,1]	146	[4,2,1,3,4,1]
23	[2,1,3]	65	[3,4,1,2]	106	[2,4,3,1,2]	147	[2,4,3,1,2,3]
24	[4,2,1]	66	[1,4,2,3]	107	[3,4,1,2,3]	148	[2,1,4,3,2,4]
25	[3,4,1]	67	[4,3,2,4]	108	[1,4,3,2,4]	149	[3,2,4,1,3,4]
26	[3,1,2]	68	[4,1,3,4]	109	[2,4,1,3,4]	150	[2,4,3,1,2,4]
27	[1,2,3]	69	[2,3,4,1,3]	110	[1,3,2,4,3]	151	[2,3,1,4,2,3]
28	[4,1,2]	70	[3,1,4,2]	111	[3,2,4,1,3]	152	[4,1,3,2,4,3]
29	[4,2,3]	71	[1,2,4,3]	112	[3,1,4,2,3]	153	[3,2,4,1,2,4]
30	[1,4,3]	72	[3,2,1,4]	113	[3,2,1,4,2]	154	[1,4,2,3,4,2,1]
31	[2,4,1,2]	73	[2,1,3,4]	114	[1,2,3,4,1,3]	155	[1,3,4,2,3,4]
32	[1,2,4]	74	[4,1,2,3,1]	115	[2,3,1,4,3]	156	[1,2,3,4,2,3,1]
33	[3,4,2]	75	[4,3,2,1,3]	116	[4,2,1,3,2,1]	157	[4,3,2,1,4,2]
34	[3,1,4]	76	[1,3,4,2,3]	117	[4,2,3,1,4]	158	[3,4,1,2,3,1,2]
35	[4,1,3]	77	[2,4,3,1,4]	118	[3,4,1,2,4,1]	159	[1,2,3,4,1,3,4]
36	[4,3,2]	78	[2,1,4,3,1]	119	[3,2,1,4,3]	160	[2,4,3,1,4,3,2]
37	[1,4,2]	79	[1,4,3,2,1]	120	[3,1,2,4,1,2]	161	[4,1,3,2,1,3,4]
38	[2,4,3]	80	[1,3,2,4]	121	[4,3,1,2,4]	162	[2,3,1,4,2,1,4]

Figure 2. Action of Symmetric Generators

39 [2,1,4]	81 [2,1,4,3]	122 [2,1,3,4,2]	163 [3,1,2,4,3,2,4]
40 [1,3,4]	82 [4,1,3,2,1]	123 [2,1,4,3,2]	164 [1,2,3,4,2,3,1,4]
41 [1,2,3,1]	83 [3,1,2,4]	124 [2,3,1,4,2]	165 [1,2,3,4,1,3,4,2]
42 [2,3,4]			

Figure 2. Action of Symmetric Generators (Continued)

where

t_1 = (1,3)(2,11)(4,8)(6,16)(7,26)(9,28)(10,21)(13,36)(14,24)(17,37)(18,33)(19,50)(20,55)(22,31)(23,45)(25,65)(27,29)(30,60)(32,49)(34,70)(35,62)(38,53)(39,67)(40,84)(41,90)(42,59)(43,88)(44,76)(46,82)(47,48)(51,106)(52,108)(54,97)(56,66)(57,99)(58,63)(61,101)(64,95)(68,94)(69,104)(71,91)(72,113)(73,122)(74,133)(75,136)(77,79)(78,112)(80,83)(81,123)(85,124)(86,98)(87,132)(89,148)(92,151)(93,144)(96,138)(100,120)(102,116)(103,140)(105,131)(107,109)(110,147)(111,142)(115,121)(117,143)(119,150)(125,134)(126,157)(127,145)(128,158)(129,162)(130,153)(135,156)(139,160)(141,154)(152,163)(159,165)(161,164);

t_2 = (1,4)(2,10)(3,12)(5,23)(6,13)(9,35)(11,27)(14,44)(15,38)(16,29)(17,30)(19,20)(21,36)(22,61)(24,64)(25,41)(26,53)(28,56)(31,78)(32,71)(33,45)(34,57)(37,66)(39,81)(40,42)(43,51)(46,96)(47,103)(48,75)(49,68)(50,91)(52,74)(54,69)(55,94)(58,92)(59,95)(60,62)(63,111)(65,107)(67,109)(70,112)(72,119)(73,86)(76,84)(77,139)(79,135)(80,110)(82,137)(83,125)(85,115)(87,89)(88,134)(90,123)(93,128)(97,145)(98,140)(99,101)(100,130)(102,143)(104,150)(105,114)(106,147)(108,149)(113,127)(117,155)(118,161)(120,156)(121,142)(122,136)(124,151)(126,131)(132,158)(144,152)(146,159)(148,163)(153,160)(154,164)(162,165);

t_3 = (1,6)(2,17)(3,15)(4,18)(5,39)(7,34)(8,49)(10,40)(11,32)(12,42)(14,72)(19,85)(20,46)(21,80)(22,25)(23,73)(24,52)(26,83)(27,86)(28,59)(29,57)(30,38)(31,98)(33,37)(35,68)(36,67)(41,110)(43,117)(44,100)(45,115)(47,89)(48,126)(50,93)(51,77)(53,113)(54,63)(55,121)(56,130)(58,153)(60,108)(61,109)(62,144)(64,127)(65,99)(66,96)(69,152)(70,84)(71,81)(74,125)(75,102)(76,155)(78,141)(79,145)(82,122)(87,142)(88,105)(90,146)(91,157)(92,118)(94,124)(95,119)(97,138)

(101,131)(103,151)(104,129)(106,150)(107,143)(111,149)(112,139)(114,159)(116,154)(123,148)(133,162)(134,163)(136,160)(137,161)(156,164)(158,165).

t_4 = (1,2)(3,5)(4,7)(6,9)(8,14)(12,19)(13,20)(15,22)(16,24)(18,25)(21,31)(23,35)(26,28)(27,41)(29,43)(30,44)(32,46)(33,47)(34,39)(36,48)(37,50)(38,51)(40,52)(42,54)(45,58)(49,63)(53,69)(55,65)(56,74)(57,75)(59,77)(60,79)(61,64)(62,82)(66,87)(67,88)(68,89)(70,90)(71,92)(72,85)(73,93)(76,97)(78,81)(80,102)(83,100)(84,104)(86,105)(91,111)(94,114)(95,116)(96,103)(98,117)(99,118)(101,120)(106,108)(107,128)(109,129)(110,126)(112,131)(113,132)(115,135)(119,137)(121,141)(122,142)(123,143)(124,133)(125,145)(127,146)(130,150)(138,154)(140,156)(144,151)(147,158)(149,159)(155,161)(157,162)(160,164)(163,165).

<u>Proof of Isomorphism</u>: The maximum possible index of N in G is calculated as follows:

$|N|/|N| + |N|/|N^{(1)}| + |N|/|N^{(12)}| + |N|/|N^{(123)}| + |N|/|N^{(124)}| + |N|/|N^{(1234)}| + |N|/|N^{(1243)}| + |N|/|N^{(1231)}| + |N|/|N^{(1241)}| + |N|/|N^{(12342)}| + |N|/|N^{(12432)}| + |N|/|N^{(12341)}| + |N|/|N^{(12431)}| + |N|/|N^{(123413)}| + |N|/|N^{(123412)}| + |N|/|N^{(1234134)}| + |N|/|N^{(1234231)}| + |N|/|N^{(12341342)}| + |N|/|N^{(12342314)}|$

$= 12/12 + 12/3 + 12/1 + 12/1 + 12/1 + 12/1 + 12/1 + 12/2 + 12/2 + 12/1 + 12/1 + 12/1 + 12/1 + 12/1 + 12/2 + 12/3 + 12/3 + 12/12 + 12/12 = 1 + 4 + 12 + 12 + 12 + 12 + 12 + 6 + 6 + 12 + 12 + 12 + 12 + 12 + 6 + 4 + 4 + 1 + 1 = 165.$

The order of the image group G is at most $|N|*165 = 1980$, i.e. $|G| \leq 1980$.

We will consider G as a permutation group on the 165 cosets of N. The action of the control group N on the cosets is:

x = (3,4,6)(5,7,9)(8,13,15)(10,17,11)(12,18,16)(14,20,22)(19,25,24)(21,30,32)(23,34,28)(26,35,39)(27,40,37)(29,42,33)(31,44,46)(36,38,49)(41,52,50)(43,54,47)(45,57,59)(48,51,63)(53,68,67)(55,61,72)(56,73,70)(58,75,77)(60,71,80)

(62,81,83)(64,85,65)(66,86,84)(69,89,88)(74,93,90)(76,96,98)(78,100,82)(79,92,102)(87,105,104)(91,110,108)(94,109,113)(95,115,99)(97,103,117)(101,119,121)(106,111,126)(107,127,124)(112,130,122)(114,129,132)(116,135,118)(120,137,141)(123,125,144)(128,146,133)(131,150,142)(134,152,148)(136,139,153)(138,140,155)(143,145,151)(147,149,157)(154,156,161)(158,159,162);

y = (2,3,4)(5,8,10)(7,11,12)(9,16,13)(14,21,23)(15,18,17)(19,26,27)(20,28,29)(22,33,30)(24,36,35)(25,37,38)(31,45,44)(32,42,34)(39,49,40)(41,50,53)(43,55,56)(46,59,57)(47,60,61)(48,62,64)(51,65,66)(52,67,68)(54,70,71)(58,76,78)(63,84,81)(69,90,91)(72,80,73)(74,88,94)(75,82,95)(77,99,96)(79,101,103)(83,86,85)(87,106,107)(89,108,109)(92,97,112)(93,113,110)(98,115,100)(102,122,119)(104,123,111)(105,124,125)(114,133,134)(116,136,137)(117,121,130)(118,138,139)(120,140,135)(126,144,127)(128,132,147)(129,148,149)(131,151,145)(141,153,155)(142,150,143)(146,157,152)(154,160,161)(159,162,163).

Moreover, the action of x and y and therefore of N on the symmetric generators is:

x: (t_1, t_2, t_3);

y: (t_4, t_1, t_2).

The order of xy is 2, so we will have $N = \langle x,y\rangle \cong A_4$. Now we will check the relation $[(4,1)(2,3)t_4]^5 = 1$ or equivalently $(4,1)(2,3) = t_4t_1t_4t_1t_4$.

$t_1^{t_4t_1t_4t_1t_4} = t_4$;

$t_2^{t_4t_1t_4t_1t_4} = t_3$;

$t_3^{t_4t_1t_4t_1t_4} = t_2$;

$t_4^{t_4t_1t_4t_1t_4} = t_1$.

This shows that $t_4t_1t_4t_1t_4$ acts as the permutation (4,1)(2,3) on the symmetric generators, which proves the relation

$(4,1)(2,3) = t_4t_1t_4t_1t_4$.

The elements x, y and t_4 generate the group $L_2(11)\times 3$, so $L_2(11)\times 3$ is an image of G. Thus, $|G| \geq |L_2(11)\times 3| = 1980$. Previously, we established that $|G| \leq 1980 = |L_2(11)\times 3|$. So, we have $|G| \leq 1980 = |L_2(11)\times 3| \leq |G|$; thus, $G \cong L_2(11)\times 3$.

CHAPTER THREE

DOUBLE COSET ENUMERATION OF $U_3(3):2$ OVER $PGL_2(7)$

The detailed manual double coset enumeration of $U_3(3):2$, $J_2:2$ and of $G_2(4):2$ are shown in [7]. In this chapter, we will replicate the double coset enumeration of $U_3(3):2$ over $PGL_2(7)$. In the following chapters we will perform the double coset enumeration of $J_2:2$ over $U_3(3):2$ and of $G_2(4):2$ over $J_2:2$. For $J_2:2$ and $G_2(4):2$, we will use different progenitors and different generators for control groups than those that were used in [7]. This will enable a nested algorithm that will be described later.

A symmetric presentation for the progenitor $2^{*(7+7)}:PGL_2(7)$ is given by:

$<x,y,t,s \mid x^7,\ y^2,\ t^2,\ (x^{-1}*t)^2,\ (y*x)^3,\ t*x^{-1} * y*x*t*y,$
$x^2*y*x^3*y*x^{-4}*y*x^{-4}*y*x,\ s^2,\ (s^{x\wedge 3},y),\ (s^{x\wedge 4},x*y)>.$

The action of the control group $N = PGL_2(7)$ on the symmetric generators is given by

$x \sim (1,2,3,4,5,6,7)(14,13,12,11,10,9,8),$

$y \sim (2,6)(4,5)(14,10)(12,13),$

$t \sim (7,14)(1,8)(2,9)(3,10)(4,11)(5,12)(6,13).$

The progenitor is factored by the relations $t=s_7s_{14}s_7$, $y=(s_8s_7)^2$ and $y=s_3s_8s_1s_7$, to obtain the finite homomorphic image (given in [6]):

$$G = \frac{2^{*(7+7)}:PGL_2(7)}{t=s_7s_{14}s_7, y=(s_8s_7)^2, y=s_3s_8s_1s_7} \cong U_3(3):2$$

The index of $PGL_2(7)$ in $U_3(3):2$ is 36.

Note, that a relator conjugated by an element of N is also a relator. It is convenient to introduce $\pi_{i,j} = t^n$ and $\delta_{i,j} = y^m$, where n and m $\in$ N. In Figure 3, MAGMA commands can be used to find all the relators. The functions Pi and P2 written in MAGMA return permutations $\pi_{i,j}$ and $\delta_{i,j}$ respectively.

```
trans := Transversal(N, Stabilizer(N, [7,14]));
prs := {@ [7,14]^x : x in trans @};
sgs := [t^x : x in trans];
Pi := func< i,j | Index(prs, [i,j]) ne 0 select
sgs[Index(prs, [i,j])] else Id(N)>;

tr2:= Transversal(N,Stabiliser(N, [7,8]));
prs2:= {@ [7,8]^x : x in tr2 @};
sgs2 := [y^x : x in tr2];

P2 := func< i,j | Index(prs2, [i,j]) ne 0
select sgs2[Index(prs2, [i,j])] else Id(N)>;
```

Figure 3. $\pi_{i,j}$ and $\delta_{i,j}$ of $U_3(3):2$

We will start with [*], the identity double coset. There are 14 symmetric generators that take from [*] to [7], since N is transitive on 14 letters.

Fixing 7 in N, we will obtain $N^7 = N^{(7)}$ that has four orbits: O(7) of size 1, O(8) of size 3, O(10) of size 4 and O(1) of size 6. We will apply one representative s_i from each orbit to find out to which double coset Ns_7s_i belongs.

$s_7s_7 = Id(G)$ or [7 7] = [*].

Ns_7s_1 belongs to the double coset [7 1].

Ns_7s_8 belongs to the double coset [7 1] since $Ns_7s_1 = Ns_3s_8$ and there exists n in N such that $N(s_7s_8)^n = Ns_3s_8$, so we will have [7 8] = [7 1].

$s_7s_{10} = \pi_{7,10}s_7$. So, Ns_7s_{10} belongs to [7] or [7 10] = [7].

Now we will fix 1 in N^7 to obtain

$N^{7,1} \geq \langle y, (2,4)(5,6)(9,11)(12,13)\rangle$ that at this point has the following orbits: O(7), O(1), O(3), O(8), O(9), O(10), O(11), O(12) and O(2). To find useful relations we will try a representative i from each orbit to find out whether $\pi_{1,i}$ exists. We have found that $\pi_{1,12}$ exists and we have the following:

$s_7s_1 = s_7s_1s_{12}s_1s_1s_{12} = s_7\pi_{1,12}s_1s_{12} = \pi_{1,12}s_{13}s_1s_{12}$

$= \pi_{1,12}\pi_{13,1}s_{13}s_{12}$.

Thus, $s_7s_1 \sim s_{13}s_{12}$ and $Ns_7s_1 = Ns_{13}s_{12}$, and there exist n in N such that $N(s_7s_1)^n = Ns_{13}s_{12}$. So, we will have $N^{(7,1)} \geq < N^{7,1}, (1,12,7,13)(2,10,4,9,5,14,6,11)(3,8)>$ whose order is at least 16, which induces $|N/ N^{(7,1)}| = 21$, the number of single cosets in the double coset [7 1]. $N^{(7,1)}$ has the orbits O(3) of size 2, O(1) of size 4 and O(2) of size 8. We will apply one representative s_i from each orbit to find out to which double coset $Ns_7s_1s_i$ belongs.

$s_7s_1s_1 = s_7$ or [7 1 1] = [7], so for all i in O(1) $Ns_7s_1s_i$ belongs to [7].

$s_7s_1s_3 = ys_3s_8s_3 = y\delta_{i,j}s_8s_3s_3 = y\delta_{i,j}s_8$,

so $s_7s_1s_3 \sim s_8$, and there exists n in N such that $N(s_8)^n = Ns_7$. Thus, [7 1 3] = [7], and for all i in O(3) $Ns_7s_1s_i$ belongs to [7].

$s_7s_1s_2 = ys_3s_8s_2 = ys_3\pi_{8,2}s_8 = y\pi_{8,2}s_{13}s_8$,

so $s_7s_1s_2 \sim s_{13}s_8$, and there exists n in N such that $N(s_{13}s_8)^n = Ns_7s_1$. Therefore, [7 1 2] = [7 1], and for all i in O(2) $Ns_7s_1s_i$ belongs to [7 1]. In the following table the above double coset enumeration is summarized.

Label [w]	Coset stabilizing subgroup $N^{(w)}$	Number of cosets
[*]	N	1
[7]	$N^{(7)} \cong S_4$, with orbits {7}, {8,9,11}, {10,12,13,14} and {1,2,3,4,5,6}.	14
[7 7] = [*]		
[7 10] = [7]		
[7 8] = [7 1]		
[7 1]		21
	$N^{(7,1)} \cong \langle y, (1,12,7,13)(2,10,4,9,5,14,6,11)(3,8)\rangle$ since 7 1 ~ 13 12. [7 1] has four names: 7 1 ~ 1 7 ~ 12 13 ~ 13 12. $N^{(7,1)}$ has orbits {3,8},{1,7,12,13} and {2,4,5,6,9,10,11,14}.	
[7 1 1] = [7]		
[7 1 3] = [7]		
[7 1 2] = [7 1]		

Figure 4. The Double Cosets [w] = NwN in $U_3(3):2$

The Cayley's graph will be as shown in Figure 5.

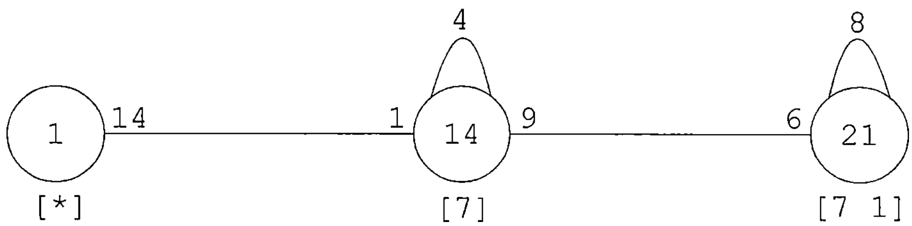

Figure 5. Cayley Graph of $U_3(3):2$ Over $PGL_2(7)$

From the Cayley's graph above, it is clear that every element of $U_3(3):2$ can be represented by an element of $PGL_2(7)$, i.e. a permutation on 14 letters, followed by a word in the symmetric generators of length at most two. Chapter 7 addresses the question of how to find a symmetric representation of an element given as a permutation.

CHAPTER FOUR

DOUBLE COSET ENUMERATION OF J_2:2 OVER $U_3(3)$:2

In this chapter we will perform the double coset enumeration of J_2:2 over $U_3(3)$:2. A symmetric presentation for the progenitor 2^{*36}: $U_3(3)$:2 is given by

$<x,y,t,s,u|x^7, y^2, t^2, (x^{-1}*t)^2, (y*x)^3,$

$t*x^{-1}*y*x*t*y, x^2*y*x^3*y*x^{-4}*y*x^{-4}*y*x, s^2, (s^{x\wedge 3},y),$

$(s^{x\wedge 4},x*y), t*s*s^t*s, y*(s*s^{t*x\wedge 6})^2, y*s^{x\wedge 3}*s^{t*x\wedge 6}*s^x *s, u^2,$

$(u,t),(u,y),(u,x)>$

and the action of x, y, t and s on the symmetric generators is given by:

x ~ (2,3,6,13,21,12,5)(4,9,19,20,10,14,7)(8,17,28,35,29,18,11)(15,16,27,25,32,22,26)(23,33,36,30,31,34,24);

y ~ (4,10)(5,6)(9,19)(11,15)(12,21)(17,26)(18,27)(22,31)(25,35)(28,30)(32,36)(33,34);

t ~ (2,4)(3,7)(5,9)(6,14)(8,18)(10,13)(12,19)(15,25)(16,27)(17,29)(20,21)(24,33)(26,32)(28,35)(30,31)(34,36);

s ~ (1,2)(3,8)(5,11)(6,15)(7,16)(12,22)(13,23)(14,24)(17,30)(18,27)(20,29)(21,31)(25,33)(26,28)(32,36)(34,35).

We will factor the progenitor by the relation $t = s_1s_2s_1$ to obtain a finite homomorphic image:

$$G = \frac{2^{*36}: U_3(3):2}{s = u_1u_2u_1}$$

G is isomorphic to J_2:2, where J_2 is the Hall-Janko group [6].The index of the control group N = $U_3(3)$:2 in G is 100. The relation $s = u_1u_2u_1$ implies $su_1 = u_1u_2$ or equivalently, $Ns_1s_2 = Ns_1$.

Since a relator conjugated by an element of N is also a relator, we present $\pi_{i,j} = s^n$, where $n \in N$. In Figure 6, MAGMA commands can be used to find all the relators. The function Pi returns the permutation $\pi_{i,j}$.

```
trans := Transversal(N, Stabilizer(N, {1,2}));
prs := {@ {1,2}^x : x in trans @};
sgs := [s^x : x in trans];

Pi := func< i,j | Index(prs, {i,j}) ne 0 select
sgs[Index(prs, {i,j})] else Id(N)>;
```

Figure 6. $\pi_{i,j}$ of J_2:2

We will start with [*], the identity double coset. There are 36 symmetric generators that take from [*] to [1], since N is transitive on 36 letters.

