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The Riemann zeta function

Ernesto Oscar Reyes

THE RIEMANN ZETA FUNCTION

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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by

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Approved by:

[redacted] 8/27/04

Charles Stanton, Committee Chair, Date

[redacted]

Zahid Hasan, Committee Member

[redacted]

Belisario Ventura, Committee Member

[redacted] [redacted]

Peter Williams, Chair
Department of Mathematics

Charles Stanton,
Graduate Coordinator
Department of
Mathematics

ABSTRACT

The Riemann zeta function has a deep connection with the distribution of primes. This expository thesis will explain the techniques used in proving the properties of the Riemann zeta function, its analytic continuation to the complex plane, and the functional equation that the Riemann zeta function satisfies. Furthermore, we will describe the connection between the Riemann zeta function and the Prime Number theorem, and state the most famous unsolved mathematical problem, the Riemann Hypothesis. One of the most important generalizations of the Riemann zeta function are Dirichlet's L-functions. Also, we will explain the techniques used in proving the properties of Dirichlet's L-functions and the functional equation that Dirichlet's L-functions satisfy.

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CHAPTER ONE

INTRODUCTION

In the study of the distribution of the prime numbers, Bernhard Riemann, in 1859, wrote an eight-page paper entitled, "On the Number of Primes Less Than a Given Magnitude" that impacted the mathematical world. In his paper, Riemann stated several unproved conjectures that were settled by Hadamard, de la Vallee Poussin, and von Mangoldt more that 50 years later. However, one of the conjectures in Riemann's paper still remains without a proof. This conjecture is now known as the famous Riemann Hypothesis, and it was listed as one of the millennium most important unsolved mathematical problem for the 21st century.

In his eight-page paper, Riemann introduced the zeta function, $\zeta(s), s \in \mathbb{C}$ and over the years, the study of the $\zeta(s)$ function has contributed immensely to mathematics. Moreover, the essence of many theorems depends on the understanding of the Riemann Zeta function. The objective of this thesis is to provide a survey on the proven properties of the $\zeta(s)$ function, its particular application to the Prime Number Theorem, and the functional equations that Dirichlet's L-functions satisfy.

In chapter 2, I will briefly go through the historical background of the Riemann Zeta function. A detailed proof will be provided for the analytic continuation of the $\zeta(s)$ function, and its functional equation. Chapter 3 will provide important facts of the product formula for the related $\xi(s)$ function. Chapter 4 will explain the application of $\zeta(s)$ function in proving the Prime Number theorem. Finally, in chapter 5, we will generalize the Riemann Zeta function to Dirichlet's L-functions and outline the proof for the functional equation that Dirichlet's L-functions satisfy.

The three major source of information used in this thesis are the following references: Edwards[6], Davenport[2], and E.C.[4]. See references for detail on these sources.

CHAPTER TWO

HISTORICAL BACKGROUND OF THE RIEMANN ZETA FUNCTION

Introduction

The study of prime numbers dates back as far as Euclid's time (300 B.C.). Moreover, Euler also provided important results about prime numbers (1737). In fact, Euler proved a famous identity,

$$\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}} \quad \text{for a fixed } s>1. \tag{2.1}$$

This identity became an important tool and a starting point in Riemann's paper. During 1792, independent works from the great Gauss and Legendre contributed major results in regard to the asymptotic density of the prime numbers which led to the statement $\pi(x)\sim x/\log x,\quad x\to\infty$, where $\pi(x)$ counts the number of primes less than given x. This statement became known as the Prime Number Theorem (PNT). However, the PNT remained unproved for almost fifty years, from 1850 to 1894.

Properties of the Riemann Zeta Function

As was mentioned above, Riemann's ingenious idea was to treat Euler's identity (2.1) as function of complex variable s. So, $\zeta(s)$, $s \in \mathbb{C}$ was adopted as the Riemann Zeta function and it became the key point in the study of the distribution of prime numbers. In fact, Riemann defined the zeta function, $\zeta(s)$, with the following formula

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x}, \quad s \in \mathbb{C}, \tag{2.2}$$

and showed that this formula is analytic throughout the whole complex s-plane except for a simple pole at $s = 1$. (Note: $\Pi(s)$ is the factorial function notation used in Edward[6], $\Gamma(s) = \Pi(s-1)$ and it is used throughout this thesis). We will explain this formula (2.2) in the following section. Moreover, another fascinating result was that the zeta function, $\zeta(s)$ satisfies the functional equation,

$$\zeta(s) = \Pi(-s)(2\pi)^{s-1} 2\sin(s\pi/2)\zeta(1-s). \tag{2.3}$$

This result follows from (2.2) and it is used to calculate the trivial zeros of the $\zeta(s)$ function. An important result that Riemann found was

$$\Pi(\frac{s}{2} - 1)\pi^{-s/2}\zeta(s) = \Pi(\frac{1-s}{2} - 1)\pi^{-(1-s)/2}\zeta(1-s), \tag{2.4}$$

which is another functional equation that remains unaltered with the substitution of $s=1-s$. Its full proof will be provided in the last section of this chapter. The zeros of the zeta function $\zeta(s)$, for $\mathrm{Re}(s)<0$ are known as the trivial zeros and they are precisely at the poles of $\Pi(\frac{s}{2}-1)$ i.e., $s=-2,-4,-6,\ldots$ Riemann went on to state that the non-trivial zeros of the zeta function, $\zeta(s)$ lie in the critical strip $0<\sigma<1$. In fact, he conjectured that the $\zeta(s)$ has infinitely many zeros in $0<\sigma<1$ that are symmetrical with respect to the real axis and to the critical line $\sigma=\frac{1}{2}$, which is now known as the famous Riemann Hypothesis. Furthermore, Riemann also stated that the number of zeros $\rho=\frac{1}{2}+it$ where $0<t\leq T$ is approximately

$$\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(\log T)\,. \tag{2.5}$$

Von Mangoldt gave a complete proof of this statement in 1905. Riemann went a step further and defined an integral function as follows:

$$\xi(s)=\Pi(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s)\ . \tag{2.6}$$

This is an analytic function of s, is defined for all values of s, and it has the product representation

$$\xi(s) = \xi(0)\prod_{\rho}(1 - \frac{s}{\rho}), \text{ where } \xi(0) = \frac{1}{2}. \qquad (2.7)$$

This product formula was proved by Hadamard in 1893 and became a very important tool for the proof of the Prime Number Theorem as we will see in chapter 4.

In Riemann's paper, he gave a difficult explicit formula for $\pi(x) \sim \mathrm{li}(x)$ for $x > 1$, namely,

$$J(x) = \mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) - \log 2$$

$$+ \sum_{\rho} \mathrm{li}(x^{\rho}) - \log 2 + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\log t}, \quad (x > 1). \qquad (2.8)$$

Here $\mathrm{li}(x) = \int_{2}^{x} \frac{dt}{\log t}$ is known as logarithmic integral. There are various important components that Riemann used to derive the above formula. For $\mathrm{Re}(s) > 1$, he showed that we can rewrite $\log\zeta(s) = \sum_{p}\sum_{n=2}^{\infty}\frac{1}{n}p^{-ns}$ as

$$\log\zeta(s) = s\int_{0}^{\infty} J(x)\frac{1}{x^{s+1}}dx, \quad \mathrm{Re}(s) > 1. \qquad (2.9)$$

$J(x)$ is a step function that jumps 1 when it encounters a prime p, $\frac{1}{2}$ for p^2, $\frac{1}{3}$ for p^3 and so on. He then obtained

$J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log\zeta(s) x^s \frac{ds}{s}$, $(a > 1)$ by applying the Fourier inversion formula to (2.9). From both formulas, (2.6) and (2.7) for $\xi(s)$, we get $\Pi(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s) = \frac{1}{2}\prod_\rho(1-\frac{s}{\rho})$. So, $\log\zeta(s)$ is expressed in terms of the non-trivial roots, ρ of the zeta function and other logarithmic terms. However, von Mangoldt provided a much simpler formula that has replaced Riemann's original formula, namely,

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} + \sum_n \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)} \quad (x > 1), \tag{2.10}$$

where $\sum_\rho \frac{x^\rho}{\rho}$ consists of a sum over the non-trivial roots ρ of the zeta function, $\zeta(s)$. (2.8) was derived by using $\log\zeta(s) = s\int_0^\infty J(x)\frac{1}{x^{s+1}}dx$, however, (2.10) is derived from the $\frac{\zeta'(s)}{\zeta(s)} = -\int_0^\infty \frac{1}{x^s}\log(x)J(x)dx = \int_0^\infty \frac{1}{x^s}\psi(x)dx$. Here, $\psi(x) = \sum_{p^n<x}\log p$ is also a step function, but $\psi(x)$ behaves better than $J(x)$ because $\frac{\zeta'(s)}{\zeta(s)}$ is analytic everywhere, except at $s=1$, the roots ρ, and the zeros of $\Pi(\frac{s}{2})$.

In 1914, Hardy succeeded in proving the existence of infinitely many zeros on the critical line. However, no one has been able to prove that all of non-trivial zeros are on the critical line.

Analytic Continuation

The goal of this section is to show the analytic continuation of the Riemann zeta function using Riemann's original techniques (See E.C.[4] and Edwards[6]). The zeta function is defined by $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$, and it converges for $\mathrm{Re}(s)=\sigma>1$, and converges uniformly for $\sigma\geq 1+\varepsilon$, where $\varepsilon>0$.

Remark: Let $s=\sigma+ti$, so $\mathrm{Re}(s)=\sigma$. We will sometimes use $\mathrm{Re}(s)$ instead of σ for convenience.

Proof: The proof is not difficult to show. Let m and n be positive integers, $m<k$ for $\sigma\geq 1+\varepsilon$, we get

$$\left|\sum_{n=m+1}^{k}\frac{1}{n^s}\right|\leq\sum_{n=m+1}^{k}\frac{1}{n^\sigma}\leq\int_m^k\frac{1}{x^\sigma}dx=\left.\frac{x^{1-\sigma}}{(1-\sigma)}\right|_m^k$$

$$\leq\frac{m^{1-\sigma}}{(\sigma-1)}<\frac{m^{-\varepsilon}}{\varepsilon}.$$

Note that the last term can be as small as we desire for a fixed $\varepsilon>0$ and large enough m. $\square$

Once the convergence has been proved, we can prove the identity,

$$\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p}\frac{1}{1-p^{-s}}, \tag{2.11}$$

for $s\in\mathbb{C}$, $\mathrm{Re}(s)>1$, where p runs through all primes.

Proof:

Note that $\frac{1}{1-p^{-s}}=1+p^{-s}+p^{-2s}+p^{-3s}+p^{-4s}+p^{-5s}...=\sum_{n=0}^{\infty}\frac{1}{p^{-ns}}$.

So, each term of the product can be expressed as $\sum_{n=0}^{\infty}\frac{1}{p^{-ns}}$.

Thus, the right-hand side of (2.11) is

$$\sum_{n_1,n_2,\ldots,n_r=0}^{\infty}\frac{1}{p_1^{n_2 s}\cdot p_2^{n_2 s}\cdots p_r^{n_r s}}=\sum_{n_1,n_2,\ldots,n_r=0}^{\infty}\frac{1}{\left(p_1^{n_2}\cdot p_2^{n_2}\cdots p_r^{n_r}\right)^s},$$ where

$p_{1,}p_{2,}\cdots,p_n$ are distinct primes and $n_1,n_2,...n_r$ are natural numbers. By the Fundamental Theorem of Arithmetic, we can conclude

that $\sum_{n_1,n_2,\ldots,n_r=0}^{\infty}\frac{1}{\left(p_1^{n_2}\cdot p_2^{n_2}\cdots p_r^{n_r}\right)^s}=\sum_{n=0}^{\infty}\frac{1}{n^s}$.

□

We will show that $\zeta(s)$ has no zeros for $\mathrm{Re}(s)>1$.

Lemma 2.1 The $\zeta(s)$ function has no zeros for $\mathrm{Re}(s)>1$.

Proof:

For $\mathrm{Re}(s)>1$, we can see that $(1-2^{-s})(1-3^{-s})\cdots(1-P^{-s})\zeta(s)=$ $1+m_1^{-s}+m_2^{-s}+...$ where $m_1,m_2,...$ are integers whose factor are greater than P. This implies that $|(1-2^{-s})\cdots(1-P^{-s})\zeta(s)| \geq 1-(P+1)^{-\sigma}-(P+2)^{-\sigma}-\ldots \geq 0$ for a large enough P. Thus, $|\zeta(s)|>0.$

$\square$

So far, we seen have that $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ holds for $\operatorname{Re}(s)>1$. However, Riemann showed $\zeta(s)$ has an analytic continuation which holds for all $s \in \mathbb{C}\setminus\{1\}$. The formula by which Riemann extended the zeta function is

$$\zeta(s)=\frac{\prod(-s)}{2\pi i}\int_{+\infty}^{+\infty}\frac{(-x)^s}{e^x-1}\cdot\frac{dx}{x}, \quad s \in \mathbb{C}. \tag{2.12}$$

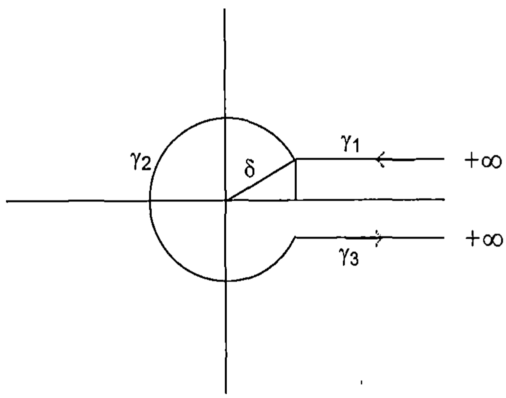
The derivation of this formula will be not discussed in depth, but we are going to mention important techniques that Riemann used to derive his formula. One important key to understand is Cauchy's Integral Theorem: If $f(z)$ is analytic in a simply connected domain D and γ is any loop (closed contour) in D, then $\int_{\gamma} f(z)dz=0$.

Let us examine the following contour integral,

$$\int_{+\infty}^{+\infty}\frac{(-x)^s}{e^x-1}\cdot\frac{dx}{x}, \quad s \in \mathbb{C}. \tag{2.13}$$

We note that $(-x)^s$ is defined as $(-x)^s=e^{s\log(-x)} = e^{s(\log|x|+iArg(-x))}$, and $\log(-x)$ is not defined on the positive real-axis since

$\log(z)$ is not defined on the negative axis. The limit of this integration refers to a path of integration that starts at $+\infty$ above the real-axis and goes around the origin counterclockwise (positive direction) and heads back to $+\infty$ below the real-axis,



Let $\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x-1} \cdot \frac{dx}{x} = \int\limits_{\gamma_1} + \int\limits_{\gamma_2} + \int\limits_{\gamma_3}$ be the contour path, and let the argument, $Arg(-x) = -\pi$ for γ_1 going from $+\infty$ to $\delta>0$, and $Arg(-x) = \pi$ for γ_3 going back to $+\infty$. The integral (2.13) converges for all s, which implies that (2.13) does not depend on the choice δ by Cauchy's Theorem. Again, (2.13) is function of s only. Thus, Cauchy's Theorem can be applied to evaluate the integral on each of the paths, $\gamma_1, \gamma_2, \gamma_3$ such that (2.13) becomes,

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x-1} \frac{dx}{x} = \int\limits_{\delta}^{+\infty} \frac{(-x)^s}{(e^x-1)} \frac{dx}{x} + \int\limits_{|x|=\delta} \frac{(-x)^s}{(e^x-1)} \frac{dx}{x} + \int\limits_{+\infty}^{\delta} \frac{(-x)^s}{(e^x-1)} \frac{dx}{x}.$$

It is not difficult to show that the middle term goes to zero as $\delta \to 0$ for $s > 1$ on the circle $|x| = \delta$, because we can write $x = \delta e^{i\theta}$ and $\frac{dx}{x} = \frac{i\delta e^{i\theta} d\theta}{\delta e^{i\theta}} = id\theta$ and note that $x(e^x - 1)^{-1}$ has a removable singularity. The remainder terms can expressed as

$$\lim_{\delta \to 0} \int_{\delta}^{+\infty} \frac{\exp(s \log(x + i\pi) dx}{(e^x - 1)x} = e^{i\pi s} \int_{0}^{+\infty} \frac{x^{s-1}}{e^x - 1} dx$$

and

$$\lim_{\delta \to 0} \int_{+\infty}^{\delta} \frac{\exp(s \log x - i\pi) dx}{(e^x - 1)x} = -e^{-i\pi s} \int_{0}^{+\infty} \frac{x^{s-1}}{(e^x - 1)} dx \, . \tag{2.14}$$

We now state the formula, $\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \Pi(s-1)\zeta(s)$ which can be obtained from the definition of the factorial function $\Pi(s) = \int_0^\infty e^{-t} t^s dt, \quad (s > -1)$ by replacing t for nx and summing over n. Adding both terms in (2.14) and the formula we stated, we get $\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x}$, as we desired.