$N^1 = N^{(1)}$ has three orbits: $O(1)$ of size 1, $O(2)$ of size 14 and $O(8)$ of size 21. We apply one representative u_i from each orbit to find out to which double coset Nu_1u_i belongs to.

$u_1u_1 = 1$, so Nu_1u_1 belongs to [*] or [1 1] = [*].

$u_1u_2 = su_1$, since the relation is $s = u_1u_2u_1$. Thus, [1 2] = [1] and $Nu_1u_i \in [1]$ for all i in the orbit $O(2)$ of $N^{(1)}$, and the order of this orbit is 14.

Now we will consider u_1u_8. To calculate how many single cosets are in the double coset [1 8], we need to find the order of $N^{(1,8)}$. First, we will fix 8 in N^1 and then find the orbits of $N^{1,8}$: $O(1)$, $O(8)$, $O(7)$, $O(2)$, $O(16)$, $O(4)$, $O(11)$, and $O(18)$. We will explore one representative from each orbit.

$u_1u_8 = u_1u_8u_7u_8u_8u_7 = u_1\pi_{8,7}u_8u_7 = \pi_{8,7}u_{13}u_8u_7$

$= \pi_{8,7}\pi_{13,8}u_{13}u_7$.

Thus, $Nu_1u_8 = Nu_{13}u_7$, and we denote it by $1\ 8 \sim 13\ 7$.

$u_1u_8 = u_1u_8u_{13}u_8u_8u_{13} = u_1\pi_{8,13}u_8u_{13} = \pi_{8,13}u_7u_8u_{13}$

$= \pi_{8,13}\pi_{7,8}u_7u_{13}$.

Thus, $Nu_1u_8 = Nu_7u_{13}$, and we conclude that $7\ 13 \sim 13\ 7$. From this established relation, we conclude that if we have Nu_iu_j in the double coset [1 8], then we have Nu_ju_i in [1 8].

$u_1u_8 = u_1u_8u_2u_8u_8u_2 = u_1\pi_{8,2}u_8u_2 = \pi_{8,2}u_3u_8u_2$

$= \pi_{8,2}\pi_{3,8}u_3u_2$.

Thus, $Nu_1u_8 = Nu_3u_2$, and we denote it by 1 8 ~ 3 2. So far, we have:

1 8 ~ 8 1 ~ 2 3 ~ 3 2 ~ 13 7 ~ 7 13.

Now, we will calculate $N^{(1,8)} \geq \langle N^{1,8}$, s, (1,7)(2,16)(3,23)(4,18)(8,13)(9,32)(10,27)(11,15)(14,29)(17,33)(19,36)(20,24)(22,31)(25,28)(26,34)(30,35)$\rangle$. The order of $N^{(1,8)}$ is at least 192, which induces the number of single cosets in the double coset [1 8] to be equal to

$|N / N^{(1,8)}| = 12096 / 192 = 63$.

$N^{(1,8)}$ has two orbits O(1) of size 12 and O(4) of size 24. We will apply one representative *i* from each orbit to find out to which double coset $Nu_1u_8u_i$ belongs.

$u_1u_8u_1 \sim u_8u_1u_1 \sim u_8$.

There exists n in N such that $(u_8)^n = u_1$, so

[1 8 1] = [8] = [1] and

for all *i* in O(1) $Nu_1u_8u_i$ belongs to [1].

$u_1u_8u_4 = u_1\pi_{8,2}u_8u_2u_4 = \pi_{8,2}u_3u_8u_2u_4 = \pi_{8,2}\pi_{3,8}u_3u_2u_4$

$= \pi_{8,2}\pi_{3,8}u_3\pi_{2,4}u_2 = \pi_{8,2}\pi_{3,8}\pi_{2,4}u_7u_2$.

Thus, $Nu_1u_8u_4 = Nu_7u_2$ or 1 8 4 ~ 7 2. There exists n in N such that $[7\ 2]^n = [1\ 8]$. So, [1 8 4] = [7 2] = [1 8] and for all *i* in O(4) $Nu_1u_8u_4$ belongs to [1 8].

Figure 7 summarizes the double coset enumeration of $J_2{:}2$ over $U_3(3){:}2$.

Label [w]	Coset stabilizing subgroup $N^{(w)}$	Number of cosets
[*]	N	1
[1]	$N^{(1)} \cong PGL_2(7)$, with orbits {1}, {2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 19, 20, 21} and {8, 11, 15, 16, 17, 18, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36}.	36
[1 1] = [*]		
[1 2] = [1]		
[1 8]		63

Figure 7. The Double Cosets [w] = NwN in $J_2{:}2$

Label [w]	Coset stabilizing subgroup $N^{(w)}$	Number of cosets
	$N^{(1,8)}$=<Stabiliser(N,[1,8]),s,(1,7)(2,16)(3,23)(4,18)(8,13)(9,32)(10,27)(11,15)(14,29)(17,33)(19,36)(20,24)(22,31)(25,28)(26,34)(30,35)> Each coset in [7 1] has twelve names: 1 8 ~ 19 9 ~ 32 36 ~ 9 19 ~ 16 23 ~ 8 1 ~ 2 3 ~ 23 16 ~ 36 32 ~ 13 7 ~ 7 13 ~ 3 2. $N^{(1,8)}$ has orbits: {1, 2, 3, 7, 8, 9, 13, 16, 19, 23, 32, 36} and {4, 5, 6, 10, 11, 12, 14, 15, 17, 18, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35}.	
[1 8 1] = [1]		
[1 8 5] = [1 8]		

Figure 7. The Double Cosets [w] = NwN in J_2:2 (Continued)

The Cayley graph for J_2:2 over $U_3(3)$:2 is shown in Figure 8.

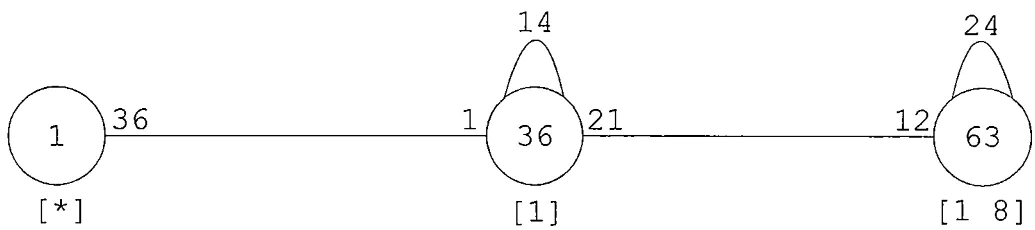

Figure 8. Cayley Graph of J_2:2 Over $U_3(3)$:2

From the Cayley's graph above, it is clear that every element of J_2:2 can be represented by an element of $U_3(3)$:2, i.e. a permutation on 36 letters, followed by a word in the symmetric generators of length at most two. In Chapter 7, an example is provided to show how to find a symmetric representation of an element given as a permutation.

CHAPTER FIVE

DOUBLE COSET ENUMERATION OF $G_2(4):2$ OVER $J_2:2$

In this chapter we will perform the manual double coset enumeration of $G_2(4):2$ over $J_2:2$, where double coset representatives will be written as words in symmetric generators. The group $G \cong G_2(4):2$ given in [6] by

$$G = \frac{2^{*100}:\ J_2:2}{u = v_1 v_2 v_1}$$

A symmetric presentation for the progenitor 2^{*100}: $J_2:2$ is given by:

<x,y,t,s,u,v|x^7, y^2, t^2, $(x^{-1}*t)^2$, $(y*x)^3$, $t*x^{-1}*y*x*t*y$, $x^2*y*x^3*y*x^{-4}*y*x^{-4}*y*x$, s^2, $(s^{x\wedge 3},y)$, $(s^{x\wedge 4},x*y)$, $t*s*s^t*s$, $y*(s*s^{t*x\wedge 6})^2$, $y*s^{x\wedge 3}*s^{t*x\wedge 6}*s^x *s$, u^2, (u,t),(u,y),(u,x), s=u*u^s*u, v^2, (v,x), (v,y), (v,t), (v,s)>, and the progenitor is factored by the relation $u = v_1v_2v_1$. The action of x, y, t, s and u on the symmetric generators is given by:

x ~ (3,4,7,14,24,13,6)(5,10,21,22,11,15,8)(9,18,33,45,34,19,12)(16,17,31,28,40,25,30)(20,36,49,59,50,37,23)(26,41,52,35,38,44,27)(29,32,48,46,56,39,47)(42,57,65,51,55,58,43)(53,66,72,89,98,86,67)(54,68,85,77,61,76,69)(60,74,84,94,97,93,75)(62,78,95,81,63,80,79)(64,82,73,90,92,91,83)(70, 87,96,100,99,88,71);

y ~ (5,11)(6,7)(10,21)(12,16)(13,24)(18,30)(19,31)(23,29)(25,38)(28,45)(33,35)(36,47)(37,48)(39,55)(40,52)(41,44)(46,59)(49,51)(53,61)(54,62)(56,65)(57,58)(60,73)(63,64)(66,84)(67,85)(68,76)(69,79)(70,74)(71,72)(75,92)(77,94)(78,80)(81,96)(82,87)(83,95)(86,97)(88,98)(91,100)(93,99);

t ~ (3,5)(4,8)(6,10)(7,15)(9,19)(11,14)(13,21)(16,28)(17,31)(18,34)(20,37)(22,24)(27,41)(29,46)(30,40)(32,48)(33,45)(35,38)(36,50)(43,57)(44,52)(47,56)(49,59)(51,55)(58,65)(61,77)(63,81)(66,67)(68,69)(70,88)(72,86)(73,91)(74,75)(76,85)(78,79)(80,95)(82,83)(84,93)(87,99)(89,98)(90,92)(94,97)(96,100);

s ~ (2,3)(4,9)(6,12)(7,16)(8,17)(13,25)(14,26)(15,27)(18,35)(19,31)(22,34)(24,38)(28,41)(30,33)(36,53)(37,54)(40,52)(44,45)(46,60)(47,61)(48,62)(49,63)(51,64)(56,70)(57,71)(58,72)(59,73)(65,74)(66,67)(68,82)(69,79)(75,86)(76,87)(77,88)(78,83)(80,95)(81,91)(84,85)(89,90)(92,97)(93,99)(94,98)(96,100);

u ~ (1,2)(9,20)(12,23)(16,29)(17,32)(18,36)(19,37)(25,39)(26,42)(27,43)(28,46)(30,47)(31,48)(33,49)(34,50)(35,51)(38,55)(40,56)(41,57)(44,58)(45,59)(52,65)(53,64)(54,62)(60,71)(61,63)(66,82)(67,83)(68,78)(69,79)(70,74)(72,73)(75,88)(76,80)(77,81)(84,87)(85,95)(86,91)(89,90)(92,98)

(93,99)(94,96)(97,100).

The index of the control group $N = J_2:2$ in $G_2(4):2$ is 416.

The relation $u = v_1v_2v_1$ implies $Nv_1v_2 = Nv_1$.

As we have mentioned above, a relator conjugated by an element of N is also a relator. For convenience, we denote a relator $\pi_{i,j} = u^n$, where $n \in N$. In Figure 9, we will use MAGMA commands to find all the relators $\pi_{i,j}$. The function Pi returns the permutation $\pi_{i,j}$.

```
trans := Transversal(N, Stabilizer(N, {1,2}));
prs := {@ {1,2}^x : x in trans @};
sgs := [u^x : x in trans];

Pi := func< i,j | Index(prs, {i,j}) ne 0 select
sgs[Index(prs, {i,j})] else Id(N)>;
```

Figure 9. $\pi_{i,j}$ of $G_2(4):2$

We will start with [*], the identity double coset. There are 100 symmetric generators that take from [*] to [1], since N is transitive on 100 letters.

$N^{(1)}$ has three orbits: O(1) of size 1, O(2) of size 36 and O(20) of size 63. We apply one representative v_i from each orbit to find out to which double coset Nv_1v_i belongs.

$v_1v_1 = 1$.

$v_1v_2 = uv_1$, since the relation is $u = v_1v_2v_1$. Thus $Nv_1v_i \in [1]$ for all *i* in the orbit O(2) of $N^{(1)}$, and $Nv_1v_i \in [1\ 20]$ for all *i* in the orbits O(20).

Now we will fix 20 in $N^{(1)}$, to find $N^{(1,20)}$. So far, there are seven orbits of $N^{1,20} \geq$ Stabiliser(N, [1,20]): O(1) and O(20) of size 1, O(32) of size 6, O(2) of size 12, O(5) of size 24, O(37) of size 24 and O(23) of size 32.

$v_1v_{20} = v_1v_{20}v_2v_{20}v_{20}v_2 = v_1\pi_{20,2}v_{20}v_2 = \pi_{20,2}v_9v_{20}v_2$

$= \pi_{20,2}\pi_{9,20}v_9v_2$.

Thus, $Nv_1v_{20} = Nv_9v_2$ and we denote it av 1 20 ~ 9 2.

$v_1v_{20} = v_1v_{20}v_9v_{20}v_{20}v_9 = v_1\pi_{20,9}v_{20}v_9 = \pi_{20,9}v_2v_{20}v_9$

$= \pi_{20,9}\pi_{2,20}v_2v_9$.

So, we have $Nv_1v_{20} = Nv_2v_9$ and 1 20 ~ 2 9 ~ 9 2.

From this established relation we conclude that if we have Nv_iv_j in the double coset [1 20], then we also have Nv_jv_i in [1 20].

Thus, $N^{(1,20)} \geq \langle N^{1,20}$, (1,9,3)(2,4,20)(5,19,62)(6,82,60)(7,87,73)(8,32,26)(11,31,54)(12, 33,61)(13,76,64)(14,42, 17)(15,90,50)(16,35,53)(18,85,39)(22,89,43)(23,41,83)(24,68,63)(25,45,72)(28,71,38)(29,44,95)(30,67,55)(36,80,59)(40,56,70)(46,47,78)(49,57,66)(51,58,84)(52,65,74)(69,100,88)(75,99,96)(79,91,98)(81,92,93)>. The order of $N^{(1,20)}$ is at least 3840, which induces at most 315 single cosets in the double coset [1 20]. $N^{(1,20)}$ has two orbits: O(1) of size 20 and O(5) of size 80. We will apply one representative *i* from each orbit to determine to which double coset $v_1v_{20}v_i$ belongs.

$v_1v_{20}v_1 \sim v_{20}v_1v_1 \sim v_{20}$. $[20] \in [1]$, since there exists n in N such that $[20]^n = [1]$. So,

$1\ 20\ 1 \sim 20 \sim 1 \Rightarrow [1\ 20\ 1] = [20] = [1]$.

$v_1v_{20}v_5 \sim v_{20}v_1v_5 \sim v_{20}\pi_{1,5}v_1 \sim \pi_{1,5}v_{32}v_1$.

Thus, $Nv_1v_{20}v_5 = Nv_{32}v_1$. There exist n in N such that $N(v_{32}v_1)^n = Nv_1v_{20}$ and $[1\ 20\ 5] = [32\ 1] = [1\ 20]$.

Figure 10 summarizes the double coset enumeration above.

Label [w]	Coset stabilizing subgroup $N^{(w)}$	Number of cosets
[*]	N	1
[1]	$N^{(1)} \cong U_3(3){:}2$, with orbits {1}, {1},{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17, 18,19,21,22,24,25,26,27,28,30,31,33,34,35,38, 40,41,44,45,52},{20,23,29,32,36,37,39,42,43, 46,47,48,49,50,51,53,54,55,56,57,58,59,60,61, 62,63,64,65,66,67,68,69,70,71,72,73,74,75,76, 77,78,79,80,81,82,83,84,85,86,87,88,89,90,91, 92,93,94,95,96,97,98,99,100}.	100
[1 1] = [*]		
[1 2] = [1]		
[1 20]		315

Figure 10. The Double Cosets [w] = NwN in $G_2(4){:}2$

Label [w]	Coset stabilizing subgroup $N^{(w)}$	Number of cosets
	$N^{(1,20)}$ = <Stabiliser(N, [1,20]), (1,9,3) (2,4,20)(5,19,62)(6,82,60)(7,87,73) (8,32,26)(11,31,54)(12, 33,61)(13,76,64) (14,42,17)(15,90,50)(16,35,53)(18,85,39) (22,89,43)(23,41,83)(24,68,63)(25,45,72) (28,71,38)(29,44,95)(30,67,55)(36,80,59) (40,56,70)(46,47,78)(49,57,66)(51,58,84) (52,65,74)(69,100,88)(75,99,96)(79,91,98) (81,92,93)>. Each coset in [1 20] has twenty different names: 17 26 ~ 14 8 ~ 4 3 ~ 56 65 ~ 32 42 ~ 65 56 ~ 10 21 ~ 74 70 ~ 52 40 ~ 20 1 ~ 9 2 ~ 21 10 ~ 26 17 ~ 8 14 ~ 42 32 ~ 40 52 ~ 2 9 ~ 70 74 ~ 1 20 ~ 3 4	

Figure 10. The Double Cosets [w] = NwN in $G_2(4)$:2 (Continued)

Label [w]	Coset stabilizing subgroup $N^{(w)}$	Number of cosets
	$N^{(1,20)}$ has orbits: {1,2,3,4,8,9,10,14,17,20,21,26,32,40,42,52, 56,65,70,74}, {5,6,7,11,12,13,15,16,18,19,22, 23,24,25,27,28,29,30,31,33,34,35,36,37,38,39, 41,43,44,45,46,47,48,49,50,51,53,54,55,57,58, 59,60,61,62,63,64,66,67,68,69,71,72,73,75,76, 77,78,79,80,81,82,83,84,85,86,87,88,89,90,91, 92,93,94,95,96,97,98,99,100}.	
[1 20 1] = [1]		
[1 20 5] = [1 20]		

Figure 10. The Double Cosets [w] = NwN in $G_2(4)$:2 (Continued)

The Cayley graph is shown in Figure 11.

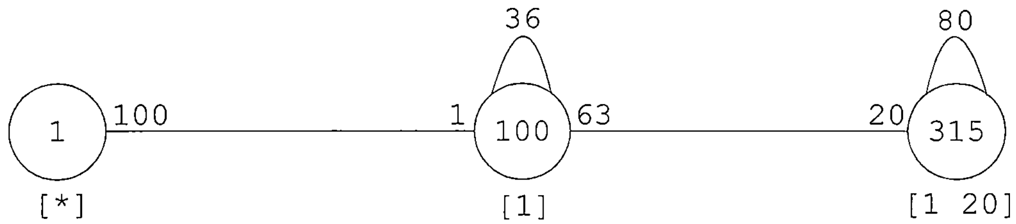

Figure 11. Cayley Graph of $G_2(4):2$ Over $J_2:2$

From the Cayley's graph above, it is clear that every element of $G_2(4):2$ can be represented by an element of $J_2:2$, i.e. a permutation on 100 letters, followed by a word in the symmetric generators of length at most two. In Chapter 7, an example is provided to show how to find a symmetric representation of an element given as a permutation and vice versa.

CHAPTER SIX

DOUBLE COSET ENUMERATION OF 3·SUZ:2 OVER $G_2(4):2$

In this chapter we will demonstrate that the elements of the group 3·Suz:2 can be written as a permutation of the group $G_2(4):2$ followed by at most four of the symmetric generators. A symmetric presentation of the progenitor 2^{*416}: $(G_2(4):2)$ is given by:

$<x,y,t,s,u,v,w \mid x^7,\ y^2,\ t^2,\ (x^{-1}*t)^2,\ (y*x)^3,\ t*x^{-1}*y*x*t*y,$ $x^2*y*x^3*y*x^{-4}*y*x^{-4}*y*x,\ s^2,\ (s^{x\hat{}3},y),\ (s^{x\hat{}4},x*y),\ t*s*s^t*s,$ $y*(s*s^{t*x\hat{}6})^2,\ y*s^{x\hat{}3}*s^{t*x\hat{}6}*s^x*s,\ u^2,\ (u,t),(u,y),(u,x),$ $s=u*u\hat{}s*u,\ v^2,\ (v,x),\ (v,y),\ (v,t),\ (v,s),\ u=v*v\hat{}u*v,\ w^2,$ $(w,x),\ (w,s),\ (w,u),\ v=w*w\hat{}v*w>,$

where the control group

$N = G_2(4):2 \cong <x,y,t,s,u,v \mid x^7,\ y^2,\ t^2,\ (x^{-1}*t)^2,\ (y*x)^3,\ t*x^{-1}*y*x*t*y,\ x^2*y*x^3*y*x^{-4}*y*x^{-4}*y*x,\ s^2,\ (s^{x\hat{}3},y),\ (s^{x\hat{}4},x*y),$ $t*s*s^t*s,\ y*(s*s^{t*x\hat{}6})^2,\ y*s^{x\hat{}3}*s^{t*x\hat{}6}*s^x*s,\ u^2,$ $(u,t),(u,y),(u,x),\ s=u*u\hat{}s*u,\ v^2,\ (v,x),\ (v,y),\ (v,t),$ $(v,s),\ u=v*v\hat{}u*v>,$

where x, y, t, s, u and v are given in Appendix B.

To find the finite homomorphic image of this progenitor, we apply Curtis' Lemma and obtain:

$C_N(N^{12}) = <w>.$

We factor the progenitor by the relation $v=w_1w_2w_1$, where $w_1=w$ and $w_2=w^u$, to obtain G (see Bray [1]):

$$G = \frac{2^{*416}:(G_2(4):2)}{v=w_1w_2w_1}$$

The index of N in G is 5346 and $G \cong 3{\cdot}Suz:2$.

Here we will perform a manual double coset enumeration of $3{\cdot}Suz:2$ over $G_2(4):2$. Since we will use the relator $v=w_1w_2w_1$, it is useful to introduce $\pi_{i,j} = v^n$, where $n \in N$. Simple code, written in MAGMA language, that gives all the permutations $\pi_{i,j}$ is presented in Figure 12. The function Pi returns the permutation $\pi_{i,j}$.

```
sg:=v;
trans := Transversal(N, Stabilizer(N, {1,2}));
prs := {@ {1,2}^x : x in trans @};
sgs := [sg^x : x in trans];

Pi := func< i,j | Index(prs, {i,j}) ne 0 select
sgs[Index(prs, {i,j})] else Id(N)>;
```

Figure 12. $\pi_{i,j}$ of $3{\cdot}Suz:2$

We will start with [*], the identity double coset. There are 416 symmetric generators that take from [*] to [1], since N is transitive on 416 letters.