□

The Functional Equation of the Zeta Function

Riemann himself gave two proofs of the functional equation for the zeta function, $\zeta(s)$. Mathematicians have found different ways of proving the functional equation for the zeta function; however, this section will be devoted in the discussion of the techniques used by Riemann in his original paper which is found in E.C.[4]. It is interesting to note that historical accounts point out that the Poisson summation formula was an extremely important tool used in Riemann's original proof. The Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \text{ where } \hat{f}(r) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i r x} dx \tag{2.15}$$

tells us that existence of the relationship between the sum over the integers of a function f and its Fourier transform. The function must be differentiable "smooth" and vanishing at ∞. The argument for the proof of the functional equation goes as follows:

By definition,

$$\Pi(s) = \int_0^{\infty} e^{-t} t^{-s} dt,$$

which implies that

$$\Pi(\frac{s}{2}-1)=\int_0^\infty e^{-t}t^{\frac{s}{2}-1}dt\,,\ \text{for}\ \sigma>1\,. \tag{2.16}$$

Now, let $t=n^2\pi x$ in (2.16) and note that $dt=n^2\pi dx$, which gives the following result: $\Pi(\frac{s}{2}-1)=\int_0^\infty e^{-n^2\pi x}(n^2\pi x)^{\frac{s}{2}-1}dx(n^2\pi)=$ $\int_0^\infty e^{-n^2\pi x}n^s\pi^{\frac{s}{2}}x^{\frac{s}{2}}dx$. This implies that $\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)n^{-s}=\int_0^\infty e^{-n^2\pi x}x^{\frac{s}{2}}dx$, for $\sigma>1$. Let us now take the summation over n and get $\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)\sum_{n=1}^\infty n^{-s}=\sum_{n=1}^\infty\int_0^\infty x^{\frac{s}{2}}e^{-n^2\pi x}dx$, and note that the right hand side converges absolutely, which allows us to bring the summation inside as follows:

$$\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)\sum_{n=1}^\infty n^{-s}=\int_0^\infty x^{\frac{s}{2}}\left(\sum_{n=1}^\infty e^{-n^2\pi x}\right)dx\,.$$

If we let $\psi(x)=\sum_{n=1}^\infty e^{-n^2\pi x}$, then we get the following expression,

$$\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)\sum_{n=1}^\infty n^{-s}=\int_0^\infty x^{\frac{s}{2}}\psi(x)dx\,. \tag{2.17}$$

This can be solved for $\zeta(s)$, $\sum_{n=1}^\infty n^{-s}=\frac{\pi^{\frac{s}{2}}}{\Pi(\frac{s}{2}-1)}\int_0^\infty x^{\frac{s}{2}}\psi(x)dx$.

Riemann then applies the Poisson summation formula, (2.15) to the function, $f(x) = e^{-n^2\pi x}$, $x > 0$ and obtains its Fourier transform to be, $\hat{f}(r) = \frac{1}{\sqrt{t}} e^{-n^2\pi/x}$ which gives us the following identity:

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi/x} .$$

Since $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$ it is clear that,

$$2\psi(x) + 1 = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x} =$$

$$\frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi/x} = \frac{1}{\sqrt{x}}\left(2\psi(\frac{1}{x}) + 1\right). \tag{2.18}$$

The interchanging of the summation and integration in the following argument is justifiable by absolute convergence of the integrands in the integrals for $\mathrm{Re}(s) > 1$. We now apply (2.18) to the right-hand side of the following formula, $\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)\sum_{n=1}^{\infty} n^{-s} = \int_1^{\infty} x^{\frac{1}{2}s-1}\psi(x)dx + \int_0^1 x^{\frac{1}{2}s-1}\psi(x)dx$. We get

$$\int_1^{\infty} x^{\frac{1}{2}s-1}\psi(x)dx + \int_0^1 x^{\frac{1}{2}s-1}(\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2})dx$$

$$= \int_1^{\infty} x^{\frac{1}{2}s-1}\psi(x)dx + \int_0^1 x^{\frac{1}{2}s-1}\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)dx + \int_0^1 x^{\frac{1}{2}s-1}\frac{1}{2\sqrt{x}}dx - \frac{1}{2}\int_0^1 (x^{\frac{1}{2}s-1})dx$$

$$= \int_1^\infty x^{\frac{1}{2}s-1}\psi(x)dx + \int_0^1 x^{\frac{1}{2}s-1}\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)dx + \frac{1}{2}\int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}dx - \frac{1}{2}\int_0^1 x^{\frac{1}{2}s-1}dx$$

$$= \int_1^\infty x^{\frac{1}{2}s-1}\psi(x)dx + \int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx + \frac{1}{s-1} - \frac{1}{s}.$$

Changing the limits of integration for the integral

$\int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}\psi(\frac{1}{x})dx$, we obtain the desired result,

$$\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)\sum_{n=1}^{\infty} n^{-s} =$$

$$\int_1^\infty (x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}})\psi(x)dx + \frac{1}{(s-1)} - \frac{1}{s}. \qquad (2.19)$$

Thus, $\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)\zeta(s) = \pi^{-\frac{s}{2}+\frac{1}{2}}\Pi(-\frac{s}{2}-\frac{1}{2})\zeta(1-s)$ is the functional

equation which remains unchanged with the substitution of

$s = 1-s$.

□

We note that $\Pi(s) = s\Pi(s-1)$, and $\Pi(\frac{s}{2}) = \frac{s}{2}\Pi(\frac{s}{2}-1)$ which implies

that $\Pi(\frac{s}{2}-1) = \frac{2}{s}\Pi(\frac{s}{2})$. Riemann then multiplies (2.19) by

$\frac{s(s-1)}{2}$ and defines

$$\xi(s) = \pi^{-\frac{s}{2}}\Pi(\frac{s}{2})(s-1)\zeta(s), \quad (\sigma > 0). \qquad (2.20)$$

We will discuss formula (2.20) in the following chapter.

CHAPTER THREE

THE PRODUCT FORMULA FOR $\xi(s)$

Introduction

In Chapter two, we stated Riemann's definition for $\xi(s)$,

$$\xi(s) = \pi^{-\frac{s}{2}}\Pi(\frac{s}{2})(s-1)\zeta(s), \tag{3.1}$$

which is an entire function and by the symmetry of the functional equation, we get

$$\xi(s) = \xi(1-s).$$

We state some of the interesting remarks for the $\xi(s)$ function. In Chapter two, we showed that $\zeta(s)$ has no zeros for $\operatorname{Re}s > 1$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel so that $s=1$ is neither a pole nor a zero. So, $\xi(s)$ has no roots ρ for the half-plane $\operatorname{Re}s > 1$. Similarly, by the functional equation, if ρ is root of the $\xi(s)$, it implies that $1-\rho$ is also a root, and so is its conjugate $1-\overline{\rho}$. This shows that $\xi(s)$ has no roots for the half-plane $\operatorname{Re}s < 0$ because the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Pi(\frac{s}{2})$. Thus, all roots ρ of $\xi(s)$ must be the nontrivial roots of the zeta function in the strip $0 \le \operatorname{Re}\rho \le 1$.

Riemann stated that the $\xi(s)$ can expanded into an infinite product,

$$\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) \tag{3.2}$$

where ρ are precisely the roots of the Riemann zeta function where the factor ρ and $1-\rho$ are paired. However, it was Hadamard who showed the validity of (3.2).

In following sections, we will follow Hadamard's line of argument. We will discuss the estimate of the distribution of the roots of $\xi(s)$, show the convergence of the product representation, and conclude with the validity of product formula for $\xi(s)$.

Estimate of $\xi(s)$

It is not difficult to show that for sufficiently large values of R the estimate $|\xi(s)| \le R^R$ is valid in the disk $\left|s-\frac{1}{2}\right| \le R$, since the largest value of $\xi(s)$ takes place at $s=\frac{1}{2}+R$, (See Edwards[4]). For sufficiently large enough R, we choose N such that $\frac{1}{2}+R \le 2N < \frac{1}{2}+R+2$. By using (3.1) and noting that $N! < N^N$ we obtain the following result;

$$\xi(\frac{1}{2}+R) \le \xi(2N) = (N!)\pi^{-N}(2N-1)\zeta(2N)$$

$$\le N^N(2N)\zeta(2N) = CN^{N+1}$$

$$\le C(\frac{1}{2}R+2)^{(R/2)+3} < R^R .$$

Where C is some constant and the result is true for a large enough R.

□

Another interesting question that we may ponder is, how many zeros does $\xi(s)$ have? Before we answer this question, let us appeal to the well-known Jensen's formula which is stated as the follow theorem.

Theorem 3.1 (Jensen's formula)

Suppose that $f(z)$ is an analytic function defined in a disk $|z| \le R$. Suppose also that $f(z)$ has no zeros on $|z| = R$ but it has zeros inside the disk, namely, $z_1, z_2, z_3, \ldots, z_n$ with their multiplicities and $f(0) \ne 0$, then

$$\log|f(0)| + \log\left|\frac{R}{z_1}\cdot\frac{R}{z_2}\cdot\frac{R}{z_3}\cdots\frac{R}{z_n}\right| = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(\mathrm{Re}^{i\theta})\right|d\theta .$$

(The proof follows from properties of analytic functions and the Cauchy integral formula).

Theorem 3.2 Let $n(R)$ be the number of zeros (with their multiplicities) of $\xi(s)$ which are located in the disk $\left|s-\frac{1}{2}\right| \le R$. Then $n(R) \le 3R\log R$ for all large enough R.

Proof:

We Apply Jensen's formula to $\xi(s)$ on a disk $\left|s-\frac{1}{2}\right| \le 2R$ and get $\log\xi(\frac{1}{2}) + \sum_{|\rho-1/2|<2R} \log\frac{2R}{|\rho-1/2|} \le \log\left((2R)^{2R}\right)$. We see that, $\sum_{|\rho-1/2|<2R} \log\frac{2R}{|\rho-1/2|} = \sum_{|\rho-1/2|<2R} (\log 2 + \log R - \log\left|\rho-\frac{1}{2}\right|)$. The summation indicates that the terms corresponding to roots ρ in the disk $\left|s-\frac{1}{2}\right| \le R$ are all at least $\log 2$. This implies that $n(R)\log 2 < 2R\log 2R - \log\xi(\frac{1}{2})$ and hence

$n(R) \le \frac{2R\log 2R}{\log 2} - \frac{\log\xi(\frac{1}{2})}{\log 2} \le 3R\log R$, for all large enough R. Thus, we have completed the proof.

□

Convergence of $\xi(s)$

This section is devoted to show that the above rough estimate is sufficient for the convergence of the product formula in (3.2). We know that if the infinite series $\sum_{n=1}^{\infty} a_n$

converges absolutely, then product $\prod_{n=0}^{\infty}(1-|a_n|)$ is also convergent. It is easy to see that we can pair the roots of the $\xi(s)$ by the noting that $\log\xi(s)=\sum_{\rho}\log(1-\frac{s}{\rho})$ and by absolute convergence of the sum, we get

$$\sum_{\operatorname{Im}\rho>0}\left[\log\left(1-\frac{s}{\rho}\right)+\log\left(1-\frac{s}{1-\rho}\right)\right],$$

where ρ ranges over all roots of $\xi(s)$. Now,

$$\sum_{\operatorname{Im}\rho>0}\left[\log\left(1-\frac{s}{\rho}\right)+\log\left(1-\frac{s}{1-\rho}\right)\right]=\sum_{\operatorname{Im}\rho>0}\left[\log\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right)\right]$$

$$=\sum_{\operatorname{Im}\rho>0}\log\left(1-\frac{s(1-s)}{\rho(1-\rho)}\right)$$

This implies that $\prod_{\operatorname{Im}\rho>0}\left(1-\frac{s}{\rho}\right)=\prod_{\operatorname{Im}\rho>0}\left(1-\frac{s(1-s)}{\rho(1-\rho)}\right)$.

To prove the convergence of the product formula in (3.2), namely $\prod\left(1-\frac{s}{\rho}\right)$, it suffices to show the convergence of $\sum\left|\frac{1}{\rho(1-\rho)}\right|$. In fact, the technique here is to write $\frac{1}{|\rho(1-\rho)|}=\frac{1}{\left|(\rho-\frac{1}{2})^2-\frac{1}{4}\right|}<\frac{1}{\left|\rho-\frac{1}{2}\right|^2}$ by completing the square. So, it suffices to show the convergence of $\sum\frac{1}{|\rho-1/2|^{1+\varepsilon}}$ for

$\varepsilon > 0$. By knowing the estimate of the ρ of $\xi(s)$ in $|s-1/2| = R_n$ and noting that $|\rho - 1/2| > R_n$, we can show that $\sum \frac{1}{|\rho - 1/2|^{1+\varepsilon}} < C + \sum \frac{1}{n^{1/2+\varepsilon}}$ and the right-hand side is convergent for $\varepsilon > 0$.

□

Validity of the Product
Formula $\xi(s)$

We now state Theorem 3.3 that will show the validity of the product formula (3.2). The proof for Theorem 3.3 requires three very important results which we state as lemmas. We are merely going to sketch the important steps in the proofs of the lemmas.

Theorem 3.3 The function $F(s) = \dfrac{\xi(s)}{\prod_\rho \left[1 - \left(s - \frac{1}{2}\right) / \left(\rho - \frac{1}{2}\right)\right]}$ is an even analytic function of $s - \frac{1}{2}$, which is defined in the entire s plane. Moreover, $F(s)$ does not have zeros, thus $\log F(0)$ is determined up to multiples of $2\pi n i$. Hence, $\log F(s) = \text{constant}$.

□

Lemma 3.1 Given $\varepsilon > 0$, then $\operatorname{Re}\log \dfrac{\xi(s)}{\prod_\rho (1 - \frac{s-1/2}{\rho - 1/2})} \leq \left| s - \frac{1}{2} \right|^{1+\varepsilon}$ for

Proof: The idea is to write $f(s) = u(s) + v(s)$ as the sum of its real and imaginary parts, and consider

$$\phi(s) = \frac{f(s)}{s[2M - f(s)]} . \tag{3.4}$$

Substitute $u(s) + v(s)$ for $f(s)$ and observe that $|2M - u(s)| \geq M \geq u(s)$ on the circle $|s| = r$; it shows that $|\phi(s)| \leq \frac{1}{r}$ in the disk $\{|s| \leq r\}$. We express (3.4) as $f(s) = \frac{2Ms\phi(s)}{1 + f(s)}$ which gives $|f(s)| \leq \frac{2Mr_1 r^{-1}}{1 - r_1 r^{-1}} = \frac{2Mr_1}{r - r_1}$ for $|s| = r_1$ and holds in a disk $\{|s| \leq r_1\}$.

$\square$

Lemma 3.3 Let $f(s)$ be an even analytic function which is defined in the entire s plane. Suppose $f(s)$ has a rate of growth less than $|s|^2$, that is; for every $\varepsilon > 0$ there exists R such that $\mathrm{Re} f(s) < \varepsilon |s|^2$ for all points s, $|s| > R$. Then f is a constant.

Proof: (sketch of the proof)

We must conclude that $f(s) = \text{constant}$. The idea here is to express $f(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots + a_n s^n$ and assume that $a_0 = 0$. The coefficients of $f(s)$, $a_n = \frac{1}{2\pi i} \int_{\partial D} \frac{f(s)}{s^{n+1}} ds$ by the

Cauchy Integral formula. Let D be the disk $\left\{|s| \leq \frac{1}{2}R\right\}$, then

$$|a_n| = \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\frac{1}{2}\mathrm{Re}^{i\theta})}{\left(\frac{1}{2}\mathrm{Re}^{i\theta}\right)^n} id\theta \right| \leq \frac{2^{n+1}\varepsilon}{R^{n-2}}$$ for $\varepsilon > 0$ by Lemma 3.2. The

result shows that $a_n = 0$ for $n > 2$ since ε can be chosen as small as we desire to. We also note that $a_0 = 0 = a_1$ follows from the fact $f(s)$ is an even function and the rate of growth of $f(s)$. Thus, $f(s) = 0$ which implies that $f(s)$ is a constant.

□

We recall the function,

$$F(s) = \frac{\xi(s)}{\prod_\rho \left[1 - \left(s - \frac{1}{2}\right)/\left(\rho - \frac{1}{2}\right)\right]} \tag{3.5}$$

from Theorem 3.1 and by the three lemmas, we can see that $\log F(s) = \mathrm{c}$ where c is constant. Thus we obtain,

$\xi(s) = c\prod_\rho \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right)$, which implies that $\xi(0) = c\prod_\rho \left(1 - \frac{-\frac{1}{2}}{\rho - \frac{1}{2}}\right)$.

Dividing $\xi(s)/\xi(0)$, we get $c\prod_\rho \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right) \cdot c^{-1}\prod_\rho \left(1 - \frac{-\frac{1}{2}}{\rho - \frac{1}{2}}\right)^{-1} =$

$\frac{\xi(s)}{\xi(0)} = \prod_{\rho}\left(1-\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right)\cdot\prod_{\rho}\left(1-\frac{-\frac{1}{2}}{\rho-\frac{1}{2}}\right)^{-1}$. The right-hand side equals 0 or 1 if $s=\rho$ or $s=0$, respectively. Since it is a function of s, we get $\xi(s) = \xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right)$ to be the desired expression.

□

CHAPTER FOUR

THE PRIME NUMBER THEOREM

Introduction

The Riemann Zeta function has many applications, and this chapter will be devoted to showing how the Riemann zeta function, $\zeta(s)$, play a key role in proving the Prime Number Theorem (PNT). Before we discuss what each section contains, it will be important to state the PNT.

The Prime Number Theorem:

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty.$$

We take a moment to explain what the theorem is actually telling us. $\pi(x)$ is a function that counts the number of primes less than or equal to x. In fact, if we were to graph $\pi(x)$, we can see that this is a step function that jumps by one unit every time it encounters a prime.

For a large value of x, $\pi(x)$ starts to look like the function of $x/\log x$. In fact, as $x \to \infty$, $\pi(x) \sim x/\log x$. This implies that $\pi(x)$ is an asymptotic function; which simply means, $\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$. However, $x/\log x$ does not really give a good approximation to $\pi(x)$, and it was great Gauss who provided the logarithmic integral, $\mathrm{li}(x)=\int_2^x \frac{dt}{\log t}$ which better approximates $\pi(x)$. Comparing Table 1 and Table 2 we can see that $\mathrm{li}(x)$ is a better approximation than $x/\log x$, (see Table 1 and Table 2).