$N^1 = N^{(1)}$ has three orbits: O(1) of size 1, O(2) of size 100 and O(39) of size 315. We apply one representative w_i from each orbit to find out to which double coset Nw_1w_i belongs.

$w_1w_1 = 1$, so [1 1] = [*].

$w_1w_2 = vw_1$, since the relation is $v=w_1w_2w_1$. Thus, [1 2] = [1] and $Nw_1w_i \in$ [1] for all i in the orbit O(2) of $N^{(1)}$, and the order of this orbit is 100.

Now we will consider the double coset [1 39].

$w_1w_{39} = w_1w_{39}w_2w_{39}w_{39}w_2 = w_1\pi_{39,2}w_{39}w_2 = \pi_{39,2}w_{21}w_{39}w_2$

$= \pi_{39,2}\pi_{21,39}w_{21}w_2$.

Thus, $Nw_1w_{39} = Nw_{21}w_2$, and we denote it by 1 39 ~ 21 2.

$w_1w_{39} = w_1w_{39}w_{21}w_{39}w_{39}w_{21} = w_1\pi_{39,21}w_{39}w_{21} = \pi_{39,21}w_2w_{39}w_{21}$

$= \pi_{39,21}\pi_{2,39}w_2w_{21}$.

Thus, $Nw_1w_{39} = Nw_2w_{21}$, and we conclude that 2 21 ~ 21 2. From this established relation we conclude that if we have Nw_iw_j in the double coset [1 39], then we have Nw_jw_i in [1 39].

$N^{(1,39)}$ is calculated by fixing 39 in $N^{(1)}$ and taking into account the relations found above; the order of $N^{(1,26)}$

is 122880, which induces 4095 single cosets in the double coset [1 26]. Applying $N^{(1,39)}$ to [1 39], we can find all the different names of the double coset [1 39]:

1 39 ~ 21 2 ~ 10 3 ~ 81 64 ~ 33 45 ~ 5 4 ~ 9 15 ~ 22 11 ~ 27 18 ~ 3 10 ~ 4 5 ~ 64 81 ~ 97 90 ~ 57 43 ~ 89 112 ~ 45 33 ~ 90 97 ~ 18 27 ~ 66 53 ~ 15 9 ~ 43 57 ~ 11 22 ~ 2 21 ~ 112 89 ~ 122 131 ~ 393 415 ~ 53 66 ~ 131 122 ~ 171 162 ~ 39 1 ~ 415 393 ~ 162 171.

There are three orbits in $N^{(1,39)}$: O(39) of size 32, O(6) of size 64 and O(218) of size 320. We proceed as follows:

$w_1w_{39}w_1 \sim w_1w_{39}w_{39} \sim w_1$.

Thus, for all j in O(39), we will have $Nv_1v_{39}v_j \in [1]$. The order of O(39) is 32.

$w_1w_{39}w_6 \sim w_{39}w_1w_6 \sim w_{39}\pi_{1,6}w_1 \sim \pi_{1,6}w_{53}w_1 \sim w_{53}w_1$.

Using MAGMA, it is easy to verify that there exists $n \in N$ such that $N(w_1w_{39})^n = Nw_{53}w_1$.; therefore, $Nw_{53}w_1 \in [1\ 39]$ and [1 39 6] = [53 1] = [1 39].

Finally, $Nw_1w_{39}w_{218} \in [1\ 39\ 218]$.

$w_1w_{39}w_{218} \sim w_{39}w_1w_{218} \sim w_{39}\pi_{1,6}w_1w_6w_{218} \sim \pi_{1,6}w_{53}w_1w_6w_{218}$
$\sim \pi_{1,6}w_{53}w_1\pi_{6,218}w_6 \sim \pi_{1,6}\pi_{6,218}w_{409}w_{87}w_6$.

So, we have obtained 1 39 218 ~ 409 87 6.

$N^{(1,39,218)}$ is calculated by fixing 218 in $N^{(1,39)}$ and adding the relation found above, i.e. $N^{(1,39,218)}$ is generated by $N^{(1,39,218)}$ and all $n \in N$ such that $N(w_1w_{39}w_{218})^n = Nw_{409}w_{87}w_6$.

The order of $N^{(1,39,218)}$ is at least 604800; thus, the number of single cosets in [1 39 218] is at most $|N / N^{(1,39,218)}| = 503193600/604800 = 832$.

$N^{(1,39,218)}$ has three orbits: O(218), O(387) and O(333) with orders of 315, 100 and 1 respectively.

$w_1w_{39}w_{218}w_{218} = w_1w_{39}$ or [1 39 218 218] = [1 39].

$w_1w_{39}w_{218}w_{387} = w_1w_{39}\pi_{218,387}w_{218} = \pi_{218,387}w_{297}w_{405}w_{218}$.

It is easy to verify that there exists $n \in N$ such that $N(w_1w_{39}w_{218})^n = w_{297}w_{405}w_{218}$. So, $Nw_{297}w_{405}w_{218} \in$ [1 39 218] and [1 39 218 387] = [297 405 218] = [1 39 218].

Finally, $Nw_1w_{39}w_{218}w_{333} \in$ [1 39 218 333].

$w_1w_{39}w_{218}w_{333} \sim w_1w_{39}\pi_{218,387}w_{218}w_{387}w_{333} \sim \pi_{218,387}w_{297}w_{405}w_{218}w_{387}w_{333}$

$\sim \pi_{218,387}w_{297}w_{405}w_{218}\pi_{387,333}w_{387} \sim \pi_{218,387}\pi_{387,333}w_{43}w_{57}w_{348}w_{387}$

$\sim w_{43}w_{57}w_{348}w_{387}$.

We will calculate $N^{(1,39,218,333)}$ by fixing 333 in $N^{(1,39,218)}$ and adding the relation found above. The order of $N^{(1,39,218,333)}$ is at least 251,596,800, so there are 2 single cosets in the double coset [1 39 218 333].

There is a single orbit of order 416 in $N^{(1,39,218,333)}$, so we can complete the double coset enumeration:

$v_1v_{39}v_{218}v_{333}v_{333} \sim v_1v_{39}v_{218} \Rightarrow [1\ 39\ 218\ 333\ 333] = [1\ 39\ 218]$.

In Appendix A in the function symrepSuz, one can find how to find $N^{(1,39)}$, $N^{(1,39,218)}$ and $N^{(1,39,218,333)}$.

The Caley's graph for double coset enumeration of $3{\cdot}Suz{:}2$ over $G_2(4){:}2$ is shown in Figure 13.

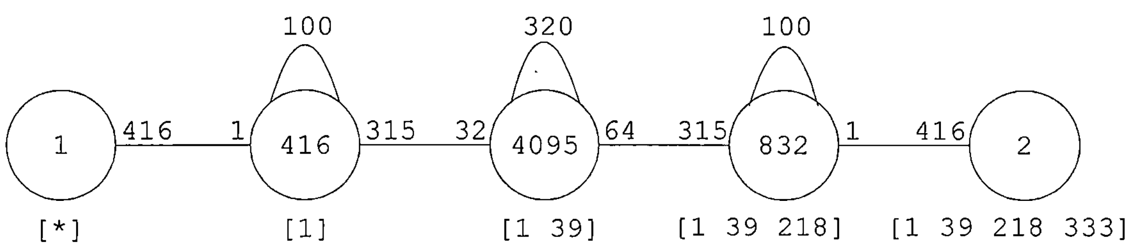

Figure 13. Cayley Graph of $3{\cdot}Suz{:}2$ Over $G_2(4){:}2$

From the Cayley's graph above, it is clear that every element of $3{\cdot}Suz{:}2$ can be represented by an element of $G_2(4){:}2$, i.e. a permutation on 416 letters, followed by a word in the symmetric generators of length at most four. In Chapter 7, an example is provided to show how to find a symmetric representation of an element given as a permutation and vice versa.

CHAPTER SEVEN

PROGRAMS AND ALGORITHMS

General Description

In this chapter we will describe the programs and algorithms developed in this paper. In [2], the program was written to manipulate the elements of the Janko group, J_1. Using the ideas for the programs and algorithms developed in [2], we wrote the algorithms and programs for the groups $U_3(3){:}2$, $J_2{:}2$, $G_2(4){:}2$ and $3{\cdot}Suz{:}2$. In particular, the functions *mult*, *sym2perm* and *perm2sym* in [2] were modified and new functions *symrep* and *canon* were written for the groups $U_3(3){:}2$, $J_2{:}2$, $G_2(4){:}2$ and $3{\cdot}Suz{:}2$ in this project.

The purpose of these programs is to enable a user to manipulate the elements of a larger group in their symmetric presentation, where each element of a larger group is represented by a permutation of a smaller group followed by a word in symmetric generators. The programs allow the conversion of elements of the groups $U_3(3){:}2$, $J_2{:}2$, $G_2(4){:}2$, and $3{\cdot}Suz{:}2$ into their symmetric representation (permutations of control groups $PGL_2(7)$, $U_3(3){:}2$, $J_2{:}2$ and $G_2(4){:}2$ respectively followed by words in symmetric generators), to perform the reverse process - to convert elements given in their symmetric representation into permutation, multiply elements given in symmetric representation, and find their inverses.

First, the function *symrep* is used to create a structure for each group that holds a working group, its control group, coset action information, symmetric generators and other information that is used by other functions. Each group $U_3(3):2$, $J_2:2$, $G_2(4):2$ and $3{\cdot}Suz:2$ has an individual *symrep* function.

The following functions are shared by all three groups: *sym2per* function converts an element of a group given in its symmetric representation into a permutation, *per2sym* function converts a permutation into its symmetric representation, *mult* function multiplies two elements of a group given in their symmetric representation and *invert* function finds an inverse of an element given in its symmetric representation.

The function *canon* is the same for the groups $J_2:2$, $G_2(4):2$, and $3{\cdot}Suz:2$, but the group $U_3(3):2$ has its own *canonU* function. These functions are used by the function *Prod* to multiply two elements given in their symmetric representations. The difference between *canon* and *mult* is that *mult* uses coset action information and elements of a larger group, and *canon* uses only permutations of a control group and relations by which we factor progenitors to reduce words in symmetric generators.

In addition to the above functions, this paper developed a nested algorithm that allows us to write an element of $3{\cdot}Suz{:}2$, $G_2(4){:}2$ or of $J_2{:}2$ as a permutation of $PGL_2(7)$ on 14 letters followed by a word in at most 10, 6 or 4 symmetric generators respectively.

The functions *sym2perNest* and *per2symNest* perform similar tasks as *sym2per* and *per2sym* respectively, except the former functions deal with nested symmetric representation, i.e. a permutation of $PGL_2(7)$ on 14 letters followed by a word in symmetric generators. The program is written in such a way that the functions *sym2perNest* and *per2symNest* take a sequence Q and an element in its nested symmetric representation or a permutation of a larger group respectively as arguments. A sequence Q holds group structures returned by *symrep* functions that are arranged in the order from the largest to the smallest group. For example, we may call symrep functions that return structures for the groups $U_3(3){:}2$, $J_2{:}2$ and $G_2(4){:}2$ like the following: GDes:=symrepG(), JDes:=symrepJ(), UDes:=symrepU(). Then we need to place these structures into Q in the order: Q:= [GDes, JDes, UDes] (since the corresponding $G_2(4):2$ is larger than $J_2:2$, and $J_2:2$ is larger than $U_3(3):2$). An element in nested symmetric representation returned or passed by these functions has the following form: <permutation of $PGL_2(7)$, [[symmetric generator #1 of $U_3(3):2$, symmetric generator #2 of $U_3(3):2$], [symmetric generator #1 of $J_2:2$, symmetric generator #2 of $J_2:2$], [symmetric generator #1 of $G_2(4):2$, symmetric generator #2 of $G_2(4):2$]] (for example, <(1,2,3,4,5,6,7)(14,13,12,11,10,9,8), [[3,5],[1,2],[5,6]]>, where 3 and 5 are indexes of symmetric generators of $U_3(3):2$, 1 and 2 of symmetric generators of $J_2:2$ and 5 and 6 of symmetric generators of $G_2(4):2$).

The functions *multNest* and *ProdNest* have the same purpose as the functions *mult* and *Prod* respectively, but the former deal with elements passed in their nested symmetric representation. The program needs not necessarily convert elements of $G_2(4):2$ into nested symmetric representation using permutations of $PGL_2(7)$; if a user needs to write elements of $G_2(4):2$ in their nested symmetric representation as permutations of $U_3(3):2$ followed by the words in symmetric generators of the length at most four, the program can perform it as well. In this case, Q will contain only GDes and Jdes structures. All the programs are given in Appendix A. Additional information about the functions and steps of algorithms for the functions can be found there as well.

Programs 1 and 3 in Appendix A are MAGMA dependent, but easily can be adopt to other group handling packages such as GAP. Program 2 can be written in any high-level language.

Examples

In this section, we will show manually how the programs work. First, we will use an element of the group $U_3(3){:}2$ to show how the programs find the element's symmetric representation. Then we will use the symmetric representation obtained in the previous example to show how the programs restore permutation representation of the same element. Then we will show how the function *canonU* works for two elements of $U_3(3){:}2$. And finally, we will show how a nested program works on the element of $3{\cdot}Suz{:}2$.

Finding Symmetric Representation

First, we will show the theory behind this procedure and then show the example. Let $\alpha \in G$, where G is a group. Let N be a control group of G. We will need to find a symmetric representation for α, i.e. we will need to find $\pi \in N$ and word w in the symmetric generators such that $\alpha = \pi w$.

$N^{\alpha} = N\alpha$ (the action is given by right multiplication)

$\Rightarrow N\alpha = N\pi w \Rightarrow N\alpha = Nt_it_j$ (since $\pi \in N$ and the length of w is at most 2) $\Rightarrow N\alpha t_jt_i = N \Rightarrow \alpha t_jt_i \in N$.

Finally, by computing the action of αt_jt_i on the set of symmetric generators, we will find π and w. To compute the action of αt_jt_i on the set of symmetric generators, we will use the information on the action of the symmetric generators on the cosets of $N = PGL_2(7)$, that is calculated in *symrep* function and stored in *cst*, an indexed set. For convenience, we will give indexed *cst* in Figure 14 (the index is shown in the rows # 1, 3, 5, 7 and 9 and the entries of *cst* are shown in the rows # 2, 4, 6, 8 and 10).

1	2	3	4	5	6	7	8
[*]	[7]	[1]	[14]	[6]	[2]	[8]	[7,1]
9	10	11	12	13	14	15	16
[13]	[10]	[13,14]	[5]	[3]	[9]	[2,7]	[3,1]
17	18	19	20	21	22	23	24
[11,12]	[6,5]	[12]	[11]	[4]	[14,12]	[10,14]	[11,8]
25	26	27	28	29	30	31	32
[9,14]	[1,6]	[10,8]	[10,11]	[8,9]	[14,11]	[10,13]	[8,13]
33	34	35	36				
[9,13]	[9,12]	[9,10]	[8,12]				

Figure 14. Action of Symmetric Generators on the Right Cosets of $N = PGL_2(7)$

Let $\alpha \in U_3(3){:}2$, and

α = (1,36,24,27)(2,26,14,30)(3,13,31,22)(4,9,35,34)(5,11)(6,7)(8,10,12,33)(15,29)(16,20)(17,32)(18,28)(19,23,25,21).

To find the action of α on [*], we notice that the index of cst corresponding to [*] is 1, and

$1^{\alpha} = 36$.

The entry of *cst* that corresponds to index 36 is [8,12]; therefore, we have:

$N\alpha = Nt_8t_{12}$ and

$\Rightarrow \alpha t_{12}t_8 \in N$.

Multiply:

$\alpha t_{12}t_8$ = (2,4,13,14,21,7,6,19)(3,9,5,20)(8,11,30,16,33,32,24,17)(10,12)(15,22,23,25,35,29,27,36)(18,28,34,31).

Now, the action of $\alpha t_{12}t_8$ on the symmetric generators (the index of a symmetric generator corresponds to the entry of *cst*) is given in Figure 15.

Entry of Cst, i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Index(cst,i),j	3	6	13	21	12	5	2	7	14	10	20	19	9	4
$k = j^{\alpha}$	9	19	14	7	10	20	4	6	21	12	3	2	5	13
cst[k]	13	12	9	8	10	11	14	2	4	5	1	7	6	3

Figure 15. Action of $\alpha t_{12}t_8$ on the Symmetric Generators

The result is represented by $Nt_i^{\alpha} = Nt_j$, where i is in the first row of the table above and j is in the last row: $Nt_1^{\alpha} = Nt_{13}$, $Nt_2^{\alpha} = Nt_{12}$, ..., $Nt_{14}^{\alpha} = Nt_3$. The corresponding $\pi \in N$ with this action is:

π = (1,13,6,11)(2,12,7,14,3,9,4,8)(5,10).

Therefore,

$\alpha = \pi t_8 t_{12}$ = (1,13,6,11)(2,12,7,14,3,9,4,8)(5,10)t_8t_{12}.

Finding the Permutation Representation

Here we will show how the programs restore the permutation representation of an element given in its symmetric representation.

Let $\beta = (1,13,6,11)(2,12,7,14,3,9,4,8)(5,10)t_8t_{12}$. The action of $\pi = (1,13,6,11)(2,12,7,14,3,9,4,8)(5,10) \in PGL_2(7)$ on the cosets Nw_i of N can be found as described in the following scheme (here, N is a subgroup of $U_3(3):2$, homomorphic image of $PGL_2(7)$):

- find w, the entry of *cst* that corresponds to i;
- find the action of π on w, call it w^π;
- find the index j of *cst* that corresponds to w^π;
- the action is described by $i \to j$.

Note, that not all w^π are in *cst;* the function *prodim* uses the symmetric generators to find the equivalent w^π. The results of these calculations are shown in Figure 16.

i	1	2	3	4	5	6	7	8
w	[*]	[7]	[1]	[14]	[6]	[2]	[8]	[7,1]
w^{π}	[*]	[14]	[13]	[3]	[11]	[12]	[2]	14,13
j	1	4	9	13	20	19	6	11
i	9	10	11	12	13	14	15	16
w	[13]	[10]	[13,14]	[5]	[3]	[9]	[2,7]	[3,1]
w^{π}	[6]	[5]	[6,3]	[10]	[9]	[4]	[12,14]	[9,13
j	5	12	30	10	14	21	22	33
i	17	18	19	20	21	22	23	24
w	[11,12]	[6,5]	[12]	[11]	[4]	[14,12]	[10,14]	[11,8
w^{π}	[1,7]	[11,1	[7]	[1]	[8]	[3,7]	[5,3]	[1,2]
j	8	28	2	3	7	23	25	17
i	25	26	27	28	29	30	31	32
w	[9,14]	[1,6]	[10,8]	[10,11]	[8,9]	[14,11]	[10,13]	[8,13
w^{π}	[4,3]	13,11	[5,2]	[5,1]	[2,4]	[3,1]	[5,6]	[2,6]
j	35	26	36	34	27	16	18	24
i	33	34	35	36				
w	[9,13]	[9,12	[9,10]	[8,12]				
w^{π}	[4,6]	[4,7]	[4,5]	[2,7]				
j	32	31	29	15				

Figure 16. The Action of $\pi \in PGL_2(7)$ on the Cosets Nw_i

The corresponding $\sigma \in N$ with this action is:

$\sigma = (2,4,13,14,21,7,6,19)(3,9,5,20)(8,11,30,16,33,32,24,17)$
$(10,12)(15,22,23,25,35,29,27,36)(18,28,34,31)$.

The permutation representation of β is calculated as following: $\alpha = \sigma t_8 t_{12}$

$= (1,36,24,27)(2,26,14,30)(3,13,31,22)(4,9,35,34)(5,11)$
$(6,7)(8,10,12,33)(15,29)(16,20)(17,32)(18,28)(19,23,25,21)$.

Finding Product

Here, we will show how the programs multiply two elements given in symmetric representation. We will show how the function *canonU* is used to multiply two symmetrically represented elements of $U_3(3):2$.

Let α, $\beta \in U_3(3):2$; α, β are given in their symmetric representation ($\alpha = \pi_\alpha t_8 t_{12}$ and $\beta = \pi_\beta t_{10} t_8$):

$\alpha = (1, 13, 6, 11)(2, 12, 7, 14, 3, 9, 4, 8)(5, 10) t_8 t_{12}$;

$\beta = (1, 10)(2, 13)(3, 9)(4, 11)(5, 14)(6, 8)(7, 12) t_{10} t_8$.

First, *unify* function is used to multiply these elements:

$\alpha * \beta = \pi_\alpha t_8 t_{12} * \pi_\beta t_{10} t_8 = \pi_\alpha * \pi_\beta \ (t_8 t_{12})^{\pi\_\beta} \ t_{10} t_8$

$= (1, 2, 7, 5)(4, 6)(8, 13)(9, 11, 10, 14) t_6 t_7 t_{10} t_8$.

Then *canon* function is used to write the result in a short canonical form following the steps:

- if the adjusted symmetric generators are the same, delete them;
- if there is a relation $t_i t_j = \pi_{i,j} t_i$ between the adjusted symmetric generators, then use it to shorten the word in symmetric generators, moving $\pi_{i,j}$ to the left (is the same as described in Chapter Three);

- if there is a relation $t_i t_j = \sigma_{i,j} t_j t_i$, then use this to shorten the word, moving $\sigma_{i,j}$ to the left ($\sigma_{i,j}$ is found by the function P3);
- if there is a relation $t_i t_j = \delta_{i,j} t_k t_l$, then use this to change the word by an equivalent one and move $\delta_{i,j}$ to the left ($\delta_{i,j}$ is found by the function P2).

Let see how it works on our example:

$\alpha * \beta = (1, 2, 7, 5)(4, 6)(8, 13)(9, 11, 10, 14) t_6 t_7 t_{10} t_8$

$= (1, 2, 7, 5)(4, 6)(8, 13)(9, 11, 10, 14) t_6 \pi_{7,10} t_7 t_{10} t_8$

$= (1, 2, 7, 5)(4, 6)(8, 13)(9, 11, 10, 14) \pi_{7,10} t_9 t_7 t_8$

$= (1, 2, 7, 5)(4, 6)(8, 13)(9, 11, 10, 14) \pi_{7,10} \delta_{9,7} t_7 t_9 t_8$

$= (1, 2, 7, 5)(4, 6)(8, 13)(9, 11, 10, 14) \pi_{7,10} \delta_{9,7} t_7 \sigma_{9,8} t_{11} t_7$

$= (1, 2, 7, 5)(4, 6)(8, 13)(9, 11, 10, 14) \pi_{7,10} \delta_{9,7} \sigma_{9,8} t_7 t_{11} t_7$

$= (1, 2, 7, 5)(4, 6)(8, 13)(9, 11, 10,14) \pi_{7,10} \delta_{9,7} \sigma_{9,8} \delta_{7,11} t_{11} t_7 t_7$

$= (1, 2, 7, 5)(4, 6)(8, 13)(9, 11, 10,14) \pi_{7,10} \delta_{9,7} \sigma_{9,8} \delta_{7,11} t_{11}$

$= (1, 4, 2, 13, 3, 12, 6, 10)(4, 9, 5, 8)(7,11) t_{11}$.

Nested Symmetric Representation

Here we will show how the programs writes a permutation on 5346 letters of 3·Suz:2 in nested symmetric representation, a permutation of $PGL_2(7)$ followed by a word in symmetric generators of length at most 10.

Let g be in $3\cdot Suz:2$ (g is given in Appendix D). To write g in nested symmetric representation, the program performs a loop. First, it converts g into symmetric representation, a permutation of $G_2(4):2$, say α, followed by a word in symmetric generators: $g = \alpha w_{50} w_{378}$

=(1,416,210,166,167,265,165,43,163,232,24,59,320,130,15,400,101,76,179,203,67,146,224,227)(2,393,211,69,44,171,207,96,155,245,107,25,335,4,177,404,93,27,64,352,17,225,343,297)(3,295,30,368,168,411,35,324,78,293,28,89,362,22,129,314,190,192, 359,100,349,26,231,237)(5,90,358,350,315,255,85,141,18,241,374,158)(6,226,219,12,386,73,347,108,172,234,7,173,309,95,36,355,83,356,150,394,98,376,378,170)(8,302,278,327,34,395,288,153,325,164,275,56,371,79,54,405,269,235,329,337,154,99,208,222)(9,260,50,334,123,413,169,112,268,230,361,407,289,63,201,115,38,134,236,258,216,330,306,133)(10,244,144,401,365,183,92,304,60,333,118,88,242,147,198,70,247,311,68,110,105,342,191,388)(11,263,220,296,363,152,19,156,215,74,284,119,377,39,102,384,274,52,262,13,176,392,354,128)(14,279,159,202,382,205,312,408)(16,131,33,373)(20,298,292,313,151,213,367,185)(21,109,338,369,249,399,266,29,353,252,94,71,322,186,409,49,267,87,380,340,189,23,390,366)(31,331,402,46,125,180,217,47,57,332,345,406,246,209,137,174,391,120,37,65,283,142,346,321)(32,305,86,212,259,51,281,257,381,303,218,370,82,41,397,193,221,299,261,336,240,264,270,229)(40,66,282,48,200,97,42,357)(45,132,254,277,307,124,116,121,238,387,111,148,122,414,285,223,294,410,398,104,233,323,344,75)(53,162,181,106,385,415,204,136,194,272,135,195)(55,184,280,188,127,379,326,117,389,143,319,178,375,160,81,256,175,372,206,273,348,360,62,291)(58,251,276,383,182,351,239,248,243,339,197,61,318,287,139,308,271,214,72,328,317,300,412,196)(77,316,140,84,310,149,341,364)(80,157,91,138,250,126,161,403)(103,113,187,253,199,114,145,286)(228,290,301,396)$w_{50}w_{378}$.

Then the program converts $\alpha \in G_2(4):2$ into its symmetric representation, i.e. a permutation of $J_2:2$, say β, followed by a word in symmetric generators of length at most 2, and g now is:

$g = \beta v_{10}v_{92}w_{50}w_{378} =$

$(1,21,84,69,34,85,62,23,15,80,60,6)(2,4,70,88,8,9,72,86,50,89,77,5)(3,58,17,53,43,19)(7,76,63,100,79,28,59,41,29,30,40,35)(10,44,52,95,67,33,64,49,78,91,46,37)(11,74,97,94,98,71)(12,13,20,82)(14,36,57,47,26,56,31,65,90,92,61,75)(18,39,48,32,68,93,83,27,45,25,51,22)(24,87,96,81)(38,73,99,54,55,66)v_{10}v_{92}w_{50}w_{378}$.

Then the program converts $\beta \in J_2{:}2$ into its symmetric representation, i.e. a permutation of $U_3(3){:}2$, say γ, followed by a word in symmetric generators of length at most 2, and g now is:

$g = \gamma u_{19}v_{10}v_{92}w_{50}w_{378}$

$=(1,3,23,13,20,17,4)(2,14,26,32,30,6,18)(5,19,33,10,16,29,25)(7,8,24,35,22,21,27)(9,34,36,28,31,11,12)u_{19}v_{10}v_{92}w_{50}w_{378}$.

Finally, the program converts $\gamma \in U_3(3){:}2$ into its symmetric representation, i.e. a permutation of $PGL_2(7)$, say η, followed by a word in symmetric generators of length at most 2, and g now is:

$g = \eta s_1u_{19}v_{10}v_{92}w_{50}w_{378}$

$=(1,8,7,9,6,12,4,14)(2,10,3,11)(5,13)s_1u_{19}v_{10}v_{92}w_{50}w_{378}$.

To restore a permutation from its nested symmetric representation, the program simply performs the steps above in the reverse order.

To multiply two elements given in nested symmetric representation, the program converts elements into their usual symmetric representation, performs multiplication, and then converts the result into nested symmetric representation.

CHAPTER EIGHT

DOUBLE COSET ENUMERATION OF O'NAN OVER M_{11}

In this chapter we will discuss possibility of double coset enumeration of O'Nan over M_{11}. In [3], it was proved that a presentation of O'Nan sporadic group is given by:

$$O'N:2 \cong \frac{2^{*12}:N}{(t_\infty t_0)^4 = (\sigma^3 t_\infty t_3)^5 = (\sigma(t_\infty t_0)^2)^5 = 1}$$

where N is a group on 12 letters isomorphic to the Mathieu group M_{11} and $\sigma = (\infty)(3,4)(0,1,8)(2,5,6,X,9,7) \in N$.

A symmetric presentation of the progenitor $2^{*12}:M_{11}$, where M_{11} is the Mathieu group is given by:

$<x,y,t \mid x^8,\ y^5,\ y^{(x\wedge 2)}*y^3,\ (x*y)^6,\ (x^3*y)^{11},\ t^2,\ (t,y),$

$(t,x^4),\ (t,(x*y)^{(y\wedge 3*x\wedge 3*y)}),\ (t,(x*y)\wedge^{(y\wedge 3*x\wedge 3*y)})>,$

where the control group

$M_{11} \cong <x,y \mid x^8,\ y^5,\ y^{(x\wedge 2)}*y^3,\ (x*y)^6,\ (x^3*y)^{11}>$ and

$x \sim (1, 2, 4, 3)(5, 8, 12, 10, 6, 7, 11, 9)$, and

$y \sim (2, 5, 9, 10, 6)(3, 7, 12, 11, 8)$, and corresponding to the above σ will be $\sigma \in M_{11}$, and

$\sigma \sim (2, 5)(3, 6, 11, 9, 7, 10)(4, 8, 12)$.

The progenitor $2^{*12}:M_{11}$ is factored by the following relations $(t_1t_{12})^4 = (\sigma^3t_1t_5)^5 = (\sigma(t_1t_{12})^2)^5 = 1$. The index of M_{11} in $G \cong O'N:2$ is 116,367,552; i.e., O'N:2 is a permutation group on 116,367,552 letters.

MAGMA does not handle such big groups. We have made an effort to build the Cayley's graph for O'Nan over M_{11}, and here we will present our results. First of all, one needs to consider the number of nodes in the Cayley's graph: since the order of M_{11} is 7920, the maximum number of single cosets in a double coset can be 7920, so the number of double cosets of M_{11} in O'Nan is at least 116,367,552 / 7920 $\approx$ 14,692. In our further discussion we refer to M_{11} as N.

Obviously, to draw the graph with this many nodes is impossible, so we had to design another way to present the graph. Each node (double coset) in the Cayley's graph was represented as a simple structure that held: a name of a double coset, the number of single cosets in this double coset, and indexed sets of "parents" and "children". "Parents" and "children" represented nodes, with which a double coset was connected. A "parent" was a node whose double coset was a descendent or a node that was a "parent" to an equivalent to a double coset node that was deleted from the graph. A "child" is a node that was a descendent of a double coset. Each "parent" or "child" had two entries of information: a name and a number that identified how many points out of 12 took from this double coset to the "parent" or the "child".

For example, the double coset [1 2] had 132 single cosets, the "parent" <[1], 1> and the "children": <[1, 2, 1], 1> and <[1, 2, 3], 10>. This information is equivalent to the graphical representation given in Figure 17.

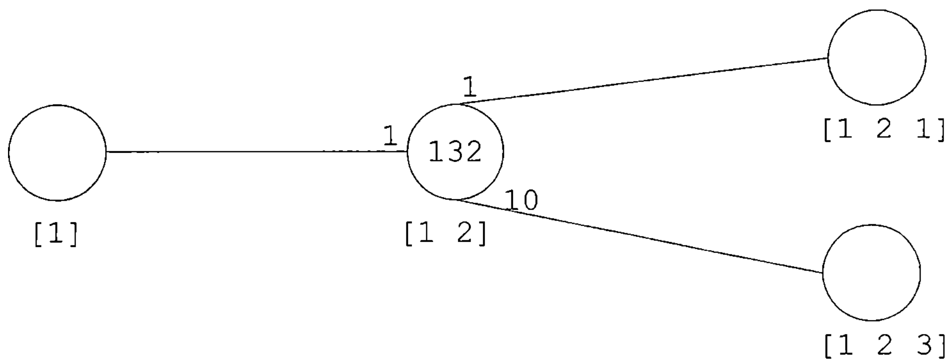

Figure 17. Node of Cayley Graph of O'Nan Over M_{11}

The next step of our design was a computer program that constructed the graph with the nodes that were represented as described above. The program is given in Appendix C. For each double coset [w] this program calculated: $N^{(w)}$ and its orbits, took a representative i from each orbit and built children from [w] and i by juxtaposition: [wi]. Then the program called the function *canon* that used the given relations to write [wi] in the short canonical form. The program ran in cycles, and in each cycle double cosets with names of the same length were calculated. Then it was a programmer's job to find the double cosets that had different names, but represented the same double coset.

Using the program described above, we calculated the following double cosets: [*], [1], [1 2], [1 2 1], [1 2 3], 6 double cosets with name of length four, 30 double cosets with name of length five, 257 double cosets with name of length six, 2611 double cosets with name of length seven, and 28,440 double cosets with name of length eight. At that point it was obvious that manually finding equivalent double cosets among double cosets with name of length eight was not practical because of time constraints.

APPENDIX A

PROGRAMS

```
Gfmt := recformat<
/*
Data structure for the symmetric representation of G.
*/
      G: GrpPerm,              //group
      N: GrpPerm,              //control group
      cst: SeqEnum,            // of integer sequences
      ts: SeqEnum,             // of elements of G
      tr: SetIndx,             // of elements of N
      tr1: SetIndx,
      tr2: SetIndx,
      prs: SetIndx,
      sgs: SeqEnum,
      ij: SeqEnum,       //name of a double coset of type [i j]
      ijk: SeqEnum,
      ijkl: SeqEnum
>;

//-----------------------------------------------------------------------
-----
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied sequentially.
*/
      v:=pt;
      for i in I do
            v:=v^(Q[i]);
      end for;
      return v;
end function;
//-----------------------------------------------------------------------
symrepSuz:=function()

G<x,y,t,s,u,v,w>:= Group<x,y,t,s,u,v,w|x^7, y^2, t^2, (x^-1*t)^2,
(y*x)^3,
t*x^-1 * y*x*t*y, x^2*y*x^3*y*x^-4*y*x^-4*y*x, s^2, (s^(x^3),y),
(s^(x^4),x*y), t*s*s^t*s, y*(s*s^(t*x^6))^2,
y*(s^(x^3))*(s^(t*x^6))*(s^x) *s, u^2, (u,t),(u,y),(u,x), s=u*u^s*u,
v^2, (v,x), (v,y), (v,t), (v,s), u=v*v^u*v, w^2, (w,x), (w,s), (w,u),
v=w *w^v*w >;
print Index(G, sub<G|x,y,t,s,u,v>);
N:=sub<G|x,y,t,s,u,v>;
s5346:=Sym(5346);
tr:=Transversal(G, N);
seq:=[i: i in [1..5346]];
for j in [1..5346] do
for r in [1..5346] do
if N*tr[j]*x eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
xx:=s5346!seq;

seq:=[i: i in [1..5346]];
for j in [1..5346] do
for r in [1..5346] do
```

```
if N*tr[j]*y eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
yy:=s5346!seq;

seq:=[i: i in [1..5346]];
for j in [1..5346] do
for r in [1..5346] do
if N*tr[j]*t eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
tt:=s5346!seq;

seq:=[i: i in [1..5346]];
for j in [1..5346] do
for r in [1..5346] do
if N*tr[j]*s eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
ss:=s5346!seq;

seq:=[i: i in [1..5346]];
for j in [1..5346] do
for r in [1..5346] do
if N*tr[j]*u eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
uu:=s5346!seq;

seq:=[i: i in [1..5346]];
for j in [1..5346] do
for r in [1..5346] do
if N*tr[j]*v eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
vv:=s5346!seq;

seq:=[i: i in [1..5346]];
for j in [1..5346] do
for r in [1..5346] do
if N*tr[j]*w eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
ww:=s5346!seq;

N:=sub<s5346|xx, yy, tt, ss, uu, vv>;
G:=sub<s5346|xx,yy,tt,ss,uu, vv,ww>;

//find symmetric generators
ts:=[ Id(s5346) : m in [1..416] ];
j:=2; i:=1;
while i ne 417 do
m:=j;
b:=0;
for g in N do //we use N here because N acts on index of symmetric
generators
if 1^(ww^g) eq j then ts[i]:=ww^g; i:=i+1; j:=j+1; break; end if;
b:=b+1;
if b ge 500 then break; end if;
end for;
```

```
if m eq j then j:=j+1; end if;
end while;

s416:=Sym(416);
x:=s416!(4,5,8,15,25,14,7)(6,11,22,23,12,16,9)(10,19,34,48,35,20,
    13)(17,18,32,29,43,26,31)(21,37,54,69,55,38,24)(27,44,57,36,
    41,47,28)(30,33,52,49,64,42,51)(39,59,77,96,78,61,40)(45,65,
    81,56,62,68,46)(50,53,75,71,89,63,72)(58,82,94,127,157,118,
    84)(60,85,117,102,73,101,87)(66,92,112,80,88,93,67)(70,97,113,
    139,156,135,99)(74,103,140,107,76,106,105)(79,109,95,128,134,
    129,111)(83,114,126,165,192,155,116)(86,119,154,137,100,136,
    121)(90,123,145,168,163,124,91)(98,131,151,176,191,174,133)(104,
    141,178,146,108,144,143)(110,148,130,167,173,169,150)(115,152,
    166,198,226,190,153)(120,158,188,177,138,175,159)(122,160,182,
    200,195,161,125)(132,171,186,206,224,204,172)(142,179,208,183,
    147,181,180)(149,184,170,199,202,201,185)(162,196,212,236,231,
    197,164)(187,217,266,346,352,271,219)(189,221,275,356,361,279,
    223)(193,227,284,368,357,276,222)(194,228,253,329,376,288,
    230)(203,239,304,392,380,306,240)(205,242,310,375,351,291,
    244)(207,246,318,332,354,321,247)(209,249,325,403,406,328,
    251)(210,252,229,286,373,330,254)(211,255,277,358,343,264,
    257)(213,259,338,312,398,340,260)(214,261,341,394,348,268,
    218)(215,262,342,409,404,326,250)(216,263,278,359,331,256,
    265)(220,272,270,350,402,323,274)(225,281,314,399,313,366,
    282)(232,290,287,374,311,243,292)(233,267,345,360,344,369,
    293)(234,294,372,285,370,383,296)(235,295,355,327,405,371,
    297)(237,299,334,397,333,389,300)(238,301,339,336,407,391,
    303)(241,307,384,415,378,396,309)(245,315,381,367,410,401,
    317)(248,322,269,349,353,273,324)(258,335,305,302,390,408,
    337)(280,362,388,386,416,413,364)(283,320,319,316,400,411,
    347)(289,377,382,393,308,395,379)(298,385,365,363,412,414,387);
y:=s416!(6,12)(7,8)(11,22)(13,17)(14,25)(19,31)(20,32)(24,30)(26,41)(29
,48)(34,36)(37,51)(38,52)(40,50)(42,62)(43,57)(44,47)(49,69)(54,
    56)(58,73)(59,72)(60,74)(61,75)(63,88)(64,81)(65,68)(70,95)(71,
    96)(76,79)(77,80)(82,113)(83,100)(84,117)(85,101)(86,104)(87,
    105)(89,112)(90,97)(91,94)(92,93)(98,130)(99,134)(102,139)(103,
    106)(107,145)(108,110)(109,123)(111,140)(114,151)(115,138)(116,
    154)(118,156)(119,136)(120,142)(121,143)(122,131)(124,157)(125,
    126)(129,168)(132,170)(133,173)(135,163)(137,176)(141,144)(146,
    182)(147,149)(148,160)(150,178)(152,186)(153,188)(155,191)(158,
    175)(159,180)(161,192)(162,171)(164,166)(169,200)(172,202)(174,
    195)(177,206)(179,181)(183,212)(184,196)(185,208)(187,216)(189,
    220)(190,224)(193,205)(194,210)(197,226)(201,236)(203,238)(204,
    231)(207,245)(209,211)(213,258)(214,232)(215,248)(217,267)(218,
    269)(219,270)(221,246)(222,277)(223,278)(225,280)(227,281)(228,
    252)(229,254)(230,253)(233,283)(237,298)(239,261)(240,305)(241,
    289)(242,255)(243,313)(244,314)(247,319)(249,299)(250,287)(251,
    310)(256,333)(257,334)(259,262)(260,339)(263,272)(264,344)(265,
    345)(266,347)(268,343)(271,351)(273,354)(274,318)(275,317)(276,
    353)(279,311)(282,365)(284,364)(285,329)(286,327)(288,370)(290,
    322)(291,380)(292,304)(293,381)(294,377)(295,307)(296,382)(297,
    384)(300,388)(301,335)(302,346)(303,341)(306,352)(308,371)(309,
    372)(312,332)(315,320)(316,368)(321,357)(323,398)(324,338)(325,
    387)(326,331)(328,402)(330,405)(336,403)(337,342)(340,406)(348,
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    369)(349,358)(350,375)(355,379)(356,363)(359,374)(360,397)(361,
    366)(362,385)(367,394)(373,395)(376,396)(378,383)(386,409)(389,
    404)(390,411)(391,410)(392,399)(393,415)(400,413)(401,412)(407,
    414)(408,416);
t:=s416!(4,6)(5,9)(7,11)(8,16)(10,20)(12,15)(14,22)(17,29)(18,32)(19,35
)(21,38)(23,25)(28,44)(30,49)(31,43)(33,52)(34,48)(36,41)(37,55)(39,
    61)(46,65)(47,57)(50,71)(51,64)(53,75)(54,69)(56,62)(59,78)(67,
    92)(68,81)(72,89)(73,102)(76,107)(77,96)(80,88)(82,84)(85,87)(90,
    124)(93,112)(94,118)(95,129)(97,99)(100,137)(101,117)(103,105)(106,
    140)(108,146)(109,111)(113,135)(114,116)(119,121)(122,161)(123,
    163)(126,155)(127,157)(128,134)(130,169)(131,133)(136,154)(138,
    177)(139,156)(141,143)(144,178)(145,168)(147,183)(148,150)(151,
    174)(152,153)(158,159)(160,195)(162,197)(165,192)(166,190)(167,
    173)(170,201)(171,172)(175,188)(176,191)(179,180)(181,208)(182,
    200)(184,185)(186,204)(187,218)(189,222)(193,223)(194,229)(196,
    231)(198,226)(199,202)(205,243)(206,224)(209,250)(210,253)(211,
    256)(212,236)(214,219)(215,251)(216,264)(217,268)(220,273)(221,
    276)(227,279)(228,252)(230,286)(232,291)(234,295)(235,294)(238,
    302)(239,240)(241,308)(242,311)(244,292)(245,316)(246,247)(248,
    323)(249,326)(254,329)(255,331)(257,265)(258,336)(259,260)(261,
    271)(262,328)(263,343)(266,348)(267,293)(269,350)(270,349)(272,
    353)(274,324)(275,357)(277,359)(278,358)(280,363)(281,282)(283,
    367)(284,361)(285,371)(287,375)(288,373)(289,378)(290,351)(296,
    355)(297,372)(298,386)(299,300)(301,305)(303,390)(304,306)(307,
    393)(309,395)(310,374)(312,398)(313,399)(314,366)(315,319)(317,
    400)(318,321)(320,381)(322,402)(325,404)(327,383)(330,376)(332,
    354)(333,397)(334,389)(335,339)(337,407)(338,340)(341,352)(342,
    406)(344,360)(345,369)(346,394)(347,410)(356,368)(362,365)(364,
    412)(370,405)(377,415)(379,396)(380,392)(382,384)(385,388)(387,
    416)(391,408)(401,411)(403,409)(413,414);
s:=s416!(3,4)(5,10)(7,13)(8,17)(9,18)(14,26)(15,27)(16,28)(19,36)(20,32
)(23, 35)(25,41)(29,44)(31,34)(37,58)(38,60)(43,57)(47,48)(49,70)(51,
    73)(52,74)(54,76)(56,79)(59,83)(61,86)(64,90)(65,91)(68,94)(69,
95)(71,98)(72,100)(75,104)(77,108)(80,110)(81,97)(82,84)(85,109)(87,
    105)(89,122)(92,125)(93,126)(96,130)(99,118)(101,123)(102,124)(103,
    111)(106,140)(107,129)(112,131)(113,117)(114,116)(119,148)(121,
    143)(127,128)(133,155)(134,156)(135,163)(136,160)(137,161)(139,
    157)(141,150)(144,178)(145,168)(146,169)(151,154)(152,187)(153,
    189)(158,193)(159,194)(165,167)(172,203)(173,191)(174,195)(175,
    205)(176,192)(177,207)(179,209)(180,210)(181,211)(182,200)(183,
    213)(184,214)(185,215)(186,216)(188,220)(190,225)(196,232)(197,
    233)(198,234)(199,235)(201,237)(202,238)(204,241)(206,245)(208,
    248)(212,258)(217,221)(218,222)(219,251)(223,229)(224,280)(226,
    283)(227,285)(228,252)(230,287)(231,289)(236,298)(239,259)(240,
    305)(242,312)(243,273)(244,292)(246,267)(247,320)(249,327)(250,
    253)(254,278)(255,332)(256,323)(257,265)(260,301)(261,262)(263,
    343)(264,308)(266,303)(268,272)(269,277)(270,310)(271,322)(274,
    355)(275,357)(276,353)(279,360)(281,329)(282,358)(284,369)(286,
    299)(288,330)(290,351)(291,378)(293,381)(294,302)(295,367)(296,
    297)(300,375)(304,314)(306,325)(307,394)(309,338)(311,397)(313,
    354)(315,319)(316,363)(317,321)(318,379)(324,372)(326,392)(328,
    402)(331,399)(333,398)(334,345)(335,339)(336,386)(337,340)(341,
    347)(342,406)(344,371)(346,377)(348,364)(349,365)(350,388)(352,
    387)(356,368)(359,385)(361,389)(362,374)(366,404)(370,405)(373,
```