Table 1. Approximation of $\pi(x)$ to $x/\log x$

x	$\pi(x)$	$x/\log x$	$\pi(x)/(x/\log x)$
10^3	168	144.8	1.16
10^4	1,229	1,086	1.13
10^5	9,592	8,686	1.10
10^6	78,498	72,382	1.08
10^7	664,579	620,420	1.07
10^8	5,761,455	5,428,681	1.06
10^9	50,847,534	48,254,942	1.05
10^{10}	455,052,511	434,294,482	1.04

Table 2. Approximation of $\pi(x)$ to $\mathrm{li}(x)$

x	$\pi(x)$	$\mathrm{li}(x)$	$\pi(x)/\mathrm{li}(x)$
10^3	168	178	0.94...
10^4	1,229	1,246	0.98...
10^5	9,592	9,630	0.993...
10^6	78,498	78,628	0.996...
10^7	664,579	664,918	0.9994...
10^8	5,761,455	5,762,209	0.99986...
10^9	50,847,534	50,849,235	0.99996...
10^{10}	455,052,511	455,055,614	0.999993...

The Prime Number theorem, can either refer to $\pi(x) \sim x/\log x$, $x \to \infty$ or $\pi(x) \sim \mathrm{li}(x)$, $x \to \infty$, because we are interested in the value of their limits.

Riemann himself outlined the proof (with no details) of it in his eight-page paper, but it was Hadamard and de la Vallée Poussin who provided the missing details, independently. Nonetheless, this chapter will primarily focus and follow Hadarmard's line of argument, but it will include de la Vallée Poussin's argument. We will discuss the proof that $\mathrm{Re}\,\rho < 1$ for the roots, ρ. Moreover, we will prove that $\psi(x) \sim x$ and conclude with the proof of the PNT.

The Riemann Zeta Function Has No Zeros on the Line $\mathrm{Re}(s)=1$

We are going to show that for all the non-trivial zero of $\zeta(s)$, their real parts are less than one. This is going to be de la Vallée Poussin's proof found in Edwards[6]. By (2.1), $\zeta(s)$ has no roots for $\mathrm{Re}(s)>1$ and from von Mangoldt's formula, this reduces to showing that $\zeta(s)$ has no roots ρ on line $\mathrm{Re}(s)=1$.

Theorem 4.1 (de la Vallée Poussin) $\zeta(s)$ has no roots ρ on line $\mathrm{Re}(s)=1$.

Proof:

From (2.1) (Euler product) and for $\sigma>1$ we get

$$\log\zeta(s)=\sum_p\sum_{m=1}^{\infty}\frac{1}{m}p^{-m\sigma}e^{-itm\log p}=\sum_\rho p^{-\sigma}+\sum_p\sum_{m=2}^{\infty}\frac{1}{m}p^{-m\sigma}e^{-itm\log p},$$

where $\sum_p\sum_{m=2}^{\infty}\frac{1}{m}p^{-m\sigma}e^{-itm\log p}$ is bounded. We see that,

$\log\zeta(s)=\sum_p\sum_{m=1}^{\infty}\frac{1}{m}p^{-m\sigma}e^{-itm\log p}$ can be rewritten as $\zeta(s)=$

$\exp\left(\sum_p\sum_{m=1}^{\infty}\frac{1}{m}p^{-m\sigma}e^{-itm\log p}\right)$. It follows that,

$$\mathrm{Re}\,\zeta(s)=\exp\left(\sum_p\sum_{m=1}^{\infty}\frac{1}{m}p^{-m\sigma}\cos(t\log p^m)\right). \tag{4.1}$$

Now, de la Vallée Poussin established a relationship between $\zeta(\sigma+it)$ and $\zeta(\sigma+2it)$ by using Merten's identity;

$$3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2 \geq 0\,, \tag{4.2}$$

for all values of θ. We are going to prove that $\zeta(1+it)$ remains bounded away from zero for all t by (4.2). By the identities (4.1) and (4.2), we see that

$3\log\zeta(\sigma)+4\operatorname{Re}\log\zeta(\sigma+it)+\ \operatorname{Re}\log\zeta(\sigma+2it)\geq 0$ for $\operatorname{Re}(s)>1$. This implies that

$$\zeta^3(\sigma)\left|\zeta^4(\sigma+it)\right|\left|\zeta(\sigma+2it)\right| =$$

$$\exp\left(\sum_p\sum_{m=1}^{\infty}\frac{3+4\cos(mt\log p)+\cos(2mt\log p)}{mp^{m\sigma}}\right). \tag{4.3}$$

The right-hand side of (4.3) is either 1 or greater than 1, which implies that

$$\zeta^3(\sigma)\left|\zeta^4(\sigma+it)\right|\left|\zeta(\sigma+2it)\right|\geq 1, \quad (\sigma>1).$$

If we fix t and let $\sigma\to 1$, $\zeta(\sigma)=\zeta(1)$ has a simple pole. If $1+it$ were a zero of the zeta function, $\zeta(s)$, the inequality tells us that the left-hand side goes to zero as $\sigma\to 1$. However, $\zeta(\sigma+2it)$ is bounded as $\sigma\to 1$, contradiction. Thus there exist no zeros of the zeta function on the line $\operatorname{Re}(s)=1$.

□

Proof that $\psi(x) \sim x$

In this section, we outline the main steps of the proof for $\psi(x) \sim x, \quad x \to \infty$. We recall the formula (2.10), namely,

$$\psi(x) = x - \sum \frac{x^\rho}{\rho} + \sum \frac{x^{-2n}}{2n} + \frac{\zeta'(0)}{\zeta(0)}, \quad (x > 1). \tag{4.4}$$

As it was mentioned in Chapter 2, the formula was obtained by applying Fourier inversion to $-\zeta'(s)/\zeta(s) = s\int_0^\infty \psi(x) x^{-s-1} dx$ (where $\psi(x) = \sum_{p^m \le x} \log p$), and from using the identity $\Pi(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s) = \frac{1}{2}\prod_\rho (1 - \frac{s}{\rho})$. The result from applying the Fourier inversion is the following equation,

$$\psi(x) = \frac{1}{2\pi i}\int_{a-it}^{a+it}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)x^s \frac{ds}{s}, \tag{4.5}$$

which is the key in evaluating (4.4). Next, Riemann found the anti-derivative of (4.4) and showed the validity of its term-wise integration to obtain another formula, namely;

$$\int_0^x \psi(t)dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + \sum \frac{x^{-2n+1}}{2n(2n+1)} + \frac{\zeta'(0)}{\zeta(0)}x + \frac{\zeta'(-1)}{\zeta(-1)} \text{ for all}$$

$x > 1$. It is not difficult to argue that the last four terms, $\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + \sum \frac{x^{-2n+1}}{2n(2n+1)} + \frac{\zeta'(0)}{\zeta(0)}x + \frac{\zeta'(-1)}{\zeta(-1)}$ converge to zero,

as $x \to \infty$ by dividing every term by $\frac{x^2}{2}$. Thus, we obtain

the desired result $\int_0^x \psi(t)dt \sim \frac{x^2}{2}$.

□

To show that $\psi(x) \sim x$, we are going to require a preliminary result that we state as Theorem 4.2 which is found in Apostol[1].

Theorem 4.2 Let $A(x) = a(1) + a(2) + ... + a(n) = \sum_{n \le x} a(n)$ and let

$A_1(x) = \int_1^x A(t)dt$. Suppose that $a(n) \ge 0$ for $n = 1,2,3...$ If

$A_1(x) \sim Lx^c$ as $x \to \infty$ is an asymptotic formula, for a

positive constant c and $L > 0$, then $A(x) \sim cLx^{c-1}$.

Proof: We are only going to provide an outline of the proof for Theorem 4.2. We first need to consider the fact that $A(x)$ is an increasing function and $a(n) \ge 0$. We let $\beta > 1$ and observe the difference of $A_1(\beta x) - A_1(x)$. We get,

$$A_1(\beta x) - A_1(x) = \int_x^{\beta x} A(u)du \ge \int_x^{\beta x} A(x)du$$

$$= A(x)(\beta x - x)$$

$$= x(\beta - 1)A(x) .$$

This gives us

$$\frac{A(x)}{x^{c-1}} \le \frac{1}{\beta - 1}\left\{\frac{A_1(\beta x)}{(\beta x)^c}\beta^c - \frac{A_1(x)}{x^c}\right\}.$$

It is not difficult to show that if we keep β fixed and let $x \to \infty$, we find

$$\limsup_{x\to\infty} \frac{A(x)}{x^{c-1}} \le L\frac{\beta^c - 1}{\beta - 1}.$$

Now, as $\beta \to 1$, we obtain

$$\limsup_{x\to\infty} \frac{A(x)}{x^{c-1}} \le cL .$$

Consider $A_1(x) - A_1(\alpha x)$ for any α with $0 < \alpha < 1$, we clearly see that

$$\liminf_{x\to\infty} \frac{A(x)}{x^{c-1}} \ge L\frac{1-\alpha^c}{1-\alpha}.$$

$L\dfrac{1-\alpha^c}{1-\alpha} \to cL$, as $\alpha \to 1$

This shows that $\dfrac{A(x)}{x^{c-1}} \to cL, \quad x \to \infty$

□

Let us denote $\psi_1(x) = \sum_{n \le x} \Lambda(n)(x-n)$ and $\psi_1(x) \sim \dfrac{x^2}{2}$ (from the previous result). By letting $a(n) = \Lambda(n), \quad A_1(x) = \psi_1(x),$ and $A(x) = \psi(x)$ as Theorem 4.2 indicates, we can easily conclude that $\psi(x) \sim x$.