414)(376,413)(380,383)(382,384)(390,412)(391,408)(393,415)(395,
407)(396,400)(401,411)(403,409)(410,416);
u:=s416!(2,3)(10,21)(13,24)(17,30)(18,33)(19,37)(20,38)(26,42)(27,45)(2
8, 46)(29,49)(31,51)(32,52)(34,54)(35,55)(36,56)(41,62)(43,64)(44,
65)(47,68)(48,69)(57,81)(58,79)(60,74)(70,91)(73,76)(82,109)(83,
115)(84,111)(85,103)(86,120)(87,105)(90,97)(94,95)(98,132)(99,
124)(100,138)(101,106)(102,107)(104,142)(108,147)(110,149)(113,
123)(114,152)(116,153)(117,140)(118,129)(119,158)(121,159)(122,
162)(125,164)(126,166)(127,128)(130,170)(131,171)(133,172)(134,
157)(135,163)(136,175)(137,177)(139,145)(141,179)(143,180)(144,
181)(146,183)(148,184)(150,185)(151,186)(154,188)(155,190)(156,
168)(160,196)(161,197)(165,198)(167,199)(169,201)(173,202)(174,
204)(176,206)(178,208)(182,212)(187,189)(191,224)(192,226)(193,
214)(194,210)(195,231)(200,236)(203,225)(205,232)(207,233)(209,
215)(211,248)(213,237)(216,220)(217,221)(218,222)(219,223)(227,
261)(228,252)(229,253)(230,254)(234,235)(238,280)(239,281)(240,
282)(241,289)(242,290)(243,291)(244,292)(245,283)(246,267)(247,
293)(249,262)(250,251)(255,322)(256,323)(257,324)(258,298)(259,
299)(260,300)(263,272)(264,273)(265,274)(266,275)(268,276)(269,
277)(270,278)(271,279)(284,341)(285,327)(286,329)(287,310)(288,
330)(294,295)(296,297)(301,362)(302,363)(303,364)(304,314)(305,
365)(306,366)(307,377)(308,378)(309,379)(311,351)(312,397)(313,
380)(315,320)(316,367)(317,347)(318,345)(319,381)(321,369)(325,
342)(326,328)(331,402)(332,360)(333,398)(334,338)(335,385)(336,
386)(337,387)(339,388)(340,389)(343,353)(344,354)(346,356)(348,
357)(349,358)(350,359)(352,361)(355,372)(368,394)(370,405)(371,
383)(373,376)(374,375)(382,384)(390,412)(391,413)(392,399)(393,

415)(395,396)(400,410)(401,411)(403,409)(404,406)(407,416)(408,414);
v:=s416!(1,2)(21,39)(24,40)(30,50)(33,53)(37,59)(38,61)(42,63)(45,66)(4
6, 67)(49,71)(51,72)(52,75)(54,77)(55,78)(56,80)(58,83)(60,86)(62,
88)(64,89)(65,92)(68,93)(69,96)(70,98)(73,100)(74,104)(76,108)(79,
110)(81,112)(82,114)(84,116)(85,119)(87,121)(90,122)(91,125)(94,
126)(95,130)(97,131)(99,133)(101,136)(102,137)(103,141)(105,143)(106,
144)(107,146)(109,148)(111,150)(113,151)(115,149)(117,154)(118,
155)(120,142)(123,160)(124,161)(127,165)(128,167)(129,169)(132,
164)(134,173)(135,174)(138,147)(139,176)(140,178)(145,182)(152,
184)(153,185)(156,191)(157,192)(158,179)(159,180)(162,171)(163,
195)(166,170)(168,200)(172,197)(175,181)(177,183)(186,196)(187,
214)(188,208)(189,215)(190,201)(193,209)(194,210)(198,199)(202,
226)(203,233)(204,231)(205,211)(206,212)(207,213)(216,232)(217,
261)(218,219)(220,248)(221,262)(222,251)(223,250)(224,236)(225,
237)(227,249)(228,252)(229,253)(230,254)(234,235)(238,283)(239,
267)(240,293)(241,289)(242,255)(243,256)(244,257)(245,258)(246,
259)(247,260)(263,290)(264,291)(265,292)(266,341)(268,271)(269,
270)(272,322)(273,323)(274,324)(275,342)(276,328)(277,310)(278,
287)(279,326)(280,298)(281,299)(282,300)(284,325)(285,327)(286,
329)(288,330)(294,295)(296,297)(301,320)(302,367)(303,347)(304,
345)(305,381)(306,369)(307,377)(308,378)(309,379)(311,331)(312,
332)(313,333)(314,334)(315,335)(316,336)(317,337)(318,338)(319,
339)(321,340)(343,351)(344,380)(346,394)(348,352)(349,350)(353,
402)(354,398)(355,372)(356,409)(357,406)(358,375)(359,374)(360,
392)(361,404)(362,385)(363,386)(364,387)(365,388)(366,389)(368,
403)(370,405)(371,383)(373,376)(382,384)(390,410)(391,411)(393,

```
    415)(395,396)(397,399)(400,407)(401,408)(412,416)(413,414);
N:=sub<s416|x,y,s,t,u,v>;

trans:=Transversal(N, Stabilizer(N, {1,2}));
prs:={@ {1,2}^x : x in trans @};
sgs:=[v^x : x in trans];

cst:=[null : i in [1..5346]] where null is [Integers() | ];

N1:=Stabiliser(N, 1);
N39:=Stabiliser(N1, 39);
g:=N!(1,21)(2,39)(6,142)(7,185)(8,208)(12,120)(13,215)(14,188)(16,198)(
17,248)(19,249)(20,228)(23,199)(24,138)(25,153)(26,220)(28,234)(29,
    329)(30,115)(31,299)(32,252)(33,66)(34,286)(35,235)(36,327)(37,
    175)(40,132)(41,189)(42,166)(44,281)(45,53)(47,227)(48,285)(50,
    170)(51,158)(58,205)(59,186)(61,75)(62,164)(63,149)(64,112)(65,
   184)(67,78)(68,196)(71,77)(72,152)(73,193)(80,96)(81,89)(83,216)(85,
    246)(86,104)(87,353)(88,147)(90,131)(91,214)(92,179)(93,181)(94,
  232)(97,122)(98,108)(99,311)(100,187)(101,221)(102,350)(105,276)(107,
    356)(109,267)(110,130)(114,141)(116,150)(118,397)(119,239)(121,
    240)(123,217)(124,388)(125,209)(126,211)(129,368)(133,282)(134,
    279)(135,293)(136,261)(137,392)(139,375)(143,305)(144,151)(145,
    363)(146,336)(148,259)(154,178)(155,358)(156,360)(157,300)(159,
    226)(160,262)(161,326)(162,171)(163,381)(165,167)(168,316)(169,
    386)(172,204)(173,365)(174,402)(176,399)(177,190)(180,197)(182,
    403)(183,212)(191,349)(192,331)(194,283)(195,328)(200,409)(201,
    236)(202,231)(203,241)(206,224)(207,225)(210,233)(213,258)(218,
    398)(219,251)(222,333)(223,254)(229,278)(230,344)(237,298)(238,
    289)(242,294)(243,273)(244,274)(245,280)(247,268)(250,308)(253,
    264)(255,377)(256,277)(257,400)(260,339)(263,315)(265,396)(266,
    340)(269,323)(270,310)(271,351)(272,320)(275,389)(284,288)(287,
    371)(290,322)(291,383)(292,355)(295,385)(296,352)(297,387)(301,
    335)(302,312)(303,337)(304,379)(306,382)(307,362)(309,408)(313,
    354)(314,318)(317,404)(319,343)(321,366)(324,410)(325,384)(330,
    369)(332,346)(334,413)(338,391)(341,342)(345,376)(347,406)(348,
    405)(357,361)(359,367)(364,370)(372,416)(373,401)(374,394)(378,
    380)(390,407)(393,415)(395,412)(411,414);
N39:=sub<N|N39,g>;
tr:=Transversal(N, N39);

N218:=Stabiliser(N39, 218);
g:=N!(1,13,19,6,218,107,60,87)(2,213,80,225,360,33,34,90)(3,355,410,
    66,271,64,126,304)(4,121,42,155,255,15,187,43)(5,169,261,133,
    205,159,123,372)(7,95,219,52,118,85,137,58)(8,65,92,294,367,
    247,140,30)(9,73,318,59,201,152,356,283)(10,257,22,40,196,190,
    246,415)(11,122,83,172,411,359,206,81)(12,305,25,233,370,177,
    371,282)(14,265,182,302,153,97,214,263)(16,266,336,125,300,211,
    277,113)(17,46,352,264,363,374,286,220)(18,35,102,110,146,120,
    392,76)(20,124,48,229,56,24,82,207)(21,210,117,203,136,183,
    285,274)(23,129,114,27,86,331,26,194)(28,287,186,44,301,165,
    243,379)(29,164,345,91,381,101,49,338)(31,375,405,99,189,242,
    256,328)(32,348,37,251,275,38,278,134)(36,296,227,142,310,135,
    100,290)(39,108,248,78,330,399,167,409)(41,161,354,386,380,349,
    62,244)(45,116,376,115,180,54,320,192)(47,130,75,308,361,272,
```

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    312,109)(50,342,378,315,128,358,154,70)(51,57,119,316,149,324,
    289,268)(53,332,401,131)(55,105,341,295,61,403,344,353)(63,223,
    387,368,252,168,94,141)(67,157,132,173,323,224,284,156)(68,297,
    398,197,111,292,217,250)(69,215,311,74,340,373,104,303)(71,390,
    98,238,347,163,321,145)(72,413,150,307,327,179,408,209)(77,299,
    191,235,402,216,325,181)(79,253,291,273,230,139,84,103)(88,
    393)(89,193,309,166,335,204,394,396)(93,388,236,319,407,162,
    148,414)(96,170,185,138,276,343,346,262)(106,143,127,222,249,
    237,228,288)(112,171)(144,306,337,389,364,365,259,281)(147,151,
    406,198,326,329,280,178)(158,397,369,293,258,260,313,391)(160,
    385)(174,245,212,416)(175,314,339,267)(176,231,400,298)(184,362,
    383,377,195,351,317,334)(188,269,382,202,239,412,221,366)(199,
    357,240,234,254,208,404,270)(200,241)(232,350)(279,322);
N218:=sub<N|N218, g>;
tr1:=Transversal(N,N218);

N333:=Stabiliser(N218,333);
g:=N!(1,81,43)(2,10,112)(3,89,21)(4,171,122)(5,162,131)(6,302,316)(7,84
,   153)(8,154,140)(12,394,356)(13,187,257)(14,178,117)(16,374,
    350)(17,101,345)(18,33,415)(19,58,304)(20,268,192)(23,312,360)(24,
    274,103)(25,111,185)(26,123,408)(27,45,393)(28,339,174)(30,338,
    220)(31,239,324)(32,319,133)(35,271,121)(36,327,80)(37,334,
    227)(38,202,293)(39,64,57)(40,119,193)(41,209,410)(42,413,248)(44,
    259,400)(46,231,300)(47,94,401)(48,285,96)(49,71,286)(50,232,
    175)(51,244,76)(52,197,353)(54,77,329)(55,180,311)(59,217,115)(60,
    206,137)(61,260,279)(62,407,82)(63,205,196)(65,390,70)(67,247,
    381)(68,391,249)(72,100,246)(73,355,108)(74,190,191)(75,351,
    388)(78,343,276)(79,149,309)(85,396,141)(86,332,169)(88,148,
    214)(91,395,98)(92,125,267)(93,221,166)(95,170,376)(102,129,
    409)(104,359,182)(105,326,402)(106,181,373)(107,139,349)(109,372,
    114)(110,262,318)(113,186,379)(118,392,168)(120,183,375)(124,305,
    331)(127,201,146)(128,212,200)(130,261,414)(132,416,281)(134,328,
    282)(142,236,397)(143,252,159)(144,211,314)(145,336,156)(147,265,
    299)(151,216,411)(152,412,189)(155,167,346)(161,235,172)(163,365,
    240)(165,367,176)(173,234,226)(177,368,198)(179,292,215)(194,255,
    290)(195,228,204)(199,224,363)(203,320,280)(207,283,263)(218,303,
    348)(219,405,250)(223,406,354)(225,307,233)(237,322,362)(238,245,
    294)(241,385,301)(242,258,335)(243,287,266)(251,380,317)(253,369,
    366)(254,352,357)(264,389,270)(273,382,291)(277,321,284)(278,310,
    288)(297,344,313)(306,323,342)(325,341,371)(333,361,387)(337,378,
    364);
N333:=sub<N|N333,g>;
tr2:=Transversal(N,N333);

for i := 1 to 416 do
      cst[prodim(1, ts, [i])] := [i];
end for;
for i := 1 to 4095 do
      ss:= [1,39]^tr[i];
      cst[prodim(1, ts, ss)] := ss;
end for;
for i := 1 to 832 do
      ss:= [1,39,218]^tr1[i];
      cst[prodim(1, ts, ss)] := ss;
```

```
end for;
for i := 1 to 2 do
      ss:= [1,39,218,333]^tr2[i];
      cst[prodim(1, ts, ss)] := ss;
end for;

ij:=[1,39]; ijk:=[1,39,218]; ijkl:=[1,39,218,333];

return rec<Gfmt | G := G, N := N, cst := cst, ts := ts, tr := tr, tr1:=
tr1, tr2:= tr2, prs:=prs, sgs:=sgs, ij:=ij, ijk:=ijk, ijkl:=ijkl>;
end function;

//----------------------------------------------------------------------
-----
symrepG:=function()
/*
Initialize the data structures for the symmetric representation of
G_2(4):2.
*/
G<x,y,t,s,u,v>:= Group<x,y,t,s,u,v|x^7, y^2, t^2, (x^-1*t)^2, (y*x)^3,
t*x^-1 * y*x*t*y, x^2*y*x^3*y*x^-4*y*x^-4*y*x, s^2, (s^(x^3),y),
(s^(x^4),x*y), t*s*s^t*s, y*(s*s^(t*x^6))^2,
y*(s^(x^3))*(s^(t*x^6))*(s^x) *s, u^2, (u,t),(u,y),(u,x), s=u*u^s*u,
v^2, (v,x), (v,y), (v,t), (v,s), u=v*v^u*v >;
print Index(G, sub<G|x,y,t,s,u>);
N:=sub<G|x, y, t,s,u>;

s416:=Sym(416);
tr:=Transversal(G, N);
seq:=[i: i in [1..416]];
for j in [1..416] do
for r in [1..416] do
if N*tr[j]*x eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
x:=s416!seq;

seq:=[i: i in [1..416]];
for j in [1..416] do
for r in [1..416] do
if N*tr[j]*y eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
y:=s416!seq;

seq:=[i: i in [1..416]];
for j in [1..416] do
for r in [1..416] do
if N*tr[j]*t eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
t:=s416!seq;

seq:=[i: i in [1..416]];
for j in [1..416] do
for r in [1..416] do
```

```
if N*tr[j]*s eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
s:=s416!seq;

seq:=[i: i in [1..416]];
for j in [1..416] do
for r in [1..416] do
if N*tr[j]*u eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
u:=s416!seq;

seq:=[i: i in [1..416]];
for j in [1..416] do
for r in [1..416] do
if N*tr[j]*v eq N*tr[r] then seq[j]:=r; break; end if; end for; end
for;
v:=s416!seq;

N:=sub<s416|x, y, t, s,u>;
G:=sub<s416|x,y,t,s,u,v>;

ts:=[ Id(s416) : m in [1..100] ];
j:=2; i:=1;
while i ne 101 do
m:=j;
b:=0;
for g in N do //we use N here because N acts on index of symmetric
generators
if 1^(v^g) eq j then ts[i]:=v^g; i:=i+1; j:=j+1; break; end if;
b:=b+1;
if b ge 10000 then break; end if;
end for;
if m eq j then j:=j+1; end if;
end while;
s100:=SymmetricGroup(100);
x:=s100!(3, 4, 7, 14, 24, 13, 6)(5, 10, 21, 22, 11, 15, 8)(9, 18, 33,
45, 34, 19, 12)(16, 17, 31, 28, 40, 25, 30)(20, 36, 49, 59, 50, 37,
23)(26, 41, 52, 35, 38, 44, 27)(29, 32, 48, 46, 56, 39, 47)(42, 57, 65,
51, 55, 58, 43)(53, 66, 72, 89, 98, 86, 67)(54, 68, 85, 77, 61, 76,
69)(60, 74, 84, 94, 97, 93, 75)(62, 78, 95, 81, 63, 80, 79)(64, 82, 73,
90, 92, 91, 83)(70, 87, 96, 100, 99, 88, 71);
y:=s100!(5, 11)(6, 7)(10, 21)(12, 16)(13, 24)(18, 30)(19, 31)(23,
29)(25, 38)(28, 45)(33, 35)(36, 47)(37, 48)(39, 55)(40, 52)(41, 44)(46,
59)(49, 51)(53, 61)(54, 62)(56, 65)(57, 58)(60, 73)(63, 64)(66, 84)(67,
85)(68, 76)(69, 79)(70, 74)(71, 72)(75, 92)(77, 94)(78, 80)(81, 96)(82,
87)(83, 95)(86, 97)(88, 98)(91, 100)(93, 99);
t:=s100!(3, 5)(4, 8)(6, 10)(7, 15)(9, 19)(11, 14)(13, 21)(16, 28)(17,
31)(18, 34)(20, 37)(22, 24)(27, 41)(29, 46)(30, 40)(32, 48)(33, 45)(35,
38)(36, 50)(43, 57)(44, 52)(47, 56)(49, 59)(51, 55)(58, 65)(61, 77)(63,
81)(66, 67)(68, 69)(70, 88)(72, 86)(73, 91)(74, 75)(76, 85)(78, 79)(80,
95)(82, 83)(84, 93)(87, 99)(89, 98)(90, 92)(94, 97)(96, 100);
s:=s100!(2, 3)(4, 9)(6, 12)(7, 16)(8, 17)(13, 25)(14, 26)(15, 27)(18,
35)(19, 31)(22, 34)(24, 38)(28, 41)(30, 33)(36, 53)(37, 54)(40, 52)(44,
45)(46, 60)(47, 61)(48, 62)(49, 63)(51, 64)(56, 70)(57, 71)(58, 72)(59,
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73)(65, 74)(66, 67)(68, 82)(69, 79)(75, 86)(76, 87)(77, 88)(78, 83)(80,
95)(81, 91)(84, 85)(89, 90)(92, 97)(93, 99)(94, 98)(96, 100);
u:=s100!(1, 2)(9, 20)(12, 23)(16, 29)(17, 32)(18, 36)(19, 37)(25,
39)(26, 42)(27, 43)(28, 46)(30, 47)(31, 48)(33, 49)(34, 50)(35, 51)(38,
55)(40, 56)(41, 57)(44, 58)(45, 59)(52, 65)(53, 64)(54, 62)(60, 71)(61,
63)(66, 82)(67, 83)(68, 78)(69, 79)(70, 74)(72, 73)(75, 88)(76, 80)(77,
81)(84, 87)(85, 95)(86, 91)(89, 90)(92, 98)(93, 99)(94, 96)(97, 100);

N:=sub<s100|x,y,t,s,u>;
cst:=[null : i in [1..416]] where null is [Integers() | ];

trans:=Transversal(N, Stabiliser(N, {1,2}));
prs:={@ {1,2}^x : x in trans @};
sgs:=[u^x : x in trans];

N1:=Stabiliser(N, 1);
N20:=Stabiliser(N1, 20);
for g in N do if [1,20]^g eq [9,2] then
N20:=sub<N| N20, g>; end if; end for;
tr:=Transversal(N, N20); //#tr=315

for i := 1 to 100 do
      cst[prodim(1, ts, [i])] := [i];
end for;
for i := 1 to 315 do
      ss:= [1,20]^tr[i];
      cst[prodim(1, ts, ss)] := ss;
end for;

ij:=[1,20];
ijk:=[0,0,0]; ijkl:=[0,0,0,0]; tr1:={@ @}; tr2:={@ @};

return rec<Gfmt | G := G, N := N, cst := cst, ts := ts, tr := tr,
tr1:=tr1, tr2:=tr2, prs:=prs, sgs:=sgs, ij:=ij, ijk:=ijk, ijkl:=ijkl>;
end function;

//-------------------------------------------------------------------
symrepJ := function()
/*
Initialize the data structures for the symmetric representation of
J2:2.
*/
      J<x,y,t,s,u>:= Group<x,y,t,s,u|x^7, y^2, t^2, (x^-1*t)^2,
(y*x)^3, t*x^-1 * y*x*t*y, x^2*y*x^3*y*x^-4*y*x^-4*y*x, s^2,
(s^(x^3),y), (s^(x^4),x*y), t*s*s^t*s, y*(s*s^(t*x^6))^2,
y*(s^(x^3))*(s^(t*x^6))*(s^x) *s, u^2, (u,t),(u,y),(u,x), s=u*u^s*u >;
print Index(J, sub<J|x,y,t,s>); //100
      f, G, k:=CosetAction(J, sub<J|x, y, t,s>);

x:=f(x); y:=f(y); t:=f(t); s:=f(s); u:=f(u);
s100:=Sym(100);
N:=sub<s100|x, y, t,s>;

ts:=[ Id(s100) : m in [1..36]];
```

```
j:=2; i:=1;
while i ne 37 do
m:=j;
for g in N do
if 1^(u^g) eq j then ts[i]:=u^g; i:=i+1; j:=j+1; break; end if;
end for;
if m eq j then j:=j+1; end if;
end while;

s36:=Sym(36);
x:=s36!(2, 3, 6, 13, 21, 12, 5)(4, 9, 19, 20, 10, 14, 7)(8, 17, 28, 35,
29, 18, 11)(15, 16, 27, 25, 32, 22, 26)(23, 33, 36, 30, 31, 34, 24);
y:=s36!(4, 10)(5, 6)(9, 19)(11, 15)(12, 21)(17, 26)(18, 27)(22, 31)(25,
35)(28, 30)(32, 36)(33, 34);
t:=s36!(2, 4)(3, 7)(5, 9)(6, 14)(8, 18)(10, 13)(12, 19)(15, 25)(16,
27)(17, 29)(20, 21)(24, 33)(26, 32)(28, 35)(30, 31)(34, 36);
s:=s36!(1, 2)(3, 8)(5, 11)(6, 15)(7, 16)(12, 22)(13, 23)(14, 24)(17,
30)(18, 27)(20, 29)(21, 31)(25, 33)(26, 28)(32, 36)(34, 35);
N:=sub<s36| x, y, t,s>;
cst:=[null : i in [1..100]] where null is [Integers() | ];

N1:=Stabiliser(N, 1);
N18:=Stabilizer(N1, 8);
for g in N do if [1,8]^g eq [7,13] then
      N18:=sub<N|N18, g>;
end if; end for;
for g in N do if [1,8]^g eq [2,3] then
      N18:=sub<N|N18, g>;
end if ; end for;
tr:=Transversal(N, N18);

for i := 1 to 36 do
      cst[prodim(1, ts, [i])] := [i];
end for;
for i := 1 to 63 do
      ss:= [1,8]^tr[i];
      cst[prodim(1, ts, ss)] := ss;
end for;

trans := Transversal(N, Stabilizer(N, {1,2}));
prs := {@ {1,2}^x : x in trans @};
sgs := [s^x : x in trans];

ij:=[1,8];
ijk:=[0,0,0]; ijkl:=[0,0,0,0]; tr1:={@ @}; tr2:={@ @};

return rec<Gfmt | G := G, N := N, cst := cst, ts := ts, tr := tr,
tr1:=tr1, tr2:=tr2, prs:=prs, sgs:=sgs, ij:=ij, ijk:=ijk, ijkl:=ijkl>;
end function;
```

```
//--------------------------------------------------------------------------
symrepU := function()
/*
Initialize the data structures for the symmetric representation of
G_2(4):2.
*/
      U<x,y,t,s>:=Group<x,y,t,s|x^7, y^2, t^2, (x^-1*t)^2, (y*x)^3,
t*x^-1 * y*x*t*y, x^2*y*x^3*y*x^-4*y*x^-4*y*x, s^2, (s^(x^3),y),
(s^(x^4),x*y), t*s*s^t*s, y*(s*s^(t*x^6))^2,
y*(s^(x^3))*(s^(t*x^6))*(s^x) *s>;
print Index(U, sub<U|x,y,t>);

      f, G, k:=CosetAction(U, sub<U|x, y, t>);

ts:=[ Id(G) : j in [1..14] ];
for j:=1 to 7 do ts[j]:=f(s^(x^j)); end for;
for j:=8 to 14 do ts[j]:=f((s^t)^(x^(14-j))); end for;

// Construct representatives cst for the control subgroup N
// as words in the symmetric generators consisting of the empty
// word,14 words of length one and 21 words of length two.

s14:=Sym(14);
x:=s14!(1,2,3,4,5,6,7)(14,13,12,11,10,9,8);
y:=s14!(2,6)(4,5)(14,10)(12,13);
t:=s14!(7,14)(1,8)(2,9)(3,10)(4,11)(5,12)(6,13);
N:=sub<s14| x, y, t >;
cst:=[null : i in [1..36]] where null is [Integers() | ];

N1:=Stabiliser(N, 1);
N71:=Stabilizer(N1, 7);
for g in N do if [7,1]^g eq [13,12] then
      N71:=sub<N|N71, g>;
end if; end for;

tr:=Transversal(N, N71);

for i := 1 to 14 do
      cst[prodim(1, ts, [i])] := [i];
end for;
for i := 1 to 21 do
      ss:= [7,1]^tr[i];
      cst[prodim(1, ts, ss)] := ss;
end for;

N7:=Stabiliser(N,7); N14:=Stabiliser(N7,14);
trans := Transversal(N, N14);
prs := {@ {7,14}^x : x in trans @};
sgs := [t^x : x in trans];

ij:=[7,1];
ijk:=[0,0,0]; ijkl:=[0,0,0,0]; tr1:={@ @}; tr2:={@ @};
```

```
return rec<Gfmt | G := G, N := N, cst := cst, ts := ts, tr := tr,
tr1:=tr1, tr2:=tr2, prs:=prs, sgs:=sgs, ij:=ij, ijk:=ijk, ijkl:=ijkl>;
end function;

//--------------------------------------------------------------------
-------
mult:=function(GDes, x, y)
/*
Return in its symmetric representation the product of elements x and y
of G themselves symmetrically represented.
*/
G := GDes`G; cst := GDes`cst; ts := GDes`ts; N := GDes`N;
rrr:=N!x[1];  sss:=N!y[1];
uu:=x[2]^sss; vv:=y[2];
tt:=&*[G|ts[uu[i]]: i in [1..#uu]] * &*[G|ts[vv[i]]: i in [1..#vv]];
ww:=cst[1^tt];
tt:=tt* &*[G|ts[ww[#ww - k +1]]:k in [1..#ww]];
zz:=N![rep{j: j in [1..#ts] | (1^ts[i])^tt eq 1^ts[j]}: i in [1..#ts]];
return <rrr*sss*zz, ww>;
end function;

//-----------------------------------------------------
per2sym:= function(GDes, p)
/*
Convert permutation p of G on 100 letters into its symmetric
representation. The image of 1 under p gives the coset representative
for Np as a word ww in the symmetric generators. Multiplication of p by
the symmetric generators of ww in reverse order yields a permutation
which can be identified with an element of N by its action on the 36
cosets of length one.
*/
G := GDes`G; cst := GDes`cst; ts := GDes`ts; N := GDes`N;
ww:=cst[1^p];
tt:=p*&*[G|ts[ww[#ww - l +1]]: l in [1..#ww]];
zz:=N![rep{j: j in [1..#ts] | (1^ts[i])^tt eq 1^ts[j]}: i in [1..#ts]];
return <zz,  ww>;
end function;

//--------------------------------------------------------------------
-------
sym2per:=function(GDes, x)
/*
Convert an element x of G in the symmetric representation into a
permutation acting on 100 letters. The image of an element of N is
determined by its action on the eleven cosets whose representatives
have length one.
*/
G := GDes`G; cst := GDes`cst; ts := GDes`ts; tr := GDes`tr;
N := GDes`N; ij:=GDes`ij; tr1:=GDes`tr1; tr2:=GDes`tr2; ijk:=GDes`ijk;
ijkl:=GDes`ijkl;

xx:=x[1];
uu:=x[2];
p:=[1: i in [1..#ts]];
```

```
for i:=1 to #ts do
      p[prodim(1,ts, [i])]:=prodim(1,ts,[i]^xx);
end for;
for i:=1 to #tr do
      ss:=ij^tr[i];
      p[prodim(1,ts, ss)]:=prodim(1,ts, ss^xx);
end for;
for i:=1 to #tr1 do
      ss:=ijk^tr1[i];
      p[prodim(1,ts, ss)]:=prodim(1,ts, ss^xx);
end for;
for i:=1 to #tr2 do
      ss:=ijkl^tr2[i];
      p[prodim(1,ts, ss)]:=prodim(1,ts, ss^xx);
end for;

aa:=(G ! p)*&*[G|ts[uu[j]]: j in [1..#uu]];
return aa;
end function;

//---------------------------------------------------------------------
cenelt := function(GDes, x)
/*
Construct the centralizer of element x of G given its symmetric
representation.
*/
cent := Centralizer(GDes`G, sym2per(GDes, x));
return <Order(cent), [per2sym(GDes, c): c in Generators(cent)]>;
end function;

/*
      Program 2
      In this program we assume detailed knowledge of the control
subgroup N, but use no representation of elements G other than their
symmetric representation. First, the procedure unify combines two
symmetrically represented elements (xx, uu) and (yy, vv) into a single
sequence ss of length (#ts + length(uu) + length(vv)), which represents
a permutation of N followed by a word in symmetric generators. The
procedure canon then takes such sequence and reduces it to its shortest
form using the following recursive algorithm. We make use of relations
ij = P(ij)i, where i,j are indexes of the sym. generators.

The algorithm:

Step I: If two adjacent symmetric generators are equal, delete them.

Step II: If P(ij) exists, replace ij by P(ij)i and move the permutation
P(ij) over the preceding symmetric generators in the standard manner.
If P(ij) does not exist, we denote it as Id(N) for the program's
reference.

Step III: If a string ijk appears with P(ij) = P(jk) = Id(N) (both do
not exist), then find such P(jm) that both P(i^P(jm),j) and P(m, k)
```

```
exist and replace j by P(jm)jm, and move the permutation Pjm over the
preceding symmetric generators in the standard manner.

Step IV: If a string ij appears with P(ij) = Id(N), then find such
P(jm) that both P(im) and P(mj) exist and replace i by P(im)im, and
move the permutation P(im) over the preceding symmetric generators in
the standard manner, then replace mj by P(mj)m and move P(mj) over the
preceding symmetric generators.

After each step recall canon.
*/
/*-----------------------------------------------------------------------
Given two symmetrically represented elements of G, where x[1] and y[1]
are permutations of N and x[2] and y[2] are words in the symmetric
generators, return a single sequence of length #ts + l(x[2]) + l(y[2])
using the above identity.
*/
Unify := func<x, y | [q[p[i]] : i in [1..#p] ] cat y[2]
where p is Eltseq(x[1]) cat x[2]
where q is Eltseq(y[1]) >;

/*----------------------------------------------------------------------
Return permutation of N given by the word ij in the symmetric
generators. Here we use the relationship t=s1s2s1 (sg as above)
*/
Pi := func<GDes,i,j | Index(GDes`prs, {i,j}) ne 0 select
GDes`sgs[Index(GDes`prs, {i,j})] else Id(GDes`N)>;

/*---------------------------------------------------------------------
Return permutation of N given the word ijk in the symmetric generators.
Here we use relationship that was explained in Step III.
*/
Pe := func<GDes,i,j,k | exists(g){g : g in [1..#GDes`ts] | Pi(GDes, j,
g) ne Id(GDes`N) and Pi(GDes, i^Pi(GDes, j, g), j) ne Id(GDes`N) and
Pi(GDes, j^Pi(GDes, j, g), k) ne Id(GDes`N)} select Pi(GDes, j, g) else
Id(GDes`N) >;

/*---------------------------------------------------------------------
Return permutation of N given the word ij in the symmetric generators.
Here we use relationship that was explained in Step IV.
*/
Ph := func<GDes,i,j | exists(g){g : g in [1..#GDes`ts] | Pi(GDes, i, g)
ne Id(GDes`N) and Pi(GDes, g, j) ne Id(GDes`N) } select Pi(GDes, i, g)
else Id(GDes`N) >;

/*-----------------------------------------------------------------------
For ss a sequence representing a permutation of N followed by a word in
the symmetric generators, return an equivalent sequence of canonically
shortest length. $$ calls canon again
*/
canon := function(GDes, ss)
s:=ss; ts:=GDes`ts;

//Step 1. If adjusted ti are the same, delete them
```

```
if exists(i){i : i in [#ts+1..#s-1] | s[i] eq s[i+1] } then
s:= $$(GDes,  s[1..i-1] cat s[i+2..#s] ); end if;

//Step 2. If adjusted titj have Pij then change titj=Pijti  and move
Pij to the left.
if exists(j){j : j in [#ts+1..#s-1] | Pi(GDes, s[j], s[j+1]) ne
Id(GDes`N)} then
s:= $$(GDes,  [ p[s[k]] : k in [1..j-1] ] cat s[j..j] cat s[j+2..#s]
           where p is Eltseq(Pi(GDes, s[j], s[j+1])));
end if;

// Step 3. If the length of the word in symmetric generators is still 3
or greater, then it means that Pij for adjusted i and j does not exist.
So, we use the relationship: ijk = iP(jg)jgk (jgj = Pjg) for the middle
element in the word of sym. gen.
if #s ge #ts+3 then
if exists(j){j : j in [#ts+2..#s-1] | Pe(GDes, s[j-1], s[j], s[j+1]) ne
Id(GDes`N)} then
H:=Pe(GDes, s[j-1], s[j], s[j+1]);
s:= $$(GDes, [ p[s[k]] : k in [1..j-1] ] cat s[j..j] cat [s[j]^H] cat
s[j+1..#s]        where p is Eltseq(H));
end if;
end if;

//Step 4. This step is for Suz. The previous three steps take care of
J_2:2 and G_2(4):2. If the length of the word in symmetric generators
is still 5 or greater then use the ijk = iP(jg)jgk (jgj = P(jg)).
if #s ge #ts+5 then
if exists(j){j : j in [#ts+2..#s-1] | Ph(GDes, s[j], s[j+1]) ne
Id(GDes`N)} then
H:=Ph(GDes, s[j], s[j+1]);
s:= [ p[s[k]] : k in [1..j-1] ] cat s[j..j] cat [s[j]^H] cat s[j+1..#s]
      where p is Eltseq(H);
s:=$$(GDes, [ q[s[k]] : k in [1..j] ] cat s[j+1..j+1] cat s[j+3..#s]
      where q is Eltseq(Pi(GDes, s[j+1], s[j+2])));
end if;
end if;

return s;
end function;

/* ------------------------------------------------------------------
canon function for the group U_3(3):2 is slightly different since it
uses two additional relations.
*/
UDes:=symrepU();
s14:=Sym(14);
y:=s14!(2,6)(4,5)(14,10)(12,13);

tr2:=Transversal(UDes`N,Stabiliser(UDes`N, [7,8]));
prs2:= {@ [7,8]^x : x in tr2 @};
sgs2 := [y^x : x in tr2];
P2 := func< i,j | Index(prs2, [i,j]) ne 0 select sgs2[Index(prs2,
[i,j])] else Id(UDes`N)>;
```

```
tr3:=Transversal(UDes`N,Stabiliser(UDes`N, [7,1]));
prs3:= {@ [7,1]^x : x in tr3 @};
prs32:=[ [3,8]^x : x in tr3 ];
sgs3 := [y^x : x in tr3];
P3 := func< i,j | Index(prs3, [i,j]) ne 0 select <sgs3[Index(prs3,
[i,j])], prs32[Index(prs3,[i,j])]> else <Id(UDes`N),[0,0]>>;

canonU := function(GDes, ss)
s:=ss; ts:=GDes`ts;

//Step 1. If adjusted ti are the same, delete them
if exists(i){i : i in [#ts+1..#s-1] | s[i] eq s[i+1] } then
s:= $$(GDes,  s[1..i-1] cat s[i+2..#s] ); end if;

//Step 2. If adjusted titj have Pij then change titj=Pijti  and move
Pij to the left.
if exists(j){j : j in [#ts+1..#s-1] | Pi(GDes, s[j], s[j+1]) ne
Id(GDes`N)} then
s:= $$(GDes,  [ p[s[k]] : k in [1..j-1] ] cat s[j..j] cat s[j+2..#s]
            where p is Eltseq(Pi(GDes, s[j], s[j+1])) );
end if;

// Step 3. If the length of the word in symmetric generators is still 3
or greater, then it means that Pij for adjusted i and j does not exist.
So, we use the relationship: ijk = iP(jg)jgk (jgj = P(jg)) for the
middle element in the word of sym. gen.
if #s ge #ts+3 then
if exists(j){j : j in [#ts+2..#s-1] | Pe(GDes, s[j-1], s[j], s[j+1]) ne
Id(GDes`N)} then
H:=Pe(GDes, s[j-1], s[j], s[j+1]);
s:= $$(GDes, [ p[s[k]] : k in [1..j-1] ] cat s[j..j] cat [s[j]^H] cat
s[j+1..#s]        where p is Eltseq(H) );
end if; end if;

//Step 4. If the length of the word in symmetric generators is still 3
or greater, then we need to use other relations. This step uses the
relation y7,1=3,8.
if #s ge #ts+3 then
if exists(j){j : j in [#ts+1..#s-1] | P3(s[j], s[j+1])[1] ne Id(GDes`N)
} then
s:= $$(GDes, [ p[s[k]] : k in [1..j-1] ] cat P3(s[j], s[j+1])[2] cat
s[j+2..#s] where p is Eltseq(P3(s[j], s[j+1])[1]));
end if;
end if;

//Step 5. If the length of the word in symmetric generators is still 3
or greater then we need to use the relation y7,8=8,7
if #s ge #ts+3 then
if exists(j){j : j in [#ts+1..#s-1] | P2(s[j], s[j+1]) ne Id(GDes`N) }
then
s:= $$(GDes, [p[s[k]] : k in [1..j-1] ] cat s[j+1..j+1] cat s[j..j] cat
s[j+2..#s] where p is Eltseq(P2(s[j], s[j+1])) );
end if;
```

```
end if;

return s;
end function;

/*-----------------------------------------------------------------
Return the product of two symmetrically represented elements of G
*/
Prod := function(GDes, x, y)
if #GDes`N eq 336 then
t:=canonU(GDes, Unify(x,y));
else
t:=canon(GDes, Unify(x, y));
end if;
return <GDes`N!t[1..#GDes`ts], t[#GDes`ts+1..#t]>;
end function;

/*-----------------------------------------------------------------
Return the inverse of a symmetrically represented element of G
*/
Invert := func< x, u | x^-1, [u[#u-i+1]^(x^-1) : i in [1..#u] ] >;

/*                        Program 3

In this program the functions deal with nested symmetric
representations of elements of a group. For example, an element of
G_2(4):2 is represented as a permutation of PGL_2(7) followed by a word
in symmetric generators of length at most 6.
*/
per2symNest:= function(Q, p)
/*
Converts permutation p of the largest G in Q into nested symmetric
representation: a permutation of the smallest group in Q followed by
symmetric generators. For example: Q:=[GDes, JDes, UDes]. The program
is written in such a way that it is imperative to arrange elements of Q
from the largest to the smallest in that order. GDes, JDes, and UDes
here are structures returned by the corresponding "symrep" functions.
The result will be given in the form <perm, [[t1,t2],
[s1,s2],[u1,u2]]>, where perm is a permutation of the smallest control
group, and [t1,t2] is a word in symmetric generators of the smallest
group of length at most two, and [u1,u2] is a word in symmetric
generators of the largest group in Q.
Right now program is designed to handle Suz, G_2(4):2, J_2:2 and
U_3(3):2 - there progenitors and generators of the control groups are
aligned to work together.
*/
n:= #Q; v:=Q[#Q]; N:=v`N; res:=<Id(N), [[0,0],[0,0],[0,0],[0,0]]>;
sr:=p;
for j in [1..#Q] do
sr:=per2sym(Q[j], sr);
res[2][n]:= sr[2];
n:=n-1;
sr:=sr[1];
end for;
```

```
res[1]:=sr;

return res;
end function;

//------------------------------------------------------------------
sym2perNest:=function(Q,w)
/*
Converts an element of the largest group G in Q given in nested
symmetric representation into a permutation.
*/
n:=#Q; x:=w[1]; u:=w[2];
for j in [1..#Q] do
if u[j] ne [0,0] then
x:=sym2per(Q[n], <x,u[j]>);
n:=n-1;
end if;
end for;
return x;
end function;

//------------------------------------------------------------------
multNest:=function(Q, x,y)
/*
multiply two elements of the largest group in Q and give the result in
the form of a permutation of the smallest group in Q followed by sym
generators of all groups of Q; x and y are given as the result of the
function, i.e. x=<smallest_permutation,
[[s1,s2],[u1,u2],[v1,v2],[w1,w2]]>.
*/

//Step 1 Convert x and y into symmetric representation of elements of
the largest group in Q.
x:=<sym2perNest(Q[2..#Q], x), x[2][#Q]>;
y:=<sym2perNest(Q[2..#Q],y), y[2][#Q]>;

//Step 2 Call mult function to perform multiplication of elements of
the largest group given in their symmetric representation.
m:=mult(Q[1], x, y);

//Step 3 Convert the result of multiplication into nested symmetric
representation
p:=per2symNest(Q[2..#Q], m[1]);
p[2][#Q]:= m[2];
res:=<p[1], p[2]>;

return res;
end function;
```

```
//--------------------------------------------------------------------
ProdNest:=function(Q, x, y)
/*
function takes x and y represented in nested symmetric form and first
converts them into symmetric representation of the largest group in Q
and then calls Prod for the result, and finally converts the result
back into nested symmetric format.
*/
//Step 1 Convert x and y into symmetric representation of elements of
the largest group in Q.
x:=<sym2perNest(Q[2..#Q], x), x[2][#Q]>;
y:=<sym2perNest(Q[2..#Q],y), y[2][#Q]>;