□

The Prime Number Theorem

The Prime Number Theorem says that

$$\pi(x) \sim x/\log x \text{ as } x \to \infty.$$

Proof: To prove the PNT, the key idea to understand is the chain of connections among $\psi(x) = \sum_{p^m \le x} \log p$, $\theta(x) = \sum_{p \le x} \log p$, and $\pi(x)$ for $x > 0$, such that $\pi(x)/(x/\log x)$, $\theta(x)/x$, and $\psi(x)/x$ have the same limits $x \to \infty$. We first find the relation between $\psi(x)$ and $\theta(x)$, and the relation of $\theta(x)$ to $\pi(x)$. This series of connections will lead to $\pi(x) \sim x/\log x$.

We note that $\psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \ldots$ by grouping together the terms of $\psi(x)$ for which the power m is the same. Moreover, by grouping the terms for which p are the same, we obtain $\psi(x) = \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p$. We get the following inequality,

$$\theta(x) \le \psi(x) \le \sum_{p \le x} \frac{\log x}{\log p} \log p = \pi(x) \log x. \tag{4.6}$$

Divide (4.6) by x, and recall that $\psi(x) \sim x$, as $x \to \infty$ from the previous section. Thus, $1 \leq \frac{\pi(x)\log x}{x}$ as $x \to \infty$. Next, if $0 < \alpha < 1$, $x > 1$, $\theta(x) \geq \sum_{x^\alpha < p < x} \log p \geq \{\pi(x) - \pi(x^\alpha)\} \log x^\alpha$. We observe that $\pi(x^\alpha) < x^\alpha$, so $\frac{\theta(x)}{x} > \alpha\left(\frac{\pi(x)\log x}{x} - \frac{\log x}{x^{1-\alpha}}\right)$. let α be fixed and $x \to \infty$, we get $\frac{\theta(x)}{x} > 1 > \frac{\pi(x)\log x}{x}$ since α can be as close to 1 as we desire, which implies that $1 > \frac{\pi(x)\log x}{x}$ as $x \to \infty$. Thus $\pi(x) \sim \frac{x}{\log x}$.

□

CHAPTER FIVE

DIRICHLET L-FUNCTIONS

Introduction

The Riemann zeta function, $\zeta(s)$ is not an isolated object but, rather comes from a family of functions called Dirichlet series, $\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$ where $f(n)$ is an arithmetical function. In fact, there exist other functions that are generalizations of the Riemann zeta function which are used in the study of algebraic number fields. Examples of such functions are the Dedekind zeta function, the Hurwitz zeta function, and the Epstein zeta function. This chapter will be focusing on the properties of Dirichlet L-functions. Moreover, just as we proved the functional equation for $\zeta(s)$ function, we will also provide the proof of functional equation that Dirichlet L-functions satisfy. Dirichlet L-functions are extremely important tool in the study of prime numbers in arithmetic progression, and their properties are somewhat related to the properties of $\zeta(s)$. In order to have a better understanding of Dirichlet L-functions, this chapter will provide definitions and properties of Dirichlet characters, primitive characters, and conclude with the proof for the functional equation. Most of the material in this chapter comes from Apostol[1] and Davenport[2].

Dirichlet's Character

Definition 5.1 (Dirichlet Characters)

Let G be the group of reduced residue classes $\bmod q$ and q be a fixed positive integer. Let $\chi' : (\mathbb{Z}/q\mathbb{Z}) \to \mathbb{C}^*$ be a homomorphism. Corresponding to each character χ' of G, we define an arithmetical function $\chi : \mathbb{Z} \to \mathbb{C}^*$ as follow:

$$\chi(n) = \begin{cases} \chi'(n+q\mathbb{Z}) & \text{if } (n,q)=1. \\ 0 & \text{if } (n,q)>1. \end{cases} \tag{5.1}$$

Moreover, we denote the Principal Dirichlet character $\bmod q$ to be

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n,q)=1. \\ 0 & \text{if } (n,q)>1. \end{cases}$$

The functions χ with the above definition are known as Dirichlet characters.

We can clearly see from our Definition 5.1 that Dirichlet characters are completely multiplicative, that is; $\chi(nm) = \chi(n)\chi(m)$ for all m and n since χ' is a homomorphism map.

Lemma 5.1 There exists $\varphi(q)$ distinct characters $\bmod q$, where $\varphi(q)$ is the Euler φ-function which gives the number of positive integers less than or equal to q and relatively prime to q.

Proof: The proof follows from the fact that $\varphi(q)$ forms a set of reduced residue system $\bmod q$. This implies that there exists $\varphi(q)$ characters χ' for the group G of reduced residue classes $\bmod q$. Thus, there exists $\varphi(q)$ characters χ $\bmod q$.

Lemma 5.2 and Lemma 5.3 follow from Lemma 5.1.

Lemma 5.2 Let $\chi_1, \chi_2, \ldots, \chi_{\varphi(q)}$ be the $\varphi(q)$ Dirichlet characters $\bmod q$. Then, the first sum is given by,

$$\sum_{n \bmod q} \chi(n) = \begin{cases} \varphi(q) & \text{if } \chi = \chi_0 \\ 0 & \text{otherwise;} \end{cases}$$

where the sum is over any representative set of residues $\bmod q$. The second sum is of the form,

$$\sum_{\chi \bmod q} \chi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv 1 (\bmod q) \\ 0 & \text{otherwise;} \end{cases}$$

where the sum is over all the $\varphi(q)$ characters.

Lemma 5.3 Let $\chi_1, \chi_2, \ldots, \chi_{\varphi(q)}$ be the $\varphi(q)$ Dirichlet characters $\bmod q$, and let $a, n \in \mathbb{Z}$ with $(a,q) = 1$. If $\overline{\chi}$ is the conjugate character to χ, then

$$\sum_{\chi \bmod q} \overline{\chi}(a)\chi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv a (\bmod q) \\ 0 & \text{otherwise.} \end{cases}$$

This sum represents the orthogonality relation of the characters. We note that $\overline{\chi}(a)\chi(n) = \chi(n')$ where $n' \equiv 1(\bmod q)$ if and only if $n \equiv a(\bmod q)$.

Note that if the order of a group G is n, $|G| = n$ and for $q \in G$ then $q^n = 1$ (1 is the identity of G). Similarly, let $\chi \in \hat{G}$ where $\hat{G} = \{\chi \mid \chi : G \to \mathbb{C}^*\}$ forms a group of order $\varphi(q)$, then we get $(\chi(n))^{\varphi(q)} = 1 = \chi_0$. So the values of χ are precisely the $\varphi(q)^{\text{th}}$ roots of unity. Moreover, we denote $\overline{\chi}$ to be the complex conjugate of χ.

We now provide the following examples to show the different Dirichlet characters $\bmod q$ for each value $q \in \{1,2,3,4,5,6,7,8\}$. For $q = 1$ and $q = 2$, $\varphi(q) = 1$, there exists only the principal character χ_0. For $q = 3$, $\varphi(q) = 2$, and $q = 4$, $\varphi(q) = 2$, we obtain two Dirichlet characters, χ_0, χ_1 for each modulus

n	1	2	3
$\chi_0(n)$	1	1	0
$\chi_1(n)$	1	-1	0

n	1	2	3	4
$\chi_0(n)$	1	0	1	0
$\chi_1(n)$	1	0	-1	0

For $q = 5$, $\varphi(q) = 4$, we obtain four Dirichlet characters,

$\chi_0, \chi_1, \chi_2, \chi_3$.

n	1	2	3	4	5
$\chi_0(n)$	1	1	1	1	0
$\chi_1(n)$	1	-1	-1	1	0
$\chi_2(n)$	1	i	$-i$	-1	0
$\chi_3(n)$	1	$-i$	i	-1	0

For $q = 6$, $\varphi(q) = 2$, we get two Dirichlet characters, χ_0, χ_1.

n	1	2	3	4	5	6
$\chi_0(n)$	1	0	0	0	1	0
$\chi_1(n)$	1	0	0	0	-1	0

With $q = 7$, $\varphi(q) = 6$ we obtain 6 Dirichlet characters,

$\chi_0, \chi_1, \chi_2, \chi_3\ \chi_4, \chi_5$.

n	1	2	3	4	5	6	7
$\chi_0(n)$	1	1	1	1	1	1	0
$\chi_1(n)$	1	1	-1	1	-1	-1	0
$\chi_2(n)$	1	ω^2	ω	$-\omega$	$-\omega^2$	-1	0
$\chi_3(n)$	1	ω^2	$-\omega$	$-\omega$	ω^2	1	0
$\chi_4(n)$	1	$-\omega$	ω^2	ω^2	$-\omega$	1	0
$\chi_5(n)$	1	$-\omega$	$-\omega^2$	ω^2	ω	-1	0

Here $\omega = e^{\frac{i\pi}{3}}$. When $q = 8$, $\varphi(q) = 4$, we obtain four Dirichlet characters, $\chi_0, \chi_1, \chi_2, \chi_3$.

n	1	2	3	4	5	6	7	8
$\chi_0(n)$	1	0	1	0	1	0	1	0
$\chi_1(n)$	1	0	1	0	-1	0	-1	0
$\chi_2(n)$	1	0	-1	0	1	0	-1	0
$\chi_3(n)$	1	0	-1	0	-1	0	1	0

However, the functional equation for $L(s,\chi)$ will be valid only for primitive characters, so we will define primitive characters.