//Step 2 Call Prod function to perform multiplication of elements of
the largest group given in their symmetric representation.
m:=Prod(Q[1], x,y);

//Step 3 Convert the result of multiplication into nested symmetric
representation
p:=per2symNest(Q[2..#Q], m[1]);
p[2][#Q]:= m[2];
res:=<p[1], p[2]>;

return res;
end function;
```

APPENDIX B

GENERATORS OF $G_2(4){:}2$

x ~(4,5,8,15,25,14,7)(6,11,22,23,12,16,9)(10,19,34,48,35,20, 13)(17,18,
32,29,43,26,31)(21,37,54,69,55,38,24)(27,44,57,36,41,47,28)(30,33,
52,49,64,42,51)(39,59,77,96,78,61,40)(45,65,81,56,62,68,46)(50,53,
75,71,89,63,72)(58,82,94,127,157,118,84)(60,85,117,102,73,101,87)
(66,92,112,80,88,93,67)(70,97,113,139,156,135,99)(74,103,140,107,
76,106,105)(79,109,95,128,134,129,111)(83,114,126,165,192,155,116)
(86,119,154,137,100,136,121)(90,123,145,168,163,124,91)(98,131,151,
176,191,174,133)(104,141,178,146,108,144,143)(110,148,130,167,173,
169,150)(115,152,166,198,226,190,153)(120,158,188,177,138,175,159)
(122,160,182,200,195,161,125)(132,171,186,206,224,204,172)(142,179,
208,183,147,181,180)(149,184,170,199,202,201,185)(162,196,212,236,
231,197,164)(187,217,266,346,352,271,219)(189,221,275,356,361,279,
223)(193,227,284,368,357,276,222)(194,228,253,329,376,288,
230)(203,239,304,392,380,306,240)(205,242,310,375,351,291,
244)(207,246,318,332,354,321,247)(209,249,325,403,406,328,
251)(210,252,229,286,373,330,254)(211,255,277,358,343,264,
257)(213,259,338,312,398,340,260)(214,261,341,394,348,268,
218)(215,262,342,409,404,326,250)(216,263,278,359,331,256,
265)(220,272,270,350,402,323,274)(225,281,314,399,313,366,
282)(232,290,287,374,311,243,292)(233,267,345,360,344,369,
293)(234,294,372,285,370,383,296)(235,295,355,327,405,371,
297)(237,299,334,397,333,389,300)(238,301,339,336,407,391,
303)(241,307,384,415,378,396,309)(245,315,381,367,410,401,
317)(248,322,269,349,353,273,324)(258,335,305,302,390,408,
337)(280,362,388,386,416,413,364)(283,320,319,316,400,411,
347)(289,377,382,393,308,395,379)(298,385,365,363,412,414,387);

y ~ (6,12)(7,8)(11,22)(13,17)(14,25)(19,31)(20,32)(24,30)(26,41)(29,
48)(34,36)(37,51)(38,52)(40,50)(42,62)(43,57)(44,47)(49,69)(54,
56)(58,73)(59,72)(60,74)(61,75)(63,88)(64,81)(65,68)(70,95)(71,
96)(76,79)(77,80)(82,113)(83,100)(84,117)(85,101)(86,104)(87,
105)(89,112)(90,97)(91,94)(92,93)(98,130)(99,134)(102,139)(103,
106)(107,145)(108,110)(109,123)(111,140)(114,151)(115,138)(116,
154)(118,156)(119,136)(120,142)(121,143)(122,131)(124,157)(125,
126)(129,168)(132,170)(133,173)(135,163)(137,176)(141,144)(146,
182)(147,149)(148,160)(150,178)(152,186)(153,188)(155,191)(158,
175)(159,180)(161,192)(162,171)(164,166)(169,200)(172,202)(174,
195)(177,206)(179,181)(183,212)(184,196)(185,208)(187,216)(189,
220)(190,224)(193,205)(194,210)(197,226)(201,236)(203,238)(204,
231)(207,245)(209,211)(213,258)(214,232)(215,248)(217,267)(218,
269)(219,270)(221,246)(222,277)(223,278)(225,280)(227,281)(228,
252)(229,254)(230,253)(233,283)(237,298)(239,261)(240,305)(241,
289)(242,255)(243,313)(244,314)(247,319)(249,299)(250,287)(251,
310)(256,333)(257,334)(259,262)(260,339)(263,272)(264,344)(265,
345)(266,347)(268,343)(271,351)(273,354)(274,318)(275,317)(276,
353)(279,311)(282,365)(284,364)(285,329)(286,327)(288,370)(290,
322)(291,380)(292,304)(293,381)(294,377)(295,307)(296,382)(297,
384)(300,388)(301,335)(302,346)(303,341)(306,352)(308,371)(309,
372)(312,332)(315,320)(316,368)(321,357)(323,398)(324,338)(325,
387)(326,331)(328,402)(330,405)(336,403)(337,342)(340,406)(348,
369)(349,358)(350,375)(355,379)(356,363)(359,374)(360,397)(361,
366)(362,385)(367,394)(373,395)(376,396)(378,383)(386,409)(389,
404)(390,411)(391,410)(392,399)(393,415)(400,413)(401,412)(407,
414)(408,416);

t~(4,6)(5,9)(7,11)(8,16)(10,20)(12,15)(14,22)(17,29)(18,32)(19,35)(21,
38)(23,25)(28,44)(30,49)(31,43)(33,52)(34,48)(36,41)(37,55)(39,
61)(46,65)(47,57)(50,71)(51,64)(53,75)(54,69)(56,62)(59,78)(67,
92)(68,81)(72,89)(73,102)(76,107)(77,96)(80,88)(82,84)(85,87)(90,
124)(93,112)(94,118)(95,129)(97,99)(100,137)(101,117)(103,105)(106,
140)(108,146)(109,111)(113,135)(114,116)(119,121)(122,161)(123,
163)(126,155)(127,157)(128,134)(130,169)(131,133)(136,154)(138,
177)(139,156)(141,143)(144,178)(145,168)(147,183)(148,150)(151,
174)(152,153)(158,159)(160,195)(162,197)(165,192)(166,190)(167,
173)(170,201)(171,172)(175,188)(176,191)(179,180)(181,208)(182,
200)(184,185)(186,204)(187,218)(189,222)(193,223)(194,229)(196,
231)(198,226)(199,202)(205,243)(206,224)(209,250)(210,253)(211,
256)(212,236)(214,219)(215,251)(216,264)(217,268)(220,273)(221,
276)(227,279)(228,252)(230,286)(232,291)(234,295)(235,294)(238,
302)(239,240)(241,308)(242,311)(244,292)(245,316)(246,247)(248,
323)(249,326)(254,329)(255,331)(257,265)(258,336)(259,260)(261,
271)(262,328)(263,343)(266,348)(267,293)(269,350)(270,349)(272,
353)(274,324)(275,357)(277,359)(278,358)(280,363)(281,282)(283,
367)(284,361)(285,371)(287,375)(288,373)(289,378)(290,351)(296,
355)(297,372)(298,386)(299,300)(301,305)(303,390)(304,306)(307,
393)(309,395)(310,374)(312,398)(313,399)(314,366)(315,319)(317,
400)(318,321)(320,381)(322,402)(325,404)(327,383)(330,376)(332,
354)(333,397)(334,389)(335,339)(337,407)(338,340)(341,352)(342,
406)(344,360)(345,369)(346,394)(347,410)(356,368)(362,365)(364,
412)(370,405)(377,415)(379,396)(380,392)(382,384)(385,388)(387,
416)(391,408)(401,411)(403,409)(413,414);

s~(3,4)(5,10)(7,13)(8,17)(9,18)(14,26)(15,27)(16,28)(19,36)(20,32)(23,
35)(25,41)(29,44)(31,34)(37,58)(38,60)(43,57)(47,48)(49,70)(51,
73)(52,74)(54,76)(56,79)(59,83)(61,86)(64,90)(65,91)(68,94)(69,95)
(71,98)(72,100)(75,104)(77,108)(80,110)(81,97)(82,84)(85,109)(87,
105)(89,122)(92,125)(93,126)(96,130)(99,118)(101,123)(102,124)(103,
111)(106,140)(107,129)(112,131)(113,117)(114,116)(119,148)(121,
143)(127,128)(133,155)(134,156)(135,163)(136,160)(137,161)(139,
157)(141,150)(144,178)(145,168)(146,169)(151,154)(152,187)(153,
189)(158,193)(159,194)(165,167)(172,203)(173,191)(174,195)(175,
205)(176,192)(177,207)(179,209)(180,210)(181,211)(182,200)(183,
213)(184,214)(185,215)(186,216)(188,220)(190,225)(196,232)(197,
233)(198,234)(199,235)(201,237)(202,238)(204,241)(206,245)(208,
248)(212,258)(217,221)(218,222)(219,251)(223,229)(224,280)(226,
283)(227,285)(228,252)(230,287)(231,289)(236,298)(239,259)(240,
305)(242,312)(243,273)(244,292)(246,267)(247,320)(249,327)(250,
253)(254,278)(255,332)(256,323)(257,265)(260,301)(261,262)(263,
343)(264,308)(266,303)(268,272)(269,277)(270,310)(271,322)(274,
355)(275,357)(276,353)(279,360)(281,329)(282,358)(284,369)(286,
299)(288,330)(290,351)(291,378)(293,381)(294,302)(295,367)(296,
297)(300,375)(304,314)(306,325)(307,394)(309,338)(311,397)(313,
354)(315,319)(316,363)(317,321)(318,379)(324,372)(326,392)(328,
402)(331,399)(333,398)(334,345)(335,339)(336,386)(337,340)(341,
347)(342,406)(344,371)(346,377)(348,364)(349,365)(350,388)(352,
387)(356,368)(359,385)(361,389)(362,374)(366,404)(370,405)(373,
414)(376,413)(380,383)(382,384)(390,412)(391,408)(393,415)(395,
407)(396,400)(401,411)(403,409)(410,416);

u~(2,3)(10,21)(13,24)(17,30)(18,33)(19,37)(20,38)(26,42)(27,45)(28,46)(29,49)(31,51)(32,52)(34,54)(35,55)(36,56)(41,62)(43,64)(44,65)(47,68)(48,69)(57,81)(58,79)(60,74)(70,91)(73,76)(82,109)(83,115)(84,111)(85,103)(86,120)(87,105)(90,97)(94,95)(98,132)(99,124)(100,138)(101,106)(102,107)(104,142)(108,147)(110,149)(113,123)(114,152)(116,153)(117,140)(118,129)(119,158)(121,159)(122,162)(125,164)(126,166)(127,128)(130,170)(131,171)(133,172)(134,157)(135,163)(136,175)(137,177)(139,145)(141,179)(143,180)(144,181)(146,183)(148,184)(150,185)(151,186)(154,188)(155,190)(156,168)(160,196)(161,197)(165,198)(167,199)(169,201)(173,202)(174,204)(176,206)(178,208)(182,212)(187,189)(191,224)(192,226)(193,214)(194,210)(195,231)(200,236)(203,225)(205,232)(207,233)(209,215)(211,248)(213,237)(216,220)(217,221)(218,222)(219,223)(227,261)(228,252)(229,253)(230,254)(234,235)(238,280)(239,281)(240,282)(241,289)(242,290)(243,291)(244,292)(245,283)(246,267)(247,293)(249,262)(250,251)(255,322)(256,323)(257,324)(258,298)(259,299)(260,300)(263,272)(264,273)(265,274)(266,275)(268,276)(269,277)(270,278)(271,279)(284,341)(285,327)(286,329)(287,310)(288,330)(294,295)(296,297)(301,362)(302,363)(303,364)(304,314)(305,365)(306,366)(307,377)(308,378)(309,379)(311,351)(312,397)(313,380)(315,320)(316,367)(317,347)(318,345)(319,381)(321,369)(325,342)(326,328)(331,402)(332,360)(333,398)(334,338)(335,385)(336,386)(337,387)(339,388)(340,389)(343,353)(344,354)(346,356)(348,357)(349,358)(350,359)(352,361)(355,372)(368,394)(370,405)(371,383)(373,376)(374,375)(382,384)(390,412)(391,413)(392,399)(393,415)(395,396)(400,410)(401,411)(403,409)(404,406)(407,416)(408,414);

v~(1,2)(21,39)(24,40)(30,50)(33,53)(37,59)(38,61)(42,63)(45,66)(46,67)(49,71)(51,72)(52,75)(54,77)(55,78)(56,80)(58,83)(60,86)(62,88)(64,89)(65,92)(68,93)(69,96)(70,98)(73,100)(74,104)(76,108)(79,110)(81,112)(82,114)(84,116)(85,119)(87,121)(90,122)(91,125)(94,126)(95,130)(97,131)(99,133)(101,136)(102,137)(103,141)(105,143)(106,144)(107,146)(109,148)(111,150)(113,151)(115,149)(117,154)(118,155)(120,142)(123,160)(124,161)(127,165)(128,167)(129,169)(132,164)(134,173)(135,174)(138,147)(139,176)(140,178)(145,182)(152,184)(153,185)(156,191)(157,192)(158,179)(159,180)(162,171)(163,195)(166,170)(168,200)(172,197)(175,181)(177,183)(186,196)(187,214)(188,208)(189,215)(190,201)(193,209)(194,210)(198,199)(202,226)(203,233)(204,231)(205,211)(206,212)(207,213)(216,232)(217,261)(218,219)(220,248)(221,262)(222,251)(223,250)(224,236)(225,237)(227,249)(228,252)(229,253)(230,254)(234,235)(238,283)(239,267)(240,293)(241,289)(242,255)(243,256)(244,257)(245,258)(246,259)(247,260)(263,290)(264,291)(265,292)(266,341)(268,271)(269,270)(272,322)(273,323)(274,324)(275,342)(276,328)(277,310)(278,287)(279,326)(280,298)(281,299)(282,300)(284,325)(285,327)(286,329)(288,330)(294,295)(296,297)(301,320)(302,367)(303,347)(304,345)(305,381)(306,369)(307,377)(308,378)(309,379)(311,331)(312,332)(313,333)(314,334)(315,335)(316,336)(317,337)(318,338)(319,339)(321,340)(343,351)(344,380)(346,394)(348,352)(349,350)(353,402)(354,398)(355,372)(356,409)(357,406)(358,375)(359,374)(360,392)(361,404)(362,385)(363,386)(364,387)(365,388)(366,389)(368,403)(370,405)(371,383)(373,376)(382,384)(390,410)(391,411)(393,415)(395,396)(397,399)(400,407)(401,408)(412,416)(413,414);

APPENDIX C

PROGRAM FOR O'NAN

```
G<x,y>:=Group<x,y|x^8,y^5,y^(x^2)*y^3,(x*y)^6,(x^3*y)^11>;
H:=sub<G|y, x^-4,x * y * x * y * x, x * y * x^-1 * y^-2 * x^-1,
  x * y^-1 * x * y^-1 * x, x * y^-2 * x * y * x^-1>;
f,M,k:=CosetAction(G,H);

sigma:=M!(2, 5)(3, 6, 11, 9, 7, 10)(4, 8, 12);
sigma3:=sigma^3;
sigma5:=sigma^5;
qq:=M!(1,4,8)(2,11,9)(3,5,10);

M1:=Stabiliser(M, 1); M12:=Stabiliser(M1,12);
trans := Transversal(M, M12);
rs:={@ [12, 1, 12, 1, 12]^x : x in trans @};

M8:=Stabiliser(M12,8);
trans3:=Transversal(M,M8);
rs3:={@ [8,1,8,12]^x : x in trans3 @};
sgs3:=[qq^x : x in trans3];
P3:=func<i,j,k | Index(rs3, [i,j,i,k]) ne 0 select sgs3[Index(rs3,
[i,j,i,k])] else Id(M)>; //i=8,j=1,k=12

M34:=Stabiliser(M1, 5); M34:=Stabiliser(M34, 2);
trans1 := Transversal(M, M34);
rs1:={@ [1, 5, 1, 2, 1, 5]^x : x in trans1 @};
sgs1 := [sigma3^x : x in trans1];
P1 := func<i,j,k | Index(rs1, [i,j,i,k,i,j]) ne 0 select
sgs1[Index(rs1, [i,j,i,k,i,j])] else Id(M)>; //i=1, j=5, k=2

M4:=Stabiliser(M12, 4); M8:=Stabiliser(M4, 8);
trans2 := Transversal(M, M8);
rs2:={@ [4,1,4,12,1,12,8,1,8]^x : x in trans2 @};
sgs2 := [sigma5^x : x in trans2];
P2 := func<i,j,k,l | Index(rs2, [i,j,i,k,j,k,l,j,l]) ne 0 select
sgs2[Index(rs2, [i,j,i,k,j,k,l,j,l])] else Id(M)>; //i=4, j=1, k=12,
l=8


canon:=function(sm, rs, rs1, rs2)
child:=sm;

//Step 1. If adjusted ti are the same, delete them
if exists(i){i : i in [1..#child-1] | child[i] eq child[i+1] } then
child:= $$(child[1..i-1] cat child[i+2..#child], rs, rs1, rs2 ); end
if;

//relation Pi8,1,8,12=12,1
if #child ge 4 then
if exists(j){j : j in [1..#child-3] | Index(rs3, child[j..j+3] ) ne 0 }
then
c:=child[j..j+3];
child:= $$( [p[child[k] ] : k in [1..j-1] ] cat c[4..4] cat c[2..2] cat
child[j+4..#child] where p is Eltseq(P3(c[1],c[2],c[4])), rs,rs1, rs2);
end if;
end if;
```

```
//relation one does not use Pi
if #child ge 5 then
      if exists(i){i : i in [1..#child-4] |Index(rs, child[i..i+4]) ne
0 } then
c:=child[i..i+4];
      child:= $$( child[1..i-1] cat c[2..4] cat child[i+5..#child], rs,
rs1, rs2);
end if;
end if;

//relation two
if #child ge 6 then
      if exists(j){j : j in [1..#child-5] | Index(rs1, child[j..j+5])
ne 0 } then
      c:= child[j..j+5];
child:= $$([ p[child[k] ] : k in [1..j-1]  ] cat c[2..5] cat
child[j+6..#child] where p is Eltseq(P1(c[1], c[2],c[4])), rs, rs1,
rs2);

else if exists(j){j : j in [1..#child-5] | child[j] eq child[j+4] and
child[j+1] eq child[j+3] and child[j+1] eq child[j+5] and Index(rs1,
child[j+3..j+5] cat child[j+2..j+4]) ne 0} then
c:= child[j..j+5];
child := $$([ p[child[k] ] : k in [1..j-1]  ] cat c[4..6] cat c[3..3]
cat child[j+6..#child] where p is Eltseq(P1(c[2], c[1],c[3])), rs, rs1,
rs2);
end if;
end if;
end if;

//relation three
if #child ge 9 then
      if exists(j){j : j in [1..#child-8] | Index(rs2, child[j..j+8])
ne 0  } then
      c:=child[j..j+8]; d:= c[4..6] cat c[2..2] cat c[1..3];
      child:=$$( [ p[child[n]] : n in [1..j-1] ] cat d cat
child[j+9..#child]      where p is Eltseq(P2(c[1], c[2],c[4],c[7])),
rs, rs1, rs2);
end if;
end if;

return child;
end function;

orderM:= #M;
//this part runs once and then need to be taken in /* */
/*
h:=<[], {@  @}, 1, {@ <[1], 12>  @}, 0>;
OO:={@ h @};
*/

for s in OO do
node:=s[1]; children:=s[4]; flag:=s[5];
```

```
if flag ne 1 then      //p:=p+1;
      for child in children do
            new_children:={@ @}; parents:={@ <node, 1> @};
            M1:=Stabiliser(M, child[1][1]);
            for j in [2..#child[1]] do
                  M1:=Stabiliser(M1, child[1][j]);
            end for;
            sizeofdoublecoset:= orderM/#M1;
            O:=Orbits(M1);
            for j in [1..#O] do
                  m:=12; for g in O[j] do if g le m then m:=g; end if;
end for;
                  new_child:= child[1] cat [m];
                  new_child:= <canon(new_child, rs, rs1, rs2), #O[j]>;

                  if new_child ne parents[1] then new_children:=
new_children join {@ new_child  @}; end if;
            end for ; //j
            new_s:= <child[1], parents, sizeofdoublecoset,
new_children, 0>;

            OO:= OO join {@ new_s @};
            PrintFile("doublecosets_Onan.txt", new_s);
      end for; //child
end if; //flag is not set to 1
h:=<s[1], s[2], s[3], s[4], 1>;
OO:=OO diff {@ s @};
OO:=OO join {@ h @};
end for; //s
```

APPENDIX D

AN ELEMENT OF 3·SUZ:2

g in Suz:
(1,3769,3797,1912,1654,4304,271,1195,133,4441,2506,1361,796,
3210)(2,3486,3137,2883,1395,4673,2072,1972,668,4834,3875,1298,
1840,5207)(3,4297,1578,4103,2700,1736,17,938,868,2180,2160,5052,
1231,1864)(4,4290,180,3173,4419,4192,1256,232,100,1662,3165,
3192,5188,4128)(5,5300,1756,165,1470,5218,1728,1344,3883,4070,
1421,1643,4526,4697)(6,4808,1874,5301,3219,4140,110,1158,880,
4243,5200,3696,364,3133)(7,1845,117,4187,3728,3487,1179,1528,
360,3395,4839,4806,3406,3454)(8,3106,708,1527,2438,1232,613,
1038,4078,1002,373,2914,1933,3863)(9,3280,3075,3461,5173,1220,
1249,3250,1624,4926,3658,1558,832,1836)(10,4591,3844,3154,315,
2948,539,1126,2644,1686,199,341,2117,4840)(11,3293,4905,1648,
1160,5039,135,906,3521,3259,2361,1769,4156,2245)(12,4940,3677,
974,2966,4248,15,3443,1417,1511,2369,4870,2783,3670)(13,4531,
4620,3015,4719,3518,3182,3249,4381,4360,5002,4007,5081,2675)(14,
2390,2156,769,3665,3068,229,4803,581,4404,1295,2035,978,
3507)(16,203,876,5127,1966,1834,2065)(18,4662,2715,438,3092,2392,
3365,4788,924,3781,3776,1898,1093,2690)(19,3339,5021,2831,83,
2087,2517,812,654,1804,2669,450,343,5334)(20,1646,3689,3967,
3449,1033,1185,1514,3618,2232,4682,3105,3954,865)(21,2702,2461,
5019,178,1993,3430,4816,1509,3304,2167,2650,3969,4741)(22,2957,
4125,973,3485,3840,1631,2323,4275,1565,3394,163,2136,3584)(23,
2706,2546,2097,5046,3817,2316,2747,2542,2303,5064,1239,1096,
3453)(24,4189,2718,2610,3072,396,2536)(25,4718,58,328,4539,4780,
1001,3922,161,2520,1321,1543,1650,4396)(26,3977,2641,3256,2268,
3221,340,3489,1037,1510,480,5080,4955,2011)(27,4226,4571,2670,
1943,1560,3060,1498,1253,2894,3044,1480,1468,5187)(28,2764,2391,
3439,4818,2834,975,2778,4701,2362,3213,1813,814,723)(29,3110,
3481,1990,1545,4522,2796,3360,627,3021,403,1546,3328,2671)(30,
2399,2270,3388,4012,4770,352,2397,32,4361,5185,1241,5111,
2141)(31,3026,4786,3302,1373,4286,4720,3760,727,4811,2748,541,
2052,3556)(33,5117,4438,5006,3878,3890,68,4789,3042,5078,4228,
1658,901,4929)(34,2986,1005,2880,102,4365,3335,120,970,3514,
1409,772,854,3223)(35,3985,4927,1905,4777,964,2367,1081,1164,
4706,4427,1484,4003,2779)(36,5063,4886,93,4737,3557,570,908,
1197,4971,635,1808,2423,811)(37,1567,816,2540,1969,3152,565,496,
4801,2282,1348,5215,222,3602)(38,2496,696,5066,5036,4001,1556,
985,3752,1851,3901,1245,963,2013)(39,4837,1812,5251,1924,4336,
2301,3909,2280,4668,4450,266,1698,2363)(40,1386,904,2412,583,
1483,187,448,2647,4765,793,430,1101,3064)(41,1380,602,3194,1970,
2674,2906,1382,62,4507,333,5148,3662,5110)(42,4899,4995,2579,
347,4417,1713,1848,1661,3995,1604,4258,1911,4692)(43,4121,3572,
526,466,2330,4694,193,2989,2850,3185,4527,4464,4062)(44,3211,
4470,3821,3545,704,3076,1024,2719,573,1964,295,4282,2982)(45,
2393,817,5104,2687,3868,636,5198,730,4173,1371,5314,224,
1341)(46,1691,2664,3647,2334,4568,2209,1205,509,2626,3301,4762,
4314,5107)(47,2469,4235,256,1492,2242,4748,1152,475,4163,456,
3637,5085,3585)(48,4546,1839,4030,5045,4093,2734,5108,695,4148,
4319,1300,3195,3325)(49,270,4392,4488,1322,3452,632,3007,827,
895,1541,4631,2980,3167)(50,408,2544,4240,1936,4982,1471,1206,
399,4191,3756,1425,3305,1561)(51,3650,1725,664,5220,4338,626,
3216,950,3402,1071,4181,4372,634)(52,4112,4868,1117,2785,4263,
1829,992,1369,3299,3236,3546,1887,4736)(53,4423,612,3516,1010,
3242,1202,3994,4456,3946,2640,3245,2294,2589)(54,4216,3527,1290,

2396,4066,3159,127,215,490,3701,3616,870,2538)(55,2840,3617,
1384,3590,669,2014,710,409,4807,5234,856,1506,3904)(56,5043,
3484,607,3953,4147,4949,2324,145,3934,899,346,1863,4966)(57,
3148,507,4462,4872,843,4964,2450,614,2871,2315,1989,1934,
3124)(59,4989,561,511,4639,4866,3307,4599,687,4827,4283,552,
4928,4916)(60,4109,143,5240,3383,4107,2803,939,1564,1831,1407,
5191,4083,4542)(61,2068,1266,1757,169,2039,4190,1035,3397,4702,
2322,140,3471,4830)(63,3612,4424,3729,1154,4889,230,3398,1766,
4428,1172,3730,1052,4997)(64,2374,2088,2838,3789,4683,1072,4332,
781,2943,4924,4035,3525,4041)(65,4797,1599,1925,2170,3895,2740,
1614,1258,4996,1562,4239,4440,2311)(66,1032,1823,647,3721,4545,
1171,331,820,262,4917,4039,952,3682)(67,579,2389,4842,245,2704,
682,137,4391,1045,1762,1473,1794,3580)(69,2927,2766,3278,1007,
4353,348,2597,3723,1519,2104,3247,4703,3292)(70,5193,1086,431,
1199,3234,267,2598,1116,1123,2884,484,1064,1997)(71,2152,437,
1569,2109,1370,2297,4709,128,3788,4276,5089,4761,4389)(72,1053,
1219,830,5199,4757,4783,1161,983,893,2235,2257,4913,3499)(73,
3699,4177,1271,5316,4925,2583,5292,1747,3924,5109,1163,3869,
4506)(74,4854,241,1869,3715,3873,1454,2893,191,3344,5204,3566,
2454,3070)(75,3433,142,2062,5287,2872,3588,4171,390,4098,4828,
1749,5280,4198)(76,4439,1697,1137,2279,4238,872,2732,2214,2085,
1785,4638,3951,5037)(77,4127,5339,421,4943,3766,513,3842,2416,
4856,5208,1351,1435,3197)(78,3655,3611,2578,1433,4993,3657,3127,
2949,2591,5324,3409,1388,3914)(79,4948,2360,4907,2709,631,3368,
2196,2246,3469,2199,3283,932,1872)(80,1493,738,3537,2694,1534,
2990,162,1049,1635,3277,660,4578,2946)(81,5154,3385,383,2329,
1674,3056,2773,4760,2019,967,1442,3887,2493)(82,181,4437,3244,
4042,1051,420)(84,2108,3014,2278,3691,3997,4160,5119,2133,3695,
1276,3603,423,871)(85,4696,3128,4167,3792,172,323,532,4110,4986,
670,3214,1287,718)(86,2044,1066,3802,4552,4647,1319,1613,91,
1982,2705,2521,2400,4861)(87,4695,4911,4492,729,354,3157,1758,
1842,4898,4135,1100,582,2272)(88,5135,3371,3524,2817,3319,2063,
4366,1760,5295,3039,4728,1981,3392)(89,1973,2561,1240,1223,3693,
458,4650,2625,2204,99,2575,2717,2289)(90,4967,2646,2417,4050,
1735,1247,1732,1781,761,4430,5283,1040,3845)(92,4426,2001,2967,
3745,2345,719,3315,2629,3375,5219,2253,3972,5013)(94,5186,368,
1028,3330,2286,862,4334,2567,2456,2985,3345,4144,4482)(95,5343,
2350,1913,3854,1402,487,2153,3262,3425,4836,2448,2959,1844)(96,
5305,714,4572,3190,1994,2455,1227,2181,2118,1267,2939,3434,
2681)(97,609,2486,2806,1600,3067,1415,255,3111,139,3570,4869,
1419,1003)(98,3200,4587,1999,1307,4903,2522,2822,2555,1475,4687,
2724,1354,5211)(101,2467,1538,1861,419,3785,2366,819,4244,717,
1461,2408,1947,4026)(103,3826,3993,3552,1774,571,2244,3862,400,
3725,4253,2818,1111,1016)(104,1977,2874,1459,5330,4954,3910,5169,
2917,1494,1652,5329,5027,2746)(105,3415,2131,282,2903,1264,3583,
3532,4331,4615,3965,275,2805,3269)(106,661,175,3886,2621,5252,
465,1186,207,453,1189,2340,4990,5273)(107,2523,3714,3291,451,
1655,918,2000,1926,2411,3586,1293,146,1638)(108,5302,2612,3352,
4065,1945,474,2415,2744,4819,2446,1439,425,685)(109,3206,435,
3407,2375,1669,2028,5272,515,3054,864,5098,842,2250)(111,4885,
1587,5311,3974,1273,1061,3338,951,4852,3911,3643,2947,4033)(112,
2309,5327,852,4752,3317,318,799,2938,1339,599,4224,4985,
606)(113,699,5271,1957,499,4773,1456,3243,185,3850,4390,366,
4627,889)(114,656,2794,3827,2440,1537,4845,1084,3744,1041,4511,

5112,1238,4063)(115,2802,3623,1021,3010,4746,941,5023,968,2568,
2891,288,4237,2143)(116,1914,1184,2956,307,3543,1553,179,518,
2165,1159,2048,1044,2171)(118,2447,1406,5341,4317,1991,380,4169,
2177,5313,3370,3903,2952,3431)(119,3684,4310,4209,3589,4879,1787,
5170,553,3706,4798,1440,176,5247)(121,1078,3285,319,4363,1837,
216,4077,3871,4451,2005,4617,1196,4076)(122,4074,3384,5106,3941,
4397,677,2451,2465,3374,5255,4707,2418,4651)(123,4379,4036,1581,
860,4612,3141,2463,302,4113,2511,3761,4326,5011)(124,3151,815,
4284,1572,3952,2656,2379,4398,2281,1772,1314,4298,4241)(125,5116,
706,577,1469,2975,257,1754,2326,4444,641,1672,3260,969)(126,
3536,3666,1283,3859,4613,417,5077,2757,3308,1657,4058,377,
4463)(129,1259,4047,2249,2089,1530,1784,1868,2428,3757,4291,1228,
942,4499)(130,4969,2351,2812,2965,791,1207,2995,3877,2179,966,
616,5137,2777)(131,4756,1596,2587,4740,3098,3470,3807,155,5146,
2107,329,4204,3422)(132,2310,3348,410,1097,4394,2222,913,2742,
2940,2557,2358,3671,744)(134,4678,4838,1165,2800,1144,3024,3333,
2580,3762,3071,4860,1463,1767)(136,4119,2828,3089,5236,3410,1827,
4814,962,3938,2006,4217,4019,4871)(138,3539,2787,3916,3764,494,
2041,1073,1687,5244,4785,2707,4624,1932)(141,2151,671,5084,3645,
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4576,3639,485,1013,1935,4193,3446,560,3393,5024)(147,1050,3596,
3526,2613,4555,1720,2931,989,1953,2341,4685,2337,2889)(148,4289,
2586,1996,1630,3651,482,4120,4015,4264,3991,1835,1127,4704)(149,
2862,3286,1065,5328,2239,1518,3055,221,4514,3379,4141,1893,
2541)(150,546,4670,1718,4524,4168,3086,462,753,2915,2563,1383,
2427,838)(151,1405,259,1422,2382,2527,3865,3096,166,4674,1653,
4436,324,2622)(152,1282,3690,2226,4932,3123,1862,4530,2017,1077,
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792,879,4318,3104,3614)(154,4778,4123,2047,2229,2122,823,3809,
2897,313,1444,2875,2384,3357)(156,2678,684,296,3102,5249,281,
1815,2036,2055,531,1424,3220,1888)(157,4183,3743,1193,2064,3155,
188,1309,1830,4693,600,1305,3947,3215)(158,3163,4887,4882,1580,
2885,4356,3538,2147,2274,559,2166,3597,4117)(159,4956,5092,3594,
1208,3925,4053,2864,160,4504,3408,1445,3567,5298)(164,1362,4541,
1162,2918,1486,1803,365,2574,4285,1449,955,971,5152)(167,2430,
322,2251,1859,4341,3569,2685,3933,1353,3963,1963,1549,4288)(168,
2441,1403,3860,722,5325,922,2913,3668,2960,4848,3053,4322,
5333)(170,885,2007,3615,1895,5322,1030,1892,414,5161,1727,3517,
2504,2387)(171,371,4254,5229,2221,4550,2852,2094,1692,3366,4893,
5177,521,698)(173,1350,4810,2247,1135,3349,2347,289,321,2186,
592,2922,5285,3199)(174,5265,5167,4230,4881,890,881,5268,4084,
1968,5005,4327,2352,2333)(177,1451,1675,594,3987,3720,2074,3879,
2112,3631,4824,5029,1257,5007)(182,2628,3992,1882,608,742,2287,
1659,4851,4656,1272,1709,639,4787)(183,1490,1589,2432,4295,4249,
5312,2813,3171,3838,3940,623,2697,2325)(184,5128,261,3713,3737,
4442,213,3805,604,4092,239,501,3913,4809)(186,1667,4790,524,
2558,2529,3080,443,2126,3350,2051,242,5291,250)(189,1291,4792,
3490,2606,2508,3736,5026,2344,4908,3912,1114,1875,701)(190,4525,
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2518,197,2726,2826,2224,4588,853,2140,5069,1896,1094,3025,
1901)(195,3889,2139,2090,1464,2394,915,1856,4465,413,4689,459,
2312,3506)(196,4534,5083,3082,5331,4388,470,1110,4732,3136,2759,
1360,3949,633)(198,3599,2815,2227,4445,5178,237,2010,508,449,
4994,4767,4410,3674)(200,2539,2537,2516,1739,4538,1121,1059,2696,

1776,1716,1087,4271,517)(201,5091,317,3382,2596,808,1408,4642,
1501,2636,1592,5213,2357,3943)(202,3077,5310,589,2632,3905,2618,
2533,4951,886,3661,1611,3870,5082)(204,5156,821,1909,4257,2866,
3451,4973,4242,1080,2861,4118,5289,5165)(205,2314,269,3139,4804,
1450,2260,2458,1031,2877,845,3073,1103,5174)(206,2992,3207,1715,
4601,3296,4277,1897,897,4052,4246,2349,1182,2331)(208,783,1022,
246,3884,2459,4059,1105,279,3968,3782,1364,1098,3268)(209,3008,
5124,3726,3750,4510,1448,4883,5261,5101,850,4311,743,442)(210,
3837,2768,2081,651,3900,2304,782,2531,1826,3508,5022,965,
3222)(211,2994,4667,2398,1956,1294,1938,1414,5233,1320,4781,3500,
260,3812)(212,2183,2789,388,2464,3202,2293,912,2308,2795,2846,
4395,5235,3313)(214,2901,2762,5202,4614,2175,2509,1145,2487,2230,
4429,1302,4984,3252)(217,2642,1663,1651,2843,2784,837,4710,1079,
4032,4593,1846,3094,2276)(218,2668,2338,3417,1019,4422,3248,690,
1547,233,2255,3052,5103,4618)(219,5143,2962,1378,1801,5051,2195,
3763,3806,688,3627,3300,5122,3740)(220,834,1620,2433,5206,5344,
902,1194,1250,2218,3872,1379,3266,4018)(223,3686,1346,4944,3835,
1009,1548,3652,640,3246,2688,1345,1146,4339)(225,4749,5149,5062,
4259,4769,297,4867,1036,4161,4771,4892,4585,3460)(226,3340,1761,
3351,1852,4894,4002,1524,3852,3145,5263,692,4991,2955)(227,4085,
441,3515,857,477,3311,2686,311,2254,605,2571,584,1151)(228,2951,
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4671,4306,4579,4087,1523,3257,737,3205,3656,1533,4676,294,
4474)(234,4820,2514,2887,2609,1647,2569,4744,1782,5227,3964,3906,
3066,5197)(235,4863,489,4602,4556,1356,1759,3078,325,3982,1626,
2836,2038,5115)(236,4846,2953,1236,1396,4106,707,775,5281,888,
3013,4129,4072,2684)(238,4669,3437,758,3793,1153,619,4904,1540,
2419,361,2672,3002,2808)(240,1818,4343,2997,4416,3921,1131,555,
3041,3784,4644,2863,958,2042)(243,2071,658,2655,809,3882,891,
386,4558,2934,2146,4340,766,1930)(244,4729,1398,826,1112,1722,
3032,578,1149,5253,1937,3048,3787,3907)(247,4473,3238,3830,869,
1593,530,3732,1799,2782,4622,263,2158,2475)(248,776,2760,1222,
4480,3610,1178)(249,4468,3990,4487,2827,1908,3320,4721,597,3607,
3560,1173,5017,4876)(251,1729,4471,3158,4537,2043,1326,2348,948,
4784,3258,4566,1132,3919)(252,5133,1880,4349,4475,4716,1048,4301,
3492,1741,896,3011,3712,3704)(253,5260,2449,1133,5038,5020,1214,
3051,385,2176,1499,1330,2296,4270)(254,2602,1060,2079,4999,5130,
1275,3999,944,2434,4921,4857,2327,1618)(258,2770,520,5338,4960,
833,3456,764,4105,5323,4268,1466,1229,2501)(264,3976,342,2611,
2080,2219,3031,3839,1047,3836,308,1954,3554,4027)(265,1106,3327,
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491,2103,4564,3634,2658,1074,933,2277,4269,3771,3463,655)(273,
4292,3172,3493,2265,5000,4176,3746,1590,1700,760,5049,2845,
1710)(274,2163,4708,2460,312,5047,2144,334,1677,807,2788,3241,
1301,4051)(276,404,416,2479,1726,2652,3848,703,1212,2935,773,
4775,4048,5088)(277,1770,1885,5018,5139,3846,802,1104,1595,1707,
401,1412,5297,1148)(278,2172,4476,916,1689,4384,3555,981,1952,
1693,1157,3927,5138,3823)(280,3120,1446,4460,3181,839,2388,4825,
4158,736,2689,3931,1890,4153)(283,1598,4175,1833,3626,284,
1603)(285,1043,1571,2925,1317,3062,2926,2892,1042,1532,1986,5183,
5307,3473)(286,2899,3948,3608,1429,3980,3012,4185,3438,2380,2409,
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3587,925,1939,3697)(290,5163,4220,3849,5189,4454,672,5212,1536,
4413,3290,1277,2902,1793)(291,1778,3009,3276,1248,4057,2944,1918,

2207,5294,1481,1270,1673,2115)(292,3162,2582,4853,1070,4629,516,
3475,407,2057,927,3847,2275,4251)(293,3548,1390,4082,3331,2736,
3326,2659,2203,3791,5059,3843,4660,3069)(298,4432,1985,3509,4983,
3960,4114,2738,2767,2771,4281,5246,4031,2810)(299,618,4149,3751,
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4799,3553,1950,2847,749,4690,3467,4902,3573,2401,2964,3803)(304,
4580,3224,875,4987,476,1717,2142,2248,2069,4037,1313,3978,
3462)(305,4049,510,2083,4448,2725,3218,3779,1058,3630,984,4901,
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3294,4822,2651)(309,2184,3530,3018,3237,900,4029,1946,1269,5015,
3016,787,903,3574)(310,3558,3337,3033,4747,1455,2923,2018,4222,
3017,5320,2498,2814,1055)(314,4227,683,2807,2588,976,1017)(316,
2472,2197,3795,4503,4877,1375,3765,2190,4561,3358,1274,1660,
4923)(320,1447,2741,5228,2138,4132,2192,5040,1685,1941,3783,1525,
935,4535)(326,2868,2020,4373,1170,2945,3323,2407,936,1359,5009,
2421,3891,2024)(327,3356,3767,335,1130,4196,3074,1745,5276,1723,
358,2474,2134,415)(330,4358,824,3466,2269,3118,954,4028,914,
2194,1434,3372,1612,5222)(332,2607,768,3774,2665,4630,3755,3187,
1011,4533,4962,3497,1242,4433)(336,3598,4517,3935,785,4060,1333,
2119,702,1376,697,4755,3028,1681)(337,3719,488,5044,2993,848,
4735,457,2564,4952,4493,4017,4022,1920)(338,4582,2756,4380,986,
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2431,3312,755,1340,4213,1218,4855,2635,2733,5140)(344,957,2066,
1262,4089,4038,4621)(345,1246,3853,429,2148,3798,2264,1629,4174,
1177,993,2722,778,4214)(349,4734,2075,2376,3899,362,4699,1822,
953,1849,3966,3876,2056,2457)(350,757,803,3005,5145,1622,2988,
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3233,1308,395,644,2976,2040,2234,2492)(353,5073,2026,3619,2634,
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2095,2592,2114,1467,3322,2524,2105,3759,3579,2895)(356,4459,2084,
997,4223,1507,3681,3748,3119,3675,3636,2034,5248,4988)(357,1579,
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2027,3733,3143,982,1323,1907,4011,4435,4888,2550,740,4548,
1644)(363,3692,2837,3513,1884,4666,4351,2123,4180,4409,2452,866,
1850,2468)(367,1062,1927,3235,2078,4759,2676,392,463,877,2723,
5337,1841,5157)(369,1495,424,4573,1608,4763,3381,4179,2876,4055,
4589,1485,4930,1814)(370,2844,2667,3000,3061,3217,1487,4188,2823,
2302,3678,2121,1279,1730)(374,1975,5100,3272,1566,2604,4649,3811,
1352,5264,3653,3117,1006,5172)(375,4043,4061,4197,1122,3364,1705,
4833,878,2549,2751,741,2711,3874)(376,2291,657,4731,2135,5113,
1389,4961,3144,2339,940,4446,1922,5042)(378,2878,3362,1965,1155,
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3851,504,2663,2855,1866,3958,5308,1436,1237,4116,4124,1217,
4858)(387,4013,1413,3255,548,5016,3628,1082,1215,3253,4081,4636,
646,4184)(389,1244,3391,716,1742,4211,840,694,1949,873,2110,
1632,3043,3888)(391,5318,4195,537,3768,4425,1027,4750,1755,2648,
1619,5168,4616,2573)(393,4758,5014,1281,2205,1343,1847,3620,2101,
2679,1750,1777,1437,1639)(394,3738,1521,3529,5126,750,3825,4203,
2372,3929,1576,4134,4157,554)(397,1695,3166,4402,1699,3550,3601,
4299,4519,5114,1020,4138,3595,2793)(398,3310,3749,568,2378,4643,
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