Definition 5.2 Let $\chi(n)$ be any character $\bmod q$ other than the principal character. The character $\chi(n)$ is said to be primitive $\bmod q$ if it has no induced modulus $q_1 < q$. That is, $\chi(n)$ is primitive $\bmod q$ if and only if for every q_1 that divides q, $0 < q_1 < q$, there exists an integer $a \equiv 1 \bmod (q_1)$,

$(a,q)=1$ such that $\chi(a)\neq 1$.

We state Lemma 5.4 without proving it.

Lemma 5.4 Every non-principal character $\chi \bmod p$ where p is prime is a primitive character $\bmod p$.

Here is an example of a character that is not primitive.

Example: Let χ be the character $\bmod 9$ given by

n	1	2	3	4	5	6	7	8	9
$\chi(n)$	1	-1	0	1	-1	0	1	-1	0

This particular character is not primitive for the following reason: By the above definition, q_1 that divides 9 such that $0<q_1<9$ are 1 and 3. The principal character, χ_0 is not primitive because 1 is an induced modulus. Now, $q_1=3$ is an induced modulus for $\bmod 9$ so let $a=4\equiv 1(\bmod 3)$, $(4,9)=1$ but $\chi(4)=1$. Thus it fails our definition 5.2 so $\chi \bmod 9$ is not primitive.

Dirichlet L-Functions

After the previous lemmas, we are in the position to officially state the definition of Dirichlet's L-functions. In this section, we are going to prove their convergence and hence the Euler product formula. The proof follows a similar argument to the case of the Riemann zeta function.

Definition 5.3 Let χ be a Dirichlet character mod q. We define the function $L(s,\chi)$ as

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \quad \text{for } s\in\mathbb{C},\ \operatorname{Re}(s)=\sigma>1. \tag{5.2}$$

Theorem 5.1 (Davenport[2]) For $\sigma>1$,

i) $L(s,\chi)$ converges absolutely.

ii) $L(s,\chi)=\prod_{p}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}$.

Proof: To prove absolute convergence, let $\operatorname{Re}(s)=1+\varepsilon$, $\varepsilon>0$; then

$$\left|\sum_{n=N+1}^{M}\frac{\chi(n)}{n^s}\right|\le\sum_{n=N+1}^{M}\left|\frac{\chi(n)}{n^s}\right|\le\sum_{n=N+1}^{M}\frac{1}{n^\sigma}$$
$$\le(N+1)^{-(1+\varepsilon)}\int_{N+1}^{M}\frac{1}{x^{1+\varepsilon}}\,dx<(N+1)^{-(1+\varepsilon)}+\varepsilon^{-1}(N+1)^{-\varepsilon}.$$

We can take ε to be as small as we desire, and by Cauchy's criterion of convergence, $\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ converges absolutely for $\operatorname{Re}(s)>1$ and uniformly for $\operatorname{Re}(s)>1+\varepsilon$. In Part ii), the argument is the same as in the case for the Riemann zeta function by using part i) and taking the fact that χ is completely multiplicative. Thus, $L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$.

□

Up to this moment, we have not acquired sufficient tools to prove the functional equation for $L(s,\chi)$, so in the following section we start by providing the tools that will be needed.

Functional Equation of $L(s,\chi)$

In this section, we are going to prove the functional equation for $L(s,\chi)$ which is valid only for primitive characters. Once we have understood the Gauss summation in relation to the characters, the proof follows similarly as the proof for the functional equation for the $\zeta(s)$.

Definition 5.4 Let $\chi(n)$ be any Dirichlet character mod q, we define the Gauss sum related with χ as follow;

$$\tau(\chi) = \chi(1)e_q(1) + \chi(2)e_q(2) + \dots\chi(m)e_q(m)$$

$$= \sum_{m=1}^{q} \chi(m)e_q(m) . \tag{5.3}$$

This is a linear combination of imaginary exponentials, where $e_q(m)$ is defined by $e_q(m) = e^{2\pi i m/q}$.

Lemma 5.5 For any primitive character, $\chi(n)$,

$$\chi(n)\tau(\overline{\chi}) = \sum_{h=1}^{q} \overline{\chi}(h) e_q(nh) . \qquad (5.4)$$

Proof: If $(n,q)=1$, by (5.3), we multiply $\chi(n)$ by $\tau(\overline{\chi})$ which gives us $\chi(n)\tau(\overline{\chi}) = \sum_{m=1}^{q} \overline{\chi}(m)\chi(n)e_q(m)$. Let $m \equiv nh (\bmod q)$, and we note that $\chi(n)\overline{\chi}(n)=1$. This implies that $\chi(n)\tau(\overline{\chi}) =$

$$\chi(n)\tau(\overline{\chi}) = \sum_{m=1}^{q} \overline{\chi}(nh)\chi(n)e_q(nh) = \sum_{h=1}^{q} \overline{\chi}(n)\chi(n)\overline{\chi}(h)e_q(nh) = \sum_{h=1}^{q} \overline{\chi}(h)e_q(nh) .$$

Thus, we get the desired result. It is not difficult to show that it holds whether $(n,q)=1$ or $(n,q)>1$.

□

Now, (5.4) can be rewritten as

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{m=1}^{q} \overline{\chi}(m) e_q(mn) \qquad (5.5)$$

by replacing h for m, and with $(n,q)=1$ and assuming $\tau(\chi) \neq 0$.

Another result we would like to prove is the following proposition.

Proposition 5.1 Let $\chi(n)$ be a primitive character $\bmod q$, then

$$|\tau(\chi)| = \sqrt{q} \qquad (5.6)$$

Proof: To prove this, we use (5.3) to get the following expression, $|\chi(n)|^2|\tau(\chi)|^2 = \sum_{h_1=1}^{q}\sum_{h_2=1}^{q}\chi(h_1)\chi(h_2)e_q(n(h_1-h_2))$. We note that each character has $|\chi(n)|=1$ and there exist $\varphi(q)$ in a complete residue class $\bmod q$ which shows that $|\chi(n)|^2=\varphi(q)$. Furthermore, the sum for $e_q n(h_1-h_2)$ is zero unless $h_1 \equiv h_2$. Thus, we get $\varphi(q)|\tau(\chi)|^2 = q\varphi(q)$ which gives (5.6).

□

We now state our goal for this chapter as a theorem, which will require the previous tools to prove it.

Theorem 5.2 The Functional Equation for an L-function: Let $\chi(n)$ be a primitive character $\bmod q$. If

$$\xi(s,\chi) = \left(\frac{\pi}{q}\right)^{-\frac{1}{2}(s+a)}\Pi\left[\frac{1}{2}(s+a-1)\right]L(s,\chi) \text{ where } a=\begin{cases}0 & \text{if } \chi(-1)=1\\ 1 & \text{if } \chi(-1)=-1\end{cases}$$

then $\xi(1-s,\overline{\chi}) = \dfrac{i^a q^{\frac{1}{2}}}{\tau(\chi)}\xi(s,\chi)$. Here, $\tau(\chi)$ is defined as in (5.3)

We note that $\chi(-1)=\pm 1$ and consider the two cases.

CASE 1: $\chi(-1)=1$.

The technique here is similar as the one used in proving the functional equation for the Riemann zeta function, $\zeta(s)$. We recall the result (2.16) from Chapter two, namely,

$\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)n^{-s} = \int_0^\infty e^{-n^2\pi x}x^{\frac{s}{2}-1}\,dx$. Let $x=\frac{t}{q}$ then $dx=\frac{dt}{q}$, and substitute into the equation then,

$$\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)n^{-s} = \int_0^\infty e^{-n^2\pi\frac{t}{q}}\left(\frac{t}{q}\right)^{\frac{s}{2}-1}\frac{dt}{q} = \int_0^\infty e^{-n^2\pi\frac{t}{q}}t^{\frac{s}{2}-1}q^{-\frac{s}{2}}q\frac{dt}{q} =$$

$$q^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)n^{-s} = \int_0^\infty e^{-n^2\pi\frac{t}{q}}t^{\frac{s}{2}-1}dt\ .$$

Replacing t by x we get

$$q^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)n^{-s} = \int_0^\infty e^{-n^2\pi\frac{x}{q}}x^{\frac{s}{2}-1}dx\ . \tag{5.7}$$

We multiply both sides of (5.7) by $\chi(n)$ and sum over n to get

$$q^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)L(s,\chi) = \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n=1}^{\infty}\chi(n)e^{-n^2\pi\frac{x}{q}}\right)dx$$ which holds for $\sigma > 1$.

We note that $\chi(-1)=1$ tells us that it is an even function.

Since $\psi(x,\chi) = \sum_{-\infty}^{\infty}\chi(n)e^{-n^2\pi x/q}$ so $\frac{1}{2}\psi(x,\chi) = \sum_{n=1}^{\infty}\chi(n)e^{-n^2\pi x/q}$, we obtain

$$q^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)L(s,\chi) = \frac{1}{2}\int_0^\infty x^{\frac{s}{2}-1}\psi(x,\chi)\,dx\ . \tag{5.8}$$

Lemma 5.6

$$\tau(\overline{\chi})\psi(x,\chi) = (q/x)\psi(x^{-1},\overline{\chi}) \tag{5.9}$$

Proof: This follows from the three elements;

$\psi(x,\chi) = \sum_{-\infty}^{\infty}\chi(n)e^{-n^2\pi x/q}$, $\chi(n) = \frac{1}{\tau(\overline{\chi})}\sum_{m=1}^{q}\overline{\chi}(m)e_q(mn)$ and the general form of the Theta function,

$$\sum_{-\infty}^{\infty} e^{-(n+\alpha)^2\pi/x} = x^{\frac{1}{2}}\sum_{-\infty}^{\infty} e^{-n^2\pi x+2\pi i n\alpha} \tag{5.10}$$

(As in Chapter two. the proof of (5.10) follows from Poisson Summation formula. Moreover, note that when $\alpha = 0$ in (5.8), we get $\psi(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x} = x^{-\frac{1}{2}}\psi(x^{-1})$ which was used in proving the functional equation for the zeta-function).

□

We can express (5.9) as $\psi(x,\chi) = \frac{(q/x)}{\tau(\overline{\chi})}\psi(x^{-1},\overline{\chi})$ and by $\frac{1}{2}\psi(x,\chi) = \sum_{n=1}^{\infty}\chi(n)e^{-n^2\pi x/q}$, we obtain $\frac{1}{2}\psi(x,\chi) + \frac{1}{2}\frac{(q/x)}{\tau(\overline{\chi})}\psi(x^{-1},\overline{\chi}) = \sum_{-\infty}^{\infty}\chi(n)e^{-n^2\pi x/q}$. This allows us to split the integral in (5.8) as

$$q^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)L(s,\chi) = \frac{1}{2}\int_1^{\infty} x^{\frac{s}{2}-1}\psi(x,\chi)\,dx + \frac{1}{2}\int_1^{\infty} x^{-\frac{s}{2}-\frac{1}{2}}\psi\left(x^{-1},\overline{\chi}\right)dx$$

$$= \frac{1}{2}\int_1^{\infty} x^{\frac{s}{2}-1}\psi(x,\chi)\,dx + \frac{1}{2}\frac{q^{\frac{1}{2}}}{\tau(\overline{\chi})}\int_1^{\infty} x^{-\frac{s}{2}-\frac{1}{2}}\psi(x,\overline{\chi})\,dx\,.$$

The second line is obtained by expressing $\psi(x^{-1},\overline{\chi})$ in terms of $\psi(x,\chi)$. Furthermore, the second expression remains unchanged when s and χ are replaced by $1-s$ and $\overline{\chi}$, respectively. The term $\Pi(\frac{s}{2}-1)$ is never zero, which tells us that the last expression is regular. We achieve the functional equation which is the analytic continuation for $L(s,\chi)$, namely;

$$q^{\frac{1}{2}(1-s)}\pi^{-\frac{1}{2}(1-s)}\Pi\left[\frac{1}{2}(-s-1)\right]L(1-s,\chi)=\frac{q^{\frac{1}{2}}}{\tau(\chi)}\pi^{-\frac{s}{2}}\Pi(\frac{s}{2}-1)L(s,\chi)\,. \quad (5.11)$$

□

We now consider.

CASE 2: $\chi(-1)=-1$.

Proof:

Since $\chi(-1)=-1$ implies that χ is an odd function, in which case, $\psi(x,\chi)=\sum_{-\infty}^{\infty}\chi(n)e^{-n^2\pi x/q}$ vanishes. The main technique here is to replace $\frac{s}{2}-1$ by $\frac{1}{2}(s+1)$ in (5.7). Once the replacement is done, the argument is similar to the above, so we are just going to mention important steps. By replacing $\frac{s}{2}-1$ by $\frac{1}{2}(s+1)$, we get;

$$q^{\frac{1}{2}(s+1)}\pi^{-\frac{1}{2}(s+1)}\Pi(\frac{s}{2})n^{-s}=\int_0^{\infty}ne^{-n^2\pi\frac{x}{q}}x^{\frac{s}{2}-\frac{1}{2}}dx\,. \quad (5.12)$$

□

In a similar way as in case 1), we multiply both sides of (5.12) by $\chi(n)$, sum over n and let $\psi_1(x,\chi)=\sum_{-\infty}^{\infty} n\chi(n)e^{-n^2\pi\frac{x}{q}}$ to obtain the following expression,

$$q^{\frac{1}{2}(s+1)}\pi^{-\frac{1}{2}(s+1)}\Pi(\frac{s}{2})L(s,\chi)=\int_0^\infty \psi_1(x,\chi)x^{\frac{s}{2}-\frac{1}{2}}dx\ . \tag{5.13}$$

Lemma 5.7

$$\tau(\overline{\chi})\psi_1(x,\chi)=i(q/x)^{3/2}\psi_1(x^{-1},\overline{\chi}) \tag{5.14}$$

Proof: The important technique here is to take the derivative of the generalized Theta function (5.10), $y^{-\frac{1}{2}}\sum_{-\infty}^{\infty} e^{-(n+\alpha)^2\pi/y}=\sum_{-\infty}^{\infty} e^{-n^2\pi y+2\pi in\alpha}$ with respect to α and note that the derivatives of its series converge uniformly. We use $\psi_1(x,\chi)$ and $\tau(\overline{\chi})$ as in Lemma 5.6 and it will not be too difficult to see that we can get $\tau(\overline{\chi})\psi_1(x,\chi)=i(q/x)^{3/2}\psi_1(x^{-1},\overline{\chi})$.

□

Similarly as in case 1), the integral in (5.13) can be separated as

$$q^{\frac{1}{2}(s+1)}\pi^{-\frac{1}{2}(s+1)}\Pi(\frac{s}{2})L(s,\chi)=\frac{1}{2}\int_1^\infty \psi_1(x,\chi)x^{\frac{s}{2}-\frac{1}{2}}dx+\frac{1}{2}\int_1^\infty \psi_1(x^{-1},\overline{\chi})x^{\frac{s}{2}-\frac{1}{2}}dx$$

$$=\frac{1}{2}\int_1^\infty \psi_1(x,\chi)x^{\frac{s}{2}-\frac{1}{2}}dx+\frac{1}{2}\frac{iq^{\frac{1}{2}}}{\tau(\overline{\chi})}\int_1^\infty \psi_1(x,\overline{\chi})x^{-\frac{s}{2}}dx$$

Thus, the last expression remains unchanged with the substitution of s for $1-s$. So, we obtain the other functional equation for the case $\chi(-1)=-1$, that is;

$$q^{\frac{1}{2}(2-s)}\pi^{-\frac{1}{2}(2-s)}\Pi(\frac{1}{2}(1-s))L(1-s,\overline{\chi})=\frac{iq^{\frac{1}{2}}}{\tau(\overline{\chi})}q^{\frac{1}{2}(s+1)}\pi^{-\frac{1}{2}(s+1)}\Pi(\frac{s}{2})L(s,\chi)\,. \qquad (5.15)$$

We recall the functional equation for the case 1), (5.11)

$$q^{\frac{1}{2}(1-s)}\pi^{-\frac{1}{2}(1-s)}\Pi\left[\frac{1}{2}(-s-1)\right]L(1-s,\chi)=\frac{q^{\frac{1}{2}}}{\tau(\chi)}\pi^{-\frac{s}{2}}q^{\frac{s}{2}}\Pi(\frac{s}{2}-1)L(s,\chi)\,.$$

So, we join both (5.11) and (5.15) by $a=\begin{cases}0 & \text{if } \chi(-1)=1\\ 1 & \text{if } \chi(-1)=-1\end{cases}$. As a result, the functional equation for Dirichlet L-function is obtained; namely, if $\xi(s,\chi)=(\frac{\pi}{q})^{-\frac{1}{2}(s+a)}\Pi\left[\frac{1}{2}s+a-1\right]L(s,\chi)$ then

$$\xi(1-s,\overline{\chi})=\frac{i^a q^{\frac{1}{2}}}{\tau(\chi)}\xi(s,\chi)\,.$$

□

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